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PARANORMAL MODAL LOGIC – PART I
The System $K_?$ and the Foundations of the Logic of Skeptical and Credulous Plausibility*

Abstract. In this two-parts paper we present paranormal modal logic: a modal logic which is both paraconsistent and paracomplete. Besides using a general framework in which a wide range of logics — including normal modal logics, paranormal modal logics and classical logic — can be defined and proving some key theorems about paranormal modal logic (including that it is inferentially equivalent to classical normal modal logic), we also provide a philosophical justification for the view that paranormal modal logic is a formalization of the notions of skeptical and credulous plausibility.

Keywords: paraconsistent logic, paracomplete logic, modal logic, inductive plausibility.

1. Introduction

The practical side of the problem of combining logics [6] is recognizably one of its most appealing aspects. Take knowledge representation for example. An agent able to interact with its external environment has to represent not only its beliefs about the external world and its internal states but also how these beliefs change during time. Supposing that the agent representation mechanism is a logic-based one, we have then to face the problem of combining doxastic logic with temporal logic. If besides this the agent is equipped with information about obligations and

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permissions, we will have the further task of adding a deontic component to the combined system.

Paraconsistent logics are logics able to formalize inconsistent but non-trivial theories [11]. They have been advertised as important tools in applications such as knowledge representation, multi-agent systems and database management. In the situation above described, for example, the agent might have evidences both to believe and not to believe something, or its normative component might both require and prohibit something, in cases of which a paraconsistent mechanism might be required. Therefore it seems a quite natural thing to combine paraconsistent logic with logics for representing modalities; this has been done in the context of relevant logics (which is one type of paraconsistent logic) [13, 26, 36], da Costa's systems [10, 23] and "truth-value gluts" paraconsistent logics [15].

Something remarkable about modal logic and paraconsistent logic, and consequently about the enterprise of combining them, is the alleged relation that exists between the two classes of systems [2, 3, 4, 25]. Take the following definition of paraconsistency: a paraconsistent negation is a unary operator that does not satisfy the principle of explosion (for any formulas α and β, $\{\alpha, \neg\alpha\} \vdash \beta$) and has enough properties to be called a negation; a paraconsistent logic then is a logic having a paraconsistent negation [4]. Now if we define '$\sim$' as '$\Diamond\neg$', we will have that S5 (meant as a consequence relation), for instance, is a paraconsistent logic, for it contains a paraconsistent negation as defined above [4]. This of course is due to the logical properties of '$\Diamond$' when used along with S5's negation '$\neg$' or, more specifically, to the fact that $\ulcorner \Diamond\neg\alpha \urcorner$ can get along with α, and with $\ulcorner \Diamond\alpha \urcorner$, without trivializing the theory. This very fact that for S5 it is not the case, for any α and β, that $\{\Diamond\neg\alpha, \Diamond\alpha\} \vdash \beta$, which we know is the basis of Jaśkowski's calculus for contradictory deductive systems [20], has led some to speak of a subtler sort of paraconsistency named "hertian" [5] and conceptual [40] paraconsistency. Of course we could speak of a true or formal paraconsistency (regarding the primitive symbol '$\neg$' in connection with '$\Diamond$') if we had, for every α and some β, something like $\{\neg\Diamond\alpha, \Diamond\alpha\} \nvdash \beta$; this would certainly make the claim that modal logic is paraconsistent even stronger.

In [32] a paraconsistent modal logic called LEI (*Logic of Epistemic Inconsistency*) was proposed to serve as the monotonic base for a version of Reiter's default logic [34]. That gave rise to a whole family of paraconsistent modal systems and a quite significant contribution to the field of non-classical logic [1, 7, 8, 27, 28, 29]. The original idea was to mark formulas obtained through the use of defaults with a modal operator — the symbol '?', which was used in a post-fixed notation — so that they might be treated paraconsistently. 'α?' was read as "α is plausible". Syntactically the paraconsistency was obtained by a slight modification on the *reductio ad absurdum* axiom

$$(\alpha \to B) \to ((\alpha \to \neg B) \to \neg\alpha)$$

with a ?-free formula B. Thus, formulas of the form such as $\ulcorner\beta?\urcorner$ and $\ulcorner\neg(\beta?)\urcorner$ are not able to trivialize theories.

Moreover, since the negative ?-marked formulas introduced through default rules are of the form $\ulcorner(\neg\alpha)?\urcorner$, in order for the paraconsistency to be really required, there had to be some way to transform $\ulcorner(\neg\alpha)?\urcorner$ into $\ulcorner\neg(\alpha?)\urcorner$, as for instance, setting $(\neg\alpha)? \to \neg(\alpha?)$ as an axiom. Since its converse side $\neg(\alpha?) \to (\neg\alpha)?$ seems uncontroversial enough, we have then the following axiom:

`K2`: $\quad (\neg\alpha)? \leftrightarrow \neg(\alpha?)$

where the negaton '$\neg$' is not classical (see Section 4.2).

Now, there were some interesting things about the axiom `K2`. First, it allows us to go from what we have called conceptual paraconsistency to a formal one. From a semantic point of view, '?' corresponds to the '$\Diamond$' operator of traditional modal logic: $\ulcorner\alpha?\urcorner$ is true iff α is true in at least one member of a set of worlds, which in this case might be called *plausible worlds*. Therefore it shares with '$\Diamond$' the property of tolerating contradictions of the form $\{\alpha?, (\neg\alpha)?\}$, or in other words, there is a model which satisfies both $\ulcorner\alpha?\urcorner$ and $\ulcorner(\neg\alpha)?\urcorner$. But since from $\ulcorner(\neg\alpha)?\urcorner$ we get $\ulcorner\neg(\alpha?)\urcorner$, we have that this model also satisfies both $\ulcorner\alpha?\urcorner$ and $\ulcorner\neg(\alpha?)\urcorner$, which is the same as saying that its paraconsistency also applies to "true contradictions" of the form $\{\alpha?, \neg(\alpha?)\}$. In other words, with respect to '?' the negation symbol '$\neg$' does not behave in the standard, classical way, as it does with respect to '$\Diamond$'.

Second, the existence of a $\Diamond$-like operator suggests a $\Box$-like operator with an axiom corresponding to `K2`. Letting this operator be represented by '!' (also used in a post-fixed notation), we would have the following formula as axiom:

`K3`: $\quad (\neg\alpha)! \leftrightarrow \neg(\alpha!)$

Notice that, since '!' is to be interpreted like '$\Box$': $\ulcorner\alpha!\urcorner$ is true iff α is true in all plausible worlds. So we have that '!' can be said to represent a sort of *strong* or *skeptical plausibility*, with '?' representing a *weak* or *credulous plausibility*. Moreover, there shall be a semantic model in which neither $\ulcorner(\neg\alpha)!\urcorner$ nor $\ulcorner\alpha!\urcorner$ are true. But since by K3, $\ulcorner\neg(\alpha!)\urcorner$ implies $\ulcorner(\neg\alpha)!\urcorner$, this model shall not satisfy $\ulcorner\neg(\alpha!)\urcorner$ either. Therefore we have a model in which neither $\ulcorner\alpha!\urcorner$ nor $\ulcorner\neg(\alpha!)\urcorner$ is satisfied, which entails that in this logic the excluded middle principle is not universally valid. With respect to the modal operator '!' then, '$\neg$' does not behave in the standard, classical way, as it does with respect to '$\Box$'.

This is interesting because the property a logic might have of not satisfying the principle of excluded middle, which has been named by some *paracompleteness* [22], is usually taken as the dual of *paraconsistency* [5]. Logics which are both paraconsistent and paracomplete have been called *paranormal* or *non-alethic logics*.[1] Therefore, if we had such a '!' operator, we would have a paranormal modal logic in which the modal operators '!' and '?' when taken along with '$\neg$' would exhibit, respectively, the dual properties of paracompleteness and paraconsistency. This logic, we believe, would be a very interesting example of combination of modal logic with paraconsistency and paracompleteness, and one which could shed some light on the relation between paraconsistency, paracompleteness and modality.

Now, despite the fact that all resources, we may say, for this paranormal modal logic were present in [32] and subsequent works, some things prevented it from arising. First, despite clearly using resources of modal logic, most systems of the LEI family were commonly presented in a non-standard way, so that the fact that they were a sort of modal logic, or more generally, a combination of modal logic with something else was not clear enough. Second, very few was said about the formal relations between LEI systems and traditional modal logic. Thirdly, no $\Box$-like operator in the way described above was proposed in the published versions of this logic. And finally, no satisfactory philosophical justification for the left-to-right side of K2, i.e. $\ulcorner(\neg\alpha)? \to \neg(\alpha?)\urcorner$, was given. This last point is important for if we take '?' as representing the notion of *plausibility* (or *weak plausibility*), we have to give very convincing arguments that K2 is an important feature of this notion, indeed one of the features which would distinguish it from the notion of *possibility*.

[1] This "para" notation is due to Miró Quesada [5].

In [40] a first attempt was made to fill in these gaps and present what has been called *paranormal modal logic*.[2] In the present paper we extend this attempt and present paranormal modal logic inside a general framework in which a wide range of logics, including classical logic and traditional (normal) modal logics, can be defined. By proceeding in this way we make it easier to achieve a couple of goals. First of all, we are able to present this new version of LEI in such a way as to make it explicit both its combining aspect as a paraconsistent (and paracomplete) modal logic and those formal features which make it, as a modal logic, depart from traditional modal logics. Second, we can easily do, inside this framework, a more precise comparative analysis with other logics, in especial with traditional modal logics. Despite contrary appearances, we show that paranormal modal logics are both from a representational as well as from an inferential point of view equivalent to normal modal logics. We can also in this framework present in a quite natural fashion the several members of the paranormal modal logic family. In the same way that the system K can be extended into D, T, B, S4, S5, etc., the most basic paranormal modal logic $K_?$ can be extended into corresponding paranormal modal systems. Besides, we also present first order paranormal modal logic and a multimodal logic which combines paranormal and normal modalities. Finally, by providing such a general framework we also intend to offer what we think to be an interesting way to introduce modal logic, so the textbook flavor of good part of the text.

In addition to that, we also provide a philosophical justification for axiom `K2`. In a nutshell, we claim that there are two equally authentic ways to deal with the problem of inductive ambiguities; and that a crucial logical aspect of the two plausibility concepts that arise from these two approaches to induction — not by chance our notions of skeptical and credulous plausibility — is exactly the one captured by axioms `K2` and `K3`. This is done in the next section. In Section 3 we lay down the definitions of our general framework. In Section 4 we make use of such definitions to introduce the most basic of paranormal modal logics: the system $K_?$. Then, in the last section, we lay down some conclusive remarks for this first part of the paper.

[2] So far as we know, the only work which resembles what we calling paranormal modal logic is [31], which gets to paranormal analogues to K by taking intuitionistic modal versions of K as starting points.

In the second part [37] of this paper we prove some theorems about $K_?$, including the soundness and completeness theorems, and show the formal relations that exist between $K_?$ and normal modal logic K as well as between $K_?$ and classical logic. We also present there some other propositional paranormal modal logics, first-order paranormal modal logics, and a multimodal paranormal modal logic.

2. Foundations of the logic of skeptical and credulous plausibility

The problem of inductive inconsistencies, studied in both artificial intelligence and philosophy of science [14, 16, 17, 19, 33], strongly suggests the existence of at least two different approaches to the formal analysis of induction: what we shall call the *skeptical* and the *credulous* approaches to induction. They can be explained as follows. Let Δ be a consistent set of statements. Supposing the existence of some inductive mechanism of inference I, which might simply be a set of inductive inference rules, to be applied to the members of Δ, we name the deductive closure of each consistent set of conclusions obtained from Δ an *inductive extension*. A trivial consequence of this definition of extension is that the cases where contradictions are obtained from the application of I to Δ, and this is something we do expect to happen, lead to more than one inductive extension. In these cases we have at least two options at our disposal: to ignore contradictions and recognize as sound only those inductive conclusions belonging to the intersection of all extensions, or to take contradictions seriously and accept as authentic inductive conclusions all formulas belonging to the union of all extensions. While the first option is a strict or skeptical approach which requires a great deal to accept an inductive conclusion as sound, the second is a tolerant or credulous approach which requires just the minimum to accept a formula as an authentic inductive conclusion.

This distinction between a skeptical approach and a credulous one is of course not new [24, 30, 32]. It has been used in the nonmonotonic literature, for instance, to classify some of the available formalisms to common sense reasoning [24]. What is new however is its being used to name two general approaches to inductive reasoning, representing we might say the result of a conceptual analysis to the notion of induction which takes seriously the phenomenon of inductive ambiguities [39]. As far as the philosophical literature is concerned, even though the existence of these two approaches to induction has not been explicitly acknowledged, it is possible to identify isolated uses of them in several discussions related to the problem of inductive ambiguities. It can be shown for instance how some of the main solutions given to the lottery paradox [21] can be seen as instances either of a skeptical approach or of a credulous one, and that when we recognize these approaches as complementary instead of competing, the whole controversy regarding the proper solution to the lottery paradox is dissolved [39].

Now, supposing we can effectively infer new conclusions from our inductive inference rules, it is natural that we qualify such conclusions in order to distinguish them from non-inductive ones. The common attitude in philosophy has been to use some probability notion to do this job. In order to distinguish such sort of probability from other probability notions, in special from his notion of logical probability, Carnap uses the term "pragmatical probability" to refer to this "detached" probability [9]. We shall use here the less controversial term "plausibility" (or "inductive plausibility"), so that what we have called so far inductive conclusions are the same as plausible conclusions, plausible statements or plausible hypotheses.

See however that according to our proposal we cannot speak of inductive conclusions *per se*. Instead, we must speak of inductive or plausible conclusions *according* to this or that approach: when α is true in all inductive extensions we say that α is plausible according to a skeptical approach, and when α is true in at least one extension we say that α is plausible according to a credulous approach. Trivially then, the skeptical and credulous approaches work as evaluation functions which assess in different ways the truthfulness of plausible statements, giving rise in fact to two plausibility notions: what we might call *skeptical plausibility* and *credulous plausibility*.

From a general point of view, we can say that the credulous and skeptical approaches represent, respectively, minimizing and maximizing strategies of truth assessing. If one adopts a credulous position, for example, he will be tolerant, not requiring too much to accept statement α as plausible. If we use 1 to represent truth and 0 to represent falsehood, this can be restated by saying that he will somehow try to *maximize* or bring close to 1 the truth-value of plausible statements. On the other hand, if one adopts a skeptical position he will be more strict in the matter of accepting α as plausible, which means that he will try to *minimize* or bring close to 0 the truth-value of plausible statements.

What has been said so far can be quite fairly represented with the aid of a Kripkean semantic framework. First of all, each inductive extension might be naturally associated with a possible world, in this case a special kind of world named by us *plausible world*. Second, following the notation introduced in the previous section and representing our notions of skeptical and credulous plausibility with the help of the modal operators '!' and '?' (if α is a formula, then $\ulcorner\alpha!\urcorner$ and $\ulcorner\alpha?\urcorner$ are also formulas) so that $\ulcorner\alpha!\urcorner$ means "α is plausible according to a skeptical approach" and $\ulcorner\alpha?\urcorner$ "α is plausible according to a credulous approach", we have that while '!' is interpreted alike to the operator '$\Box$', '?' is interpreted alike to '$\Diamond$'. In other words, $\ulcorner\alpha!\urcorner$ is true iff α is true in all plausible worlds, and $\ulcorner\alpha?\urcorner$ is true iff α is true in at least one plausible world.

Of course, since the key semantic notion is replaced by the notion of *plausible worlds*, there will be important conceptual differences between a logic of plausibility so conceived and the logic of possibility and necessity as formalized, say, by S5. But there will be important similarities and relations too. For instance, since every plausible world is a possible world, we might set the following relations between the notions of *necessity*, *possibility*, *skeptical plausibility* and *credulous plausibility*: $\ulcorner\Box\alpha \rightarrow \alpha!\urcorner$, $\ulcorner\alpha! \rightarrow \alpha?\urcorner$ and $\ulcorner\alpha? \rightarrow \Diamond\alpha\urcorner$ (see [7]). More important however is that this logic of plausibility seems to have the same formal structure as traditional modal logic, so that the task of building a logic of plausibility would be reduced to the task of deciding which one of the normal modal systems, say, is more adequate to our needs. This in fact would be so if it were not for the following fact: having a skeptical and a credulous approach to evaluate the truth value of plausible formulas causes the negation operator to behave in a way that traditional modal logic simply cannot handle.

To start with, let us examine how the notion of implausibility would be represented inside our sketched framework. First of all, for all intends and purposes, the notion of implausibility might be seen simply as the negation of plausibility, so that "α is implausible" can be taken as an abbreviation to "it is not the case that α is plausible". But since here the concept of plausibility is being taken obligatorily according either to a credulous view or to a skeptical view, the same should be done to all notions derived from it, in special the notion of implausibility. Therefore we shall have something like (I) and (II) below:

(I) it is not the case that the statement α is plausible (α is implausible) according to a skeptical position.
(II) it is not the case that the statement α is plausible (α is implausible) according to a credulous position.

According to our notation, (I) and (II) are trivially represented as $\neg(\alpha!)$ and $\neg(\alpha?)$, respectively.

Note however that there is an ambiguity in the reading of these two sentences. Are we negating the plausibility of α according to such and such approach; or are we negating, according to that approach, the plausibility of α? This can be better seen with the help of brackets, where (i) or (ii) below correspond to each one of the two possible ways we can read (I) and (II):

(i) it is not the case that [α is plausible according to a skeptical (credulous) position],
(ii) [it is not the case that α is plausible] according to a skeptical (credulous) position.

In the skeptical case, for example, while (i) means that we were not able to take "α is plausible" as truth according to a rigid, strict posture, (ii) means that we *did* succeed in the task of attributing "true" to the sentence "α is not plausible" according to that posture. Similarly for the credulous case: while (i) means that adopting a tolerant posture concerning truth-assignment we were not able to classify "α is plausible" as true, all that (ii) says is that "α is not plausible" is true according to that posture.

Now, (i) clearly involves a negation pretty much alike to the negation of traditional modal logic: (i) is true iff α is false in at least one world, in the case of the skeptical approach; and iff α is false in all worlds, in the case of the credulous one. Regarding (ii), however, the situation seems to be quite different: instead of denying that α is plausible according to a specific position, (ii) is in fact classifying the whole sentence "it is not the case that α is plausible" as true according to a specific position. We can therefore see the negation involved in (ii) as meaning something like "it is not the case according to the position from which a given statement is uttered." As we shall try to show below, this reading allows us to philosophically justify axioms `K2` and `K3`.

According to what we have explained above, to evaluate "α is not plausible" according to a skeptical position means to be very strict, requiring the maximum we can to classify "α is not plausible" as true; and to evaluate "α is not plausible" according to a credulous position means to be tolerant, requiring the minimum we can to classify "α is not plausible" as true. Given the semantic framework sketched here, clearly to require the *maximum* we can to classify "α is not plausible" as true means to require α to be false in all plausible worlds, and to require the *minimum* we can to classify "α is not plausible" as true is tantamount to requiring α to be false in at least one world. This means that the skeptical version of (ii), or in symbols $\ulcorner\neg(\alpha!)\urcorner$, is true iff α is false in all plausible worlds; and the credulous version of (ii), or in symbols $\ulcorner\neg(\alpha?)\urcorner$, is true iff α is false in at least one plausible world.

As already mentioned, a trivial presupposition present in analyses such as the one we are doing here is that the notion of implausibility, is to be analyzed, represented or described in terms of the concepts of negation and plausibility. As consequence of that, it can be claimed that a fundamental step in the task of formally disambiguating statements (I) and (II) involves having two different negations, one for each reading of (I) and (II). Let us use the symbol '$\sim$' to refer to the negation involved in (i) and '$\neg$' to the negation involved in (ii), so that the first reading of (I) and (II) might be formally represented as $\ulcorner{\sim}(\alpha!)\urcorner$ and $\ulcorner{\sim}(\alpha?)\urcorner$, respectively, and the second reading of (I) and (II) as $\ulcorner\neg(\alpha!)\urcorner$ and $\ulcorner\neg(\alpha?)\urcorner$, respectively. While '$\sim$' is a negation which interprets $\ulcorner{\sim}(\alpha!)\urcorner$ and $\ulcorner{\sim}(\alpha?)\urcorner$, respectively, in exactly the same way as $\ulcorner\sim\Box\,\alpha\urcorner$ and $\ulcorner\sim\Diamond\,\alpha\urcorner$ in traditional modal logic, '$\neg$' has a different, non-classical behavior, according to which $\ulcorner\neg(\alpha!)\urcorner$ is true iff α is false in all plausible worlds; and $\ulcorner\neg(\alpha?)\urcorner$ is true iff α is false in at least one plausible world. From a general perspective, $\ulcorner\neg\alpha\urcorner$ means "it is not the case that α according to the position from which it is being uttered." About the relations between these two negations, it is easy to see that neither $\ulcorner\neg\alpha \rightarrow \sim\alpha\urcorner$ nor $\ulcorner\sim\alpha \rightarrow \neg\alpha\urcorner$ are generally valid: even though $\ulcorner\neg(\alpha!) \rightarrow {\sim}(\alpha!)\urcorner$ holds, $\ulcorner\neg(\alpha?) \rightarrow {\sim}(\alpha?)\urcorner$ is not valid; and even though $\ulcorner{\sim}(\alpha?) \rightarrow \neg(\alpha?)\urcorner$ holds, $\ulcorner{\sim}(\alpha!) \rightarrow \neg(\alpha!)\urcorner$ is not valid.

One might ask now how $\ulcorner(\neg\alpha)!\urcorner$ and $\ulcorner(\neg\alpha)?\urcorner$ are to be analyzed. Well, according to the way we are reading '$\neg$', $\ulcorner(\neg\alpha)!\urcorner$ shall mean something like "it is skeptically plausible that [it is not the case that α according to the position from which it is being uttered]". But α is being uttered according to no position at all (the skeptical reading is being applied to the whole of $\neg\alpha$). Therefore '$\neg$' must in this case behave in the usual way: $\neg\alpha$ is true iff α is false. We have thus as follows: $\ulcorner(\neg\alpha)!\urcorner$

is true iff α is false in all plausible worlds and $\ulcorner(\neg\alpha)?\urcorner$ is true iff α is false in at least one plausible world. This however is the same evaluation which, we have agreed above, should be given to $\ulcorner\neg(\alpha!)\urcorner$ and $\ulcorner\neg(\alpha?)\urcorner$ in order to account for the second reading of (I) and (II). Therefore $\ulcorner\neg(\alpha!)\urcorner$ is semantically equivalent to $\ulcorner(\neg\alpha)!\urcorner$ and $\ulcorner(\neg\alpha)?\urcorner$ is semantically equivalent to $\ulcorner\neg(\alpha?)\urcorner$, or in symbols: $\ulcorner\neg(\alpha!) \leftrightarrow (\neg\alpha)!\urcorner$ and $\ulcorner(\neg\alpha)? \leftrightarrow \neg(\alpha?)\urcorner$. Needless to say, these are exactly the axioms `K2` and `K3`.

An important point is that according to the interpretation which was given here, it might happen that neither $\ulcorner\alpha!\urcorner$ nor $\ulcorner\neg(\alpha!)\urcorner$ are true. Regarding !-marked formulas, thus, '$\neg$' has a *paracomplete* behavior. Also due to this interpretation, we might have a model that satisfies both $\ulcorner\alpha?\urcorner$ and $\ulcorner\neg(\alpha?)\urcorner$, making '$\neg$' correspond to what we have called earlier a true paraconsistent negation. Finally, as we saw above, regarding non-modal formulas, that is, formulas dissociated from both of our two approaches, '$\neg$' behaves classically. We have then a negation with a sort of plural behavior: in connection with ?-marked formulas '$\neg$' behaves paraconsistently, in connection with !-marked ones it behaves like a paracomplete negation, and along with non-modal formulas it behaves classically. We call such a negation a *modality-dependent paranormal negation*.

3. General Definitions

In this section we introduce the basic syntactic, semantic and axiomatic notions to be used in the course of the paper. They are intent to provide a basic framework in which several sorts of modal logics can be formulated. Such general approach will make the movement from one system to another as well as the comparison between them more natural. Regarding the semantic and axiomatic definitions, we are in general following the standard style of semantic and syntactic definitions of modal logics found in textbooks such as [12, 18].

3.1. Syntactic Definitions

Definition 3.1. By a *language* we mean any set $\mathfrak{L}$ of expressions such that given two expressions $\alpha, \beta \in \mathfrak{L}$: $\ulcorner\neg\alpha\urcorner$, $\ulcorner(\alpha \to \beta)\urcorner$, $\ulcorner(\alpha \wedge \beta)\urcorner$, $\ulcorner(\alpha \vee \beta)\urcorner \in \mathfrak{L}$. Elements of $\mathfrak{L}$ are called its formulas while the monadic operator '$\neg$' and the dyadic operators '$\to$', '$\wedge$' and '$\vee$' are called *logical symbols of* $\mathfrak{L}$.

DEFINITION 3.2. Let P be a countable set of entities called *propositional symbols*. The language L_P is the smallest language such that $P \subseteq \mathrm{L}_P$; thus it is defined as follows:

- $P \subseteq \mathrm{L}_P$;
- if $\alpha, \beta \in \mathrm{L}_P$, then $\ulcorner\neg\alpha$, $\ulcorner(\alpha \to \beta)\urcorner$, $\ulcorner(\alpha \wedge \beta)\urcorner$, $\ulcorner(\alpha \vee \beta)\urcorner$ belong to L_P;
- nothing else belongs to L_P.

The logical symbols '$\to$', '$\wedge$' and '$\vee$' are interpreted according to their usual meaning. Depending on whether the modal logic which is using L_P is a normal or paranormal one, '$\neg$' will behave either as a classical negation or as a modality-dependent paranormal negation, respectively. When writing down formulas, we will use the standard rules for omitting parentheses. In all mentioned below definitions let $\mathfrak{L}$ be any language.

DEFINITION 3.3. We say that $\mathfrak{L}$ is a *propositional language built upon a set of propositional symbols* P iff $\mathrm{L}_P \subseteq \mathfrak{L}$.

DEFINITION 3.4. Let $\mathfrak{L}$ be a propositional language built upon a set of propositional symbols P. We define the tautology and contradiction symbols as follows:

$$\top := p \vee \neg p$$
$$\bot := p \wedge \neg p$$

where $p \in P$, i.e., it is an arbitrary propositional symbol.

Let L be the language L_P built upon an arbitrary set of propositional symbols P (whenever we mention P without further qualification we are meaning this arbitrary set of propositional symbols).

DEFINITION 3.5. A *vocabulary* U is a quadruple $\langle U_C, U_V, U_F, U_R\rangle$, where U_C is a countable set of individual constant symbols, U_V a countable set of variable symbols (variables for short), U_F a countable set of function symbols and U_R a countable set of predicate or relation symbols. Moreover, these sets are pairwise disjoint. Each element u of $U_F \cup U_R$ has associated with it a number which we call *the arity* of u, in the case that arity of u equals n we say u is an n-ary symbol.

In all following standard definitions let $U = \langle U_C, U_V, U_F, U_R\rangle$ be a vocabulary.

DEFINITION 3.6. A *term* in U is defined as follows:

- if $t \in U_C \cup U_V$ then t is a term in U;

- if $t_1, \ldots, t_n$ are terms in U and $f \in U_F$ is a function symbol of arity n, then $\ulcorner f(t_1, \ldots, t_n)\urcorner$ is a term in U;
- nothing else is a term in U.

Definition 3.7. The language L_U is defined as follows:

- if $t_1, \ldots, t_n$ are terms in U and $r \in U_R$ is a relation symbol of arity n, then $\ulcorner r(t_1, \ldots, t_n)\urcorner \in \mathrm{L}_U$;
- if $\alpha, \beta \in \mathrm{L}_U$, then $\ulcorner\neg\alpha\urcorner$, $\ulcorner(\alpha \to \beta)\urcorner$, $\ulcorner(\alpha \wedge \beta)\urcorner$, $\ulcorner(\alpha \vee \beta)\urcorner \in \mathrm{L}_U$;
- if $\alpha \in \mathrm{L}_U$ and $x \in U_V$, then $\ulcorner\forall x\alpha\urcorner \in \mathrm{L}_U$;
- nothing else belongs to L_U.

Besides '$\neg$', '$\to$', '$\wedge$' and '$\vee$', '$\forall$' is also a logical symbol of L_U.

The elements of P and formulas of L_U of the form $\ulcorner r(t_1, \ldots, t_n)\urcorner$, where $t_1, \ldots, t_n$ are terms in U and $r \in U_R$ is a relation symbol of arity n, are called *atomic formulas*.

If α is an atomic formula, then α and $\neg\alpha$ are called *basic formulas*.

We define a variable x as being free in α in the usual way. $\alpha(x)$ means that formula α contains (possibly zero) free occurrences of variable x. If, subsequently, we write $\alpha(t)$, we mean the formula that is like $\alpha(x)$ except that occurrences of the term t have been substituted for all free occurrences of x. We say that such a substitution is admissible if and only if no variable symbol $z \in U_V$ occurring in t is such that a free occurrence of x in $\alpha(x)$ is within the scope of a quantifier $\ulcorner\forall z\urcorner$.

Definition 3.8. We say that $\mathfrak{L}$ is a *first-order language* iff $\mathrm{L}_U \subseteq \mathfrak{L}$, for some vocabulary U. Then we also say that $\mathfrak{L}$ is *built upon U*.

Definition 3.9. Let $\mathfrak{L}$ be a first-order language built upon a vocabulary U. We define the tautology and contradiction symbols as follows:[3]

$$\top := (r(t_1, \ldots, t_n) \vee \neg r(t_1, \ldots, t_n)),$$
$$\bot := (r(t_1, \ldots, t_n) \wedge \neg r(t1, \ldots, t_n)),$$

where $r \in U_R$ is an arbitrary n-ary relation symbol and $t_1, \ldots, t_n$ are terms in U.

Let $\mathscr{L}$ be the language L_U built upon a vocabulary U. We will refer to the components of the vocabulary U upon which $\mathscr{L}$ is based, without further mention, simply by the symbols U_C, U_V, U_F and U_R.

[3] Despite the fact that we have already used the same symbols $\top$ and $\bot$ for propositional languages there is no risk here of ambiguity.

Definition 3.10. We define the derivate symbols '$\leftrightarrow$' and '$\sim$' for any $\alpha, \beta \in \mathfrak{L}$ as follows:

$$\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$$
$$\sim \alpha := \alpha \to \bot$$

In paranormal modal logic, '$\sim$' will be used to simulate classical negation. While '$\neg$' is a modality-dependent paranormal negation, '$\sim$' behaves, for any sort of formula, both syntactically and semantically, exactly like classical negation. From now on, when introducing the several non-classical logics to be described in this paper, we will refer to '$\neg$' as the (modality-dependent) paranormal negation of the logic at hand, and '$\sim$' as its classical negation.

Definition 3.11. A *modal logic basis* ϑ is a pair $\langle \Theta, \Theta_{\mathrm{d}} \rangle$, where Θ and Θ_{d} are two possibly empty sets of modal monadic operators[4] such that $\Theta_{\mathrm{d}} \subseteq \Theta$. Θ_{d} is called the *set of distinguished modal operators*. Letting n be the number of elements of Θ_{d}, we say that ϑ is an n-modal logic basis.

Definition 3.12. Let $\vartheta = \langle \Theta, \Theta_{\mathrm{d}} \rangle$ be a modal logic basis and let $\mathfrak{L}$ be a language. The *modal language* $\mathfrak{L}_\vartheta$ *based on* $\mathfrak{L}$ *and* ϑ, is defined as follows:

- If $\alpha \in \mathfrak{L}$ is such that α contains no one of $\mathfrak{L}$'s logical symbols, then $\alpha \in \mathfrak{L}_\vartheta$.
- If $\forall$ is a logical symbol of $\mathfrak{L}$, a is a variable symbol of $\mathfrak{L}$, and $\alpha \in \mathfrak{L}_\vartheta$, then $\ulcorner \forall a \alpha \urcorner \in \mathfrak{L}_\vartheta$.
- If $\circ$ is a monadic logical symbol of $\mathfrak{L}$ and $\alpha \in \mathfrak{L}_\vartheta$, then $\ulcorner \circ \alpha \urcorner \in \mathfrak{L}_\vartheta$.
- If $\circ$ is a dyadic logical symbol of $\mathfrak{L}$ and $\alpha, \beta \in \mathfrak{L}_\vartheta$, then $\ulcorner (\alpha \circ \beta) \urcorner \in \mathfrak{L}_\vartheta$.
- If $\theta \in \Theta$ and $\alpha \in \mathfrak{L}_\vartheta$, then $\ulcorner \theta \alpha \urcorner \in \mathfrak{L}_\vartheta$ (or $\ulcorner \alpha \theta \urcorner \in \mathfrak{L}_\vartheta$, if a post-fixed notation is used).
- Nothing else belongs to $\mathfrak{L}_\vartheta$.

The purpose of a modal logic basis is both syntactic and meta-logical. On the syntactic side, Θ determines how a language $\mathfrak{L}$ will be extended in order to accommodate modal sentences. On the meta-logical side, Θ_{d} determines whether the logic in question is a mono-modal logic or a multi-modal one. From an axiomatic point of view, this is done by

[4] The term 'modal operator' is been taken here according to its traditional meaning (which has already been done in previous sections).

using the elements of Θ_{d} in the formulation of necessitation rules: for each $\theta \in \Theta_{\mathrm{d}}$ there is a rule saying that from any α one may conclude $\ulcorner\theta\alpha\urcorner$ (or $\ulcorner\alpha\theta\urcorner$). From a semantic point of view, this is related to the definition of the accessibility relation to be used by the elements of Θ_{d}.

An important point concerns the reason why Θ may be different from Θ_{d}. The whole idea of Θ is to contain several pairs of corresponding modal operators in the style of '$\Box$' and '$\Diamond$' (and of '!' and '?'), where one of the members of each pair belongs to Θ_{d}. It is possible, however, as it happens with '!' and '?', that we cannot define any one of the members of a specific pair of corresponding operators through the other. Therefore, both have to be introduced as primitive symbols (and belong to Θ), even though only one of them will be taken as a distinguished modal operator.

3.2. Semantic Definitions

Definition 3.13. A *frame* F of arity n (or simply an n-frame F), $n \geqslant 1$, is a $n{+}1$-tuple $\langle W, R_1, \dots, R_n\rangle$, where W is a non-empty countable set of entities called *worlds* and $R_1, \dots, R_n$ are binary relations on W called *accessibility relations*.

Let Rel be the set of names {'reflexive', 'transitive', 'symmetric'}. Let $F = \langle W, R_1, \dots, R_n\rangle$ be an n-frame and $P_1, \dots, P_n$ be the sets of *all* names from Rel of the classes of relations to which $R_1, \dots, R_n$, respectively, belong. We then say that F is a $\langle P_1\text{-}\cdots\text{-}P_n\rangle$ *frame*. If $P_1 = \cdots = P_n$, we call F *simply a* P_1 *frame*. For a given $\langle P_1\text{-}\cdots\text{-}P_n\rangle$ frame we will skip brackets in P_i's. If $n = 1$, we also drop the use of '$\langle$' and '$\rangle$'. Moreover, an n-frame $F = \langle W, R_1, \dots, R_n\rangle$ is called *serial* iff for every $i = 1, \dots, n$ and for every $w \in W$ there is at least one $w' \in W$ such that $w\, R_i\, w'$.

In all mentioned below definitions let $\mathfrak{L}$ be any language and $F = \langle W, R_1, \dots, R_n\rangle$ be any n-frame.

Definition 3.14. A *modal interpretation* of $\mathfrak{L}$ in F is a structure[5] which, along with other possible parameters, evaluate the truth-value of the atomic formulas of $\mathfrak{L}$ in each world $w \in W$. If $\mathfrak{L}$ is a propositional language, a modal interpretation of $\mathfrak{L}$ in F, called simply a *proposi-*

[5] We are here using the word 'structure' in a broad sense, so as to encompass structures in the sense of n-tuples as well as functions.

tional modal interpretation in F, is a function ν mapping elements of the Cartesian product $W \times P$ to truth-values 0 and 1.

In representing a function we shall often abbreviate its value, writing only its last parameter among parentheses; the others shall be written without parentheses and in subscript. For instance, instead of writing $\nu(w, p)$ we shall write $\nu_w(p)$.

DEFINITION 3.15. A *model* M of arity n (or simply an n-model M) in $\mathfrak{L}$ is a n+2-tuple $\langle W, R_1, \ldots, R_n, \nu \rangle$, where ν is a modal interpretation of $\mathfrak{L}$ in F. We say that the model M is *based on* F and that w in W is a world of M. If ν is a propositional modal interpretation we call M a propositional model of arity n (or simply a propositional n-model).

The consideration of more than one accessibility relation is needed to contemplate multi-modal logics, that is, logics that have more than one distinguished modal operator. In general, to each distinguished modal operator θ_i it will correspond an accessibility relation R_i. From now on, we may refer to some model or frame without any qualification concerning their arity. When this happens, we are either referring to a model or frame of arbitrary arity or to a model or frame of arity unambiguously determined by the context where the reference is made.

The actual meaning of the worlds of a model M will depend on the other components of the logical system and on the aimed application. Since our main purpose is to present paranormal modal logic as well as to formalize the notions of credulous and skeptical plausibility, we will generally refer to W as a set of *plausible worlds*.

In all following definitions let $\vartheta = \langle \Theta, \Theta_{\mathrm{d}} \rangle$ be an n-modal logic basis.

DEFINITION 3.16. A *modal valuation of arity* n (or simply an n-modal valuation) *in* $\mathfrak{L}$ *and* ϑ is a function Ψ which, given an n-model $M = \langle W, R_1, \ldots, R_n, \nu \rangle$ in $\mathfrak{L}$, maps tuples built of a world of M and a formula from $\mathfrak{L}_\vartheta$ (and possibly another parameters) to truth values 0 and 1.

In the case of a propositional modal valuation in ϑ, there will be no other parameters besides M, w and a formula α. We represent Ψ applied to M, w and α by $\Psi_{M,w}(\alpha)$.

In all mentioned below definitions let Ψ be any n-modal valuation in $\mathfrak{L}$ and ϑ, and let $\mathcal{F}$ be any class of n-frames.

DEFINITION 3.17. A *semantic modal system* Λ° *of arity* n (or simply *a semantic* n*-modal system*) *based on* $\mathfrak{L}$ is any triple $\langle \vartheta, \Psi, \mathcal{F} \rangle$.

Definition 3.18. For any n-model $M = \langle W, R_1, \ldots, R_n, \nu \rangle$, $w \in W$ and $\alpha \in \mathfrak{L}_\vartheta$:

- α *is Ψ-satisfied by M at w* (in symbols: $M, w \Vdash_\Psi \alpha$) iff $\Psi_{M,w}(\alpha) = 1$;
- α *is Ψ-satisfied by M* (in symbols: $M \Vdash_\Psi \alpha$) iff, for all $w' \in W$, $M, w' \Vdash_\Psi \alpha$.

In all following definitions let $\mathcal{M}$ be the class of all n-models based on a frame F, for each F in a class $\mathcal{F}$, and let $\Lambda^\circ = \langle \vartheta, \Psi, \mathcal{F} \rangle$ be any semantic n-modal system based on the language $\mathfrak{L}$.

Definition 3.19. For any set of formulas $A \subseteq \mathfrak{L}_\vartheta$ the function $\mathcal{M}_{\Lambda^\circ}$ is defined as follows:

$$\mathcal{M}_{\Lambda^\circ}(A) := \{M \mid M \in \mathcal{M} \text{ and } M \Vdash_\Psi \alpha, \text{ for all } \alpha \in A\}.$$

Definition 3.20. For any sets of formulas $A, B \subseteq \mathfrak{L}_\vartheta$ and any formula $\varphi \in \mathfrak{L}_\vartheta$, we say that φ *is a Λ°-logical consequence of A and B*, where A being a set of global premises and B a set of local premises (in symbols: $A \oplus B \vDash_{\Lambda^\circ} \varphi$) iff for every n-model $M \in \mathcal{M}_{\Lambda^\circ}(A)$ and every world w of M such that for every $\beta \in B$, $M, w \Vdash_\Psi \beta$, also $M, w \Vdash_\Psi \varphi$.

Here we are making use of the important distinction between global and local premises [12]. As we will see below, on the axiomatic side the same distinction is made by restricting the use of the necessitation rule only to global premises.

Definition 3.21. For any set of formulas $A \subseteq \mathfrak{L}_\vartheta$ and any formula $\varphi \in \mathfrak{L}_\vartheta$, we say that φ *is a Λ°-logical consequence of A* (in symbols: $A \vDash_{\Lambda^\circ} \varphi$) iff $A \oplus \emptyset \vDash_{\Lambda^\circ} \varphi$.

Moreover, we say that φ *is Λ°-valid*, or valid in Λ° (in symbols: $\vDash_{\Lambda^\circ} \alpha$), iff $\emptyset \oplus \emptyset \vDash_{\Lambda^\circ} \alpha$.

3.3. Axiomatic Definitions

In all definitions in this subsection let $\mathfrak{L}$ be any language and $\vartheta = \langle \Theta, \Theta_\mathrm{d} \rangle$ be any n-modal logic basis.

Definition 3.22. The *axioms of positive logic Σ_P in $\mathfrak{L}_\vartheta$* is the set composed by all formulas of $\mathfrak{L}_\vartheta$ falling under one of the following schemas:

P1: $\alpha \to (\beta \to \alpha)$
P2: $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

P3: $\alpha \wedge \beta \to \alpha$
P4: $\alpha \wedge \beta \to \beta$
P5: $\alpha \to (\beta \to \alpha \wedge \beta)$
P6: $\alpha \to \alpha \vee \beta$
P7: $\beta \to \alpha \vee \beta$
P8: $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$

Definition 3.23. An *axiomatic modal system* Λ^* *of arity* n (or simply an axiomatic n-modal system or, still, a modal calculus) *based on* $\mathfrak{L}$ is a pair $\langle \vartheta, \Sigma \rangle$, where $\Sigma \subseteq \mathfrak{L}_\vartheta$ is a set of axioms. We also say that Λ^* *is based on* ϑ.

Definition 3.24. Let $\Lambda^* = \langle \vartheta, \Sigma \rangle$ be an axiomatic modal system based on $\mathfrak{L}$ with $\vartheta = \langle \Theta, \Theta_{\mathrm{d}} \rangle$, A and B be any sets of formulas of $\mathfrak{L}_\vartheta$, φ be a formula of $\mathfrak{L}_\vartheta$ and $S = \langle \lambda_1, \ldots, \lambda_k \rangle$ be a sequence of formulas of $\mathfrak{L}_\vartheta$. Let S be divided into two parts: the global part $S_{\mathrm{G}} = \langle \lambda_1, \ldots, \lambda_n \rangle$, $n \leqslant k$, and the local part $S_{\mathrm{L}} = \langle \lambda_{n+1}, \ldots, \lambda_k \rangle$. We say that S is a Λ^*-*derivation* of φ from A and B, where A being the global premises and B the local premises, iff $\lambda_k = \varphi$ and for every $1 \leqslant i \leqslant k$, one of the following conditions is satisfied:

a) $\lambda_i \in A$, in the case that $1 \leqslant i \leqslant n$,
b) $\lambda_i \in B$, in the case that $n+1 \leqslant i \leqslant k$,
c) $\lambda_i \in \Sigma$,
d) there are $r, s < i$ such that $\lambda_r = \lambda_s \to \lambda_i$,
e) (just in the case $\mathfrak{L}$ is a first-order language) there is $r < i$ such that $\lambda_r = \alpha \to \beta$, $\lambda_i = \alpha \to \forall x \beta$ and x has no free occurrences in α,
f) (in the case that $1 \leqslant i \leqslant n$) there is $r < i$ and $\theta \in \Theta_{\mathrm{d}}$ such that $\lambda_i = \theta\lambda_r$ (or $\lambda_i = \lambda_r\theta$, if a post-fixed notation is used).

If $B = \emptyset$, then we simply say that S is a Λ^*-derivation of φ from A. If $A = B = \emptyset$, then we say that S is a Λ^*-derivation of φ.

As one might suspect, items d), e), and f) correspond respectively to *modus ponens*, *generalization*, and *necessitation* rules. Concerning the latter, three points should be mentioned. First, since we allow more than one distinguished modal operator, we also allow more than one necessitation rule, one for each distinguished operator. Second, as we have remarked earlier, the necessitation rule is applied only to global premises. And third, even though our primary goal with these definitions is to provide a framework where modal logics can be defined, we can define non-modal logics as well. In this case there will be of course no necessitation rule (since Θ_{d} will be empty, item f) will never be satisfied).

In the following two definitions let $\Lambda^* = \langle \vartheta, \Sigma \rangle$ be an axiomatic modal system based on $\mathfrak{L}$.

Definition 3.25. For any sets of formulas $A, B \subseteq \mathfrak{L}_\vartheta$ and any formula $\varphi \in \mathfrak{L}_\vartheta$, we say that φ *is Λ^*-deducted from A and B*, where A being the set of global premises and B the set of local premises (in symbols: $A \oplus B \vdash_{\Lambda^*} \varphi$), iff there is a Λ^*-derivation of φ from A and B.

Definition 3.26. For any set of formulas $A \subseteq \mathfrak{L}_\vartheta$ and any formula $\varphi \in \mathfrak{L}_\vartheta$, we say that φ *is Λ^*-deducted from A* (in symbols: $A \vdash_{\Lambda^*} \alpha$) iff $A \oplus \emptyset \vdash_{\Lambda^*} \alpha$.

We say that φ *is a Λ^*-theorem*, or a theorem of Λ^* (in symbols: $\vdash_{\Lambda^*} \varphi$), iff $\emptyset \oplus \emptyset \vdash_{\Lambda^*} \varphi$.

3.4. Modal Systems

In all definitions in this subsection let $\mathfrak{L}$ be any language.

Definition 3.27. A *modal system Λ of arity n* (or simply an n-modal system Λ) *based on* $\mathfrak{L}$ is a quadruple $\langle \vartheta, \Psi, \mathcal{F}, \Sigma \rangle$, where $\Lambda^\circ = \langle \vartheta, \Psi, \mathcal{F} \rangle$ is a semantic n-modal system based on $\mathfrak{L}$ and $\Lambda^* = \langle \vartheta, \Sigma \rangle$ is an axiomatic n-modal system based on $\mathfrak{L}$. We also say that Λ is the modal system based on Λ° and Λ^*.

In all mentioned below definitions let Λ be any modal system based on a semantic n-modal system $\Lambda^\circ = \langle \vartheta, \Psi, \mathcal{F} \rangle$ and an axiomatic n-modal system $\Lambda^* = \langle \vartheta, \Sigma \rangle$, both based on $\mathfrak{L}$.

Definition 3.28. For any sets of formulas $A, B \subseteq \mathfrak{L}_\vartheta$ and any formula $\varphi \in \mathfrak{L}_\vartheta$, we say that:

- φ *is a Λ-logical consequence of A and B* (in symbols: $A \oplus B \vDash_\Lambda \varphi$) iff $A \oplus B \vDash_{\Lambda^\circ} \varphi$;
- φ *is a Λ-logical consequence of A* (in symbols: $A \vDash_\Lambda \varphi$) iff $A \oplus \emptyset \vDash_\Lambda \varphi$;
- φ *is Λ-valid*, or valid in Λ (in symbols: $\vDash_\Lambda \varphi$), iff $\emptyset \oplus \emptyset \vDash_\Lambda \varphi$;
- φ *is Λ-deducted from A and B* (in symbols: $A \oplus B \vdash_\Lambda \varphi$) iff $A \oplus B \vdash_{\Lambda^*} \varphi$;
- φ *is a Λ-deducted from A* (in symbols: $A \vdash_\Lambda \varphi$) iff $A \oplus \emptyset \vdash_\Lambda \varphi$;
- φ *is a Λ-theorem*, or a theorem of Λ (in symbols: $\vdash_\Lambda \varphi$), iff $\emptyset \oplus \emptyset \vdash_\Lambda \varphi$.

Definition 3.29. Λ *is sound* iff for any $A, B \subseteq \mathfrak{L}_\vartheta$ and $\varphi \in \mathfrak{L}_\vartheta$, if $A \oplus B \vdash_\Lambda \varphi$ then $A \oplus B \vDash_\Lambda \varphi$.

Λ *is complete* iff for any $A, B \subseteq \mathfrak{L}_\vartheta$ and $\varphi \in \mathfrak{L}_\vartheta$, if $A \oplus B \vDash_\Lambda \varphi$ then $A \oplus B \vdash_\Lambda \varphi$.

A modal system Λ based on $\mathfrak{L}$ is meant to contain all elements of a specific modal logic, both syntactic and semantic. It is defined in such a way that, given two n-modal systems Λ and Λ', it will be clear from their very components what makes them different from each other. For instance, normal propositional modal logics K, T, S4, etc. are based on the same language $\mathfrak{L}$, have the same modal logic basis and propositional modal valuation, but differ in their sets of frames and axioms. As we show in the second part of the paper [37], what makes propositional modal system K different from propositional paranormal modal system $K_?$, for instance, is their modal logic basis, modal valuation and axioms. But since the syntactic shape of a modal symbol is logically irrelevant for the logic which uses it, what in fact makes K and $K_?$ (and T and $T_?$, D and $D_?$, S4 and $S4_?$, and so on) different from each other is their corresponding modal valuations and axioms.

4. Paranormal Modal Logic: the System $K_?$

In this section we present, both syntactically and semantically, the basic paranormal modal system $K_?$. Since all other paranormal modal logics are extensions of $K_?$, this section contains the most fundamental notions of paranormal modal logic.

4.1. The Language of Paranormal Modal Logic

The notions of ?-modal logic basis, paranormal modal logic basis and paranormal modal language are defined as follows[6]:

Definition 4.1. (i) A ?-*modal logic basis* is any pair $\langle \Theta, \Theta_\mathrm{d} \rangle$ in which $\{!, ?\} \subseteq \Theta$ and $! \in \Theta_\mathrm{d}$. The notation adopted for the operators ! and ? is a post-fixed one.

[6] In naming several components of paranormal logic, such as what we are calling ?-*modal logic basis*, we are using ? instead of !, which would be the most natural choice. This has to with the decision of ours to follow the historical primacy given to ? in the development of paranormal modal logic

(ii) We call the ?-modal logic basis $\vartheta_? = \langle \{!, ?\}, \{!\} \rangle$ the *paranormal modal logic basis*.

(iii) Letting $\mathfrak{L}$ be a language, we call the modal language based on $\mathfrak{L}$ and $\vartheta_?$, which we will refer to by the symbol $\mathfrak{L}_?$, a *paranormal modal language*.

Given formula α,

- $\alpha?$ means "α is credulously plausible",
- $\alpha!$ means "α is skeptically plausible".

As we noticed above, ! and ? have both to be introduced as primitive symbols, for no one can be defined through the other, neither with the help of paranormal negation $\neg$ nor with the help of classical negation $\sim$.

Definition 4.2. Let $\mathfrak{L}$ be a language, ϑ a ?-modal logic basis and $\alpha \in \mathfrak{L}_\vartheta$ a formula. We say that α is ?-*free* (resp. !-*free*) iff ? (resp. !) does not occur in α. We say α is ?!-*free* iff α is both ?-free and !-free.

4.2. Paranormal Modal Semantics

In this subsection let $\mathfrak{L}$ be any language, ϑ be any ?-modal logic basis of arity n, and let k be any natural number such that $1 \leqslant k \leqslant n$.

Definition 4.3. A Ω_k-*modal valuation in* $\mathfrak{L}$ *and* ϑ, and a $\mho_k$-*modal valuation in* $\mathfrak{L}$ *and* ϑ, which will also be referred to as the max-min k-modal valuations in $\mathfrak{L}$ and ϑ, are n-modal valuations $\Omega_{M,w,\ldots}$ and $\mho_{M,w,\ldots}$ in $\mathfrak{L}$ and ϑ which, given an n-model $M = \langle W, R_1, \ldots, R_k, \ldots, R_n, \nu \rangle$, a world $w \in W$, any two formulas $\alpha, \beta \in \mathfrak{L}_\vartheta$ and possibly other parameters, satisfy the following conditions:

(i) $\Omega_{M,w,\ldots}(\neg\alpha) = 1$ iff $\mho_{M,w,\ldots}(\alpha) = 0$,
(ii) $\mho_{M,w,\ldots}(\neg\alpha) = 1$ iff $\Omega_{M,w,\ldots}(\alpha) = 0$,
(iii) $\Omega_{M,w,\ldots}(\alpha \to \beta) = 1$ iff $\Omega_{M,w,\ldots}(\alpha) = 0$ or $\Omega_{M,w,\ldots}(\beta) = 1$,
(iv) $\mho_{M,w,\ldots}(\alpha \to \beta) = 1$ iff $\Omega_{M,w,\ldots}(\alpha) = 0$ or $\mho_{M,w,\ldots}(\beta) = 1$,
(v) $\Omega_{M,w,\ldots}(\alpha \wedge \beta) = 1$ iff $\Omega_{M,w,\ldots}(\alpha) = 1$ and $\Omega_{M,w,\ldots}(\beta) = 1$,
(vi) $\mho_{M,w,\ldots}(\alpha \wedge \beta) = 1$ iff $\mho_{M,w,\ldots}(\alpha) = 1$ and $\mho_{M,w,\ldots}(\beta) = 1$,
(vii) $\Omega_{M,w,\ldots}(\alpha \vee \beta) = 1$ iff $\Omega_{M,w,\ldots}(\alpha) = 1$ or $\Omega_{M,w,\ldots}(\beta) = 1$,
(viii) $\mho_{M,w,\ldots}(\alpha \vee \beta) = 1$ iff $\mho_{M,w,\ldots}(\alpha) = 1$ or $\mho_{M,w,\ldots}(\beta) = 1$,
(ix) $\Omega_{M,w,\ldots}(\alpha?) = 1$ iff for some $w' \in W$ such that $w \, R_k \, w'$, $\Omega_{M,w',\ldots}(\alpha) = 1$,

(x) $\mho_{M,w,\ldots}(\alpha?) = 1$ iff for any $w' \in W$ such that $w\, R_k\, w'$, $\mho_{M,w',\ldots}(\alpha) = 1$,

(xi) $\Omega_{M,w,\ldots}(\alpha!) = 1$ iff for any $w' \in W$ such that $w\, R_k\, w'$, $\Omega_{M,w',\ldots}(\alpha) = 1$,

(xii) $\mho_{M,w,\ldots}(\alpha!) = 1$ iff for some $w' \in W$ such that $w\, R_k\, w'$, $\mho_{M,w',\ldots}(\alpha) = 1$.

The purpose of the number k is to allow the same structure to be used in multimodal logics where ? and ! are interpreted with the help of the k-th accessibility relation R_k.

Definition 4.4. A *propositional Ω_k-modal valuation in ϑ* and a *propositional $\mho_k$-modal valuation in ϑ*, which will also be referred to as the propositional max-min k-modal valuations in ϑ, are the max-min k-modal valuations $\Omega_{M,w}$ and $\mho_{M,w}$ in L[7] and ϑ which, given a propositional n-model $M = \langle W, R_1, \ldots, R_k, \ldots, R_n, \nu\rangle$, a world $w \in W$, and any propositional symbol $p \in P$, satisfy the following condition:

$$\Omega_{M,w}(p) = 1 \text{ iff } \nu_w(p) = 1 \text{ iff } \mho_{M,w}(p) = 1\,.$$

Ω and $\mho$ are called max-min valuations because depending on the plausible formula they have as parameter they act either as a maximizing or as a minimizing valuation function. Concerning ?!-free formulas however, the skeptical and credulous positions are ineffective: for them paranormal modal logic behaves just like classical logic.[8] Regarding modal formulas, Ω evaluates the truth-value of ?-marked formulas according to a maximal or credulous position and of !-marked ones according to a skeptical or minimal posture. Because of this conformity with the meaning we want to give to ! and ?, Ω is the valuation to be used in the definition of the notion of logical consequence. Formally speaking then, paranormal logic can be characterized as those modal systems whose modal valuation is a Ω_k-modal valuation.

Concerning $\mho$ however, it may be surprising to note that it behaves exactly in the opposite way: while it evaluates ?-marked formulas according to a minimal posture, !-marked formulas are evaluated according to a maximal one. In order to understand the need of such an strange

[7] L is the propositional language built upon an arbitrary set of propositional symbols P. See Section 3.

[8] This observation will be made precise in the second part of the paper [37], when we shall compare $K_?$ with classical logic.

function, we have to look a bit closer at how formulas of the form $\neg\alpha$ are to be appraised by a function which is supposed to recursively define the truth-value of sentences sometimes according to a maximizing position and sometimes according to a minimizing one.

If we want to recursively define the truth-value of formulas we have to define the truth-value of a complex formula in function of its less complex components. More specifically, in order to have a recursive definition of the truth of $\neg\alpha$ we have to take into account the truth of α. In our case however we have two different ways to assess the truth of formulas, a skeptical or minimizing one and a credulous or maximizing one, both of which are incorporated in Ω: depending on the form of the formula, Ω evaluates it skeptically or credulously. Now, how would we define the interpretation of $\neg\alpha$ for Ω? If we follow the usual path we would have something like $\Omega_{M,w}(\neg\alpha) = 1$ iff $\Omega_{M,w}(\alpha) = 0$, which in the case of ? would lead us to the following: $\Omega_{M,w}(\neg(\alpha?)) = 1$ iff $\Omega_{M,w}(\alpha?) = 0$ iff for all for all $w' \in W$ such that $w\, R_k\, w'$, $\Omega_{M,w'}(\alpha) = 0$. This however is not what we mean by α being implausible according to a tolerant position: evaluating the truth-value of "it is not the case that α is plausible" according to a tolerant position means to require just a little to accept it as true. As we have concluded in Section 2, this meaning is achieved by requiring α to be false in at least one plausible world.

One may wonder however why $\Omega_{M,w}(\alpha?) = 0$, for instance, does not mean the same as "it is not the case that α is plausible" according to a credulous position. First of all, we can see the result of a function applied to a specific value as parameter as a sort of qualification over this specific value. For instance, $\Omega_{M,w}(\alpha) = 0$ may be seen as a qualification over α, namely one which qualifies it as false. Now, when we add one of our modal operators to α, we have an additional qualification: if we write $\Omega_{M,w}(\alpha?) = 0$, for instance, we are qualifying not only the plausibility of α as false, but the plausibility of α according to a maximal position. Putting in terms of negation of statements, $\Omega_{M,w}(\alpha?) = 0$ means something like "it is not the case that [α is plausible according to a maximal position]". This is quite different from "[it is not the case that α is plausible] according to a maximal position", which is our intended meaning for $\Omega_{M,w}(\neg(\alpha?)) = 1$. A similar reasoning can be made on !.

We have already digressed on that[9]: the result achieved by interpreting $\Omega_{M,w}(\neg\alpha) = 1$ as $\Omega_{M,w}(\alpha) = 0$ is exactly what we expect to achieve

[9] See Section 2.

by classical negation $\sim$. But how then to define $\Omega_{M,w}(\neg\alpha)$ in such a way as to capture our modality-dependent paranormal negation $\neg$?

As we have seen, $\Omega_{M,w}(\neg(\alpha!)) = 1$ means "[α is not plausible] is true according to a posture which is rigid in the matter of attributing 1 (true) to [α is not plausible]". Two points are important here. First, saying that [α is not plausible] is true is of course the same as saying that [α is plausible] is false; second, if an approach arrives at "[α is not plausible] is true" by being rigid in the matter of attributing 1 to [α is not plausible], it shall arrive at the same result by being rigid in the matter of attributing 0 (false) to [α is plausible]. Therefore, $\Omega_{M,w}(\neg(\alpha!)) = 1$ also means "[α is plausible] is false according to a posture which is rigid in the matter of attributing 0 to [α is plausible]", or equivalently, "we did succeed in the task of evaluating [α is plausible] as false according to a criterion that will be very strict in the matter of qualifying formulas as false". However, to be very strict in the matter of qualifying α as false means to require a lot (of evidences, if you wish) to qualify α as false. In our bivalent framework, this also means that very little will be enough for us to classify α as true. We can then rephrase our last translation to $\Omega_{M,w}(\neg(\alpha!)) = 1$ as "we did succeed in the task of evaluating [α is plausible] as false according to a posture which is very tolerant in the matter of qualifying [α is plausible] as true or attributing the value 1 to it". But this is exactly the meaning of $\mho_{M,w}(\alpha!) = 0$. Therefore $\Omega_{M,w}(\neg(\alpha!)) = 1$ is equivalent to $\mho_{M,w}(\alpha!) = 0$.

We can use the same reasoning to conclude that $\Omega_{M,w}(\neg(\alpha?)) = 1$ is equivalent to $\mho_{M,w}(\alpha?) = 0$, as well as to generalize such result and conclude that, from a conceptual point of view, to say that $\neg\alpha$ is true according to a skeptical or credulous position is the same as saying that α is false according to the opposite position. Since Ω and $\mho$ are such that given a modal formula α, if $\Omega_{M,w}(\alpha)$ returns a truth-value according to maximal posture, $\mho_{M,w}(\alpha)$ returns a truth-value according to a minimal posture; and if $\Omega_{M,w}(\alpha)$ returns a truth-value according to minimal posture, $\mho_{M,w}(\alpha)$ returns a truth-value according to a maximal posture, we have that $\Omega_{M,w}(\neg\alpha) = 1$ iff $\mho_{M,w}(\alpha) = 0$, and $\mho_{M,w}(\neg\alpha) = 1$ iff $\Omega_{M,w}(\alpha) = 0$.

At this point it shall be clear why we need function $\mho$ in order for Ω to take account of formulas of the form $\neg\alpha$. If we want really to have a function that recursively defines the truth-value of $\neg\alpha$ from a skeptical (credulous) point of view, we have no choice but to also have a function that defines the truth-value of α from a credulous (skeptical) point of view. Regarding $\neg$ (and, as we will see below, also $\rightarrow$) one view is not complete without the other. Therefore, if we want to analyze $\neg(\alpha!)$ from a skeptical point of view we have to have a way to analyze $\alpha!$ from a credulous point of view; similarly for $\neg(\alpha?)$.

Concerning the classical negation, $\sim(\alpha!)$ means that it is not the case that [α is plausible according to a skeptical position], and $\sim(\alpha?)$ means that it is not the case that [α is plausible according to a credulous position]. Since $\sim(\alpha!)$ is an abbreviation for $\alpha! \to \bot$, we have the following: $\Omega_{M,w}(\sim(\alpha!)) = 1$ iff $\Omega_{M,w}(\alpha! \to \bot) = 1$ iff $\Omega_{M,w}(\alpha!) = 0$ or $\Omega_{M,w}(\bot) = 1$, which in its turn is the case iff $\Omega_{M,w}(\alpha!) = 0$ iff for at least one $w' \in W$ such that $w\ R_k\ w'$, $\Omega_{M,w'}(\alpha) = 0$. Similarly for $\sim(\alpha?)$: $\Omega_{M,w}(\sim(\alpha?)) = 1$ iff $\Omega_{M,w}(\alpha? \to \bot) = 1$ iff $\Omega_{M,w}(\alpha?) = 0$ or $\Omega_{M,w}(\bot) = 1$, which in its turn is the case iff $\Omega_{M,w}(\alpha?) = 0$ iff for all $w' \in W$ such that $w\ R_k\ w'$, $\Omega_{M,w'}(\alpha) = 0$.

A last and crucial point concerns item (iv), which is asymmetric with respect to (v). It can be very easily shown that according to our conceptual framework the correct form of (iv) should be $\Omega_{M,w}(\alpha \to \beta) = 1$ iff $\mho_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\beta) = 1$. The reason why we did not incorporate such a view in our formulation lies in two words: *modus ponens*. If we equate $\Omega_{M,w}(\alpha \to \beta) = 1$ with $\mho_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\beta) = 1$, given a model M and a world w of M, we would have possibly that $M, w \Vdash_\Omega \alpha \to \beta$ and $M, w \Vdash_\Omega \alpha$ but $M, w \nVdash_\Omega \beta$. As a consequence of this, on the semantic side, *modus ponens* would not be valid. In order to have *modus ponens* as a valid principle of paranormal modal logic, we have no choice but to define $\Omega_{M,w}(\alpha \to \beta)$ exclusively in terms of Ω: $\Omega_{M,w}(\alpha \to \beta) = 1$ iff $\Omega_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\beta) = 1$, which is tantamount to interpreting $\to$ in terms of $\sim$.

An immediate consequence of this unavoidable decision is that many logical laws such as $(\alpha \to \beta) \leftrightarrow \neg\alpha \vee \beta$ and $(\alpha \to \beta) \leftrightarrow (\neg\beta \to \neg\alpha)$ are not valid in paranormal modal logic. On the other hand, since $\Omega_{M,w}(\sim\alpha) = 1$ iff $\Omega_{M,w}(\alpha) = 0$, all these laws are still be valid if we consider them along with the classical negation $\sim$. (Given any model M, we have that $M \Vdash_\Omega (\alpha \to \beta) \leftrightarrow \sim\alpha \vee \beta$ and $M \Vdash_\Omega (\alpha \to \beta) \leftrightarrow (\sim\beta \to \sim\alpha)$.) In fact, not only these two laws but all other classical laws are valid in paranormal modal logic when $\neg$ is replaced by $\sim$; and if we consider ! and ? exclusively in connection with $\sim$, we will have two modalities indistinguishable from normal modalities $\Box$ and $\Diamond$.[10]

[10] These points will be made precise when we compare $K_?$ with classical logic and normal modal logic K in the second part of the paper [37].

4.3. Paranormal Modal Calculus

The calculus of paranormal modal logic is a modal extension of positive classical logic built in such a way as to consider a modality-dependent paranormal negation: it is therefore built by adding extra axioms to the set of classical positive axioms in $L_?$ (as stated in Definition 3.22). In all definitions of this subsection $\mathfrak{L}$ is a language and ϑ is a ?-modal logic basis.

Definition 4.5. The *paranormal classical axioms* Σ_A *in* $\mathfrak{L}_\vartheta$ is the set composed by all formulas of $\mathfrak{L}_\vartheta$ falling under one of the schemas below:

`A1`:	$(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$	wherein β is ?-free and α is !-free
`A2`:	$\neg\alpha \to (\alpha \to \beta)$	wherein α is ?-free
`A3`:	$\alpha \vee \neg\alpha$	wherein α is !-free

The schemas of formula `A1`–`A3` correspond to the negative axioms of classical logic. Along with the axiom schemas `P1`–`P8` of positive logic (see Definition 3.22), they strongly resemble a quite standard axiomatization for classical logic. The difference is that, as defined above, schemas `A1`–`A3` are not universally applied to any formula: there are restrictions concerning modal formulas. The reason for such restrictions lies on the paraconsistency of ? and on the paracompleteness of !. Actually, they are the very key of the paracomplete behavior of ! and the paraconsistent behavior of ?. Since the set $\{\alpha?, \neg(\alpha?)\}$ is intent to be an inconsistent but non-trivial theory, schemas `A1` and `A2` should restrict their use only to ?-free formulas: from $\alpha \to \beta?$ and $\alpha \to \neg(\beta?)$ one should not be able to use `A1` to conclude $\neg\alpha$, and from $\alpha?$ and $\neg(\alpha?)$ one should not be able to use `A2` to conclude β. Similarly, since we may have both $\alpha!$ and $\neg(\alpha!)$ as false, `A3` shall have the same sort of restriction concerning !-marked formulas: we shall not be able to use `A3` to conclude $\alpha! \vee \neg(\alpha!)$. The reason why `A1` has also a restriction concerning !-free formulas is very simple. Consider an instance of `A1` where β is a propositional symbol: $(\alpha \to p) \to ((\alpha \to \neg p) \to \neg\alpha)$. From this we can derive $(\alpha \to p \wedge \neg p) \to \neg\alpha$, which is the same as $(\alpha \to \bot) \to \neg\alpha$, which in its turn is the unabbreviated form of $\sim\alpha \to \neg\alpha$. But, as we have in Section 2, this formula should not hold universally in paranormal modal logic. In special, it is should not hold for !-marked formulas: according to our conceptual analysis of $\neg$ and $\sim$, $\sim(\alpha!) \to \neg(\alpha!)$ is not a valid principle of paranormal modal logic (even though $\sim(\alpha?) \to \neg(\alpha?)$ is). As a consequence of these restrictions, many logical principles whose derivations depend on one of these schemas are not theorems in paranormal modal calculus. However, since for ?!-free formulas schemas A1–A3 can be used freely, we have that for non-modal formulas all principles of classical logic are theorems in paranormal modal calculus.

DEFINITION 4.6. The *additional classical axioms* Σ_{N} *in* $\mathfrak{L}_\vartheta$ is the set composed by all formulas of $\mathfrak{L}_\vartheta$ of the form:

N1: $\neg(\alpha \to \beta) \leftrightarrow (\alpha \wedge \neg\beta)$
N2: $\neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$
N3: $\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$
N4: $\neg\neg\alpha \leftrightarrow \alpha$
N5: $((\alpha \to \beta) \to \alpha) \to \alpha$

Axiom schemas N1–N5 are meant to restore the deductive power of paranormal modal logic weakened by the restrictions imposed to axioms A1–A3.

DEFINITION 4.7. The *paranormal modal axioms* Σ_{M} *in* $\mathfrak{L}_\vartheta$ is the set composed by all formulas of $\mathfrak{L}_\vartheta$ falling under one of the following schemas:

K1: $\alpha? \leftrightarrow \sim((\sim\alpha)!)$
K2: $(\neg\alpha)? \leftrightarrow \neg(\alpha?)$
K3: $(\neg\alpha)! \leftrightarrow \neg(\alpha!)$

The paranormal modal axioms K1–K3 set the basic properties of the modal operators ! and ?. K1 states that in connection with classical negation $\sim$, ? and ! are the dual operators of each other, in the same way that $\Box$ is the dual operator of $\Diamond$ and $\Diamond$ the dual operator of $\Box$. The difference is that traditionally $\Diamond$ is taken as a derived operator ($\Diamond\alpha := \neg\Box\neg\alpha$). Independently however of the way we build normal modal logic, K1 and axiom schema $\mathrm{K}_?$ (to be defined below) set ! and ? in connection with $\sim$ as indistinguishable from $\Box$ and $\Diamond$ of normal modal logic.

The axioms K2 and K3 have been discussed in sections 1 and 2.

DEFINITION 4.8. The $\mathtt{K}_?$-*axioms* $\Sigma_{\mathtt{K}_?}$ *in* $\mathfrak{L}_\vartheta$ is the set composed by all formulas of $\mathfrak{L}_\vartheta$ falling under the following schema:

$\mathtt{K}_?$: $(\alpha \to \beta)! \to (\alpha! \to \beta!)$

4.4. The System $\mathrm{K}_?$

With the semantic and syntactic elements we have defined in Section 3 and so far in this section we can give a precise definition of paranormal modal logic. We define here the basic system upon which all other paranormal modal systems are based: $\mathrm{K}_?$. This name is a direct reference both to the way it is axiomatically obtained — by adding $\Sigma_{\mathrm{K}_?}$ to the system obtained by Σ_{P}, Σ_{A}, Σ_{N}, Σ_{M}, necessitation rule and *modus ponens* — as well as to normal modal system K, which is obtained by adding axiom

K: $\quad \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$

and necessitation rule to classical logic.

By Definition 4.1, $\vartheta_? := \langle\{!, ?\}, \{!\}\rangle$ (the paranormal modal logic basis); moreover $\mathrm{L}_?$ is what we can call propositional paranormal modal language, i.e., the modal language based on propositional language L and $\vartheta_?$.

Let $\Omega_?$ be the propositional Ω_1-modal valuation in $\vartheta_?$ (see Definition 4.4).

Definition 4.9. The propositional paranormal modal logic $\mathrm{K}_?$ is the propositional modal system $\langle\vartheta_?, \Omega_?, \mathcal{F}_{\mathrm{K}}, \Sigma^\star_{\mathrm{K}_?}\rangle$, where $\mathcal{F}_{\mathrm{K}}$ is the class of all frames and

$$\Sigma^\star_{\mathrm{K}_?} := \Sigma_{\mathrm{P}} \cup \Sigma_{\mathrm{A}} \cup \Sigma_{\mathrm{N}} \cup \Sigma_{\mathrm{M}} \cup \Sigma_{\mathrm{K}_?}\,,$$

where Σ_{P} contains the axioms of positive logic in $\mathrm{L}_?$, Σ_{A} — the paranormal classical axioms in $\mathrm{L}_?$, Σ_{N} — the additional classical axioms in $\mathrm{L}_?$, Σ_{M} — the paranormal modal axioms in $\mathrm{L}_?$ and $\Sigma_{\mathrm{K}_?}$ — the $\mathrm{K}_?$-axioms in $\mathrm{L}_?$.

5. Conclusion

We have presented in this paper the first part of a two-parts paper intent to introduce a paraconsistent and paracomplete modal logic called by us paranormal modal logic. Besides giving a philosophical justification to paranormal modal logic as a logic of skeptical and credulous plausibility, we introduced a general framework in which a wide range of logics can be defined. By making use of this framework we defined the most basic of all paranormal modal logics: propositional system $\mathrm{K}_?$. In the second part of the paper [37] we shall prove some key theorems about paranormal modal logic in general and system $K_?$ in particular, including its soundness and completeness and that it is inferentially equivalent to classical normal modal logic K. We also introduce other paranormal modal systems, including some propositional extensions of $K_?$, first-order paranormal modal logic and a multi-modal paranormal logic.

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