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NEGATION IN WEAK POSITIONAL CALCULI

Abstract. Four weak positional calculi are constructed and examined. They refer to the use of the connective of negation within the scope of the positional connective "$\mathcal{R}$" of realization. The connective of negation may be fully classical, partially analogical or independent from the classical, truth-functional negation. It has been also proved that the strongest system, containing fully classical connective of negation, is deductively equivalent to the system MR from Jarmużek and Pietruszczak.

Keywords: negation, distribution, positional, many-valued.

1. Introduction

Jerzy Łoś delivered the first positional calculus in his master's thesis, supervised by Jerzy Słupecki, completed and accepted as early as 1947 and published in 1948 [4]. Łoś' main concern was to refer sentences of physical discourse to positions in time. To achieve this Łoś introduced a connective of *realization*, referring a formula φ to a term α, to the effect that the formula φ is realized in the point denoted by α, e.g. the formula "the moon is waxing" may be referred to particular dates: the moon is waxing on t, where t is a date. Nicolas Rescher and Alasdair Urquhart reviewed other domains of reference, like positions in space [7]. Generally, realization or occurence at a point is considered in positional logic, instead of mere occurence. In 2004 Tomasz Jarmużek and Andrzej Pietruszczak examined a very weak and very general positional calculus MR [2]. The system is designed to be the minimal one which allows to prove that the connective of realization is distributive over all classical connectives. The objective of this paper is to generalize Jarmużek and

Pietruszczak's result on some cases even weaker than MR with respect to the connective of negation.

2. The System MR

As it has been just mentioned, the system MR comes from Tomasz Jarmużek and Andrzej Pietruszczak [2]. The letter "M" in "MR" stands for "minimal". The system is in fact minimal in the sense that it guarantees the connective of realization "$\mathcal{R}$" to be distributive over all classical connectives [2, p. 148].

The set of well formed formulas of the system MR may be called $\mathbb{FM}$. The alphabet consists of (a) an infinite, but denumerable, set $\mathbb{SL}$ of schematic sentence letters, (b) an infinite, but denumerable, set $\mathbb{IN}$ of schematic positional letters, also known as schematic indicators, (c) the connectives: "$\mathcal{R}$" of realization, "$\neg$" of negation, "$\wedge$" of conjunction, "$\vee$" of disjunction, "$\rightarrow$" of conditional and "$\equiv$" of equivalence, as well as (d) parentheses, serving for punctuation signs. All the connectives, but "$\mathcal{R}$" are *propositional*, whereas the connective "$\mathcal{R}$" is *positional*. The indicators are usually to be understood as proper names of positions, however, on the purely formal level we only require the sets $\mathbb{SL}$ and $\mathbb{IN}$ to be mutually exclusive, i.e. $\mathbb{SL} \cap \mathbb{IN} = \varnothing$. The set $\mathbb{QF}$ of quasi-formulas is characterized as the smallest collection, containing the set $\mathbb{SL}$ of schematic sentence letters, and closed under the following operations: $\ulcorner(\neg\varphi)\urcorner \in \mathbb{QF}$, provided $\varphi \in \mathbb{QF}$, and $\ulcorner(\varphi \wedge \psi)\urcorner$, $\ulcorner(\varphi \vee \psi)\urcorner$, $\ulcorner(\varphi \rightarrow \psi)\urcorner$, $\ulcorner(\varphi \equiv \psi)\urcorner \in \mathbb{QF}$, provided $\varphi, \psi \in \mathbb{QF}$. An atomic formula of $\mathbb{FM}$ is any sign cluster

$$\ulcorner\mathcal{R}_\alpha\varphi\urcorner, \tag{1}$$

in which $\alpha \in \mathbb{IN}$ and $\varphi \in \mathbb{QF}$. An atomic formula (1) is to be read: at the point α it is the case that φ — or similarly. The set $\mathbb{FM}$ of formulas of the language we consider is characterized as the smallest collection containing the set of all atomic formulas of $\mathbb{FM}$ and closed under the following operations: $\ulcorner(\neg\varphi)\urcorner \in \mathbb{FM}$, provided $\varphi \in \mathbb{FM}$, and $\ulcorner(\varphi \wedge \psi)\urcorner$, $\ulcorner(\varphi \vee \psi)\urcorner$, $\ulcorner(\varphi \rightarrow \psi)\urcorner$, $\ulcorner(\varphi \equiv \psi)\urcorner \in \mathbb{FM}$, provided $\varphi, \psi \in \mathbb{FM}$. Having excluded any ambiguity, we traditionally allow to omit parentheses. In such cases the following order of connectives: "$\mathcal{R}$", "$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\equiv$" is to be preserved.

Notice that in the language $\mathbb{FM}$ all schematic letters appear always within the scope of the connective "$\mathcal{R}$" and that no nested occurences of

the connecive "$\mathcal{R}$" are allowed. Those features constitute *weak* positional calculi.

Single-sentence-letter formulas are such members of $\mathbb{FM}$ that involve exactly one occurence of an element of $\mathbb{SL}$ and no iteration of any connective, i.e. any formulas:

$$\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner \text{ and } \ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner, \tag{2}$$

in which $\alpha \in \mathbb{IN}, \varphi \in \mathbb{SL}$. Such formulas make the set $\mathbb{SSL}$. The set $\mathbb{EF}$ of elementary formulas is the smallest collection containing all elements of $\mathbb{SSL}$ and such that $\ulcorner \varphi \vee \psi \urcorner \in \mathbb{EF}$, provided $\varphi, \psi \in \mathbb{EF}$. The set $\mathbb{NF}$ of conjunctive normal forms is the smallest collection containing all elements of $\mathbb{EF}$ and such that $\ulcorner \varphi \wedge \psi \urcorner \in \mathbb{NF}$, provided $\varphi, \psi \in \mathbb{NF}$.

The system MR has been originally axiomatized as follows. Let $\mathbb{CPC}$ be the set of all tautologies of the classical propositional calculus, let $\alpha \in \mathbb{IN}, \varphi, \psi \in \mathbb{QF}$ and let $e \colon \mathbb{QF} \longrightarrow \mathbb{FM}$ be any substitution of $\mathbb{FM}$-formulas for all schematic letters of $\mathbb{QF}$. Then the following formulas are axioms:

$$e(\varphi), \text{ provided } \varphi \in \mathbb{CPC}, \tag{3}$$
$$\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \text{ provided } \varphi \in \mathbb{CPC}, \tag{4}$$
$$\mathcal{R}_\alpha \neg \varphi \equiv \neg \mathcal{R}_\alpha \varphi, \tag{5}$$
$$\mathcal{R}_\alpha \varphi \wedge \mathcal{R}_\alpha \psi \to \mathcal{R}_\alpha(\varphi \wedge \psi). \tag{6}$$

The set of axioms is the smallest collection meeting the above formulated conditions. The rule of *Modus Ponens* is the unique primitive rule of inference:

$$\begin{array}{l} \varphi, \\ \underline{\varphi \to \psi,} \\ \psi, \end{array} \tag{MP}$$

for all $\varphi, \psi \in \mathbb{FM}$. The set of theorems of MR is the smallest collection containing all the axioms and closed under the rule (MP) [2, p. 149–150].

Calculi in the set $\mathbb{FM}$ of well formed formulas may be considered *weak positional calculi*. It may be asked, what other than MR, interesting weak positional calculi there exist. In this paper I provide a general model template for such calculi and examine some interesting examples.

As it has been already mentioned, in the system MR the connective "$\mathcal{R}$" is distributive over all sentential connectives, i.e., for all $\alpha \in$

$\mathbb{IN}, \varphi, \psi \in \mathbb{QF}$, the following *distributive laws* are provable in MR:

$$\mathcal{R}_\alpha \neg\varphi \equiv \neg\mathcal{R}_\alpha\varphi, \tag{5}$$

$$\mathcal{R}_\alpha(\varphi \wedge \psi) \equiv \mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi, \tag{7}$$

$$\mathcal{R}_\alpha(\varphi \vee \psi) \equiv \mathcal{R}_\alpha\varphi \vee \mathcal{R}_\alpha\psi, \tag{8}$$

$$\mathcal{R}_\alpha(\varphi \to \psi) \equiv \mathcal{R}_\alpha\varphi \to \mathcal{R}_\alpha\psi, \tag{9}$$

$$\mathcal{R}_\alpha(\varphi \equiv \psi) \equiv (\mathcal{R}_\alpha\varphi \equiv \mathcal{R}_\alpha\psi). \tag{10}$$

The proofs have been delivered in the referred paper [2, p. 151–153]. Jarmużek and Pietruszczak have also proved the adequacy (i.e. soundness and completeness) result for the system MR with respect to a very simple structure [2, p. 154–159].

However, there are good reasons to consider positional calculi even weaker than MR. Such calculi lack of some of the distributive laws (5), (7)–(10). For example, in an earlier work of mine I claimed that temporal realization in physical discourse, prominently that of Relativity Theory, lacks of distribution with respect to negation. The formula

$$\neg\mathcal{R}_\alpha\varphi \to \mathcal{R}_\alpha\neg\varphi \tag{11}$$

is not valid in such a discourse, for — according to the principles of relativity — the antecedent means that it is not the case that α and φ occur simultaneously, while the consequent means that α occurs simultaneously to $\ulcorner\neg\varphi\urcorner$. Hence, the formula (11) is clearly false, provided α does not occur at all.

Lack of some distributive laws makes the connective in question, when it appears within the scope of "$\mathcal{R}$", actually non-classical, deviant. Consequently in the sysem MR all the connectives within the scope of "$\mathcal{R}$" are perfectly classical.

3. A general model template

The objective of this paper is to provide a possibly general and uniform template of formal semantics for weak positional calculi. What is to be delivered is actually a generalization of Jarmużek and Pietruszczak's result. Within such a template non-standard negation is to be considered. So, a model in general is any structure $\mathfrak{M}$ of the sort

$$\mathfrak{M} = \langle \mathbb{U}, \Omega, \Omega^\star, \mathfrak{d}, \mathfrak{s} \rangle,$$

where

$$\mathbb{U}, \Omega, \Omega^\star \text{ are non-empty sets,} \tag{12}$$

$$\Omega^\star \subseteq \Omega, \tag{13}$$

$$\mathfrak{d} \colon \mathbb{IN} \longrightarrow \mathbb{U}, \tag{14}$$

$$\mathfrak{s} \colon \mathbb{U} \times \mathbb{QF} \longrightarrow \Omega. \tag{15}$$

The set $\mathbb{U}$ may be understood as a set of positions of any kind, e.g. positions in time, space, persons, worlds or whatever, the set Ω is the set of truth-values, the set $\Omega^\star$ is the set of designated truth-values, the denotation function $\mathfrak{d}$ attributes positions to indicators and the satisfaction function $\mathfrak{s}$ attributes truth-values to elements of $\mathbb{QF}$, relative to positions. Formulas of weak positional language, i.e. elements of the set $\mathbb{FM}$, are true or false in such models. To be false in a model means exactly not to be true in it. So, for any model $\mathfrak{M}$ and any $\varphi \in \mathbb{FM}$, either $\mathfrak{M} \vDash \varphi$, or $\mathfrak{M} \nvDash \varphi$. Let $\alpha \in \mathbb{IN}$ and $\varphi \in \mathbb{QF}$:

$$\mathfrak{M} \vDash \ulcorner \mathcal{R}_\alpha \varphi \urcorner \text{ if and only if } \mathfrak{s}(\mathfrak{d}(\alpha), \varphi) \in \Omega^\star. \tag{16}$$

So, an atomic formula $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ is true in a model $\mathfrak{M}$ if and only if φ takes on a designated value at $\mathfrak{d}(\alpha)$. Truth conditions for compound formulas are classical:

$$\mathfrak{M} \vDash \ulcorner \neg \varphi \urcorner \text{ if and only if } \mathfrak{M} \nvDash \varphi, \tag{17}$$

$$\mathfrak{M} \vDash \ulcorner \varphi \wedge \psi \urcorner \text{ if and only if } \mathfrak{M} \vDash \varphi \text{ and } \mathfrak{M} \vDash \psi, \tag{18}$$

$$\mathfrak{M} \vDash \ulcorner \varphi \vee \psi \urcorner \text{ if and only if } \mathfrak{M} \vDash \varphi \text{ or } \mathfrak{M} \vDash \psi, \tag{19}$$

$$\mathfrak{M} \vDash \ulcorner \varphi \rightarrow \psi \urcorner \text{ if and only if } \mathfrak{M} \nvDash \varphi \text{ or } \mathfrak{M} \vDash \psi, \tag{20}$$

$$\mathfrak{M} \vDash \ulcorner \varphi \equiv \psi \urcorner \text{ if and only if } \mathfrak{M} \vDash \varphi, \psi \text{ or } \mathfrak{M} \nvDash \varphi, \psi, \tag{21}$$

for any $\varphi, \psi \in \mathbb{FM}$. Formulas may be considered as valid in a sense if and only if they are true in a set of models related to the sense.

4. Four-Cornered Models

Let $\mathfrak{X}$ be a set of models determined by the sets $\Omega_{\mathfrak{X}}$, $\Omega_{\mathfrak{X}}^\star$ and the function $\mathfrak{s}_{\mathfrak{X}}$. In the models to be considered in this paper the function $\mathfrak{s}_{\mathfrak{X}}$ is described by means of matrix valuations. Depending on the class of models, in $\Omega_{\mathfrak{X}}$ there appear at most four truth-values:

    1 — truth and nothing but the truth,

    0 — falsehood and nothing but the falsehood,

X — both truth and falsehood,

Y — neither truth nor falsehood.

The values 1 and 0 are obviously identical with the classical truth and falsehood respectively, whereas X is known as *truth-value glut* and Y is known as *truth-value gap*. The values admitted refer to the so called *four corners of truth*, described by a pre-sixth century Indian philosopher Sanjaya [6]. In the description of $\mathfrak{s}_{\mathfrak{x}}$ the following operations on the truth-values play an auxiliary rôle:

$$\mathfrak{f}_{\mathfrak{x}}^{-} : \Omega_{\mathfrak{x}} \longrightarrow \Omega_{\mathfrak{x}}, \tag{22}$$

$$\mathfrak{f}_{\mathfrak{x}}^{\sharp} : \Omega_{\mathfrak{x}} \times \Omega_{\mathfrak{x}} \longrightarrow \Omega_{\mathfrak{x}}, \tag{23}$$

for $\mathfrak{f}_{\mathfrak{x}}^{\sharp}$ being $\mathfrak{f}_{\mathfrak{x}}^{\wedge}, \mathfrak{f}_{\mathfrak{x}}^{\vee}, \mathfrak{f}_{\mathfrak{x}}^{\rightarrow}$ or $\mathfrak{f}_{\mathfrak{x}}^{\equiv}$. If $\varphi \in \mathbb{SL}$, the only requirement imposed on $\mathfrak{s}_{\mathfrak{x}}$ is

$$\mathfrak{s}_{\mathfrak{x}}(x, \varphi) \in \Omega_{\mathfrak{x}}, \tag{24}$$

In other elements of $\mathbb{QF}$, i.e. compound quasi-formulas, the function $\mathfrak{s}$ takes on the following values:

$$\mathfrak{s}_{\mathfrak{x}}(x, \neg\varphi) = \mathfrak{f}_{\mathfrak{x}}^{-}(\mathfrak{s}_{\mathfrak{x}}(x, \varphi)), \tag{25}$$

$$\mathfrak{s}_{\mathfrak{x}}(x, \varphi \wedge \psi) = \mathfrak{f}_{\mathfrak{x}}^{\wedge}(\mathfrak{s}_{\mathfrak{x}}(x, \varphi), \mathfrak{s}_{\mathfrak{x}}(x, \psi)), \tag{26}$$

$$\mathfrak{s}_{\mathfrak{x}}(x, \varphi \vee \psi) = \mathfrak{f}_{\mathfrak{x}}^{\vee}(\mathfrak{s}_{\mathfrak{x}}(x, \varphi), \mathfrak{s}_{\mathfrak{x}}(x, \psi)), \tag{27}$$

$$\mathfrak{s}_{\mathfrak{x}}(x, \varphi \rightarrow \psi) = \mathfrak{f}_{\mathfrak{x}}^{\rightarrow}(\mathfrak{s}_{\mathfrak{x}}(x, \varphi), \mathfrak{s}_{\mathfrak{x}}(x, \psi)), \tag{28}$$

$$\mathfrak{s}_{\mathfrak{x}}(x, \varphi \equiv \psi) = \mathfrak{f}_{\mathfrak{x}}^{\equiv}(\mathfrak{s}_{\mathfrak{x}}(x, \varphi), \mathfrak{s}_{\mathfrak{x}}(x, \psi)). \tag{29}$$

An element of the set $\mathbb{FM}$ of formulas is $\mathfrak{X}$-valid if and only if it is true in all models of the set $\mathfrak{X}$. A calculus is $\mathfrak{X}$-sound if and only if all the calculus' theorems are $\mathfrak{X}$-valid. A calculus is $\mathfrak{X}$-complete if and only in all the $\mathfrak{X}$-valid elements of the set $\mathbb{FM}$ are provable in the calculus. A calculus is $\mathfrak{X}$-adequate if and only if it is both $\mathfrak{X}$-sound and $\mathfrak{X}$-complete.

5. The System $\mathsf{R}_{\mathfrak{B}}$

The system $\mathsf{R}_{\mathfrak{B}}$ is algebraically analogical to Nuel D. Belnap's four-valued logic, originally concerned with the way a computer should work on possibly incomplete or inconsistent data [1]. The set $\mathfrak{B}$ of models is determined by the sets

$$\Omega_{\mathfrak{B}} = \{1, 0, X, Y\}, \ \Omega_{\mathfrak{B}}^{\star} = \{1, X\}, \tag{30}$$

as well as by the collection of operations $\mathfrak{f}_{\mathfrak{B}}^{\neg}, \mathfrak{f}_{\mathfrak{B}}^{\wedge}, \mathfrak{f}_{\mathfrak{B}}^{\vee}, \mathfrak{f}_{\mathfrak{B}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{B}}^{\equiv}$ on $\Omega_{\mathfrak{B}}$, presented on the table 1. A formula $\varphi \in \mathbb{FM}$ is $\mathfrak{B}$-valid if and only if φ is

	¬
1	0
X	X
Y	Y
0	1

∧	1	X	Y	0
1	1	X	Y	0
X	X	X	0	0
Y	Y	0	Y	0
0	0	0	0	0

∨	1	X	Y	0
1	1	1	1	1
X	1	X	1	X
Y	1	1	Y	Y
0	1	X	Y	0

→	1	X	Y	0
1	1	X	Y	0
X	1	X	1	X
Y	1	1	Y	Y
0	1	1	1	1

≡	1	X	Y	0
1	1	X	Y	0
X	X	X	1	X
Y	Y	1	Y	Y
0	0	X	Y	1

Table 1. The operations $\mathfrak{f}_{\mathfrak{B}}^{\neg}, \mathfrak{f}_{\mathfrak{B}}^{\wedge}, \mathfrak{f}_{\mathfrak{B}}^{\vee}, \mathfrak{f}_{\mathfrak{B}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{B}}^{\equiv}$

true in every $\mathfrak{B}$-model. An axiomatics of the system $\mathsf{R}_{\mathfrak{B}}$ consists of the axioms (3), the rule (MP) and the axioms:

$$\mathcal{R}_\alpha(\varphi \wedge \psi) \equiv \mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi, \tag{31}$$

$$\mathcal{R}_\alpha\neg(\varphi \wedge \psi) \equiv \mathcal{R}_\alpha\neg\varphi \vee \mathcal{R}_\alpha\neg\psi, \tag{32}$$

for any $\alpha \in \mathbb{IN}, \varphi, \psi \in \mathbb{QF}$, as well as the following rules of mutual interchange:

$$\ulcorner\neg\neg\varphi\urcorner \parallel \varphi, \tag{33}$$

$$\ulcorner\varphi \vee \psi\urcorner \parallel \ulcorner\neg(\neg\varphi \wedge \neg\psi)\urcorner, \tag{34}$$

$$\ulcorner\varphi \rightarrow \psi\urcorner \parallel \ulcorner\neg(\varphi \wedge \neg\psi)\urcorner, \tag{35}$$

$$\ulcorner\varphi \equiv \psi\urcorner \parallel \ulcorner(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)\urcorner, \tag{36}$$

for any $\varphi, \psi \in \mathbb{QF}$. Quasi-formulas appearing on oposite sides of the sign "$\parallel$" are mutually interchangebile *salva dogmate*. The system $\mathsf{R}_{\mathfrak{B}}$ is adequate (i.e. both sound and complete) with respect to the set $\mathfrak{B}$ of models, i.e. any formula $\varphi \in \mathbb{FM}$ is a theorem of $\mathsf{R}_{\mathfrak{B}}$ if and only if φ is $\mathfrak{B}$-valid. The system $\mathsf{R}_{\mathfrak{B}}$ is also decidable.

THEOREM 1. *All theorems of the system $\mathsf{R}_{\mathfrak{B}}$ are $\mathfrak{B}$-valid.*

PROOF. The axioms (3) are $\mathfrak{B}$-valid and the rule (MP) preserves $\mathfrak{B}$-validity due to the truth-condtitions (17)–(21). To become convinced

that all the axioms (31) are $\mathfrak{B}$-valid, notice that $\mathfrak{f}_\mathfrak{B}^\wedge(x,y) \in \Omega_\mathfrak{B}^\star$ if and only if $x, y \in \Omega_\mathfrak{B}^\star$. In relation to the axioms (32) notice that $\mathfrak{f}_\mathfrak{B}^\neg(\mathfrak{f}_\mathfrak{B}^\wedge(x,y)) \in \Omega_\mathfrak{B}^\star$ if and only if $\mathfrak{f}_\mathfrak{B}^\wedge(x,y)$ equals X or 0, which is the case if and only if either x or y equals X or 0, which is the case if and only if either $\mathfrak{f}_\mathfrak{B}^\neg(x)$ or $\mathfrak{f}_\mathfrak{B}^\neg(y)$ belongs to $\Omega_\mathfrak{B}^\star$. The rules (33)–(36) preserve $\mathfrak{B}$-validity due to classical mutual definability of the operations $\mathfrak{f}_\mathfrak{B}^\neg, \mathfrak{f}_\mathfrak{B}^\wedge, \mathfrak{f}_\mathfrak{B}^\vee, \mathfrak{f}_\mathfrak{B}^\rightarrow$ and $\mathfrak{f}_\mathfrak{B}^\equiv$. It follows that all the theorems of $\mathsf{R}_\mathfrak{B}$ are $\mathfrak{B}$-valid. $\square$

THEOREM 2. *For every $\varphi \in \mathbb{FM}$ there exists such a $\varphi' \in \mathbb{NF}$ that the equivalence $\ulcorner \varphi \equiv \varphi' \urcorner$ is a theorem of the system $\mathsf{R}_\mathfrak{B}$.*

PROOF. Consider any formula $\varphi \in \mathbb{FM}$. All sentential connectives but conjunction and negation are eliminable out of the scopes of the positional connective "$\mathcal{R}$" by means of the rules (34)–(36). So are interated occurences of the connective of negation — by means of the rule (33). All conjunctions and and negated conjunctions are eliminable out of the scopes of the positional connective "$\mathcal{R}$" by means of the axioms (31) and (32). Hence, a formula $\bar\varphi$ is achievable, such that single sentence letters or negations of such letters appear within scopes of the positional connective "$\mathcal{R}$" in $\bar\varphi$, and the equivalence $\ulcorner \varphi \equiv \bar\varphi \urcorner$ is a theorem of the system $\mathsf{R}_\mathfrak{B}$. Consider now all atomic formulas in $\bar\varphi$ as indivisible units. By means of the axioms (3), analogically to the classical propositional calculus, a conjunctive normal form of $\bar\varphi$ may be found and the conjunctive normal form is the formula $\varphi' \in \mathbb{NF}$ to be found. $\square$

THEOREM 3. *A formula $\psi \in \mathbb{EF}$ is $\mathfrak{B}$-valid if and only if there are such letters $\alpha \in \mathbb{IN}$ and $\varphi \in \mathbb{SL}$ that either of the following pairs of formulas*

(a) $\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner$,
(b) $\ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$

appears among the disjuncts of ψ.

PROOF. Pick any $\mathfrak{B}$-model $\mathfrak{M}$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 1$, then $\mathfrak{M} \vDash \ulcorner \mathcal{R}_\alpha \varphi \urcorner$, $\ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 0$, then $\mathfrak{M} \vDash \ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = \mathtt{X}$, then $\mathfrak{M} \vDash \ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = \mathtt{Y}$, then $\mathfrak{M} \vDash \ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner$, $\ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$. Hence, if either the case (a) or the case (b) occurs, at least one disjunct of ψ is true in $\mathfrak{M}$. Under the condition (19) so is the whole formula ψ. Suppose neither (a) nor (b) occurs. For every letters $\alpha \in \mathbb{IN}, \varphi \in \mathbb{SL}$ appearing in ψ, put $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = \mathtt{Y}$, if both $\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$ are disjuncts of ψ, $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = \mathtt{X}$, if both $\ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$ are disjuncts of ψ, otherwise put $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 1$, if either $\ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$ or $\ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner$

is a disjunct of ψ, $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 0$, if either $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ or $\ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$ are disjuncts of ψ. In such case no disjunct of ψ is true in the model $\mathfrak{M}$, and hence, under the rule (19), neither is ψ itself. $\qquad\square$

THEOREM 4. *Every $\mathfrak{B}$-valid elementary formula $\varphi \in \mathbb{EF}$ is a theorem of the system $\mathsf{R}_\mathfrak{B}$.*

PROOF. Suppose $\varphi \in \mathbb{EF}$ is $\mathfrak{B}$-valid. Due to the theorem 3 either the case (a) or the case (b) occurs. In the case (a) use the axiom (3):

$$\mathcal{R}_\alpha \varphi \vee \neg \mathcal{R}_\alpha \varphi.$$

Add other disjuncts of φ, using axioms (3) and the rule (MP). In the case (b) use the axiom (3):

$$\mathcal{R}_\alpha \neg \varphi \vee \neg \mathcal{R}_\alpha \neg \varphi.$$

Add other disjuncts of φ, using axioms (3) and the rule (MP). $\qquad\square$

THEOREM 5. *Every $\mathfrak{B}$-valid formula $\varphi \in \mathbb{FM}$ is a theorem of the system $\mathsf{R}_\mathfrak{B}$.*

PROOF. Let $\varphi \in \mathbb{FM}$ be $\mathfrak{B}$-valid. Due to the theorem 2 such an equivalence

$$\varphi \equiv \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \tag{37}$$

is a theorem of $\mathsf{R}_\mathfrak{B}$, that $n \geq 1$ is a natural number and $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathbb{EF}$. Due to the theorem 1 the equivalence (37) is also valid, and hence, because of the rules (18) and (21) all the formulas: $\varphi_1, \varphi_2, \ldots, \varphi_n$ are $\mathfrak{B}$-valid. Consequently, because of the theorem 4, all the formulas: $\varphi_1, \varphi_2, \ldots, \varphi_n$ are provable in $\mathsf{R}_\mathfrak{B}$, and, by the axiom (3), so is their conjunction. From that and from the provability of the equivalence (37) it follows that the formula φ is itself provable in the system $\mathsf{R}_\mathfrak{B}$. $\qquad\square$

THEOREM 6. *The system $\mathsf{R}_\mathfrak{B}$ is decidable.*

The theorem follows from the theorems: 1, 2, 3 and 5. Search for normal forms constitutes a decision procedure.

6. The System $\mathsf{R}_\mathfrak{K}$

The system $\mathsf{R}_\mathfrak{K}$ is algebraically analogical to Stephen Cole Kleene's strong three-valued logic. Kleene was originally concerned with mathematical sentences that are neither true nor false, but indeterminate in

the sense of being unprovable as well as undisprovable [3] The set $\mathfrak{K}$ of
models is determined by the sets

$$\Omega_{\mathfrak{K}} = \{\mathtt{1}, \mathtt{0}, \mathtt{Y}\}, \ \Omega_{\mathfrak{K}}^{\star} = \{\mathtt{1}\}, \tag{38}$$

as well as by the collection of operations $\mathfrak{f}_{\mathfrak{K}}^{\neg}, \mathfrak{f}_{\mathfrak{K}}^{\wedge}, \mathfrak{f}_{\mathfrak{K}}^{\vee}, \mathfrak{f}_{\mathfrak{K}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{K}}^{\equiv}$ on $\Omega_{\mathfrak{K}}$, pre-
sented on the table 2. A formula $\varphi \in \mathbb{FM}$ is $\mathfrak{K}$-valid if and only if φ is

	¬		∧	1	Y	0		∨	1	Y	0		→	1	Y	0		≡	1	Y	0
1	0		1	1	Y	0		1	1	1	1		1	1	Y	0		1	1	Y	0
Y	Y		Y	Y	Y	0		Y	1	Y	Y		Y	1	Y	Y		Y	Y	Y	Y
0	1		0	0	0	0		0	1	Y	0		0	1	1	1		0	0	Y	1

Table 2. The operations $\mathfrak{f}_{\mathfrak{K}}^{\neg}, \mathfrak{f}_{\mathfrak{K}}^{\wedge}, \mathfrak{f}_{\mathfrak{K}}^{\vee}, \mathfrak{f}_{\mathfrak{K}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{K}}^{\equiv}$

true in every $\mathfrak{K}$-model. The axiomatics of the system $\mathsf{R}_{\mathfrak{K}}$ consists of all
the axioms and rules of the system $\mathsf{R}_{\mathfrak{B}}$ as well as the additional axioms:

$$\mathcal{R}_{\alpha} \neg \varphi \rightarrow \neg \mathcal{R}_{\alpha} \varphi, \tag{39}$$

for any $\alpha \in \mathbb{IN}, \varphi, \psi \in \mathbb{QF}$. The system $\mathsf{R}_{\mathfrak{B}}$ is adequate (i.e. both sound
and complete) with respect to the set $\mathfrak{B}$ of models, i.e. any formula
$\varphi \in \mathbb{FM}$ is a theorem of $\mathsf{R}_{\mathfrak{B}}$ if and only if φ is $\mathfrak{B}$-valid. The system $\mathsf{R}_{\mathfrak{K}}$
is also decidable.

THEOREM 7. *All theorems of the system $\mathsf{R}_{\mathfrak{K}}$ are $\mathfrak{K}$-valid.*

PROOF. The axioms (3) are $\mathfrak{K}$-valid and the rule (MP) preserves $\mathfrak{K}$-
validity due to the truth-condtitions (17)–(21). To become convinced
that all the axioms (31) are $\mathfrak{K}$-valid, notice that $\mathfrak{f}_{\mathfrak{K}}^{\wedge}(x, y) \in \Omega_{\mathfrak{K}}^{\star}$ if and only
if $x, y \in \Omega_{\mathfrak{K}}^{\star}$. In relation to the axioms (32) notice that $\mathfrak{f}_{\mathfrak{K}}^{\neg}(\mathfrak{f}_{\mathfrak{K}}^{\wedge}(x, y)) \in \Omega_{\mathfrak{K}}^{\star}$
if and only if $\mathfrak{f}_{\mathfrak{K}}^{\wedge}(x, y)$ equals $\mathtt{0}$, which is the case if and only if either
x or y equals $\mathtt{0}$, which is the case if and only if either $\mathfrak{f}_{\mathfrak{K}}^{\neg}(x)$ or $\mathfrak{f}_{\mathfrak{K}}^{\neg}(y)$
belongs to $\Omega_{\mathfrak{K}}^{\star}$. In relation to the axioms (39) suppose that $\mathfrak{f}_{\mathfrak{K}}^{\neg}(x) \in \Omega_{\mathfrak{K}}^{\star}$.
It follows that $x = \mathtt{0}$, so $x \notin \Omega_{\mathfrak{K}}^{\star}$. The rules (33)–(36) preserve $\mathfrak{K}$-validity
due to classical mutual definability of the operations $\mathfrak{f}_{\mathfrak{K}}^{\neg}, \mathfrak{f}_{\mathfrak{K}}^{\wedge}, \mathfrak{f}_{\mathfrak{K}}^{\vee}, \mathfrak{f}_{\mathfrak{K}}^{\rightarrow}$ and
$\mathfrak{f}_{\mathfrak{K}}^{\equiv}$. It follows that all the theorems of $\mathsf{R}_{\mathfrak{K}}$ are $\mathfrak{K}$-valid.   □

THEOREM 8. *For every $\varphi \in \mathbb{FM}$ there exists such a $\varphi' \in \mathbb{NF}$ that the
equivalence $\ulcorner \varphi \equiv \varphi' \urcorner$ is a theorem of the system $\mathsf{R}_{\mathfrak{K}}$.*

The proof is quite analogical to the proof of the theorem 2.

THEOREM 9. *A formula $\psi \in \mathbb{EF}$ is provable in the system $\mathsf{R}_{\mathfrak{K}}$ if and only if there are such letters $\alpha \in \mathbb{IN}$ and $\varphi \in \mathbb{SL}$ that either of the following pairs of formulas*

(a) $\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner$,
(b) $\ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$,
(c) $\ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$

appears among the disjuncts of ψ.

PROOF. Pick any $\mathfrak{K}$-model $\mathfrak{M}$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 1$, then $\mathfrak{M} \vDash \ulcorner \mathcal{R}_\alpha \varphi \urcorner$, $\ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 0$, then $\mathfrak{M} \vDash \ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = \mathsf{Y}$, then $\mathfrak{M} \vDash \ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$. Hence, if either the case (a), (b) or (c) occurs, at least one disjunct of ψ is true in $\mathfrak{M}$. Under the condition (19) so is the formula ψ itself. Suppose neither of the cases (a), (b), (c) occurs. For every letters $\alpha \in \mathbb{IN}, \varphi \in \mathbb{SL}$ appearing in ψ, put $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = \mathsf{Y}$, if both $\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$ are disjuncts of ψ, otherwise put $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 1$, if either $\ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$ or $\ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner$ is a disjunct of ψ, $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 0$, if either $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ or $\ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$ are disjuncts of ψ. In such case no disjunct of ψ is true in the model $\mathfrak{M}$, and hence, under the rule (19), neither is ψ itself. □

THEOREM 10. *Every $\mathfrak{K}$-valid elementary formula $\varphi \in \mathbb{EF}$ is a theorem of the system $\mathsf{R}_{\mathfrak{K}}$.*

PROOF. Suppose $\varphi \in \mathbb{EF}$ is $\mathfrak{K}$-valid. Due to the theorem 9 one of the cases (a)–(c), described in the theorem, occurs. In the cases (a) and (b) act analogically to the theorem 4. In the case (c) use the axioms (39) and (3) to derive the theorem:

$$\neg \mathcal{R}_\alpha \neg \varphi \vee \neg \mathcal{R}_\alpha \varphi.$$

Add other disjuncts of φ, using axioms (3) and the rule (MP). □

THEOREM 11. *Every $\mathfrak{K}$-valid formula $\varphi \in \mathbb{FM}$ is a theorem of the system $\mathsf{R}_{\mathfrak{K}}$.*

The proof is based on the theorems: 7, 8 and 10, and is analogical to the proof of the theorem 5.

THEOREM 12. *The system $\mathsf{R}_{\mathfrak{K}}$ is decidable.*

The theorem follows from the theorems: 7, 8, 9 and 11. Search for normal forms constitutes a decision procedure.

7. The System $R_{\mathfrak{P}}$

The system $R_{\mathfrak{P}}$ is algebraically analogical to Graham Priest's dialetheic logic of paradox, advanced to deal with antinomies [5]. The set $\mathfrak{P}$ of models is determined by the sets

$$\Omega_{\mathfrak{P}} = \{1, 0, \mathtt{X}\}, \ \Omega_{\mathfrak{P}}^{\star} = \{1, \mathtt{X}\}, \tag{40}$$

as well as by the collection of operations $\mathfrak{f}_{\mathfrak{P}}^{\neg}, \mathfrak{f}_{\mathfrak{P}}^{\wedge}, \mathfrak{f}_{\mathfrak{P}}^{\vee}, \mathfrak{f}_{\mathfrak{P}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{P}}^{\equiv}$ on $\Omega_{\mathfrak{P}}$, presented on the table 3. A formula $\varphi \in \mathbb{FM}$ is $\mathfrak{P}$-valid if and only if φ is

	¬		∧	1	X	0		∨	1	X	0		→	1	X	0		≡	1	X	0
1	0		1	1	X	0		1	1	1	1		1	1	X	0		1	1	X	0
X	X		X	X	X	0		X	1	X	X		X	1	X	X		X	X	X	X
0	1		0	0	0	0		0	1	X	0		0	1	1	1		0	0	X	1

Table 3. The operations $\mathfrak{f}_{\mathfrak{P}}^{\neg}, \mathfrak{f}_{\mathfrak{P}}^{\wedge}, \mathfrak{f}_{\mathfrak{P}}^{\vee}, \mathfrak{f}_{\mathfrak{P}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{P}}^{\equiv}$

true in every $\mathfrak{P}$-model. The axiomatics of the system $R_{\mathfrak{P}}$ consists of all the axioms and rules of the system $R_{\mathfrak{B}}$ as well as the additional axioms:

$$\neg \mathcal{R}_\alpha \varphi \rightarrow \mathcal{R}_\alpha \neg \varphi, \tag{41}$$

for any $\alpha \in \mathbb{IN}, \varphi, \psi \in \mathbb{QF}$. The system $R_{\mathfrak{B}}$ is adequate (i.e. both sound and complete) with respect to the set $\mathfrak{P}$ of models, i.e. any formula $\varphi \in \mathbb{FM}$ is a theorem of $R_{\mathfrak{B}}$ if and only if φ is $\mathfrak{P}$-valid. The system $R_{\mathfrak{P}}$ is also decidable.

THEOREM 13. *All theorems of the system $R_{\mathfrak{P}}$ are $\mathfrak{P}$-valid.*

PROOF. The axioms (3) are $\mathfrak{P}$-valid and the rule (MP) preserves $\mathfrak{P}$-validity due to the truth-condtitions (17)–(21). To become convinced that all the axioms (31) are $\mathfrak{P}$-valid, notice that $\mathfrak{f}_{\mathfrak{P}}^{\wedge}(x, y) \in \Omega_{\mathfrak{P}}^{\star}$ if and only if $x, y \in \Omega_{\mathfrak{P}}^{\star}$. In relation to the axioms (32) notice that $\mathfrak{f}_{\mathfrak{P}}^{\neg}(\mathfrak{f}_{\mathfrak{P}}^{\wedge}(x, y)) \in \Omega_{\mathfrak{P}}^{\star}$ if and only if $\mathfrak{f}_{\mathfrak{P}}^{\wedge}(x, y)$ equals X or 0, which is the case if and only if either x or y equals X or 0, which is the case if and only if either $\mathfrak{f}_{\mathfrak{P}}^{\neg}(x)$ or $\mathfrak{f}_{\mathfrak{P}}^{\neg}(y)$ belongs to $\Omega_{\mathfrak{P}}^{\star}$. In relation to the axioms (41) suppose that $x \notin \Omega_{\mathfrak{P}}^{\star}$, so $x = 0$. It follows that $\mathfrak{f}_{\mathfrak{P}}^{\neg}(x) \in \Omega_{\mathfrak{P}}^{\star}$. The rules (33)–(36) preserve $\mathfrak{P}$-validity due to classical mutual definability of the operations $\mathfrak{f}_{\mathfrak{P}}^{\neg}, \mathfrak{f}_{\mathfrak{P}}^{\wedge}, \mathfrak{f}_{\mathfrak{P}}^{\vee}, \mathfrak{f}_{\mathfrak{P}}^{\rightarrow}$ and $\mathfrak{f}_{\mathfrak{P}}^{\equiv}$. It follows that all the theorems of $R_{\mathfrak{P}}$ are $\mathfrak{P}$-valid. $\square$

THEOREM 14. *For every $\varphi \in \mathbb{FM}$ there exists such a $\varphi' \in \mathbb{NF}$ that the equivalence $\ulcorner \varphi \equiv \varphi' \urcorner$ is a theorem of the system $\mathsf{R}_\mathfrak{P}$.*

The proof is quite analogical to the proof of the theorem 2.

THEOREM 15. *A formula $\psi \in \mathbb{EF}$ is $\mathfrak{P}$-valid if and only if there are such letters $\alpha \in \mathbb{IN}$ and $\varphi \in \mathbb{SL}$ that either of the following pairs of formulas*

(a) $\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner$,
(b) $\ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$,
(c) $\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$

appears among the disjuncts of ψ.

PROOF. Pick any $\mathfrak{P}$-model $\mathfrak{M}$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 1$, then $\mathfrak{M} \vDash \ulcorner \mathcal{R}_\alpha \varphi \urcorner$, $\ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 0$, then $\mathfrak{M} \vDash \ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = \mathsf{X}$, then $\mathfrak{M} \vDash \ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$. Hence, if either the case (a), (b) or (c) occurs, at least one disjunct of ψ is true in $\mathfrak{M}$. Under the condition (19) so is the formula ψ itself. Suppose neither of the cases (a), (b), (c) occurs. For every letters $\alpha \in \mathbb{IN}, \varphi \in \mathbb{SL}$ appearing in ψ, put $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = \mathsf{X}$, if both $\ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$ are disjuncts of ψ, otherwise put $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 1$, if either $\ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$ or $\ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner$ is a disjunct of ψ, $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 0$, if either $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ or $\ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$ are disjuncts of ψ. In such case no disjunct of ψ is true in the model $\mathfrak{M}$, and hence, under the rule (19), neither is ψ itself. □

THEOREM 16. *Every $\mathfrak{P}$-valid elementary formula $\varphi \in \mathbb{EF}$ is a theorem of the system $\mathsf{R}_\mathfrak{P}$.*

PROOF. Suppose $\varphi \in \mathbb{EF}$ is $\mathfrak{P}$-valid. Due to the theorem 15 one of the cases (a)–(c), described in the theorem, occurs. In the cases (a) and (b) act analogically to the theorem 4. In the case (c) use the axioms (41) and (3) to derive the theorem:

$$\mathcal{R}_\alpha \varphi \vee \mathcal{R}_\alpha \neg \varphi.$$

Add other disjuncts of φ, using axioms (3) and the rule (MP). □

THEOREM 17. *Every $\mathfrak{P}$-valid formula $\varphi \in \mathbb{FM}$ is a theorem of the system $\mathsf{R}_\mathfrak{P}$.*

The proof is based on the theorems: 13, 14 and 16, and is analogical to the proof of the theorem 5.

THEOREM 18. *The system $\mathsf{R}_\mathfrak{P}$ is decidable.*

The theorem follows from the theorems: 13, 14, 15 and 17. Search for normal forms constitutes a decision procedure.

8. The System $\mathsf{R}_{\mathfrak{C}}$

The system $\mathsf{R}_{\mathfrak{C}}$ is algebraically analogical to the classical propositional calculus and turnes out to be identical to the above described system MR. Practically one has to do here with the classical matrix. The set $\mathfrak{C}$ of models is determined by the sets

$$\Omega_{\mathfrak{C}} = \{1, 0\}, \ \Omega_{\mathfrak{C}}^{\star} = \{1\}, \tag{42}$$

as well as by the collection of operations $\mathfrak{f}_{\mathfrak{C}}^{\neg}, \mathfrak{f}_{\mathfrak{C}}^{\wedge}, \mathfrak{f}_{\mathfrak{C}}^{\vee}, \mathfrak{f}_{\mathfrak{C}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{C}}^{\equiv}$ on $\Omega_{\mathfrak{C}}$, presented on the table 4. A formula $\varphi \in \mathbb{FM}$ is $\mathfrak{C}$-valid if and only if φ is true

	$\neg$		$\wedge$	1	0		$\vee$	1	0		$\rightarrow$	1	0		$\equiv$	1	0
1	0		1	1	0		1	1	1		1	1	0		1	1	0
0	1		0	0	0		0	1	0		0	1	1		0	0	1

Table 4. The operations $\mathfrak{f}_{\mathfrak{C}}^{\neg}, \mathfrak{f}_{\mathfrak{C}}^{\wedge}, \mathfrak{f}_{\mathfrak{C}}^{\vee}, \mathfrak{f}_{\mathfrak{C}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{C}}^{\equiv}$

in every $\mathfrak{C}$-model. The axiomatics of the system $\mathsf{R}_{\mathfrak{C}}$ consists of all the axioms and rules of the system $\mathsf{R}_{\mathfrak{B}}$, except for (32), which gets derivable, as well as of the additional axioms (5). The system $\mathsf{R}_{\mathfrak{C}}$ is adequate (i.e. both sound and complete) with respect to the set $\mathfrak{C}$ of models, i.e. any formula $\varphi \in \mathbb{FM}$ is a theorem of $\mathsf{R}_{\mathfrak{C}}$ if and only if φ is $\mathfrak{C}$-valid. The system $\mathsf{R}_{\mathfrak{C}}$ is also decidable.

THEOREM 19. *All theorems of the system $\mathsf{R}_{\mathfrak{C}}$ are $\mathfrak{C}$-valid.*

PROOF. The axioms (3) are $\mathfrak{C}$-valid and the rule (MP) preserves $\mathfrak{C}$-validity due to the truth-condtitions (17)–(21). To become convinced that all the axioms (31) are $\mathfrak{C}$-valid, notice that $\mathfrak{f}_{\mathfrak{C}}^{\wedge}(x, y) \in \Omega_{\mathfrak{C}}^{\star}$ if and only if $x, y \in \Omega_{\mathfrak{C}}^{\star}$. In relation to the axioms (5) notice that $\mathfrak{f}_{\mathfrak{C}}^{\neg}(x) \in \Omega_{\mathfrak{C}}^{\star}$ if and only if $x \notin \Omega_{\mathfrak{C}}^{\star}$. The rules (33)–(36) preserve $\mathfrak{C}$-validity due to classical mutual definability of the operations $\mathfrak{f}_{\mathfrak{C}}^{\neg}, \mathfrak{f}_{\mathfrak{C}}^{\wedge}, \mathfrak{f}_{\mathfrak{C}}^{\vee}, \mathfrak{f}_{\mathfrak{C}}^{\rightarrow}$ and $\mathfrak{f}_{\mathfrak{C}}^{\equiv}$. It follows that all the theorems of $\mathsf{R}_{\mathfrak{C}}$ are $\mathfrak{C}$-valid. $\square$

THEOREM 20. *For every $\varphi \in \mathbb{FM}$ there exists such a $\varphi' \in \mathbb{NF}$ that the equivalence $\ulcorner\varphi \equiv \varphi'\urcorner$ is a theorem of the system $\mathsf{R}_{\mathfrak{C}}$.*

The proof is quite analogical to the proof of the theorem 2.

THEOREM 21. *A formula $\psi \in \mathbb{EF}$ is provable in the system $\mathsf{R_C}$ if and only if there are such letters $\alpha \in \mathbb{IN}$ and $\varphi \in \mathbb{SL}$ that either of the following pairs of formulas*

(a) $\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner$,
(b) $\ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$,
(c) $\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$,
(d) $\ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$

appears among the disjuncts of ψ.

PROOF. Pick any $\mathfrak{C}$-model $\mathfrak{M}$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 1$, then $\mathfrak{M} \models \ulcorner \mathcal{R}_\alpha \varphi \urcorner$, $\ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$. If $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 0$, then $\mathfrak{M} \models \ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$. Hence, if either the case (a), (b), (c) or (d) occurs, at least one disjunct of ψ is true in $\mathfrak{M}$. Under the condition (19) so is the formula ψ itself. Suppose neither of the cases (a)–(d) occurs. For every letters $\alpha \in \mathbb{IN}, \varphi \in \mathbb{SL}$ appearing in ψ, put $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 1$, if either $\ulcorner \mathcal{R}_\alpha \neg \varphi \urcorner$ or $\ulcorner \neg \mathcal{R}_\alpha \varphi \urcorner$ is a disjunct of ψ, $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = 0$, if either $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ or $\ulcorner \neg \mathcal{R}_\alpha \neg \varphi \urcorner$ are disjuncts of ψ. In such case no disjunct of ψ is true in the model $\mathfrak{M}$, and hence, under the rule (19), neither is ψ itself. □

Notice that, due to the perfect distributivity of negation in the system $\mathsf{R_C}$, the cases (b)–(c) of the theorem 21 are practically reducible to the case (a). It is the case in neither of the calculi $\mathsf{R_B}$, $\mathsf{R_K}$, $\mathsf{R_P}$.

THEOREM 22. *Every $\mathfrak{C}$-valid elementary formula $\varphi \in \mathbb{EF}$ is a theorem of the system $\mathsf{R_C}$.*

PROOF. Suppose $\varphi \in \mathbb{EF}$ is $\mathfrak{C}$-valid. Due to the theorem 21 one of the cases (a)–(d) there described occurs. In the cases (a) and (b) act analogically to the theorem 4. In the case (c) use the left conditional, derived from the axiom (5), and act analogically to the theorem 16. In the case (d) use the right conditional, derived from the axiom (5), and act analogically to the theorem 10. □

THEOREM 23. *Every $\mathfrak{C}$-valid formula $\varphi \in \mathbb{FM}$ is a theorem of the system $\mathsf{R_C}$.*

The proof is based on the theorems: 19, 20 and 22, and is analogical to the proof of the theorem 5.

THEOREM 24. *The system $\mathsf{R_C}$ is decidable.*

The theorem follows from the theorems: 19, 20, 21 and 23. Search for normal forms constitutes a decision procedure.

9. Four Corners of Positinal Calculi

The weak positional calculi we described form a four-cornered lattice analogical to the four corners of truth-values. Let $E(\mathfrak{X})$ be the set of all $\mathfrak{X}$-valid formulas, of course, identical to the set of theorems of the system $\mathcal{R}_{\mathfrak{X}}$. It turns out that the following interrelations hold:

$$E(\mathfrak{B}) \subset E(\mathfrak{P}) \subset E(\mathfrak{C}), \tag{43}$$

$$E(\mathfrak{B}) \subset E(\mathfrak{K}) \subset E(\mathfrak{C}), \tag{44}$$

$$E(\mathfrak{P}) \nsubseteq E(\mathfrak{K}) \text{ and } E(\mathfrak{K}) \nsubseteq E(\mathfrak{P}), \tag{45}$$

$$E(\mathfrak{B}) = E(\mathfrak{K}) \cap E(\mathfrak{P}) \tag{46}$$

and finally

$$E(\mathfrak{C}) = E(\mathfrak{K}) \cup E(\mathfrak{P}). \tag{47}$$

Those theorems are easily justifiable due to decidability of all the calculi in question. To justify the theorems (43)–(47) it is practically enough to carefully observe the axioms concerning the connective of negation: (5), (39) and (41). Furthermore, the system $\mathsf{R}_{\mathfrak{C}}$ is deductively equivalent to the system MR from Jarmużek and Pietruszczak.

THEOREM 25. *The system* $\mathsf{R}_{\mathfrak{C}}$ *is deductively equivalent to the system* MR.

PROOF. It is enough to show that the axiom collection (4) is derivable in the system $\mathsf{R}_{\mathfrak{C}}$. We are then about to show that $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ is provable in $\mathsf{R}_{\mathfrak{C}}$, provided $\varphi \in \mathbb{CPC}$. Define such a substitution $e\colon \mathbb{QF} \longrightarrow \mathbb{FM}$, that

$$e(\psi) = \ulcorner \mathcal{R}_\alpha \psi \urcorner,$$

for all $\psi \in \mathbb{SL}$. Let $\varphi \in \mathbb{FM}$ be any classical tautology. Due to the axiom collection (3) the formula $e(\varphi)$ is a theorem of $\mathsf{R}_{\mathfrak{C}}$, provided $\varphi \in \mathbb{CPC}$. Now, by use of the distribution laws of $\mathsf{R}_{\mathfrak{C}}$, the formula $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ is trivially deducible from the formula $e(\varphi)$.   □

Analogical interrelations apply to the distributive laws that hold in particular calculi.

Finally, a question seems to be posed. What is the minimal formal tool that would allow to construct algorithmically arbitrary sound and complete weak positional calculi, containing some, all or none distributive laws? As far as I am aware the question remains unanswered.

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