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SOME NEW RESULTS ON PCL1 AND ITS RELATED SYSTEMS*

Abstract. In [Waragai & Shidori, 2007], a system of paraconsistent logic called PCL1, which takes a similar approach to that of da Costa, is proposed. The present paper gives further results on this system and its related systems. Those results include the concrete condition to enrich the system PCL1 with the classical negation, a comparison of the concrete notion of "behaving classically" given by da Costa and by Waragai and Shidori, and a characterisation of the notion of "behaving classically" given by Waragai and Shidori.

Keywords: paraconsistent logic, classical negation, da Costa's system C_1.

1. Introduction

We shall first recall some basic definitions used in the study of paraconsistent logics. Then, the position of the system PCL1 in the systems of paraconsistent logic will be given. Finally, the aim and the outline of the present paper will be clarified.

1.1. Basic Definitions

In this paper, we understand by *logic L* an ordered pair of a set of formulas (written as *For*) and a consequence relation ⊢ defined over *For*.

*We would like to thank the referee for his/her advice to make some differences between da Costa's system C_n and PCL1 clear.

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We shall write the consequence relation of the logic L as $\vdash_L$. Also any subset Γ of *For* will be called a *theory* of L.

Definition 1. Let L be a logic and Γ be a theory of L. Then,

- Γ is said to be *contradictory* when for some formula A, $\Gamma \vdash_L A$ and $\Gamma \vdash_L \neg A$ both hold.
- Γ is said to be *explosive with respect to* A *and* $\neg$ when $\Gamma \vdash_L A \supset (\neg A \supset B)$ holds for any formula B.
- Γ is said to be *trivial* when $\Gamma \vdash_L A$ holds for any formula A.
- L is said to be *contradictory, explosive with respect to* A *and* $\neg$*, or trivial* when any theories of L are contradictory, explosive with respect to A and $\neg$, or trivial respectively.
- L is said to be *paraconsistent* when there exists a theory which is contradictory but not trivial.

It should be noted here that two logical notions, that a theory Γ being contradictory and a theory Γ being trivial, are *different* logical notions. In many of the familiar logics such as classical propositional calculus or intuitionistic propositional calculus these two notions *do* coincide, though in logics which are paraconsistent, it does not. In general, to treat these two notions as different logical notions must be the common objective in the study of paraconsistent logic. As it is known, however, there are several approaches to this objective such as that of Jaśkowski, da Costa, Priest, Batens, etc.

Since the systems to be treated in this paper are in the tradition of da Costa's approach, we shall next recall his idea briefly.

1.2. Recalling da Costa's Approach

In da Costa's systems of paraconsistent logic C_n ($1 \leqslant n < \omega$), the following definition is given:

$$A^\circ =_{\mathrm{def}} \mathsf{N}(A \wedge \mathsf{N}\, A)$$

$$A^n =_{\mathrm{def}} A^{\overbrace{\circ\circ\cdots\circ}^{n}}$$

$$A^{(n)} =_{\mathrm{def}} A^1 \wedge \cdots \wedge A^n$$

Formula A is said to be *behaving classically in* C_n when $\vdash_{\mathrm{C}_n} A^{(n)}$ holds.

Keeping this definition in mind, the following results hold in C_n:

$$\nvdash_{\mathrm{C}_n} A \supset (\mathsf{N}\, A \supset B)$$
$$\vdash_{\mathrm{C}_n} A^{(n)} \supset (A \supset (\mathsf{N}\, A \supset B))$$

That is, C_n is not necessarily explosive with respect to A and N, though if it turns out that A behaves classically in C_n, then C_n is explosive with respect to A and N.

As we can see here, the idea to internalise the notion regulating the explosion of the system into the object-language level is one of the main features of da Costa's system, and this idea was enlarged in a more general setting by Carnielli, Coniglio and Marcos.[1] Taking their treatment into account, we shall next give a definition.

Definition 2. Let L be a logic, Γ be a theory of L, and $\Delta(A)$ be a formula which depends only on A. Then Γ is said to be *gently explosive with respect to* A *and* $\neg$ when the following conditions hold:

(a) Neither $\{\Delta(A), A\}$ nor $\{\Delta(A), \neg A\}$ are trivial.

(b) For any formula B, the following holds:

(gOEF) $\qquad \Gamma \vdash_L \Delta(A) \supset (A \supset (\neg A \supset B))$

Also the logic L is said to be *gently explosive with respect to* A *and* $\neg$ when any theories of L are gently explosive with respect to A and $\neg$.

Remark. In the system PCL1, which is to be treated in the present paper, $\Delta(A)$ is the formula $\ulcorner A \supset \mathsf{N}\,\mathsf{N}\, A \urcorner$ and will be written as A^{I}.

As we pointed out above, da Costa's idea is generalised by the work given by Carnielli, Coniglio and Marcos, but still the concrete treatment of the notion of "behaving classically" must be an important task in order to make an evaluation of da Costa's approach. That is to say, if we are to develop a system of paraconsistent logic by da Costa's appraoch, then it must also be necessary to introduce some concrete notions of "behaving classically" defined differently from the one given by da Costa, and to compare the new definitions with da Costa's original definition.

[1] See [Carnielli & Marcos, 2002] and [Carnielli *et al*, 2005].

1.3. Aim and Outline of the Present Paper

With all the observations given above in mind, we shall now make the aim and the outline of the present paper clear. The objective of the present paper is to discuss the following three points:

- to clarify the conditions needed to enrich the system PCL1 with the classical negation.
- to give a justification of the notion of "behaving classically" given by Waragai and Shidori.
- to examine the concrete notions of "behaving classically" given by da Costa and by Waragai and Shidori.[2]

We shall add some words for each of the points. The first point is important since if we take a different notion of "behaving classically" from that of da Costa's, then we won't be able to define the classical negation as it is done in da Costa's systems. The second point must help us to grasp the idea of the new definition. Finally, the third point is necessary in order to see the differences between the original definition and the new definition of the notion of "behaving classically".

Now, in order to reach these three points, the following three steps will be taken beforehand:

1. to state the axiom schemata and the rules of inference of PCL1.
2. to give some basic results of PCL1.
3. to define the strong negation and show some of its results.

These three steps will be the content of the coming three sections from 2 to 4 respectively and are followed by three sections from 5 to 7 treating the above three points respectively.

2. The Axiom Schemata and the Rules of Inference of PCL1

We shall now give the axiom schemata and the rules of inference of PCL1. As da Costa's systems C_n are based on the axiomatisation of classical propositional calculus in [Kleene, 1967], PCL1 is based on the

[2]There is another concrete notion of "behaving classically" proposed by Guillaume in his [Guillaume, 2007]. However, we shall discuss his definition in another paper.

axiomatisation of it given in [Rasiowa & Sikorski, 1970]. In this section, we shall first recall the formulation of classical propositional calculus in [Rasiowa & Sikorski, 1970], and then give the formulation of PCL1.

2.1. Formulation of Classical Propositional Calculus in [Rasiowa & Sikorski, 1970]

The formulation of classical propositional calculus given in [Rasiowa & Sikorski, 1970] consists of the following axiom schemata and a rule of inference:

Axiom Schemata

$A_{PCL1}1$	$(A \supset B) \supset ((B \supset C) \supset (A \supset C))$
$A_{PCL1}2$	$A \supset (A \vee B)$
$A_{PCL1}3$	$B \supset (A \vee B)$
$A_{PCL1}4$	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
$A_{PCL1}5$	$(A \wedge B) \supset A$
$A_{PCL1}6$	$(A \wedge B) \supset B$
$A_{PCL1}7$	$(C \supset A) \supset ((C \supset B) \supset (C \supset (A \wedge B)))$
$A_{PCL1}8$	$(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$
$A_{PCL1}9$	$((A \wedge B) \supset C) \supset (A \supset (B \supset C))$
A10	$(A \wedge \neg A) \supset B$
A11	$(A \supset (A \wedge \neg A)) \supset \neg A$
A12	$A \vee \neg A$

Rule of Inference

MP $\quad A, A \supset B \ / \ B$

Remark. It should be noted here that if we eliminate the last axiom scheme A12 then a formulation of intuitionistic propositional calculus will be obtained. We will make use of this fact later in this paper.

2.2. Formulation of PCL1

Keeping the formulation of classical propositional calculus given above in mind, we shall now state the axiom schemata, two rules of inference and two definitions of the system PCL1 which will be the main system to be discussed in this paper.

Axiom Schemata1: $\mathrm{A}_{\mathrm{PCL1}}1$–$\mathrm{A}_{\mathrm{PCL1}}9$ are in common with those stated above,

$\mathrm{A}_{\mathrm{PCL1}}10$	$\mathsf{N}\,\mathsf{N}\,A \supset A$
$\mathrm{A}_{\mathrm{PCL1}}11$	$A \vee \mathsf{N}\,A$
D1	$A^{\mathrm{I}} =_{\mathrm{def}} A \supset \mathsf{N}\,\mathsf{N}\,A$

Axiom Schemata2:

$\mathrm{A}_{\mathrm{PCL1}}12.1$	$A^{\mathrm{I}} \supset ((\mathsf{N}\,A \vee B) \supset (A \supset B))$
$\mathrm{A}_{\mathrm{PCL1}}13$	$(A^{\mathrm{I}} \wedge B^{\mathrm{I}}) \supset (\mathsf{N}(A \wedge B) \supset (\mathsf{N}\,A \vee \mathsf{N}\,B))$
$\mathrm{A}_{\mathrm{PCL1}}14$	$(A^{\mathrm{I}} \wedge B^{\mathrm{I}}) \supset (\mathsf{N}(A \vee B) \supset (\mathsf{N}\,A \wedge \mathsf{N}\,B))$
$\mathrm{A}_{\mathrm{PCL1}}15$	$A^{\mathrm{I}} \supset ((A \wedge \mathsf{N}\,B) \supset \mathsf{N}(A \supset B))$

Rules of Inference

MP	$A, A \supset B \;/\; B$
RA15	$A^{\mathrm{I}} \;/\; \mathsf{N}(A \supset B) \supset (A \wedge \mathsf{N}\,B)$
D2	$A \equiv B =_{\mathrm{def}} (A \supset B) \wedge (B \supset A)$

Remark. Several modifications, which will be given below, have been made from the point of view of the previous formulation of PCL1 given in [Waragai & Shidori, 2007].

1. $A \vee \mathsf{N}\,A$ is substituted for $\mathsf{N}\,A \supset \mathsf{N}\,\mathsf{N}\,\mathsf{N}\,A$ in A11 of the previous formulation.
2. $\supset$ is substituted for $\equiv$ in A12 of the previous formulation and A12 is relabelled as $\mathrm{A}_{\mathrm{PCL1}}12.1$.
3. $\supset$ is substituted for $\equiv$ in D2 of the previous formulation and D2 is relabelled as D1.

We shall prove the inferential equivalence of these two formulations in the end of the next section.

Another point to be noted is that two axiom schemata $\mathrm{A}_{\mathrm{PCL1}}13$, $\mathrm{A}_{\mathrm{PCL1}}14$ and one of the rules of inference RA15, which are stated with the help of A^{I}, enable us to derive the formulas expressing the propagation of the notion of "behaving classically".[3]

Finally, a justification of the two axioms, $\mathrm{A_{PCL1}}10$ and $\mathrm{A_{PCL1}}11$, together with the justification of A^{I} will be given in the later section.

3. Some Basic Results on PCL1

In this section, some of the basic results on PCL1 will be presented. We shall first point out some positive theses to be made use of in the present paper, and then prove some results related with the negation N. And in the last subsection, the inferential equivalence of the two formulations of PCL1 mentioned above will be given.

3.1. Preliminaries on "Positive" Formulas

Since the system PCL1 has intuitionistic positive calculus as a subsystem, we can certainly prove the following theses:

$\mathrm{T_{PCL1}}1$	$\vdash_{\mathrm{PCL1}} A \supset A$
$\mathrm{T_{PCL1}}2$	$\vdash_{\mathrm{PCL1}} A \supset (B \supset A)$
$\mathrm{T_{PCL1}}3$	$\vdash_{\mathrm{PCL1}} (A \supset (B \supset C)) \supset (B \supset (A \supset C))$
$\mathrm{T_{PCL1}}4$	$\vdash_{\mathrm{PCL1}} (A \supset (A \supset B)) \supset (A \supset B)$
$\mathrm{T_{PCL1}}5$	$\vdash_{\mathrm{PCL1}} (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

As is shown by $\mathrm{T_{PCL1}}2$ and $\mathrm{T_{PCL1}}5$, we have Simplification and Fregean Syllogism available. So we can make use of the so-called *Deduction Theorem*, which will be abbreviated to (DT) hereafter, if the rule of inference RA15 is not involved in the proof.

We can also prove the following two theses; the former is quite useful in the present paper:

$\mathrm{T_{PCL1}}6$	$\vdash_{\mathrm{PCL1}} (A \vee B) \supset ((B \supset C) \supset (A \vee C))$
$\mathrm{T_{PCL1}}7$	$\vdash_{\mathrm{PCL1}} (A \vee (B \wedge C)) \equiv ((A \vee B) \wedge (A \vee C))$

The theses appearing here are all "positive" ones which we shall make use of in the following sections. Now, we shall start to observe some theses in which negation N is involved.

[3]This is one of the main results presented in [Waragai & Shidori, 2007], and will be examined further in [Omori & Waragai, 2010].

3.2. Results Related with the Negation N

Since we have the law of the excluded middle with respect to N, we can prove the following:

$\mathrm{T_{PCL1}}8$ $\quad \vdash_{\mathrm{PCL1}} (A \supset \mathsf{N}\, A) \supset \mathsf{N}\, A$ $\quad \mathrm{A_{PCL1}}4$, $\mathrm{T_{PCL1}}3$, $\mathrm{T_{PCL1}}1$, $\mathrm{A_{PCL1}}11$

$\mathrm{T_{PCL1}}9$ $\quad \vdash_{\mathrm{PCL1}} (\mathsf{N}\, A \supset A) \supset A$ $\quad \mathrm{A_{PCL1}}4$, $\mathrm{T_{PCL1}}1$, $\mathrm{T_{PCL1}}3$, $\mathrm{A_{PCL1}}11$

The next thesis $\mathrm{T_{PCL1}}10$ shows that PCL1 is not necessarily explosive with respect to A and N.

$\mathrm{T_{PCL1}}10$ $\quad \nvdash_{\mathrm{PCL1}} A \supset (\mathsf{N}\, A \supset B)$

Proof. Use Waragai and Shidori's matrix (see [Waragai & Shidori, 2007], p. 184), given below, and assign the value ½, 0 to A, B respectively. Then the formula we are considering takes the value 0 which gives the desired result.

$\wedge$	1	½	0
1	1	½	0
½	½	½	0
0	0	0	0

$\vee$	1	½	0
1	1	1	1
½	1	1	½
0	1	½	0

$\supset$	1	½	0
1	1	½	0
½	1	½	0
0	1	1	1

	N
1	0
½	1
0	1

Note that 1 and ½ are the distinguished values. ⊣

3.2.1. Explosion and Reductio ad Absurdum

We shall here offer an useful proof-mechanism, which can be applied if we are not making use of the rule (RA15). As we saw in the previous subsection, we are free to use the well-known Deduction Theorem, if the rule (RA15) is not concerned.

Theorem 1. *The following metatheorem holds in* PCL1, *if we are not making use of the rule* (RA15):

(RA) $\quad \forall B\ (\Gamma, A \vdash_{\mathrm{PCL1}} B) \Rightarrow \Gamma \vdash_{\mathrm{PCL1}} \mathsf{N}\, A$

PROOF. 1. $\forall B\ (\Gamma, A \vdash_{\mathrm{PCL1}} B)$ sup.

2. $\Gamma, A \vdash_{\mathrm{PCL1}} \mathsf{N}\, A$ 1 $(B/\mathsf{N}\, A)$

3. $\Gamma \vdash_{\mathrm{PCL1}} A \supset \mathsf{N}\, A$ 2, (DT)

4. $\Gamma \vdash_{\mathrm{PCL1}} (A \supset \mathsf{N}\, A) \supset \mathsf{N}\, A$ $\mathrm{T}_{\mathrm{PCL1}}8$

5. $\Gamma \vdash_{\mathrm{PCL1}} \mathsf{N}\, A$ 3, 4 ⊣

This (RA) says that if we succeed in showing that A causes logical explosion, then we can derive that $\mathsf{N}\, A$ holds. This offers us a kind of *reductio ad absurdum* method to prove $\mathsf{N}\, A$.

3.2.2. Bottom Particles in PCL1

Now we shall observe some bottom particles in PCL1.

$\mathrm{T}_{\mathrm{PCL1}}11$ $\vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \supset (A \supset (\mathsf{N}\, A \supset B))$ $\mathrm{A}_{\mathrm{PCL1}}12.1$, $\mathrm{A}_{\mathrm{PCL1}}2$, (DT)

As we saw in $\mathrm{T}_{\mathrm{PCL1}}10$, PCL1 is not always explosive with respect to A and N, but this $\mathrm{T}_{\mathrm{PCL1}}11$ tells us that if it turns out that A behaves classically in PCL1, then PCL1 *is* explosive with respect to A and N. In other words, PCL1 is gently explosive with respect to A and N for any formula A.

By a simple calculation based on $\mathrm{T}_{\mathrm{PCL1}}11$, we reach the following:

$\mathrm{T}_{\mathrm{PCL1}}12$ $\vdash_{\mathrm{PCL1}} (A^{\mathrm{I}} \wedge A \wedge \mathsf{N}\, A) \supset B$ $\mathrm{T}_{\mathrm{PCL1}}11$

This $\mathrm{T}_{\mathrm{PCL1}}12$ shows that $A^{\mathrm{I}} \wedge A \wedge \mathsf{N}\, A$ is a bottom particle in PCL1. We actually have another bottom particle in PCL1 and in order to prove this fact, we need the help of $\mathrm{T}_{\mathrm{PCL1}}15$ below:

$\mathrm{T}_{\mathrm{PCL1}}13$ $\vdash_{\mathrm{PCL1}} \mathsf{N}\,\mathsf{N}\, A \supset (A \wedge A^{\mathrm{I}})$ $\mathrm{A}_{\mathrm{PCL1}}10$, $\mathrm{T}_{\mathrm{PCL1}}2$, D1, $\mathrm{A}_{\mathrm{PCL1}}7$

$\mathrm{T}_{\mathrm{PCL1}}14$ $\vdash_{\mathrm{PCL1}} (A \wedge A^{\mathrm{I}}) \supset \mathsf{N}\,\mathsf{N}\, A$ $\mathrm{T}_{\mathrm{PCL1}}12$, (RA)

$\mathrm{T}_{\mathrm{PCL1}}15$ $\vdash_{\mathrm{PCL1}} \mathsf{N}\,\mathsf{N}\, A \equiv (A \wedge A^{\mathrm{I}})$ $\mathrm{T}_{\mathrm{PCL1}}13$, $\mathrm{T}_{\mathrm{PCL1}}14$

According to this $\mathrm{T}_{\mathrm{PCL1}}15$ it is shown that double negation of A in PCL1 can be split into two parts: A itself and A^{I}.

Now, $\mathrm{T}_{\mathrm{PCL1}}12$ together with $\mathrm{T}_{\mathrm{PCL1}}15$ enables us to derive the following thesis:

$\mathrm{T}_{\mathrm{PCL1}}16$ $\vdash_{\mathrm{PCL1}} (\mathsf{N}\, A \wedge \mathsf{N}\,\mathsf{N}\, A) \supset B$ $\mathrm{T}_{\mathrm{PCL1}}12$, $\mathrm{T}_{\mathrm{PCL1}}15$

Therefore, $\mathsf{N}\, A \wedge \mathsf{N}\,\mathsf{N}\, A$ is also a bottom particle in PCL1.[4]

[4] Note that $\nvdash_{\mathrm{C}_n} \mathsf{N}\,\mathsf{N}\, A \equiv (A \wedge A^{(n)})$ and $\nvdash_{\mathrm{C}_n} (\mathsf{N}\, A \wedge \mathsf{N}\,\mathsf{N}\, A) \supset B$.

3.3. Equivalence between the two formulations of PCL1

We shall here prove the equivalence between the previous formulation of PCL1 developed in [Waragai & Shidori, 2007] and the present formulation of PCL1. It will be sufficient to prove the following two formulas in the present PCL1:

$\mathrm{T}_{\mathrm{PCL1}}17$ $\quad\vdash_{\mathrm{PCL1}} \mathsf{N}\, A \supset \mathsf{N}\,\mathsf{N}\,\mathsf{N}\, A$

$\mathrm{T}_{\mathrm{PCL1}}18$ $\quad\vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \supset ((A \supset B) \supset (\mathsf{N}\, A \vee B))$

Proof. For $\mathrm{T}_{\mathrm{PCL1}}17$:

1.1.	$\mathsf{N}\, A$	sup.
1.2.	$\mathsf{N}\,\mathsf{N}\, A$	sup.
1.3.	$\mathsf{N}\, A \wedge \mathsf{N}\,\mathsf{N}\, A$	1.1,1.2,$\mathrm{A}_{\mathrm{PCL1}}7$, MP
1.4.	B	1. 3, $\mathrm{T}_{\mathrm{PCL1}}16$, MP
1.	$\mathsf{N}\, A, \mathsf{N}\,\mathsf{N}\, A \vdash_{\mathrm{PCL1}} B$	1.1–1.4
2.	$\mathsf{N}\, A \vdash_{\mathrm{PCL1}} \mathsf{N}\,\mathsf{N}\,\mathsf{N}\, A$	1, (RA)
3.	$\vdash_{\mathrm{PCL1}} \mathsf{N}\, A \supset \mathsf{N}\,\mathsf{N}\,\mathsf{N}\, A$	2, (DT)

For $\mathrm{T}_{\mathrm{PCL1}}18$:

1.	$(\mathsf{N}\, A \vee A) \supset ((A \supset B) \supset (\mathsf{N}\, A \vee B))$	$\mathrm{T}_{\mathrm{PCL1}}6$
2.	$\mathsf{N}\, A \vee A$	$\mathrm{A}_{\mathrm{PCL1}}4$, $\mathrm{A}_{\mathrm{PCL1}}3$, $\mathrm{A}_{\mathrm{PCL1}}2$, $\mathrm{A}_{\mathrm{PCL1}}11$, MP
3.	$(A \supset B) \supset (\mathsf{N}\, A \vee B)$	1, 4, MP
4.	$A^{\mathrm{I}} \supset ((A \supset B) \supset (\mathsf{N}\, A \vee B))$	5, $\mathrm{T}_{\mathrm{PCL1}}2$, MP

Therefore, the desired result is proved. ⊣

4. Strong Negation in PCL1

In this section, the strong negation in PCL1 will be defined. Also some of the results showing the properties of the strong negation and its relation with the paraconsistent negation N and A^{I} will be given.

4.1. Definition of Strong Negation and Some Results

In addition to the fact that we have the bottom particle $\mathsf{N}\,A \wedge \mathsf{N}\,\mathsf{N}\,A$ at hand in PCL1, we can also prove the following thesis:

$\mathrm{T_{PCL1}19}$ $\quad \vdash_{\mathrm{PCL1}} (\mathsf{N}\,A \wedge \mathsf{N}\,\mathsf{N}\,A) \equiv (\mathsf{N}\,B \wedge \mathsf{N}\,\mathsf{N}\,B)$ $\quad$ $\mathrm{T_{PCL1}16}$

Therefore, with this result in mind, let us define the strong negation $\neg$ with respect to A as follows:

Definition 3. For a certain formula X, we put the formula $\mathsf{N}\,X \wedge \mathsf{N}\,\mathsf{N}\,X$ as f, and define strong negation of A as follows:

DSN $\quad \neg A =_{\mathrm{def}} A \supset \mathsf{f}$

We shall now prove some results on the strong negation using the definition given above.

$\mathrm{T_{PCL1}20}$ $\quad \vdash_{\mathrm{PCL1}} (A \supset \neg A) \supset \neg A$ $\quad$ $\mathrm{T_{PCL1}4}$, DSN

$\mathrm{T_{PCL1}21}$ $\quad \vdash_{\mathrm{PCL1}} (A \supset (A \wedge \neg A)) \supset \neg A$ $\quad$ $\mathrm{T_{PCL1}20}$

With the help of $\mathrm{T_{PCL1}20}$, we also have (RA) stated in terms of strong negation, i.e. the following theorem holds:

Theorem 2. *If we are not making use of the rule* (RA15)*, then the following metatheorem holds in* PCL1:

(RAS) $\quad \forall B(\Gamma, A \vdash_{\mathrm{PCL1}} B) \Rightarrow \Gamma \vdash_{\mathrm{PCL1}} \neg A$

$\mathrm{T_{PCL1}22}$ $\quad \vdash_{\mathrm{PCL1}} A \supset (\neg A \supset B)$ $\quad$ $\mathrm{A_{PCL1}1}$, $\mathrm{T_{PCL1}3}$, $\mathrm{T_{PCL1}16}$, $\mathrm{T_{PCL1}3}$, DSN

This $\mathrm{T_{PCL1}22}$ shows that PCL1 is explosive with respect to A and $\neg$.

$\mathrm{T_{PCL1}23}$ $\quad \vdash_{\mathrm{PCL1}} (A \wedge \neg A) \supset B$ $\quad$ $\mathrm{T_{PCL1}22}$, $\mathrm{A_{PCL1}8}$

Since we have this $\mathrm{T_{PCL1}23}$, we immediately reach the following results.

$\mathrm{T_{PCL1}24}$ $\quad \vdash_{\mathrm{PCL1}} \mathsf{N}(A \wedge \neg A)$ $\quad$ $\mathrm{T_{PCL1}23}$, (RA)

$\mathrm{T_{PCL1}25}$ $\quad \vdash_{\mathrm{PCL1}} \neg(A \wedge \neg A)$ $\quad$ $\mathrm{T_{PCL1}23}$, (RAS)

It should also be noted that the following theorem holds for the strong negation in PCL1:

Theorem 3. *The strong negation in* PCL1 *is not the classical negation.*

Proof. It would be sufficient to prove $\nvdash_{\text{PCL1}} A \vee \neg A$, and this can be proved by using Alves' matrix given below (see [Carnielli & Marcos, 2002], pp. 34–35). Just assign the value ½ to A.

$\wedge$	1	½	0
1	1	½	0
½	½	½	0
0	0	0	0

$\vee$	1	½	0
1	1	1	1
½	1	½	½
0	1	½	0

$\supset$	1	½	0
1	1	½	0
½	1	1	0
0	1	1	1

	N
1	0
½	1
0	1

Note that 1 is the only distinguished value. ⊣

Remark. It should be noted here that this result is showing a difference of the behaviour of strong negation in da Costa's C_n and PCL1, i.e. the strong negation in C_n *is* the classical negation.[5]

We can also prove the following theorem.

Theorem 4. PCL1 *contains intuitionistic propositional calculus as a subsystem.*

Proof. "Positive" axiom schemata of PCL1 from $\mathrm{A}_{\text{PCL1}}1$ to $\mathrm{A}_{\text{PCL1}}9$ together with the two theses for the strong negation $\mathrm{T}_{\text{PCL1}}21$ and $\mathrm{T}_{\text{PCL1}}23$ show that PCL1 has intuitionistic propositional calculus as a subsystem. ⊣

4.2. Strong Negation, Paraconsistent Negation and A^{I}

We now relate the strong negation to the paraconsistent negation and A^{I}.

$\mathrm{T}_{\text{PCL1}}26$ $\quad\vdash_{\text{PCL1}} \neg A \supset \mathsf{N}\, A$ $\qquad$ $\mathrm{T}_{\text{PCL1}}23$, (RA)

$\mathrm{T}_{\text{PCL1}}27$ $\quad\nvdash_{\text{PCL1}} \mathsf{N}\, A \supset \neg A$

Proof. Use Waragai and Shidori's matrix and assign the value ½ to A. This gives the desired result. ⊣

These two results show the basic relations between the strong negation $\neg$ and the paraconsistent negation N.

$\mathrm{T}_{\text{PCL1}}28$ $\quad\vdash_{\text{PCL1}} \neg A \supset A^{\mathrm{I}}$ $\qquad$ $\mathrm{T}_{\text{PCL1}}22$, $\mathrm{T}_{\text{PCL1}}3$, D1

$\mathrm{T}_{\text{PCL1}}29$ $\quad\vdash_{\text{PCL1}} \neg A \supset (\mathsf{N}\, A \wedge A^{\mathrm{I}})$ $\qquad$ $\mathrm{T}_{\text{PCL1}}26$, $\mathrm{T}_{\text{PCL1}}28$, $\mathrm{A}_{\text{PCL1}}7$

[5] Cf. [da Costa, 1974], p. 500, Theorem 5.

$\mathrm{T}_{\mathrm{PCL1}}30 \quad \vdash_{\mathrm{PCL1}} (\mathsf{N}\,A \wedge A^{\mathrm{I}}) \supset \neg A$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}12$, (RAS)

$\mathrm{T}_{\mathrm{PCL1}}31 \quad \vdash_{\mathrm{PCL1}} \neg A \equiv (\mathsf{N}\,A \wedge A^{\mathrm{I}})$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}29$, $\mathrm{T}_{\mathrm{PCL1}}30$

In the literature, $\mathrm{T}_{\mathrm{PCL1}}31$ is often taken as the *definition* of strong negation[6], though in PCL1 this can be *proved* as is demonstrated above.

$\mathrm{T}_{\mathrm{PCL1}}32 \quad \vdash_{\mathrm{PCL1}} \neg\,\mathsf{N}\,A \supset \mathsf{N}\,\mathsf{N}\,A$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}26$

$\mathrm{T}_{\mathrm{PCL1}}33 \quad \vdash_{\mathrm{PCL1}} \mathsf{N}\,\mathsf{N}\,A \supset \neg\,\mathsf{N}\,A$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}16$, (RAS)

$\mathrm{T}_{\mathrm{PCL1}}34 \quad \vdash_{\mathrm{PCL1}} \neg\,\mathsf{N}\,A \equiv \mathsf{N}\,\mathsf{N}\,A$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}32$, $\mathrm{T}_{\mathrm{PCL1}}33$

This $\mathrm{T}_{\mathrm{PCL1}}34$ says that when N is iterated, the outer N can be replaced by the strong negation.[7]

Now, as we saw in $\mathrm{T}_{\mathrm{PCL1}}27$, $\mathsf{N}\,A \supset \neg A$ is not a thesis of PCL1, but this is actually equivalent to the formula A^{I} as it is given in $\mathrm{T}_{\mathrm{PCL1}}37$.

$\mathrm{T}_{\mathrm{PCL1}}35 \quad \vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \supset (\mathsf{N}\,A \supset \neg A)$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}12$, (RAS)

$\mathrm{T}_{\mathrm{PCL1}}36 \quad \vdash_{\mathrm{PCL1}} (\mathsf{N}\,A \supset \neg A) \supset A^{\mathrm{I}}$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}23$, (RA), (DT), D1

$\mathrm{T}_{\mathrm{PCL1}}37 \quad \vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \equiv (\mathsf{N}\,A \supset \neg A)$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}35$, $\mathrm{T}_{\mathrm{PCL1}}36$

We can also derive the following result:

$\mathrm{T}_{\mathrm{PCL1}}38 \quad \vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \equiv (\neg A \equiv \mathsf{N}\,A)$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}37$, $\mathrm{T}_{\mathrm{PCL1}}26$

Remark. This $\mathrm{T}_{\mathrm{PCL1}}38$ together with $\mathrm{T}_{\mathrm{PCL1}}34$ enables us to derive

$$\vdash_{\mathrm{PCL1}} (\mathsf{N}\,A)^{\mathrm{I}}.$$

The same fact can be proved by D1 and $\mathrm{T}_{\mathrm{PCL1}}17$. Again, note that $\nvdash_{\mathrm{C}_n} (\mathsf{N}\,A)^{(n)}$.

It is obvious by the definition of A^{I} that the following holds:

$\mathrm{T}_{\mathrm{PCL1}}39 \quad \vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \equiv (\mathsf{N}\,\mathsf{N}\,A \equiv A)$ $\qquad$ D1, $\mathrm{A}_{\mathrm{PCL1}}10$

But if we replace N with $\neg$, then it is no longer a thesis of PCL1:

$\mathrm{T}_{\mathrm{PCL1}}40 \quad \vdash_{\mathrm{PCL1}} A \supset \neg\neg A$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}23$, (RAS)

$\mathrm{T}_{\mathrm{PCL1}}41 \quad \vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \supset (\neg\neg A \supset A)$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}35$, $\mathrm{T}_{\mathrm{PCL1}}32$, $\mathrm{A}_{\mathrm{PCL1}}10$

$\mathrm{T}_{\mathrm{PCL1}}42 \quad \vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \supset (\neg\neg A \equiv A)$ $\qquad$ $\mathrm{T}_{\mathrm{PCL1}}40$, $\mathrm{T}_{\mathrm{PCL1}}41$

$\mathrm{T}_{\mathrm{PCL1}}43 \quad \nvdash_{\mathrm{PCL1}} (\neg\neg A \equiv A) \supset A^{\mathrm{I}}$

Proof. Assign ½ to A in Waragai and Shidori's matrix. ⊣

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[6] See, for example, [da Costa, 1974], p. 500, Definition 1.

[7] Note that $\nvdash_{\mathrm{C}_1} \neg\,\mathsf{N}\,A \equiv \mathsf{N}\,\mathsf{N}\,A$.

5. PCL1 and PCL1C

The objective of this section is to give some conditions which enable us to enrich the system PCL1 with the classical negation. We shall begin with the following theorem which shows that the strong negation defined in the previous section plays an important role for the present purpose.

Theorem 5. *The following two conditions are equivalent*:

1. *The system* PCL1 *is equipped with the classical negation.*
2. *The strong negation in the system* PCL1 *is the classical negation.*

Proof. Assume that PCL1 is equipped with the classical negation which we shall write as $\neg_c$. Then based on the formulation of classical propositional calculus given in [Rasiowa & Sikorski, 1970], this classical negation $\neg_c$ satisfies the following conditions:

(CN1) $(A \wedge \neg_c A) \supset B$

(CN2) $(A \supset (A \wedge \neg_c A)) \supset \neg_c A$

(CN3) $A \vee \neg_c A$

By (CN1) and (RAS), we have $\vdash_{\mathrm{PCL1}} \neg_c A \supset \neg A$. So this together with (CN3) and $\mathrm{T}_{\mathrm{PCL1}}6$ enables us to derive $\vdash_{\mathrm{PCL1}} A \vee \neg A$ which shows that the strong negation in PCL1 is the classical negation.

For the other way around, just regard the strong negation which is the classical negation by the assumption as the classical negation with which PCL1 is equipped. ⊣

With the help of this result, we can prove the following:

Theorem 6. *The system* PCL1 *itself does not have the classical negation.*[8]

Proof. It is an immediate consequence of theorems 3 and 5. ⊣

This Theorem 6 shows that there is room to find out some conditions to enrich the system PCL1 with the classical negation and, for this purpose, Theorem 5 tells us that the strong negation is the key.

With all these observations in mind, we shall define the system PCL1C as follows:

[8]Note that da Costa's C_n does have the classical negation since the strong negation in C_n is the classical negation as we remarked after Theorem 3.

Definition 4. A system is called PCL1C if that system is the minimal extension of PCL1 such that the strong negation is the classical negation.

Remark. Note that the definition of PCL1C given here is different from the one given in [Waragai & Shidori, 2007]. Details will be discussed in another paper.

Now, on the basis of Theorem 5, the idea is to make the strong negation in PCL1 work as the classical negation by adding a certain formula which is either logically equivalent to the law of the excluded middle with respect to the strong negation or adequate to derive the law of the excluded middle with respect to the strong negation.

In this section, we shall give two ways of extending the system PCL1. The first one is to add Dummett's formula (i.e. $A \vee (A \supset B)$, abbreviated to DF hereafter) and the other is to add Peirce's law.

5.1. Extending PCL1 by Dummett's Formula

First of all, we shall find out the formula which is logically equivalent to the law of the excluded middle with respect to the strong negation.

$\mathrm{T}_{\mathrm{PCL1}}44$ $\vdash_{\mathrm{PCL1}} (A \vee \neg A) \equiv (A \vee A^{\mathrm{I}})$

Proof. 1. $(A \vee \neg A) \equiv (A \vee (\mathsf{N}\, A \wedge A^{\mathrm{I}}))$ $\mathrm{T}_{\mathrm{PCL1}}31$

2. $(A \vee (\mathsf{N}\, A \wedge A^{\mathrm{I}})) \equiv ((A \vee \mathsf{N}\, A) \wedge (A \vee A^{\mathrm{I}}))$ $\mathrm{T}_{\mathrm{PCL1}}7$

3. $((A \vee \mathsf{N}\, A) \wedge (A \vee A^{\mathrm{I}})) \equiv (A \vee A^{\mathrm{I}})$ $\mathrm{A}_{\mathrm{PCL1}}11$

4. $(A \vee \neg A) \equiv (A \vee A^{\mathrm{I}})$ 1, 2, 3 $\dashv$

With this result in mind, we reach the following theorem:

Theorem 7. *The system obtained from* PCL1 *by adding the formula* $A \vee A^{\mathrm{I}}$ *has the classical negation.*

The formula which we added here is $A \vee A^{\mathrm{I}}$ i.e. $A \vee (A \supset \mathsf{N}\,\mathsf{N}\, A)$ and this formula is a special case of Dummett's formula $A \vee (A \supset B)$. Therefore, we reach the following theorem:

Theorem 8. *The system obtained from* PCL1 *by adding the formula* $A \vee (A \supset B)$ *has the classical negation.*

Now, it is obvious that the system obtained in Theorem 7 is a subsystem of the one obtained in Theorem 8, but we can also prove the inverse of this fact. That is to say, we can prove the following theorem:

Theorem 9. *The systems obtained in theorems 7 and 8 are inferentially equivalent.*

This can be proved by using the following result of PCL1:

$\mathrm{T}_{\mathrm{PCL1}}45$ $\quad \vdash_{\mathrm{PCL1}} (A \vee A^{\mathrm{I}}) \supset (A \vee (A \supset B))$

Proof. 1.1. $A \vee A^{\mathrm{I}}$ sup.

1.2. $A \vee \neg A$ 1.1, $\mathrm{T}_{\mathrm{PCL1}}44$

1.3. $\neg A \supset (A \supset B)$ $\mathrm{T}_{\mathrm{PCL1}}22$

1.4. $A \vee (A \supset B)$ 1.2, 1.3

1. $A \vee A^{\mathrm{I}} \vdash_{\mathrm{PCL1}} A \vee (A \supset B)$ 1.1–1.4

2. $\vdash_{\mathrm{PCL1}} (A \vee A^{\mathrm{I}}) \supset (A \vee (A \supset B))$ 1, (DT) ⊣

Proof of Theorem 9. It is an immediate consequence of $\mathrm{T}_{\mathrm{PCL1}}45$. ⊣

On the basis of the result of Theorem 9, we shall hereafter refer to the system obtained in theorems 7 and 8 as PCL1DF.

It should also be noted that we have the following result:

$\mathrm{T}_{\mathrm{PCL1}}46$ $\quad \nvdash_{\mathrm{PCL1}} (A \vee (A \supset B)) \supset (A \vee A^{\mathrm{I}})$

Proof. Assign ½ and 1 to A and B respectively in Alves' matrix. ⊣

5.2. Extending PCL1 by Peirce's Law

This time, we shall begin with a definition.

Definition 5. Let PCL1PL be the system which can be obtained from PCL1 by adding the formula $(((A \supset f) \supset A) \supset A)$, which can be written as $((\neg A \supset A) \supset A)$ with the strong negation.

Then the following fact can be proved:

THEOREM 10. *The following metatheorem holds in* PCL1PL *if we are not making use of the rule* (RA15):

(RAS′) $\forall B\ (\Gamma, \neg A \vdash_{\text{PCL1PL}} B) \Rightarrow \Gamma \vdash_{\text{PCL1PL}} A$

that is, if we succeed in showing that $\neg A$ *causes logical explosion, then we can derive that* A *holds.*

PROOF. 1. $\forall B\ (\Gamma, \neg A \vdash_{\text{PCL1PL}} B)$ sup.

2. $\Gamma, \neg A \vdash_{\text{PCL1PL}} A$ 1 (B/A)

3. $\Gamma \vdash_{\text{PCL1PL}} \neg A \supset A$ 2, (DT)

4. $\Gamma \vdash_{\text{PCL1PL}} (\neg A \supset A) \supset A$ def. of PCL1PL

5. $\Gamma \vdash_{\text{PCL1PL}} A$ 3, 4 ⊣

Making use of this theorem, we can prove the law of the excluded middle with respect to the strong negation in PCL1PL as follows:

$\text{T}_{\text{PCL1}}47$ $\vdash_{\text{PCL1PL}} A \vee \neg A$

PROOF. 1.1. $\neg(A \vee \neg A)$ sup.

1.2. $\neg A \wedge \neg\neg A$ 1

1.3. B 2, $\text{T}_{\text{PCL1}}23$

1. $A \vee \neg A$ 1.1–1.3, (RAS′) ⊣

We therefore reach the following theorem:

THEOREM 11. PCL1DF *is a subsystem of* PCL1PL.

The inverse of this theorem, i.e. the following theorem holds:

THEOREM 12. *The systems* PCL1DF *and* PCL1PL *are inferentially equivalent.*

In order to prove this theorem, we need the following result of the system PCL1DF:

$\text{T}_{\text{PCL1}}48$ $\vdash_{\text{PCL1DF}} (\neg A \supset A) \supset A$

PROOF. This follows immediately since the strong negation in PCL1DF is the classcal negation. ⊣

PROOF OF THEOREM 12. It is a consequence of $\text{T}_{\text{PCL1}}48$. ⊣

It should also be noted that as $A \vee A^{\mathrm{I}}$ is a special form of Dummett's formula, $(((A \supset f) \supset A) \supset A)$ is a special form of Peirce's law. We shall here point out that the general form of Peirce's law can be proved in PCL1PL since we have the following result in PCL1:

$\mathrm{T_{PCL1}}49 \qquad \vdash_{\mathrm{PCL1}} ((\neg A \supset A) \supset A) \supset (((A \supset B) \supset A) \supset A)$

Proof. 1. $(\neg A \supset (A \supset B)) \supset (((A \supset B) \supset A) \supset (\neg A \supset A))$ $\qquad \mathrm{A_{PCL1}}1$

2. $\neg A \supset (A \supset B)$ $\qquad \mathrm{T_{PCL1}}22$, $\mathrm{T_{PCL1}}3$

3. $((A \supset B) \supset A) \supset (\neg A \supset A)$ $\qquad$ 1, 2, MP

4. $(((A \supset B) \supset A) \supset (\neg A \supset A)) \supset$
$(((\neg A \supset A) \supset A) \supset (((A \supset B) \supset A) \supset A))$ $\qquad \mathrm{A_{PCL1}}1$

5. $((\neg A \supset A) \supset A) \supset (((A \supset B) \supset A) \supset A)$ $\qquad$ 3 , 4, MP $\quad \dashv$

We therefore reach the following theorem:

Theorem 13. *The systems* PCL1PL *and* PCL1 *strengthened by the formula* $((A \supset B) \supset A) \supset A$ *are inferentially equivalent.*

It should be noted that we also have the following result:

$\mathrm{T_{PCL1}}50 \qquad \nvdash_{\mathrm{PCL1}} (((A \supset B) \supset A) \supset A) \supset ((\neg A \supset A) \supset A)$

Proof. Assign ½ and 1 to A and B respectively in Alves' matrix. $\dashv$

5.3. Summary of Extending PCL1

Now, as a matter of fact, the system PCL1DF is the system PCL1C since we have the following result:

Theorem 14. PCL1DF *is contained in any extended system of* PCL1 *which has the classical negation.*

Proof. What we have to prove is that DF can be proved in the extended system. Now, let us write the classical negation in the extended system as $\neg_{\mathrm{c}}$ as we did in the proof of Theorem 5. Then, we certainly have $A \vee \neg_{\mathrm{c}} A$ and $\neg_{\mathrm{c}} A \supset (A \supset B)$ as theses in the extended system so we can prove DF with the help of $\mathrm{T_{PCL1}}6$. $\dashv$

Therefore in order to extend PCL1 to PCL1C, we may add any of the following four formulas[9]:

- $A \vee (A \supset B)$
- $A \vee A^{\mathrm{I}}$
- $((A \supset B) \supset A) \supset A$
- $(\neg A \supset A) \supset A$

6. Justification of $\mathbf{A_{PCL1}10}$, $\mathbf{A_{PCL1}11}$ and A^{I}

Since we have the system PCL1C at hand by the results of the previous section, we are now in the position to give a justification of the notion of "behaving classically" using A^{I} and two axiom schemata $\mathrm{A_{PCL1}10}$, $\mathrm{A_{PCL1}11}$ of PCL1. The outline of our justification given here will be as follows:

- First, we shall take a system, called W, which includes classical propositional calculus as a subsystem and has an unary operator N *without* any syntactical constraints.
- We then give a semantical condition for N which seems to be appropriate and also natural under a certain situation.
- Finally, we will show that the syntactical condition which follows by the semantical condition corresponds to the formulas we took as axiom schemata.

We shall begin with the first step.

6.1. The First Step —System W—

We take the system of classical propositional calculus and enrich the system with an unary operator N by taking the smallest set W which is closed under the following two conditions:

(a) $A : \text{PC-wff} \Longrightarrow A \in \mathrm{W}$,

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[9] Since the strong negation in da Costa's system C_n is the classical negation, the results of this section trivially holds for C_n. The key fact is Lemma 2 of [Urbas, 1989], p. 593, which states that $A \vee A^{(\mathrm{n})}$ is derivable in C_ω, for $1 \leq n < \omega$. Obviously, $A \vee A^{(\mathrm{n})}$ in C_n corresponds to $A \vee A^{\mathrm{I}}$ in PCL1.

(b) $A, B \in \mathrm{W} \Longrightarrow \mathsf{N}\,A, \neg A, A \supset B, A \wedge B, A \vee B \in \mathrm{W}$.

It should be noted that here we are writing the classical negation as $\neg$.

6.2. The Second Step —Setting of the Justification—

We shall now make the semantical condition for the system W clear.

Definition 6. Let Λ be a set of formulas of W. Then,

- Λ is said to be *W-inconsistent with respect to* $\neg$ when there are some formulas $A_1, \ldots, A_n \in \Lambda$ such that

$$\vdash_{\mathrm{W}} \neg(A_1 \wedge \cdots \wedge A_n)$$

- Λ is said to be *W-consistent with respect to* $\neg$ when it is not W-inconsistent with respect to $\neg$.
- Λ is said to be *maximal with respect to* $\neg$ when for every formula A, either $A \in \Lambda$ or $\neg A \in \Lambda$.
- Λ is said to be *maximally consistent with respect to* $\neg$ when Λ is both W-consistent with respect to $\neg$ and maximal with respect to $\neg$.

With this definition in mind, take a set of all the theorems of classical propositional calculus which we shall denote as CPC. Then, apply Lindenbaum's Lemma to CPC in order to obtain maximal consistent sets. After that, collect all the maximal consistent sets, and denote this as $\{\Gamma_\lambda\}_{\lambda\in\Lambda}$. Finally, we assume an equivalence relation R over $\{\Gamma_\lambda\}_{\lambda\in\Lambda}$. Within this setting, we shall make two more definitions as follows:

D$_{\mathrm{W}}$1 $\qquad x \models_{\mathrm{W}} A \overset{\text{def}}{\Longleftrightarrow} A \in \Gamma_x$

D$_{\mathrm{W}}$2 $\qquad x \models_{\mathrm{W}} \mathsf{N}\,A \overset{\text{def}}{\Longleftrightarrow} \exists y\ (xRy\ \ \&\ \ y \not\models_{\mathrm{W}} A)$

Two remarks for the above definition D$_{\mathrm{W}}$2. The first point is that the semantical condition for $\mathsf{N}\,A$ has the same form as the one for $\Diamond\neg A$ where $\Diamond$ is the possibility operator in the modal logic S5. The other point is that the above definition D$_{\mathrm{W}}$2 actually appears in one of the Marcos' work in a more general way (see [Marcos, 2005b]). The point which we want to emphasise here is that we can also reach an interesting result if we consider not the general case but one of the special cases.

Now, before giving the justification we shall derive some results which will be used in the justification.

$\text{T}_{\text{W}}1$ $\quad x \models_{\text{W}} \neg A \iff x \not\models_{\text{W}} A$

PROOF. If $x \models_{\text{W}} \neg A$ holds, then $\neg A \in \Gamma_x$ by $\text{D}_{\text{W}}1$. So, by the consistency of Γ_x, we have $A \notin \Gamma_x$ which is equal to $x \not\models_{\text{W}} A$. For the other way around, we assume $x \not\models_{\text{W}} A$ which is $A \notin \Gamma_x$ by $\text{D}_{\text{W}}1$. Then this time, by the maximality of Γ_x, we have $\neg A \in \Gamma_x$ and this is $x \models_{\text{W}} \neg A$ by $\text{D}_{\text{W}}1$. ⊣

$\text{T}_{\text{W}}2$ $\quad x \models_{\text{W}} A \;\&\; x \models_{\text{W}} (A \supset B) \Longrightarrow x \models_{\text{W}} B$

PROOF. Suppose that $x \models_{\text{W}} A$ and $x \models_{\text{W}} (A \supset B)$ but not $x \models_{\text{W}} B$. Then we have $x \models_{\text{W}} \neg B$, so $\{A, A \supset B, \neg B\}$ as a subset of Γ_x. But this again makes Γ_x inconsistent since $\vdash_{\text{W}} \neg(A \wedge (A \supset B) \wedge \neg B)$ holds in W. ⊣

$\text{T}_{\text{W}}3$ $\quad (x \models_{\text{W}} A \Longrightarrow x \models_{\text{W}} B) \Longrightarrow x \models_{\text{W}} (A \supset B)$

PROOF. Suppose that $(x \models_{\text{W}} A \Longrightarrow x \models_{\text{W}} B)$ and not $x \models_{\text{W}} (A \supset B)$. Then we have $x \models_{\text{W}} \neg(A \supset B)$. Now we split the case by the validity of A. If $\models_{\text{W}} A$, then we have $\{B, \neg(A \supset B)\}$ as a subset of Γ_x, but this makes Γ_x inconsistent since we have $\vdash_{\text{W}} \neg(B \wedge \neg(A \supset B))$. Also if $\not\models_{\text{W}} A$, then we have $\{\neg A, \neg(A \supset B)\}$ as a subset of Γ_x, but this makes Γ_x inconsistent since we have $\vdash_{\text{W}} \neg(\neg A \wedge \neg(A \supset B))$. ⊣

$\text{T}_{\text{W}}4$ $\quad x \models_{\text{W}} \mathsf{N}\,\mathsf{N}\,A \Longrightarrow \forall z\,(xRz \;\Rightarrow\; z \models_{\text{W}} A)$

PROOF. We can prove this as follows:

$$
\begin{aligned}
x \models_{\text{W}} \mathsf{N}\,\mathsf{N}\,A &\iff \exists y (xRy \;\&\; y \not\models_{\text{W}} \mathsf{N}\,A) \quad (\text{D}_{\text{W}}2)\\
&\iff \exists y\,(xRy \;\&\; \text{not}(\exists z\,(yRz \;\&\; z \not\models_{\text{W}} A))) \quad (\text{D}_{\text{W}}2)\\
&\Longrightarrow xRy_0 \;\&\; \text{not}(\exists z\,(y_0Rz \;\&\; z \not\models_{\text{W}} A))\\
&\iff xRy_0 \;\&\; (\forall z\,(y_0Rz \;\Rightarrow\; z \models_{\text{W}} A))\\
&\iff \forall z\,(xRy_0 \;\&\; (y_0Rz \;\Rightarrow\; z \models_{\text{W}} A))\\
&\iff \forall z\,((xRy_0 \;\&\; \text{not } y_0Rz) \text{ or } (xRy_0 \;\&\; z \models_{\text{W}} A))\\
&\Longrightarrow \forall z\,((\text{not } xRz) \text{ or } (xRy_0 \;\&\; z \models_{\text{W}} A))\\
&\qquad\qquad (R : \text{symmetric, transitive})
\end{aligned}
$$

$$\Longrightarrow \forall z \; ((\text{not } xRz) \text{ or } z \models_{\mathrm{W}} A)$$
$$\Longleftrightarrow \forall z \; (xRz \;\Rightarrow\; z \models_{\mathrm{W}} A) \qquad \dashv$$

$\mathrm{T_W}5$ $\quad \forall z \; (xRz \;\Rightarrow\; z \models_{\mathrm{W}} A) \Longrightarrow x \models_{\mathrm{W}} \mathsf{N}\,\mathsf{N}\, A$

Proof. The proof runs as follows:

$$\begin{aligned} \forall z \; (xRz \;\Rightarrow\; z \models_{\mathrm{W}} A) &\Longleftrightarrow xRx \;\;\&\;\; \forall z \; (xRz \;\Rightarrow\; z \models_{\mathrm{W}} A) && R : \text{reflexive} \\ &\Longrightarrow \exists y \; (xRy \;\;\&\;\; \forall z \; (yRz \;\Rightarrow\; z \models_{\mathrm{W}} A)) \\ &\Longleftrightarrow \exists y \; (xRy \;\;\&\;\; y \not\models_{\mathrm{W}} \mathsf{N}\, A)) && (\mathrm{D_W}2) \\ &\Longleftrightarrow x \models_{\mathrm{W}} \mathsf{N}\,\mathsf{N}\, A && (\mathrm{D_W}2) \quad \dashv \end{aligned}$$

$\mathrm{T_W}6$ $\quad x \models_{\mathrm{W}} \mathsf{N}\,\mathsf{N}\, A \Longleftrightarrow \forall z \; (xRz \;\Rightarrow\; z \models_{\mathrm{W}} A)$

Proof. This is an immediate consequence of $\mathrm{T_W}4$ and $\mathrm{T_W}5$. $\dashv$

This $\mathrm{T_W}6$ plays an important role in deriving the syntactical condition for N in the following subsection.

6.3. The Third Step —Syntactical Conditions We Reach—

In this final step, we shall see that under the semantical conditions we posed in the previous step, we are forced to accept the formulas we stated as an axiom schemata and also that it is quite natural to read the formula A^{I} as "behaving classically".

We shall begin with the case of $\mathrm{A_{PCL1}}10$, i.e. the formula $\mathsf{N}\,\mathsf{N}\, A \supset A$.

$$\begin{aligned} x \models_{\mathrm{W}} \mathsf{N}\,\mathsf{N}\, A &\Longleftrightarrow \forall z \; (xRz \;\Rightarrow\; z \models_{\mathrm{W}} A) && \mathrm{T_W}6 \\ &\Longrightarrow xRx \;\Rightarrow\; x \models_{\mathrm{W}} A \\ &\Longleftrightarrow x \models_{\mathrm{W}} A && R : \text{reflexive} \end{aligned}$$

Therefore, with the help of $\mathrm{T_W}3$ we have $x \models_{\mathrm{W}} (\mathsf{N}\,\mathsf{N}\, A \supset A)$ for all the "worlds" x which forces us to accept $\mathsf{N}\,\mathsf{N}\, A \supset A$ as an axiom scheme.

For the case of $\mathrm{A_{PCL1}}11$, i.e. the formula $A \vee \mathsf{N}\, A$ runs as follows:

$$x \models_{\mathrm{W}} \neg A \Longleftrightarrow x \not\models_{\mathrm{W}} A \qquad \mathrm{T_W}1$$

$$\begin{aligned} &\iff xRx \;\&\; x \not\models_{\mathrm{W}} A && R : \text{reflexive} \\ &\Longrightarrow \exists y\, (xRy \;\&\; y \not\models_{\mathrm{W}} A) && \\ &\iff x \models_{\mathrm{W}} \mathsf{N}\, A && \mathrm{D_W}2 \end{aligned}$$

So, again with the help of $\mathrm{T_W}3$ we have $x \models_{\mathrm{W}} (\neg A \supset \mathsf{N}\, A)$ and thus $x \models_{\mathrm{W}} (A \vee \mathsf{N}\, A)$ for all the "worlds" x. This means that we are again forced to accept $A \vee \mathsf{N}\, A$ as an axiom scheme.

Finally, the case of A^{I} runs as follows:

$$\begin{aligned} \vdash_{\mathrm{CPC}} A &\Longrightarrow \forall y\, (y \models_{\mathrm{W}} A) && \text{definition of } \{\Gamma_\lambda\}_{\lambda\in\Lambda} \\ &\iff \forall y\, (xRy \Rightarrow y \models_{\mathrm{W}} A) && R : \text{reflexive} \\ &\iff x \models_{\mathrm{W}} \mathsf{N}\,\mathsf{N}\, A && \mathrm{T_W}6 \\ &\Longrightarrow x \models_{\mathrm{W}} (A \supset \mathsf{N}\,\mathsf{N}\, A) && \mathrm{T_W}2 \end{aligned}$$

This shows that if A is a thesis of classical propositional calculus, then we have $x \models_{\mathrm{W}} (A \supset \mathsf{N}\,\mathsf{N}\, A)$ for all "worlds" x. In other words, it is necessary for $(A \supset \mathsf{N}\,\mathsf{N}\, A)$ to hold in every "world" if A is a thesis of classical propositional calculus. This seems to be giving a reason for us to read the formula A^{I} as "behaving classically".

7. On the Concrete Notions of "behaving classically"

In this final section we shall first compare the notions of "behaving classically" given by da Costa and by Waragai and Shidori. After that, we shall discuss a characterisation of the notion of "behaving classically" given by Waragai and Shidori.

7.1. On da Costa's Notion of "behaving classically"

We shall here examine the notion of "behaving classically" given by da Costa.

Da Costa's notion of "behaving classically" in C_1 is defined as $\mathsf{N}(A \wedge \mathsf{N}\, A)$, but for this notion we have the following theses $\mathrm{T_{C_1}}1$ and $\mathrm{T_{C_1}}2$ which show that the formulas we reach by switching the places of A and $\mathsf{N}\, A$ are not logically equivalent:

$\mathrm{T_{C_1}}1$	$\vdash_{\mathrm{C_1}} \mathsf{N}(A \wedge \mathsf{N}\, A) \supset \mathsf{N}(\mathsf{N}\, A \wedge A)$
$\mathrm{T_{C_1}}2$	$\nvdash_{\mathrm{C_1}} \mathsf{N}(\mathsf{N}\, A \wedge A) \supset \mathsf{N}(A \wedge \mathsf{N}\, A)$

The latter result $T_{C_1}2$ was first shown in [Urbas, 1989], and Marcos says that there seems to be no particular reason to accept this result (see [Marcos, 2005c]). Now we shall see what is the case for the same formulas in PCL1.

$T_{PCL1}51$ $\nvdash_{PCL1} \mathsf{N}(A \wedge \mathsf{N}A) \supset \mathsf{N}(\mathsf{N}A \wedge A)$

Proof. Use the matrix given below, and assign the value ½ to A. Then the formula we are considering takes the value 0 which gives the desired result.

∧	1	½	0
1	1	1	0
½	½	½	0
0	0	0	0

∨	1	½	0
1	1	1	1
½	1	½	½
0	1	½	0

⊃	1	½	0
1	1	½	0
½	1	½	0
0	1	1	1

	N
1	0
½	1
0	1

Note that 1 and ½ are the distinguished values. ⊣

$T_{PCL1}52$ $\nvdash_{PCL1} \mathsf{N}(\mathsf{N}A \wedge A) \supset \mathsf{N}(A \wedge \mathsf{N}A)$

Proof. Use the matrix given below, and assign the value ½ to A. Then the formula we are considering takes the value 0 which gives the desired result.

∧	1	½	0
1	1	½	0
½	1	½	0
0	0	0	0

∨	1	½	0
1	1	1	1
½	1	½	½
0	1	½	0

⊃	1	½	0
1	1	½	0
½	1	½	0
0	1	1	1

	N
1	0
½	1
0	1

Note that 1 and ½ are the distinguished values. ⊣

Remark. The three-valued matrix we made use of here also enables us to give a simple proof of $T_{C_1}2$ compared with the one given in Theorem 4 of [Urbas, 1989].

Now above two results show that the case is actually worse in PCL1. But here, it should be noted that in PCL1, these two formulas do *not* play the same role as it does in C_1. Therefore, these results are not so serious as it was in C_1.

However, da Costa's approach, which tries to internalise the notion regulating the explosion of the system by making use of a certain form of the law of non-contradiction, seems to be quite a natural one. This is because there are relations between the law of non-contradiction and paraconsistency. We won't of course be able to say that the system is always explosive with respect to A and N, though if we were sure, to some extent, that it is not the case that both A and $\mathsf{N}\,A$ hold, then it must be reasonable to require the system to be explosive with respect to A and N in such a case. And actually, as for this point we have the following results in PCL1:

$\mathrm{T_{PCL1}}53$	$\vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \supset \neg(A \wedge \mathsf{N}\,A)$	$\mathrm{T_{PCL1}}12$, (RAS)
$\mathrm{T_{PCL1}}54$	$\vdash_{\mathrm{PCL1}} \neg(A \wedge \mathsf{N}\,A) \supset A^{\mathrm{I}}$	$\mathrm{T_{PCL1}}23$, (RA), (DT), D1
$\mathrm{T_{PCL1}}55$	$\vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \equiv \neg(A \wedge \mathsf{N}\,A)$	$\mathrm{T_{PCL1}}53$, $\mathrm{T_{PCL1}}54$
$\mathrm{T_{PCL1}}56$	$\vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \equiv \neg(\mathsf{N}\,A \wedge A)$	$\mathrm{T_{PCL1}}55$

$\mathrm{T_{PCL1}}55$ says that the notion of "behaving classically" in PCL1 actually has a relation with a certain type of the law of non-contradiction differing from da Costa's notion only in the outer negation. In the formula $\neg(A \wedge \mathsf{N}\,A)$, we won't be troubled by switching the place of A and $\mathsf{N}\,A$, as is shown in $\mathrm{T_{PCL1}}56$, since the strong negation has at least the power of intuitionistic negation.

With this result in PCL1 at hand, we shall come back to see and analyse the notion given by da Costa in C_1.

$\mathrm{T_{C_1}}3$	$\vdash_{\mathrm{C_1}} \mathsf{N}(A \wedge \mathsf{N}\,A) \equiv \neg(A \wedge \mathsf{N}\,A)$
$\mathrm{T_{C_1}}4$	$\nvdash_{\mathrm{C_1}} \mathsf{N}(\mathsf{N}\,A \wedge A) \equiv \neg(A \wedge \mathsf{N}\,A)$
$\mathrm{T_{C_1}}5$	$\nvdash_{\mathrm{C_1}} \mathsf{N}(\mathsf{N}\,A \wedge A) \equiv \neg(\mathsf{N}\,A \wedge A)$

As $\mathrm{T_{C_1}}3$ shows, the same result as $\mathrm{T_{PCL1}}55$ holds in C_1 but with the switched version, as is proved in $\mathrm{T_{C_1}}4$ and $\mathrm{T_{C_1}}5$, the outer N cannot be replaced with the strong negation, which is the classical negation in C_1. So, these results seem to be the root of the undesired result for da Costa's notion.

7.2. A Characterisation of A^{I} in PCL1C

In this subsection, we shall go on a little further to see the notion of "behaving classically" in PCL1 and PCL1C and try to give a characterisation of A^{I} in PCL1C. We shall start by examining A^{I} in PCL1.

$\mathrm{T_{PCL1}}57$	$\vdash_{\mathrm{PCL1}} (A \wedge \mathsf{N}\,A) \supset \neg(A^{\mathrm{I}})$	$\mathrm{T_{PCL1}}13$, (RAS)
$\mathrm{T_{PCL1}}58$	$\nvdash_{\mathrm{PCL1}} \neg(A^{\mathrm{I}}) \supset (A \wedge \mathsf{N}\,A)$	

Proof. Assign ½ to A in Alves' matrix which we have already mentioned. ⊣

From these two results $\mathrm{T_{PCL1}}57$ and $\mathrm{T_{PCL1}}58$, we can regard $\neg(A^{\mathrm{I}})$ as expressing "inconsistency" not being "contradictory" in the sense given by Carnielli, Coniglio and Marcos in their Logics of Formal Inconsistency. However, what seems to be more interesting to us is the case of PCL1C as we shall see in the following.

$\mathrm{T_{PCL1}}59$ $\quad \vdash_{\mathrm{PCL1C}} \neg(A^{\mathrm{I}}) \supset (A \wedge \mathsf{N}\, A)$ [$\mathrm{T_{PCL1}}34$, $\mathrm{A_{PCL1}}10$]

This $\mathrm{T_{PCL1}}59$ is saying that the inverse direction of $\mathrm{T_{PCL1}}57$, which was falsified in PCL1, is a thesis of PCL1C. So, combining this with $\mathrm{T_{PCL1}}57$, we have $\mathrm{T_{PCL1}}60$ below which shows that the strong negation of A^{I} is exactly equivalent to $A \wedge \mathsf{N}\, A$.

$\mathrm{T_{PCL1}}60$ $\quad \vdash_{\mathrm{PCL1C}} \neg(A^{\mathrm{I}}) \equiv (A \wedge \mathsf{N}\, A)$ $\mathrm{T_{PCL1}}57$, $\mathrm{T_{PCL1}}59$

With this result in mind, we propose a way of reading A^{I} in PCL1C. For this purpose, we make some definitions first.

Definition 7. Let L be a logic and Γ be a theory of L. Then,

- A is said to be *pre-normal in* Γ *with respect to* $\neg$ when both $\Gamma \nvdash_L A$ and $\Gamma \nvdash_L \neg A$ hold.
- A is said to be *normal in* Γ *with respect to* $\neg$ when only one of $\Gamma \nvdash_L A$ or $\Gamma \nvdash_L \neg A$ holds.
- A is said to be *non-normal in* Γ *with respect to* $\neg$ when both $\Gamma \vdash_L A$ and $\Gamma \vdash_L \neg A$ hold.

Since we have the following results in PCL1C,

$\mathrm{T_{PCL1}}55$ $\quad \vdash_{\mathrm{PCL1}} A^{\mathrm{I}} \equiv \neg(A \wedge \mathsf{N}\, A)$

$\mathrm{T_{PCL1}}60$ $\quad \vdash_{\mathrm{PCL1C}} \neg(A^{\mathrm{I}}) \equiv (A \wedge \mathsf{N}\, A)$

we can give a way of reading of A^{I} in PCL1C as follows:

Theorem 15. *Let* Γ *be a non-trivial theory of* PCL1C. *Then,*

- *A is pre-normal or normal in Γ with respect to* N, *if* $\Gamma \vdash_{\mathrm{PCL1C}} A^{\mathrm{I}}$.
- *A is non-normal in Γ with respect to* N *iff* $\Gamma \vdash_{\mathrm{PCL1C}} \neg(A^{\mathrm{I}})$.

Proof. For the former, the proof runs as follows:

$$\begin{aligned}\Gamma \vdash_{\mathrm{PCL1C}} A^{\mathrm{I}} &\Longleftrightarrow \Gamma \vdash_{\mathrm{PCL1C}} \neg(A \wedge \mathsf{N}\, A) \quad (\mathrm{T}_{\mathrm{PCL1}}55)\\ &\Longrightarrow \Gamma \nvdash_{\mathrm{PCL1C}} (A \wedge \mathsf{N}\, A)\\ &\Longleftrightarrow \Gamma \nvdash_{\mathrm{PCL1C}} A \ \text{ or } \ \Gamma \nvdash_{\mathrm{PCL1C}} \mathsf{N}\, A\\ &\Longleftrightarrow A \text{ is pre-normal or normal in } \Gamma \text{ with respect to } \mathsf{N}\end{aligned}$$

And for the latter, we can prove it as follows:

$$\begin{aligned}\Gamma \vdash_{\mathrm{PCL1C}} \neg(A^{\mathrm{I}}) &\Longleftrightarrow \Gamma \vdash_{\mathrm{PCL1C}} A \wedge \mathsf{N}\, A \quad (\mathrm{T}_{\mathrm{PCL1}}60)\\ &\Longleftrightarrow \Gamma \vdash_{\mathrm{PCL1C}} A \ \ \& \ \ \Gamma \vdash_{\mathrm{PCL1C}} \mathsf{N}\, A\\ &\Longleftrightarrow A \text{ is non-normal in } \Gamma \text{ with respect to } \mathsf{N}\end{aligned}$$
⊣

As a matter of fact, we can actually prove a refined result of Theorem 15 by adding some conditions for the non-trivial theory Γ. In order to acheive this, we make use of the following two lemmas:

Lemma 1. *Let* Γ *be a maximal consistent theory of* PCL1C. *Then the following holds*:

$$\Gamma \nvdash_{\mathrm{PCL1C}} A \Longleftrightarrow \Gamma \vdash_{\mathrm{PCL1C}} \neg A$$

Proof. From the left to the right follows by the maximality of Γ, and the other way around follows by the consistency of Γ. ⊣

Lemma 2. *Let* Γ *be a maximal consistent theory of* PCL1C. *Then, it is not the case that both* $\Gamma \nvdash_{PCL1C} A$ *and* $\Gamma \nvdash_{PCL1C} \mathsf{N}\, A$ *hold, i.e. either* $\Gamma \vdash_{PCL1C} A$ *or* $\Gamma \vdash_{PCL1C} \mathsf{N}\, A$ *holds.*

Proof. 1. $\Gamma \nvdash_{\mathrm{PCL1C}} A$ sup.

2. $\Gamma \nvdash_{\mathrm{PCL1C}} \mathsf{N}\, A$ sup.

3. $\Gamma \vdash_{\mathrm{PCL1C}} \neg A$ 1, Lemma 1

4. $\Gamma \vdash_{\mathrm{PCL1C}} A \vee \mathsf{N}\, A$ $\mathrm{A}_{\mathrm{PCL1}}11$

5. $\Gamma \vdash_{\mathrm{PCL1C}} \mathsf{N}\, A$ 3, 4

6. Contradiction 2, 5 ⊣

With the help of above two lemmas, we can see that A^{I} in a maximally consistent theory of PCL1C can be characterised as follows:

THEOREM 16. *Let* Γ *be a maximally consistent theory of* PCL1C. *Then*

- A *is normal in* Γ *with respect to* N *iff* $\Gamma \vdash_{PCL1C} A^{\mathrm{I}}$.
- A *is non-normal in* Γ *with respect to* N *iff* $\Gamma \vdash_{PCL1C} \neg(A^{\mathrm{I}})$.

PROOF. For the former, the proof runs as follows:

$$
\begin{aligned}
\Gamma \vdash_{\mathrm{PCL1C}} A^{\mathrm{I}} &\Longleftrightarrow \Gamma \vdash_{\mathrm{PCL1C}} \neg(A \wedge \mathsf{N}\, A) \quad (\mathrm{T_{PCL1}}55)\\
&\Longleftrightarrow \Gamma \nvdash_{\mathrm{PCL1C}} (A \wedge \mathsf{N}\, A) \quad (\text{Lemma 1})\\
&\Longleftrightarrow \Gamma \nvdash_{\mathrm{PCL1C}} A \;\text{ or }\; \Gamma \nvdash_{\mathrm{PCL1C}} \mathsf{N}\, A\\
&\Longleftrightarrow \Gamma \nvdash_{\mathrm{PCL1C}} A \;\text{ or }\; \Gamma \nvdash_{\mathrm{PCL1C}} \mathsf{N}\, A) \;\&\\
&\qquad\quad (\Gamma \vdash_{\mathrm{PCL1C}} A \;\text{ or }\; \Gamma \vdash_{\mathrm{PCL1C}} \mathsf{N}\, A) \quad (\text{Lemma 2})\\
&\Longleftrightarrow (\Gamma \nvdash_{\mathrm{PCL1C}} A \;\&\; \Gamma \vdash_{\mathrm{PCL1C}} \mathsf{N}\, A) \;\text{ or}\\
&\qquad\quad (\Gamma \vdash_{\mathrm{PCL1C}} A \;\&\; \Gamma \nvdash_{\mathrm{PCL1C}} \mathsf{N}\, A)\\
&\Longleftrightarrow A \text{ is normal in } \Gamma \text{ with respect to } \mathsf{N}
\end{aligned}
$$

As for the latter, the proof is already given in Theorem 15. ⊣

Note that theorems 15 and 16 still hold even if we replace PCL1C, A^{I} with C_1, A° respectively.

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