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MENGER AND NÖBELING ON POINTLESS TOPOLOGY

Abstract. This paper looks at how the idea of pointless topology itself evolved during its pre-localic phase by analyzing the definitions of the concept of topological space of Menger and Nöbeling. Menger put forward a topology of lumps in order to generalize the definition of the real line. As to Nöbeling, he developed an abstract theory of posets so that a topological space becomes a particular case of topological poset. The analysis emphasizes two points. First, Menger's geometrical perspective was superseded by an algebraic one, a lattice-theoretical one to be precise. Second, Menger's bottom–up approach was replaced by a top–down one.

Keywords: pointless topology, Karl Menger, Georg Nöbeling, lattice theory, abstraction, generalization

1. Introduction

Historically, Karl Menger appears to have been the first to raise the idea of a point-free definition of the concept of topological space. In his 1928 book *Dimensionstheorie* [16], he indeed proposed to define the general concept of space without referring to the points of an underlying set, but rather using pieces or, as he liked to say, lumps.[1] In doing so, Menger

[1] Menger was also interested in geometry, especially in what he at times called metrical, set-theoretical or even general geometry. See [16, 17, 18, 21]. He also made suggestions towards pointless geometry, the most notable of which was an algebra of geometry based on the operations of join and meet. See [15, 17, 19]. Von Neumann would refer to it in his work on continuous geometry. See [24].

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was formulating for topology an idea that had already been put forward for geometry, most notably by de Laguna and Whitehead.[2]

This is in itself interesting for Johnstone, whose papers [11] and [12] constitute standard references on the history of pointless topology, does not mention Menger. According to him, the idea that points are secondary in a topological space required the prior development of lattice theory. In this regard, the key step occurred in the mid 1930s when Marshall H. Stone established a formal connection between general topology and algebra (see [29, 30]). It initiated a lattice-theoretical approach to general topology according to which topological spaces are to be understood as lattices and studied by means of the methods of abstract algebra, that is independently of points.

This lattice-theoretical approach to general topology would find its most achieved presentation in Georg Nöbeling's 1954 book *Grundlagen der analytischen Topologie* [27].[3] In fact, the review in the *Bulletin of the American Mathematical Society* opened with the following sentence: "This carefully written monograph has a good chance of becoming the definitive text on the subject it treats." [2, p. 594] Furthermore, in light of the advent of locales in the late 1950s (see [11]), Nöbeling's book could be seen as marking the end of the lattice-theoretical phase of pointless topology.

The present paper successively looks at the work of Menger and Nöbeling in order to highlight how the idea itself of pointless topology changed. More specifically, the emphasis will be on two aspects. First, Menger's geometrical perspective was superseded by an algebraic one. Second, Menger's idea of a generalized theory inspired by the definition of real numbers was replaced by an abstract one based on lattice theory. In other words, his bottom-up approach was replaced by a top-down one.

2. Menger or the topology of lumps

The theory of curves and dimension is one of Menger's many contributions to mathematics. Indeed, he is responsable, along with Pavel Urysohn who developed it independently, for the definition of the con-

[2] It should be noted however that Menger only refers to Huntington's foundation of Euclidean geometry and to Nicod's 1923 thesis which he describes as a continuation of Whitehead's ideas. See [20, p. 84]. Menger [19, p. 41] also refers to Pieri.

[3] Johnstone [12, p. 838] refers to him as *Gustav* Nöbeling.

cept of dimension.[4] His interest for these notions goes back to the first lecture of Hans Hahn's 1921 seminar "*Neueres über den Kurvenbegriff*" at *Universität Wien*.[5]

As mentioned in the introduction, Menger first argued for a definition of the concept of topological space independent of points in his book *Dimensionstheorie* (see [16, §4]). The book presented the concept of dimension for spaces in general — starting with the elementary geometrical objects — and in such a way as to be suitable for set-theoretical geometry. Actually, Menger's idea of a set-theoretical geometry led him to look at spaces in terms of subsets.

Menger presented again his point-free definition of space in a talk entitled "Topology Without Points" [20], the last of a series of three he gave at the Rice Institute in December 1939.[6]

Central to Menger's conception is a comparison between the processes by means of which topological spaces and the real line are respectively defined.

2.1. Topological spaces and the real line

Menger starts by recalling the three classic definitions of the concept of topological space.

The first is the concept of a limit class or, to use the terminology of Fréchet who first defined it in his thesis, of a (L) class. Informally, a limit class is a set of elements with a notion of convergent sequence.[7]

The second is based on neighborhoods and is obviously due to Hausdorff. According to Hausdorff, a topological space is a set whose elements are associated to certain subsets called neighborhoods.[8]

The third is due to Kuratowski and uses the closure operator. A topological space is a set X and a function, called closure, assigning to each set $X \subset T$ a set $\overline{X}$ satisfying some axioms.[9]

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[4] For more on the history of the theory of dimension, see [9, 10].

[5] For further biographical information, see [13].

[6] Menger also gave a talk entitled "Topology of Lumps" at the fourth annual mathematical symposium at the University of Notre-Dame in April 1940. See [1, p. 599].

[7] For Fréchet's definition, see [6, p. 5]

[8] For Hausdorff's definition, see [8, p. 213].

[9] For the definition, see [14, p. 182].

Menger insists that all three definitions are set-theoretical, meaning that a space is to be thought of as a set of points. The points themselves are pre-existent and indivisible elementary particles that need to be organized. From this point of view, to know a space, one must look at its points, their properties and the relationships between them. In this sense, the points could be said to determine the identity of the space.

> These three foundations of topology have in common that they are what may be called point set theoretical [*sic*]. By this we mean that each of them considers the space as a set of elements. Of course, it is a set with special properties distinguishing the space from an abstract set, viz. a set in which certain sequences of elements are distinguished, or a set in which subsets are associated with elements or with subsets in a certain way. But in all three cases it is assumed that the elements of the space, the points, are somehow given individually, and that the space character of the set consists in relation between, and properties of, certain sets of these elements. [20, p. 82]

Before going any further, it should be stressed that, while Menger's analysis certainly holds for Fréchet's and Hausdorff's concepts of topological space, it hardly does for Kuratowski's. Indeed, Kuratowski's axioms make absolutely no reference to the points of the underlying set, only to arbitrary subsets. Moreover, his definition contains no separation axiom. This implies that, in contradistinction to Hausdorff, it is not essential to be able to tell the points of a topological space apart. The reason is simple: the topological nature of a space depends on the algebraic structure defined by the closure operator. In other words, to know about a topological space is to know about the properties of the closure operator. The points are simply irrelevant from this point of view. If anything, Kuratowski's approach is not set-theoretical, but algebraic.

Menger next looks at the three standard definitions of the real numbers or, from a geometrical point of view, the straight line. Each is put in correspondence with one of the definitions of the concept of topological space previously presented.[10]

- To the concept of (L) class corresponds Cantor's definition of real numbers as limits of converging sequences of rational numbers.
- To Hausdorff's neighborhood definition corresponds the definition of real numbers as limits of nested sequences of rational intervals of decreasing length.

[10] For the detailed constructions of real numbers, see [20, p. 82–83].

- To Kuratowski's definition in terms of the closure operator corresponds Dedekind's cuts.[11]

Menger points out that a common feature of these definitions of the straight line is that any talk of the set of real numbers is preceded by the explicit introduction of the real numbers themselves. In other words, the points are not assumed to exist; they have to be explicitly constructed. Only then can the set of real numbers be formed.

> These three ways of introducing the straight line or the set of all real numbers do not presuppose the concept of a point or of a real number. On the basis of a denumerable set (the set of rational numbers or rational intervals) for whose elements certain relations are assumed to be given, (a <-relation for the rational numbers, a relation of containing for the rational intervals) they introduce the individual real numbers or points. The set of all of them is formed in order to enable us to talk, if necessary, about the straight line as a whole. [20, p. 84]

This means that, despite the correspondence between them, the definitions of the concept of topological space and of the straight line are fundamentally different. In fact, they represent two different procedures to define a space. The first presupposes the existence of points and consists in organizing them so that certain properties are satisfied. It is exemplified by topological spaces. The second starts with pieces and consists in explicitly constructing points in terms of such pieces as illustrated by the case of the real line.

2.2. The idea of a generalization of the real line

Menger was perfectly aware of the difference between the definition of the concept of topological space and that of the real line and thought that the first should be modified so to be analogous to the second.

> In my book *Dimensionstheorie* I pointed out the desirability of an introduction of the general concept of space in topology which is not point set theoretical [*sic*] in the sense of Section 1, but rather analogous to the introductions of the straight line in arithmetic, outlined in Section 2. [20, p. 85]

[11] Compared to the other two cases, Menger here seems to nuance the correspondence as he writes: "A way seemingly related to the concept of closure class is Dedekind's definition of a real number as a cut or upper section in the set of rational numbers." [20, p. 83].

As to why it would be desirable for topological spaces to be defined analogously to the real line, Menger seems to have two things in mind.

The first is what Menger refers to as an abstract definition. In *Dimensionstheorie*, Menger makes a parallel with the transition from number fields to abstract fields. This implies that Menger wanted a definition that would induce an abstract concept of space. This is puzzling for Fréchet, Hausdorff and Kuratowski's concepts of space were already abstract, that is they are to number and function spaces what abstract fields are to number fields. Furthermore, in the historical notes at the end of §4, Menger states that Fréchet was the first to adopt an abstract point of view, that is one that makes abstraction of the arithmetic nature of the points of $\mathbb{R}^n$ (see [16, p. 23]).

This concern for abstraction is still present in the conference, but in a different form. Menger insists that it is the abstract properties of topological spaces that should be similar to those of the real line: "I should like to mention that in order to obtain a space whose abstract properties are related to those of a straight line [. . .]" [20, p. 88]

This suggests that when Menger talked of an abstract definition, what he actually meant was a generalization. Indeed, he is not looking for a definition that would subsume many cases under a single abstract one, but rather appears to want to reproduce the construction of the real numbers in the more general context of topological spaces. This also explains why he was not satisfied with the definitions of Fréchet, Hausdorff and Kuratowski; despite being abstract, they did not provide the generalization he had in mind.

The second aspect is that the definition of the general concept of space should have the same dignity as that of the real line. Menger writes:

> To make it clear that the general concept of space of set-theoretical geometry can be introduced so to have a dignity fully equivalent to that of the methods employed in analysis, we will now consider a foundation of the concept of space which is absolutely analog to the reasoning that was used in the previous section for the definition of the straight line and Cartesian space.[12] [16, p. 16]

[12] *Um deutlich zu machen, daß der allgemeine Raumbegriff der mengentheoretischen Geometrie durch Betrachtungen eingeführt werden kann, die an Dignität den in der gesamten Analysis angewandten Methoden völlig gleichstehen, gründen wir im folgenden den Raumbegriffe auf eine Überlegung, welche durchaus analog ist dem Rä-*

In the conference, the value of pointless topology is once again related to such a dignity: "To supply the [general concept of space] with a foundation whose logical dignity equals that of the basis of the concept of the straight line, is the purpose of our theory." [20, p. 85]

Unfortunately, neither *Dimensionstheorie* nor "Topology Without Points" go deeper into the question of what is that logical dignity that Menger attributes to the definition of real numbers and that the set-theoretical definitions of the concept of topological space does not have.

There is a third aspect that Menger does not explicitly touch on, but that seems to underlie the importance he gives to a definition of topological spaces generalizing that of the real line. Indeed, Menger's fundamental concern appears to be with continuity.

In the case of topological spaces, continuity is defined by means of the points of a set. In the case of the real line, points are explicitly constructed in order to form a continuum. This means that there is a conceptual tension that could be described as follows. On the one hand, continuity's most basic representation is given by the real line. As such, it does not rely on points. On the other hand, continuity in general can only be defined, and as a result completely understood as a mathematical phenomenon, in the context of topological spaces, that is on the basis of a point-based concept of space. Simply put, there are two treatments of continuity at play.

2.3. Lumps

In section 3 of "Topology Without Points", Menger presents the point-free definition of the concept of topological space first exposed in *Dimensionstheorie*. As stated in the previous section, his goal is to generalize the construction of the real line in order to obtain topological spaces. In particular, he takes as model the definition of real numbers as nested sequences of rational intervals.

> I especially aimed to introduce the points of a space as nested sequences of what may be called pieces or lumps — analogous to the introduction of real numbers as nested sequences of rational intervals, and related to Hausdorff's concept of a neighborhood. [20, p. 85]

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sonnement, das im vorigen Abschnitt zur Definition der Zahlengeraden und der Cartesischen Räume durchgeführt wurde.

To achieve this, Menger considers nested sequences of "generalized intervals". As indicated in the quotation above, these generalized intervals are what he calls lumps.

Now, this raises the obvious question of what exactly is a lump. First, Menger sees the words "piece" and "lump" as synonyms. In fact, the words seem to be used in the sense of R. L. Moore in his axiomatization of the plane topology of the place (see [23]). By a "piece" of the plane, Moore means any limited piece of the plane with an interior: "[...] the word ‚piece' is interpreted to mean any limited piece (in the ordinary sense) of the plane". [23, p. 14] In a footnote, he adds: "It would seem to be accordance with ordinary usage to refrain from applying the term ‚piece of space' to a point or a straight line or anything else which has no interior." [23, p. 14, n. 3]

Second, in the conclusion of his conference, Menger gives an indication of what he means by a lump: "[...] by a lump, we mean something with a well defined boundary." [20, p. 107]

While those indications do not provide a precise definition of the notion of lump, they do suggest that a lump must be thought of a part of space that is well delimitated and has an interior.

Because the central notion of his theory is that of lump, Menger actually refers to it as a topology of lumps.

2.4. The topology of lumps

Menger starts with a partially ordered system of lumps U, V, W, ... The order relation is "completely contained in" and is denoted by $\subset\subset$. Menger [20, p. 86] says he uses the relation "completely contained in" instead of "contained in" so that lumps U and V such that $U \subset\subset V$ behave like open sets such that the closure of U is a subset of V.[13]

The order relation gives rise to a criterion of identity for the elements of the partially ordered system. Two elements U and V are *identical* if for each element W, $W \subset\subset U$ if and only if $W \subset\subset V$. In other words, two lumps are identical if they completely contain the same lumps.

A *point* of this partially ordered system is a sequence of lumps U_1, U_2, ... such that $U_{k+1} \subset\subset U_k$ for each k. Simply put, a point is a strictly decreasing sequence of lumps.

[13] Interestingly, Menger would later use the relation "contained in". See [21, pp. 27–28].

An equality relation is defined for points. Two points U_1, U_2, ... and V_1, V_2, ... are *equal* if each U_i completely contains a V_j and each V_i completely contains a U_k. The idea is that U_i's completely contains almost all of the V_j and reciprocally.

An inclusion relation for points can also be defined. A point $U_1, U_2, \ldots$ *lies in* the lump U if there exist an i such that $U_i \subset\subset U$.

The set of all the points so defined is called a *space*.

Menger's topology of lumps unfortunately suffers from some problems. The first occurs when the lumps are the open sets of a topological space. Given a strictly decreasing sequences U_1, U_2, ... of open sets, four cases are possible: (i) the sequence contracts to a point p; (ii) the sets of the sequence have only one point in common, but they do not contract to this point; (iii) the sets of the sequence have many points in common; and (iv) the sets of the sequence have no points in common.

Seen as a particular case of the theory of lumps, the definition of real numbers as nested sequences of rational intervals works because the requirement that the length of the intervals converges towards 0 is sufficient to exclude case 3. As to cases 2 and 4, they "are automatically excluded" says Menger [20, p. 87]

This entails that for Menger's generalization of the real line to work or, in other words, for the concept of space at the core of the topology of lumps to be a suitable one, a general criteria that would exclude cases 2, 3 and 4 is required. In this respect, Menger mentions that in *Dimensionstheorie*, he did not give such a general criteria. Furthermore, he suggests that he only realized its necessity later on.

> In my book I suggested the considerations of points of a space as certain decreasing sequences of lumps, without giving a criterion as to which sequences should be called points. I pointed out the desirability of formulating such a criterion in my colloquium in Vienna. [20, p. 88]

A somewhat related problem is the following. As mentioned above, the topology of lumps is a device to define the concept of topological space as a generalization of the real line. In particular, Menger wanted topological spaces to have the same abstract properties as the real line. Menger says that this prompted him to consider a denumerable set of lumps. Now, this entails that the class of topological spaces so defined does not contain all topological spaces.

> [...] it is clear that if we start with a set of lumps that is altogether denumerable (as I originally did), then we can not possibly get all topo-

> logical spaces on the mere basis of the order relation for the lumps. For, the set of all partially ordered denumerable sets which are distinct (that is to say, no two of which are isomorphic) has the power of the continuum. Consequently, if we wish to describe the points of a space in terms of the order relation alone, we cannot get more than continuously many spaces — while the set of all types of topological spaces satisfying the second denumerability axiom, and even the set of all topological types of subsets of the straight line has a greater power than the continuum. [20, p. 88]

In the rest of the conference, Menger looks at attempts to define the concept of topological space without referring to the points of an underlying set. He identifies a fundamental difference between, on the one hand, the theories of Wald, Moore and himself and, on the other, those of Stone, Wallman and Milgram.[14]

> Another remark should clear up the relation between the theories discussed in sections 3–5, and those studied in sections 6–8. The former ones introduce points as nested sequences of lumps after the model of the introduction of real numbers as sequences of rational intervals — the latter ones introduce points as sets of lumps which if applied to the case of the straight line would yield a definition of a real number as the set of all open rational intervals containing the number. [20, p. 105]

This suggests that the difference between both classes of theories amounts to that between generalization and abstraction. On the one hand, the theories of Menger, Wald and Moore would define the points of a topological space as nested sequences of lumps by generalizing the construction of real numbers as nested sequences of rational intervals. On the other hand, the theories of Stone, Wallman and Milgram would be abstract ones from which a definition of real numbers could be deduced as a particular case. In other words, Menger, Wald and Moore adopt a bottom–up approach while Stone and his successors favor a top–down one.

While he recognizes that the latter approach has the advantage of being applicable to a wider class of spaces, Menger still prefers the former because of its simplicity or, to paraphrase him, its minimal logical machinery.

[14] See [20, §4–8]. The theories themselves are in [31], [23], [29], [30] and [22] respectively.

3. Nöbeling or lattice-theoretical topology

Before looking at *Grundlagen der analytichen Topologie*, a historical remark should be made. Nöbeling's PhD advisor was none other than Menger. Nöbeling also worked as a research assistant with Menger until 1933 when he left Vienna for Erlangen to work with Otto Haupt.[15]

This obviously raises the question of whether Nöbeling's research on pointless topology is indebted to Menger. Considering the working relationship from the late 1920s to the early 1930s, but also the fact that *Dimensionstheorie* is listed in the bibliography of *Grundlagen der analytischen Topologie*, Nöbeling had to know about Menger's idea of defining spaces independently of points. However, nothing suggests that Nöbeling looked at pointless topology in the continuity of Menger. In fact, Nöbeling's papers of 1948 [25] and 1953 [26] in which he had previously published parts of his theory make no reference to Menger while *Grundlagen der analytischen Topology* makes only one, but in relation to continuous curve (see [27, p. 128]).

Putting this historical question aside, the title of Nöbeling's book naturally raises the question of what analytical topology is. Nöbeling uses the expression as a synonym for general topology. Considering that by 1954 the denominations "general topology" or even "set-theoretical topology" were well-established, Nöbeling's choice of terminology is at the very least surprising, but intents to highlight the fact that analysis is based on topology. In a footnote, he writes:

> The name *general* topology is also usual. However, for a theory of the foundations of analysis, the name *analytic* topology appears clearer to us. Besides, the latter establishes a clearer delimitation with *algebraic* topology.[16] [27, p. VIII, n. 1]

This said, Nöbeling's book is not about general topology in the usual sense. The introduction actually makes it clear that the goal of the book is to develop topology independently of points. Nöbeling starts by pointing out that, because the fundamental concepts of analysis are that of convergence of a sequence and continuity of a function, topological

[15] For further biographical information on Nöbeling, see [7].

[16] *Es ist auch der Name* allgemeine *Topologie gebräuchlich. Für eine Theorie der Grundbegriffe der Analysis erscheint uns jedoch der Name* analytische *Topologie deutlicher. Außerdem liegt im letzteren eine klarere Abgrenzung gegenüber der* algebraische *Topologie.*

spaces have traditionally been approached by means of the notions of point and point-set. He then suggests that making abstraction of the points would provide clearer foundations for topology: "To clarify this situation and also from a methodological point of view, it seems justified to ask whether one could not initially do without the points and only introduce them when they are really needed.[17]" [27, p. IX]

Nöbeling's solution is to resort to the theory of posets and lattices. Indeed, using the closure operator, topologies can be defined on posets. So, for Nöbeling, the concept of topological space is derived from that of topological poset and the theory of topological spaces is a particular case of the theory of posets.

3.1. The theory of posets and lattices

Because it is a preamble to topology, the theory of posets and lattices is the object of the first chapter of *Grundlagen der analytischen Topologie* which, incidentally, is aptly titled "Preliminaries"[18].

Before going any further, it must be said that the following does not pretend to give an exhaustive presentation of that first chapter, but simply to present notions that will be directly involved in the theory of topological spaces.

Nöbeling starts by defining the concept of poset. A set $\mathfrak{B}$ is a *poset* if there is a binary relation $\leq$ such that, for all elements A, B and C, the following axioms are satisfied:

Axiom V_1. if $A \leq B \leq C$, then $A \leq C$;

Axiom V_2. if $A \leq B \leq A$, then $A = B$ and conversely.[19]

An element of a poset is called a *soma*. Nöbeling takes the word from Constantin Carathéodory to whom he refers in his 1948 paper *Topologie der Vereine und Verbänte* (see [25, p. 1]).

Now, this terminological choice is in itself revealing. Carathéodory coined the term "soma" in the late 1930s in the context of algebraic theory of measure to designate elements of a Boolean algebra forming sets that themselves had to be considered as elements of a set. These

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[17] *Um diese Verhältnisse zu klären und auch vom methodischen Standpunkt aus erscheint es gerechtfertig zu fragen, ob man nicht auf die Punkte zunächst verzichten und sie erst dann einführen kann, wenn man sie wirklich braucht.*

[18] *Vorbereitungen* in German.

[19] In contradistinction to the contemporary definition, Nöbeling does not require $\leq$ to be reflexive, but points out in a footnote that it a consequence of his axiom V_2.

elements were intended to be the equivalent of the points in the theory of abstract spaces. However, Carathéodory did not want to call them points because they did not agree with Euclid's conception of a point as that which is indivisible. He also did not want to refer to them as sets because they have different properties. In particular, the sum of uncountably many somas is not necessarily a soma.

> For mathematical objects with properties so fundamental and important, one must have a name that is neutral and suggests no false associations. We will see that these objects are not always sets so the word "set" cannot be used. The name "body" is also excluded because its use would lead to misunderstandings; the "structures" introduced by *Ore* are far too general. However, nothing prevents to talk of *soma* (τὸ σῶμα= body).[20] [4, p. 304]

By designating the elements of posets as somas, Nöbeling is clearly taking an abstract stance.[21]

As he will do with many notions introduced in the first chapter, Nöbeling gives a set-theoretical example of poset which he says is the most important for his purposes. Let $\mathfrak{B}$ be a system of subsets. The inclusion $\subseteq$ between subsets defines a relation satisfying axioms V_1 and V_2. The resulting poset is called the set-theoretical poset.[22]

Various notions related to posets are next introduced. For the definition of the concepts of lattice and, by extension, of topological space, the following are especially important.

First, let $\mathfrak{B}$ be a poset and $(A_i)_{i\in I}$ be a family of somas. A soma S is an *upper bound* if for all $i \in I$, $A_i \leq S$. A soma S is called a *lower bound* if for all $i \in I$, $S \leq A_i$. Each poset contains a *unit* and a *zero*, that is two somas that are respectively an upper bound and a lower bound for all the somas of the poset

Second, let $\mathfrak{B}$ be a poset and $(A_i)_{i\in I}$ be a family of somas. A *least upper bound* of the family $(A_i)_{i\in I}$ is a soma V such that (i) $A_i \leq V$ for

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[20] *Für mathematische Objekte, die so grundlegende und wichtige Eigenschaften besitzen, muß man einen Namen haben, der ganz neutral ist und keine falschen Assoziationen erweckt. Wir werden sehen, daß diese Objekte nicht immer Mengen sind, so daß die Bezeichnung „Menge“nicht benutzt werden kann. Der Name „Körper“, der vieles für sich hätte, ist ebenfalls ausgeschlossen, weil sein Gebrauch zu Mißverständnissen führen würde; die „Strukturen”, die* Ore *eingefürht hat, sind viel zu allgemein. Nichts hindert aber* Soma *(τὸ σῶμα = Körper) zu sagen.*

[21] For more on Carathéodory's theory of somas, see [4, 5].

[22] *Mengenverein* in German.

all $i \in I$ and (ii) if $A_i \leq B$ for all $i \in I$, then $V \leq B$. V is then called the *join* of the family and is noted $\bigvee A_i$. A *greatest lower bound* of the family is a soma D such that (i) $D \leq D_i$ for all $i \in I$ and (ii) if $B \leq A_i$ for all $i \in I$, then $B \leq D$. D is called the *meet* of the A_i's and is noted by $\bigwedge A_i$.

For example, if $\mathfrak{B}$ is the set-theoretical poset whose elements are the subsets of a set, the least upper bound and greatest lower bound are simply the set-theoretical union $\cup$ and intersection $\cap$.

Third, let $\mathfrak{B}$ a poset. A soma A of $\mathfrak{B}$ is an *atom* if A is different from zero and there is no soma B different from zero such that $B < A$. A poset is *atomic* if every soma A of $\mathfrak{B}$ can be represented as a join of atoms.

The notion of lattice is introduced by means of $\vee$- and $\wedge$-posets. Let $\mathfrak{B}$ be a poset. $\mathfrak{B}$ is a $\vee$-*poset* if, for any somas A and B of $\mathfrak{B}$, the join $A \vee B$ exists and is in $\mathfrak{B}$. Likewise, $\mathfrak{B}$ is a $\wedge$-*poset* if the meet $A \wedge B$ exists and is in $\mathfrak{B}$ for any somas A and B of $\mathfrak{B}$. A *lattice* is simply a $\vee$- and $\wedge$-poset.

Some properties of lattices are then defined. Let $\mathfrak{B}$ be a poset. $\mathfrak{B}$ is *complete* if, for any family of somas $(A_i)_{i \in I}$, the join $\bigvee A_i$ and the meet $\bigwedge A_i$ exist. $\mathfrak{B}$ is *distributive* if $A_1 \wedge (A_2 \vee A_3) = (A_1 \wedge A_2) \vee (A_1 \wedge A_3)$. Assuming that $\mathfrak{B}$ has a unit E and a zero 0, then $\mathfrak{B}$ is complemented if, for each soma A, there is a soma cA such that $A \vee cA = E$ and $A \wedge cA = 0$.

In particular, a lattice that is distributive and complemented is *Boolean*.

Another important notion is that of poset homomorphism. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be two posets. An *homomorphism* from $\mathfrak{B}$ into $\mathfrak{B}'$ is an assignment Φ that associates to each soma A of $\mathfrak{B}$ a soma $A' = \Phi A$ of $\mathfrak{B}'$ such that $A_1 \leq A_2$ entails $\Phi A_1 \leq \Phi A_2$. A *onto* homomorphism is defined as expected. A homomorphism $\Phi \colon \mathfrak{B}$ to (onto) $\mathfrak{B}'$ is an *isomorphism* from $\mathfrak{B}$ to (onto) $\mathfrak{B}'$ if $\Phi A_1 \leq \Phi A_2$ entails $A_1 \leq A_2$.

As an example, Nöbeling defines the notion of function between lattices, specifying again that he considers it the most important. Let E and E' be two sets and $\mathfrak{E}$ and $\mathfrak{E}'$ be the respective complete lattices of their subsets. Let every element $p \in E$ be assigned to an element $p' = \phi p \in E'$, that is let ϕ be a function $E \to E'$. This function ϕ determines a homomorphism, also denoted ϕ, from $\mathfrak{E}$ to $\mathfrak{E}'$ by assigning to each set A of $\mathfrak{E}$ the set A' of $\mathfrak{E}'$ whose elements are $p' = \phi p$ for $p \in A$. This homomorphism is called a *function* from $\mathfrak{E}$ to $\mathfrak{E}'$. Reciprocally, the function $\phi \colon \mathfrak{E} \to \mathfrak{E}'$ determines the function $\phi \colon E \to E'$ in a unique manner.

3.2. Topological spaces as lattices

The 148-page long second chapter is dedicated to topological structures. As hinted at previously, Nöbeling is able to define the concepts of topological space and, as a result, to develop topology on the sole basis of the poset- and lattice-theoretical notions introduced in the first chapter. From this perspective, general topology becomes part of the theory of lattices and is developed independently of points. This would prompt A. H. Stone to write in his review of *Grundlagen der analystischen Topologie* that "on the whole the essential fundamentals of 'point-set' topology have been fitted into this 'pointless' setting" [28]. As the considerations below shall make clear, Nöbeling does so by adopting a top–down approach.

Nöbeling starts by defining the notion of a topology on a poset.[23] Let $\mathfrak{B}$ be a poset. A *topological structure* or *topology* on $\mathfrak{B}$ is an endomorphism $\top$ assigning to each soma A a soma $\top A = \overline{A}$ of $\mathfrak{B}$ called its closure such that

Axiom H_0. if $A_1 \leq A_2$, then $\overline{A_1} \leq \overline{A_2}$;
Axiom H_1. $A \leq \overline{A}$;
Axiom H_2. $\overline{\overline{A}} = \overline{A}$.[24]

A poset $\mathfrak{B}$ equipped with such an endomorphism is called a *topological poset*.

Let $\mathfrak{B}$ be a topological poset. A soma A of $\mathfrak{B}$ is *closed* if $\overline{A} = A$. Closed somas have the desired properties in that both the maximal element of a topological poset and the meet of an arbitrary number of closed somas, if they exist, are closed.

Let $\mathfrak{B}$ be a $\vee$-poset with a minimal element 0. $\mathfrak{B}$ has a *classic topology* if it satisfies the following axioms which, as Nöbeling points out, are nothing but Kuratowski's closure axioms:

Axiom H_1. $A \leq \overline{A}$;
Axiom H_2. $\overline{\overline{A}} = \overline{A}$;

[23] The notion of a topology is actually defined for the first time in the section on homomorphisms as an example of a reversible one. See [27, p. 17].

[24] It should be pointed out that Nöbeling's definition of topology marks a departure within the lattice-theoretical approach in so far as Stone and his followers defined the notion by means of prime ideals. Also, Nöbeling's axioms are not some early version of the conditions defining a Lawvere–Tierney topology in elementary topos.

Axiom H_3. $\overline{A_1 \vee A_2} = \overline{A_1} \vee \overline{A_2}$;
Axiom H_4. $\overline{0} = 0$.

In a classic topological poset, the join of a finite number of closed somas is closed.

At this point, Nöbeling turns his attention to topological Boolean lattices. Let $\mathfrak{B}$ be a topological Boolean lattice. For any soma A of $\mathfrak{B}$, the soma $\underline{A} = c\overline{cA}$ is the *interior* of A. A soma A of $\mathfrak{B}$ is *open* if $A = \underline{A}$ and an open soma U is a *neighborhood* of A if $A \leq U$. As the notations $\overline{A}$ and $\underline{A}$ imply, the closure and interior operators are dual.

In particular, topological spaces are special instances of topological Boolean lattices. In this regard, Nöbeling writes that "[t]he most important examples of our theory are the topological spaces.[25]" [27, p. 47] A *topological space* is a topological atomic complete Boolean lattice $\mathfrak{B}$. If the topology of $\mathfrak{B}$ is classic, then $\mathfrak{B}$ is a *classic topological space*.

Having previously shown, using Stone's representation theorem, that any atomic Boolean lattice is isomorphic to a lattice of sets[26], Nöbeling points out that to any topological space corresponds a lattice of sets whose subsets form a fixed set E. This set E and its elements are respectively called the *support* and the *points* of the space.

Nöbeling's study of topological spaces is oriented towards recuperating what he has identified in the introduction as the fundamental notions of analysis, to wit convergence of a sequence and continuity of a function. For the first, he develops a theory of limits adapted to topological posets.[27]

For the second, he uses poset homomorphisms which he defines as follows. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be two topological posets. A homomorphism $\Phi\colon \mathfrak{B} \to \mathfrak{B}$ is *continuous* if for any somas A and B of $\mathfrak{B}$ such that $A \leq \overline{B}$, $\Phi A \leq \overline{\Phi B}$.

In the case where Φ has an inverse, the above definition is equivalent to the usual one, that is Φ is continuous if an only if, for any closed soma A' of $\mathfrak{B}'$, $\Phi^{-1}A'$ is a closed soma of $\mathfrak{B}$.

Continuous homomorphisms allow Nöbeling to define homeomorphisms. An isomorphism $\Phi\colon \mathfrak{B} \to \mathfrak{B}'$ is a *homeomorphism* if Φ and its inverse Φ^{-1} are continuous. As expected, properties invariant under homeomorphisms are called *topologically invariant*.

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[25] *Die wichtigsten Beispiele unserer Theorie sind die topologischen Räume.*

[26] See [27, Theorem 5.3, p. 31].

[27] For this, see [27, §8].

Since the notion of function is defined by means of that of homomorphism, a continuous function between two topological spaces is simply a particular case of continuous homomorphism between topological posets. Likewise, a function is said to be *topological* if it is a homeomorphism.

Nöbeling then gives a local condition for a function to be continuous. Let $\mathfrak{E}$ and $\mathfrak{E}'$ be two topological spaces. A function $\phi\colon \mathfrak{E} \to \mathfrak{E}'$ is *continuous* if and only if for each point $p \in \mathfrak{E}$ and each neighborhood U' of ϕp in $\mathfrak{E}'$, there exists a neighborhood of p in $\mathfrak{E}$ such that $\phi(U) \subseteq U'$. In other words, a function is continuous if and only if it is at every point of its domain. This is remarkable because it reverses the usual order. In classic analysis, continuity in general is defined by means of continuity at a point. In the lattice-theoretical framework, continuity at a point is deduced from the definition of continuity in general.

In the third and last chapter of *Grundlagen der analytischen Topologie*, Nöbeling presents uniform structures as particular cases of the theory of topological structures developed in the second chapter.

3.3. Pointless topology: an abstract algebraic theory

As seen in section 2, Menger's interest in a point-free concept of topological space was motivated by analysis. More specifically, he wanted to eliminate the conceptual tension generated by the two different treatments of continuity — one for the real line and one for topological spaces. To do so, he proposed to define topological spaces by generalizing the construction procedure of the real numbers as nested sequences of rational intervals. In this sense, his approach was bottom-up.

Nöbeling is also motivated by analysis. Indeed, he sees pointless topology as a mean to provide analysis with better foundations. However, in contradistinction to Menger, Nöbeling takes a top–down approach in that he develops an abstract theory from which can be deduced a theory of topological spaces.

Another aspect that sets apart Nöbeling's approach to pointless topology from Menger's is its algebraic character. In this regard, it puts forward a radically different conceptualization of space.

First, in *Grundlegen der analytischen Topologie*, a topological space is a lattice, a topological atomic complete Boolean one to be exact. So, a topological space is not so much to be thought of as set of points with a spatial structure, but as a particular case of topological poset and, by extension, of poset. Now, posets and lattices being algebraic structures, topological spaces have to be seen, not as geometrical or even analytical objects, but as genuine algebraic objects.

Consequently, topological spaces no longer need to have a geometrical inspiration. This disconnects topology from the geometrical background it rose from historically and, simultaneously, widens its field of applicability.

Second, general topology itself becomes algebrized insofar as the methods by means of which topological spaces are studied are purely algebraic. Historically, this idea certainly was not new and should be attributed to Stone. Indeed, Stone duality had shown that the methods of modern algebra could be transferred to the study of topology spaces. This possibility is a consequence of the equivalence of the theories of topological spaces and of Boolean algebras by virtue of which any proposition about the latter can be translated into a proposition about the former and vice-versa. In this sense, algebraic methods are only applied indirectly to topological spaces; what is really being manipulated algebraically are Boolean algebras known to be equivalent (see [29, 30]). In comparison, Nöbeling's approach is much stronger for the inclusion of topology into poset theory allows him to take the methods of poset and lattice theory as the natural methods of general topology.

4. Conclusion

The analysis of Menger's and Nöbeling's respective idea of a point-free definition of the concept of topological space attest of the evolution of pointless topology during its pre-localic phase.

Menger's topology of lumps was motivated by the double treatment of continuity at play in the contexts of topological spaces and the real line and was intended to allow for a generalization of the construction of the real line. As such, Menger's approach was a bottom-up one.

Working in the continuity of Stone, Nöbeling adopted an algebraic point of view instead of Menger's geometric one. Indeed, he derived a theory of topological spaces from an abstract theory of posets and lattices. To be specific, he used the closure operator to define the notion of topological poset, that is a poset equipped with a topology, and then defined the concept of topological space as a particular case. For this reason, Nöbeling's approach is top-down.

Historically, the idea that the concept of topological space should be independent from points was further developed in the context of locale theory in the late 1950s and, a few years later, in the context of category theory. Now, one remarkable aspect is that, even though lattices remained relevant and the approach was just as abstract, these developments mark at least a partial departure from the lattice-theoretical point of view, and especially from Nöbeling's work. Indeed, Bénabou's referring to Nöbeling appears to be the exception rather the rule.[28]

This being said, to properly understand the transition from the lattice-theoretical approach to the localic one, it should be kept in mind, as Johnstone emphasizes (see [11, 12]), that the goals were considerably different in both enterprises. Whereas Nöbeling was trying to recover the classic concept of topological space which goes back to Hausdorff, locale and category theorists were searching for a generalized concept of space. The best example in this respect has to be Grothendieck's concept of topos.

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