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SOME REMARKS ON AXIOMATIZING LOGICAL CONSEQUENCE OPERATIONS*

Abstract. In this paper we investigate the relation between the axiomatization of a given logical consequence operation and axiom systems defining the class of algebras related to that consequence operation. We show examples which prove that, in general there are no natural relation between both ways of axiomatization.

1. Introduction

Most of the results presented in this paper are rather natural consequences of strong algebraic results achieved in the last few decades. The aim of this paper is to interpret some recent algebraic results and illustrate them by the unexpected properties of logical consequences determined by means of ortholattices.

The aim of this paper is to analyze the relation between two different notions of the axiomatizability: L-axiomatizability and A-axiomatizability. We will give precise definitions of these notions in the next section. Let's start however with the most important case - classical sentential logic.

It is well known that the classical logic can be axiomatized by means of the following Hilbert style proofs: P is provable from X ($P \in \mathsf{C}(X)$) iff there are $P_1, \ldots, P_n$ such that $P_n = P$, for $1 \leqslant i \leqslant n-1$, P_i is a substitution of axiom, or $P_i \in X$, or there are $j, k \in \{1, \ldots, i\}$ such that $P_k = \neg P_j \vee P_i$.

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Axioms:

(L1) $p \to (q \to p)$,

(L2) $((p \to (q \to r)) \to ((p \to q) \to (p \to r))$,

(L3) $(p \wedge q) \to p$,

(L4) $(p \wedge q) \to q$,

(L5) $(r \to p) \to ((r \to q) \to (r \to (p \wedge q)))$,

(L6) $p \to (p \vee q)$,

(L7) $q \to (p \vee q)$,

(L8) $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$,

(L9) $(p \to \neg q) \to (q \to \neg p)$,

(L10) $\neg(p \to p) \to q$,

(L11) $\neg\neg p \to p$

The axioms (L1)–(L11) determine the classical sentential logic. We will call them L-axioms.

It is also well known that the classical logic can be semantically defined by means of the class of Boolean algebras. The classical sentential logic as well as the terms of Boolean algebras are expressed in the same language. Any sentence P of the classical sentential logic correspond in the natural way to the identity $P = 1$.

A very natural question arises: Given a logical consequence operation C determined in the way presented above by means of the set T of *L-axioms.* Let $T_1 = \{P = 1 : P \in T\}$. The set T_1 determines some class $\boldsymbol{K}$ (a variety) of algebras. We will say that T_1 form an *A-axiom system* for $\boldsymbol{K}$. The class $\boldsymbol{K}$ determines semantically logical consequence operation $\mathsf{Cn}_{\boldsymbol{K}}$ in the following well known way. Thus if we consider any algebras A from $\boldsymbol{K}$ as a logical matrix $\langle A, \{1\}\rangle$, where 1 is the unit element of A, then the logical consequence operation:

> $P \in \mathsf{C}(X)$ if and only if for any matrix of the from $\langle A, \{1\}\rangle$, where A is a Boolean algebra and any valuation v if for any sentences Q from X, $v(Q) = 1$, then $v(P) = 1$.

medskip is a structural consequence operation.

One might have expected that C is equal to $\mathsf{Cn}_{\boldsymbol{K}}$. The aim of this paper is to show that such an expectation is mistaken. C is different from $\mathsf{Cn}_{\boldsymbol{K}}$ even if C is the classical sentential logic.

Let Tc denote the set of L-axioms (L1)–(L11). They correspond in the sense presented above to the following set Tc_1 of A-axioms:

(A1) $x \to (y \to x) = 1,$

(A2) $((x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1,$

(A3) $(x \wedge y) \to x = 1,$

(A4) $(x \wedge y) \to y = 1,$

(A5) $(z \to x) \to ((z \to y) \to (z \to (x \wedge y))) = 1,$

(A6) $x \to (x \vee y) = 1,$

(A7) $y \to (x \vee y) = 1,$

(A8) $(x \to z) \to ((y \to z) \to ((x \vee y) \to z)) = 1,$

(A9) $(x \to \neg y) \to (y \to \neg x) = 1,$

(A10) $\neg(x \to x) \to y = 1,$

(A11) $\neg\neg x \to x = 1$

We will show that algebras satisfying the identities from Tc_1 don't need to be Boolean algebras.

2. Logics versus algebras

By S we mean a sentential langauge generated by the set $\mathrm{Var} = \{p_1, \ldots, p_n\}$ of sentential variables and finite set of sentential connectives $\{F_1, \ldots, F_n\}$. A sentential language is defined as absolutely free algebra. Thus a sentential language (as an absolutely free algebra) is an abstract algebra with the following property: For any algebra A, any function $f\colon \mathrm{Var} \longrightarrow A$ can be extended to homomorphism $g\colon S \longrightarrow A$.

All the examples of logical consequence operation considered in this paper are formulated in the language S_0 of classical sentential logic. Thus, the language S_0 has the connectives $\wedge$, $\vee$, and $\neg$. Those connectives will not, in general, satisfy the properties of the classical connectives of conjunction, disjunction and negation. Although the connective of implication $\to$ plays an important role, we don't consider it as a primitive connective of the language S and will just consider a sentence $P \to Q$ as shorthand for $\neg P \vee Q$.

The definition of the language as a free algebra has the advantage that there are no structural differences between a sentence of S, for example

$P \wedge \neg P$ and an element $a \wedge a'$ of a given algebra. Although usually for sentences the symbol $\neg P$ is used and for algebras we usually use the symbol a', we understand that both of them have the same structure and mean a kind of complementation of a given sentence or of an element of a given algebra. In just this sense we are enabled to say about a given object P that it correspond to a sentence of S or to an element of given algebra.

Both the language S as well as any algebra with the same operations as S are the objects of the same type. For this very reason any sentence $P \in S$ can be considered (to be precise: have an natural counterpart in) some element of a given algebra. Given a sentence γ of S. Then by γ_A we denote the realization of γ in the algebra A. If the sentential variables p_i and p_j occurs in γ and a and b are the elements of some algebra A then $\gamma(a, b)$ denotes the realization of γ in A.

We will often say that K is a set of sentences of the language S and at the same time that K is the set of elements of given algebra without giving more explanation.

The logical notion of substitution corresponds to the algebraic notion of homomorphism of the language S into itself. The logical notion of valuation corresponds to the notion of homomorphism of the language S into a given algebra. This allows us to use algebraic techniques and results for purely logical investigations.

By a *logic* we mean a structural consequence operation, i.e. a function:

$$C\colon S \supseteq X \longmapsto C(X) \subseteq S,$$

satisfying the conditions: $X \subseteq C(X)$, if $X \subseteq Y$ then $C(X) \subseteq C(Y)$, $CC(X) = C(X)$, and closed with respect to substitutions, i.e. such that if $P \in C(X)$ then any substitution of P belong to $C(Y)$, where Y is a set all respective substitutions of sentences from the set X.

It is easy to check that any operation defined by means of Hilbert style proofs (also with any other axiom system than the one above) is a logic in the sense of the definition above.

By a *logical matrix* we mean a pair $M = \langle A, D\rangle$ where A is an abstract algebra and D is a subset of A—set of designated elements of M. By a *valuations* of S a matrix M we mean any homomorphism of the language S into the algebra A of the matrix M.

Every class of matrices $\boldsymbol{K}$ determines a function:

$$\mathsf{Cn}_{\boldsymbol{K}}\colon S \supseteq X \longmapsto \mathsf{Cn}_{\boldsymbol{K}}(X) \subseteq S$$

defined in the following way: $P \in \mathsf{Cn}_{\boldsymbol{K}}(X)$ if and only if for every matrix $M = \langle A, D\rangle \in \boldsymbol{K}$ and every valuation v, if $v(X) \subseteq D$, then $v(P) \in D$.

It is easy to check that $\mathsf{Cn}_{\boldsymbol{K}}$ is a logic i.e., a structural consequence operation.

We use the symbol Cn for denoting a logic defined semantically, by means of some class of matrices while C (with indices if needed) is reserved for syntactic definitions (like Hilbert-style proofs) or for general considerations concerning logic as a structural consequence operation.

We say that a logic C is *strongly complete* relative to a semantics $\boldsymbol{K}$ for S, if $C = \mathsf{Cn}_{\boldsymbol{K}}$. A semantics $\boldsymbol{K}$ is *strongly adequate* for a logic C if C is strongly complete relative to $\boldsymbol{K}$.

Let C be a logic. For any $X \subseteq S$ a matrix:

$$\mathrm{L}_X = \langle S, C(X)\rangle$$

is called a *Lindenbaum matrix* for C. A class of matrices:

$$\mathbf{L}_C = \{\mathrm{L}_X : X \subseteq S\}$$

will be called a *Lindenbaum bundle* for C. One can show that any logic is strongly complete relative to the Lindenbaum bundle $\mathbf{L}_C$. To express it in more general way any logic is strongly complete with respect to some class of matrices. Given a logic C let $\mathrm{Mod}(C)$ denote the following set of matrices

$$\mathrm{Mod}(C) = \{M = \langle A, D\rangle : \forall_{X \subseteq S}\, C(X) \subseteq C_M(X)\}.$$

One can prove that C is strongly complete with respect to $\mathrm{Mod}(C)$. For full references and a more detailed analysis of the theory of logical consequence we refer the reader to the monograph (Wójcicki 1987).

Given a set of sentences Z of the langauge S_0, by C_Z we mean the following consequence operation:

> $P \in \mathsf{C}_Z(X)$ iff there are sentences $P_1, \ldots, P_n$ such that $P_n = P$, for $1 \leqslant i \leqslant n-1$, P_i is a substitution of some sentence from Z, or $P_i \in X$, or there are $j, k \in \{1, \ldots, i\}$ such that $P_k = \neg P_j \vee P_i$.

We will say that a given structural consequence operation C is *L-axiomatizable* if and only if there exists a set of sentences Z such that $C = \mathsf{C}_Z$. The set Z will be called then *L-axiom system for C*.

Given a set $T = \{t_i = u_i : i \in I\}$ where t_i and u_i are the sentences of the language S_0. Let $\boldsymbol{K}$ denotes the class of all algebras satisfying for $i \in I$ the condition $t_i = u_i$. We will then call T an *A-axiom system* for $\boldsymbol{K}$. We will then also call $\boldsymbol{K}$ an *A-axiomatizable class of algebras*.

It is clear that not every class of algebras is A-axiomatizable. However, given a class $\boldsymbol{K}$ of algebras and the set $T = \{t_i = u_i : i \in I\}$ of identities which are satisfied in all the algebras from $\boldsymbol{K}$, then there exists the class of algebras $\mathrm{V}(\boldsymbol{K})$ which is A-axiomatizable by means of the set T. Using algebraic terminology, $\mathrm{V}(\boldsymbol{K})$ is the least variety containing the class $\boldsymbol{K}$.

A technical notion important for this paper is the notion of matrix congruence. Given a logical matrix, we can paste some of its elements and then consider pasting classes as some independent objects - elements of some new logical matrix. The notion of matrix congruence and its properties tell us under what circumstances such a procedure leads us to—a logical matrix defining the same logical consequence.

Given a logical matrix $M = \langle A, D\rangle$, a binary relation $\equiv$ on A will be called a *matrix congruence* on M if and only if it is a congruence on A which doesn't paste together designated elements with non-designated ones. More precisely: $\equiv$ is an equivalence relation on A which agrees with the operations on A and moreover $a \equiv b$ and $a \in D$ if and only if $b \in D$. For a given logical matrix $M = \langle A, D\rangle$, and matrix congruence $\equiv$ on A we can define the new *factor matrix* $M_\equiv$. Its algebra and its set of designated elements consist of classes of equivalence of the elements of A with respect to $\equiv$. Thus, $M_\equiv = \langle A_\equiv, D_\equiv\rangle$.

Let $M = \langle A, D\rangle$ be a matrix for S, v being a valuation S in M. For every $p \in \mathrm{Var}(S)$ and every $a \in A$ we shall define on the set $\mathrm{Var}(S)$ a valuation $v(a/p)$ of the language S into a matrix M as follows:

$$v(a/p)(q) = \begin{cases} v(q) & \text{when } p \neq q, \\ a & \text{when } p = q, \end{cases}$$

where q ranges over the set of all sentential variables of the language S.

The following theorem gives the main properties of the matrix congruences:

Theorem 2.1. (a) *Given a matrix $M = \langle A, D\rangle$. Let $\equiv$ be a matrix congruence on M. Then $C_M = C_{M_\equiv}$.*

(b) (Shoesmith & Smiley 1978) *Let $M = \langle A, D\rangle$ be a matrix for S, θ_M the binary relation on A defined as follows: $a\, \theta_M\, b$ if and only if for every $\phi \in S$, every variable $p \in \mathrm{Var}(\phi)$ and every valuation $v \in \mathrm{Hom}(S, A)$ the following condition is fulfilled:*

$$v(a/p)(\phi) \in D \quad \textit{iff} \quad v(b/p)(\phi) \in D\,.$$

Then $\theta_M \in \mathrm{Con}(M)$. Moreover, θ_M is the greatest matrix congruence on A (in the sense of the relation of inclusion).

(c) *Let $\langle S, C\rangle$ be a logic. Then for every $X \subseteq S$ the greatest matrix congruence $\Omega_S X$ on a Lindenbaum matrix $\mathrm{L}_X = \langle S, C(X)\rangle$ is defined as follows: for $\alpha, \beta \in S$ $\alpha\,\theta_X\,\beta$ if and only if for every $\phi \in S$, every variable $p \in \mathrm{Var}(\phi)$ we have $\phi(\alpha/p) \in C(X)$ if and only if $\phi(\beta/p) \in C(X)$.*

(d) *For every matrix $M = \langle A, D\rangle$ the set $\langle \mathrm{Con}(M), \subseteq\rangle$ partially ordered by the inclusion relation $\subseteq$ is a complete sublattice of a lattice $\langle \mathrm{Con}(A), \subseteq\rangle$. More precisely, $\mathrm{Con}(M) = \{\theta \in \mathrm{Con}(A) : \theta \subseteq \Omega_A D\}$.*

Proof. (a) Let k denote the canonical function: $A \ni a \longmapsto [a] \in A/\equiv$. By the definition of matrix congruence we have that $a \in D$ if and only if $k(a) = [a] \in D$. Moreover for any valuation $\overline{v}\colon S \longrightarrow A/\equiv$ there exists a valuation $v\colon S \longrightarrow A$ such that $\overline{v}(P) = k(v(P))$.

Suppose that $P \in C_M(X)$, then for any valuation $v\colon S \longrightarrow A$ such that $v(X) \subseteq D$ we have $v(P) \in D$. We will show that $P \in C_{M/\equiv}(X)$. Let $\overline{v}\colon S \longrightarrow A/\equiv$ denote any valuation such that $\overline{v}(X) \subseteq D/\equiv$. There exists a valuation $v\colon S \longrightarrow A$ such that $\overline{v}(P) = k(v(P))$. Obviously we have $v(X) \subseteq k^{-1}(\overline{v}(X)) \subseteq D$. From the assumption $v(P) \in D$ we have $\overline{v}(P) \in D/\equiv$.

Suppose that $P \in C_{M/\equiv}(X)$, then for any valuation $\overline{v}\colon S \longrightarrow A/\equiv$ such that $\overline{v}(X) \subseteq D/\equiv$ we have $\overline{v}(P) \in D/\equiv$. We will show that $P \in C_M(X)$. Let $v\colon S \longrightarrow A$ denote any valuation such that $v(X) \subseteq D$. We obviously have $\overline{v}(X) = k(v(X)) \subseteq D/\equiv$. From the assumption, $\overline{v}(P) \in D/\equiv$. As $a \in D$ iff $k(a) = [a] \in D$, hence $v(P) \in D$ □.

(b) First we shall show, that θ_M is a congruence on A. It is clear that θ_M is an equivalence relation. Let $\#_A$ be the interpretation of a m-ary connective $\#$ in A. Assume, that $a\,\theta_M\,b$. Let $e_1, \ldots, e_i, \ldots, e_m \in A$. Denote:

$$d_1 = \#_A(e_1, \ldots, e_{i-1}, a, e_{i+1}, \ldots, e_m),$$
$$d_2 = \#_A(e_1, \ldots, e_{i-1}, b, e_{i+1}, \ldots, e_m).$$

We shall show that $d_1\,\theta_M\,d_2$.

Let $\phi \in S$, $p \in \mathrm{Var}(\phi)$ and $v \in \mathrm{Hom}(S, A)$. We have to prove, that $v(d_1/p)(\phi) \in D$ if and only if $v(d_2/p)(\phi) \in D$. Assume, that $\mathrm{Var}(\phi) = \{p, q_1, \ldots, q_n\}$. Consider the formula $\alpha = \#(r_1, \ldots, r_{i-1}, p, r_{i+1}, \ldots, r_m)$, where $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_m$ are any sentential variables such that $\{r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_m\} \cap \mathrm{Var}(\phi) = \emptyset$. Let ϕ' be the sentence: $\phi(\alpha/p)$,

v' will then be a valuation into a matrix M such that:

$$v'(p) = \begin{cases} e_k & \text{when } p = r_k \text{ for a certain } k \in \{1, ..., i-1, i+1, ..., m\} \\ v(q_k) & \text{when } p = q_k \text{ for a certain } k \in \{1, ..., n\} \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Notice, that $v(d_1/p)(\phi) = v'(a/p)(\phi')$ and $v(d_2/p)(\phi) = v'(b/p)(\phi')$. Then $v(d_1/p)(\phi) \in D$ if and only if $v'(a/p)(\phi') \in D$ if and only if $v'(b/p)(\phi') \in D$ if and only if $v'(d_2/p)(\phi) \in D$. Thus $d_1 \, \theta_M \, d_2$.

Now let us assume that $c_1 \, \theta_M \, c_2$. Let $p \in \mathrm{Var}(S)$. Let $v \in \mathrm{Hom}(S, A)$. Then $c_1 \in D$ if and only if $v(c_1/p)(p) \in D$ if and only if $v(c_2/p)(p) \in D$ if and only if $c_2 \in D$. So we have shown $\theta_M \in \mathrm{Con}(M)$.

Let $\theta \in \mathrm{Con}(M)$. Assume, $a \, \theta \, b$ (where $a, b \in A$). We shall show that $a \, \theta_M \, b$. Let $\phi \in S$, $\mathrm{Var}(\phi) = \{p, q_1, \ldots, q_n\}$ and let $v \in \mathrm{Hom}(S, A)$. We denote: $d = v(a/p)(\phi)$ and $d‘ = v(b/p)(\phi)$. As $a \, \theta \, b$ and $\theta \in \mathrm{Con}(A)$ we have, that $d \, \theta \, d'$. Moreover, because of $\theta \in \mathrm{Con}(M)$, we have $v(a/p)(\phi) \in D$ if and only if $d \in D$ if and only if $d' \in D$ if and only if $v(b/p)(\phi) \in D$. Then $a \, \theta_M \, b$. So, $\theta \subseteq \theta_M$, which was to be proved.

(c) and (d) are immediate consequence of b). □

Corollary 2.2. *Let $\langle S, C \rangle$ be a logic. Then for every $X \subseteq S$ the greatest matrix congruence θ_X on a Lindenbaum matrix $\mathrm{L}_X = \langle S, C(X) \rangle$ is defined as follows: for $\alpha, \beta \in S$, $\alpha \, \theta_X \, \beta$ if and only if for every $\phi \in S$, every variable $p \in \mathrm{Var}(\phi)$ we have $\phi(\alpha/p) \in C(X)$ if and only if $\phi(\beta/p) \in C(X)$.* □

Corollary 2.3 (Porte 1965). *For every matrix $M = \langle A, D \rangle$ the set $\langle \mathrm{Con}(M), \subseteq \rangle$ partially ordered by the inclusion relation $\subseteq$ is a complete sublattice of a lattice $\langle \mathrm{Con}(A), \subseteq \rangle$. More precisely,* $\mathrm{Con}(M) = \{\theta \in \mathrm{Con}(A) : \theta \subseteq \theta_M\}$. □

For a matrix M we will use the symbol θ_M to denote the greatest matrix congruence on M. We shall call a matrix M *simple* if and only if θ_M is an identity relation. In a simple matrix the identity is the only matrix congruence. Let $\langle S, C \rangle$ be a logic. Then the class of all simple matrices from the class $\mathrm{Mod}(C)$ will be denoted by $\mathrm{Mod}^\star(C)$. It is a immediate consequence of a) that C is strongly complete with respect to $\mathrm{Mod}^\star(C)$.

Let $M = \langle A, D \rangle$ and $N = \langle B, E \rangle$ be similar matrices. The matrix M will be called a *submatrix* of the matrix N, in symbols $M \subseteq N$, if A is a subalgebra of the algebra B and $D = A \cap E$.

A logic C is called *equivalential (finitely equivalential)* if there exists a set (a finite set, resp.) $\mathrm{E}(p, q)$ of sentences of the language S, in which two variables occur and such that for any sentences P, Q, ϕ, any variable p occurring in ϕ and any sentences the following conditions hold:

(R) $\quad \mathrm{E}(P, P) \subseteq C(\emptyset)$

(MP) $\quad Q \in C(\mathrm{E}(P, Q), P)$

(RP) $\quad \mathrm{E}(\phi(p/P), \phi(p/Q)) \subseteq C(\mathrm{E}(P, Q))$.

The set $\mathrm{E}(\alpha, \beta)$ is called a *a system of equivalence sentences for* C, or C*–equivalence* for short.

The above definition is due to (Prucnal and Wroński 1974). Czelakowski 2001 is a monograph on the subject of equivalential logic.

Theorem 2.4. *Every equivalence* $\mathrm{E}(p, q)$ *for* C *satisfies the following conditions for any* P, Q, R, *and any* n*-ary connective* f *and any sentences* P_1, $\ldots$, P_n, Q_1, $\ldots$, Q_n

(S) $\quad \mathrm{E}(P, Q) \subseteq C(\mathrm{E}(Q, P))$,

(T) $\quad \mathrm{E}(P, R) \subseteq C(\mathrm{E}(P, Q), \mathrm{E}(Q, R))$,

(RP′) $\quad \mathrm{E}(f(P_1, \ldots, f(P_n), f(Q_1, \ldots, f(Q_n) \subseteq C(\mathrm{E}(P_1, Q_1), \mathrm{E}(P_n, Q_n)).$ □

Directly from the definition one can show that if C' is stronger than C than C-equivalence (if it exists) is also a C'-equivalence.

Theorem 2.5. (a) *Let* $\mathrm{E}(p, q)$ *be an equivalence for* C*. Then for any reduced* C*-matrix* $M = \langle A, D\rangle$ *the greatest matrix congruence* θM *on* M *is equal to the following relation* $\equiv_E(D)$*:*

$$a \equiv_E b \quad \textit{iff} \quad \delta(a, b) \in D \textit{ for all } \delta(p, q) \in \mathrm{E}(p, q).$$

(b) *If* C *is a equivalential logic then any submatrix of a matrix from* $\mathrm{Mod}^\star(C)$ *also belong to* $\mathrm{Mod}^\star(C)$.

Proof. (a) It is easy to show that $\equiv_E$ is a matrix congruence on $\langle A, D\rangle$. To finish the proof it is then enough to show that $\equiv_E$ is greatest or equal to the greatest matrix congruence θ_M on M.

Take any $a, b \in A$ such that $a\, \theta_M\, b$. According to Theorem 2.1b for every $\delta(p, q) \in \mathrm{E}(p, q)$ we have $\delta_M(a, b) \in D$ if and only if $\delta_M(b, b) \in D$. Since $b\, \theta_M\, b$, it follows that for any $\gamma \in E$ $\gamma_M(a, b) \in D$, i.e. $a \equiv_M b$.

(b) The formula $\mathrm{E}(a,b) \subseteq D$ will be used as a shorthand for the following formula:

$$\forall_{\delta(p,q)\in \mathrm{E}(p,q)}\ \delta(a,b) \in D$$

Thus for any matrix $\langle A, D\rangle$ we have:

$$a \equiv_E b \quad \text{iff} \quad \mathrm{E}(a,b) \subseteq D.$$

Let $M = \langle A, D\rangle$ be a submatrix of a matrix $N = (B, E)$ and $N \in \mathrm{Mod}^\star(C)$ then $a \equiv_D b$ iff $\mathrm{E}(a,b) \subseteq D$ iff $\mathrm{E}(a,b) \subseteq A \cap E$ iff $\mathrm{E}(a,b) \subseteq A$ and $\mathrm{E}(a,b) \subseteq E$ iff $(a,b) \in A \times A$ and $a \equiv_E b$. Since N is simple then $\equiv_E$ is the identity relation and then $a \equiv_D b$ is identity relation. Hence, M is simple. $\square$

Remark 2.6. The proof above is based on the following formula

$$\star \qquad \equiv_D = (A \times A) \cap \equiv_E,$$

where $\equiv$ determines the greatest congruence on any C-matrix for a given equivalential logic C. So, $\equiv_D$ and $\equiv_E$ are just the greatest matrix congruences respectively on M and N. Let us note however that if we replace $\equiv_D$ by the greatest congruence θ_M on M then the resulting formula

$$\star\star \qquad \theta_M = (A \times A) \cap \theta_N$$

is not always true (see Example 3.3). $\square$

The notion of an implicative logic was introduced by Rasiowa in (1974). A logic C is called *implicative* if there exists a sentence $\sigma(p,q) \in S$ such that $\mathrm{Var}(\sigma(p,q)) = \{p,q\}$ and for any sentences α, β, γ, $\phi \in S$ and any variable $p \in \mathrm{Var}(\phi)$ the following conditions are satisfied:

(i) $\sigma(\alpha,\alpha) \in C(\emptyset)$,
(ii) $\sigma(\alpha,\beta) \in C(\beta)$,
(iii) $\sigma(\alpha,\gamma) \in C(\sigma(\alpha,\beta), \sigma(\beta,\gamma))$,
(iv) $\beta \in C(\sigma(\alpha,\beta), \alpha)$,
(v) $\sigma(\phi(\alpha/p, \beta/q) \in C(\sigma(\alpha,\beta), \sigma(\beta,\alpha))$.

It is an obvious consequence of definitions that every implicative logic is equivalential. Moreover, if $\sigma(p,q)$ is an implication for C then $\{\sigma(p,q), \sigma(q,p)\}$ is a C-equivalence.

A class of matrices $\boldsymbol{K}$ is called *an algebraic semantics* if the set of distinguished elements of every matrix from the class $\boldsymbol{K}$ is one–element. We say that a logic $\langle S, C\rangle$ *has an algebraic semantics* if it has a strongly adequate algebraic semantics.

THEOREM 2.7 (Suszko). *For a logic $\langle S, C\rangle$ fulfilling the condition $C(\emptyset) \neq \emptyset$ to have an algebraic semantics it is necessary and sufficient that the following rule is a rule of C:*

$$(\star) \qquad \alpha,\ \beta,\ \phi(\alpha/p)/\phi(\beta/p),$$

where $\alpha, \beta, \phi \in S$. In consequence, every strengthening of a logic with an algebraic semantics has an algebraic semantics.

PROOF. Assume that C has an algebraic semantics $\boldsymbol{K}$. Let $\langle A, \{1\}\rangle \in \boldsymbol{K}$ and let $v\colon S \longrightarrow A$ be any valuation such that $v(\alpha) = v(\beta) = v(\phi(\alpha/p)) = 1$. Then $v(\phi(\beta/p)) = \phi(v(\beta)/p) = \phi(v(\alpha)/p) = 1$. So, the rule $(\star)$ is a rule for C. □

3. Orthologic – a couple examples

By an *ortholattice* we mean an algebra in $\langle A, \wedge, \vee, ' \rangle$ with two binary operators $\wedge$ and $\vee$ and one unary operation $'$ such that $\langle A, \wedge, \vee\rangle$ which is a bounded lattice i.e. a lattice with the greatest and the smallest elements 1 and 0 satisfying the following conditions:

$$\begin{aligned} x &= x'', \\ x \wedge x' &= 0, \\ x \vee x' &= 1, \\ (x \wedge y)' &= x' \vee y'. \end{aligned}$$

It is worth mentioning that the last condition is equivalent to either of the following two:

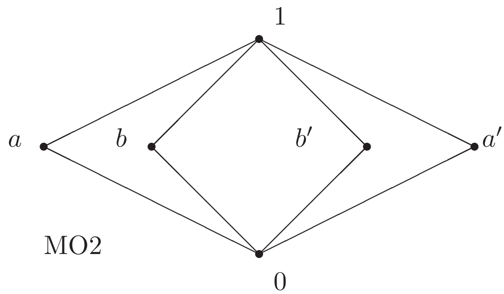
$$\begin{gathered} (x \vee y)' = x' \wedge y', \\ \text{if } x \leq y \text{ then } y' \leq x'. \end{gathered}$$

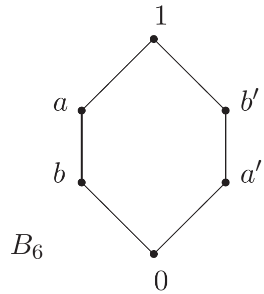
If moreover for $x \leq y$, we have $y \vee (y' \wedge x) = x$, then A is called an *orthomodular lattice*. The class of all ortholattices as well as the class of all orthomodular lattices are equationally definable—they constitute varieties which we shall denote by **OL** and **OML**, respectively.

The set of all sentences P of S_0 such that the realization P_A of P in an ortholattice A satisfies the condition $P_A = 1$ will be called the content of A and denoted by $\mathrm{E}(A)$. By the content $\mathrm{E}(A)$ of a class K of ortholattices we mean the meet of contents of elements of K. Thus $\mathrm{E}(\mathbf{OL})$ (respectively

E(**OML**)) is a class of all sentence satisfied in all ortholattices (respectively in all orthomodular lattices). In this way any class of algebras determines a logical system. However classes of algebras can also serve as a tool to define a consequence operations.

Lets consider a matrix $M = \langle A, \{1\}\rangle$, where A is an ortholattice and 1 is its unit element. Obviously $C_M(\emptyset) = \mathrm{E}(A)$. Trusting that this will not lead to any misunderstanding, we identify an ortholattice A with the matrix$\langle A, \{1\}\rangle$ and any class of ortholattices $\boldsymbol{K}$ with the class of matrices $\{\langle A, \{1\}\rangle : A \in \boldsymbol{K}\}$.





The logic $\mathsf{Cn}_{\mathbf{OL}}$ in the language $S = \langle S, \wedge, \vee, ' \rangle$ has an algebraic semantics, hence any strengthening of it has accordingly to Theorem 2.7, an algebraic semantics. Any strengthening of the logic $\mathsf{Cn}_{\mathbf{OL}}$ ($\mathsf{Cn}_{\mathbf{OML}}$ respectively) will be called an *orthologic* (*orthomodular logic*, respectively)

THEOREM 3.1. *If an ortholattice A is not orthomodular, then it contains as a sublattice the ortholattice B_6 from the picture above.*

PROOF. Since A is not orthomodular then there exist $x, y \in A$ such that: $y \leq x$ and

$$x \neq y \vee (y' \wedge x).$$

It is easy to see that $x \neq y$ and hence

$$x > y \vee (y' \wedge x).$$

Put: $a = x$, $b = y \vee (y' \wedge x)$. It is easy to check that the set $\{1, 0, a, b, a', b'\}$ is the universe of the lattice isomorphic to B_6. □

THEOREM 3.2. *Let C be an orthologic. The following conditions are equivalent:*

(i) *C is an orthomodular logic.*
(ii) *C is an implicative logic.*

(iii) C *is an finitely equivalential logic.*

(iv) C *is an equivalential logic.*

Proof. (i) $\Rightarrow$ (ii) Consider the following terms in two variables x, y:

$$\begin{aligned} p_1(x,y) &= (x' \wedge y) \vee (x' \wedge y') \vee (x \wedge (x' \vee y)), \\ p_2(x,y) &= (x' \wedge y) \vee (x \wedge y) \vee ((x' \vee y) \wedge y'), \\ p_3(x,y) &= x' \vee (x \wedge y), \\ p_4(x,y) &= y \vee (x' \wedge y'), \\ p_5(x,y) &= (x' \wedge y) \vee (x \wedge y) \vee (x' \wedge y'). \end{aligned}$$

Kotas (1967) (see also (Kalmbach 1983)) proved that the terms $p_1, \ldots, p_5$ are the only terms having the property that for any orthomodular lattice A and any $a, b \in A$, $p(a,b) = 1$ if and only if $a \leq b$. As a consequence, the logic $\mathsf{Cn}_{\mathbf{OML}}$ is implicative and each of (and only for) the following implication connectives:

$$\begin{aligned} \alpha \to_1 \beta &= (\alpha' \wedge \beta) \vee (\alpha' \wedge \beta') \vee (\alpha \wedge (\alpha' \vee \beta)), \\ \alpha \to_2 \beta &= (\alpha' \wedge \beta) \vee (\alpha \wedge \beta) \vee ((\alpha' \vee \beta) \wedge \beta'), \\ \alpha \to_3 \beta &= \alpha' \vee (\alpha \wedge \beta), \\ \alpha \to_4 \beta &= \beta \vee (\alpha' \wedge \beta'), \\ \alpha \to_5 \beta) &= (\alpha' \wedge \beta) \vee (\alpha \wedge \beta) \vee (\alpha' \wedge \beta') \end{aligned}$$

is its implication. As a consequence, every orthomodular logic is implicative.

The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i) We will show that if C is an orthologic which is not orthomodular, then C is not equivalential. Thus, Let $\boldsymbol{K}$ be an algebraic semantics for C. Put:

$$\boldsymbol{K}^\star = \{A/\theta_A : A \in \boldsymbol{K}\},$$

where θ_A is the greatest matrix congruence in the matrix $\langle A, \{1\}\rangle$. Surely, for $A \in \boldsymbol{K}^\star$ the matrix $\langle A, \{1\}\rangle$ is simple; moreover in accordance with Theorem 2.1a we have:

$$C = \mathsf{Cn}_{\boldsymbol{K}} = \mathsf{Cn}_{\boldsymbol{K}^\star}.$$

C is not an orthomodular logic, so there exists an algebra $A \in \boldsymbol{K}^\star$ which is not orthomodular. From theorem (3.1) the ortholattice B_6 is a subalgebra of the algebra A. Consequently the matrix $(B_6, \{1\})$ is a submatrix of the matrix $(A, \{1\})$. However the matrix $(B_6, \{1\})$ is not simple since the principal congruence $\theta(a, b)$ is a matrix congruence. Consequently the class $\mathrm{Mod}^\star(C)$ is not closed under the submatrices operation and hence from Theorem 2.5b C is not equivalential. □

Theorem 3.3. *For any L-axiom system $T = \{P_i : i \in I\}$ for classical logic the set $T_1 = \{P_i = 1 : i \in I\}$ is not an A-axiom system for the class of Boolean Algebras.*

Proof. Let M denote the matrix $M = \langle B_6, \{1\}\rangle$. The binary relation on $B_6 \equiv = \{(a, b), (a', b')\}$ is a matrix congruence. Let $N = M/\equiv$ then $N = \langle A, \{1\}\rangle$ where A is just the four-element Boolean algebra. Obviously Cn_N is the classical logic. Then, by Theorem 2.1a, Cn_M is the classical logic. Let $T = \{P_i : i \in I\}$ denotes any L-axiom system for (classical logic) $\mathsf{Cn}_{\mathbf{M}}$. Then all the identities from the set $T_1 = \{P_i = 1 : i \in I\}$ are satisfied in M. On the other hand M is non-distributive and hence it is not a Boolean algebra. □

To end let us consider an example illustrating Theorem 2.5 and Remark 2.7.

Remark 3.4. Lets consider the logic determined by the matrix $M = (B_6, \{1, a\})$. It is easy to check that the only non-trivial congruence on B_6 pastes together a and b, and this is not matrix congruence and hence M is simple. Let A denote the sublattice of B_6 generated by the set $\{1, a, b, 0\}$. It is clear that $N = \langle A, \{1, a\})$ is a submatrix of M. However it is not simple. The greatest matrix congruence on N paste together a and 1 and also b and 0. For matrices M and N the formula ($\star\star$) from the Remark 2.6 fails because we have $\theta_M = \mathrm{id} \neq \{(a, 1), (b, 0)\} = (A \times A) \cap \{(a, 1), (b, 0)\} = (A \times A) \cap \theta_N$. It proves that:

(a) the formula ($\star\star$) doesn't need to be valid if the greatest matrix congruence θ_M on M is not definable by means of an equivalence system,

(b) the logic determined by M is not equivalential.

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