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IN WHAT SENSE IS KANTIAN PRINCIPLE OF CONTRADICTION NON-CLASSICAL?

Abstract. On the ground of Kant's reformulation of the principle of contradiction, a non-classical logic **KC** and its extension **KC+** are constructed. In **KC** and **KC+**, $\neg(\phi \wedge \neg\phi), \phi \to (\neg\phi \to \psi)$, and $\phi \vee \neg\phi$ are not valid due to specific changes in the meaning of connectives and quantifiers, although there is the explosion of derivable consequences from $\{\phi, \neg\phi\}$ (the deduction theorem lacking). **KC** and **KC+** are interpreted as fragments of an **S5**-based first-order modal logic **M**. The quantification in **M** is combined with a "subject abstraction" device, which excepts predicate letters from the scope of modal operators. Derivability is defined by an appropriate labelled tableau system rules. Informally, **KC** is mainly ontologically motivated (in contrast, for example, to Jaśkowski's discussive logic), relativizing state of affairs with respect to conditions such as time.

Keywords: Kant, paracompleteness, paraconsistency, principle of contradiction, square of oppositions, subject abstraction, labelled tableau.

1. Introduction

According to Kant's formulation of the principle of contradiction, a contradiction can occur only in the relation between the "subject" and the "predicate" of a categorical proposition (a, e, i, and o propositions in the logical square). Starting from that formulation of the principle of contradiction, we want to describe a non-classical logic **KC** where $\phi \wedge \neg\phi$ and $\neg(\phi \vee \neg\phi)$ are satisfiable as predicates of a categorical proposition, even under the existential import of the subject term of the proposition. **KC** is

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half-paraconsistent in that $\neg(\phi \wedge \neg\phi)$ and $\phi \to (\neg\phi \to \psi)$ are not theorems of **KC**, but, at the same time, an explosion of consequences is derivable form $\{\phi, \neg\phi\}$. We will disregard the non-explosiveness of Kant's syllogistic, where the derivability is constricted to the traditional syllogistic figures (for example, $\{\forall x(Mx \to Px), \neg\forall x(Mx \to Px)\} \nvdash \exists x(Sx \wedge Px)$ [14, p. 293]).

We start with the analysis of Kant's examples that illustrate his formulation of the principle of contradiction. Second, we describe a non-classical logic **KC** as a shorthand for a fragment of **S5**-based first-order modal logic, and define an appropriate tableau system. Next, we extend **KC** to **KC+** with strict conditional and strict disjunction. Finally, we sketch a proof of the adequacy of the tableau system for **KC+**.

2. The principle of contradiction

2.1. Kant's formulation of the principle

In the Section 'The highest principle of all analytic judgments' of the *Critique of Pure Reason*, Kant criticizes the following formulation of the principle of contradiction originating from Aristotle and Plato:

> **PC***: It is impossible that something should at one and the same time both be and not be. (B 191)[1]

PC* is not acceptable for Kant as a principle of logic because it contains a time condition ('at one and the same time'), which is not for him a logical concept. Kant adopts the principle of contradiction in the following formulation, which is based solely on the logical form of a categorical proposition, i.e., on the relation of a predicate to the subject of a proposition:[2]

> **PC**: No predicate contradictory of a thing can belong to it. (B 190)[3]

[1]As usual, 'B' denotes the 2nd edition of Kant's 'Critique of Pure Reason' [7, vol. 3], which is quoted according to the English translation by N. Kemp Smith [8].

[2]To avoid ambiguity, we will use 'subject term', 'predicate term', 'sentence' ('formula') for subject, predicate, and proposition in the linguistic sense, respectively. Note that "predicate letters" of quantificational logic occur both within subject terms and within predicate terms. As is known, Kant himself conceived logical forms as forms of "thinking" (of "bringing our representations under the unity of consciousness").

[3]Nulli subiecto competit praedicatum ipsi oppositum [7, vol. 2, *Die falsche Spitzfindigkeit der vier syllogistischen Figuren*, p. 60]. In historical or informal logical texts we often encounter an analogous general scheme: *A is not non-A*. But that scheme is often

"Thing" is referred to by the subject of a proposition. Thus, it is not a violation of **PC** if two predicates of a subject negate each other, if only they do not negate the subject. On Kant's presuppositions, this leads straightforwardly to the non-classicality of logic in the way to be illustrated by examples and more precisely explained below.

According to Kant, the principle of identity, A is A, is already contained in **PC**, since Kantian contradiction relates to negation in a classical way: the negation of a self-contradictory proposition (where the predicate contradicts the subject) is valid. As Kant says: "since the opposite of the concept would contradict the object, the concept itself must necessarily be affirmed of it" (B 190–191).

2.2. Some examples

Let us, first, pause on an example by which Kant illustrates the need of his reformulation of the principle of contradiction:

A man who is unlearned is not learned. (B 192) (1)

In (1), 'unlearned' is (universally) affirmed as a predicate of the subject 'man', and 'learned' is denied as a predicate of the same subject. Since 'unlearned' and 'learned' are merely predicates not being included or excluded by the subject itself, (1) is not true by **PC**, according to Kant.

In other words and according to Kant's nomenclature, (1) is not an *analytic* proposition, because for Kant a categorical proposition is analytic if and only if "its truth can always be adequately known in accordance with the principle of contradiction" (B 190[4]). That means that in each analytic proposition either what contradicts the subject is denied, or what is contained in the subject must be affirmed, which is obviously not the case in (1) if only 'man' is conceived as the subject term of (1).

Consequently, the following proposition can be stated without Kantian contradiction:

Some men who are unlearned are learned, (2)

---

interpreted indistinguishably to the Aristotelian way: what is A cannot at the same time be non-A. For example, according to Leibniz, it is included in the principle of contradiction "that the same proposition cannot be true and false at the same time [à la fois]", and hence "what is A cannot be non A [11, vol. 5, p. 343]; cf. also a letter to S. Clarke (with 'en même temps' instead of 'à la fois') [10, vol. 6, p. 355].

[4]In other words, "the principle of contradiction must ... be recognized as being the universal and completely sufficient principle of all analytic knowledge" (B 191).

since neither predicate 'unlearned', nor predicate 'learned' contradict the subject 'men'. To put it another way, the same man could be unlearned (at one time), as well as learned (at another time). Generalizing Kant's conception (where only time as a predicate relativizing condition is mentioned), we could say that the same man can be in one sense, or in one possible world, unlearned, and in another sense, or in another possible world, learned:

$$\exists x(Mx \land (\Diamond\neg Lx \land \Diamond Lx))$$

(where M^1 stands for '... is a man', and L^1 for '... is learned').

In distinction to (2), the following proposition:

$$\text{Some unlearned men are learned (B 192)} \tag{3}$$

according to Kant, disobeys **PC**, in that it predicates to the subject ('unlearned man') what contradicts the subject, and therefore in no way could be true.

On the other hand, the proposition

$$\text{No unlearned man is learned} \qquad \text{(B 192)} \tag{4}$$

is true precisely by **PC**. That is, it is analytic, 'unlearned man' being the subject, and the predicate 'learned' (which contradicts the subject) being denied of the subject.

3. Logic KC

On the ground of Kant's conception of contradiction, we can describe a Kantian non-classical quantificational logic **KC** (a logic of "Kantian contradiction"), with analytic propositions, as defined in the previous section, among its theorems.

A crucial point for **KC** will be to distinguish a subject term from the predicate term in the way that the constituents of the subject term can be clearly traced within the predicate term. This cannot be accomplished by translating categorical sentences of the logical square (a, e, i, o) into a familiar language of classical first-order logic. In the latter approach, we could not clearly distinguish, for example, between sentence (1) (where the article 'a' should be understood as 'each') and sentence (4), since both could be classically translated as $\forall x((Mx \land \neg Lx) \to \neg Lx)$, although (1) is non-analytic and (4) analytic. Similarly, both sentence (2) and sentence (3) could be classically translated as $\exists x((Mx \wedge \neg Lx) \wedge \neg Lx)$, although (2) is non-contradictory and (3) contradictory.

To distinguish the subject term and the predicate term within a formula of **KC**, we will use the angle notation for quantified conditionals and conjunctions:

$$\forall x\langle Mx \to Lx\rangle, \quad \exists x\langle Mx \wedge Lx\rangle \tag{5}$$

where Mx and Lx are the subject term and the predicate term, respectively. The predicate letter L in (5), which does not occur in the subject term, is treated as logically independent of the subject term in the sense that it cannot produce a Kantian contradiction (according to **PC**). However, the following sentence contains a contradiction:

$$\exists x\langle(Mx \wedge \neg Lx) \wedge Lx\rangle \tag{6}$$

being in fact a translation of (3). In (6) the predicate letter L is subject dependent (since it occurs in the subject term, not only in the predicate term), and is both affirmed (in the predicate term) and denied (in the subject term).

For evaluating the same predicate letter in the same sentence in different ways, a modal semantics can be applied, where values of predicate letters are relativized with respect to different semantic points ("possible world"). Accordingly, the satisfaction conditions of formulas will be redefined in a modal way. In the result, the subject term of a formula will be rigid regarding the predicate letters occurring in the subject term, in the sense that those predicate letters will be evaluated with respect to the same point throughout the formula. At the same time, a predicate term will be non-rigid regarding its constituent predicate letters that do not occur in the subject term, that is, they could be evaluated with respect to different semantic points.

Remark 1. We note that, in fact, Kant's idea was not to advocate a kind of non-classical logic. Obviously, Kant did not regard "copulative proposition" (a kind of conjunction) as a logical form (logical "unity") and hence did not include it in his well known table of propositions. Apparently, "copulative proposition" was for him a plurality of psychologically associated propositions merely linguistically put together in one sentence (thus, a sentence with many predicates of one subject stands for many categorical propositions). Therefore, Kant reformulated the principle of contradiction in order to make it independent of what were for him non-logical conditions. We aim to show that Kant's "copulative proposition" is a kind of non-classical conjunction, which can be combined with Kant's principle of contradiction in one (non-classical) logic.[5]

Let us now see the details of the formal description of the logic **KC**.

3.1. Languages $\mathscr{KC}$ and $\mathscr{M}$

The *vocabulary* of $\mathscr{KC}$ consists of n-place predicate letters (P^n, Q^n, R^n, with or without subscripts; informally, we also use other latin capital letters), individual terms (individual variables u, x, y, z, and individual constants c, d, e, with or without subscripts), connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$), quantifier symbols ($\forall, \exists$), parentheses and brackets.

The set $Form_{\mathscr{KC}}$ of formulas of $\mathscr{KC}$ consists of atomic formulas $\Phi t_1 \dots t_n$, where t_i is an individual term, compound formulas $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$, $\langle\phi \rightarrow \psi\rangle$, $\langle\phi \wedge \psi\rangle$, and quantified formulas $\forall x\langle\phi \rightarrow \psi\rangle$, $\exists x\langle\phi \wedge \psi\rangle$, where ϕ and ψ are formulas. There are also formulas with a square bracket indication, $\phi[\Phi_1, \dots, \Phi_n]$, where $\Phi_1, \dots, \Phi_n$ are predicate letters. As usual, we can omit outer parentheses of ϕ when ϕ is not a subformula of another formula. Closed formulas are called sentences.

Square bracket indication is not an operator (e.g., an atomic formula with a square bracket indication is still an atomic formula), so that the main operator of $\phi[\Phi_1, \dots, \Phi_n]$ is the same as the main operator of ϕ. The purpose of the square bracket indication will be explained below. In fact, a formula without a square bracket indication can be conceived as having empty square bracket notation. Note a special use of angle brackets to delimit immediate subformula in a quantified formula. In addition, only a conjunction and a conditional can be the immediate subformula of a quantified formula, since $\mathscr{KC}$ is designed primarily to formalize quantification in Kantian logic, which is restricted to the sentences of the logical square. For the sake of generality, we allow n-place predicate letters, although Kantian quantification logic is merely monadic. We say that ϕ and ψ are the subject term and the predicate term, respectively, of $\forall x\langle\phi \rightarrow \psi\rangle$, $\exists x\langle\phi \wedge \psi\rangle$, $\neg\forall x\langle\phi \rightarrow \psi\rangle$, and $\neg\exists x\langle\phi \wedge \psi\rangle$. In addition, a quantified and negated quantified formula will be called an *SP*-formula.

In what follows, a translation from $\mathscr{KC}$ to a first-order modal language $\mathscr{M}$ will be important. $\mathscr{M}$ is familiar except that it has predicate variables

[5] Some other examples of bringing Kant in connection with non-classical (especially paraconsistent) logic are reasoning in moral dilemmas [15], and reasoning about the "limits of thought" [13]. The opinion that Kant's (and Leibniz's) ideas "can be restated" in the setting of Jaśkowski's discussive logic is expressed in [16, p. 487].

$X_1^n, X_2^n, \ldots$, atomic formulas $Xt_1 \ldots t_n$, and *subject abstraction* formulas of the form

$$(X.\phi)(\Phi) \tag{7}$$

Subject abstraction in $\mathscr{M}$ is meant to except each atomic formula in which a predicate letter Φ occurs from the scope of any modal operator in ϕ. We say that Φ and $\Phi t_1 \ldots t_n$ are *bound* by subject abstraction $(X.\phi)(\Phi)$.[6] For example, in

$$(X.\Box(Px \wedge Xx))(Q)$$

Q, substituted for X in $\Box(Px \wedge Xx)$, is excepted from the scope of $\Box$. In

$$\Diamond(X.\Box(Px \wedge Xx))(Q)$$

Q, substituted for X in $\Box(Px \wedge Xx)$, is excepted from the scope of $\Box$ (as in the example above), but it remains within the scope of $\Diamond$. Nested subject abstractions,

$$(X_1.(X_2.\ldots(X_n.\phi)(\Phi_n)\ldots)(\Phi_2))(\Phi_1) \tag{8}$$

will be abbreviated in the following way:

$$(X_1 \ldots X_n.\phi)(\Phi_1, \ldots, \Phi_n) \tag{9}$$

Let $\mathit{Form}_{\mathscr{M}}$ be the set of formulas of $\mathscr{M}$. We will now define a translation function T from $\mathscr{KC}$ into $\mathscr{M}$. The translation T does not grammatically change formulas inside a subject term, except for angle brackets, which become parentheses. Outside a subject term, T modalizes immediate subformulas of compounds if compounds are not bound by a subject abstraction.

Definition 1 (Translation from $\mathscr{KC}$ into $\mathscr{M}$). The translation from $\mathscr{KC}$ into $\mathscr{M}$ is a function $T : \mathit{Form}_{\mathscr{KC}} \longrightarrow \mathit{Form}_{\mathscr{M}}$ such that

$T(\phi) = \phi$, with parentheses in the right side ϕ
instead of angle brackets in the left side ϕ,

[6]Subject abstraction should be distinguished from the predicate (or λ-) abstraction as used, for example, by Melvin Fitting (see [2, 3]). Predicate abstraction excepts individual terms or function symbols, not predicate letters, from the scope of modal operators. Further, predicate abstraction produces a predicate term from a formula, whereas subject abstraction does not produce any new term.

if the left side ϕ is a subformula of a subject term, otherwise:

$$
\begin{array}{lcl}
T(\phi) & = & \phi, \text{ if } \phi \text{ is atomic} \\
T(\neg\phi) & = & \neg T(\phi) \\
T(\phi \wedge \psi) & = & \Diamond T(\phi) \wedge \Diamond T(\psi) \\
T(\phi \vee \psi) & = & \Box T(\phi) \vee \Box T(\psi) \\
T(\phi \to \psi) & = & \Diamond T(\phi) \to \Box T(\psi) \\
T(\phi \leftrightarrow \psi) & = & \Box(\Diamond T(\phi) \wedge \Diamond T(\psi)) \vee \Box(\Diamond\neg T(\phi) \wedge \Diamond\neg T(\psi))
\end{array}
$$

$$T(\langle\phi \to \psi\rangle) = T(\phi) \to (X_1 \ldots X_n.T(\psi[X_1/\Phi_1, \ldots, X_n/\Phi_n]))(\Phi_1, \ldots, \Phi_n)$$

where $\Phi_1, \ldots, \Phi_n$ are all and only predicate letters occurring in ϕ

$$T(\langle\phi \wedge \psi\rangle) = T(\phi) \wedge (X_1 \ldots X_n.T(\psi[X_1/\Phi_1, \ldots, X_n/\Phi_n]))(\Phi_1, \ldots, \Phi_n)$$

where $\Phi_1, \ldots, \Phi_n$ are all and only predicate letters occurring in ϕ

$$
\begin{array}{lcl}
T(\forall x\langle\phi \to \psi\rangle) & = & \forall x T(\langle\phi \to \psi\rangle) \\
T(\exists x\langle\phi \wedge \psi\rangle) & = & \exists x T(\langle\phi \wedge \psi\rangle) \\
T(\phi[\Phi_1, \ldots, \Phi_n]) & = & (X_1 \ldots X_n.T(\phi[X_1/\Phi_1, \ldots, X_n/\Phi_n]))(\Phi_1, \ldots, \Phi_n)
\end{array}
$$

$\phi[X/\Phi]$ is a formula ϕ in which each occurrence of Φ is replaced by an occurrence of X.

Remark 2 (Square bracket indication). Sometimes we need to translate a formula ϕ as if each of the predicate letters $\Phi_1, \ldots, \Phi_n$ occurs in the subject term of ϕ, although some of $\Phi_1, \ldots, \Phi_n$ may actually not occur in that term or ϕ even may not have a subject term at all. To that end, we have introduced a new kind of formulas, $\phi[\Phi_1, \ldots, \Phi_n]$, and apply the translation function T in accordance with the last case in Definition 1.

$\Gamma[\Phi_1, \ldots, \Phi_n]$ will be like a set Γ except that to each $\phi \in \Gamma$ a square bracket indication $[\Phi_1, \ldots, \Phi_n]$ is added.

3.2. Semantics for $\mathscr{KC}$ and $\mathscr{M}$

The main semantic concepts for the language $\mathscr{KC}$ and its logic **KC** are defined by means of the semantics for the formerly introduced modal language $\mathscr{M}$ and its modal logic **M**. We, first, semantically outline the logic **M**, and after that, we define the main semantic concepts of **KC**.

3.2.1. Outline of the semantics of M

A *model* $\mathfrak{M}$ of the logic **M** is an **S5**-based quantificational model with (for simplicity) a universal accessibility relation and a constant domain. It is a quintuple $\langle W, R, D, I\rangle$, where W is a non-empty set (of worlds, moments of time), R the universal relation on W (i.e., $W \times W$),[7] D a non-empty set (of objects), and I an interpretation such that $I(\mathrm{c}) \in D$ and $I(\Phi^n, w \in W) \in \wp D^n$ (c is an individual constant, and Φ^n an n-placed predicate letter). *Variable assignment* v maps each individual variable to a member of D, and each n-place predicate variable at a world to an n-ary relation on D. *Variants* $v_{[d/x]}$ and $v_{[r/X]}$ of a variable assignment v differ from v at most in assigning $d \in D$ to x and an n-ary relation r on D to the n-place predicate variable X (at w), respectively.

New satisfaction cases for $\mathscr{M}$ to be defined are the satisfaction of an atomic formula with a predicate variable:

$$\mathfrak{M}, w \models^{\mathbf{M}}_{v} Xt_1 \dots t_n \text{ iff } \langle [\![t_1]\!]^{\mathfrak{M},w}_{v}, \dots, [\![t_n]\!]^{\mathfrak{M},w}_{v} \rangle \in [\![X]\!]^{\mathfrak{M},w}_{v}$$

and the satisfaction of a subject abstraction formula:

$$\mathfrak{M}, w \models^{\mathbf{M}}_{v} (X.\phi)(\Phi) \text{ iff } \mathfrak{M} \models^{\mathbf{M}}_{v[[\![\Phi]\!]^{\mathfrak{M},w}_{v}/X]} \phi.$$

That is, Φ remains evaluated at w through the whole ϕ, regardless of the modal context in ϕ in which Φ may occur. The other cases of the satisfaction of translated formulas are defined in a way familiar in first-order modal logic with a constant domain.

3.2.2. The semantics of KC

The models of **KC** are precisely the models of **M**. The following semantic concepts of **KC** and its language are defined locally, that is, with respect to a world in a model, and with the help of the translation function T.

Definition 2 (Satisfaction in **KC**).

$$\mathfrak{M}, w \models^{\mathbf{KC}}_{v} \phi \text{ iff } \mathfrak{M}, w \models^{\mathbf{M}}_{v} T(\phi).$$

It is easy to see that under universal accessibility relation R in **M**, for each compound formula $(\phi * \psi)$ without a square bracket indication ($*$ is a two-place connective), it holds that $\mathfrak{M}, w \models^{\mathbf{KC}}_{v} (\phi * \psi)$ iff for each w, $\mathfrak{M}, w \models^{\mathbf{KC}}_{v} (\phi * \psi)$.

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[7] Models with a universal relation R can for present purposes sufficiently represent Kantian time. For, as is known, asymmetry, transitivity and comparability for time operators H ('always in the past') and G ('always in the future') give equivalence relation for the operator A ('always'). Universal relation is a special case of equivalence relation, with only one equivalence class of worlds (moments of time).

DEFINITION 3 (Satisfiability in **KC**). A set Γ of formulas of $\mathscr{KC}$ is satisfiable iff there are $\mathfrak{M}$, w, and v such that for each $\phi \in \Gamma$, $\mathfrak{M}, w \models_v^{\mathbf{KC}} \phi$.

DEFINITION 4 (Consequence in **KC**). $\Gamma \models^{\mathbf{KC}} \psi$ iff for each $\mathfrak{M}, w$, and v, $\mathfrak{M}, w \models_v^{\mathbf{KC}} \psi$ whenever for each $\phi \in \Gamma, \mathfrak{M}, w \models_v^{\mathbf{KC}} \phi$.

DEFINITION 5 (Semantic equivalence in **KC**). ϕ and ψ are semantically equivalent iff for each $\mathfrak{M}, w$, and v, $\mathfrak{M}, w \models_v^{\mathbf{KC}} \phi, \psi$ or $\mathfrak{M}, w \not\models_v^{\mathbf{KC}} \phi, \psi$.

DEFINITION 6 (Analytic formula in **KC**). $\forall x\langle\psi \to \psi\rangle$, $\exists x\langle\psi \wedge \psi\rangle$, and their equivalents are analytic iff $\models^{\mathbf{KC}} \langle\psi \to \chi\rangle$.

DEFINITION 7 (Self-contradictory formula in **KC**). $\forall x\langle\psi \to \psi\rangle$, $\exists x\langle\psi \wedge \psi\rangle$, and their equivalents are self-contradictory iff $\models^{\mathbf{KC}} \langle\psi \to \neg\chi\rangle$.

The validity of traditional inferences on *SP*-sentences ("immediate consequences", "categorical syllogisms") is preserved if each subject term in an inference is treated as if it is a subject term of each sentence in the inference, in the way defined by the following *special* cases of the consequence relation and of the corresponding satisfiability property:

DEFINITION 8 (*SP*-consequence in **KC**). $\Gamma[\Phi_1, \dots, \Phi_n] \models^{\mathbf{KC}} \psi[\Phi_1, \dots, \Phi_n]$ iff for each $\mathfrak{M}, w$, and v, $\mathfrak{M}, w \models_v^{\mathbf{KC}} \psi[\Phi_1, \dots, \Phi_n]$ whenever for each $\phi \in \Gamma, \mathfrak{M}, w \models_v^{\mathbf{KC}} \phi[\Phi_1, \dots, \Phi_n]$, where $\Phi_1, \dots, \Phi_n$ are all and only predicate letters occurring in the subject terms of the members of $\Gamma \cup \{\psi\}$.

DEFINITION 9 (*SP*-satisfiability in **KC**). A set $\Gamma[\Phi_1, \dots, \Phi_n]$ is satisfiable iff there are $\mathfrak{M}$, w, and v such that for each $\phi \in \Gamma$, $\mathfrak{M}, w \models_v^{\mathbf{KC}} \phi[\Phi_1, \dots, \Phi_n]$, where $\Phi_1, \dots, \Phi_n$ are all and only predicate letters occurring in the subject terms of the members of Γ.

Example 1. If we consider the translations of the sentences (1)–(4) into $\mathscr{M}$, we can easily see that they are non-analytic, non-selfcontradictory, self-contradictory, and analytic, respectively:

1. non-analytic ("synthetic") sentence:

   A man who is unlearned is not learned,

   $\mathscr{KC}: \forall x\langle Mx \to (\neg Lx \to \neg Lx)\rangle$,

   $\mathscr{M}: \forall x(Mx \to (X.(\Diamond\neg Lx \to \Box\neg Lx))(M))$

   (the sentence is conceived universally, as an *e* sentence),

2. non-selfcontradictory sentence:

   Some men who are unlearned are learned,

   $\mathscr{KC}: \exists x\langle Mx \wedge (\neg Lx \wedge Lx)\rangle$,

   $\mathscr{M}: \exists x(Mx \wedge (X.(\Diamond\neg Lx \wedge \Diamond Lx))(M))$,

3. self-contradictory sentence:

   Some unlearned men are learned,
   $\mathscr{KC}: \exists x\langle (Mx \wedge \neg Lx) \wedge Lx\rangle$,
   $\mathscr{M}: \exists x((Mx \wedge \neg Lx) \wedge (XY.Yx)(M, L))$,

4. analytic sentence:

   No unlearned man is learned,
   $\mathscr{KC}: \forall x\langle (Mx \wedge \neg Lx) \rightarrow \neg Lx\rangle$,
   $\mathscr{M}: \forall x((Mx \wedge \neg Lx) \rightarrow (XY.\neg Yx)(M, L))$.

In the first and the second of the above translations, subject abstraction could be omitted because it appears only vacuously, and in the third and the fourth translation the subject abstraction for M and X could be omitted for the same reason.

The above examples clearly indicate that **KC** is paraconsistent in the sense that $\phi \wedge \neg\phi$ is satisfiable (see sentence 2) without the explosion of consequences. **KC** is paracomplete too, since $\phi \vee \neg\phi$ of $\mathscr{KC}$ (being interpreted in $\mathscr{M}$ as $\Box\phi \vee \Box\neg\phi$) is non-valid. We will return to the paraconsistency and paracompleteness of **KC** at the end of this section.

Remark 3. There are some similarities (as well as differences) between the propositional fragment of **KC** and Jaśkowski's discussive logic $\mathbf{D_2}$ [4, 5, 6] (see also, for example, [12]), first of all in the modal approach to the semantics, and, in particular, in the similar interpretation of the conditional and the conjunction (cf. $\Diamond\phi \rightarrow \psi$ and $\phi \wedge \Diamond\psi$ of $\mathbf{D_2}$). The propositional fragment of **KC** is, in a sense, more classical then discussive logic, in that, for example, in **KC** conjunction, disjunction, and conditional are classically interdefinable. In addition, theorems of **KC** can be modally interpreted as necessities, not merely as possibilities (like "theses", which are discussive counterparts of classical theorems). Note that discussive conjunction as a thesis becomes equivalent to $\Diamond\phi \wedge \Diamond\psi$ (which is the modal interpretation of the conjunction of $\mathscr{KC}$).

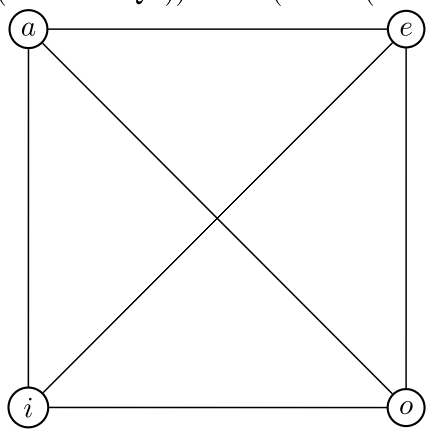
Philosophically, the difference between discussive logic and **KC** lies in the fact that discussive logic is primarily subjective and methodologically motivated by the cases where there is a lack of a uniform opinion or of a uniform meaning of terms, while **KC** is primarily objective and ontologically motivated by the non-uniformity of the state of affairs through time.[8]

[8] Cf. Perzanowski's observation about the "lack of an ontological motivation" in Jaśkowski's logic [12, p.19].

3.3. The principle of contradiction in the square of oppositions

Although **KC** is paraconsistent, all the oppositions of the logical square of categorical propositions hold in **KC** if we presuppose the *existential import* for the subject term. For the proof, compare the semantics of the translations of categorical propositions of $\mathscr{KC}$ into the modal language $\mathscr{M}$. The oppositions hold in each Kripke frame. See as an example the figure below with a compound predicate term, and $\exists x\ Sx$ as a general assumption for all propositions in the square.

$\forall x(Sx \to (\Diamond Px \wedge \Diamond Qx))$ $\qquad$ $\forall x(Sx \to (\Box\neg Px \vee \Box\neg Q))$



$\exists x(Sx \wedge (\Diamond Px \wedge \Diamond Qx))$ $\qquad$ $\exists x(Sx \wedge (\Box\neg Px \vee \Box\neg Q))$

Figure: the square of translations in $\mathscr{M}$, $\exists x\ Sx$ presupposed

3.4. Conversion and categorical syllogism

In a categorical inference of **KC**, a subject term of a premise or a conclusion is a subject term of the whole inference (see Definition 8). In that way, familiar conversions and categorical syllogisms remain valid.

Example 2.

$$\dfrac{\begin{array}{l}\forall x\langle Mx \to (Px \vee Qx)\rangle[M,S]\\ \forall x\ \langle Sx \to Mx\rangle\ [M,S]\end{array}}{\forall x\ \langle Sx \to (Px \vee Qx)\rangle\ [M,S]}\ \mathbf{KC} \qquad \text{iff} \qquad \dfrac{\begin{array}{l}\forall x(Mx \to (\Box Px \vee \Box Qx))\\ \forall x(Sx \to (X.Xx)(M))\end{array}}{\forall x(Sx \to (\Box Px \vee \Box Qx)).}\ \mathbf{M}$$

Since predicate terms of the first premise and of the conclusion do not contain any predicate letter that occurs in their respective subject terms, we omitted the subject abstraction from the first premise and the conclusion of the **M** inference (right side) as unnecessary. The right side inference is valid in each Kripke frame.

3.5. Tableau system for KC

We present the tableau rules for the logic **KC**, and conceive the rules as a formal proof system. Each sentence in a tableau is labelled. As usual, we start a tableau from a set of assumptions with the same label n, and try to prove the inconsistency of the set by obtaining, in each path, both $m\ \phi$ and $m\ \neg\phi$ for an atomic ϕ. As usual, α refers to non-branching rules for connectives, β refers to branching rules for connectives, γ refers to universal instantiation rules, and δ refers to existential instantiation rules.

α	α_1	α_2	*label properties*
$m\ \phi \wedge \psi$	$n\ \phi$	$o\ \psi$	n, o new
$m\ \neg(\phi \vee \psi)$	$n\ \neg\phi$	$o\ \neg\psi$	n, o new
$m\ \neg(\phi \to \psi)$	$n\ \phi$	$o\ \neg\psi$	n, o new
$m\ \neg\neg\phi$	$m\ \phi$		
$m\ \neg\langle\phi \to \psi\rangle$	$m\ \phi$	$m\ \neg\psi$	literals with predicate letters of ϕ are bound to m in the whole α
$m\ \langle\phi \wedge \psi\rangle$	$m\ \phi$	$m\ \psi$	literals with predicate letters of ϕ are bound to m in the whole α

β	β_1	β_2	*label properties*
$m\ \neg(\phi \wedge \psi)$	$n\ \phi$	$o\ \psi$	n, o any
$m\ \phi \vee \psi$	$n\ \phi$	$o\ \psi$	n, o any
$m\ \phi \to \psi$	$n\ \neg\phi$	$o\ \psi$	n, o any
$m\ \phi \leftrightarrow \psi$	$n\ \phi \wedge \psi$	$o\ \neg\phi \wedge \neg\psi$	n, o any
$m\ \neg(\phi \leftrightarrow \psi)$	$n\ \phi \wedge \neg\psi$	$o\ \neg\phi \wedge \psi$	n, o any
$m\ \langle\phi \to \psi\rangle$	$m\ \neg\phi$	$m\ \psi$	literals with predicate letters of ϕ are bound to m in the whole β
$m\ \neg\langle\phi \wedge \psi\rangle$	$m\ \neg\phi$	$m\ \neg\psi$	literals with predicate letters of ϕ are bound to m in the whole β

γ	$\gamma(\mathrm{c}/x)$	*term properties*
$m\ \forall x\phi$	$m\ \phi(\mathrm{c}/x)$	any c
$m\ \neg\exists x\phi$	$m\ \neg\phi(\mathrm{c}/x)$	any c

δ	$\delta(\mathrm{c}/x)$	*term properties*
$m\ \neg\forall x\phi$	$m\ \neg\phi(\mathrm{c}/x)$	new c
$m\ \exists x\phi$	$m\ \phi(\mathrm{c}/x)$	new c

> Rules for $m\ \phi[\Phi_1, \ldots, \Phi_n]$ are as above, depending on the type of ϕ, with the difference that the results of the application of the rule preserve the square bracket indication $[\Phi_1, \ldots, \Phi_n]$ (literals with $\Phi_1, \ldots, \Phi_n$ are bound to m in the whole ϕ).

In justifications of lines on the right side of a tableau, there are also annotations about the label to which a subject term predicate letter is associated. The number of a line and the name of a rule in a justification are separated by a slash. *Closure* and *openness* conditions for a tableau are as in classical tableaux, with the addition that literals that cause the closing of a path should have the same label, no matter whether any of those literals has a square bracket indication appended or not.[9] Finally, the proof of ϕ from Γ, i.e., $\Gamma \vdash^{\mathbf{KC}} \phi$ (Γ and ϕ unlabelled), is a closed tableau for $\Gamma \cup \{\neg\phi\}$.

Example 3. $\nvdash^{\mathbf{KC}} \forall x\langle Mx \to (\neg Lx \to \neg Lx)\rangle$.

1	$1\ \neg\forall x\langle Mx \to (\neg Lx \to \neg Lx)\rangle\ \checkmark$	*ass.*
2	$1\ \neg\langle Mc \to (\neg Lc \to \neg Lc)\rangle\ \checkmark$	$1/\neg\forall.$
3	$1\ Mc$	$2/\neg\langle\to\rangle, 1\ M$
4	$1\ \neg(\neg Lc \to \neg Lc)\checkmark$	$2/\neg\langle\to\rangle$
5	$2\ \neg Lc$	$4/\neg\to$
6	$3\ \neg\neg Lc\ \checkmark$	$4/\neg\to$
7	$3\ Lc$	$6/\neg\neg$
	$\circ$	

Example 4. $\vdash^{\mathbf{KC}} \forall x\langle (Mx \wedge \neg Lx) \to \neg Lx\rangle$.

1	$1\ \neg\forall x\langle (Mx \wedge \neg Lx) \to \neg Lx\rangle\ \checkmark$	*ass.*
2	$1\ \neg\langle (Mx \wedge \neg Lx) \to \neg Lx\rangle\ \checkmark$	$1/\neg\forall$
3	$1\ Mc \wedge \neg Lc$	$2/\neg\langle\to\rangle$
4	$1\ \neg\neg Lc\ \checkmark$	$2/\neg\langle\to\rangle$
5	$1\ Mc$	$3/\wedge, 1\ M$
6	$1\ \neg Lc$	$3/\wedge, 1\ L$
7	$1\ Lc$	$6/\neg\neg$
	$\times$	

In proving categorical inferences we will treat the premises and the conclusion analogously to Definition 8 (and Example 2).

[9] For the use of labelled tableaux in discussive logic see [1].

Example 5. In the following tableau, square bracket indications are added where that makes a difference in the application of a rule. Each assumption has a square bracket indication.

$$\frac{\begin{array}{l}\forall x\langle Mx \to (Px \wedge Qx)\rangle[M,S]\\ \forall x\,\langle Sx \to Mx\rangle\,[M,S]\end{array}}{\forall x\,\langle Sx \to (Px \wedge Qx)\rangle\,[M,S]}\ \mathbf{KC}$$

$$\begin{array}{rcl}
1 & 1\ \forall x\langle Mx \to (Px \wedge Qx)\rangle[M,S]\ ^{c\checkmark} & ass.\\
2 & 1\ \forall x\,\langle Sx \to Mx\rangle\,[M,S]^{c\checkmark} & ass.\\
3 & 1\ \neg\forall x\langle Sx \to (Px \wedge Qx)\rangle[M,S]\ \checkmark & ass.\\
4 & 1\ \neg\langle Sc \to (Pc \wedge Qc)\rangle\ \checkmark & 3/\neg\forall\\
5 & 1\ Sc & 4/\neg\langle\to\rangle\\
6 & 1\ \neg(Pc \wedge Qc) & 4/\neg\langle\to\rangle\\
7 & 1\ \langle Mc \to (Pc \wedge Qc)\rangle\ \checkmark & 1/\forall\\
8 & 1\ \langle Sc \to Mc\rangle[M,S] & 2/\forall\\
 & / \quad \backslash & \\
9 & 1\ \neg Mc \quad 1\ Pc \wedge Qc & 7/\langle\to\rangle, 1\ M\\
 & / \quad \backslash \qquad\quad | & \\
10 & 1\ \neg Sc \quad Mc[M] \qquad | & 8/\langle\to\rangle\\
 & \times \qquad \times \qquad\quad | & \\
11 & 2\ Pc & 9/\wedge\\
12 & 3\ Qc & 9/\wedge\\
 & / \quad \backslash & \\
13 & 2\ \neg Pc \quad 3\ \neg Qc & 6/\neg\wedge\\
 & \times \qquad \times &
\end{array}$$

PROPOSITION 1. *All traditional categorical syllogisms that are valid without the existential import for the subject term are valid with respect to* **KC** *tableau system, provided premises and the conclusion are expressed by a square bracket indication* $[\Phi_1, \dots, \Phi_n]$, *where* $\Phi_1, \dots, \Phi_n$ *are all and only predicate letters occurring in the premises and the conclusion.*

PROOF. The proof is obvious from the fact that all immediate subformulas of quantified formulas in a syllogism and all predicate letters occurring in subject terms come under the same label. All subject terms of a syllogism behave therefore in a classical way (as if they are not labelled). It is easy to check that the tableau decomposition of both (classical) subject terms and (possibly non-classical) predicate terms result with a classical contradiction (path closure condition). $\dashv$

PROPOSITION 2 (Theorems of **KC**).

$$\begin{array}{l}
\vdash^{\mathbf{KC}} \phi \to (\psi \to (\phi \wedge \psi)) \\
\vdash^{\mathbf{KC}} (\phi \to \psi) \leftrightarrow (\neg\phi \vee \psi) \\
\vdash^{\mathbf{KC}} (\phi \wedge \psi) \to (\psi \wedge \phi),\ \vdash^{\mathbf{KC}} (\phi \vee \psi) \to (\psi \vee \phi) \\
\vdash^{\mathbf{KC}} (\phi \wedge (\psi \wedge \chi)) \leftrightarrow ((\phi \wedge \psi) \wedge \chi) \\
\vdash^{\mathbf{KC}} (\phi \vee (\psi \vee \chi)) \leftrightarrow ((\phi \vee \psi) \vee \chi) \\
\vdash^{\mathbf{KC}} (\phi \wedge (\psi \vee \chi)) \to ((\phi \wedge \psi) \vee (\phi \wedge \chi)) \\
\vdash^{\mathbf{KC}} \neg(\phi \wedge \psi) \leftrightarrow (\neg\phi \vee \neg\psi),\ \vdash^{\mathbf{KC}} \neg(\phi \vee \psi) \leftrightarrow (\neg\phi \wedge \neg\psi) \\
\vdash^{\mathbf{KC}} (\neg\psi \to \neg\phi) \to (\phi \to \psi) \\
\vdash^{\mathbf{KC}} ((\phi \to \psi) \wedge \phi) \to \psi \\
\vdash^{\mathbf{KC}} \neg\exists x \langle \phi \wedge \neg\phi \rangle \\
\vdash^{\mathbf{KC}} \forall x \langle \phi \to \phi \rangle \\
\vdash^{\mathbf{KC}} \forall x(\phi \to (\psi \to \phi)) \\
\vdash^{\mathbf{KC}} \forall x((\neg\psi \to \neg\phi) \to (\phi \to \psi)) \\
\vdash^{\mathbf{KC}} \forall x(((\phi \to \psi) \wedge \phi) \to \psi) \\
\vdash^{\mathbf{KC}} \forall x \langle (\phi \wedge \psi) \to \phi \rangle \\
\vdash^{\mathbf{KC}} \forall x \langle \phi \to (\phi \vee \psi) \rangle \\
\vdash^{\mathbf{KC}} \forall x(((\phi \wedge \psi) \vee (\phi \wedge \chi)) \to (\phi \wedge (\psi \vee \chi))) \\
\vdash^{\mathbf{KC}} \forall x \langle \neg\neg\phi \to \phi \rangle \\
\vdash^{\mathbf{KC}} (\exists x\langle \phi \wedge \psi \rangle \vee \exists x\langle \phi \wedge \chi \rangle) \to \exists x\langle \phi \wedge (\psi \vee \chi) \rangle
\end{array}$$

PROPOSITION 3 (Non-theorems of **KC**).

$$\begin{array}{l}
\nvdash^{\mathbf{KC}} \neg(\phi \wedge \neg\phi) \\
\nvdash^{\mathbf{KC}} \phi \to (\neg\phi \to \psi) \\
\nvdash^{\mathbf{KC}} \phi \vee \neg\phi \\
\nvdash^{\mathbf{KC}} \phi \to \phi \\
\nvdash^{\mathbf{KC}} \phi \to (\psi \to \phi) \\
\nvdash^{\mathbf{KC}} (\phi \wedge \psi) \to \phi \\
\nvdash^{\mathbf{KC}} \phi \to (\phi \vee \psi) \\
\nvdash^{\mathbf{KC}} ((\phi \wedge \psi) \vee (\phi \wedge \chi)) \to (\phi \wedge (\psi \vee \chi)) \\
\nvdash^{\mathbf{KC}} \neg\neg\phi \to \phi \\
\nvdash^{\mathbf{KC}} \forall x\langle \phi \to \psi \rangle \to \exists x\langle \phi \wedge \psi \rangle \\
\nvdash^{\mathbf{KC}} \neg\forall x\langle \phi \to \psi \rangle \leftrightarrow \exists x\langle \phi \wedge \neg\psi \rangle \\
\nvdash^{\mathbf{KC}} (\forall x\langle \phi \to \psi \rangle \wedge \forall x\langle \phi \to \chi \rangle) \to \forall x\langle \phi \to (\psi \wedge \chi) \rangle
\end{array}$$

PROPOSITION 4 (Derivability in **KC**).

$$\{\phi, \psi\} \vdash^{\mathbf{KC}} (\phi \wedge \psi)$$
$$\{\phi, \neg\phi\} \vdash^{\mathbf{KC}} \psi$$
$$\{\phi \to \psi, \phi\} \vdash^{\mathbf{KC}} \psi$$
$$\{\phi \to \psi, \neg\psi\} \vdash^{\mathbf{KC}} \neg\phi$$
$$\{\neg\neg\phi\} \vdash^{\mathbf{KC}} \phi$$
$$\{\exists x\langle\phi \wedge \psi\rangle \vee \exists x\langle\phi \wedge \chi\rangle\} \vdash^{\mathbf{KC}} \exists x\langle\phi \wedge (\psi \vee \chi)\rangle$$

PROPOSITION 5 (Non-derivability in **KC**).

$$\{\phi \wedge \psi\} \nvdash^{\mathbf{KC}} \phi, \psi$$
$$\{\phi\} \nvdash^{\mathbf{KC}} \phi \vee \psi$$
$$\{(\phi \wedge \psi) \vee (\phi \wedge \chi)\} \nvdash^{\mathbf{KC}} \phi \wedge (\psi \vee \chi)$$
$$\{\forall x\langle\phi \to \psi\rangle\} \nvdash^{\mathbf{KC}} \exists x\langle\phi \wedge \psi\rangle$$
$$\{\forall x\langle\phi \to \psi\rangle \wedge \forall x\langle\phi \to \chi\rangle\} \nvdash^{\mathbf{KC}} \forall x\langle\phi \to (\psi \wedge \chi)\rangle$$

PROPOSITION 6 (Deductive equivalences in **KC**). *Beside the commutativity and the associativity of $\wedge$ and $\vee$, we mention the following equivalences.*

$$\neg(\phi \wedge \psi) \dashv\vdash^{\mathbf{KC}} \neg\phi \vee \neg\psi$$
$$\neg(\phi \vee \psi) \dashv\vdash^{\mathbf{KC}} \neg\phi \wedge \neg\psi$$
$$\phi \to \psi \dashv\vdash^{\mathbf{KC}} \neg\phi \vee \psi$$
$$\neg(\phi \to \psi) \dashv\vdash^{\mathbf{KC}} \phi \wedge \neg\psi$$
$$\neg\forall x\langle\phi \to \psi\rangle \dashv\vdash^{\mathbf{KC}} \exists x\langle\phi \wedge \neg\psi\rangle$$

Propositions 2–6 can be checked by the **KC** tableau system. Note that some quantificational theorems from Proposition 2 become non-theorems when the quantifier is omitted (see Proposition 3) (e.g., the classical principle of contradiction, the principle of double negation). Further, for example, the classical definition of the conditional (by $\neg$ and $\vee$), the commutativity, associativity, and De Morgan's laws for $\wedge$ and $\vee$ hold as theorems as well as derivability relations. In contrast, the law of distribution does not hold neither as a theorem, nor as a derivability relation. Note the asymmetry between theoremhood and derivability: for example, *ex contradictione quodlibet* and the principle of double negation hold as derivability relations, but not as theorems.

COROLLARY 1. *The deduction theorem for* **KC** *does not hold.*

PROOF. According to propositions 4 and 3, $\{\phi, \neg\phi\} \vdash^{\mathbf{KC}} \psi$, but $\nvdash^{\mathbf{KC}} \phi \to (\neg\phi \to \psi)$. See the same propositions for the principle of double negation.

Compare also the second last non-theorem in Proposition 3 and the last equivalence in Proposition 6. ⊣

COROLLARY 2. **KC** *is paraconsistent and paracomplete regarding theoremhood.*

PROOF. See in Proposition 3 the first and the second non-theorem for paraconsistency, and the third non-theorem for paracompleteness. ⊣

COROLLARY 3. *Regarding derivability,* **KC** *is explosive.*

PROOF. See Proposition 4, the second case. ⊣

COROLLARY 4. **KC** *is adjunctive regarding theoremhood and derivability.*

PROOF. See Proposition 2 case 1, and Proposition 4 case 1. ⊣

4. An extension of KC

Beside categorical propositions, Kant introduces "hypothetical" and "disjunctive" propositions. In order to express such propositions, we extend the language $\mathscr{KC}$ to a language $\mathscr{KC}+$ with the following new kinds of formulas: $\phi \multimap \psi$ and $\phi \veebar \psi$. The first kind is the familiar strict conditional, and the second one is strict exclusive disjunction.[10] They translate into $\mathscr{M}$ in the following way:

$$T(\phi \multimap \psi) = \Box(\phi \to \psi),$$
$$T(\phi \veebar \psi) = \Box\neg(\phi \leftrightarrow \psi).$$

By $\Box$ we want to express that hypothetical and disjunctive propositions, unlike categorical proposition, are meant time independently. For instance, when Kant gives an example of the following disjunctive proposition

> The world exists either through blind chance, or through inner necessity, or through an external cause

[10] Kant describes the "relation" of the antecedent to the consequent in a hypothetical proposition as a "consequence" ("Consequenz", B 98), that is, as a "ground – consequent" ("Grund – Folge", B 98) relation. Further, Kant describes the "relation" that constitutes a disjunctive proposition as a complementation of parts (disjuncts) to a whole sphere of a possible knowledge in a chosen respect to the subject: "In disjunctive judgments we consider *all possibility* as divided in respect to a particular concept" [9, p. 95] (emphasis in [7, vol. 4, *Prolegomena*, p. 330]).

(B 99), he plainly does not relativize the meaning of the statement only to one moment of time (in the sense in which "A man is learned" means that the man is learned at one time, but can be unlearned at another time). The same holds for Kant's example of the hypothetical proposition: "If there is a perfect justice, the obstinately wicked are punished" (B 98).

In $\mathscr{K}\mathscr{C}+$ we allow $\forall x$ and $\exists x$ to be applied to $\phi \multimap \psi$ and $\phi \veebar \psi$, so that we get the formulas of the form $\forall x(\phi \multimap \psi)$ and $\forall x(\phi \veebar \psi)$, and similarly for $\exists x$. What Kant meant under "hypothetical proposition" ("judgment") were not only the propositions of the form of the strict conditional, but also the propositions of the form $\forall x(\phi \multimap \psi)$. Similarly, under (exclusively) disjunctive proposition Kant meant not only the propositions of the form $\phi \veebar \psi$, but also the propositions of the form $\forall x(\phi \veebar \psi)$.

Kant's idea was to strengthen the logical "unity" of a proposition in comparison to categorical propositions. According to Kant's theory of propositions, we can distinguish three grades of the logical unity of propositions regarding the possible truth of mutually contradictory predications[11]:

1. the weakest unity is the "relation" of a predicate to the subject in a *categorical* proposition: mutually contradictory predications to the same (non-contradictory) subject can both be true if only they do not contradict the subject (Kant's principle of contradiction);

2. the next, stronger, unity is the "relation" of a consequent to the antecedent in a *hypothetical* proposition: mutually contradictory predications cannot both be true consequents of the same true antecedent, but can both be its false consequents; a true antecedent has simply a plurality of mutually non-contradictory predications as its true consequents (the principle of sufficient reason);

3. the strongest unity is the "relation" of the whole to the members of an *exclusive disjunctive* proposition: precisely one of the contradictory predications must be true as a member of a whole of mutually exclusive predications (the principle of excluded middle).

Corresponding to the language $\mathscr{K}\mathscr{C}+$ there is a logic **KC+**, with semantic concepts (satisfaction, satisfiability, consequence, etc.) defined in an obvious way and analogously to **KC**. The tableau system of **KC+** includes the

[11]See, for example, B 98–100, a letter to Reinhold (12.05.1789) in [7, vol. 9], and reflections 2178 [7, vol. 16], 5562 and 5734 [7, vol. 18].

tableau rules of **KC** with the addition of the following rules:

α	α_1	α_2	*label property*
$m\ \neg(\phi \multimap \psi)$	$n\ \phi$	$n\ \neg\psi$	any n

β	β_1	β_2	*label property*
$m\ \phi \multimap \psi$	$n\ \neg\phi$	$n\ \psi$	any n
$m\ \phi \veebar \psi$	$n\ \phi, \neg\psi$	$n\ \neg\phi, \psi$	any n
$m\ \neg(\phi \veebar \psi)$	$n\ \phi, \psi$	$n\ \neg\psi, \neg\psi$	any n

Regarding our discussion of paraconsistency and paracompleteness of Kantian logic, it is obvious that it is not paraconsistent with respect to $\multimap$. For the proof, we first introduce a weak conjunction, $\phi \barwedge \psi$, by the following translation to $\mathscr{M}$:

$$T(\phi \barwedge \psi) = \Diamond(\phi \wedge \psi).$$

Thus $\vdash^{\mathbf{KC}+} \neg(\phi \barwedge \neg\phi)$. Since $\vdash^{\mathbf{KC}+} (\phi \multimap \psi) \leftrightarrow \neg(\phi \barwedge \neg\psi)$, and $\vdash^{\mathbf{KC}+} (\phi \multimap \psi)$, paraconsistency is excluded with respect to the weak conjunction, and hence with respect to the strict conditional. In addition, it is obvious that **KC+** is not paracomplete with respect to $\veebar$, since $\vdash^{\mathbf{KC}+} \phi \veebar \neg\phi$.

5. Soundness and completeness

As a preliminary to the sketch of the soundness and completeness proofs for the **KC+** tableau system, we define the satisfaction of a labelled formula and the satisfiability of a set of labelled formulas in **KC+**.

DEFINITION 10 (**KC+**-satisfaction of a labelled formula).

$$\mathfrak{M} \models_v^{\mathbf{KC}+} n\ \phi \text{ iff } \mathfrak{M}, w_n \models_v^{\mathbf{M}} T(\phi).$$

COROLLARY 5.

$$\mathfrak{M}, w_n \models_v^{\mathbf{KC}+} \phi \textit{ iff } \mathfrak{M} \models_v^{\mathbf{KC}+} n\ \phi.$$

PROOF. The corollary is obvious from the generalization of Definition 2 to **KC+**, and Definition 10. ⊣

DEFINITION 11 (**KC+**-satisfiability of a labelled set). A set Γ of labelled formulas of $\mathscr{KC}+$ is **KC+**-satisfiable iff there are $\mathfrak{M}$ and v such that for each $n\ \phi \in \Gamma$, $\mathfrak{M} \models_v^{\mathbf{KC}^+} n\ \phi$.

Since each tableau path (which extends from the beginning of the tableau to the end of a branch) is a set of labelled formulas of $\mathscr{KC}+$, we can speak of the **KC+**-*satisfiability of a tableau path* too.

5.1. Soundness

First, we show that **KC+** tableau rules preserve the satisfiability in a tableau.

Proposition 7 (Satisfiability preservation). *If a tableau T has at least one satisfiable path, and T' is the extension of T by a tableau rule for members of T, then T' too has a satisfiable path.*

Proof. For α rules, observe that, according to the semantics of **KC+**, both $n\ \alpha_1$ and $o\ \alpha_2$ are true in each model that satisfies a path p (of a tableau T) containing $m\ \alpha$, for indices n and o new to p. For example, if T has a satisfiable path p, and $m\ \phi \wedge \psi \in T$, then the extension T' containing $n\ \phi$ and $o\ \psi$ has a satisfiable path too if n and o are new to the path on which they occur. For, if $\mathfrak{M} \models_v^{\mathbf{KC+}} p$ and $m\ \phi \wedge \psi \in p$, then $\mathfrak{M} \models_v^{\mathbf{KC+}} p \cup \{n\ \phi, o\ \psi\}$, where n and o are new to p (i.e., with respect to the modal logic **M**, w_n and w_o need not to be identical with any world corresponding to the indices of the members of the path p). If $m\ \phi \wedge \psi \in p$ occurs in T outside the path p, p trivially remains satisfied after the extension of T.

For β rules, observe that either $n\ \beta_1$ or $o\ \beta_2$ (for any n and o) is true in each model that satisfies a path p (in T) containing $m\ \beta$. For example, if T has a satisfiable path p, and $m\ \langle \phi \to \psi \rangle \in p$, then either p extended by $m\ \neg\phi$ or p extended by $m\ \psi$ is satisfiable. If $m\ \langle \phi \to \psi \rangle \notin p$, T' trivially continues to have a satisfiable path. Therefore, the extension T' containing $n\ \beta_1$ and $o\ \beta_2$ has a satisfiable path.

Similarly, the results of applying γ and δ rules are true for models that satisfy a path p in T, p containing γ and δ, respectively.

In an analogous way, the application of α, β, γ, and δ rules to the formulas with a square bracket indication also preserves a satisfiable path in a tableau. ⊣

Theorem 1 (Soundness). *If* $\Gamma \vdash^{\mathbf{KC+}} \phi$, *then* $\Gamma \models^{\mathbf{KC+}} \phi$.

Proof. It follows from Proposition 7 that, if a tableau for a set $n\ \Gamma \cup \{n\ \neg\phi\}$ of labelled formulas eventually has on each path a pair $m\ \phi$ and $m\ \neg\phi$ for an atomic ϕ, then $n\ \Gamma \cup \{n\ \neg\phi\}$ itself cannot be a tableau with a satisfiable path. That is, if $\Gamma \cup \{\neg\phi\}$ has a closed tableau, then $\Gamma \cup \{\neg\phi\}$ is not satisfiable (see Corollary 5). (Note that a closed tableau for a finite set Δ is also a closed tableau for any infinite superset of Δ). Since the tableau method is conceived as a formal proof method, where $\Gamma \vdash^{\mathbf{KC+}} \phi$ iff $\Gamma \cup \{\neg\phi\}$ has a closed tableau, the theorem obviously follows. ⊣

5.2. Completeness

DEFINITION 12 (Labelled Hintikka set). A labelled Hintikka set H is defined according to the tableau rules for **KC**+: if $\alpha \in H$ then $\alpha_1, \alpha_2 \in H$ (α_1 for at least one label n and α_2 for at least one label o), if $\beta \in H$ then $\beta_1 \in H$ or $\beta_2 \in H$ (β_1 for each label n and β_2 for each label o), if $\gamma \in H$ then $\gamma(\mathrm{c}/\mathrm{x}) \in H$ for each c occurring in the members of H and for at least one c, if $\delta \in H$ then $\delta(\mathrm{c}/\mathrm{x}) \in H$ for at least one c. Also, if ϕ is atomic, then $n\ \phi \notin H$ or $n\ \neg\phi \notin H$. In addition, if $\phi[\Phi_1, \ldots, \Phi_n] \in H$ then the above conditions for the membership in H of atomic, α, β, γ, and δ formulas apply in dependence of the form of ϕ and with $[\Phi_1, \ldots, \Phi_n]$ appended to α_1 and α_2, β_1 and β_2, $\gamma(\mathrm{c}/x)$, $\delta(\mathrm{c}/x)$, and literals $n\ \phi$ and $n\ \neg\phi$.

PROPOSITION 8. *Each open path of a tableau is a subset of at least one labelled Hintikka set.*

PROOF. Each formula entered into an open path of a tableau would also be entered in a variant of the application of the formation rules for a labelled Hintikka set (Definition 12). Thus, each formula of an open path is also a formula of a labelled Hintikka set. ⊣

PROPOSITION 9 (Satisfiability of a labelled Hintikka set). *Each labelled Hintikka set is satisfiable.*

PROOF. We construct a canonical model $\mathfrak{M}^H = \langle W, R, D, I\rangle$ for a given labelled Hintikka set H in the following way: W is a non-empty set (of labels in H); R is the universal relation on W (universal accessibility); D is a set of individual constants occurring in the members of H ($D = \{c\}$ if no individual constant occurs in H); $I(\mathrm{c}) = \mathrm{c}$ if an individual constant c occurs in H, otherwise $I(\mathrm{c}) = c$; $\langle \mathrm{c}_1, \ldots, \mathrm{c}_n\rangle \in I(\Phi^n, m)$ iff $m\ \Phi^n \mathrm{c}_1 \ldots \mathrm{c}_n \in H$. Now, it follows by mathematical induction that if $m\ \phi \in H$ then $\mathfrak{M}^H \models^{\mathbf{KC}+} m\ \phi$. For example, let $m\ \phi \vee \psi \in H$. Then $n\ \phi \in H$ or $o\ \psi \in H$ for each n and o in H. According to the inductive hypothesis, $\mathfrak{M}^H \models^{\mathbf{KC}+} n\ \phi$ or $\mathfrak{M}^H \models^{\mathbf{KC}+} o\ \psi$, for each n and o in H, and thus $\mathfrak{M}^H \models^{\mathbf{KC}+} m\ \phi \vee \psi$ (see Corollary 5). Further (to take another example), let $m\ \langle\phi \rightarrow \psi\rangle \in H$. Then $m\ \neg\phi \in H$ or $m\ \psi \in H$ with predicate letters of ϕ bound to m. According to the inductive hypothesis, $\mathfrak{M}^H \models^{\mathbf{KC}+} m\ \neg\phi$ or $\mathfrak{M}^H \models^{\mathbf{KC}+} m\ \psi$, where each literal subformula with a predicate letter of ϕ is bound to m. Hence, $\mathfrak{M}^H \models^{\mathbf{KC}+} m\ \langle\phi \rightarrow \psi\rangle$ (see Corollary 5). ⊣

THEOREM 2 (Completeness). *If* $\Gamma \models^{\mathbf{KC}+} \phi$, *then* $\Gamma \vdash^{\mathbf{KC}+} \phi$.

PROOF. If $\Gamma \cup \{\neg\phi\}$ has a tableau with an open path, then $n\ \Gamma \cup \{n\ \neg\phi\}$ is a subset of a labelled Hintikka set (Proposition 8), and therefore $n\ \Gamma \cup \{n\ \neg\phi\}$ is satisfiable (see Proposition 9). Hence, $\Gamma \cup \{\neg\phi\}$ is also satisfiable (see Corollary 5). After the contraposition, the theorem follows, since a tableau without an open path is closed, and hence is a proof of ϕ from Γ. $\dashv$

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