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INTERPOLATION AND IMPLICIT DEFINABILITY IN EXTENSIONS OF THE PROVABILITY LOGIC*

Abstract. The provability logic GL was in the field of interest of A.V. Kuznetsov, who had also formulated its intuitionistic analog—the intuitionistic provability logic—and investigated these two logics and their extensions.

In the present paper, different versions of interpolation and of the Beth property in normal extensions of the provability logic GL are considered. It is proved that in a large class of extensions of GL (including all finite slice logics over GL) almost all versions of interpolation and of the Beth property are equivalent. It follows that in finite slice logics over GL the three versions CIP, IPD and IPR of the interpolation property are equivalent. Also they are equivalent to the Beth properties B1, PB1 and PB2.

Keywords: provability logic, interpolation, Beth property.

1. Introduction

We study interpolation problem for the family of normal extensions of the well known provability logic GL. This logic has evoked a special interest due to its completeness with respect to an arithmetical provability interpretation [23]. A.V. Kuznetsov had formulated a proof-intuitionistic logic, which is an intuitionistic analog of the provability logic GL, and investigated these two provability logics and their extensions [5, 6, 7].

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Interpolation theorem proved by W. Craig [3] in 1957 for the classical first order logic was a source of a lot of investigation devoted to interpolation problem in various logical theories. Now interpolation is considered as a standard property of logics and calculi like consistency, completeness and so on. It is closely connected with so-called Beth definability properties. The Beth properties have as their source the theorem on implicit definability proved by E. Beth in 1953 [1] for the classical first order logic: Any predicate implicitly definable in a first order theory is explicitly definable.

The original definition of interpolation admits various analogs which are equivalent in the classical logic but are not equivalent in other logics. The same is true for the Beth property. In [4, 20] several versions CIP, IPD, IPR and WIP (of interpolation) and B1, B2, PB1, PB2 (of the Beth property) are considered and a picture of their inter-relations in different logics is presented. For any normal modal logic we have

$$\begin{aligned}&\mathrm{CIP} \iff \mathrm{B1} \iff \mathrm{PB1} \Rightarrow \mathrm{IPD} \Rightarrow \mathrm{IPR} \Rightarrow \mathrm{WIP}\,,\\&\mathrm{CIP} \Rightarrow (\mathrm{B2}+\mathrm{IPD}) \Rightarrow \mathrm{PB2} \Rightarrow \mathrm{B2},\ \mathrm{PB2} \Rightarrow \mathrm{IPR}\,,\end{aligned}$$

and in general all the arrows are strict; moreover, the pairs IPD and PB2, IPD and B2, IPR and B2, WIP and B2 are independent. All normal extensions of the modal logic K4 have the property B2 and in all normal logics over K4 we have

$$\mathrm{CIP} \Rightarrow \mathrm{IPD} \Rightarrow \mathrm{PB2} \Rightarrow \mathrm{IPR} \Rightarrow \mathrm{WIP}\,,$$

and all the arrows are strict.

It was proved in [19] that the restricted interpolation property IPR implies the projective Beth property PB2 in positive logics, in extensions of a superintuitionistic logic KC and of a modal logic Grz.2, and so IPR is equivalent to PB2 in those logics. It was stated in [18] that over the modal S5 all the properties IPR, IPD, CIP, PB2 and WIP are equivalent. On the contrary, all extensions of the Grzegorczyk logic Grz and all superintuitionistic logics have WIP, and PB2 implies neither IPD nor CIP, which are equivalent for these logics.

In the present paper we consider the family of normal extensions of the provability logic GL and study interrelations of various versions of interpolation and of the Beth property over GL.

It is known that all normal extensions of the provability logic GL possess B2 and WIP [10, 18]. As for the other properties under consideration, we have already seen that CIP is the strongest of them in the family of normal modal logics and IPR is implied by B1, PB1, PB2 and IPD. Over GL, there is a continuum of logics with CIP [9] and there is a continuum of logics without IPR [14].

We define a large class of extensions of GL (containing all finite slice logics over GL), in which IPR implies CIP. As a consequence, for all logics of this class the variants CIP, IPD and IPR of interpolation, the Beth property B1 and the projective Beth properties PB1 and PB2 are equivalent.

There is a duality between normal logics over GL and varieties of diagonalizable algebras. Also interpolation properties have the corresponding variants of the amalgamation property as their algebraic equivalent. An algebraic counterpart of our results is obtained: for any locally finite variety of diagonalizable algebras, the following properties are equivalent: the amalgamation property, the strong amalgamation property, the superamalgamation property, the restricted amalgamation property and the strong epimorphisms surjectivity.

2. Beth properties and interpolation

If $\mathbf{p}$ is a list of variables, let $A(\mathbf{p})$ denote a formula whose all variables are in $\mathbf{p}$, and $\mathcal{F}(\mathbf{p})$ the set of all such formulas.

Let L be a logic, $\vdash_L$ deducibility relation in L. We mean that the language contains at least one propositional constant $\top$ ("true") or $\bot$ ("false"). Suppose that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{q}'$ are disjoint lists of variables that do not contain x and y, $\mathbf{q}$ and $\mathbf{q}'$ are of the same length, and $A(\mathbf{p},\mathbf{q},x)$ is a formula. We define two variants PB1 and PB2 of *the projective Beth property*:

PB1 If $\vdash_L A(\mathbf{p},\mathbf{q},x)\ \&\ A(\mathbf{p},\mathbf{q}',y) \to (x \leftrightarrow y)$, then $\vdash_L A(\mathbf{p},\mathbf{q},x) \to (x \leftrightarrow B(\mathbf{p}))$, for some formula $B(\mathbf{p})$.

PB2 If $A(\mathbf{p},\mathbf{q},x), A(\mathbf{p},\mathbf{q}',y) \vdash_L x \leftrightarrow y$, then $A(\mathbf{p},\mathbf{q},x) \vdash_L x \leftrightarrow B(\mathbf{p})$, for some $B(\mathbf{p})$.

The Beth properties B1 and B2 arise from PB1 and PB2 respectively by omitting $\mathbf{q}$ and $\mathbf{q}'$:

B1 If $\vdash_L A(\mathbf{p},x)\ \&\ A(\mathbf{p},y) \to (x \leftrightarrow y)$, then $\vdash_L A(\mathbf{p},x) \to (x \leftrightarrow B(\mathbf{p}))$, for a suitable formula $B(\mathbf{p})$.

B2 If $A(\mathbf{p},x), A(\mathbf{p},y) \vdash_L x \leftrightarrow y$, then $A(\mathbf{p},x) \vdash_L x \leftrightarrow B(\mathbf{p})$, for a formula $B(\mathbf{p})$.

The Beth properties are closely connected with *the Craig interpolation property*, CIP, and *the deductive interpolation property*, IPD, defined as follows (where the lists $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are disjoint):

CIP If $\vdash_L A(\mathbf{p},\mathbf{q}) \to B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $\vdash_L A(\mathbf{p},\mathbf{q}) \to C(\mathbf{p})$ and $\vdash_L C(\mathbf{p}) \to B(\mathbf{p},\mathbf{r})$.

IPD If $A(\mathbf{p},\mathbf{q}) \vdash_L B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$ and $C(\mathbf{p}) \vdash_L B(\mathbf{p},\mathbf{r})$.

We also consider a *restricted interpolation property* introduced in [16]:

IPR If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$.

A *weak interpolation property* WIP [18] is a particular case of IPR:

WIP If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$, then there is a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$.

The most known modal logics such as Lewis' systems S4 and S5, Grzegorczyk's logic Grz, the logic GL of provability, the logic K4 and the least normal modal logic K have the properties CIP and PB1.

A *normal modal logic* is any set of modal formulas containing all the tautologies of the two-valued propositional logic and the axiom $\Box(A \to B) \to (\Box A \to \Box B)$, and closed under the inference rules R1: $A, A \to B/B$ and R2: $A/\Box A$, and the substitution rule. The set of normal modal logics containing a normal modal logic L is denoted by NExt(L). If L is a normal modal logic, by $\vdash_L$ we denote deducibility in L by the rules R1 and R2.

Recall the standard denotations for some normal modal logics:

$$\begin{aligned}
\mathrm{K4} &= \mathrm{K} + (\Box p \to \Box\Box p),\\
\mathrm{GL} &= \mathrm{K4} + (\Box(\Box p \to p) \to \Box p),\\
\mathrm{S4} &= \mathrm{K4} + (\Box p \to p),\\
\mathrm{S4.1} &= \mathrm{S4} + (\Box\Diamond p \to \Diamond\Box p),\\
\mathrm{Grz} &= \mathrm{S4} + (\Box(\Box(p \to \Box p) \to p) \to p),\\
\mathrm{Grz.2} &= \mathrm{Grz} + (\Diamond\Box p \to \Box\Diamond p),\\
\mathrm{S5} &= \mathrm{S4} + (p \to \Box\Diamond p).
\end{aligned}$$

Interrelations of interpolation with Beth properties are presented in the following

Proposition 2.1 ([12, 14, 16, 18]). *In the family of normal modal logics*

(1) PB1 *is equivalent to each of the properties* CIP *and* B1,

(2) PB1 *implies the conjunction of* B2 *and* IPD *but the converse does not hold,*

(3) *the conjunction of* B2 *and* IPD *implies* PB2*,*

(4) PB2 *implies* B2 *but the converse is not true,*

(5) PB2 *and* IPD *are independent,* B2 *and* IPD *are independent,*

(6) PB2 *implies* IPR *and* IPR *implies* WIP*,*

(7) WIP *and* B2 *are independent.*

Since all logics over K4 have B2 [10], we obtain

PROPOSITION 2.2. *For any logic in* NExt (K4)*:*

$$\mathrm{CIP} \Rightarrow \mathrm{IPD} \Rightarrow \mathrm{PB2} \Rightarrow \mathrm{IPR} \Rightarrow \mathrm{WIP}.$$

In logics over S5 all these properties are equivalent [18]. On the contrary, over S4, neither CIP is equivalent to IPD nor IPD to PB2 [4, 17]. Further, all logics over S4.1 and over GL have WIP [18]. On the other hand, IPR is equivalent to PB2 over Grz.2 and only finitely many logics over Grz.2 have IPR [19], although IPR is not equivalent to IPD in NExt(Grz.2) [17].

Recall that the family of logics with CIP in NExt(GL) has the cardinality of continuum but there is also a continuum of logics without CIP in this family [9]. In [8] we have found a logic Gγ in NExt(GL), which possesses CIP but is neither finitely axiomatizable nor finitely approximable. At the same time the logic Gγ is decidable and is the greatest among infinite slice logics with IPR in NExt(GL) [14]. Recall that $L \in$ NExt(GL) is a logic *of finite slice* if $L \vdash \Box^n\bot$ for some $n \geqslant 0$, and *of infinite slice* otherwise. It is known that L is of finite slice of the number n if the length of chains in Kripke frames satisfying L is bounded by n; L is of infinite slice if it admits chains of any length.

We divide the family NExt(GL) into two parts: the lower part consisting of all logics contained in Gγ and the upper part

$$\mathrm{U(GL)} = \{L \in \mathrm{NExt(GL)} \mid L \not\subseteq \mathrm{G}\gamma\}.$$

In this paper we prove that for any logic in U(GL), IPR implies IPD, so for all such logics we have:

$$\mathrm{CIP} \iff \mathrm{IPD} \iff \mathrm{IPR} \iff \mathrm{B1} \iff \mathrm{PB1} \iff \mathrm{PB2}\,.$$

Of course, on U(GL) all the six properties differ from WIP and from B2, which are satisfied in all logics of this class.

3. Interpolation and amalgamation

An algebraic equivalent of IPR was found in [16]. For the properties B1 and B2, and also for CIP and IPD it was done in [11], and for PB2 in [14].

It is well known that there exists a duality between normal modal logics and varieties of modal algebras. *A modal algebra* is an algebra $\mathbf{A} = (A, \to, \neg, \Box)$ that is a boolean algebra with respect to $\to$ and $\neg$ and, moreover, satisfies the conditions $\Box\top = \top$ and $\Box(x \to y) \le \Box x \to \Box y$. A modal algebra $\mathbf{A}$ is called *transitive* if it satisfies the inequality $\Box x \le \Box\Box x$; *a topoboolean algebra*, or *interior algebra* if it satisfies $\Box x \le x$; a *diagonalizable* algebra if it satisfies $\Box(\Box x \to x) = \Box x$. It is well known that the modal logic K4 can be characterized by the variety of all transitive algebras, S4 by the variety of topoboolean algebras, GL by diagonalizable algebras.

By a *valuation* in an algebra $\mathbf{A}$, we mean, as usual, a homomorphism from the algebra of all formulas into $\mathbf{A}$.

If A is a formula, $\mathbf{A}$ a modal algebra, then $\mathbf{A} \models A$ denotes that the identity $A = \top$ is satisfied in $\mathbf{A}$. We write $\mathbf{A} \models L$ instead of $(\forall A \in L)(\mathbf{A} \models A)$. We denote $\mathrm{V}(L) = \{\mathbf{A} \mid \mathbf{A} \models L\}$. Each normal modal logic L is characterized by the variety $\mathrm{V}(L)$.

All varieties of modal algebras possess such important properties as congruence-distributivity and congruence extension property:

CEP If $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ then every congruence Φ on $\mathbf{A}$ can be extended to a congruence Ψ on $\mathbf{B}$ such that $\Psi \cap \mathbf{A}^2 = \Phi$.

We recall the definitions.

A class V has *Amalgamation Property* if it satisfies the condition

AP For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and monomorphisms $\delta\colon \mathbf{B} \to \mathbf{D}$ and $\epsilon\colon \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \epsilon(x)$, for all $x \in \mathbf{A}$.

A class V has *Strong Amalgamation Property* (StrAP) if it satisfies AP and, moreover, $\delta(\mathbf{B}) \cap \epsilon(\mathbf{C}) = \delta(\mathbf{A})$.

A class V has *Super-Amalgamation Property* (SAP) if it satisfies AP and, in addition:

$$\delta(x) \le \epsilon(y) \iff (\exists z \in \mathbf{A})(x \le z \text{ and } z \le y),$$
$$\delta(x) \ge \epsilon(y) \iff (\exists z \in \mathbf{A})(x \ge z \text{ and } z \ge y).$$

A class V has *Strong Epimorphisms Surjectivity* if it satisfies

SES For each $\mathbf{A}$, $\mathbf{B}$ in V, for every monomorphism $\alpha\colon \mathbf{A} \to \mathbf{B}$ and for every $x \in \mathbf{B} - \alpha(\mathbf{A})$ there exist $\mathbf{C} \in V$ and homomorphisms $\beta\colon \mathbf{B} \to \mathbf{C}$, $\gamma\colon \mathbf{B} \to \mathbf{C}$ such that $\beta\alpha = \gamma\alpha$ and $\beta(x) \neq \gamma(x)$.

A class V has *Restricted Amalgamation Property* if it satisfies the condition:

RAP For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$ there exist an algebra $\mathbf{D}$ in V and homomorphisms $\delta\colon \mathbf{B} \to \mathbf{D}$, $\varepsilon\colon \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$, for all $x \in \mathbf{A}$ and the restriction δ' of δ onto $\mathbf{A}$ is a monomorphism.

Theorem 3.3. *Let L be a normal modal logic.*

(1) *L has* CIP *if and only if* $\mathrm{V}(L)$ *has* SAP.

(2) *L has* IPD *if and only if* $\mathrm{V}(L)$ *has* AP.

(3) *L has* PB2 *if and only if* $\mathrm{V}(L)$ *has* SES.

(4) *L has* IPR *if and only if* $\mathrm{V}(L)$ *has* RAP.

Proof. (1) and (2) are proved in [11], (3) in [12] and (4) in [16]. ⊣

It follows that B1 and PB1 are also equivalent to the super-amalgamation property of the corresponding variety.

We need some criteria for AP and RAP in varieties of modal algebras. Recall that a modal algebra is *subdirectly irreducible* (*finitely indecomposable*) if it can not be represented as a subdirect product (finite subdirect product) of its proper quotient algebras. For any class V of algebras, by $\mathrm{FI}(V)$ and $\mathrm{SI}(V)$ we denote the classes of finitely indecomposable and subdirectly irreducible algebras in V respectively, $\mathrm{FG}(V)$ stands for finitely generated algebras in V.

Theorem 3.4 ([11]). *For any logic L in* $\mathrm{NExt}(\mathrm{K})$ *the following are equivalent:*

(a) *L has* IPD,

(b) $\mathrm{V}(L)$ *has* AP,

(c) *for any finitely indecomposable $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ in $\mathrm{V}(L)$ such that $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$ there exist $\mathbf{D}$ in $\mathrm{V}(L)$ and monomorphisms $g\colon \mathbf{B} \to \mathbf{D}$, $h\colon \mathbf{C} \to \mathbf{D}$ such that $g(z) = h(z)$, for all z in $\mathbf{A}$.*

Denote $[*]x = x \,\&\, \Box x$. It is known that a transitive algebra is finitely indecomposable iff it satisfies the condition:

$$[*]x \vee [*]y = \top \Rightarrow ([*]x = \top \text{ or } [*]y = \top).$$

We say that an element a of a transitive algebra $\mathbf{A}$ is an *opremum* of $\mathbf{A}$ if $a \neq \top$, $[*]a = a$ and $(\forall x \in \mathbf{A})(x \neq \top \Rightarrow [*]x \leq a)$. Recall that a transitive algebra is subdirectly irreducible iff it has an opremum; it is easy to see that an opremum is unique.

Say that a monomorphism $\alpha\colon \mathbf{A} \to \mathbf{B}$ of two subdirectly irreducible transitive algebras is *o-preserving* if it maps the opremum of $\mathbf{A}$ into opremum of $\mathbf{B}$. Another variant of the *Restricted Amalgamation Property* of a class V was defined in [14] as follows:

RAP* for any o-preserving monomorphisms $\beta\colon \mathbf{A} \to \mathbf{B}$ and $\gamma : \mathbf{A} \to \mathbf{C}$ of subdirectly irreducible algebras in V there exist an algebra $\mathbf{D}$ in V and monomorphisms $\delta\colon \mathbf{B} \to \mathbf{D}$, $\varepsilon\colon \mathbf{C} \to \mathbf{D}$ such that $\delta\beta = \varepsilon\gamma$.

In [16] we proved

Theorem 3.5. *For any logic L in* NExt(K4) *the following are equivalent*:

(a) *L has* IPR,

(b) V(L) *has* RAP,

(c) V(L) *has* RAP*.

4. Equivalence of IPR and CIP in some extensions of GL

In this section we prove that IPR implies CIP for any logic in the family U(GL) defined in Section 1, and so CIP, IPD, RAP and PB2 are equivalent in this family. Remember that these four properties are equivalent over S5 but non-equivalent over S4 and even over Grz. More exactly, CIP does not follow from IPD over S4, IPD does not follow from PB2 in finite slice logics over Grz (and so over S4). Also PB2 differs from IPD in superintuitionistic and positive logics[13, 15]. On the other hand, IPR implies PB2 over Grz.2 [19].

Recall that a modal logic L containing K4 is a logic *of finite slice* if L contains a formula φ_n for some n, where $\varphi_0 = \bot$, $\varphi_{n+1} = p_{n+1} \to \Box(\Box\neg p_{n+1} \to \varphi_n)$. For logics over GL, one can take some simpler formulas. Namely, a logic L in NExt(GL) is a finite slice logic iff it contains a formula $\Box^n\bot$ for some n. It is known that L is a logic of infinite slice if and only if it is contained in the logic

$$\mathrm{G.3} = \mathrm{GL} + ([*]([*]p \to q) \vee [*]([*]q \to p)).$$

Recall that a logic L is said to be *locally tabular* if for any finite set of propositional variables there are only finitely many formulas of these variables non-equivalent in L. The logic L is locally tabular if and only if the variety $\mathrm{V}(L)$ is locally finite, i.e. any finitely generated algebra in $\mathrm{V}(L)$ is finite.

LEMMA 4.6 ([2, 22]). *A logic L in* NExt(K4) *is locally tabular iff it is a logic of finite slice.*

Remember that we defined an upper part of NExt(GL) as follows:

$$\mathrm{U(GL)} = \{L \in \mathrm{NExt(GL)} \mid L \not\subseteq \mathrm{G}\gamma\}.$$

The logic $\mathrm{G}\gamma$ is defined by

$$\mathrm{G}\gamma = \mathrm{GL} + \{\gamma_k(p) \mid k < \omega\},$$

where for any $k \geqslant 0$:

$$\begin{aligned}
\alpha_k &= \Box^{k+1}\bot \mathbin{\&} \neg\Box^k\bot,\\
\beta_k(p) &= [*](\alpha_k \to p) \vee [*](\alpha_k \to \neg p),\\
\gamma_k(p) &= \beta_0(p) \mathbin{\&} \cdots \mathbin{\&} \beta_k(p).
\end{aligned}$$

The logic $\mathrm{G}\gamma$ is a logic of infinite slice, so the family U(GL) contains all finite slice logics in NExt(GL).

Our interest to the logic $\mathrm{G}\gamma$ arises from

PROPOSITION 4.7 ([8, 14]). *The logic* $\mathrm{G}\gamma$ *has* CIP *and is the greatest among infinite slice logics with* IPR *in* NExt(GL).

It was proved in [8] (see also [4]) that the logic $\mathrm{G}\gamma$ is neither finitely axiomatizable nor finitely approximable. At the same time it is Kripke complete and decidable. Its characterization by so-called ω-linear frames is presented in [4]. We recall the definition. It is known [22] that the logic GL is complete with respect to the class of GL-frames. A frame $W = (W, R)$ is a GL-*frame* if R is irreflexive and transitive and there is no infinite increasing chain in W. A GL-frame W is said to be ω-*linear* if the set $\mathrm{Fin}(W) = \{x \in W \mid h(x) < \omega\}$ forms a chain. Here $h(x)$ is the supremum of the cardinalities of chains in W with origin at x.

We note that the class U(GL) is decidable, i.e. there is an algorithm which, for any finite system Ax of new axiom schemes, decides if the logic $\mathrm{GL} + Ax$ belongs to U(GL). In order to show that, we use the following lemma, which is a part of Theorem 12.10 of [4].

LEMMA 4.8. *Let A_0 be any formula, r the number of subformulas of A_0, and let $s = 2^r$. Then the following are equivalent:*

(a) $A_0 \in \mathrm{G}\gamma$,

(b) *A_0 is valid in all ω-linear frames,*

(c) $\mathrm{GL} \vdash (\gamma_s(\bot) \ \&\ \gamma_s(p_1) \ \&\ \cdots \ \&\ \gamma_s(p_k)) \to A_0$, *where $p_0, \ldots, p_k$ are all the variables of A_0.*

PROPOSITION 4.9. (1) *There is an algorithm which, for any finite system Ax of new axiom schemes, decides if the logic $\mathrm{GL} + Ax$ belongs to U(GL).*

(2) *There is an algorithm which, for any finite system Ax of new axiom schemes, decides if the logic $\mathrm{GL} + Ax$ is a logic of finite slice.*

PROOF. Let $L = \mathrm{GL} + Ax$. Denote by A_0 the conjunction of all formulas in Ax. We have $L \in \mathrm{U(GL)}$ if and only if $A_0 \notin \mathrm{G}\gamma$. By Lemma 4.8 and decidability of GL [22], we obtain the statement (1). To prove (2), we note that a logic L over GL is a logic of finite slice if and only if $\mathrm{G.3} \nvdash A_0$; moreover, the logic G.3 is decidable [22]. ⊣

Now we show that IPR implies CIP in all logics of U(GL). First we prove the following statements.

LEMMA 4.10. *Let $\mathbf{A}$ be a subdirectly irreducible diagonalizable algebra and $a \in \mathbf{A}$. Then a is an opremum if and only if $a < \top$ and $\Box a = \top$.*

PROOF. Let $\mathbf{A}$ be subdirectly irreducible and Ω its opremum. Then $\Omega < \top$ and $[*]\Omega = \Omega \ \&\ \Box\Omega = \Omega$, i.e. $\Omega \leq \Box\Omega$.

Suppose that $\Box\Omega < \top$. Then $[*]\Box\Omega \leq \Omega$ by the definition of opremum. On the other hand, $[*]\Box\Omega = \Box\Omega \ \&\ \Box\Box\Omega = \Box\Omega$, so $\Box\Omega \leq \Omega$. Remember that $\mathrm{GL} \vdash \Box(\Box p \to p) \to \Box p$. It follows that $\Box(\Box x \to x) \leq \Box x$ for any diagonalizable algebra and any x. We obtain

$$\top = \Box(\Box\Omega \to \Omega) \leq \Box\Omega\,,$$

so $\Box\Omega = \top$—a contradiction. Thus $\Box\Omega = \top$.

For the converse, let a be any element in $\mathbf{A}$ such that $a < \top$ and $\Box a = \top$. We show that $a = \Omega$. As Ω is an opremum, it is clear that $a = [*]a \leq \Omega$. Assume that $a < \Omega$. Then we have $\Omega \to a < \top$, $[*](\Omega \to a) = (\Omega \to a)\ \&\ \Box(\Omega \to a) = (\Omega \to a)\ \&\ \top = (\Omega \to a) \leq \Omega$, so $\neg\Omega \leq \Omega$ and $\Omega = \top$ – a contradiction. Thus $a = \Omega$. ⊣

Lemma 4.11. *Let a diagonalizable algebra* $\mathbf{A}$ *be finitely indecomposable. Then* $\mathbf{A}$ *is subdirectly irreducible if and only if there is* $a \in \mathbf{A}$ *such that* $a < \top$ *and* $\Box a = \top$.

Proof. If $\mathbf{A}$ is subdirectly irreducible, its opremum a satisfies the required conditions by Lemma 4.10.

Assume that $\mathbf{A}$ is finitely indecomposable and there is $a \in \mathbf{A}$ such that $a < \top$ and $\Box a = \top$. We show that a is an opremum of $\mathbf{A}$. It is clear that $[*]a = a$. We note that for any $x \in \mathbf{A}$:

$$[*]x \vee [*]([*]x \to a) = \top. \tag{†}$$

Indeed, by distributivity we have

$$[*]x \vee [*]([*]x \to a) = ([*]x \vee ([*]x \to a))\ \&\ ([*]x \vee \Box([*]x \to a)) \geq \\ \geq \top\ \&\ \Box a = \top.$$

Take any $x \neq \top$. Since $\mathbf{A}$ is finitely indecomposable, by (†) we have $[*]([*]x \to a) = \top$ and so $[*]x \leq a$. Thus $\mathbf{A}$ has an opremum and is subdirectly irreducible. ⊣

Lemma 4.12. *Let* $\mathbf{A}$ *be a subdirectly irreducible diagonalizable algebra,* $\mathbf{B}$ *finitely indecomposable diagonalizable algebra and* α *a monomorphism of* $\mathbf{A}$ *into* $\mathbf{B}$. *Then* $\mathbf{B}$ *is subdirectly irreducible and* α *is o-preserving, i.e.* $\alpha(\Omega_{\mathbf{A}}) = \Omega_{\mathbf{B}}$.

Proof. Let $b = \alpha(\Omega_{\mathbf{A}})$. Then, by Lemma 4.10, $b < \top$ and $\Box b = \alpha(\Box\Omega_{\mathbf{A}}) = \top$. By Lemma 4.11, $\mathbf{B}$ is subdirectly irreducible. ⊣

Moreover, immediately by Lemma 4.11 we obtain

Lemma 4.13. *Let a diagonalizable algebra* $\mathbf{A}$ *be finitely indecomposable,* $\mathbf{A} \models \Box^{n+1}\bot$ *and* $\mathbf{A} \not\models \Box^n\bot$. *Then* $\mathbf{A}$ *is subdirectly irreducible and* $\Omega = \Box^n\bot$ *is its opremum.*

Remember that $\mathrm{U(GL)} = \{L \in \mathrm{NExt(GL)} \mid L \not\subseteq \mathrm{G}\gamma\}$.

Proposition 4.14. *Let* $L \in \mathrm{U(GL)}$ *be a logic with* IPR. *Then* L *has* CIP.

Proof. Assume that $L \in \mathrm{U(GL)}$ possesses IPR. Then $\mathrm{V}(L)$ has RAP. We consider two cases: (1) L is a logic of infinite slice and (2) L is a logic of finite slice.

Case 1. It was proved in [14], Theorem 5.6, that for any infinite slice logic $L' \in \mathrm{NExt(GL)}$ with IPR, the variety $\mathrm{V}(L')$ contains $\mathrm{V(G}\gamma)$. Then $\mathrm{V}(L)$ contains $\mathrm{V(G}\gamma)$ and $L \subseteq \mathrm{G}\gamma$. Thus $L = \mathrm{G}\gamma$, so L possesses CIP by Proposition 4.7.

Case 2. Let L be a logic of finite slice. There is a k such that $L \vdash \Box^k\bot$. We prove that $\mathrm{V}(L)$ has the amalgamation property AP. Take any finitely indecomposable algebras $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$. Take the least n such that $\mathbf{A} \models \Box^n\bot$. If $n = 0$, then all the three algebras are degenerate, so there is an amalgam of these algebras. If $n > 0$, then $\mathbf{A}$ is subdirectly irreducible and $\Omega = \Box^{n-1}\bot$ is an opremum of $\mathbf{A}$ by Lemma 4.13. By Lemma 4.12 it follows that $\mathbf{B}$ and $\mathbf{C}$ are subdirectly irreducible and the identity monomorphisms from $\mathbf{A}$ into $\mathbf{B}$ and $\mathbf{C}$ are o-preserving. By RAP* there are some $\mathbf{D} \in \mathrm{V}(L)$ and monomorphisms $\delta\colon \mathbf{B} \to \mathbf{D}$ and $\varepsilon\colon \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$. Thus, by Theorem 3.4, $\mathrm{V}(L)$ has AP and L has IPD.

It was proved in [9] that IPD implies CIP in all finite slice logics in NExt(GL). ⊣

Theorem 4.15. *For any logic in* U(GL), *all the properties* CIP, IPD, IPR, B1, PB1, PB2 *are equivalent.*

Proof. By propositions 2.1 and 2.2, over GL we have

$$\mathrm{CIP} \iff \mathrm{B1} \iff \mathrm{PB1} \Rightarrow \mathrm{IPD} \Rightarrow \mathrm{PB2} \Rightarrow \mathrm{IPR}.$$

The equivalence follows from Proposition 4.14. ⊣

Having in mind the duality between normal modal logics and varieties of modal algebras, we can reformulate Theorem 4.15 in algebraic terms. Immediately by Lemma 4.6 and Theorem 3.3 we obtain

Theorem 4.16. *For any locally finite variety of diagonalizable algebras, the following properties are equivalent: the amalgamation property, the super-amalgamation property, the strong amalgamation property, the restricted amalgamation property, the strong epimorphisms surjectivity.*

Problem. What are relations between IPR, PB2, IPD and CIP in infinite slice extensions of GL? Is it true that IPR ⇒ PB2, PB2 ⇒ IPD or IPD ⇒ CIP in NExt(GL)?

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