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THE CONTRIBUTION OF A.V. KUZNETSOV TO THE THEORY OF MODAL SYSTEMS AND STRUCTURES

In memory of Alexander Vladimirovich Kuznetsov

Abstract. We will outline the contributions of A. V. Kuznetsov to modal logic. In his research he focused mainly on semantic, i.e. algebraic, issues and lattices of extensions of particular modal logics, though his proof of the Full Conservativeness Theorem for the proof-intuitionistic logic **KM** (Theorem 17 below) is a gem of proof-theoretic art.

Keywords: intuitionistic propositional logic and its extensions, modal logic S4 and its extensions, algebraic semantics for modal logics (**S4**-, **Grz**-, **GL**-, **KM**-algebras), lattice of the extensions of a logic.

1. Introduction

Alexander Vladimirovich Kuznetsov contributed to several areas of mathematical logic, including the theory of recursive functions, general problems of decidability for propositional logics, problems of expressiveness for propositional and predicate many-valued logics, algebraic analysis of superintuitionistic logics, structural analysis of the lattice of those logics, and investigation of particular modal systems, as well as comparative investigation of lattices of extensions of such systems. I will only discuss here Kuznetsov's contribution to the two last areas, with which I am most familiar and where

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Kuznetsov and I closely collaborated. On a personal note, I want to say that throughout this collaboration I learned not merely mathematical logic, since I never took a university course in the subject, but also Kuznetsov's manner of thinking in solving logic problems so that I had emulated his way of handling problems long in my work until I developed my own style.

I was introduced to Kuznetsov in about 1969, but our collaboration began only in 1973, when I returned to Chişinău, Moldova, after military service. Prof. V. Ja. Gerčiu was granted a doctoral degree just a year before, in 1972.[1] Thus it was a time, I believe, when Kuznetsov felt that he needed to make some changes in his research to explore other themes in logic, so to speak.

Whichever course his thought was taking at that time, Kuznetsov always focused on a triangle: *Computability-Algebra-Logic*. He saw the interactions of these three fields, explained in [Kuz 87], though the idea could be traced back to [Kuz 79a], as follows.

In [Kuz 79a] Kuznetsov writes that for detecting incompleteness or inconsistency of a calculus, it suffices to find a formula that is not derivable from the axioms of the calculus. An analogous task is needed to show that two calculi are not equal in extension; then it suffices to find a formula derivable in one calculus and refutable in the other. Thus, it would be advantageous to find means for detecting refutable formulas. For most known calculi, such means are known as *logical matrices, that is, universal algebras with a predicate for designated elements* (cf. [Men 97, Cze 80]). Then the complexity of derivability in a calculus can be investigated through the complexity of the matrices, by which the formulas non-derivable in the calculus could be refuted. In general, Kuznetsov calls a *separating means* an object that stands to the logic in question in some relation R but a refutable formula does not.

One vertex, *Logic*, in the triangle above was a priority for Kuznetsov. Focusing on a logic system, he concentrated mostly on semantic, namely algebraic, issues and, when it was possible, gave computable estimates of separating means. Although he considered topological or relational models as well (see e.g. [Kuz 79b]), algebraic models were a universal tool because many logics can be associated with *varieties, or equational classes*, [MMT 87] of similar algebras.

The present paper is organized as follows. In Section 2, I give an expository framework of what Kuznetsov had known about modal logic before he started his research, as well as the results he learned from other researchers

[1]See the list of A. V. Kuznetsov's Ph. D. students in A. Y. Muravitsky, "Alexander Vladimirovich Kuznetsov" (this issue).

in the field, which influenced him. Section 3 contains his own and joint contributions to the subject. I also include there additional references to the results that acted as direct creative impulses for him. Finally, in Section 4 I point out some further and recent developments that have been based on or inspired by those discussed in Section 3 and obtained after the untimely death of A. V. Kuznetsov.

2. Kuznetsov's knowledge obtained from outside

There was an area, comprised of Kuznetsov's research interests, which included only logical-algebraic context. In this relation, his attention was directed to the propositional intuitionistic logic, **Int**, and its extensions. Originated in the papers of Gödel [Göd 32] and Jaśkowski [Jaś 33], the research of the lattice of the extensions of the intuitionistic propositional logic, *Ext* **Int**, reached its highest point in the mid-1970s. Kuznetsov and his school contributed considerably to this field. Actually, he was one of the pioneers in investigating *Ext* **Int** among the Soviet logicians in 1960s.[2] The resulting paper of those contributions, [Kuz 75], was the text of his invited lecture[3] submitted to the International Mathematical Congress in Vancouver, Canada in 1975. Gradually, modal logic **S4** and other modal systems had become involved in the circle of his interests so that approaches developed for *Ext* **Int** were applied to the lattice of the normal[4] extensions of **S4**, *Ext* **S4**, and other logics.

Now it is a well-known fact that logics **Int** and **S4** are interrelated via Gödel-McKinsey-and-Tarski's Theorem on Embedding (Theorem 1 below; cf. [Göd 33], [MT 48], and also [Mur 06].), To explain this theorem we have to introduce two propositional languages. One, *assertoric* language, is based on the denumerable set of propositional variables p, q, r, ... (with or without indices) and the (assertoric) connectives: $\wedge$ (*conjunction*), $\vee$ (*disjunction*), $\rightarrow$ (*implication*) and $\neg$ (*negation*); the other, a *modal* language, is the expansion of the former by adding modality $\Box$ understood as a unary connective. The notion of a formula for both languages is defined in a usual manner. We will be using A, B, ... throughout as metavariables for assertoric formulas

[2] Two other Soviet logicians who contributed significantly in the beginning were V. A. Jankov (or Yankov) and Ya. S. Smetanich.

[3] Kuznetsov could not participate at the Congress for medical reasons. His talk, which is an exact copy of the published version, was read by Yuri L. Ershov.

[4] A modal logic is called *normal*, if it allows the rule $\alpha/\Box\alpha$.

and α, β, ... as those for modal formulas. The Greek letters Σ and Γ will be used as metavariables for sets of such formulas, respectively.

We also introduce a special operation of embedding of the assertoric language into the modal language, which for any formula A returns the modal formula $A^{\boldsymbol{t}}$ by placing modality $\Box$ in front of every subformula of A. Accordingly, we define $\Sigma^{\boldsymbol{t}} = \{A^{\boldsymbol{t}} \mid A \in \Sigma\}$.

THEOREM 1 (On embedding). *For any formula A, the following equivalence holds:*

$$\mathbf{Int} \vdash A \ \textit{ if and only if } \ \mathbf{S4} \vdash A^{\boldsymbol{t}}.$$

This theorem has been significantly generalized since the first proof of it had appeared in print in [MT 48]. Although Kuznetsov did not contribute to this matter, he was an active participant in discussions, which occurred sporadically at a number of All-Soviet conferences on logic and algebra in the 1970s. These discussions led, on the one hand, to the generalization of the Theorem on Embedding (see Theorem 2 below) and, on the other hand, drew Kuznetsov's attention to provability interpretation of **Int** and, then, to the definition of Proof-Intuitionistic Logic, **KM**, and finally to his Full Conservativeness Theorem (see Theorem 17 below).

The Theorem on Embedding demonstrated that **Int** can be embedded into a classical modal context with respect to **S4**. Almost two decades later, Grzegorczyk showed in [Grz 67] that **S4** in the equivalence above can be replaced with its proper extension, **Grz**, which is usually defined as **S4** augmented with the axiom:[5]

$$\mathbf{grz} = \Box(\Box(p \to \Box p) \to p) \to p\,.$$

In view of Grzegorczyk's result, it was very natural to ask about other logics among the normal extensions of **S4**: Which of them stand in the same relation to **Int** as **S4** and **Grz**? A partial answer to the question gave the following:

THEOREM 2 (generalized on embedding [MT 48]+[Mak 75]). *For any modal logic M such that $\mathbf{S4} \subseteq M \subseteq \mathbf{Grz}$ and any set of classical tautologies Σ,*

$$\mathbf{Int} + \Sigma \vdash A \textit{ if and only if } M + \Sigma^{\boldsymbol{t}} \vdash A^{\boldsymbol{t}}\,.$$

[5] In fact, Grzegorczyk used a two-variable formula in his axiomatization of **Grz**. Sobociński was possibly the first who discussed the formula **grz**. Segerberg in [Seg 71], p. 169, mentions that **Grz** equals Grzegorczyk's original logic from [Grz 67].

Theorem 2 indicated that there is an interconnection between *Ext* **Int** and *Ext* **S4**. Indeed, in 1974 Maksimova and Rybakov wrote an interesting paper [MR 74] on interrelation of these lattices, which was a pioneering comparative investigation of lattices of extensions of different logics. Although at first Kuznetsov's receipt of some of the results of [MR 74] was rather cool, his very last paper [KM 86] was similarly about functional connections between pairs of lattices of the extensions of logics, developing further the topic begun in [MR 74] and involving four lattices of extensions of logics. However, he came to the last point of his research career more through the equivalence in the Theorem on Embedding than through [MR 74]. From [MR 74] Kuznetsov, as well as I myself, learned important mappings: τ, ρ, and σ.

The logics in *Ext* **Int** are called *intermediate*, since $L \in$ *Ext* **Int** if and only if $\mathbf{Int} \subseteq L \subseteq \mathbf{Cl}$, where **Cl** is the classical propositional logic.[6] We regard *Ext* **S4** as comprising only consistent normal extensions of **S4**. Just as *Ext* **Int** has the greatest element, which is **Cl**, the lattice *Ext* **S4** has the greatest element $\mathbf{S4}+p \to \Box p$. In general, for any set of classical tautologies Σ, we denote by $\mathbf{Int}+\Sigma$ the intermediate logic obtained by adding Σ to **Int** as additional axioms. In the same manner we define an extension $\mathbf{S4}+\Gamma$ for any set of modal formulas Γ valid in $\mathbf{S4}+p \to \Box p$.

For any logic $L \in$ *Ext* **Int**, we define

$$\tau(L) = \mathbf{S4} + \{A^{\boldsymbol{t}} \mid A \in L\}.$$

Also, we define for any logic $M \in$ *Ext* **S4** (cf. [MR 74]),

$$\rho(M) = \{A \mid A^{\boldsymbol{t}} \in M\}\,.$$

As established in [MR 74], $\rho(M) \in$ *Ext* **Int**; moreover, ρ is an epimorphism from *Ext* **S4** onto *Ext* **Int**.

Thus Theorem 1 can be written by the equation $\rho(\mathbf{S4}) = \mathbf{Int}$.

Now we define

$$\sigma(L) = \tau(L) \oplus \mathbf{Grz}\,,$$

where $\oplus$ is the *join* operation in *Ext* **S4**, as well as in other lattices of logics being considered here.

[6] Kuznetsov included the *absolutely inconsistent* logic, that is, the set of assertoric formulas, into consideration and preferred the term *superintuitionistic logic*, referring it to any logic containing **Int** and closed under substitution and *modus ponens*.

The following two statements probably indicate the highest point of Kuznetsov's knowledge that originated in Theorem on Embedding.

Based on some results of Maksimova and Rybakov in [MR 74] and [Mak 75] and especially on one crucial result proved independently by Blok and Esakia in [Blo 76] and [Esa 76], respectively, it was possible to establish the following property:

THEOREM 3 (Blok-Esakia inequality). *For every modal logic* $M \in \mathit{Ext}\,\mathbf{S4}$, $\tau \circ \rho(M) \subseteq M \subseteq \sigma \circ \rho(M)$.

COROLLARY 3.1 (Folklore). *Let* $M \in \mathit{Ext}\,\mathbf{S4}$. *Then the equivalence*

$$\mathbf{Int} \vdash A \text{ if and only if } M \vdash A^t\,,$$

that is, $\rho(M) = \mathbf{Int}$, *is equivalent to the inequality* $M \subseteq \mathbf{Grz}$. *Or more general: the equivalence*

$$\mathbf{Int} + \Sigma \vdash A \text{ if and only if } M + \Sigma^t \vdash A^t$$

is equivalent to the inequality $M \subseteq \sigma(\mathbf{Int} + \Sigma)$.

3. Provability logic, proof-intuitionistic logic and the lattices of their extensions

3.1. Calculi related to provability interpretation

Kuznetsov learned the Theorem on Embedding (Theorem 1) from the lecture course on *Constructive* (read: intuitionistic) *Mathematical Logic* that P.S. Novikov taught at the Moscow State University in 1955.[7] Novikov in his course devoted some time to interpretation of **Int** in terms of constructive steps that are to establish the intuitionistic validity of a statement. Unlike Kolmogorov's interpretation that was based on a *problem-be-solved* notion, Novikov used terminology of finding approximations when one weighs to determine the heaviness of something.[8] According to Kuznetsov, Novikov

[7]In 1977, i.e. long after Novikov's death, the course was published as a book [Nov 77]. The editors of the book, F. A. Kabakov and B. A. Kushner, used lecture notes taken by a number of those attended Novikov's lectures, including the notes of Kuznetsov's. The present author studied Theorem on Embedding first from the latter source.

[8]He probably did not elaborate in detail in classroom so that the idea was discussed on merely two or three pages of Kuznetsov's notes. It is no wonder that the editors of [Nov 77] decided to omit it.

did interpret modality in **S4** as "provability", denoting this modality by a D letter,[9] but Novikov's provability was based on approximations of weights. Thus in 1974, Kuznetsov hoped to find a provability interpretation for **Int** in terms Novikov's approach, by making it more mathematically precise, and via the Theorem on Embedding.

In 1974, Kuznetsov clearly was not aware of Gödel's remark in [Göd 33] that a direct provability interpretation of **S4** through the standard provability predicate with respect to the formal Peano Arithmetic (**PA**) faces Gödel's Second Incompleteness Theorem as an obstacle. In about 1975, he learned from me Löb's Theorem for **PA** and we defined calculus $\mathbf{D_0}$ with the axioms: Axioms of Classical Logic, $\Box(p \to q) \to (\Box p \to \Box q)$, $\Box p \to \Box\Box p$, and the rules of inference: substitution, *modus ponens*, $\alpha/\Box\alpha$ (*Necessitation*), $\Box\alpha/\alpha$ (*Weakening*) and $\Box\alpha \to \alpha/\alpha$ (*Löb Rule*). We realized very quickly that we get a calculus (named later **D**) equal to $\mathbf{D_0}$ in extension, when we replace *Löb Rule* with the axiom:

$$\Box(\Box p \to p) \to \Box p\,. \qquad \textit{(Löb Formula)}$$

Because of Theorem 4(a), below, we abandoned $\mathbf{D_0}$ for the sake of **D**. Also, we started considering the calculus $\mathbf{D}^-$ obtained from **D** by removing the *Weakening* from the list of postulated rules of inference. The justification for these moves was as follows.

Two calculi, $\mathbf{C_1}$ and $\mathbf{C_2}$, (of the same language) are *deductively equivalent* if

$$\mathbf{C_1} + \alpha \vdash \beta \text{ if and only if } \mathbf{C_2} + \alpha \vdash \beta.$$

Assuming that both $\mathbf{C_1}$ and $\mathbf{C_2}$ contain **Cl**, if $\mathbf{C_1}$ and $\mathbf{C_2}$ are deductively equivalent then they are *equal in extension*, that is, the calculi $\mathbf{C_1}$ and $\mathbf{C_2}$ determine the same set of theorems.

Theorem 4 ([KM 80]). (a) *Calculi* $\mathbf{D_0}$ *and* **D** *are deductively equivalent.*

(b) *Calculi* **D** *and* $\mathbf{D}^-$ *are equal in extension, though they are not deductively equivalent.*

Remark 1. It should be noted that logic $\mathbf{D}^-$ is the same as **K4W** considered by Segerberg as early as in [Seg 71]. This logic attracted much attention after publication of Solovay's paper [Sol 76], where the completeness of it

[9] Russian "Dokazuemo" (Доказуемо) means "Provable". Note that Russian and English D letters look alike in script.

with respect to arithmetical interpretation was proven. Shortly thereafter, this logic received a new name, **GL**, after Gödel and Löb. The abstract [KM 76] also appeared in 1976, but, despite its title, *The Logic of Provability*, it did not deal with any precise interpretation of modality and the term "provability" was used rather as an inclination to use logic **D** somehow to interpret **Int** through the former.

In the spring of 1974, following a suggestion from my friend Nikolai Shakenko, Kuznetsov and I defined the *split operation* $\boldsymbol{s}$ as follows:

$$\begin{aligned} p^{\boldsymbol{s}} &= p, \\ (\alpha \wedge \beta)^{\boldsymbol{s}} &= \alpha^{\boldsymbol{s}} \wedge \beta^{\boldsymbol{s}}, \\ (\alpha \vee \beta)^{\boldsymbol{s}} &= \alpha^{\boldsymbol{s}} \vee \beta^{\boldsymbol{s}}, \\ (\alpha \to \beta)^{\boldsymbol{s}} &= \alpha^{\boldsymbol{s}} \to \beta^{\boldsymbol{s}}, \\ (\Box\alpha)^{\boldsymbol{s}} &= \alpha^{\boldsymbol{s}} \wedge \Box\alpha^{\boldsymbol{s}}. \end{aligned}$$

Remark 2. In our discussions with Kuznetsov, this definition preceded the following *Working Hypothesis*: There is a formula of one variable, say $\gamma(p)$, of the modal language, which expresses the actual provability in **PA** by means of calculus **D**. Then, we found it natural to expect $\gamma(p)$ to satisfy the following conditions:

(i) $\mathbf{D} \vdash \gamma(p) \to p$,

(ii) $\mathbf{D} \vdash \gamma(p) \to \Box p$,

(iii) $\mathbf{D} \vdash \gamma(p \to p)$.

THEOREM 5 ([KM 80]). *Let $\gamma(p)$ be a formula of one variable p, satisfying the conditions* (i)–(iii) *above. Then $\mathbf{D} \vdash \gamma(p) \leftrightarrow p \wedge \Box p$.*

THEOREM 6 ([KM 76, KM 80]). *$\{A \mid \mathbf{D} + (A^t)^{\boldsymbol{s}}$ is consistent$\} = \mathbf{LC}$, where **LC** is Dummett's Logic*[10] *from* [Dum 59].

THEOREM 7 ([KM 80]). *For any formula α,*

$$\mathbf{Grz} \vdash \alpha \text{ if and only if } \mathbf{GL} \vdash \alpha^{\boldsymbol{s}}.$$ [11]

By virtue of Theorem 2 (or Grzegorczyk's result mentioned above), the following holds:

[10] We remind that $\mathbf{LC} = \mathbf{Int} + (p \to q) \vee (q \to p)$.

[11] This theorem was proved by the present author in 1976 and was included in [KM 80].

COROLLARY 7.1 ([KM 80]). *For any formula A,*

$$\mathbf{Int} \vdash A \text{ if and only if } \mathbf{GL} \vdash (A^{\boldsymbol{t}})^{\boldsymbol{s}}.$$ [12]

For any logic $L \in Ext\,\mathbf{Int}$, we call L a ***t**-fragment* of some logic $M \in Ext\,\mathbf{S4}$ if $L = \{A \mid M \vdash A^{\boldsymbol{t}}\}$. For any modal logic $M \in Ext\,\mathbf{S4}$, we call M an ***s**-fragment* of an extension of **GL** (or **D**), say M', if $M = \{\alpha \mid M' \vdash \alpha^{\boldsymbol{s}}\}$. Finally, for any logic $L \in Ext\,\mathbf{Int}$, we call L an ***st**-fragment* of an extension of **GL** (or **D**), say M, if $L = \{A \mid M \vdash A^{\boldsymbol{s} \circ \boldsymbol{t}}\}$.

Thus, according to Theorem 2, any logic $\mathbf{Int} + \Sigma$ is a $\boldsymbol{t}$-fragment of any logic $M + \Sigma^{\boldsymbol{t}}$, where $\mathbf{S4} \subseteq M \subseteq \mathbf{Grz}$. Then, by virtue of Theorem 7, **Grz** is an $\boldsymbol{s}$-fragment of **GL**, as well as of **D**. Hence, (Corollary 7.1) **Int** is an $\boldsymbol{st}$-fragment of **GL**, as well as of **D**. Finally, Theorem 6 implies that if $L \in Ext\,\mathbf{Int}$ is a $\boldsymbol{t}$-fragment of some extension of **D**, then $L \subseteq \mathbf{LC}$. However, as the reader will see in Section 3.3, any logic $L \in Ext\,\mathbf{Int}$ is a $\boldsymbol{t}$-fragment of some extension of **GL**.

THEOREM 8 ([KM 80]). *The modal logic of the frame $(\mathbb{N}, >)$, where $\mathbb{N}$ is the set of natural numbers, is the greatest consistent extension of* **D**. *Moreover, the **st**-fragment of this extension equals* **LC**.

3.2. Magari algebras and KM-algebras

A universal algebra $(B, \wedge, \vee, \neg, \mathbf{1}, \Box)$ is called a *Magari*[13] (or *diagonalizable*) *algebra* if $(B, \wedge, \vee, \neg, \mathbf{1})$ is a Boolean algebra with the unit $\mathbf{1}$ and the operation $\Box$ is subjected to the following identities:

$$\begin{aligned} \Box(x \wedge y) &= \Box x \wedge \Box y, \\ \Box x \vee \Box\Box x &= \Box\Box x, \\ \Box(\neg\Box x \vee x) \vee \Box x &= \Box x, \\ \Box\mathbf{1} &= \mathbf{1}\,. \end{aligned}$$

Both logics **GL** and **D** are determined by Magari algebras. There is a one-to-one correspondence between the normal extensions of **GL** and the varieties of Magari algebras.

Now we define

$$\Box^{\boldsymbol{s}} x = x \wedge \Box x.$$

[12] This result was also obtained in 1976.

[13] Magari algebras are named after Roberto Magari who was the first to attempt in the mid-1970s to analyze Diagonal Lemma in **PA** by algebraic means (cf. [Mag 75]).

Let us be reminded that a universal algebra $(B, \wedge, \vee, \neg, \mathbf{1}, \Box)$ is called an **S4**-*algebra* (or an *interior algebra*) if it is a Boolean algebra with respect to assertoric operations and the unit $\mathbf{1}$ and the unary operation $\Box$ satisfies the identities:

$$\begin{aligned}
\Box(x \wedge y) &= \Box x \wedge \Box y,\\
\Box x \wedge x &= \Box x,\\
\Box x \vee \Box\Box x &= \Box\Box x,\\
\Box \mathbf{1} &= \mathbf{1}.
\end{aligned}$$

An **S4**-algebra is a **Grz**-*algebra* (or *Grzegorczyk algebra*) if, in addition, it has the following property:

$$\Box(\neg\Box(\neg x \vee \Box x) \vee x) = \Box x.$$

We denote by $\mathbf{B}^t$ the Heyting algebra of the open elements of an **S4**-algebra $\mathbf{B}$. Thus if we denote the set of this set of open elements by B^t, then $\mathbf{B}^t = (\mathrm{B}^t, \wedge, \vee, \to, \mathbf{1}, \Box)$, where operation $\wedge$ and $\vee$ are the same as in $\mathbf{B}^t$, and $x \to y = \Box(\neg x \vee y)$.

Theorem 9 ([KM 77]). *If* $\mathbf{B} = (B, \wedge, \vee, \neg, \mathbf{1}, \Box)$ *is a Magari algebra, then* $\mathbf{B}^s = (B, \wedge, \vee, \neg, \mathbf{1}, \Box^s)$ *is a* **Grz**-*algebra.*

Thus, starting from a Magari algebra $\mathbf{B}$, we, according to Theorem 9, get first a **Grz**-algebra $\mathbf{B}^s$and on the next step a Heyting algebra $\mathbf{B}^{st}$.

Theorem 10 ([KM 77]). *All the algebras of the form* $\mathbf{B}^{st}$*, where* $\mathbf{B}$ *is a Magari algebra, generate the variety of Heyting algebras.*

Theorem 11 ([KM 77]). *Suppose a* **Grz**-*algebra* $(B, \wedge, \vee, \neg, \mathbf{1}, \Box^s)$ *was obtained from a Magari algebra* $(B, \wedge, \vee, \neg, \mathbf{1}, \Box)$*. Then the operation* $\Box$ *can be uniquely recovered from the operation* $\Box^s$ *by defining* $\Box x$ *as the greatest element* y *such that* $\Box^s x \leq y$*, and* $\Box^s x \leq z \leq y$ *implies* $\Box^s z = z$*. Also, any finite* **Grz**-*algebra* $\mathbf{B} = (B, \wedge, \vee, \neg, \mathbf{1}, \boxdot)$ *can be converted to a Magari algebra* $\mathbf{B}^* = (B, \wedge, \vee, \neg, \mathbf{1}, \Box)$ *by the definition of* $\Box x$ *as the greatest element* y *such that* $\boxdot x \leq y$*, and* $\boxdot x \leq z \leq y$ *implies* $\boxdot z = z$*, in which case* $\boxdot x = \Box^s x$*.*

For the rest of this subsection, we limit ourselves with logics containing **GL**. We say that a class Θ of similar algebras *corresponds* to a calculus $\mathbf{C}$ if for any formula α, $\mathbf{C} \vdash \alpha$ if and only if α is valid on all the algebras in Θ.

Also, we say that Θ *fully corresponds* to $\mathbf{C}$ if for any formulas α and β, $\mathbf{C} + \alpha \vdash \beta$ if and only if β is valid on an algebra in Θ whenever α is valid on it (cf. [KM 80], p. 214).

It is clear that if Θ fully corresponds to $\mathbf{C}$ then the former corresponds to the latter, since $\mathbf{C} \vdash p \to p$.

Let $\mathsf{M}\mathbf{C}$ be the variety of algebras, determining $\mathbf{C}$, and let $\mathsf{N}\mathbf{C}$ be the (abstract) class of the algebras, whose logics are extensions of $\mathbf{C}$, that is, containing the axioms of $\mathbf{C}$ and closed under all the postulated rules of inference of $\mathbf{C}$.

THEOREM 12 ([KM 80]). *The following items hold*:

(i) $\mathsf{N}\mathbf{C} \subseteq \mathsf{M}\mathbf{C}$.

(ii) *Class* $\mathsf{N}\mathbf{C}$ *fully corresponds to* $\mathbf{C}$.

(iii) *If a class* Θ *fully corresponds to* $\mathbf{C}$, *then* $\Theta \subseteq \mathsf{N}\mathbf{C}$.

(iv) *If a class* Θ *fully corresponds to* $\mathbf{C}$ *and is a variety, then* $\Theta = \mathsf{M}\mathbf{C} = \mathsf{N}\mathbf{C}$.

THEOREM 13 ([KM 80]). *The variety of Magari algebras fully corresponds to* $\mathbf{GL}$, *that is,* $\mathsf{M}\mathbf{GL} = \mathsf{N}\mathbf{GL}$, *and it corresponds to* $\mathbf{D}$.

THEOREM 14 ([KM 80]). $\mathsf{N}\mathbf{D} \subset \mathsf{N}\mathbf{GL}$. *Hence, no variety fully corresponds to* $\mathbf{D}$.[14]

An element x of a Magari algebra $\mathbf{B} = (B, \wedge, \vee, \neg, \mathbf{1}, \Box)$ is called *open*, if $\Box^s x = x$, that is, if x is open in the $\mathbf{Grz}$-algebra $\mathbf{B}^s$.

It was noticed in [KM 80] that if x is an open element of a Magari algebra $\mathbf{B} = (B, \wedge, \vee, \neg, \mathbf{1}, \Box)$ then $\Box x$ is open as well. This observation suggested to consider the algebra $\mathbf{B}^\circ = (\mathbf{B}^t, \Box)$, where $\mathbf{B}^t$ is the Heyting of the open elements of the $\mathbf{Grz}$-algebra $\mathbf{B}^s$. Thus $\mathbf{B}^\circ$ is a Heyting algebra with modal operation $\Box$.

THEOREM 15 ([KM 80]). *Abstract class* $\{\mathbf{B}^\circ \mid \mathbf{B} \in \mathsf{M}\mathbf{GL}\}$ *forms a variety given by the identities*:

(i) $x \leq \Box x$,

(ii) $\Box x \to x = x$,

(iii) $\Box x \leq y \vee (y \to x)$.

[14] Almost nothing is known about classes fully corresponding to $\mathbf{D}$. The question in [KM 80], whether $\mathsf{N}\mathbf{D}$ is a quasi-variety, was answered negatively in [Mur 83]: Class $\mathsf{N}\mathbf{D}$ is not universally axiomatizable.

A universal algebra $(H, \wedge, \vee, \rightarrow, \mathbf{1}, \Box)$, where $(H, \wedge, \vee, \rightarrow, \mathbf{1})$ is a Heyting algebra and $\Box$ is subjected to the identities i)–iii) above, is called **KM**-*algebra*[15].

Having borrowed a central idea from [Esa 78], Kuznetsov made the following observation:

Theorem 16 ([Kuz 79b]). *Let C be the Heyting algebra of a topological space $(X, \mathbf{I})$, endowed by the operation $\Box A = \bigcup\{\mathbf{I}(A \cup \{x\}) \mid x \in X\}$. Then C is a* **KM**-*algebra if and only if $(X, \mathbf{I})$ is a scattered space. If X is finite, then C is a* **KM**-*algebra if and only if $(X, \mathbf{I})$ is a T_0-space.*

3.3. Lattices of extensions

Logic **KM**[16] can be defined by adding to **Int**, now understood in the modal language, the following three axioms:

$$p \rightarrow \Box p,$$
$$(\Box p \rightarrow p) \rightarrow p,$$
$$\Box p \rightarrow (q \vee (q \rightarrow p)).$$

Kznetsov noticed in [Kuz 78] (see also [Kuz 85]) that **KM**can also be axiomatized by the calculus, where the last formula in the above definition of **KM**is replaced with

$$((p \rightarrow q) \rightarrow p) \rightarrow (\Box q \rightarrow p).$$

We denote the latter calculus by $\boldsymbol{I}^{\Delta}$. It is especially convenient when we ask about the Separation Property, which will be discussed in Section 4.

Since **Int** has the Finite Model Property and, by virtue of Theorem 11, one can prove that **KM**is a conservative extension of **Int**. Developing this idea, Kuznetsov proved his *Full Conservativeness*

Theorem 17 ([Kuz 85]). *For any assertoric formulas A and B, the following equivalence holds*:

$$\mathbf{Int} + A \vdash B \text{ if and only if } \mathbf{KM} + A \vdash B,$$

or in a more general form:

$$\mathbf{Int} + \Sigma \vdash B \text{ if and only if } \mathbf{KM} + \Sigma \vdash B.$$

[15]This term is due to Esakia (cf., e.g., [Esa 06]). The original name was Δ-*pseudo-Boolean algebra*.

[16]This current name was given by Esakia (cf., e.g., [Esa 06]). The original name was $\boldsymbol{I}^{\Delta}$.

A Heyting algebra $(\mathsf{H}, \wedge, \vee, \to, \mathbf{1})$ is called *enrichable*, if it is possible to define somehow a unary operation $\Box$ so that the expanded algebra $(\mathsf{H}, \wedge, \vee, \to, \mathbf{1}, \Box)$ is a **KM**-algebra. It should be noted that in an enrichable Heyting algebra the modality $\Box$ can be defined uniquely (cf.[Mur 90]).

The following statement is equivalent to Theorem 17:

COROLLARY 17.1 ([Kuz 85]). *Any variety of Heyting algebras is generated by its enrichable algebras.*

Theorem 17 has another interesting algebraic reading. It is quite easy to see, though some knowledge of model theory is needed, that Theorem 17 is equivalent to the

COROLLARY 17.2 ([Kuz 85]). *Any Heyting algebra C is embedded into an enrichable Heyting algebra C^* such that the latter generates the same variety of Heyting algebras, as does the former.*

Remark 3. There is merely a sketch of the proof of Theorem 17 in [Kuz 85], though Kuznetsov presented a proof of that theorem in full detail in the Seminar on Mathematical Logic at the Institute of Mathematics of Moldova Academy of Sciences in 1984. In a conversation with Kuznetsov before [Kuz 85] had been handed in for publication, I pointed out that Corollary 17.2 (Corollary 2 in [Kuz 85]) is in fact equivalent to Theorem 17. A proof of it can be obtained as follows. Let C be a Heyting algebra and A be the class of enrichable algebras in the variety generated by C. Then, by virtue of Corollary 17.1, $\mathsf{C} \in \mathbf{HSP}(\mathsf{A})$. Since every Heyting algebra has the Congruence Extension Property, $\mathbf{HSP}(\mathsf{A}) = \mathbf{SHP}(\mathsf{A})$. Now we notice that "being enrichable" is a $\forall\exists\forall$-property, where $\forall\exists\forall$-prefix is applied to the conjunction of equalities, that is, to a Horn predicate. Therefore, this property is strictly multiplicatively stable in the sense of [Mal 73, Ch. 7.5]. Also, it is a well-known fact that the first-order properties are preserved by the homomorphism.[17]

Let *Ext***Grz** and *Ext***KM** be the lattices of consistent normal extensions of **Grz** and extensions of **KM**, respectively.

Now we generalize operation $\boldsymbol{s} \circ \boldsymbol{t}$, expanding it to the formulas of modal language. We denote this new operation by $\boldsymbol{tr}$ and define it as follows.

[17] The problem of obtaining more "direct" and "visible" algebraic embedding construction remains open. The construction proposed in [Mur 86] gives merely an embedding of a Heyting algebra into an enrichable one, though it does allow one to see that the latter generates the same variety as does the former.

Taking a formula α as an argument, we first place $\Box^{s}$ in front of every subformula of α and, then, transform each subformula $\Box^{s}\beta$ into $(\beta \wedge \Box\beta)$.

We introduce the following four mappings $\lambda\colon \mathit{Ext}\,\mathbf{GL} \longrightarrow \mathit{Ext}\,\mathbf{KM}$, $\kappa\colon \mathit{Ext}\,\mathbf{KM} \longrightarrow \mathit{Ext}\,\mathbf{GL}$, $\mu\colon \mathit{Ext}\,\mathbf{GL} \longrightarrow \mathit{Ext}\,\mathbf{Grz}$ and $\chi\colon \mathit{Ext}\,\mathbf{KM} \longrightarrow \mathit{Ext}\,\mathbf{Int}$ as follows:

$$\lambda(M) = \{\alpha \mid M \vdash \boldsymbol{tr}(\alpha)\}, \kappa(M) = GL + \{\boldsymbol{tr}(\alpha) \mid M \vdash \boldsymbol{tr}(\alpha)\},$$
$$\mu(M) = \{\alpha \mid M \vdash \alpha^{\boldsymbol{s}}\} \text{ and } \chi(M) = \{A \mid M \vdash A\}.$$

It was proved in [Mur 85] that λ and κ are isomorphisms and inverses of one another. Also, by using Theorem 3, one can derive that ρ (reduced to $\mathit{Ext}\,\mathbf{Grz}$) and σ establish inverse isomorphisms between $\mathit{Ext}\,\mathbf{Int}$ and $\mathit{Ext}\,\mathbf{Grz}$, respectively.

THEOREM 18 ([KM 86]). *For the mappings λ, κ, μ, χ above, the following equation holds*:

$$\chi \circ \lambda = \kappa \circ \mu\,.$$

Moreover, χ and μ are meet semilattice epimorphisms that are not commutative with $\oplus$.

This theorem immediately implies the

COROLLARY 18.1 ([KM 86]). *Any normal consistent extension of* **Grz** *is the $\boldsymbol{s}$-fragment of some normal extension of* **GL** *and any intermediate logic is the $\boldsymbol{st}$-fragment of some normal extension of* **GL**.

4. Some further developments after A. V. Kuznetsov

First, I would like to mention the Separation Property for the calculus $\boldsymbol{I}^{\Delta}$ (= **KM** in extension), proved in [Mur 86] and [Sim 87] independently and at about the same time. Simonova also proved in [Sim 90] the Interpolation Property for **KM**and constructed a continuum of extensions of **KM**, having this property, as well as an extension without it.

Further, it was proved that the following statements are equivalent:

(i) $\mathbf{KM} + \Sigma \vdash A$,

(ii) $\mathbf{GL} + \Sigma^{\boldsymbol{s}\circ\boldsymbol{t}} \vdash A^{\boldsymbol{s}\circ\boldsymbol{t}}$,

(iii) $\mathbf{Grz} + \Sigma^{\boldsymbol{t}} \vdash A^{\boldsymbol{t}}$.

Also, for any set Γ of modal formulas and any α [Mur 88],

$$\mathbf{Grz} + \Gamma \vdash \alpha \text{ if and only if } \mathbf{GL} + \Gamma^{\boldsymbol{s}} \vdash \alpha^{\boldsymbol{s}},$$

and [Mur 89]

$$\mathbf{GL} + \Gamma \vdash \alpha \text{ if and only if } \mathbf{KM} + \boldsymbol{tr}(\Gamma) \vdash \boldsymbol{tr}(\alpha).$$

Esakia defined the calculus **mHC** as follows:

$$\mathbf{mHC} = \mathbf{Int} + (\Box(p \to q) \to (\Box p \to \Box q) + p \to \Box p + \Box p \to (q \vee (q \to p)),$$

where **Int** should be understood in the modal language.

It is clear that $\mathbf{KM} = \mathbf{mHC} + (\Box p \to p) \to p$. Now let

$$\mathbf{K4.Grz} = \mathbf{K4} + \Box(\Box(p \to \Box p) \to p) \to \Box p.$$

Then

$$\mathbf{mHC} \vdash \alpha \text{ if and only if } \mathbf{K4.Grz} \vdash \boldsymbol{tr}(\alpha).\text{[18]}$$

Also,

$$\mathbf{Grz} \vdash \alpha \text{ if and only if } \mathbf{K4.Grz} \vdash \alpha^{\boldsymbol{s}}.$$

Moreover, **K4.Grz** is the least normal extension of **K4**, for which the last equivalence holds (see [Esa 06]).

For algebraic developments, one can mention the following continuation of the theme of Theorem 11.

We call a **Grz**-algebra **B** *enrichable* if the Heyting algebra $\mathbf{B}^{\boldsymbol{t}}$ is enrichable.

The following statements hold:

(i) An **S4**-algebra is embedded into an enrichable **Grz**-algebra if and only if the former is a **Grz**-algebra;

(ii) The class of enrichable **Grz**-algebras is an $\forall\exists$-class but not an $\forall$-class (cf. [Mur 90]).

[18] In [Esa 06] Esakia uses an embedding operation $\#(\alpha)$, which is different from $\boldsymbol{tr}(\alpha)$ in the clauses of definition for disjunction and conjunction; namely, $\#(\alpha \circ \beta) = \#(\alpha) \circ \#(\beta)$, where $\circ \in \{\wedge, \vee\}$, whereas $\boldsymbol{tr}(\alpha \circ \beta) = \Box^{\boldsymbol{s}}(\boldsymbol{tr}(\alpha) \circ \boldsymbol{tr}(\beta))$. However, one can prove that the formulas $\#(\alpha)$ and $\boldsymbol{tr}(\alpha)$ are equivalent in **K4**.

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