Logic and Logical Philosophy
Volume 13 (2004), 147–162
DOI: 10.12775/LLP.2004.009

Tomasz Jarmużek
Andrzej Pietruszczak

COMPLETENESS OF MINIMAL POSITIONAL CALCULUS

1. Introduction

In the article *Podstawy analizy metodologicznej kanonów Milla* [2] Jerzy Łoś proposed an operator that refered sentences to temporal moments. Let us look, for example, at a sentence 'It is raining in Toruń'. From a logical point of view it is a propositional function, which does not have any logical value, unless we point at a temporal context from a fixed set of such contexts. If the sentence was considered today as a description of a state of affairs, it could be true. If it was considered yesterday, it could be false.[1] The operator enables us to connect any sentence p with any temporal context t. Such a complex sentence we read as: a sentence p is realized at a temporal context t (a point of time, an interval of some kind, etc).

The operator of realization can be applied more widely than only to temporal contexts. A review of these applications one can find in the book of Rescher and Urquhart [5]. It is why we shall, considering some very general axioms in the further part of our paper, write merely about positions, without deciding about their nature. These restrictions concern also sentential letters. They do not have to represent indexical sentences, but also so called eternal sentences. There is no obstacle to think about a logical value of the sentence 'American soldiers are bombing

[1]Hence, as an indexical sentence, it expresses various logical propositions at different contexts of utterance, denotes different parts of a world, and can change a logical value.

Received December 12, 2004
 ISSN: 1425-3305

Baghdad in 2003' at such a temporal context as, for example, 2000. Maybe it was then already true, but maybe it had the third logical value?

In the mentioned article Łoś proposed axioms that characterized not only a behavior of the operator, but also a domain of parametr, so a formal representation of time. Later, basing on Łoś' axioms, many authors developed different variants of deductive systems that described a relationship between sentences and time in an analogous way as Łoś did it (a laconic review of them can be found in the following papers [1, 4]).

In this article we shall show some basic properties of a very poor system of axioms that are not intended to be interpreted in a temporal way, because they are too general.

The system of considered axioms will be called minimal one, because it allows us to prove that the operator of realization is distributive over all classical connectives, what with the established, intuitive semantics enables us to prove the Completeness Theorem. Adding quantifiers, positional variables and other axioms, we obtain an extension of the minimal system, but it does not give any new, essential theorems.

At the end of the paper we will suggest how to change a meaning of the operator, what can take place, when we establish a structure of a set of positions. For example, we could treat a set of positions as a partial order and assume that the realization of a sentence at a context depends on additional factors, not only on its logical value at the context. Such an approach to the operator of realization makes us change not only semantics, but also initial, basic axioms. Hence, considering this situation, one should form a minimal system in another way, opening a possibility for further extensions.

2. Grammar and deductive tools of MR

The system described here and also its various extensions and modifications are not simple extensions of classical logic. Although an alphabet is an extension of propositional logic alphabet, its grammar is very specific. The considered system will be denoted by '**MR**'.

2.1. Language of MR

The alphabet of **MR** (in short: Alf) can be in an usual way described as a sum of separate sets: logical connectives: $\mathsf{Con} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \mathsf{R}\}$, sentential letters: $\mathsf{SL} = \{p_1, p_2, p_3, \ldots\}$, positional letters: $\mathsf{PL} = \{a_1, a_2, a_3, \ldots\}$, moreover, we need some auxiliary signs: ')', '(', ']', '['. We assume also that the sets SL and PL are infinite, but countable.

Before we present a definition of a formula, we will define an auxiliary notion:

DEFINITION 2.1. To the set of atomic expressions (in short: AE) belongs every and only such an expression A that satisfies exactly one of the conditions:[2]

1. $A \in \mathsf{SL}$,
2. A has one of the forms: $\ulcorner\neg B\urcorner$, $\ulcorner(B \wedge C)\urcorner$, $\ulcorner(B \vee C)\urcorner$, $\ulcorner(B \rightarrow C)\urcorner$, $\ulcorner(B \leftrightarrow C)\urcorner$, where $B, C \in \mathsf{AE}$. □

Obviously, the set AE is the set of all formulas of Classical Propositional Calculus (in short: CPC).

DEFINITION 2.2. To the set of formulas (in short: For) belongs every and only such an expression φ that has exactly one of the forms:[3]

1. $\ulcorner \mathrm{R}_\alpha A\urcorner$, where $A \in \mathsf{AE}$ and $\alpha \in \mathsf{PL}$,
2. $\ulcorner\neg\psi\urcorner$, where $\psi \in \mathsf{For}$,
3. $\ulcorner(\psi \wedge \chi)\urcorner$, where $\psi, \chi \in \mathsf{For}$,
4. $\ulcorner(\psi \vee \chi)\urcorner$, where $\psi, \chi \in \mathsf{For}$,
5. $\ulcorner(\psi \rightarrow \chi)\urcorner$, where $\psi, \chi \in \mathsf{For}$,
6. $\ulcorner(\psi \leftrightarrow \chi)\urcorner$, where $\psi, \chi \in \mathsf{For}$.

Any member of For will be called a formula. □

2.2. Axioms and rules of MR

The only rule of inference is the detachment rule MP:

$$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$$

(Ax0): An axiom of **MR** is any substitution of any tautology of CPC with formulas from For.

Except the above tools, we introduce specific axiom for any atomic expressions $A, B \in \mathsf{AE}$ and any positional letter $\alpha \in \mathsf{PL}$:

(Ax1) $$\mathrm{R}_\alpha \neg A \leftrightarrow \neg \mathrm{R}_\alpha A\,,$$

(Ax2) $$\mathrm{R}_\alpha A \wedge \mathrm{R}_\alpha B \rightarrow \mathrm{R}_\alpha (A \wedge B)\,,$$

[2] The letters 'A', 'B' and 'C' are metavariables which take values from the set of atomic expressions.

[3] The letters 'φ', 'ψ' and 'χ' are metavariables which take values from the set of formulas. Moreover, the letter 'α' is a metavariable which takes a value from the set of positional letters.

Moreover, if $A \in \mathsf{AE}$ is a theorem of CPC, the following formula is also an axiom:

(Ax3) $$\mathrm{R}_\alpha A\,.$$

Now, we will define an usual notion of proof:

DEFINITION 2.3. Let Φ be any set of formulas. Let φ be any formula. We say that φ has a proof on the ground of Φ (in short: $\Gamma \vdash \varphi$) iff there is a sequence of formulas $\psi_1, \ldots, \psi_n$ such that $\psi_n = \varphi$ and for any $i \leqslant n$ a formula ψ_i satisfies one of the conditions:

1. $\psi_i \in \Phi$;
2. ψ_i is an axiom which is a substitution of a CPC theorem schema;
3. ψ_i is one of the axioms (Ax1), (Ax2) or (Ax3);
4. ψ_i arises by use of MP, i.e., there are $j, k < i$ such that $\psi_k = \ulcorner(\psi_j \to \psi_i)\urcorner$. □

In a case when we have proved a formula φ on the ground of the empty set of assumptions Φ, this fact will be denoted by one of the following expressions: '$\vdash \varphi$' or '$\varnothing \vdash \varphi$' and the formula φ will be called theorem. On the other hand, if a formula φ can not be proved on the ground of Φ, this fact we will be denoted by: '$\Phi \nvdash \varphi$'.

2.3. The deduction theorem and derivable rules of MR

By CPC we have the following derivable rules:

$$\frac{\neg\psi \quad \varphi \to \psi}{\neg\varphi} \qquad \frac{\varphi \to \psi}{\neg\psi \to \neg\varphi} \qquad \frac{\varphi \leftrightarrow \psi}{\psi \leftrightarrow \varphi} \qquad \frac{\varphi \leftrightarrow \psi}{\varphi \to \psi} \qquad \frac{\varphi \to \psi \quad \psi \to \varphi}{\varphi \leftrightarrow \psi}$$

$$\frac{\varphi \to \chi \quad \chi \to \psi}{\varphi \to \psi} \qquad \frac{\varphi \to \psi \quad \varphi \to \chi}{\varphi \to (\psi \wedge \chi)} \qquad \frac{\neg(\varphi \wedge \psi)}{\varphi \to \neg\psi}$$

Using the introduced notions and properties of axioms one can prove the deduction theorem (in short: DT) and the rule of extensionality (in short: RE) for our system. We omit its proof, because it is analogous as in CPC.

LEMMA 2.1. *Let $\Phi \subseteq \mathsf{For}$ and $\varphi, \psi \in \mathsf{For}$. Then $\Phi \cup \{\varphi\} \vdash B$ iff $\Phi \vdash \varphi \to \psi$.* □

LEMMA 2.2. *The following rule is derivable:*

$$\frac{\varphi \leftrightarrow \psi}{\chi \leftrightarrow \chi(\varphi//\psi)}\ \text{RE}$$

where $\chi(\varphi//\psi)$ is a formula arising from χ as a result of replacement of the subformula φ with the formula ψ. □

2.4. Inconsistent sets of formulas in MR

We shall introduce some additional helpful notions. The first one is a notion of inconsistent set:

DEFINITION 2.4. We say that a set $\Phi \subseteq \mathsf{For}$ is *inconsistent* iff for every formula φ: $\Phi \vdash \varphi$.

By this definition, we can prove the following lemma:

LEMMA 2.3 (Consistency Lemma). *Let $\Phi \subseteq \mathsf{For}$ and $\varphi \in \mathsf{For}$. If $\Phi \nvdash \varphi$, then the set $\Phi \cup \{\neg\varphi\}$ is consistent.*

PROOF. Let $\Phi \subseteq \mathsf{For}$, $\varphi \in \mathsf{For}$, and $\Phi \nvdash \varphi$. For every $\psi \in \mathsf{For}$ we have: $\Phi \vdash \neg(\psi \wedge \neg\psi)$, so $\Phi \nvdash \neg(\psi \wedge \neg\psi) \to \varphi$. Hence, by CPC, $\Phi \nvdash \neg\varphi \to (\psi \wedge \neg\psi)$. So $\Phi \cup \{\neg\varphi\} \nvdash (\psi \wedge \neg\psi)$, by DT. Hence, the set $\Phi \cup \{\neg\varphi\}$ is consistent. □

2.5. Distributivity laws in MR

Now we prove that R operator is distributive over all classical connectives.

FACT 2.1. *For any $A \in \mathsf{AE}$ and $\alpha \in \mathsf{PL}$ the following formula:*

$$\mathrm{R}_\alpha A \leftrightarrow \neg\mathrm{R}_\alpha\neg A$$

is a theorem.

PROOF. It is justified by the following sequence:

1. $\mathrm{R}_\alpha\neg A \leftrightarrow \neg\mathrm{R}_\alpha A$ (Ax1)
2. $(\mathrm{R}_\alpha\neg A \leftrightarrow \neg\mathrm{R}_\alpha A) \to (\mathrm{R}_\alpha A \leftrightarrow \neg\mathrm{R}_\alpha\neg A)$
   the CPC thesis $(\varphi \leftrightarrow \neg\psi) \to (\psi \leftrightarrow \neg\varphi)$
3. $\mathrm{R}_\alpha A \leftrightarrow \neg\mathrm{R}_\alpha\neg A$ 1, 2 and MP □

FACT 2.2. *For any $A, B \in \mathsf{AE}$ and $\alpha \in \mathsf{PL}$ the following formula:*

$$\mathrm{R}_\alpha(A \to B) \to (\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B)$$

is a theorem.

PROOF. It is justified by the following sequence:

1. $\mathrm{R}_\alpha\neg((A \to B) \wedge A \wedge \neg B)$ (Ax3)
2. $\mathrm{R}_\alpha\neg((A \to B) \wedge A \wedge \neg B) \leftrightarrow \neg\mathrm{R}_\alpha((A \to B) \wedge A \wedge \neg B)$ (Ax1)
3. $\mathrm{R}_\alpha\neg((A \to B) \wedge A \wedge \neg B) \to \neg\mathrm{R}_\alpha((A \to B) \wedge A \wedge \neg B)$ 2 and CPC
4. $\neg\mathrm{R}_\alpha((A \to B) \wedge A \wedge \neg B)$ 1, 3 and MP
5. $[\mathrm{R}_\alpha(A \to B) \wedge \mathrm{R}_\alpha(A \wedge \neg B)] \to \mathrm{R}_\alpha((A \to B) \wedge A \wedge \neg B)$ (Ax2)

6.	$\neg[\mathrm{R}_\alpha(A \to B) \wedge \mathrm{R}_\alpha(A \wedge \neg B)]$	4, 5 and CPC
7.	$\mathrm{R}_\alpha(A \to B) \to \neg\mathrm{R}_\alpha(A \wedge \neg B)$	6 and CPC
8.	$(\mathrm{R}_\alpha A \wedge \mathrm{R}_\alpha \neg B) \to \mathrm{R}_\alpha(A \wedge \neg B)$	(Ax2)
9.	$\neg\mathrm{R}_\alpha(A \wedge \neg B) \to \neg(\mathrm{R}_\alpha A \wedge \mathrm{R}_\alpha \neg B)$	8 and CPC
10.	$\mathrm{R}_\alpha(A \to B) \to \neg(\mathrm{R}_\alpha A \wedge \mathrm{R}_\alpha \neg B)$	7, 9 and CPC
11.	$\mathrm{R}_\alpha(A \to B) \to (\mathrm{R}_\alpha A \to \neg\mathrm{R}_\alpha \neg B)$	10 and CPC, RE
12.	$\neg\mathrm{R}_\alpha \neg B \to \mathrm{R}_\alpha B$	Fact 2.1 and CPC
13.	$\mathrm{R}_\alpha(A \to B) \to (\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B)$	11, 12 and CPC □

Fact 2.3. *For any $A, B \in \mathsf{AE}$ and $\alpha \in \mathsf{PL}$ the following formula:*

$$\mathrm{R}_\alpha(A \wedge B) \leftrightarrow (\mathrm{R}_\alpha A \wedge \mathrm{R}_\alpha B)$$

is a theorem.

Proof. It is justified by the following sequence:

1.	$\mathrm{R}_\alpha A \wedge \mathrm{R}_\alpha B \to \mathrm{R}_\alpha(A \wedge B)$	(Ax2)
2.	$\mathrm{R}_\alpha((A \wedge B) \to A)$	(Ax3)
3.	$\mathrm{R}_\alpha(A \wedge B) \to \mathrm{R}_\alpha A$	2, Fact 2.2 and MP
4.	$\mathrm{R}_\alpha((A \wedge B) \to B)$	(Ax3)
5.	$\mathrm{R}_\alpha(A \wedge B) \to \mathrm{R}_\alpha B$	4, Fact 2.2 and MP
6.	$\mathrm{R}_\alpha(A \wedge B) \to (\mathrm{R}_\alpha A \wedge \mathrm{R}_\alpha B)$	3, 5 and CPC
7.	$\mathrm{R}_\alpha(A \wedge B) \leftrightarrow (\mathrm{R}_\alpha A \wedge \mathrm{R}_\alpha B)$	1, 6 and CPC □

Fact 2.4. *For any $A, B \in \mathsf{AE}$ and $\alpha \in \mathsf{PL}$ the following formula:*

$$\mathrm{R}_\alpha(A \to B) \leftrightarrow (\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B)$$

is a theorem.

Proof. It is justified by the following sequence:

1.	$\mathrm{R}_\alpha(A \to B) \to (\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B)$	Fact 2.2
2.	$(\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B) \leftrightarrow \neg(\mathrm{R}_\alpha A \wedge \neg\mathrm{R}_\alpha B)$	CPC
3.	$(\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B) \leftrightarrow \neg(\mathrm{R}_\alpha A \wedge \mathrm{R}_\alpha \neg B)$	(Ax1) and RE
4.	$(\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B) \leftrightarrow \neg\mathrm{R}_\alpha(A \wedge \neg B)$	Fact 2.3 and RE
5.	$(\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B) \leftrightarrow \mathrm{R}_\alpha \neg(A \wedge \neg B)$	(Ax1) and RE
6.	$\mathrm{R}_\alpha[\neg(A \wedge \neg B) \to (A \to B)]$	(Ax3)
7.	$\mathrm{R}_\alpha \neg(A \wedge \neg B) \to \mathrm{R}_\alpha(A \to B)$	6, Fact 2.2 and MP
8.	$(\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B) \to \mathrm{R}_\alpha(A \to B)$	5, 8 and CPC
9.	$\mathrm{R}_\alpha(A \to B) \leftrightarrow (\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B)$	1, 8 and CPC □

Fact 2.5. *For any* $A, B \in \mathsf{AE}$ *and* $\alpha \in \mathsf{PL}$ *the following formula:*

$$\mathrm{R}_\alpha(A \leftrightarrow B) \leftrightarrow (\mathrm{R}_\alpha A \leftrightarrow \mathrm{R}_\alpha B)$$

is a theorem.

Proof. It is justified by the following sequence:

1.	$\mathrm{R}_\alpha[(A \leftrightarrow B) \to (A \to B)]$	(Ax3)
2.	$\mathrm{R}_\alpha(A \leftrightarrow B) \to \mathrm{R}_\alpha(A \to B)$	1, Fact 2.2 and MP
3.	$\mathrm{R}_\alpha(A \to B) \to (\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B)$	Fact 2.2
4.	$\mathrm{R}_\alpha(A \leftrightarrow B) \to (\mathrm{R}_\alpha A \to \mathrm{R}_\alpha B)$	2, 3 and CPC
5.	$\mathrm{R}_\alpha[(A \leftrightarrow B) \to (B \to A)]$	(Ax3)
6.	$\mathrm{R}_\alpha(A \leftrightarrow B) \to \mathrm{R}_\alpha(B \to A)$	1, Fact 2.2 and MP
7.	$\mathrm{R}_\alpha(A \to B) \to (\mathrm{R}_\alpha B \to \mathrm{R}_\alpha A)$	Fact 2.2
8.	$\mathrm{R}_\alpha(A \leftrightarrow B) \to (\mathrm{R}_\alpha B \to \mathrm{R}_\alpha A)$	6, 7 and CPC
9.	$\mathrm{R}_\alpha(A \leftrightarrow B) \to (\mathrm{R}_\alpha B \leftrightarrow \mathrm{R}_\alpha A)$	4, 8 and CPC
10.	$(\mathrm{R}_\alpha A \leftrightarrow \mathrm{R}_\alpha B) \to \mathrm{R}_\alpha(A \to B)$	Fact 2.4 and CPC
11.	$(\mathrm{R}_\alpha A \leftrightarrow \mathrm{R}_\alpha B) \to \mathrm{R}_\alpha(B \to A)$	Fact 2.4 and CPC
12.	$(\mathrm{R}_\alpha A \leftrightarrow \mathrm{R}_\alpha B) \to [\mathrm{R}_\alpha(A \to B) \wedge \mathrm{R}_\alpha(B \to A)]$	10, 11 and CPC
13.	$(\mathrm{R}_\alpha A \leftrightarrow \mathrm{R}_\alpha B) \to \mathrm{R}_\alpha[(A \to B) \wedge (B \to A)]$	12, Fact 2.3 and RE
14.	$\mathrm{R}_\alpha[((A \to B) \wedge (B \to A)) \to (A \leftrightarrow B)]$	(Ax3)
15.	$\mathrm{R}_\alpha[(A \to B) \wedge (B \to A)] \to \mathrm{R}_\alpha(A \leftrightarrow B)$	14, Fact 2.2 and MP
16.	$(\mathrm{R}_\alpha A \leftrightarrow \mathrm{R}_\alpha B) \to \mathrm{R}_\alpha(A \leftrightarrow B)$	13, 15 and CPC
17.	$\mathrm{R}_\alpha(A \leftrightarrow B) \leftrightarrow (\mathrm{R}_\alpha B \leftrightarrow \mathrm{R}_\alpha A)$	9, 16 and CPC □

Fact 2.6. *For any* $A, B \in \mathsf{AE}$ *and* $\alpha \in \mathsf{PL}$ *the following formula:*

$$\mathrm{R}_\alpha(A \vee B) \leftrightarrow (\mathrm{R}_\alpha A \vee \mathrm{R}_\alpha B)$$

is a theorem.

Proof. It is justified by the following sequence:

1.	$\mathrm{R}_\alpha[(A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B)]$	(Ax3)
2.	$\mathrm{R}_\alpha(A \vee B) \leftrightarrow \mathrm{R}_\alpha\neg(\neg A \wedge \neg B)$	1 and Fact 2.5
3.	$\mathrm{R}_\alpha(A \vee B) \leftrightarrow \neg\mathrm{R}_\alpha(\neg A \wedge \neg B)$	2, (Ax1) and RE
4.	$\mathrm{R}_\alpha(A \vee B) \leftrightarrow \neg(\mathrm{R}_\alpha\neg A \wedge \mathrm{R}_\alpha\neg B)$	3, Fact 2.3 and RE
5.	$\mathrm{R}_\alpha(A \vee B) \leftrightarrow \neg(\neg\mathrm{R}_\alpha A \wedge \neg\mathrm{R}_\alpha B)$	4, (Ax1) and RE
6.	$\mathrm{R}_\alpha(A \vee B) \leftrightarrow (\mathrm{R}_\alpha A \vee \mathrm{R}_\alpha B)$	5, CPC, RE □

2.6. Similar formulas in MR

Definition 2.5. By $\mathrm{s}\colon \mathsf{PL} \times \mathsf{AE} \longrightarrow \mathsf{For}$ we will understand a function that satisfies the conditions for any $i, j = 1, 2, \ldots$:

1. $\mathrm{s}(a_i, p_j) = \ulcorner \mathsf{R}_{a_i} p_i \urcorner$,
2. $\mathrm{s}(a_i, \neg A) = \ulcorner \neg \mathrm{s}(a_i, A) \urcorner$,
3. $\mathrm{s}(a_i, A \wedge B) = \ulcorner \mathrm{s}(a_i, A) \wedge \mathrm{s}(a_i, B) \urcorner$,
4. $\mathrm{s}(a_i, A \vee B) = \ulcorner \mathrm{s}(a_i, A) \vee \mathrm{s}(a_i, B) \urcorner$,
5. $\mathrm{s}(a_i, A \rightarrow B) = \ulcorner \mathrm{s}(a_i, A) \rightarrow \mathrm{s}(a_i, B) \urcorner$,
6. $\mathrm{s}(a_i, A \leftrightarrow B) = \ulcorner \mathrm{s}(a_i, A) \leftrightarrow \mathrm{s}(a_i, B) \urcorner$. □

Using the above definition, we can now define the following notion:

Definition 2.6. By $\mathrm{S}\colon \mathsf{For} \longrightarrow \mathsf{For}$ we will understand a function that is determined by the following conditions $i = 1, 2, \ldots$:

1. $\mathrm{S}(\mathsf{R}_{a_i} A) = \mathrm{s}(a_i, A)$,
2. $\mathrm{S}(\neg\varphi) = \ulcorner \neg \mathrm{S}(\varphi) \urcorner$,
3. $\mathrm{S}(\varphi \wedge \psi) = \ulcorner \mathrm{S}(\varphi) \wedge \mathrm{S}(\psi) \urcorner$,
4. $\mathrm{S}(\varphi \vee \psi) = \ulcorner \mathrm{S}(\varphi) \vee \mathrm{S}(\psi) \urcorner$,
5. $\mathrm{S}(\varphi \rightarrow \psi) = \ulcorner \mathrm{S}(\varphi) \rightarrow \mathrm{S}(\psi) \urcorner$,
6. $\mathrm{S}(\varphi \leftrightarrow \psi) = \ulcorner \mathrm{S}(\varphi) \leftrightarrow \mathrm{S}(\psi) \urcorner$. □

Definition 2.7. Formulas φ and ψ are *similar* iff $\mathrm{S}(\varphi) = \mathrm{S}(\psi)$. □

From the above notions we obtain some new conclusions.

Theorem 2.1. *Let φ and ψ be similar. Then $\vdash \varphi \leftrightarrow \psi$.*

Proof. By (Ax1) and facts 2.3–2.6. □

Corollary 2.1. *Let φ and ψ be similar formulas. Then $\vdash \varphi$ iff $\vdash \psi$.* □

3. Semantics of MR

Definition 3.1. Let W be a nonempty set of positions. Then an *evaluation* is called any function $\mathfrak{v}\colon W \times \mathsf{SL} \longrightarrow \{0, 1\}$.

Definition 3.2. An *extension of the evaluation* $\mathfrak{v}$ is a function $\bar{\mathfrak{v}}\colon W \times \mathsf{AE} \longrightarrow \{0, 1\}$ that satisfies the conditions for all $p_i \in \mathsf{SL}$, $w \in W$ and $A, B \in \mathsf{AE}$:

i) $\bar{\mathfrak{v}}(w, p_i) = 1$ iff $\mathfrak{v}(w, p_i) = 1$,
ii) $\bar{\mathfrak{v}}(w, \neg A) = 1$ iff $\bar{\mathfrak{v}}(w, A) = 0$,
iii) $\bar{\mathfrak{v}}(w, A \wedge B) = 1$ iff $\bar{\mathfrak{v}}(w, A) = 1$ and $\bar{\mathfrak{v}}(w, B) = 1$,

iv) $\bar{\mathfrak{v}}(w, A \vee B) = 1$ iff $\bar{\mathfrak{v}}(w, A) = 1$ or $\bar{\mathfrak{v}}(w, B) = 1$,
v) $\bar{\mathfrak{v}}(w, A \rightarrow B) = 1$ iff $\bar{\mathfrak{v}}(w, A) = 0$ or $\bar{\mathfrak{v}}(w, B) = 1$,
vi) $\bar{\mathfrak{v}}(w, A \leftrightarrow B) = 1$ iff $\bar{\mathfrak{v}}(w, A) = \bar{\mathfrak{v}}(w, B)$. □

From the above definitions we have an obvious conclusion that any evaluation can be extended in the unique way.

DEFINITION 3.3. A *model* $\mathfrak{M}$ for the set For is any triple $\langle W, \mathfrak{d}, \mathfrak{v}\rangle$, where W is a nonempty set of positions (the universe of $\mathfrak{M}$), $\mathfrak{d}$ is a mapping from the set of positional letters PL into the set W, and $\mathfrak{v}$ is an evaluation.

DEFINITION 3.4. Let $\mathfrak{M}$ be a model. We say that a *formula* φ *is true* in $\mathfrak{M}$ (in short: $\mathfrak{M} \models \varphi$) iff it satisfies one of the conditions:

i) $\varphi = \ulcorner \mathrm{R}_\alpha A \urcorner$ for some $A \in$ AE and $\bar{\mathfrak{v}}(\mathfrak{d}(\alpha), A) = 1$,
ii) $\varphi = \ulcorner \neg\psi \urcorner$ for some $\psi \in$ For and $\mathfrak{M} \not\models \psi$,
iii) $\varphi = \ulcorner \psi \wedge \chi \urcorner$ for some $\psi, \chi \in$ For and both $\mathfrak{M} \models \psi$ and $\mathfrak{M} \models \chi$,
iv) $\varphi = \ulcorner \psi \vee \chi \urcorner$ for some $\psi, \chi \in$ For and either $\mathfrak{M} \models \psi$ or $\mathfrak{M} \models \chi$,
v) $\varphi = \ulcorner \psi \rightarrow \chi \urcorner$ for some $\psi, \chi \in$ For and either $\mathfrak{M} \not\models \psi$ or $\mathfrak{M} \models \chi$,
vi) $\varphi = \ulcorner \psi \leftrightarrow \chi \urcorner$ for some $\psi, \chi \in$ For and either $\mathfrak{M} \models \psi$, $\mathfrak{M} \models \chi$ or $\mathfrak{M} \not\models \psi$, $\mathfrak{M} \not\models \chi$. □

DEFINITION 3.5. Let $\Phi \subseteq$ For. The *Φ is true in* $\mathfrak{M}$ (in short: $\mathfrak{M} \models \Phi$) iff for every formula $\varphi \in \Phi$: $\mathfrak{M} \models \varphi$. □

Now, in the standard way, we define the relation of logical consequence:

DEFINITION 3.6. We say that a *formula φ logically follows from a set of formulas Φ* iff for every model $\mathfrak{M}$: if $\mathfrak{M} \models \Phi$, then $\mathfrak{M} \models \varphi$. □

DEFINITION 3.7. We say that φ is a *tautology* iff $\varnothing \models A$, i.e., for any $\mathfrak{M}$: $\mathfrak{M} \models \varphi$. □

4. Correctness of MR

Theorem on the Correctness expresses the fact that any formula which is provable on the ground of a set of formulas φ, also logically follows from φ. Formally, we should formulate this theorem as follows:

THEOREM 4.1. *Let $\Phi \subseteq$ For and $\varphi \in$ For. If $\Phi \vdash \varphi$, then $\Phi \models \varphi$.*

PROOF. One can easily prove that each axiom is true in any model. First, if φ is a substitution of some CPC thesis then it is obviously true in any model. For other cases we have.

(Ax1): $\mathfrak{M} \models \mathrm{R}_\alpha\neg A$ iff $\bar{\mathfrak{v}}(\mathfrak{d}(\alpha), \neg A) = 1$ iff $\bar{\mathfrak{v}}(\mathfrak{d}(\alpha), A) = 0$ iff $\mathfrak{M} \not\models \mathrm{R}_\alpha A$ iff $\mathfrak{M} \models \neg\mathrm{R}_\alpha A$. So $\mathfrak{M} \models \mathrm{R}_\alpha\neg A \leftrightarrow \neg\mathrm{R}_\alpha A$.

(Ax2): If $\mathfrak{M} \models \mathrm{R}_\alpha A \wedge \mathrm{R}_\alpha B$, then $\mathfrak{M} \models \mathrm{R}_\alpha A$ and $\mathfrak{M} \models \mathrm{R}_\alpha B$, so $\bar{\mathfrak{v}}(\mathfrak{d}(\alpha), A) = 1$ and $\bar{\mathfrak{v}}(\mathfrak{d}(\alpha), B) = 1$, thus $\bar{\mathfrak{v}}(\mathfrak{d}(\alpha), A \wedge B) = 1$, and therefore $\mathfrak{M} \models \mathrm{R}_\alpha(A \wedge B)$.

(Ax3): If A is theorem of CPC, then for any evaluation $\mathfrak{v}$ we have: $\bar{\mathfrak{v}}(w, A) = 1$. Hence $\mathfrak{M} \models \mathrm{R}_\alpha A$ for any models $\mathfrak{M}$.

Moreover, when the rule MP is applied to true formulas, it yields a true formula. □

5. Completeness of MR

The proof of The Completeness Theorem is more sophisticated and interesting.

THEOREM 5.1. *Let $\Phi \subseteq \mathsf{For}$ and $\varphi \in \mathsf{For}$. If $\Phi \models \varphi$, then $\Phi \vdash \varphi$.*

Let $\mathsf{Alf}^{\mathrm{MPC}^-}$ mean a subset of the alphabet of Monadic Predicate Calculus (in short: MPC), without quantifiers and individual variables, in which there are the infinite, but countable set of individual letters $\mathsf{IL} = \{c_1, c_2, c_3, \ldots\}$ and the infinite, but countable set of unary predicates $\mathsf{Pred} = \{P_1, P_2, P_3, \ldots\}$. The symbol $\mathsf{For}^{\mathrm{MPC}^-}$ will stand for the set of all MPC formulas which are built with $\mathsf{Alf}^{\mathrm{MPC}^-}$.

Let $h\colon \mathsf{PL} \longrightarrow \mathsf{IL}$, where $h(a_i) := c_i$, and $g\colon \mathsf{SL} \longrightarrow \mathsf{Pred}$, where $g(p_i) := P_i$, for $i = 1, 2, \ldots$.

We still need some helpful notions.

DEFINITION 5.1. By $\mathrm{t}\colon \mathsf{PL} \times \mathsf{AE} \longrightarrow \mathsf{For}^{\mathrm{MPC}^-}$ we will understand a function that satisfies the conditions for any $i, j = 1, 2, \ldots$:

1. $\mathrm{t}(a_i, p_j) = g(p_j)(h(a_i)) = \ulcorner P_j(c_i)\urcorner$,
2. $\mathrm{t}(a_i, \neg A) = \ulcorner\neg\mathrm{t}(a_i, A)\urcorner$,
3. $\mathrm{t}(a_i, A \wedge B) = \ulcorner\mathrm{t}(a_i, A) \wedge \mathrm{t}(a_i, B)\urcorner$,
4. $\mathrm{t}(a_i, A \vee B) = \ulcorner\mathrm{t}(a_i, A) \vee \mathrm{t}(a_i, B)\urcorner$,
5. $\mathrm{t}(a_i, A \rightarrow B) = \ulcorner\mathrm{t}(a_i, A) \rightarrow \mathrm{t}(a_i, B)\urcorner$,
6. $\mathrm{t}(a_i, A \leftrightarrow B) = \ulcorner\mathrm{t}(a_i, A) \leftrightarrow \mathrm{t}(a_i, B)\urcorner$. □

Using the above definition, we can now define the following notion:

DEFINITION 5.2. By $\mathrm{T}\colon \mathsf{For} \longrightarrow \mathsf{For}^{\mathrm{MPC}^-}$ we will understand a function that is determined by the following conditions $i = 1, 2, \ldots$:

1. $\mathrm{T}(\mathrm{R}_{a_i} A) = \mathrm{t}(a_i, A)$,
2. $\mathrm{T}(\neg\varphi) = \ulcorner\neg\mathrm{T}(\varphi)\urcorner$,

3. $\mathrm{T}(\varphi \wedge \psi) = \ulcorner \mathrm{T}(\varphi) \wedge \mathrm{T}(\psi) \urcorner$,
4. $\mathrm{T}(\varphi \vee \psi) = \ulcorner \mathrm{T}(\varphi) \vee \mathrm{T}(\psi) \urcorner$,
5. $\mathrm{T}(\varphi \rightarrow \psi) = \ulcorner \mathrm{T}(\varphi) \rightarrow \mathrm{T}(\psi) \urcorner$,
6. $\mathrm{T}(\varphi \leftrightarrow \psi) = \ulcorner \mathrm{T}(\varphi) \leftrightarrow \mathrm{T}(\psi) \urcorner$. □

In the further considerations we will use also a function from $\mathsf{For}^{\mathrm{MPC}^-}$ to For.

DEFINITION 5.3. $\mathrm{m}\colon \mathsf{For}^{\mathrm{MPC}^-} \longrightarrow \mathsf{For}$ is a function that satisfies the conditions $i, j = 1, 2, \ldots$:

1. $\mathrm{m}(P_j(c_i)) = \ulcorner \mathsf{R}_{h^{-1}(c_i)}(g^{-1}(P_j)) \urcorner = \ulcorner \mathsf{R}_{a_i} p_j \urcorner$,
2. $\mathrm{m}(\neg\varphi) = \ulcorner \neg \mathrm{m}(\varphi) \urcorner$,
3. $\mathrm{m}(\varphi \wedge \psi) = \ulcorner \mathrm{m}(\varphi) \wedge \mathrm{m}(\psi) \urcorner$,
4. $\mathrm{m}(\varphi \vee \psi) = \ulcorner \mathrm{m}(\varphi) \vee \mathrm{m}(\psi) \urcorner$,
5. $\mathrm{m}(\varphi \rightarrow \psi) = \ulcorner \mathrm{m}(\varphi) \rightarrow \mathrm{m}(\psi) \urcorner$,
6. $\mathrm{m}(\varphi \leftrightarrow \psi) = \ulcorner \mathrm{m}(\varphi) \leftrightarrow \mathrm{m}(\psi) \urcorner$. □

From the above definitions it follows an obvious conclusion:

COROLLARY 5.1. *For any $\varphi \in \mathsf{For}^{\mathrm{MPC}^-}$: $\mathrm{T}(\mathrm{m}(\varphi)) = \varphi$.* □

Now we will present some more complicated lemmas:

LEMMA 5.1. *Let $\varphi \in \mathsf{For}$. If $\mathrm{T}(\varphi)$ is a substitution of CPC theorem, then $\vdash \varphi$.*

PROOF. Let $\mathrm{m}(\mathrm{T}(\varphi))$ be a substitution of some CPC theorem. Hence $\vdash \mathrm{m}(\mathrm{T}(\varphi))$. Since $\mathrm{T}(\mathrm{m}(\mathrm{T}(\varphi))) = \mathrm{T}(\varphi)$, so $\mathrm{S}(\mathrm{m}(\mathrm{T}(\varphi))) = \mathrm{S}(\varphi)$. Thus φ and $\mathrm{m}(\mathrm{T}(\varphi))$ are similar. Thus $\vdash \varphi$, by Corollary 2.1. □

LEMMA 5.2. *Let $\Phi \subseteq \mathsf{For}$ be consistent in* **MR**. *Then $\mathrm{T}(\Phi)$ is also consistent.*

PROOF. Let us take any consistent set of formulas Φ and assume that $\mathrm{T}(\Phi)$ is inconsistent. According to the definition, it means one can prove any MPC^- formula on its ground, using only the CPC deductive tools, in particular, the formula of the schema $\chi \wedge \neg\chi$ for any $\chi \in \mathsf{For}^{\mathrm{MPC}^-}$. Because a proof is finite, so there is a finite set of formulas $\Psi \subseteq \Phi$, that $\mathrm{T}(\Psi) = \{\mathrm{T}(\psi_1), \ldots, \mathrm{T}(\psi_n)\}$ and $\mathrm{T}(\Psi) \vdash_{\mathrm{CPC}} \chi \wedge \neg\chi$. Using n-times the CPC deduction theorem, the schema $(A \rightarrow (B \rightarrow C)) \rightarrow (A \wedge B \rightarrow C)$ and MP, we can prove: $\vdash_{\mathrm{CPC}} \mathrm{T}(\psi_1) \wedge \cdots \wedge \mathrm{T}(\psi_n) \rightarrow \chi \wedge \neg\chi$. Applying the law $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ and MP, we obtain: $\vdash_{\mathrm{CPC}} \neg(\chi \wedge \neg\chi) \rightarrow \neg(\mathrm{T}(\psi_1) \wedge \cdots \wedge \mathrm{T}(\psi_n))$. The predecessor of it is a CPC theorem, so using MP, we have: $\vdash_{\mathrm{CPC}} \neg(\mathrm{T}(\psi_1) \wedge \cdots \wedge \mathrm{T}(\psi_n))$, what, by definition of T, is equivalent to

$\vdash_{\mathrm{CPC}} \mathrm{T}(\neg(\psi_1 \wedge \cdots \wedge \psi_n))$. From the Lemma 5.1 it follows: $\vdash \neg(\psi_1 \wedge \cdots \wedge \psi_n)$. Hence, $\Psi \vdash \neg(\psi_1 \wedge \cdots \wedge \psi_n)$ and $\Psi \vdash (\psi_1 \wedge \cdots \wedge \psi_n) \wedge \neg(\psi_1 \wedge \cdots \wedge \psi_n)$. Because $\Psi \subseteq \Phi$, we have $\Phi \vdash (\psi_1 \wedge \cdots \wedge \psi_n) \wedge \neg(\psi_1 \wedge \cdots \wedge \psi_n)$. However, the schema of a CPC theorem is the following schema: $\vdash (B_1 \wedge \cdots \wedge B_n) \wedge \neg(B_1 \wedge \cdots \wedge B_n) \to D$, a substitution of Duns Scotus law. Hence, we have: $\Phi \vdash \varphi$, for any formula φ from For. According to Definition 2.4, it means that the set Φ is inconsistent, which contradicts the assumption. □

Definition 5.4. A *model* $\mathcal{M}$ for the set $\mathsf{For}^{\mathrm{MPC}^-}$ is any pair $\langle U, d\rangle$, where U is a nonempty set (the universe of $\mathcal{M}$), d is a mapping from the set of individual letters IL into the set U, and from the set of unary predicates Pred into the set $\mathcal{P}(U)$. □

Let us notice that because the MPC^- language consists only of sentences (i.e., formulas not containing any free variables), the truth of any formula in any interpretation does not depend on the assignment of free variables.

Lemma 5.3. *Let $\Phi \subseteq \mathsf{For}$. The following facts are equivalent:*

(a) *there is a model $\mathfrak{M}$ for For such that $\mathfrak{M} \models \Phi$,*

(b) *there is a model $\mathcal{M}$ for $\mathsf{For}^{\mathrm{MPC}^-}$ such that $\mathcal{M} \models \mathrm{T}(\Phi)$.*

Proof. For any model $\mathfrak{M} = \langle W, \mathfrak{d}, \mathfrak{v}\rangle$ for For we put, $U := W$, $d(c_i) := \mathfrak{d}(a_i)$ and $d(P_i) := \{w \in W : \mathfrak{v}(w, p_i) = 1\}$, for any $i = 1, 2, \ldots$. Let $\mathcal{M} := \langle U, d\rangle$; $\mathcal{M}$ is a model for $\mathsf{For}^{\mathrm{MPC}^-}$.

By induction on the construction of $A \in \mathsf{AE}$ we prove that for any $i = 1, 2, \ldots$:

$$(*) \qquad \mathcal{M} \models \mathrm{T}(\mathsf{R}_{a_i} A) \iff \mathfrak{M} \models \mathsf{R}_{a_i} A\,.$$

If $A = p_j$ for some $j = 1, 2, \ldots$: $\mathcal{M} \models \mathrm{T}(\mathsf{R}_{a_i} p_j)$ iff $\mathcal{M} \models P_j(c_i)$ iff $d(c_i) \in d(P_j)$ iff $\mathfrak{v}(d(c_i), p_j) = 1$ iff $\mathfrak{v}(\mathfrak{d}(a_i), p_j) = 1$ iff $\mathfrak{M} \models \mathsf{R}_{a_i} p_j$.

As inductive hypothesis, let us assume that B and C be such expressions from AE that satisfy the condition $(*)$. We have the following cases:

$\mathcal{M} \models \mathrm{T}(\mathsf{R}_{a_i} \neg B)$ iff $\mathcal{M} \models \mathrm{t}(a_i, \neg B)$ iff $\mathcal{M} \models \neg \mathrm{t}(a_i, B)$ iff $\mathcal{M} \not\models \mathrm{t}(a_i, B)$ iff $\mathcal{M} \not\models \mathrm{T}(\mathsf{R}_{a_i} B)$ iff $\mathfrak{M} \not\models \mathsf{R}_{a_i} B$ iff $\bar{\mathfrak{v}}(a_i, B) = 0$ iff $\bar{\mathfrak{v}}(a_i, \neg B) = 1$ iff $\mathfrak{M} \models \neg \mathsf{R}_{a_i} B$.

$\mathcal{M} \models \mathrm{T}(\mathsf{R}_{a_i}(B \wedge C))$ iff $\mathcal{M} \models \mathrm{t}(a_i, B \wedge C)$ iff $\mathcal{M} \models \mathrm{t}(a_i, B) \wedge \mathrm{t}(a_i, C)$ iff $\mathcal{M} \models \mathrm{t}(a_i, B)$ and $\mathcal{M} \models \mathrm{t}(a_i, C)$ iff $\mathcal{M} \models \mathrm{T}(\mathsf{R}_{a_i} B)$ and $\mathcal{M} \models \mathrm{T}(\mathsf{R}_{a_i} C)$ iff $\mathfrak{M} \models \mathsf{R}_{a_i} B$ and $\mathfrak{M} \models \mathsf{R}_{a_i} C$ iff $\bar{\mathfrak{v}}(a_i, B) = 1$ and $\bar{\mathfrak{v}}(a_i, C) = 1$ iff $\bar{\mathfrak{v}}(a_i, B \wedge C) = 1$ iff $\mathfrak{M} \models \mathsf{R}_{a_i}(B \wedge C)$.

Analogously, we prove $(*)$ for '$\vee$', '$\to$' and '$\leftrightarrow$'. Moreover, by induction on the construction of $\varphi \in \mathsf{For}$ we prove that for any $i = 1, 2, \ldots$:

$$(**) \qquad \mathcal{M} \models \mathrm{T}(\varphi) \iff \mathfrak{M} \models \varphi\,.$$

Let $\Phi \subseteq \mathsf{For}$. We assume that there is a model for For such that $\mathfrak{M} \models \Phi$. Then $\mathcal{M} \models \mathrm{T}(\Phi)$, by $(**)$.

Moreover, for any model $\mathcal{M} = \langle U, d\rangle$ of $\mathsf{For}^{\mathrm{MPC}^-}$ we put, $W := U$, $\mathfrak{d}(a_i) := d(c_i)$ and: $\mathfrak{v}(w, p_i) = 1$ if $w \in d(P_i)$, and $\mathfrak{v}(w, p_i) = 0$ if $w \notin d(P_i)$, for any $i = 1, 2, \ldots$. Let $\mathfrak{M} := \langle W, \mathfrak{d}, \mathfrak{v}\rangle$; $\mathfrak{M}$ is a model for For.

By induction on the construction of $A \in \mathsf{AE}$ and $\varphi \in \mathsf{For}$ we prove $(*)$ and $(**)$. Thus, by $(**)$, if there is a $\mathcal{M}$ of $\mathsf{For}^{\mathrm{MPC}^-}$ such that $\mathcal{M} \models \mathrm{T}(\Phi)$, then there is a model for For such that $\mathfrak{M} \models \Phi$. □

Now we can come back to the Completeness Theorem.

PROOF ON THEOREM 5.1. Let us take any set of formulas Φ and any formula φ. Let us assume that $\Phi \models \varphi$, but simultaneously $\Phi \nvdash \varphi$. So, from Lemma 2.3, we know that $\Phi \cup \{\neg\varphi\}$ is a consistent set. Hence, from the Lemma 5.2 we obtain that the set $\mathrm{T}(\Phi \cup \{\neg\varphi\})$ is also consistent one. Since this is a set of MPC formulas, we can apply to it the Gödel-Malcev Theorem: *Let Ψ be a set of sentences of first-order logic. If Ψ is consistent, then there is a model $\mathcal{M}$ such that $\mathcal{M} \models \Psi$.* Hence, there is a model $\mathcal{M}$ such that $\mathcal{M} \models \mathrm{T}(\Phi \cup \{\neg\varphi\})$. Therefore, from the Lemma 5.3, there is a model $\mathfrak{M}$ such that $\mathfrak{M} \models \Phi \cup \{\neg\varphi\}$, what contradicts the assumption that $\Phi \models \varphi$. □

6. Increasing the power of expression

The system **MR** can describe aspects of only such reasonings, in which only individual terms of positions occur. However, we would like to take into account also the sentences of the following type:

At some time the Polish-Lithuanian army won Grunwald Battle.

At any time it is raining in Toruń.

These sentences do not say anything about any specific positions in which the sentence is realized, but say that at some or at all positions something is realized. Shortly speaking, they include quantifiers.

If we would like to describe this linguistic phenomenon, we should extend the language, adding to the alphabet: *i*) quantifiers '∀' and '∃', *ii*) positional variables: $\mathsf{Var} = \{x, x_1, x_2, x_3, \ldots\}$, and allow the new grammar forms: '$\mathrm{R}_{x_i}A$', '$\forall_{x_i}\varphi$', '$\exists_{x_i}\varphi$' etc. Let us denote the outlined language by $\mathsf{For}^{\mathsf{q}}$. It needs some new deductive tools, concerning rules of quantifying. As axioms, we could, for example, add all schemas of MPC theorems and the generalization rule.

We have also to modify the specific axioms, allowing positional variables. Taking into consideration the new tools and modifying the notion of proof, we obtain the system $\mathbf{MR}^{\mathsf{q}}$ with quantifiers.

($\mathrm{Ax0}_x$) An axiom of $\mathbf{MR^q}$ is any formula from $\mathsf{For^q}$ which is built with exactly one variable 'x' (without positional letters) and is a substitution of some tautology of CPC.

We introduce specific axiom for any atomic expressions $A, B \in \mathsf{AE}$:

($\mathrm{Ax1}_x$) $$\mathsf{R}_x\neg A \leftrightarrow \neg\mathsf{R}_x A\,,$$

($\mathrm{Ax2}_x$) $$\mathsf{R}_x A \wedge \mathsf{R}_x B \to \mathsf{R}_x(A \wedge B)\,,$$

Moreover, if $A \in \mathsf{AE}$ is a theorem of CPC, the following formula is an axiom:

($\mathrm{Ax3}_x$) $$\mathsf{R}_x A\,.$$

One can easily see that the constructed system $\mathbf{MR^q}$ is an extension of the initial system **MR**. Indeed, by the generalization rule and the axiom $\ulcorner\forall_x\,\varphi(x) \to \varphi(a_i)\urcorner$ we obtain the axioms (Ax0)–(Ax3) of **MR**.

The description of $\mathbf{MR^q}$ from semantic aspects would need some new notions, such as an assignment of positional variables and being satisfied in a model under an assignment. After consideration of new cases with quantifiers and positional variables, we could define an interpretation $\mathfrak{I}$ as a pair $\langle\mathfrak{M}, \mathfrak{a}\rangle$, where $\mathfrak{M}$ is a model and $\mathfrak{a}$ is an assignment of variables from Var into the universe of $\mathfrak{M}$. Truth in an interpretation $\mathfrak{I} = \langle\mathfrak{M}, \mathfrak{a}\rangle$ means being satisfied in the model $\mathfrak{M}$ under the assignment $\mathfrak{a}$. Modifying prior proofs from earlier sections, in the obvious way, we could prove the Theorem on the Correctness as well as the Completeness Theorem of the extended system $\mathbf{MR^q}$.

We shall show that $\mathbf{MR^q}$ is a conservative extension of **MR**.

THEOREM 6.1. *For any $\varphi \in \mathsf{For}$: φ is a theorem of* **MR** *iff φ is a theorem of* $\mathbf{MR^q}$.

PROOF. Let $\varphi \in \mathsf{For}$. If φ is a theorem of **MR**, then φ is also a theorem of $\mathbf{MR^q}$. On the other hand, let φ is a theorem of $\mathbf{MR^q}$. Then, by the Completeness Theorem for $\mathbf{MR^q}$, φ is a tautology, i.e., φ is true in any interpretation $\mathfrak{I} = \langle\mathfrak{M}, \mathfrak{a}\rangle$. Since φ includes neither quantifiers nor positional variables from Var, so φ is true in any model $\mathfrak{M}$. Hence, by the Completeness Theorem for **MR**, we have that φ is a theorem of **MR**. □

Theorem 6.1 says that there are not any specific, new theorems expressed in the old language, that can not be proved in the system **MR**.

It is possible to express some other interesting property of the system $\mathbf{MR^q}$. It has analogous features like every open first-order theory.

At first, we remind some facts concerning first-order theories. We use the following well-known symbols: a first-order theory $\mathscr{T} = \langle \mathscr{L}, \mathscr{C}, Ax \rangle$, where $\mathscr{L}$ is a language, $\mathscr{C}$ is the consequence operation on $\mathscr{L}$ determined by deductive tools of Classical Predicate Calculus and, finally, Ax is a set of specific axioms. The language is an ordered triple $\mathscr{L} = \langle A, T, F \rangle$, where A is alphabet, T is the set of all terms and F is the set of all formulas.

The symbol A^{o} will denote the alphabet that we obtain from A, when we omit quantifiers and individual letters. The set of all formulas built with A^{o} will be denoted by F^{o} and its members will be called open formulas. The language of open formulas $\mathscr{L}^{\mathrm{o}} = \langle A^{\mathrm{o}}, T^{\mathrm{o}}, F^{\mathrm{o}} \rangle$. The sign $\mathscr{C}^{\mathrm{o}}$ will denote consequence operation on $\mathscr{L}^{\mathrm{o}}$ determined by MP, the rule of substitution for free individual variables and a set of schemas of CPC axioms, where metavariables represent members of F^{o}.

Preserving the above symbols, there is a generally true theorem for all first-order theories:

Theorem ([3]). *If* $\Psi \subseteq F^{\mathrm{o}}$ *then* $\mathscr{C}^{\mathrm{o}}(\Psi) = \mathscr{C}(\Psi) \cap F^{\mathrm{o}}$. □

The very analogous theorem we can obtain for the consequence relations determined by the system $\mathbf{MR^q}$.

7. Summary

The R operator had almost the extensional interpretation. The statement was realized at a point, when it was true at this point. As we said at the beginning, we can interpret it in a more intentional way. If one assumes that the set of positions has a structure, R can be interpreted as follows: a statement is realized at some position a, when from a is accessible a position b (not necessarily different from a) at which the sentence is true. This interpretation makes us weaken the axioms. For example, in some structure, from one position can be accessible positions, such that at the former one the sentence $\neg p$ is true, but at the latter one the sentence p is true. This example makes the axiom schema (Ax1) be false. The approach provides new interesting questions about minimal axioms for the intensional interpretation of R and the further extensions following from the class of structures that can be defined by some formulas.

Acknowledgments. We would like to thank Marek Nasieniewski for his many valuable remarks. We do of course bear all responsibility for all logical as well as philosophical mistakes in this paper.

References

[1] Garson, J., and N. Rescher, "Topological logic", *Journal of Symbolic Logic*, 33 (1968): 537–548.

[2] Łoś, J., "Podstawy analizy metodologicznej kanonów Milla", *Annales Universitatis Mariæ Curie-Skłodowska*, 2 (1948): 269–301.

[3] Rasiowa, H., and R. Sikorski, *Mathematics of Metamathematics*, Warszawa, second edition, PWN, 1968.

[4] Rescher, N., "On the logic of chronological propositions", *Mind*, 77 (1966): 75–96.

[5] Rescher, N., and Urquhart A., *Temporal Logic*, Vienna, Springer-Verlag, 1971.

Tomasz Jarmużek
Andrzej Pietruszczak
Nicolaus Copernicus University
Department of Logic and Semiotics
ul. Asnyka 2
PL 87-100 Toruń, Poland
`{jarmuzek,pietrusz}@uni.torun.pl`