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# SPHERES, CUBES AND SIMPLEXES IN MEREOGEOMETRY

**Abstract.** In 1929 Tarski showed how to construct points in a region-based first-order logic for space representation. The resulting system, called *the geometry of solids*, is a cornerstone for region-based geometry and for the comparison of point-based and region-based geometries. We expand this study of the construction of points in region-based systems using different primitives, namely hyper-cubes and regular simplexes, and show that these primitives lead to equivalent systems in dimension $n \geq 2$. The result is achieved by adopting a single set of definitions that works for both these classes of figures. The analysis of our logics shows that Tarski's choice to take sphere as the geometrical primitive might be intuitively justified but is not optimal from a technical viewpoint.

**Keywords**: mereogeometry, geometry of solids, hyper-cube, regular simplex, sphere, mereology.

## 1. Introduction

The term *mereogeometry* is constructed from mereology, the theory of parthood, and geometry. It indicates a system which is region-based, that is, in which variables range over extended regions of space, and is an extension of mereology suitable to model (at least some) geometrical properties. A mereogeometrical system is generally obtained from a system of mereology or of mereotopology (the theory of connection) by adding one or more geometrical primitives like the convexity operator, the sphere predicate or the congruence relation.

Since the goal of mereogeometry is to talk about (commonsense) space, the comparison with systems of Euclidean geometry is essential. In this regard, Tarski laid the framework for the formal and logical

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study to first-order region-based geometries as well as first-order point-based geometries. The mereogeometrical system he developed in 1929, reprinted in [Tar56b], called *the geometry of solids*, and the axiomatization of Euclidean geometry he developed in [Tar59] are paradigmatic and have been taken as cornerstones for the study of geometry. The correspondence between these two systems is formally well understood [Tar56b, Ben01, GP08]. In [BM10] the term *full mereogeometry* was chosen to characterize the class of mereogeometrical systems that are expressively equivalent to Tarski's geometry of solids, i.e., the mereological correspondents of Euclidean geometry.

This paper is about Tarski's geometry of solids; it gives an alternative way of building the system by exploiting two different primitives, namely hyper-cubes and regular simplexes. The study applies to mereogeometry of any dimension $n \geq 2$.

The paper is structured as follows. The next section gives some references to the literature in this domain. Section 3 reports results on full mereogeometries from [BM10]. Section 4 describes the definitions, based on the sphere predicate, used by Tarski to characterize points as second-order entities. This step is needed to give the formal correspondence between the geometry of solids and Euclidean geometry. The following section, Section 5, shows how to obtain an equivalent mereogeometry by taking hyper-cube as the only geometrical primitive. Section 6 proves that the system is categorical by adapting the axiomatization and technique in [Ben01]. Section 7 compares Tarski's geometry of solids and the mereogeometry based on hyper-cubes. Section 8 shows how to obtain another equivalent mereogeometry by taking regular simplex as the only geometrical primitive. In Section 9 the axiomatization is given and categoricity is established for this second system. Finally, Section 10 suggests how to generalize the point construction methodology by merging the approach presented in this paper and Tarski's set of definitions.

## 2. Literature

Tarski's system of primitives (sphere and parthood) "was motivated not only by its simplicity, but also by the fact that the notion of ball seems to be much more intuitive than the notion of point." [Szcz86, p. 911]. Szczerba reports an interesting observation about Tarski's attitude: "I remember Szmielew complaining jokingly that the more one worked with

Tarski, the result tended to look less and less laborious. In fact Tarski would work over a mathematical presentation until it achieved an elegance and simplicity which disguised the difficulties hidden beneath the surface" [Szcz86, p. 910]. We agree and add that, in our opinion, none of the mereogeometries discussed so far in the literature is as intuitive and clean as Tarski's system.

Since Tarski's development of mereogeometry, several elaborations on the topic have been developed. These systems depart from that of Tarski in different ways, e.g., by taking other standard geometrical notions as primitives. The geometries thus developed are unsatisfactory in our view since they make a direct use of points (as a different sort) or regard some of the individuals as point-like entities. The last case is exemplified by [Gli69], which uses the notion of (finite) segment and congruence, and [GV85] based on the notions of solid (among which the so-called $\varepsilon$-points), parthood (here called inclusion) and distance. Among the first type of systems instead we cite [Pam04] that uses as primitives points and triangles (or squares) and the relation of point-triangle (or point-square), and [Pra83, Pam00] based on points, circles and the relation of point-circle incidence. More in the spirit of Tarski's mereogeometry is the result in [Pam03] where the author uses spheres and sphere tangency as only primitives as well as [Pra99, Coh95b] based on parthood (or connection) and convex hull.

Generalizations of mereological approaches to other non-Euclidean spaces have been investigated only in a limited way, for example in hyperbolic spaces [Pam03], affine spaces [Sul71], projective spaces [Sul72] and Hilbert spaces [Sul73].

Regarding the choice of geometrical primitives, most systems in literature use primitives that are inspired by distance considerations like relations "region $x$ can connect region $y$ and $z$" [DL22] and "region $x$ is closer to region $y$ than to region $z$" [VB83]. In other cases the primitives are chosen because already successful to develop Euclidean geometry like congruence, or useful to model forms of connection like tangency.

Another way to look at Tarski's work is to focus on the construction of points in region-based theories or within classes of their structures. In this case the goal is to find which points are forced to exist by a mereological system or can be "defined" within it. From this perspective, much effort has been put in the isolation of points in topological system, thus making point existence dependent on the topology of the space and not vice versa. Literature on this topic from the viewpoint of mereogeometry is reviewed in [Ger95], a modal approach ca be found in [MV95] and other investigations in [Roe97] and [For10].

## 3. Mereogeometrical systems

In [BM10] the authors compared the following mereogeometrical systems discussed in literature:

T1. Tarski's geometry of solids based on (the binary relation of) parthood and the unary predicate '$x$ is a sphere' [Tar56b], [Ben01], [BCTH00];

T2. Borgo, Guarino and Masolo's system based on parthood, the unary predicate '$x$ is a simple region' and the binary relation of congruence [BGM96];

T3. Nicod's system based on parthood and the 4-ary relation '$x, y$ and $z, w$ are conjugates' [Nic24];

T4. De Laguna's approach based on the ternary relation '$x$ can connect $y$ and $z$' [DL22, Don01];

T5. van Benthem's system based on (the binary relation of) connection and the ternary relation '$x$ is closer to $y$ than to $z$' [VB83, AVB97]; and

T6. Cohn and colleagues' theory based on connection and the binary relation '$x$ is the convex hull of $y$' [Coh95a, CBGG97b, CBGG97a].

The comparison concentrates on expressiveness of the primitives and aims to overcome a general problem: most of the mereogeometries available in literature are only weakly formalized and thus cannot be compared as axiomatic systems. The outcome of the comparison shows that the systems T1–T5, when interpreted in the most used structures, namely those based on the regular open (or closed) sets in $\mathbb{R}^n$ (including a variety of restrictions of these), are equivalent from the expressive viewpoint. [Pra99, Dav06] already showed that mereogeometry T6, based on parthood and convexity, is not full and corresponds to the Euclidean subsystem of affine geometry.

We can restate these results as follows

Theorem 1 ([BM10]). (a) T1–T5 *are expressively equivalent in all the studied structures* (*regular open sets, regular closed sets and their restriction to finite and/or connected sets*);

(b) T6 *is expressively a subtheory of the others.*

The theorem holds in $\mathbb{R}^n$ (actually, $\mathbb{E}^n$) for any $n \geq 2$.

Since theory T5, differently from the other theories, has been developed for a non-homogeneous domain, in [BM10] only theories T1–T4 are considered equivalent as the result of logical analysis and comparison of natural (or intended) models. This general notion of equivalence is there called *conceptual*. Thus, recognizing the historical, formal and technical relevance of Tarski's system, we have the following characterization

Definition 1 ([BM10]). A *full mereogeometry* is a theory that is conceptually equivalent to T1.

Corollary 1. T1–T4 *are full mereogeometries.*

As of today, Tarski's geometry of solids remains *the* bridge system between point-based and region-based geometries in the sense that if one aims to compare geometries across these categories, the comparison goes through the Tarski's system or its revisions, e.g., [Ben01].

While recognizing the achievement of the geometry of solids and its role in the study of geometry, in this paper we challenge the intuition that Tarski's choice of primitives is optimal and aim to show that a mereogeometrical study of other standard shapes may provide interesting novelties.

## 4. Tarski's geometry of solids

Whitehead noted that an infinite set of nested regions which converge to a point, can be used to provide a definition of that very point. Tarski builds on Whitehead suggestion to develop his geometry of solids and to give solid logical grounds to the development of space representation in mereological terms.

Tarski starts from Leśniewski's mereology [Les91], the theory of the relation of parthood, writing $P(x, y)$ for "$x$ is part of $y$"; and exploits the expressivity of the geometrical primitive 'being a sphere', writing $S(x)$ for "$x$ is a sphere". Here variables range over regular open sets in the Euclidean space $\mathbb{E}^3$. (We insist that points are not in the domain of quantification.) Note that these primitives and Tarski's definitions work for any Euclidean space $\mathbb{E}^n$ with integer $n \geq 2$; for this reason we will refer to a space of dimension $n$ without further specifications. In terms of $P$ and $S$ a few geometrical relations are defined. This set of definitions allows to characterize a point as *the* set of spheres (that is, the set of $n$-spheres) which are centered at it. One thus take the maximal set of concentric spheres to stand for their center point [Tar56b] and any single sphere in the set as representative of this point [Ben01]. Once one has proven that these identifications are consistent and that concentric spheres form equivalence classes of representatives, sets of spheres can be used to introduce points as defined entities in mereogeometry. It follows that the Euclidean axiom system can be used to axiomatize mereogeometry as well by constraining the previously defined points (sets of regular regions).

We report here the relations given by Tarski in his method to construct points, namely: *external tangency* (ET), *internal tangency* (IT), *external diametricity* (ED), *internal diametricity* (ID), *concentricity* (CC). We do not discuss their correctness but exemplify them via a few figures in dimension 2. Tarski's definitions are fairly intuitive, we hope these figures suffice to gain the reader's trust that the system is well constructed. For an analysis of Tarski's mereogeometry and clarifications on its classes of models see also [GP08]. First, a few standard mereological definitions are introduced:

(D1) $PP(a,b) \stackrel{\text{def}}{=} P(a,b) \wedge \neg P(b,a)$ (*proper part*)

(D2) $O(a,b) \stackrel{\text{def}}{=} \exists c\ [P(c,a) \wedge P(c,b)]$ (*overlap*)

(D3) $DR(a,b) \stackrel{\text{def}}{=} \neg O(a,b)$ (*disjoint*)

(D4) $PO(a,b) \stackrel{\text{def}}{=} O(a,b) \wedge \neg P(a,b) \wedge \neg P(b,a)$ (*proper overlap*)

(D5) provided $O(a,b)$, then
$PROD(a,b,c) \stackrel{\text{def}}{=} \forall w\ [(P(w,a) \wedge P(w,b)) \leftrightarrow P(w,c))]$ (*product*)

(D6) provided $\exists a\, X(a)$, then
$SUM(X,x) \stackrel{\text{def}}{=} \forall a\ [X(a) \to P(a,x)] \wedge \neg\exists b\ [P(b,x) \wedge \forall c[X(c) \to DR(c,b)]]$ (*generalized sum*)

In (D6), $X$ is a second-order variable which can denote any subset of the domain of regions [Ben01]. In this case, $X(y)$ stands for "$y$ belongs to $X$". Note that we use both $a$, $b$, $c$, $\ldots$ and $x$, $y$, $z$ $\ldots$(possibly decorated) as variables on regular regions. At times we also make use of these symbols to refer to Euclidean points; we will clarify these cases in the text.

Tarski's definitions are as follows.

(D7) $ET(a,b) \stackrel{\text{def}}{=} S(a) \wedge S(b) \wedge DR(a,b) \wedge \forall x,y\ [(S(x) \wedge S(y) \wedge P(a,x) \wedge P(a,y) \wedge DR(b,x) \wedge DR(b,y)) \to (P(x,y) \vee P(y,x))]$ (*externally tangent spheres*)

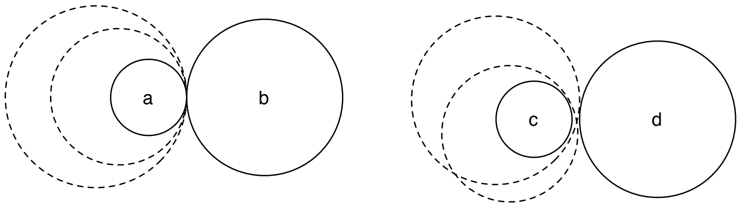



Figure 1. $ET(a, b)$ and $\neg ET(c, d)$, see definition (D7).

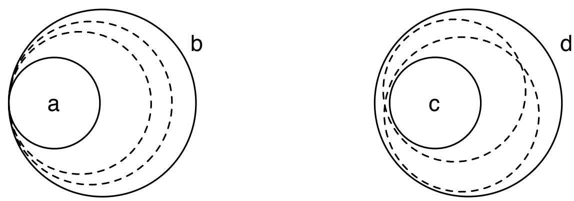



Figure 2. $IT(a, b)$ and $\neg IT(c, d)$, see definition (D8).

(D8) $IT(a,b) \stackrel{\text{def}}{=} S(a) \wedge S(b) \wedge PP(a,b) \wedge$
$\forall x, y\ [(S(x) \wedge S(y) \wedge P(a,x) \wedge P(a,y) \wedge P(x,b) \wedge P(y,b)) \rightarrow$
$(P(x,y) \vee P(y,x))]$ (*internally tangent spheres*)

(D9) $ED(a,b,c) \stackrel{\text{def}}{=} ET(a,c) \wedge ET(b,c) \wedge$
$\forall x, y\ [(S(x) \wedge S(y) \wedge DR(x,c) \wedge DR(y,c) \wedge P(a,x) \wedge P(b,y)) \rightarrow$
$DR(x,y)]$ (*externally diametric spheres*)

(D10) $ID(a,b,c) \stackrel{\text{def}}{=} IT(a,c) \wedge IT(b,c) \wedge$
$\forall x, y\ [(S(x) \wedge S(y) \wedge DR(x,c) \wedge DR(y,c) \wedge ET(a,x) \wedge ET(b,y)) \rightarrow$
$DR(x,y)]$ (*internally diametric spheres*)

(D11) $S_{\odot}(a,b) \stackrel{\text{def}}{=} S(a) \wedge S(b) \wedge (a = b$
$\vee (PP(a,b) \wedge \forall x, y\ [(ED(x,y,a) \wedge IT(x,b) \wedge IT(y,b)) \rightarrow ID(x,y,b)])$
$\vee (PP(b,a) \wedge \forall x, y\ [(ED(x,y,b) \wedge IT(x,a) \wedge IT(y,a)) \rightarrow ID(x,y,a)]))$
(*concentric spheres*)

## 5. Hyper-cubes as geometrical primitives: the system $\mathbf{MG}^n_{HC}$

It is known that Euclidean geometry can be generated starting from a variety of different primitives and we saw that a similar situation holds in mereology in Section 3. Our aim here is to study whether building mereogeometry from the primitive 'being a sphere' gives some advantage with respect to other predicates. In particular we show two things: (1) how to define points in a Tarskian fashion independently of the sphere

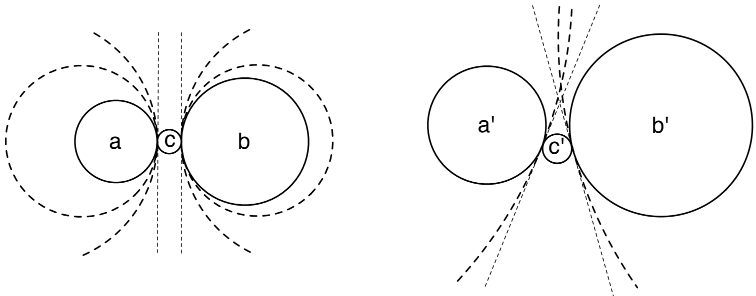



Figure 3. $ED(a, b, c)$ and $\neg ED(a', b', c')$, see definition (D9). Dotted lines mark the tangents at the points of contact.

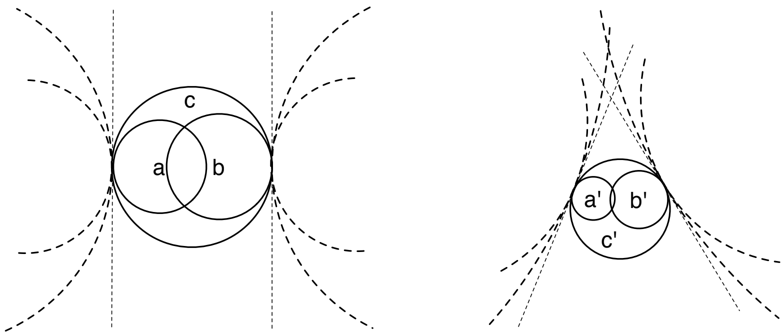



Figure 4. $ID(a, b, c)$ and $\neg ID(a', b', c')$, see definition (D10). Dotted lines mark the tangents at the points of contact.

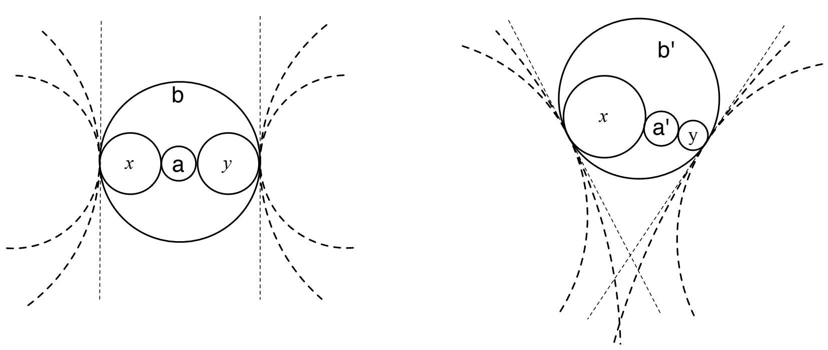



Figure 5. $S_{\odot}(a, b)$ and $\neg S_{\odot}(a', b')$, see definition (D11). Dotted lines mark the tangents at the points of contact.

primitive and (2) that the sphere primitive is not the best choice (for several reasons) for this kind of construction. It is worth noting that by relinquishing spheres we do not have at our disposal an important geometrical feature of spheres: the simple and direct definition of (ternary) equidistance. This feature, key element in Euclidean geometry, has likely played an important role in Tarski's selection of 'being a sphere' as the primitive of his system.

D. Scott [Sco56] showed that a ternary symmetric relation $\mathcal{S}_{Euc}(x, y, z)$, holding if the points $x, y$ and $z$ are the vertices of a right triangle, suffices as a primitive for Euclidean geometry. Since, as mentioned earlier, Euclidean geometry and mereogeometry can be seen as two geometrical languages which aim to model the *same* commonsense space but with different ontological commitments, it makes sense to investigate Scott's result in mereogeometry and see if it is possible to build mereogeometry starting from a symmetric figure based on right angles like the hyper-cube, perhaps even as the only primitive for mereogeometry.

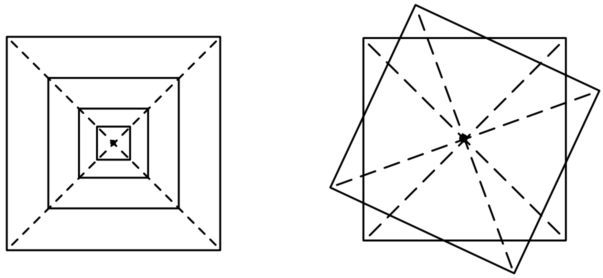
Our first goal is to investigate the notion of hyper-cube as geometrical primitive and provide a suitable way to define points in the resulting system according to Whitehead's intuition. We thus assume the standard first-order logic language with $P$ (binary) and $HCb$ (unary) as the only non-logical primitives with the following informal reading: $P(x, y)$ stands for '$x$ *is part of* $y$' and $HCb(x)$ for '$x$ *is a hyper-cube*.'

As noted in Section 4, the goal of the technique developed by Tarski is to provide a definition of points as maximal sets of concentric spheres, i.e. as second order entities. In our case, these would be maximal sets of concentric hyper-cubes. For this, we need to characterize concentric hyper-cubes in a unique way, i.e., we need to be able to define when two hyper-cubes have the same center which, in Euclidean lingo, means that the diagonals of both hyper-cubes intersect in exactly the same point. For dimension 2 this is illustrated in Figure 6.

Our first definition will consider the case in Figure 6(a), that is, concentric hyper-cubes with parallel sides hereafter called *concentric aligned hyper-cubes.* We begin with an auxiliary definition to characterize *co-vertex hyper-cubes* as depicted by Figure 7 in the two-dimensional cases.

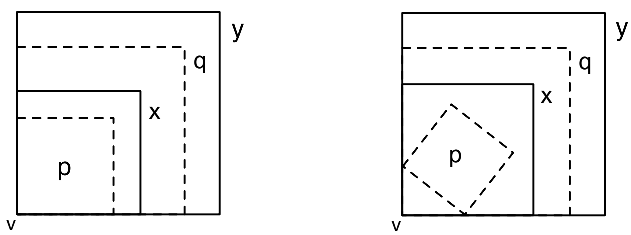
(D12) $HCb_{\boxdot}(x, y) \overset{\text{def}}{=} HCb(x) \wedge HCb(y) \wedge PP(x, y) \wedge \exists p\, [HCb(p) \wedge PP(p, x) \wedge \quad \forall q\, [(HCb(q) \wedge P(p, q) \wedge P(q, y)) \rightarrow (P(q, x) \vee P(x, q))]]$

(*co-vertex hyper-cubes*)



(a) (b)

Figure 6. Aligned (a) and rotated (b) concentric hyper-cubes in dimension 2 ($HCb_{\boxdot}$).




(a) (b)

Figure 7. Co-vertex (at v) hyper-cubes in dimension 2.

Definition (D12) says that hyper-cube $x$ shares a vertex with hyper-cube $y$ whenever $x$ is a proper part of $y$ and there exists a hyper-cube $p$, proper part of $x$, such that any hyper-cube containing $p$ and contained in $y$ either contains $x$ or is part of $x$. By forcing an ordering on hyper-cubes, (D12) ensures that we can select hyper-cubes which have a common 'corner' region.

The following definition is much in the spirit of Tarski's construction and corresponds to the case in Figure 6(a). The way the definition works is depicted in Figure 8.

(D13) $HCb_{\boxtimes}(x, y) \stackrel{\text{def}}{=} HCb(x) \wedge HCb(y) \wedge (x = y \vee$
$(P(x, y) \wedge \forall q, z\, [(HCb_{\boxed{\llcorner}}(q, y) \wedge O(q, x) \wedge PROD(q, x, z)) \rightarrow HCb(z)]) \vee$
$(P(y, x) \wedge \forall q, z\, [(HCb_{\boxed{\llcorner}}(q, x) \wedge O(q, y) \wedge PROD(q, y, z)) \rightarrow HCb(z)]))$
(*aligned concentric hyper-cubes*)

The definition states that two hyper-cubes are concentric when identical or one (say $x$) is part of the other (say $y$) and the region in common between $x$ and a hyper-cube co-vertex with $y$, if any, is a hyper-cube.

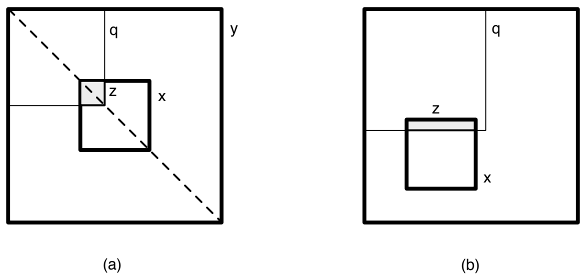



Figure 8. (a) Aligned concentric hyper-cubes in dimension 2 ($HCb_{\boxtimes}$). (b) non concentric hyper-cubes in dimension 2.

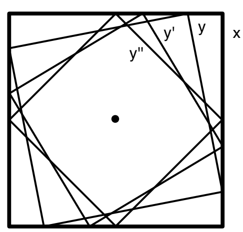



Figure 9. Concentric hyper-cubes with vertices on the boundary of $x$ in dimension 2.

Briefly stated, definition (D13) forces the inside square to have center on the hyper-cube's diagonal relative to the vertex common to both $y$ and $q$ (see Figure 8). Since the quantification is on any co-vertex hyper-cube of $y$, the inner square must be centered where the diagonals meet.

Points must be uniquely identified in the filter construction, thus the construction needs to include any hyper-cube centered at the 'same' Euclidean point. It remains to characterize those hyper-cubes that, according to our terminology, are concentric to the given one but not aligned to it. We now focus on the subclass of concentric non-aligned hyper-cubes (Figure 6(b)) whose vertices are on the boundary of a given hyper-cube, see Figure 9. Call these *concentric rotated hyper-cubes.*

(D14) $HCb_{\circlearrowright}(x, y) \stackrel{\text{def}}{=} HCb(x) \wedge HCb(y) \wedge PP(x, y) \wedge$
$\forall q\ [(HCb(q) \wedge PP(q, y) \wedge P(x, q)) \rightarrow P(q, x)]$
(*rotated concentric hyper-cubes*)

This definition, (D14), says that two hyper-cubes are concentric and rotated, in the sense described earlier, whenever there is no hyper-cube proper part of the largest and properly containing the smallest. (This definition takes advantage of a well-known property at the core of one standard proof of Pythagoras' theorem).

At this point, we have all elements to define concentric hyper-cubes. The following relation holds for all and only the pairs of concentric hyper-cubes; it is thus suitable to define points as filters in the spirit of Whitehead's suggestion.

$$\text{(D15)}\quad HCb_{\boxdot}(x,y) \stackrel{\text{def}}{=} HCb_{\boxtimes}(x,y) \vee \exists z\ [HCb_{\circlearrowright}(z,x) \wedge HCb_{\boxtimes}(z,y)] \vee \exists z\ [HCb_{\circlearrowright}(z,y) \wedge HCb_{\boxtimes}(z,x)]$$
$$(\textit{concentric hyper-cubes})$$

Definition (D15) says that two hyper-cubes are concentric whenever they are aligned concentric or there is a hyper-cube rotated concentric with respect to one of them which is at the same time aligned concentric with the other.

In the rest of this paper, we call $\mathrm{MG}^n_{HC}$ *the* full mereogeometry whose language is the language of first-order logic with $P$ (parthood) and $HCb$ ($n$-hyper-cube) as the only non logical primitives. The fixed value $n$ ($n \geq 2$) is the dimension of the space. Our next goal is to make the logical system precise by providing an axiomatization.

## 6. A direct axiomatization of $\mathrm{MG}^n_{HC}$

In this section we give the axiomatization of our version of region-based geometry and discuss the proof of categoricity of the system. Our axiom system and the verification of its properties follow closely the work of Bennett [Ben01] with some changes aimed to take into account the specificity of our geometric primitive. We concentrate on the parts where the proof departs from [Ben01] and gives only an outline of the rest. For further discussions on Tarski's axiomatization and related properties, in particular on the categorization for different classes of structures, we refer the reader to [GP08].

We use the expression 'Euclidean geometry' to mean the axiomatic system called *elementary Euclidean geometry* in [Tar59]. The specific layout of Tarski's system is not important for the comprehension of the material in this paper. It will suffice to note that, in comparison to the work of Hilbert [Hil71], Tarski's Euclidean system (including a weak version of the continuity axiom) is first order and the domain of quantification contains only points.

The guiding idea to interlace point-based and region-based geometries takes a sphere as representative of its center point. This choice, taken in [Ben01], allows to use any standard axiomatization of Euclidean geometry, in first order logic (with some version of the axiom of continuity) as a guideline for the axiomatization of region-based geometry. Here, of course, we take hyper-cubes to play the role of Bennett's spheres while following the axiomatization provided by Tarski in [Tar59]. It should be noticed that Tarski's axiomatization has been later improved by simplifying the axiomatization [Gup65, TG99] and that one could further reduce the language to use only ternary relations [Tar56a, Rob59]. Notwithstanding these observations and some criticisms ("Geometrically, Tarski-elementary plane geometry certainly seems mysterious" [Gre10, p. 215]) the formal system proposed in [Tar59] is fundamental on three aspects: it is known to be consistent (Hilbert and Bernays proved consistency for their geometry but without the continuity axiom [HB70]), deductively complete and decidable [TG99, Gre10]. Indeed, it remains the reference work in first-order Euclidean geometry and, since later improvements are irrelevant to our work, we will follow it in its original formulation.

We first give an auxiliary definition to characterize two $hyper-cubes$ each having a diagonal laying on the line through the two $hyper-cubes$'s centers.

(D16) $$HCb_{\diamond\!\!\diamond}(x,y) \stackrel{\text{def}}{=} \neg HCb_{\boxdot}(x,y) \wedge \forall x', y', z\ [(HCb_{\boxtimes}(x,x') \wedge HCb_{\boxtimes}(y,y') \wedge PO(x',y') \wedge PROD(x',y',z)) \rightarrow HCb(z)]$$
(*hyper-cubes with diagonals on same line*)

In short definition (D16) considers two non-concentric hyper-cubes such that any pair of overlapping hyper-cubes, each aligned concentric to one of them, has a hyper-cube as product.

We proceed by defining some relations inspired by elementary Euclidean geometry like betweenness and equidistance, plus other auxiliary notions. To keep the presentation simple, from now on we also use some standard operators, e.g., we will write $x \cdot y$ for a region $z$ such that $PROD(x,y,z)$ so that the previous definition would be written: $HCb_{\diamond\!\!\diamond}(x,y) \stackrel{\text{def}}{=} \neg HCb_{\boxdot}(x,y) \wedge \forall x', y'\ [(HCb_{\boxtimes}(x,x') \wedge HCb_{\boxtimes}(y,y') \wedge PO(x',y')) \rightarrow HCb(x' \cdot y')]$

(D17) $BTW(x,y,z) \stackrel{\text{def}}{=} HCb_{\boxdot}(x,y) \vee HCb_{\boxdot}(y,z) \vee$
$\exists x', y', z' \; [HCb_{\boxdot}(x,x') \wedge HCb_{\boxdot}(y,y') \wedge HCb_{\boxdot}(z,z') \wedge DR(x',z') \wedge$
$PO(x',y') \wedge PO(y',z') \wedge HCb_{\otimes}(x',y') \wedge HCb_{\otimes}(y',z') \wedge HCb_{\otimes}(x',z')]$
(*hyper-cube betweenness*)

Whenever the squares have distinct centers in (D17), condition $HCb(x' \cdot y')$, enforced by $HCb_{\otimes}(x',y')$, ensures that the line from the centers of $x'$ and $y'$ is one of their diagonals. Then, since diagonals in hyper-cubes are orthogonal, conditions $HCb(y' \cdot z')$ and $HCb(x' \cdot z')$ ensure that the center of $z$ is also on the same diagonal.

The following definition is self-explicative (adapted from [Ben01]).

(D18) $COB(x,y) \stackrel{\text{def}}{=} HCb(x) \wedge \forall x' \; [HCb_{\boxdot}(x,x') \rightarrow (O(x',y) \wedge \neg P(x',y))]$
(*center of $x$ on the boundary of $y$*)

Next we define when two hyper-cubes have centers equidistant from that of a third hyper-cube. First, we cover the linear case by defining when the center of a hyper-cube is the middle point of the centers of two other hyper-cubes.

(D19) $MID(x,y,z) \stackrel{\text{def}}{=} (HCb_{\boxdot}(x,y) \wedge HCb_{\boxdot}(y,z)) \vee$
$(BTW(x,y,z) \wedge \exists y'[HCb_{\boxdot}(y,y') \wedge COB(x,y') \wedge COB(z,y')])$
(*center of $y$ is aligned and equidistant from those of $x, z$*)

(D20) $EQD(x,y,z) \stackrel{\text{def}}{=} HCb_{\boxdot}(x,y) \vee MID(x,z,y) \vee$
$\exists w, x', y', z' \; [MID(x,w,y) \wedge HCb_{\boxdot}(x,x') \wedge HCb_{\boxdot}(y,y') \wedge$
$HCb_{\boxdot}(z,z') \wedge HCb_{\otimes}(x',w) \wedge HCb_{\otimes}(y',w) \wedge HCb_{\otimes}(z',w) \wedge$
$\neg HCb_{\otimes}(x',z') \wedge \neg HCb_{\otimes}(y',z')]$
(*centers of $x, y$ equidistant from that of $z$*)

According to (D20), the centers of $x$ and $y$, when distinct, are equidistant from $z$ if the latter is at their midpoint or is aligned with a diagonal of the hyper-cube at the midpoint which has $x, y$ aligned with another diagonal.

Following [Ben01], we now define the quaternary equidistance and nearer relations: $EQD(w,x,y,z)$ states that the distance of the centers of hyper-cubes $w, x$ is equal to that of the centers of $y, z$; $NEARER(w,x,y,z)$ holds when the centers of the first two hyper-cubes are at a closer distance than the other two hyper-cubes.

(D21) $EQD(w,x,y,z) \stackrel{\text{def}}{=} \exists u, v \; [MID(w,u,y) \wedge MID(x,u,v) \wedge$
$EQD(v,z,y)]$ (*centers of $w, x$ and of $y, z$ are equidistant*)

(D22) $NEARER(w, x, y, z) \overset{\text{def}}{=} \exists x' \; [BTW(w, x, x') \wedge \neg HCb_{\boxdot}(x, x') \wedge EQD(w, x', y, z)]$ (*centers of* $w, x$ *are closer than those of* $y, z$)

Finally, we need a relation to constrain the existence of all hyper-cubes one can define in $\mathbb{E}^n$. [Ben01] relies on the *COI* relation which has two roles: to constrain the existence of spheres and to state which spheres have center within a region. This definition will play a similar role in our system.

(D23) $COI(x, y) \overset{\text{def}}{=} \exists x' [HCb_{\boxdot}(x, x') \wedge P(x', y)]$ (*center of* $x$ *in interior of* $y$)

The next step is to adapt the axiomatization in [Tar59] along the lines of [Ben01]. The idea is to axiomatize the system based on hyper-cubes by taking the axiomatization of elementary Euclidean geometry as a guideline. The main change with respect to [Tar59] and [Ben01] is in the range of the variables: quantifiers in Tarski's system range over maximal sets of concentric spheres (informally, points); in Bennett's system over spheres (informally, representatives of their center points); in the system below they range over hyper-cubes (informally, representatives of their centers).

The axioms given below (*AM* stands for 'Axiom of Mereogeometry') furnish a partial axiomatization and are listed without comments since they do not introduce any relevant novelty compared to [Ben01]. The axioms of the following sections will present relevant changes and will be discussed in detail. Note that equality relations among points stated in [Tar59] are here expressed via the concentric relation ($HCb_{\boxdot}$) among hyper-cubes.

(AM1) $\forall x, y, z \; [(P(x, y) \wedge P(y, z)) \rightarrow P(x, z)]$
(AM2) $\forall X \; [\exists x \; [X(x)] \rightarrow \exists! x \; [SUM(X, x)]]$
(AM3) $\forall x, y \; (BTW(x, y, x) \rightarrow HCb_{\boxdot}(x, y))$
(AM4) $\forall x, y, z, u \; [(BTW(x, y, u) \wedge BTW(y, z, u)) \rightarrow BTW(x, y, z)]$
(AM5) $\forall x, y, z, u \; [(BTW(x, y, z) \wedge BTW(x, y, u) \wedge \neg HCb_{\boxdot}(x, y)) \rightarrow (BTW(x, z, u) \vee BTW(x, u, z))]$
(AM6) $\forall x, y \; [(HCb(x) \wedge HCb(y)) \rightarrow EQD(x, y, y, x)]$
(AM7) $\forall x, y \; [EQD(x, y, z, z) \rightarrow HCb_{\boxdot}(x, y)]$
(AM8) $\forall x, y, z, u, v, w \; [(EQD(x, y, z, u) \wedge EQD(x, y, v, w)) \rightarrow EQD(z, u, v, w)]$
(AM9) $\forall t, x, y, z, u \exists v \; [(BTW(x, t, u) \wedge BTW(y, u, z)) \rightarrow (BTW(x, v, y) \wedge BTW(z, t, v))]$

(AM10) $\forall t, x, y, z, u \exists v, w\ [(BTW(x,t,u) \wedge BTW(y,u,z) \wedge \neg HCb_{\boxdot}(x,y)) \rightarrow (BTW(x,z,v) \wedge BTW(x,y,w) \wedge BTW(v,t,w))]$

(AM11) $\forall x, x', y, y', z, z', u, u'\ [(BTW(x,y,x) \wedge BTW(x',y',z') \wedge \neg HCb_{\boxdot}(x,y) \wedge EQD(x,y,x',y') \wedge EQD(y,z,y',z') \wedge EQD(x,u,x',u') \wedge EQD(y,u,y',u')) \rightarrow EQD(z,u,z',u')]$

(AM12) $\forall x_1, x_2, x_3, x_4 \exists y\ [\bigwedge_i HCb(x_i) \wedge BTW(x_1,x_2,y) \wedge EQD(x_2,y,x_3,x_4)]$

(AM13) $\forall X, Y\ [\exists z \forall x, y\ [(HCb(x) \wedge HCb(y) \wedge X(x) \wedge Y(y)) \rightarrow BTW(z,x,y)] \rightarrow \exists z \forall x, y\ [(HCb(x) \wedge HCb(y) \wedge X(x) \wedge Y(y)) \rightarrow BTW(w,z,y)]]$

(AM14) $\forall x, y, z\ [(HCb_{\boxdot}(x,y) \wedge HCb_{\boxdot}(y,z)) \rightarrow HCb_{\boxdot}(x,z)]$

(AM15) $\forall x, x', y, z, w\ [(EQD(x,y,z,w) \wedge HCb_{\boxdot}(x,x')) \rightarrow EQD(x',y,z,w)]$

(AM16)$^n$ $\exists x_0, ..., x_n\ [\bigwedge_{0 \le i \ne j \ne k \le n}(HCb(x_i) \wedge \neg HCb_{\boxdot}(x_i,x_j) \wedge EQD(x_i,x_j,x_j,x_k))] \wedge \neg \exists x_0, ..., x_{n+1}\ [\bigwedge_{0 \le i \ne j \ne k \le n+1}(HCb(x_i) \wedge \neg HCb_{\boxdot}(x_i,x_j) \wedge EQD(x_i,x_j,x_j,x_k))]$

### 6.1. Geometrical models

Given the above partial axiomatization, we begin to discuss the models of the system. Here our job is fairly easy since, after proving that our approach leads to a definition of point equivalent to that given by Tarski and after adding some changes already in [Ben01], we can safely mimic these works to interpret our system. The specificity of our geometrical primitive will jump in when dealing with hyper-cube models, the models that in this work have a role similar to sphere models in [Ben01].

Let $\mathrm{MG}^n_{HC(\downarrow)}$ be the mereogeometrical system consisting of the axioms (AM3)-(AM15) plus the axiom (AM16)$^n$ adopted to fix the dimension $n$ ($\geq 2$) of the space.

DEFINITION 2. Let $\mathcal{HC}!^n = \langle C, HCb_{\boxdot}, BTW, EQD \rangle$ be a structure satisfying the set of axioms $\mathrm{MG}^n_{HC(\downarrow)}$ plus the following:

(A1) $\forall x\ [HCb(x)]$

(A2) $\forall x, y\ [HCb_{\boxdot}(x,y) \leftrightarrow x = y]$

Following [Ben01], we call $\mathcal{HC}!^n$ a *(hyper-cube) geometrical model.*

Thus, every element in the domain $C$ of $\mathcal{HC}!^n$ is a hyper-cube and there are no concentric hyper-cubes. Informally, one can build such a model by taking as domain a restriction of the set of hyper-cubes in $\mathbb{R}^n$. This intuition is formalized in our first Lemma (the Lemmas in this section are restated from [Ben01] and the proofs require only trivial adaptations).

LEMMA 1. *Every structure $\mathcal{HC}!^n$ is isomorphic to the structure $\langle \mathbb{R}^n, HCb_{\boxdot}, BTW, EQD\rangle$, where each element is identified with a (coordinate) tuple in $\mathbb{R}^n$, $HCb_{\boxdot}$ is the identity relation, $BTW$ is the betweenness relation and $EQD$ the equidistance relation with their usual algebraic definitions in terms of the coordinate tuples.*

DEFINITION 3. Let $\mathcal{HC}^n_r = \langle D, HCb_{\boxdot}, BTW, EQD\rangle$ be a structure satisfying the set of axioms $\mathrm{MG}^n_{HC(\downarrow)}$, (A1) and:

(A3) $\forall x, y, z\ [(HCb_{\boxdot}(x, y) \wedge HCb_{\boxdot}(y, z)) \to HCb_{\boxdot}(x, z)]$

Following [Ben01], we call $\mathcal{HC}^n_r$ a *(hyper-cube) relaxed geometrical model.*

From definition (D15), relation $HCb_{\boxdot}$ is reflexive and symmetric. Then, $HCb_{\boxdot}$ in $\mathcal{HC}^n_r$ is an equivalence relation.

LEMMA 2. *Given a structure $\mathcal{HC}^n_r$, let $(d_1, \ldots, d_i, \ldots, d_j) \in D^j$ and $HCb_{\boxdot}(d_i, d'_i)$. For any first-order formula $\phi(x_1, \ldots, x_j)$ in the signature of $\mathcal{HC}^n_r$ without equality:*

$$\mathcal{HC}^n_r \models \phi(d_1, \ldots, d_i, \ldots, d_j) \text{ if and only if } \mathcal{HC}^n_r \models \phi(d_1, \ldots, d'_i, \ldots, d_j).$$

Given a relaxed geometrical model $\mathcal{HC}^n_r = \langle D, HCb_{\boxdot}, BTW, EQD\rangle$, let $\mathcal{HC}^n_{\equiv}$ be the structure $\langle D_{\equiv}, HCb_{\boxdot}, BTW, EQD\rangle$ obtained from $\mathcal{HC}^n_r$ by restricting the structure to the quotient of $D$ over relation $HCb_{\boxdot}$, i.e., $D_{\equiv}$ contains only one element (a representative) for each $HCb_{\boxdot}$-equivalence class in $D$.

DEFINITION 4. Given a structure $\mathcal{HC}^n_r$ with domain $D$ and a structure $\mathcal{HC}^n_{\equiv}$ with domain $D_{\equiv}$, a surjective function $\mu : D \to D_{\equiv}$ such that $\mu(x) \in \{y \in D \mid HCb_{\boxdot}(x, y)\}$ and $\mu(x) = \mu(y)$ whenever $HCb_{\boxdot}(x, y)$, is called *minimisation* function and $\mathcal{HC}^n_{\equiv}$ a *minimal (geometrical) substructure* of $\mathcal{HC}^n_r$ [Ben01].

Note that $\mathcal{HC}^n_r$ and $\mathcal{HC}^n_{\equiv}$ have the same signature.

LEMMA 3. *Given a structure $\mathcal{HC}^n_r$ and a minimal substructure $\mathcal{HC}^n_{\equiv}$ with minimisation function $\mu$, then for any first-order formula $\phi(x_1, \ldots, x_j)$ in the signature of $\mathcal{HC}^n_r$ without equality:*

$$\mathcal{HC}^n_r \models \phi(d_1, \ldots, d_j) \text{ if and only if } \mathcal{HC}^n_{\equiv} \models \phi(\mu(d_1), \ldots, \mu(d_j)).$$

Now we can make formal the reason to select these models. Geometrical, relaxed and minimal models differ only in the set of concentric hyper-cubes they include per point and there is a surjective map to any structure for Tarski's Euclidean geometry in $\mathbb{R}^n$.

LEMMA 4. *Any minimal substructure $\mathcal{HC}^n_{\equiv}$ of a structure $\mathcal{HC}^n_r$ is isomorphic to $\mathcal{HC}!^n$.*

Let $\mathcal{E}^n = \langle \mathbb{R}^n, B, Eqd \rangle$, be a structure for the Tarskian theory of Euclidean geometry with $B$ the betweenness relation and $Eqd$ equidistance.

DEFINITION 5. Given a structure $\mathcal{E}^n$ and a structure $\mathcal{HC}^n_r$ with domain $D$, a surjective function $\pi \colon D \to \mathbb{R}^n$ such that

- if $\mathcal{HC}^n_r \models HCb_{\boxdot}(x_1, x_2)$ then $\mathcal{E}^n \models \pi(x_1) = \pi(x_2)$
- if $\mathcal{HC}^n_r \models BTW(x_1, x_2, x_3)$ then $\mathcal{E}^n \models B(\pi(x_1), \pi(x_2), \pi(x_3))$
- if $\mathcal{HC}^n_r \models EQD(x_1, x_2, x_3, x_4)$ then $\mathcal{E}^n \models Eqd(\pi(x_1), \pi(x_2), \pi(x_3), \pi(x_4))$

is called *Cartesian centre-point interpretation* or *CCPI-function* for short [Ben01].

LEMMA 5. *Fix a geometrical model $\mathcal{HC}^n_r$ with domain $D$, there is a CCPI-function $\pi \colon D \to \mathbb{R}^n$.*

Given a CCPI-function $\pi$, we will write $x^\pi$ for the point $\pi(x) \in \mathcal{E}^n$.

### 6.2. Hyper-Cube models

This section is more detailed since we depart at times from the work in [Ben01]. The changes are due to the characterization of the class of models for hyper-cubes based on our definition of point, and to the point-based geometrical description of hyper-cubes. The axioms we will include at this stage are more complicated than those in [Ben01] since the geometrical properties of hyper-cubes are harder to capture via the Euclidean relations $B$ and $Eqd$ available in structure $\mathcal{E}^n$.

More specifically, below we introduce two new axioms, (AM17) and (AM18), corresponding to (A17) and (A18) in [Ben01], with the aim to constrain the existence and construction of hyper-cubes. Differently from spheres, hyper-cubes cannot be directly constrained via the relative distance relation *NEARER* alone, thus we exploit a larger set of constraints. This effort leads to the notion of hyper-cube as captured by relation $HCVerts_{B,Eqd}$ in definition (D24). Axioms (AM17) and (AM18) build on this relation and relay on (a) the convexity property of hyper-cubes and (b) the properties of the projection of (internal) points to the hyper-cube's surface.

In this section we write $\sigma$ for a permutation of set $\{1, \ldots, 2^n\}$ and $\sigma_{(1 \to i)}$ for a permutation of the set such that $\sigma(1) = i$. Also, sometimes we write $\sigma_j$ for $\sigma(j)$, i.e., the image of index $j$ under a given permutation $\sigma$.

The following definition characterizes the vertices of a hyper-cube in dimension $n$ via the use of relations $BTW$ and $EQD$ only. (The restriction to language $\{BTW, EQD\}$ is important to refer to models $\mathcal{E}^n$. Also, recall definition (D22) showing that $NEARER$ is syntactically defined in this language.) The correctness of this definition is the goal of Lemma 7, see below.

(D24) $$HCVerts_{B,Eqd}(y_1, \ldots, y_{2^n}) \stackrel{\text{def}}{=} \bigwedge_{1 \leq i \leq 2^n - 2} EQD(y_i, y_{i+1}, y_{i+1}, y_{i+2}) \wedge EQD(y_1, y_2, y_1, y_{2^n}) \wedge \bigwedge_{i \neq j} \neg NEARER(y_i, y_j, y_1, y_2) \wedge \exists w [\bigwedge_{2 \leq i \leq 2^n} EQD(w, y_1, w, y_i) \wedge \bigwedge_i \bigvee_{j \neq i} BTW(y_i, w, y_j)] \wedge \bigwedge_i \exists \sigma_{(1 \to i)} \ [\bigwedge_{2 \leq i \neq j \neq k \leq n+1} EQD(y_{\sigma_i}, y_{\sigma_j}, y_{\sigma_i}, y_{\sigma_k}) \wedge \bigwedge_{n+2 \leq l \leq 2^n} NEARER(y_{\sigma_1}, y_{\sigma_2}, y_{\sigma_1}, y_{\sigma_k}) \wedge \bigwedge_{2 \leq j \leq n+1} EQD(y_{\sigma_1}, y_{\sigma_2}, y_{\sigma_1}, y_{\sigma_j}) \wedge \bigwedge_{2 \leq j \neq k \leq n+1} \exists u [BTW(y_{\sigma_j}, u, y_{\sigma_k}) \wedge EQD(u, y_{\sigma_1}, u, y_{\sigma_j}) \wedge EQD(u, y_{\sigma_1}, u, y_{\sigma_k})]]$$
($y_1, \ldots, y_{2^n}$ *are ordered vertices of a hyper-cube*)

Definition (D24) needs an explanation. Let us assume informally that $dist$ is a distance function with $dist(a, b)$ the distance between points $a, b$, then the definition says that points $y_1, \ldots, y_{2^n}$ (in this order) are suitable vertices of a $hyper - cube$ in dimension $n$ provided: (a) none is closer than $dist(y_1, y_2)$ and there is a path covering all of them with edges of length exactly $dist(y_1, y_2)$; (b) there exists a point $w$ equidistant to all the points $y_i$ and for each $y_i$ there is $y_j$ such that $w$ is between them (thus $w$ is midpoint for all these pairs $y_i, y_j$ and so, informally, $w$ is the center of the hyper-cube); (c) for each $i$, there is a permutation that reorder the points so that $y_i$ is the first and the $n$ following points (positions from 2 to $n + 1$) are all (and only) the points at distance $dist(y_1, y_2)$ from $y_i$ (informally speaking, this says that any vertex has exactly $n$ adjacent vertices); finally (d) the midpoint of any two of these closest vertices $y_j, y_k$ is the center of a disk through $y_i, y_j, y_k$, which forces the corresponding triangle $y_i, y_j, y_k$ to have a right angle at $y_i$.

DEFINITION 6. We call *hyper-cube model* any structure of type $\mathfrak{G}^n_{HC} = \langle C, HCb_{\boxdot}, BTW, EQD, COI\rangle$ which satisfies the conditions for $\mathcal{HC}^n_r$ and the following:

$$\text{(AM17)}\ \forall y_1, y_2, ..., y_{2^n}\ [HCVerts_{B,Eqd}(y_1, \ldots, y_{2^n}) \to \exists w \forall u\ [HCb(u) \to [COI(u,w) \leftrightarrow \forall \sigma \exists m_1, ..., m_{2^n-1}\ [\neg HCb_{\boxdot}(u, m_1) \wedge BTW(y_{\sigma(1)}, u, m_1) \wedge \bigwedge_{2\leq i\leq 2^n-1} BTW(y_{\sigma(i)}, m_{i-1}, m_i) \wedge \bigvee_j HCb_{\boxdot}(y_{\sigma(j)}, m_{2^n-1})]]]]$$

$$\text{(AM18)}\ \forall x \exists y_1, \ldots, y_{2^n}\ [HCb(x) \to (HCVerts_{B,Eqd}(y_1, \ldots, y_{2^n}) \wedge \forall u[HCb(u) \to [COI(u,x) \leftrightarrow \forall \sigma \exists m_1, ..., m_{2^n-1}\ [\neg HCb_{\boxdot}(u, m_1) \wedge BTW(y_{\sigma_1}, u, m_1) \wedge \bigwedge_{2\leq i\leq 2^n-1} BTW(y_{\sigma_i}, m_{i-1}, m_i) \wedge \bigvee_j HCb_{\boxdot}(y_{\sigma_j}, m_{2^n-1})]]])]$$

$$\text{(A4)}\ \forall x, y\ [\forall z\ [COI(z,x) \leftrightarrow COI(z,y)] \to x = y]$$

LEMMA 6. *In any hyper-cube model with a CCPI-function* $\pi$*, the defined relation* $NEARER(w,x,y,z)$ *holds for tupla* $\langle a,b,c,d\rangle$ *just in case, for* $\delta$ *a distance relation in* $\mathbb{R}^n$*, we have* $\delta(a^\pi, b^\pi) < \delta(c^\pi, d^\pi)$*.*

PROOF. By definition, a hyper-cube model is a relaxed geometrical model as well. By Lemma 5 there exists a CCPI-function $\pi\colon \mathcal{C} \to \mathbb{R}^n$ such that any $j$-relation $R$ in the language $\{HCb_{\boxdot}, BTW, EQD\}$ holds for $\langle x_1, \ldots, x_j\rangle$ in $(\mathfrak{G}^n_{HC})^j$ only if $\langle x_1^\pi, \ldots, x_j^\pi\rangle$ in $(\mathbb{R}^n)^j$ satisfies the standard interpretation of $R$ in $\mathbb{R}^n$. By (D22), relation $NEARER(w,x,y,z)$ holds if and only if $\exists x'\ [BTW(w,x,x') \wedge \neg HCb_{\boxdot}(x,x') \wedge EQD(w,x',y,z)]$, that is, if and only if there exists $x'$ such that $B(w^\pi, x^\pi, x'^\pi) \wedge x^\pi \neq x'^\pi \wedge Eqd(w^\pi, x'^\pi, y^\pi, z^\pi)$ which, in the Euclidean system $\mathcal{E}^n$, implies the claim. □

By definition, a hyper-cube model $\mathfrak{G}^n_{HC}$ is a $\mathcal{HC}^n_r$ model expanded with a new relation. Thus, $\mathfrak{G}^n_{HC}$ has a CCPI-function.

LEMMA 7. *Relation* $HCVerts_{B,Eqd}$ *holds in a hyper-cube model* $\mathfrak{G}^n_{HC}$ *with CCPI-function* $\pi$ *for tupla* $\langle b_1, \ldots, b_{2^n}\rangle$ *just in case* $b_1^\pi, \ldots, b_{2^n}^\pi$ *are the vertices of a hyper-cube.*

PROOF. As before, a CCPI-function $\pi\colon C \to \mathbb{R}^n$ exists since $\mathfrak{G}^n_{HC}$ is a relaxed geometrical model. From Definition 5, $\pi$ constraints the interpretation of $HCb_{\boxdot}$, $BTW$ and $EQD$, and from Lemma 6, this tells us that relation $NEARER$ has the intended interpretation. Thus, whether a tuple $\langle b_1, \ldots, b_{2^n}\rangle$ satisfies $HCVerts_{B,Eqd}$ is also completely determined by the corresponding points $b_1^\pi, \ldots, b_{2^n}^\pi$.

Let now show that relation $HCVerts_{B,Eqd}$ holds for $\langle b_1, \ldots, b_{2^n}\rangle$ only if the points $b_1^\pi, \ldots, b_{2^n}^\pi$ are vertices of a hyper-cube. We divide the constraints in (D24) in five conditions, from (a) to (e):

(a) The initial conditions $\bigwedge_{1\leq i\leq 2^n-2} EQD(b_i, b_{i+1}, b_{i+1}, b_{i+2})$, $EQD(b_1, b_2, b_1, b_{2^n})$ and $\bigwedge_{i\neq j} \neg NEARER(b_i, b_j, b_1, b_2)$ state that there is a close (simple) path through all points $b_i^\pi$ such that each pair of connected $b_i^\pi$ are at the same distance and this is the minimal distance between any two points $b_i^\pi$.

(b) Next, the existence of $a$ satisfying $\bigwedge_{2\leq i\leq 2^n} EQD(a, b_1, a, b_i)$ is ensured. This simply states that there is a point $a^\pi$ such that all points $b_i^\pi$ are on the $(n-1)$-sphere centered at $a^\pi$.

(c) Subformula $\bigwedge_i \bigvee_{j\neq i} BTW(b_i, a, b_j)$, where $a$ is given by the previous condition, says that for each point $b_i^\pi$ there is a point $b_j^\pi$ such that $a^\pi$ is midpoint of segment $\overline{b_i^\pi\, b_j^\pi}$. In other words, $\overline{b_i^\pi\, b_j^\pi}$ is a diameter of the $(n-1)$-sphere centered at $a^\pi$.

(d) The following constraints are a key point. Here permutation $\sigma_{(1\to i)}$ forces the following condition to hold for each point $b_i^\pi$. Fix a point $b_i^\pi$ and let $\sigma_{(1\to i)}$ be the sought permutation of $\{1, \ldots, 2^n\}$ with $\sigma_1 = i$ (recall that we write $\sigma_j$ for $\sigma(j)$). Conditions $\bigwedge_{2\leq i\neq j\neq k\leq n+1} EQD(b_{\sigma_i}, b_{\sigma_j}, b_{\sigma_i}, b_{\sigma_k})$, $\bigwedge_{n+2\leq k\leq 2^n} NEARER(b_{\sigma_1}, b_{\sigma_2}, b_{\sigma_1}, b_{\sigma_k})$, $\bigwedge_{2\leq j\leq n+1} EQD(b_{\sigma_1}, b_{\sigma_2}, b_{\sigma_1}, b_{\sigma_j})$ state (in order): there are exactly $n$ points $b_j^\pi$ at same distance from $b_i^\pi$; all other points $b_k^\pi$ are farther from $b_i^\pi$; these $n$ points $b_j^\pi$ are at the same distance to each other. From these constraints we obtain that the points closer to $b_i^\pi$ must form a regular $(n-1)$-simplex on the $(n-1)$-sphere centered at $a^\pi$, see Figure 10 for a 2-simplex in dimension 3. Since each side of these simplexes is obtained starting from different vertices, this condition forces all simplexes to have sides of same length, i.e., all simplexes associated to some point $b_i^\pi$ must be congruent. (Such a simplex is generally called *vertex figure* of its reference point, here $b_i^\pi$.)

(e) The last set of conditions $\bigwedge_{2\leq j\neq k\leq n+1} \exists u[BTW(b_{\sigma_j}, u, b_{\sigma_k}) \wedge EQD(u, b_{\sigma_1}, u, b_{\sigma_j}) \wedge\ EQD(u, b_{\sigma_1}, u, b_{\sigma_k})]$ says that the middle point of any segment $\overline{b_j^\pi\, b_k^\pi}$, which is an edge of $b_i^\pi$'s vertex figure, is equidistant from $b_i^\pi, b_j^\pi$ as well as $b_k^\pi$, i.e. these three points lay on the circumference with center $u$ and with $\overline{b_j^\pi\, b_k^\pi}$ as diameter. This forces the angle at $b_i^\pi$ to be a right angle.

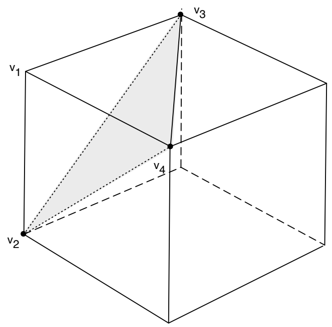



Figure 10. Points $v_2$, $v_3$, $v_4$ are the vertices of the regular 2-simplex associated with vertex $v_1$ (vertex figure of $v_1$) in the depictied hyper-cube in dimension 3.

We have seen that $2^n$ points satisfying formula $HCVerts_{B,Eqd}$ must lay on a $(n-1)$-sphere, they can be paired so that the center of the sphere is midpoint for each pair, each point has exactly $n$ adjacent points all at the same distance (while all other points are farther away) and forms right angles with all of these. It follows that these points are the vertices of a hyper-cube in dimension $n$. $\square$

Given Lemma 7, axiom (AM17) states that for any $2^n$ hyper-cubes satisfying $HCVerts_{B,Eqd}$ (in particular, centered at the vertices of a hyper-cube) there exists a hyper-cube $w$ such that a hyper-cube $z$ has center in the interior of $w$ (the condition forced by $COI$) if and only if any iterative projections of the center of $z$ from the vertices of $w$ lead to a vertex of $w$. To ensure that $z$ itself is not on the surface of the hyper-cube, it is required that the first projection always gives a hyper-cube not concentric with $z$. See Figure 11 for an example in dimension 3.

Axiom (AM18) states that for any hyper-cube $x$ one can find $2^n$ hyper-cubes, each with center at one of $x$'s vertices, such that $z$ has center in the interior of $w$ if and only if the center of $z$ satisfies the conditions on the projection from vertices as in (AM17).

Finally, axiom (A4) says that distinct regions can be distinguished just by looking at their interior points.

Definition 7. Let $\pi$ be a CCPI-function for $\mathfrak{G}^n_{HC}$. A *Cartesian open hyper-cube interpretation* or COCI-function for $\mathfrak{G}^n_{HC}$ is a function $\Pi : C \to \wp(\mathbb{R}^n)$ defined by:

$$\Pi(q) \stackrel{\text{def}}{=} \{\pi(q_i) \mid \langle q_i, q\rangle \text{ satisfies } COI(q_i, q) \text{ in } \mathfrak{G}^n_{HC}\}.$$

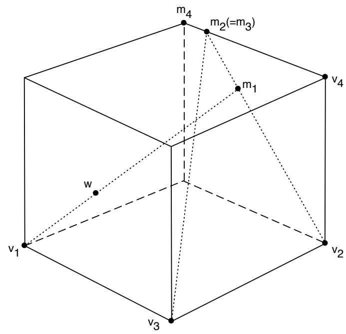



Figure 11. Some iterated projections (from vertices $v_1$, ..., $v_4$ in this order) of a point $w$ internal to a hyper-cube to the faces, edges and vertices of the hyper-cube itself, see (AM17) and (AM18).

Lemma 8. *Let $\mathfrak{G}^n_{HC}$ be a hyper-cube model and $\pi$ a CCPI-function for it. Let relation $\phi(z, y_1, \ldots, y_{2^n})$ be defined by formula:*

$$\forall u\ [HCb(u) \to [COI(u,w) \leftrightarrow \forall\sigma\exists m_1, ..., m_{2^n-1}\ [\neg HCb_{\boxdot}(u,m_1) \wedge BTW(y_{\sigma(1)}, u, m_1) \wedge \bigwedge_{2\le i\le 2^n-1} BTW(y_{\sigma(i)}, m_{i-1}, m_i) \wedge \bigvee_j HCb_{\boxdot}(y_{\sigma(j)}, m_{2^n-1})]]].$$

*Then, any tupla $\langle a, v_1, ..., v_{2^n}\rangle \in C^{2^n+1}$, such that $HCVerts_{B,Eqd}$ holds for $v_1$, ... $v_{2^n}$, satisfies $\phi(z, y_1, \ldots, y_{2^n})$ just in case $\Pi(a)$ is the open hyper-cube with the $v_i^\pi$ as vertices.*

Proof. Assume the formula holds for $\langle a, v_1, \ldots, v_{2^n}\rangle$, we need to show that $\Pi(a)$ is an open hyper-cube with vertices $v_i^\pi$. From Lemma 7 and $HCVerts_{B,Eqd}(v_1, \ldots, v_{2^n})$, the centers of $v_i, \ldots, v_{2^n}$ are the vertices of a hyper-cube, i.e., the convex hull of points $v_1^\pi, \ldots, v_{2^n}^\pi$ is a hyper-cube (and of course convex). We now prove that, if for all hyper-cube $u$ and all permutations $\sigma$ there exist regions $m_i$ that satisfy subformulas $\neg HCb_{\boxdot}(u, m_1)$, $BTW(v_{\sigma(1)}, u, m_1)$, $\bigwedge_{2\le i\le 2^n-1} BTW(v_{\sigma(i)}, m_{i-1}, m_i)$ and $\bigvee_j HCb_{\boxdot}(y_{\sigma(j)}, m_{2^n-1})$, then $u^\pi$ is in the interior of $\Pi(a)$. More specifically, given an arbitrary permutation $\sigma$, we show that it is possible to find the requested $m_1, \ldots, m_{2^n-1}$ if and only if $u^\pi$ is in the interior of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$.

Case 1): assume $u^\pi$ is in the interior of the complement of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$. If $m_1$ is such that $B(v_{\sigma(1)}^\pi, u^\pi, m_1^\pi)$, then $m_1^\pi$ is also in the interior of the complement of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$. Given this, if $m_2$ is such that $B(v_{\sigma(2)}^\pi, m_1^\pi, m_2^\pi)$, then $m_2^\pi$ is also in the interior of the complement of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$ since $m_1$ is. By finite iteration of this argument, all $m_i^\pi$ are in the interior of the complement of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$. In particular, this holds for $m_{2^n-1}^\pi$. Since any $v_i^\pi$ is obviously in the boundary of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$, condition $\bigvee_j HCb_\boxdot(y_{\sigma(j)}, m_{2^n-1})$ cannot hold.

Case 2): If $u^\pi$ is in the interior of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$, then we can take $m_1$ with center on the boundary of that convex hull so that $B(v_{\sigma(1)}^\pi, u^\pi, m_1^\pi)$ holds. Let $Bd_{\leq n-1}$ be the smallest face of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$ where $m_1^\pi$ lives. Clearly $Bd_{\leq n-1}$ is convex, is part of the boundary of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$, and has dimension at most $n-1$. Now we can take $m_2$ with center on the (manifold) boundary of $Bd_{\leq n-1}$ such that $B(v_{\sigma(2)}^\pi, m_1^\pi, m_2^\pi)$ holds. Again, the smallest component of $Bd_{\leq n-1}$ where $m_2^\pi$ lives, call it $Bd_{\leq n-2}$, is convex, part of the (manifold) boundary of $Bd_{\leq n-1}$, and of dimension at most $n-2$. By iteration, we construct a series of hyper-cubes $m_1, \ldots, m_{2^n-1}$ (possibly concentric and possibly concentric with some $v_i$) satisfying the condition $BTW(v_{\sigma(i)}^\pi, m_{i-1}^\pi, m_i^\pi)$. Finally, since $m_{2^n-1}^\pi$ is on a boundary region $Bd_{\leq 0}$ which, by construction, is a vertex of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$, then for some $j$ we have $HCb_\boxdot(y_{\sigma(j)}, m_{2^n-1})$.

Case 3): If $u^\pi$ is on the boundary of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$, then let $\sigma$ be such that $\sigma(1)$ is not on the face where $u^\pi$ lays. Any $m_1$ not concentric with $u$ and satisfying $BTW(y_{\sigma(1)}, u, m_1)$ must be in the interior of the complement of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$ and Case 1) applies.

We have thus shown that, with the given conditions on tupla $\langle a, v_1, \ldots, v_{2^n}\rangle$, relation $COI(u, z)$ holds if and only if $u^\pi$ is in the interior of the convex hull of $v_1^\pi, \ldots, v_{2^n}^\pi$. By Definition 7, this implies that $\Pi(a)$ is the open hyper-cube with vertices $v_i^\pi$. □

Lemma 9. *Given a hyper-cube model $\mathfrak{G}_{HC}^n$ with CCPI-function $\pi$ and corresponding COCI-function $\Pi$, for all $x \in C$, $\Pi(x)$ is an open hyper-cube of $\mathbb{R}^n$.*

Proof. From the previous lemma and axiom (AM18). □

Lemma 10. *Every COCI-function $\Pi$ on a hyper-cube model $\mathfrak{G}_{HC}^n$ is a bijection onto the set of hyper-cubes in $\mathbb{R}^n$.*

PROOF. From the previous lemma, the range of $\Pi$ is a subset of hyper-cubes in $\mathbb{R}^n$. Since $\pi$ is surjective, for any $2^n$ points $p_1, \ldots, p_{2^n}$, we can find hyper-cubes $x_1, \ldots, x_{2^n} \in C$ such that $x_i^\pi = p_i$ and, if the $p_i$ are vertices of a hyper-cube, then from (AM17) there exists a hyper-cube $w \in C$ which is the interior of their convex hull. Then, $\Pi(w)$ is the hyper-cube in $\mathbb{R}^n$ with vertices $p_1, \ldots, p_{2^n}$. Since any hyper-cube in $\mathbb{R}^n$ is characterized in this way by $2^n$ points (its vertices), $\Pi$ is surjective. From (AM4), we conclude that $\Pi$ is injective and thus a bijection. □

LEMMA 11. *Let $Cube_{\mathbb{R}^n}$ be the set of hyper-cubes in $\mathbb{R}^n$, all hyper-cube models $\mathfrak{G}^n_{HC}$ are isomorphic to $\langle Cube_{\mathbb{R}^n}, HCb_{\boxdot}, BTW, EQD, COI\rangle$ where the predicates have the usual intended interpretations.*

PROOF. This is ensured by the properties of the CCPI-function $\pi$ and the fact that the COCI-function $\Pi$ is a bijection (lemmas 9 and 10). □

### 6.3. Region-based geometry models

The rest of the proof that the $n$-dimensional mereogeometry $\mathrm{MG}^n_{HC}$ is categorical proceeds again along the lines of Bennett's work. This part does not dependent on the specific primitives we have used to build points and the specialized axioms we introduced, we can thus repeat Bennett's results without much details, the reader can find all these proofs in [Ben01].

DEFINITION 8. An $n$-dimensional $\mathrm{MG}^n_{HC}$-model is a structure $\langle \mathcal{R}, P, HCb\rangle$, where $P$ and $HCb$ are respectively binary and unary relations satisfying axioms (AM1)–(AM18) and the following:

(AM19) $\forall x, y\ [P(x, y) \leftrightarrow \forall z\ [COI(z, x \rightarrow COI(z, y)]]$
(AM20) $\forall x \exists y\ [HCb(y) \wedge P(y, x)]$

LEMMA 12. *Let $\mathcal{M}^n = \langle \mathcal{R}, P, HCb, HCb_{\boxdot}, BTW, EQD, COI\rangle$ be a structure that satisfies the axioms* (AM1)–(AM20). *If $C = \{r \in \mathcal{R} \mid HCb(r)\}$, then the substructure $\langle C, HCb_{\boxdot}, BTW, EQD, COI\rangle$ is a hyper-cube model.*

PROOF. $\mathcal{M}^n$ satisfies axioms (AM3)-(AM18) and since these are all restricted to hyper-cubes, they must hold in the substructure with domain $C$ as well. Regarding (A4), it is a consequence of (AM2) and (AM19), and it holds in the substructure because it is an universal formula. □

The above proof is taken from [Ben01] with the only adaptation of the axioms' references. Analogously, the proofs of the following propositions are obtained by following the proofs of the corresponding propositions in [Ben01]. For this reason, we do not report the proofs directly.

DEFINITION 9. A *Cartesian regular open set interpretation function*, CROSI-function, for a $\mathrm{MG}^n_{HC}$-model $\mathcal{M}^n$ is defined by $\Pi(r) = \{\pi(r_i) \mid \langle r_i, r\rangle$ satisfies $COI(x, y)$ in $\mathcal{M}^n\}$ with $\pi$ a CCPI-function for the hyper-cube substructure of $\mathcal{M}^n$.

LEMMA 13. *The following hold*:

- *Each $r \in \mathcal{R}$ satisfies $HCb(r)$ if and only if $\Pi(r)$ is a hyper-cube.*
- *A pair $\langle r_1, r_2\rangle \in \mathcal{R}^2$ satisfies $P(x, y)$ in $\mathcal{M}^n$ just in case $\Pi(r_1) \subseteq \Pi(r_2)$.*
- *For any $r_1, r_2 \in \mathcal{R}$ such that $HCb(r_1)$ and $HCb(r_2)$ hold in $\mathcal{M}^n$, the pair $r_1, r_2$ satisfies $COI(x, y)$ in $\mathcal{M}^n$ just in case the center point of the open hyper-cube $\Pi(r_1)$ lies within the open hyper-cube $\Pi(r_2)$.*

LEMMA 14. *For any $r_1, r_2 \in \mathcal{R}$, if $\Pi(r_1) \cap \Pi(r_2) = \emptyset$ then $\langle r_1, r_2\rangle \in \mathcal{R}^2$ satisfies $DR(x, y)$ in $\mathcal{M}^n$.*

LEMMA 15. *For every regular open set $O \subseteq \mathbb{R}^n$, there is an element $r \in \mathcal{R}$ such that $\Pi(r) = O$.*

LEMMA 16. *For every $r \in \mathcal{R}$, $\Pi(r)$ is a non-empty regular open subset of $\mathbb{R}^n$.*

LEMMA 17. *$\Pi$ is a bijection from $\mathcal{R}$ onto the non-empty regular open subsets of $\mathbb{R}^n$.*

THEOREM 2. *Axioms* (AM1)–(AM20) *form a categorical axiom system for* $\mathrm{MG}^n_{HC}$. *Any* $\mathrm{MG}^n_{HC}$*-model is isomorphic to the structure $\langle \mathcal{R}_{\mathbb{R}^n}, P, HCb\rangle$ where $\mathcal{R}_{\mathbb{R}^n}$ is the set of non-empty open regular subsets of $\mathbb{R}^n$.*

It follows that

COROLLARY 2. *The system* $\mathrm{MG}^n_{HC}$ *is categorical.*

## 7. Sphere vs Hyper-Cube

We have seen that Whitehead's intuition to construct points as sets of concentric geometric figures can be implemented in different ways as exemplified by Tarski's technique applied to spheres and our technique to hyper-cubes. In Section 8 we will see that our technique is even more general since it applies to another class of figures which is geometrically most fundamental: (regular) simplexes.

Here we cast a few observations on the result so far.

First. By taking hyper-cubes as primitive entities we build mereogeometry on a set of definitions which is syntactically comparable to that proposed by Tarski for the geometry of solids. This result is surprising since the approach in [Tar56b] and the given definitions rely on the rich symmetry system that characterizes spheres. Our exploitation of the properties of hyper-cubes to provide a relative ordering for those figures shows that one can reach the same result by combining symmetry with other geometrical features, namely, the presence of right angles. Perhaps it is even more surprising that we obtain an improvement over [Tar56b]. The number of 'distinct concepts', those relevant both formally and cognitively (and cast by the definitions), is smaller in the exploitation of hyper-cubes. Tarski's uses four notions: external and internal tangency and external and internal diametricity. Our system uses three: co-verticity, aligned concentricity and rotated concentricity.

An interesting result from the syntactic viewpoint is that Tarskian definition of concentric spheres ($S_{\odot}$) is a formula of type $\forall\exists\forall$ in terms of Tarskian primitives. Instead, our work on hyper-cubes leads to a definition ($HCb_{\boxdot}$) which is a $\exists\forall\exists$-formula in terms of our primitives. Furthermore, if the dimension $n$ of the space is fixed, the formula on hyper-cubes reduces to a $\exists\forall$-formula by substituting the following definition for the definition of aligned concentric hyper-cubes (D5.13)

$$\text{(D5.13')}\quad HCb_{\boxtimes}(x,y) \stackrel{\text{def}}{=} HCb(x) \wedge HCb(y) \wedge (x = y \vee \\ (P(x,y) \wedge \exists z_1, z'_1, \ldots, z_{2^{n-1}}, z'_{2^{n-1}}\ [\textstyle\bigwedge_{i\neq j}(PO(z_i,z_j) \wedge PO(z_i,z'_j)) \wedge \\ \textstyle\bigwedge_i PO(z_i,z'_i) \wedge \bigwedge_i (HCb_{\boxdot}(z_i,y) \wedge HCb_{\boxdot}(z'_i,y) \wedge PROD(z_i,z'_i,x))]) \vee \\ (P(y,x) \wedge \exists z_1, z'_1, \ldots, z_{2^{n-1}}, z'_{2^{n-1}}\ [\textstyle\bigwedge_{i\neq j}(PO(z_i,z_j) \wedge PO(z_i,z'_j)) \wedge \\ \textstyle\bigwedge_i PO(z_i,z'_i) \wedge \bigwedge_i (HCb_{\boxdot}(z_i,x) \wedge HCb_{\boxdot}(z'_i,x) \wedge PROD(z_i,z'_i,y))]))$$

(*aligned concentric hyper-cubes*)

Definition (D5.13') states that, given that the space has dimension $n$ and that $x$ and $y$ are two hyper-cubes with $x$ properly contained in $y$, these are aligned and concentric when there are $2^n$ *properly overlapping* hyper-cubes in $y$ and co-vertex with $y$ (thus, one per vertex since properly overlapping) such that $x$ is the product region of each pair. Tarski's definitions are not suitable for this type of simplifications.

Second. Tarski defines some geometrically complex relations, namely, *ED* and *ID* to formalize the notions of externally and internally diametrical spheres. These relations are *necessarily* ternary. Our system for hyper-cubes, and similarly the system for regular simplexes in the next section, uses only geometrical *binary* relations. Indeed, the only use of ternary relations in our approach is restricted to the definition of the product of two regions, which is a mereological notion. That is, we proved that it is possible to build a system of mereogeometry *without referring to any ternary purely geometrical relation*. This result makes even more remarkable the difference with Euclidean geometry: in standard point-based geometry it is impossible to build the Euclidean system without explicitly adding as primitive at least one ternary relation [Rob59]. That is, it does not suffice to introduce a derived ternary relation as in Tarski's mereogeometry. (Note, however, that this observation is limited to the use of primitives and does not count as a full syntactic comparison for the use of a special operator, i.e. $SUM$ used in (AM2), in the axiomatization.)

Third. While Tarski's original definitions necessarily quantify over spheres of any size, our set of definitions characterize concentric hyper-cubes by quantifying only on hyper-cubes bounded by the given regions: fix a pair of hyper-cubes $x, y$, only hyper-cubes contained in these needs to be taken into account to establish whether they are concentric.

Fourth. While the direct axiomatization of the system for spheres and that for hyper-cubes are very similar, our axioms (AM17) and (AM18) are much more complicated, both conceptually and formally, than their corresponding axioms for spheres in [Ben01]. The reason is that these axioms depend on the features of the geometrical figures we are using and the properties of hyper-cubes are harder to model in the language of the euclidean relations $BTW$ and $EQD$ only. On this aspect, we anticipate that the axiomatization of the system resulting from regular simplexes is less complicated, although even this system is not really comparable to Bennett's axiom system. Indeed, regular simplexes are easier to model than hyper-cubes in Euclidean geometry but the reconstruction of points from them is slightly more complicated with respect to hyper-cubes as we will see in next section. It seems plausible to claim that by relying on a direct axiomatization of Scott's primitive in Euclidean geometry instead of the Tarskian set of axioms, this difference between Bennett's axiom system and ours may be reversed. This, however, remains to be verified.

Fifth. Both Tarski's and our constructions are dimension independent in the sense that each system characterizes concentric figures in any space of finite dimension $n \geq 2$. The axiomatizations of both systems include an axiom, in our case (AM16), to constrain the dimension of the space. It suffices to modify this axiom to obtain an axiomatization in another dimension.

## 8. Regular simplex as geometrical primitive: the system $\mathbf{MG}^n_{SX}$

The approach developed in section 5 is not specific to the primitive 'being a hyper-cube' and works for regular simplexes as well. Recall that a geometrical $n$-simplex is the convex hull (the smallest convex set) of $n+1$ independent points. In particular a 2-simplex is a triangle and a 3-simplex a tetrahedron. A *regular $n$-simplex* is a $n$-simplex with all edges of same length.

Let us write $SX(x)$ to mean '$x$ is a regular simplex' in the dimension of the space (as usual, $n \geq 2$).

We proceed along the lines of section 5. The first definition is about *co-vertex regular simplexes* and follows the very same idea implemented for hyper-cubes.

(D25) $$SX_{\triangle}(x,y) \stackrel{\text{def}}{=} SX(x) \wedge SX(y) \wedge PP(x,y) \wedge \exists p\ [SX(p) \wedge PP(p,x) \wedge \forall q\ [(SX(q) \wedge P(p,q) \wedge P(q,y)) \rightarrow (P(q,x) \vee P(x,q))]]$$
(*co-vertex regular simplexes*)

The next constraint, adapted from (D13), turns out to be too weak when applied to regular simplexes; it holds for two regular simplexes which are aligned in the sense that they have parallel sides but does not force them to be concentric.

(D26) $$SX_{/\!/}(x,y) \stackrel{\text{def}}{=} SX(x) \wedge SX(y) \wedge (x = y \vee (P(x,y) \wedge \forall q,z\ [(SX_{\triangle}(q,y) \wedge O(q,x) \wedge PROD(q,x,z)) \rightarrow SX(z)]) \vee (P(y,x) \wedge \forall q,z\ [(SX_{\triangle}(q,x) \wedge O(q,y) \wedge PROD(q,y,z)) \rightarrow SX(z)]))$$
(*aligned regular simplexes*)

Next, we define (aligned) concentric regular simplexes by a new condition specific for regular simplexes. The general idea is the following: two aligned concentric regular simplexes $x$ and $y$, with $x$ proper part of

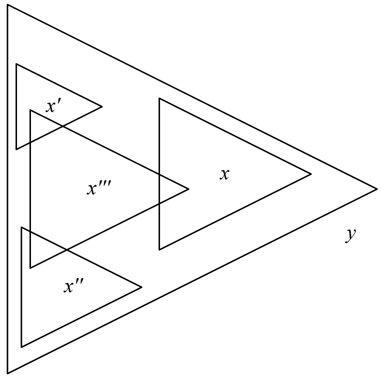



Figure 12. Definition (D26) ensures alignment of the simplexes' sides but not their concentricity. All depicted simplexes ($x$, $x'$, $x''$, $x'''$ as well as their intersections) satisfy $SX_{/\!/}$ when paired with $y$.

$y$, are concentric if the regular simplexes containing $x$ and co-vertex with it leave out at each vertex of $y$ a region which is the sum of two regular simplexes, see Figure 13 for an example in dimension 2.

(D27) $SX_{\triangle}(x,y) \stackrel{\text{def}}{=} SX_{/\!/}(x,y) \wedge (x = y \vee$
$(\exists r, w\ [SX_{\triangle}(r,y) \wedge PO(r,x)] \wedge \forall r, w[(SX_{\triangle}(r,y) \wedge PO(r,x) \wedge$
$\forall v[(SX_{\triangle}(x,v) \wedge P(v,y)) \rightarrow P(v,w)]) \rightarrow \exists z, z'[SX_{\triangle}(z,r) \wedge$
$SX_{\triangle}(z,y) \wedge SX(z') \wedge P(z',y) \wedge \neg O(z,z') \wedge \neg O(w, z+z') \wedge P(r, z+z'+$
$w)]]) \vee (\exists r, w\ [SX_{\triangle}(r,x) \wedge PO(r,y)] \wedge \forall r, w\ [(SX_{\triangle}(r,x) \wedge PO(r,y) \wedge$
$\forall v[(SX_{\triangle}(y,v) \wedge P(v,x)) \rightarrow P(v,w)]) \rightarrow \exists z, z'[SX_{\triangle}(z,r) \wedge SX_{\triangle}(z,x)$
$\wedge\ SX(z') \wedge P(z',x) \wedge \neg O(z,z') \wedge \neg O(w, z+z') \wedge P(r, z+z'+w)]]))$
(*aligned concentric regular simplexes*)

Note that definition (D27) can be used, mutatis mutandis, for hyper-cubes as well. Yet, we used definition (D13) since it is simpler. Note also, recalling the discussion in Section 7, that even (D27) can be simplified if we fix the dimension $n$ of the space. In this case it suffices to claim that there exist $2n + 2$ regular simplexes (one pair of regions $z, z'$ for each vertex) and $n + 1$ regular simplexes (these are those that form the gray region $w$) which cover all $y$.

Our next goal is to include the case of concentric regular simplexes that are not aligned, i.e., that are rotated with respect to the common center, called *concentric rotated regular simplexes.* As for hyper-cubes,

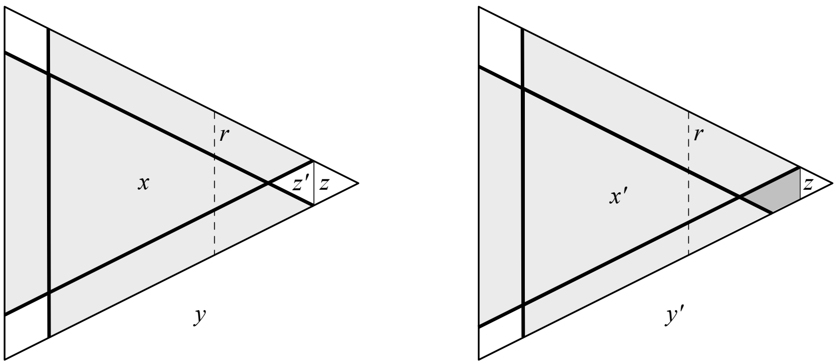



Figure 13. In the shown cases of dimension 2, (D27) holds only for the concentric aligned simplexes $x$ and $y$ on the left where suitable regions $z$, $z'$ are also shown for one vertex. On the right, $x'$ and $y'$ are aligned but not concentric and any $z$ bigger than the one shown would overlap the light gray region (region $w$ in (D27)). Clearly no regular simplex $z'$ in $y'$ can cover the dark gray part of $r$. (D27) fails for $x', y'$.

it suffices to consider the case in which the vertices of the smaller regular simplex are on the sides of the larger regular simplex, thus we simply adapt the definition used for hyper-cubes.

(D28) $SX_{\circlearrowleft}(x,y) \overset{\text{def}}{=} SX(x) \wedge SX(y) \wedge PP(x,y) \wedge$
$$\forall q\ [(SX(q) \wedge PP(q,y) \wedge P(x,q)) \rightarrow P(q,x)]$$
(*rotated concentric regular simplexes*)

We can finally give the relation (also taken from that on hyper-cubes) suitable to define points as maximal sets of concentric regular simplexes.

(D29) $SX_{\triangle}(x,y) \overset{\text{def}}{=} SX_{\triangle\kern-0.6em\triangle}(x,y) \vee \exists z\ [SX_{\circlearrowleft}(z,x) \wedge SX_{\triangle\kern-0.6em\triangle}(z,y)] \vee$
$$\exists z\ [SX_{\circlearrowleft}(z,y) \wedge SX_{\triangle\kern-0.6em\triangle}(z,x)]$$
(*concentric regular simplexes*)

## 9. Axiomatization of $\mathbf{MG}^{n}_{SX}$

The material presented in section 6 does not depend in any relevant aspect on the use of the primitive hyper-cube as opposed to regular simplex. However, it turns out that the axiomatization of the system built out of regular simplexes is much easier. For this reason, instead of repeating the overall proof, here we briefly report the steps where the axioms and proofs vary.

Let us start from the axiomatization given earlier for hyper-cubes, (AM1)–(AM20), where each occurrence of $HCb$ is substituted by $SX$, of $HCb_{\boxdot}$ by $SX_{\dot{\triangle}}$ and so on. We use (AM1')–(AM20') for the resulting axioms. Thus, given axiom (AM3), i.e., $\forall x, y(BTW(x,y,x) \to HCb_{\boxdot}(x,y))$, we now call (AM3') the axiom $\forall x, y(BTW(x,y,x) \to SX_{\dot{\triangle}}(x,y))$.

Also, we take for granted that earlier definitions are modified accordingly unless otherwise specified. For instance, from (D16) the definition of co-oriented simplexes is given by

(D30) $$SX_{\triangleright\!\triangleright}(x,y) \stackrel{\text{def}}{=} \neg SX_{\dot{\triangle}}(x,y) \wedge \forall x', y' \; [(SX_{\bigtriangleup\!\!\!\!\triangle}(x,x') \wedge SX_{\bigtriangleup\!\!\!\!\triangle}(y,y') \wedge PO(x',y')) \to SX(x' \cdot y')]$$
(*co-oriented regular simplex*)

Note that this definition corresponds syntactically to that of $HCb_{\diamond\!\diamond}$ in (D16) but the constraint is quite different. While the definition on hyper-cubes ensures that each hyper-cube has one diagonal on the very same line, the definition on regular simplexes forces the simplexes to have parallel sides. It is thus a generalization of (D26). As a consequence of the different 'meaning' of this definition, in the system based on regular simplexes definition $BTW$ (D17) must be modified.

We now introduce a new relation $COV$ holding when the center of a regular simplex is the vertex of another regular simplex. We can define $BTW$ from $COV$ without using relation $MID$ (D19) which, in turn, is defined from $BTW$ and ternary $EQD$. First, we need to restate relation $COB$; this is the relation stated for hyper-cubes (D18) with $SX$ substituted for $HCb$.

(D31) $$COB(x,y) \stackrel{\text{def}}{=} SX(x) \wedge \forall x' \; [SX_{\dot{\triangle}}(x,x') \to (O(x',y) \wedge \neg P(x',y))]$$
(*center of $x$ on the boundary of $y$*)

(D32) $$COV(x,y) \stackrel{\text{def}}{=} SX(y) \wedge COB(x,y) \wedge \forall z \; [SX_{\circlearrowright}(z,y) \to \neg COB(x,z)]$$
(*center of $x$ is on vertex of $y$*)

Definition (D32) holds when regular simplex $x$ has center on the boundary of regular simplex $y$ but the center of $x$ is never on the boundary of any regular simplex concentric and properly contained in $y$. (Recall that $SX_{\circlearrowright}(z,y)$ implies $z \neq y$.)

(D33) $BTW(x,y,z) \stackrel{\text{def}}{=} SX(x) \wedge SX(y) \wedge SX(z) \wedge (SX_{\triangle}(x,y) \vee SX_{\triangle}(y,z) \vee \forall u\, [(COV(x,u) \wedge COV(z,u)) \to COB(y,u)])$ (*regular simplex betweenness*)

(D34) $EQD(x,y,z) \stackrel{\text{def}}{=} SX_{\triangle}(x,y) \vee \exists u,v\ [COV(x,u) \wedge COV(y,u) \wedge COV(x,v) \wedge COV(y,v) \wedge \neg O(u,v) \wedge (BTW(z,u,v) \vee BTW(u,z,v) \vee BTW(u,v,z))]$ (*centers of $x,y$ equidistant from that of $z$*)

(D35) $MID(x,y,z) \stackrel{\text{def}}{=} EQD(x,z,y) \wedge BTW(x,y,z)$ (*$z$ centered at the midpoint of $x,y$*)

From this, the quaternary $EQD$ is similar to (D21)

(D36) $EQD(w,x,y,z) \stackrel{\text{def}}{=} \exists u,v[MID(w,u,y) \wedge MID(x,u,v) \wedge EQD(v,z,y)]$ (*centers of $w,x$ and of $y,z$ are equidistant*)

while the $NEARER$ relation is adapted from (D22) as usual, i.e.,

(D37) $NEARER(w,x,y,z) \stackrel{\text{def}}{=} \exists x'\ [BTW(w,x,x') \wedge \neg SX_{\triangle}(x,x') \wedge EQD(w,x',y,z)]$ (*centers of $w,x$ are closer than those of $y,z$*)

Finally, definition (D23) becomes

(D38) $COI(x,y) \stackrel{\text{def}}{=} \exists x'\, [SX_{\triangle}(x,x') \wedge P(x',y)]$ (*center of $x$ in interior of $y$*)

Moving to study the structures, the following definition of relaxed geometrical model for regular simplexes corresponds to Definition 3 for hyper-cubes.

DEFINITION 10. Let $\mathcal{SX}^n_r = \langle D, SX_{\triangle}, BTW, EQD\rangle$ be a structure satisfying the set of axioms $\mathrm{MG}^n_{SX(\downarrow)}$ given by (AM3')–(AM15') plus the axiom (AM16')$^n$ fixing the dimension of the space and:

(A1′) $\forall x\ [SX(x)]$
(A3′) $\forall x,y,z\ [(SX_{\triangle}(x,y) \wedge SX_{\triangle}(y,z)) \to SX_{\triangle}(x,z)]$

We call $\mathcal{SX}^n_r$ a (*regular simplex*) *relaxed geometrical model.*

DEFINITION 11. Given a structure $\mathcal{E}^n$ and a structure $\mathcal{SX}^n_r$ with domain $D$, a surjective function $\pi\colon D \to \mathbb{R}^n$ such that

- if $\mathcal{SX}_r^n \models SX_{\triangle}(x_1, x_2)$ then $\mathcal{E}^n \models \pi(x_1) = \pi(x_2)$
- if $\mathcal{SX}_r^n \models BTW(x_1, x_2, x_3)$ then $\mathcal{E}^n \models B(\pi(x_1), \pi(x_2), \pi(x_3))$
- if $\mathcal{SX}_r^n \models EQD(x_1, x_2, x_3, x_4)$ then $\mathcal{E}^n \models Eqd(\pi(x_1), \pi(x_2), \pi(x_3), \pi(x_4))$

is called *Cartesian centre-point interpretation* or *CCPI-function* for short.

Then, one follows the previous steps to reach the following result.

LEMMA 18. *Fix a geometrical model $\mathcal{SX}_r^n$ with domain $D$, there is a CCPI-function $\pi\colon D \to \mathbb{R}^n$.*

As anticipated, the characterization of regular simplexes in terms of $BTW$ and $EQD$ is easier than that of hyper-cubes as one can see comparing the following definition for regular simplexes with (D24). The main reason is that simplexes are characterized by $n + 1$ non-collinear points (for $n$ the dimension of the space).

(D39) $$SxVerts_{B,Eqd}(y_1, \ldots, y_{n+1}) \stackrel{\text{def}}{=} \bigwedge_i SX(y_i) \wedge \bigwedge_{i \neq j \neq k} \neg BTW(y_i, y_j, y_k) \wedge \bigwedge_{i \neq j \neq k} EQD(y_i, y_j, y_i, y_k)$$
($y_1, \ldots, y_{n+1}$ *are centered at the vertices of a regular $n$-simplex*)

In this setting, the role of Definition 6 of Section 6.2 is taken by the following

DEFINITION 12. We call *regular simplex model* any structure of type $\mathfrak{G}_{SX}^n = \langle C, SX_{\triangle}, BTW, EQD, COI\rangle$ which satisfies the conditions for $\mathcal{SX}_r^n$ and the following:

(AM17′) $$\forall y_1, y_2, ..., y_{n+1} [SxVerts_{B,Eqd}(y_1, \ldots, y_{n+1}) \to \exists w \forall u\, [SX(u) \to [COI(u, w) \leftrightarrow \forall \sigma \exists m_1, ..., m_n\, [\neg SX_{\triangle}(u, m_1) \wedge BTW(y_{\sigma(1)}, u, m_1) \wedge \bigwedge_{2 \leq i \leq n} BTW(y_{\sigma(i)}, m_{i-1}, m_i) \wedge \bigvee_j SX_{\triangle}(y_{\sigma(j)}, m_n)]]]])$$

(AM18′) $$\forall x \exists y_1, \ldots, y_{n+1}\, [SX(x) \to (SxVerts_{B,Eqd}(y_1, \ldots, y_{n+1}) \wedge \forall u\, [SX(u) \to [COI(u, x) \leftrightarrow \forall \sigma \exists m_1, ..., m_n\, [\neg SX_{\triangle}(u, m_1) \wedge BTW(y_{\sigma(1)}, u, m_1) \wedge \bigwedge_{2 \leq i \leq n} BTW(y_{\sigma(i)}, m_{i-1}, m_i) \wedge \bigvee_j SX_{\triangle}(y_{\sigma(j)}, m_n)]]]])]$$

(A4′) $\forall x, y[\forall z[COI(z, x) \leftrightarrow COI(z, y)] \to x = y]$

We now get

LEMMA 19. *Let $RegSX_{\mathbb{R}^n}$ be the set of regular simplexes in $\mathbb{R}^n$, all regular simplex models $\mathfrak{G}_{SX}^n$ are isomorphic to $\langle RegSX_{\mathbb{R}^n}, SX_{\triangle}, BTW, EQD, COI\rangle$ where the predicates have the intended interpretations.*

Consider the structure

Definition 13. An $n$-dimensional $\mathrm{MG}^n_{SX}$-model is a structure $\langle \mathcal{R}, P, SX\rangle$, where $P$ and $SX$ are respectively binary and unary relations satisfying axioms (AM1′)–(AM18′) and the following:

(AM19′) $\forall x, y\ [P(x, y) \leftrightarrow \forall z\ [COI(z, x \rightarrow COI(z, y)]]$
(AM20′) $\forall x \exists y\ [SX(y) \wedge P(y, x)]$

and repeat the steps as before to conclude

Theorem 3. *Axioms* (AM1′)–(AM20′) *form a categorical axiom system for* $\mathrm{MG}^n_{SX}$. *Any* $\mathrm{MG}^n_{SX}$*-model is isomorphic to the structure* $\langle \mathcal{R}_{\mathbb{R}^n}, P, SX\rangle$, *where* $\mathcal{R}_{\mathbb{R}^n}$ *is the set of non-empty open regular subsets of* $\mathbb{R}^n$.

Finally, it follows that

Corollary 3. *The system* $\mathrm{MG}^n_{SX}$ *is categorical.*

## 10. Conclusions

In this paper we have contributed to a line of research that started with the first rigorous formalizations of Euclidean geometry and of mereogeometry by Tarski, almost a century ago. The focus is the study of primitives and their relationships especially in terms of expressivity, see [Pam01, SST83] for reviews in Euclidean geometry and [Ger95, Dav06, CR07, BM10] (including references in these) for studies in mereogeometry.

We have shown that one can construct points in region-based structures from simple regular polygons, namely squares and equilateral triangles, and their generalizations to any dimension. From our results, it follows that these primitives not only are as expressive as spheres but can even be of interest for the properties of the theories they generate. In this way we have made evident that the expressive power of classical geometrical figures is independent of the point-region dichotomy or their ‘resemblance’ of points. This observation can be related to some results known in two-sorted Euclidean geometry, that is, in axiomatization of standard geometry with domain including both points and some class of geometrical figures (circle, equilateral triangles, squares or right triangles [Pam04, Pra83, Sco56]) interacting via the incidence relation.

We conclude with a final observation on the material in this paper. The method we have applied to regular simplexes and hyper-cubes applies to any dimension but, as given, is limited to these figures. The reason is that it focuses on one angle at a time (we called it 'co-vertex') to provide a local ordering on pairs of figures. As a result, this approach requires that, given a figure $A$, the intersection of a co-vertex figure of $A$ with a co-centered figure of $A$ is also a figure of the same type. This condition holds only when all features of the figure opposite to the co-vertex angle vary proportionally to the size of the intersecting regions. Since this happens only for regular simplexes and hyper-cubes, the method is not directly applicable to other (hyper-) polygons. The method of Tarski applies also to any dimension but covers spheres only and is definitely not applicable to (hyper-)polygons. Nonetheless, a combination of the two methods can lead to more freedom and makes possible to build mereogeometries from other classes of polygons. For instance, one can provide a local ordering of regular hexagons by the co-vertex approach (this is used to give a limited version of Tarskian $IT$ relation); then use inclusion in a larger hexagon to define a kind of $ED$ relation, and adapt Tarski's $ID$ relation to finally define concentric aligned hexagons in the way Tarski defined concentric spheres. At this point, the way we defined concentric rotated hyper-cubes gives the method to generalize the definition to any concentric hexagon. How far we can go with this combined method has not been explored yet. We also ignore how to deal with other important classes of figures, like that of ellipses.

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