Nonlocal correlations in an asymmetric quantum network

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The nonlocality revealed in a multiparty multisource network Bell experiment is conceptually different than the standard multiparty Bell nonlocality involving a single common source. Here, by introducing variants of asymmetric bilocal as well as trilocal network scenarios, we go beyond the typical bilocal network scenario where both the edge parties have an equal number of measurement settings. We first introduce an asymmetric bilocal network where one of the edge parties (say, Alice) receives 2^{n-1} inputs and the other edge party (say, Charlie) receives n inputs. We derive two variants of asymmetric bilocality inequalities and demonstrate their optimal quantum violations. Further, we explore two types of asymmetric trilocal scenarios: (i) when two edge parties receive 2^{n-1} inputs each and the other edge party receives n inputs, and (ii) when one edge party receives 2^{n-1} inputs and the other two edge parties have n inputs each. We use an elegant sum-of-squares technique that enables us to evaluate the quantum optimal values of the proposed network inequalities without assuming the dimension of the systems for both the asymmetric bilocal as well as the trilocal scenarios. Further, we demonstrate the robustness of the quantum violations of the proposed inequalities in the presence of white noise.

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I. INTRODUCTION

The study of quantum nonlocality in the network scenario [1] has recently been receiving considerable attention. Such a form of nonlocality is conceptually different from the standard Bell nonlocality [2]. While a multiparty Bell experiment involves a single common source, the multiparty network Bell experiment involves several independent sources. Each source distributes a physical system to subsequent parties, and each party performs a measurement on their subsystem prepared from different sources.

The simplest nontrivial network scenario [3–5] features three parties and two independent sources, commonly referred to as the bilocality scenario. The quantum nonlocality in a network is demonstrated through the quantum violation of suitably formulated nonlinear bilocality inequality. A straightforward generalization of the bilocality scenario is the n-locality scenario [6–8] involving an arbitrary n number of sources. For example, a star network may have n number of sources and edge parties. Each edge party shares the physical system with a central party. In recent times, the network nonlocality has been extensively studied in various topologies [8–23].

The reported interesting results such as possibility of observing quantum nonlocality without inputs [24,25] or showing the nonlocality of certain entangled states which do not exhibit nonlocality in the usual Bell scenario [26,27] establishes the fundamental importance of viewing nonlocality in terms of the symmetric network scenario in contrast to the standard Bell scenario. Characterization of network nonlocality and its correspondence with the bipartite Bell nonlocality has been studied [7]. Several theoretical proposals have also been experimentally verified [7,28–34]. Recently, genuine network nonlocality has also been introduced that cannot be traced back to Bell nonlocality [23,35,36]. Self-testing protocols using the quantum network have recently been proposed [37–40]. Further, by using a quantum network, it has been established [38,41,42] that the real quantum theory can be experimentally falsified, i.e., quantum theory inevitably needs complex numbers. To this end, different forms of network scenarios like the star network [6], chain-shaped network [19], and cycle network [24] have been explored.

We note here that while most of the studies concerning the star-network scenario have been investigated for the symmetric input scenario, i.e., each edge party performs the same number of measurements, a generalised study of network nonlocality in asymmetric input scenarios remains unexplored. In this regard, by introducing the asymmetric bilocal network scenario that comprises two edge parties performing three and six measurements, respectively, and the central party performing four measurements, a couple of recent works [38,42] have shown that complex numbers are necessary for quantum predictions. It is crucial to remark here that such a study is based on the two-qubit system. Here, the purpose of this work is to probe hitherto unexplored generalized asymmetric network nonlocality in a device-independent way. In particular, by considering the bilocality scenario, we first derive asymmetric nonlinear inequality for the scenario in which one of the two edge parties (say, Alice) has four measurement settings and the other edge party (say, Charlie) has three measurement settings. In addition to that, we extend the study

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from the asymmetric bilocality scenario to the asymmetric trilocality scenario involving three independent sources. In the trilocal network scenario, we explore two particular variants of asymmetric trilocality. First, we consider one edge party (Alice) has four measurement settings and the other two edge parties (Charlie and Diana) have three measurement settings each. Then we consider one edge party has three measurement settings (Charlie) and the other two edge parties have four measurement settings each (Alice and Diana).

Then, by using an elegant sum-of-squares (SOS) approach developed in [43–45], we analytically obtain the optimal quantum violations of the asymmetric bilocality as well as the asymmetric trilocality inequalities. It is important to note that we evaluate such optimal quantum bounds without assuming the dimension of the systems. In this process, we evaluate the required constraints on the observables of each party for achieving the optimal quantum violation. For the asymmetric bilocality scenario, we demonstrate that the quantum optimal value will be achieved if each of the both Alice-Bob and Bob-Charlie shares at least a single copy of maximally entangled two-qubit state. Moreover, for the considered asymmetric trilocality scenario, as similar to the bilocality scenario, we find that the optimal quantum value will be obtained if each of all the edge parties shares at least a single copy of maximally entangled two-qubit state with the central party Bob.

Furthermore, we extend our study for any arbitrary number of measurement settings. In particular, we consider an asymmetric bilocal network where one of the edge parties receives 2^{n-1} inputs and the other edge party receives n inputs. We explore two types of asymmetric trilocal scenarios for arbitrary inputs: (i) when two edge parties receive 2^{n-1} inputs each and the other edge party receives n inputs, and (ii) when one edge party receives 2^{n-1} inputs and the other two edge parties have n inputs each. Finally, we illustrate the robustness of the quantum violations of the proposed inequalities in the presence of the white noise for both the cases of bilocality and trilocality scenarios. We find that the proposed asymmetric inequality is most robust to white noise in the simplest bilocality network scenario.

This paper is organized as follows. To begin with, in Sec. II, by invoking the SOS approach [43,44], we derive the optimal quantum violation of the standard bilocal network inequality without assuming the dimension of the system. Next, in Secs. III and IV, we introduce two variants of asymmetric bilocal scenario and propose two different bilocal network inequalities. Then, we obtain the optimal quantum bounds along with the states and observables corresponding to the optimal quantum values (Secs. III A and III B). In Secs. V and VI, by going beyond the bilocality network scenario, we introduce asymmetricity in the trilocal network and propose two different types of asymmetric trilocal inequalities. We also evaluate the corresponding optimal quantum bounds as well as the states and observables required for attaining such optimal quantum values (Secs. V A and V B). In particular, we illustrate that in order to achieve the optimal quantum bound for both the asymmetric bilocal and trilocal cases, each of all the edge parties must share at least a single copy of maximally entangled state with the central party Bob. Then, in Sec. VII, we have generalized the asymmetric bilocal and trilocal network scenario for arbitrary number of measurement settings.

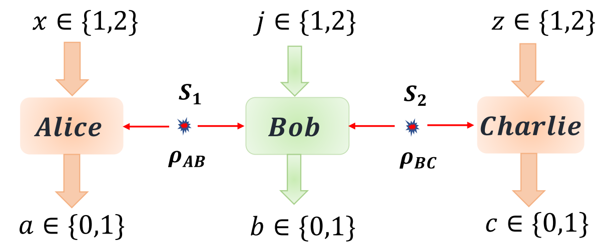

FIG. 1. The standard bilocal scenario featuring two edge parties (Alice and Charlie) and the central party Bob. The source S_1 (S_2) emits physical system for Alice (Charlie) and Bob. The sources are assumed to be independent to each other.

Further, in Sec. VIII, we provide an analysis regarding the robustness of quantum violations of the proposed inequalities to white noise. Finally, in Sec. IX, we discuss the salient features of our work and propose some interesting open questions.

II. PRELIMINARIES: THE STANDARD BILOCAL NETWORK SCENARIO

The standard bilocal network scenario (see Fig. 1) comprises of three spatially separated parties: Alice, Bob, and Charlie. Two independent sources S_1 and S_2 prepare a bipartite physical system for Alice-Bob and Bob-Charlie, respectively. Upon receiving the system from the respective source, Alice performs one of m_A local measurements, denoted by $A_{n,x} \in \{A_{n,1}, A_{n,2}, \dots, A_{n,m_A}\}$. Similarly, for Bob and Charlie the respective measurements are denoted by $B_{n,j} \in \{B_{n,1}, B_{n,2}, \dots, B_{n,m_B}\}$ and $C_{n,z} \in \{C_{n,1}, C_{n,2}, \dots, C_{n,m_C}\}$. The outcomes for Alice, Bob, and Charlie are denoted by $a, b, c \in \{0, 1\}$. The standard bilocal scenario is a symmetric scenario that implies an equal number of measurement settings for the edge parties, i.e., $m_A = m_C$. On the other hand, in this work we consider two types of asymmetric bilocal network scenario: (i) Alice performs one of $m_A = 2^{n-1}$ measurements; Charlie and the central party Bob perform one of $m_C = m_B = n$ measurements. (ii) Alice and Bob perform one of $m_A = m_B = 2^{n-1}$ measurements; Charlie performs one of $m_C = n$ measurements. The index n appearing in the subscript denotes the scenario involving the number of measurement settings considered. For example, $n = 2$ corresponds the standard bilocal network scenario comprising two measurement settings for each party.

Now, in the ontological model of the tripartite standard Bell scenario, it is assumed that the source prepares a common hidden variable λ. Then the reproducibility condition is given by

$$P(a, b, c|x, j, z) = \int d\lambda\ \mu(\lambda)P(a|x, \lambda)P(b|j, \lambda)P(c|z, \lambda). \tag{1}$$

In contrast to the ontological model of tripartite standard Bell scenario, the ontological model of bilocality scenario is that each source S_1 and S_2 prepares the physical system in the state $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$ with a probability distribution $\mu(\lambda_1)$ and $\mu(\lambda_2)$, respectively, with $\int_{\Lambda_k} \mu(\lambda_k)d\lambda_k = 1$. The crucial assumption here is that the sources S_1 and S_2 are independent to each other. This means that the joint probability distribution $\mu(\lambda_1,\lambda_2)$ can be written in the factorized form as $\mu(\lambda_1,\lambda_2)=\mu(\lambda_1)\mu(\lambda_2)$. Now, in order to reproduce the quantum theoretical prediction from the bilocal ontic model, the following reproducibility condition needs to be satisfied:

$$P(a,b,c|x,j,z)=\iint d\lambda_1 d\lambda_2\ \mu(\lambda_1)\mu(\lambda_2)P(a|x,\lambda_1)\times P(b|j,\lambda_1,\lambda_2)P(c|z,\lambda_2). \tag{2}$$

Note that in the two measurements per party scenario ($m_A=m_B=m_C=2$), it has been shown [5] that any operational theory satisfying the above Eq. (2) satisfies the nonlinear inequality

$$\mathcal{B}_2\equiv\sqrt{|\mathcal{I}_{2,1}|}+\sqrt{|\mathcal{I}_{2,2}|}\leqslant(\mathcal{B}_2)_{bl}=2, \tag{3}$$

where $\mathcal{I}_{2,2}$ are suitably defined linear combinations of the tripartite correlations, given by

$$\mathcal{I}_{2,1}=\langle(A_{2,1}+A_{2,2})\otimes B_{2,1}\otimes(C_{2,1}+C_{2,2})\rangle, \tag{4}$$

$$\mathcal{I}_{2,2}=\langle(A_{2,1}-A_{2,2})\otimes B_{2,2}\otimes(C_{2,2}-C_{2,1})\rangle \tag{5}$$

with[1] $\langle A_{n,x}B_{n,j}C_{n,z}\rangle=\sum_{a,b,c}(-1)^{a+b+c}P(a,b,c|x,j,z)$ and $P(a,b,c|x,j,z)$ is the joint probability for obtaining the outcomes (a,b,c) corresponding to the dichotomic measurements performed by Alice, Bob, and Charlie. In quantum theory, the joint probability $P(a,b,c|x,j,z)$ is given as

$$P(a,b,c|x,j,z)=\mathrm{Tr}\big[(\rho_{AB}\otimes\rho_{BC})\ \Pi^a_{A_{n,x}}\otimes\Pi^b_{B_{n,j}}\otimes\Pi^c_{C_{n,z}}\big], \tag{6}$$

where ρ_{AB} and ρ_{BC} are bipartite states produced from two independent sources S_1 and S_2, respectively.

It has been shown [5,44] that the maximum quantum value $(\mathcal{B}_2)_Q^{\mathrm{opt}}=2\sqrt{2}$ is obtained when Alice's and Charlie's observables are mutually anticommuting and Bob's observables are mutually commuting. An example of such choices of observables in two-dimensional Hilbert space ($\mathcal{H}^2$) is given as follows:

$$A_{2,1}=C_{2,1}=(\sigma_z+\sigma_x)/\sqrt{2},\quad B_{2,1}=\sigma_z\otimes\sigma_z,$$
$$A_{2,2}=C_{2,2}=(\sigma_z-\sigma_x)/\sqrt{2},\quad B_{2,2}=\sigma_x\otimes\sigma_x. \tag{7}$$

Note that while the optimal quantum value of $(\mathcal{B}_2)_Q^{\mathrm{opt}}$ was earlier derived [5] by taking a pair of the two-qubit entangled state, the dimension-independent derivation of $(\mathcal{B}_2)_Q^{\mathrm{opt}}$ has recently been proposed [44]. Throughout this work, we adopt the SOS approach introduced in [43] and derive the optimal quantum bound without assuming the dimension of the system. Thus, our optimal value possesses the the potential to be used as device-independent certification of quantum correlations.

Optimal quantum bound for $\mathcal{B}_2$

Here, by invoking the elegant SOS approach [44], we evaluate the optimal quantum value of $(\mathcal{B}_2)_Q$. Without loss of generality, one can always construct a suitable operator γ_2 satisfying $\langle\gamma_2\rangle=\beta_2-(\mathcal{B}_2)_Q$ such that $\langle\gamma_2\rangle\geqslant 0$. The existence of such operator γ_2 can be shown by suitably considering a set of operators $M_{2,j},\ \forall\ j\in\{1,2\}$, which are polynomial functions of $A_{2,x}$, $B_{2,j}$, and $C_{2,z}$:

$$\langle\gamma_2\rangle=\sum_{j=1}^{2}\frac{\sqrt{\omega_{2,j}}}{2}|M_{2,j}\,|\psi\rangle\,|^2, \tag{8}$$

where $\omega_{2,j}\geqslant 0$ are suitable positive numbers that will be specified soon. We choose the operator $M_{2,j}$ as follows:[2]

$$|M_{2,1}\,|\psi\rangle\,|=\sqrt{\left|\left(\frac{A_{2,1}+A_{2,2}}{\omega^A_{2,1}}\otimes\mathbb{I}_d\otimes\frac{C_{2,1}+C_{2,2}}{\omega^C_{2,1}}\right)|\psi\rangle\right|}-\sqrt{|\mathbb{I}_d\otimes B_{2,1}\otimes\mathbb{I}_d\ \,|\psi\rangle\,|},$$
$$|M_{2,2}\,|\psi\rangle\,|=\sqrt{\left|\left(\frac{A_{2,1}-A_{2,2}}{\omega^A_{2,2}}\otimes\mathbb{I}_d\otimes\frac{C_{2,1}-C_{2,2}}{\omega^C_{2,2}}\right)|\psi\rangle\right|}-\sqrt{|\mathbb{I}_d\otimes B_{2,2}\otimes\mathbb{I}_d\ \,|\psi\rangle\,|}, \tag{9}$$

$$\omega^A_{2,1}=||(A_{2,1}+A_{2,2})\,|\psi\rangle\,||_2=\sqrt{2+\langle\{A_{2,1},A_{2,2}\}\rangle},$$
$$\omega^A_{2,2}=||(A_{2,1}-A_{2,2})\,|\psi\rangle\,||_2=\sqrt{2-\langle\{A_{2,1},A_{2,2}\}\rangle},$$
$$\omega^C_{2,1}=||(C_{2,1}+C_{2,2})\,|\psi\rangle\,||_2=\sqrt{2+\langle\{C_{2,1},C_{2,2}\}\rangle},$$
$$\omega^C_{2,2}=||(C_{2,1}-C_{2,2})\,|\psi\rangle\,||_2=\sqrt{2-\langle\{C_{2,1},C_{2,2}\}\rangle}, \tag{10}$$

where $||\cdot||_2$ denotes the Frobenious norm given by $||\,\mathcal{O}\ \,||_2=\sqrt{\langle\psi|\,\mathcal{O}^\dagger\mathcal{O}\,|\psi\rangle}$.

Now, since Alice, Bob, and Charlie are spacelike separated, their observables are mutually commuting. Thus, the operators $(A_{2,j}\otimes\mathbb{I}_d\otimes\mathbb{I}_d)$, $(\mathbb{I}_d\otimes B_{2,j}\otimes\mathbb{I}_d)$, and $(\mathbb{I}_d\otimes\mathbb{I}_d\otimes C_{2,j})$ are also mutually commuting. Hence, these three observables must have at least one common eigenstate. Without loss of generality, $|\psi\rangle$ is taken to be one of the common eigenstates. Therefore, by evaluating the quantity $|M_{2,j}\,|\psi\rangle\,|^2$ from Eq. (9), a simple algebraic manipulation gives us from Eq. (8) the following:

$$\langle\gamma_2\rangle=(\sqrt{\omega_{2,1}}+\sqrt{\omega_{2,2}})-(\mathcal{B}_2)_Q, \tag{11}$$

where $\omega_{2,j}=\omega^A_{2,j}\ \omega^C_{2,j}$.

Now, since by construction $\langle\gamma_2\rangle\geqslant 0$, it is evident that from Eq. (11) that the optimal quantum value of $\mathcal{B}_2$ corresponds to $\langle\gamma_2\rangle=0$. Therefore, the quantum optimal value is given as

$$(\mathcal{B}_2)_Q^{\mathrm{opt}}=\sqrt{\omega^A_{2,1}\ \omega^C_{2,1}}+\sqrt{\omega^A_{2,2}\ \omega^C_{2,2}}. \tag{12}$$

Next, using the inequality $\sqrt{r_1s_1}+\sqrt{r_2s_2}\leqslant\sqrt{r_1+r_2}\sqrt{s_1+s_2}\ \ \forall\ r_1,s_1,r_2,s_2\geqslant 0$, we can write Eq. (12) as

$$(\mathcal{B}_2)_Q^{\mathrm{opt}}=\sqrt{\left(\omega^A_{2,1}+\omega^A_{2,2}\right)\left(\omega^C_{2,1}+\omega^C_{2,2}\right)}$$
$$=\sqrt{\left(\sqrt{2+\langle\{A_{2,1},A_{2,2}\}\rangle}+\sqrt{2-\langle\{A_{2,1},A_{2,2}\}\rangle}\right)}\times\sqrt{\left(\sqrt{2+\langle\{C_{2,1},C_{2,2}\}\rangle}+\sqrt{2-\langle\{C_{2,1},C_{2,2}\}\rangle}\right)}. \tag{13}$$

[1] From now on we will denote $\langle A_{n,x}\otimes B_{n,j}\otimes C_{n,z}\rangle$ as $\langle A_{n,x}B_{n,j}C_{n,z}\rangle$.

[2] From now on, we will write $A_{n,x}\otimes\mathbb{I}\otimes C_{n,z}$ as $A_{n,x}\otimes C_{n,z}$ and $\mathbb{I}_d\otimes B_{n,j}\otimes\mathbb{I}_d$ as $B_{n,j}$.

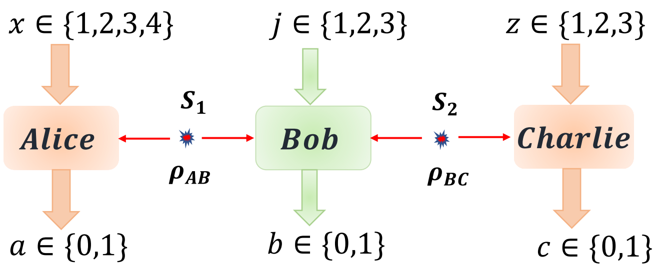

FIG. 2. Asymmetric bilocal network scenario featuring two edge parties (Alice and Charlie) and a central party Bob. The independent sources S_1 and S_2 emit physical systems for Alice-Bob and Charlie-Bob, respectively.

Equation (13) is optimised when $\{A_{2,1}, A_{2,2}\} = \{C_{2,1}, C_{2,2}\} = 0$ and the optimal quantum value is then given by

$$(\mathcal{B}_2)_Q^{\text{opt}} = 2\sqrt{2}. \tag{14}$$

Note that the evaluated optimal quantum value $(\mathcal{B}_2)_Q^{\text{opt}} = 2\sqrt{2}$ is same as that obtained when the state between three parties is assumed to be pair of maximally entangled two-qubit state. This means that for the two-settings bilocality scenario, the optimal quantum violation of the bilocal inequality remains the same even if one considers higher-dimensional maximally entangled states.

III. ASYMMETRIC BILOCAL NETWORK: SCENARIO I

Here we introduce a variant of asymmetric bilocal network scenario for $n = 3$, in which Alice performs one of the four dichotomic measurements and the other edge party (Charlie) performs one of three dichotomic measurements (see Fig. 2). For our purpose, we consider that the central party performs one of three dichotomic measurements. Note that the two-settings bilocality scenario is the same as the symmetric bilocal scenario, where each party has an equal number (two) of measurement settings. In the following, we derive a family of nonlinear asymmetric bilocality inequalities and also evaluate their optimal quantum violations.

In this scenario, let us consider the nonlinear bilocal inequality of the form

$$\mathcal{B}_3 = \sum_{j=1}^{3} \sqrt{|\mathcal{I}_{3,j}|} \leqslant (\mathcal{B}_3)_{bl}, \tag{15}$$

where $(\mathcal{B}_3)_{bl}$ is the bilocal bound of $\mathcal{B}_3$ and the quantity $\mathcal{I}_{3,j} = \langle \tilde{\mathcal{A}}_{3,j} B_{3,j} \tilde{C}_{3,j} \rangle$. We define the quantities $\tilde{\mathcal{A}}_{3,j}$ and $\tilde{C}_{3,j}$ as

$$\begin{aligned}
\tilde{\mathcal{A}}_{3,1} &= A_{3,1} + A_{3,2} + A_{3,3} - A_{3,4},\\
\tilde{\mathcal{A}}_{3,2} &= A_{3,1} + A_{3,2} - A_{3,3} + A_{3,4},\\
\tilde{\mathcal{A}}_{3,3} &= A_{3,1} - A_{3,2} + A_{3,3} + A_{3,4},\\
\tilde{C}_{3,j} &= C_{3,j} + C_{3,j+1} \quad \text{with} \quad C_{3,4} = -C_{3,1}.
\end{aligned} \tag{16}$$

In an ontological model $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$ completely determine the statistics of all the measurements. This means that in the ontological model, we can write the following:

$$\begin{aligned}
\langle A_{3,x}\rangle_{\lambda_1} &= \sum_a (-1)^a P(a|x, \lambda_1), \quad \forall\, x \in \{1,2,3,4\}\\
\langle C_{3,z}\rangle_{\lambda_2} &= \sum_c (-1)^c P(c|z, \lambda_2), \quad \forall\, z \in \{1,2,3\}\\
\langle B_{3,j}\rangle_{\lambda_1,\lambda_2} &= \sum_b (-1)^b P(b|j, \lambda_1, \lambda_2), \quad \forall\, j \in \{1,2,3\}.
\end{aligned} \tag{17}$$

Now, invoking the reproducibility condition given by Eq. (2) it follows that

$$\begin{aligned}
\mathcal{I}_{3,1} = &\iint d\lambda_1 d\lambda_2\, \mu(\lambda_1)\mu(\lambda_2)\big[\langle C_{3,1}\rangle_{\lambda_2}\\
&+ \langle C_{3,2}\rangle_{\lambda_2}\big]\, \langle B_{3,1}\rangle_{\lambda_1,\lambda_2}\big[\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1} + \langle A_{3,3}\rangle_{\lambda_1}\\
&- \langle A_{3,4}\rangle_{\lambda_1}\big].
\end{aligned} \tag{18}$$

Since $|\langle B_{3,1}\rangle_{\lambda_1,\lambda_2}| \leqslant 1$, we obtain

$$\begin{aligned}
|\mathcal{I}_{3,1}| \leqslant &\iint d\lambda_1 d\lambda_2\, \mu(\lambda_1)\mu(\lambda_2)\, \big|\langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2}\big|\\
&\times \big|\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1} + \langle A_{3,3}\rangle_{\lambda_1} - \langle A_{3,4}\rangle_{\lambda_1}\big|.
\end{aligned} \tag{19}$$

The terms $|\mathcal{I}_{3,2}|$ and $|\mathcal{I}_{3,3}|$ can also be written in a similar manner as Eq. (19). Now, for our purpose, by utilizing the inequality[3] proved in [6], we obtain the following:

$$(\mathcal{B}_3)_{bl} \leqslant \left(\iint d\lambda_1 d\lambda_2\, \mu(\lambda_1)\mu(\lambda_2)\, \delta_1\, \delta_2\right)^{\frac{1}{2}}, \tag{20}$$

where $\delta_1 = [|\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1} + \langle A_{3,3}\rangle_{\lambda_1} - \langle A_{3,4}\rangle_{\lambda_1}| + |\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1} - \langle A_{3,3}\rangle_{\lambda_1} + \langle A_{3,4}\rangle_{\lambda_1}| + |\langle A_{3,1}\rangle_{\lambda_1} - \langle A_{3,2}\rangle_{\lambda_1} + \langle A_{3,3}\rangle_{\lambda_1} + \langle A_{3,4}\rangle_{\lambda_1}|]$ and $\delta_2 = [|\langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2}| + |\langle C_{3,2}\rangle_{\lambda_2} + \langle C_{3,3}\rangle_{\lambda_2}| + |\langle C_{3,3}\rangle_{\lambda_2} - \langle C_{3,1}\rangle_{\lambda_2}|]$.

Since all the observables are dichotomic with eigenvalues ± 1, it is straightforward to derive that $\delta_1 \leqslant 6$ and $\delta_2 \leqslant 4$. Therefore, from Eq. (20), integrating over λ_1 and λ_2 we obtain

$$(\mathcal{B}_3)_{bl} \leqslant 2\sqrt{6} \approx 4.89. \tag{21}$$

We show that there are suitable states and observables for which the bilocal bound can be violated in quantum theory. Now, in the following, we evaluate the quantum optimal bound of $\mathcal{B}_3$.

A. Optimal quantum bound of the asymmetric bilocality inequality in scenario I

To derive the optimal quantum bound of $\mathcal{B}_3$ without assuming the dimension of the system, we again invoke the

[3] The inequality proved in Appendix A of [6]

$$\sum_{i=1}^{t}\left(\prod_{k=1}^{r} z_k^i\right)^{\frac{1}{r}} \leqslant \prod_{k=1}^{r}\left(\sum_{i=1}^{t} z_k^i\right)^{\frac{1}{r}}, \quad \forall\, z_k^i \geqslant 0.$$

Here r is the number of edge party in a star network. Note that for our bilocal case $r = 2$.

SOS approach discussed in the preceding Sec. II. Following the similar argument presented earlier in Sec. II, we first show that there exists a suitable operator γ_3 satisfying $\langle\gamma_3\rangle = \beta_3 - (\mathcal{B}_3)_Q$ such that $\langle\gamma_3\rangle \geqslant 0$. Note that the power of SOS approach in evaluating the optimal quantum bound of a particular Bell functional lies in the suitable construction of the operator γ_3 in such a way so that it is a semidefinite operator and can be reduced to the form of the concerned Bell functional. Now, in order to evaluate the quantum optimal bound of $\mathcal{B}_3$, we construct γ_3 in terms of a set of operators $M_{3,j}$, $\forall\ j \in \{1, 2, 3\}$, in the following way:

$$\langle\gamma_3\rangle = \sum_{j=1}^{3} \frac{\sqrt{\omega_{3,j}}}{2} |M_{3,j}\,|\psi\rangle\,|^2, \tag{22}$$

where $\omega_{3,j} \geqslant 0$ are suitable positive numbers and $\omega_{3,j} = \omega_{3,j}^A \omega_{3,j}^C$ that will be specified soon. In the following $\forall\ j \in \{1, 2, 3\}$, we choose the operator $M_{3,j}$ and the quantity $\omega_{3,j}$:

$$|M_{3,j}\,|\psi\rangle\,| = \sqrt{\left|\left(\frac{\tilde{\mathcal{A}}_{3,j}}{\omega_{3,j}^A} \otimes \frac{\tilde{C}_{3,j}}{\omega_{3,j}^C}\right)|\psi\rangle\right|} - \sqrt{|B_{3,j}\,|\psi\rangle\,|}, \tag{23}$$

$$\omega_{3,j}^A = ||\tilde{\mathcal{A}}_{3,j}\,|\psi\rangle\,||_2; \quad \omega_{3,j}^C = ||\tilde{C}_{3,j}\,|\psi\rangle\,||_2. \tag{24}$$

Putting $M_{3,j}$ from Eq. (23) into Eq. (22), after a simple algebraic evaluation, we obtain $\langle\gamma_3\rangle = \sum_{j=1}^{3} \sqrt{\omega_{3,j}} - (\mathcal{B}_3)_Q$. Then, it follows that the quantum optimal value corresponds $\langle\gamma_3\rangle = 0$. Therefore,

$$(\mathcal{B}_3)_Q^{\text{opt}} = \max \sum_{j=1}^{3} \sqrt{\omega_{3,j}^A \omega_{3,j}^C} \tag{25}$$

which in turn gives the optimization condition as follows:

$$|M_{3,j}\,|\psi\rangle\,| = 0 \ \Rightarrow M_{3,j}\,|\psi\rangle = 0, \quad \forall\ j \in \{1, 2, 3\}. \tag{26}$$

Now, in order to evaluate the optimal quantum value $(\mathcal{B}_3)_Q^{\text{opt}}$ and thus the quantity $\sum_{j=1}^{3} \sqrt{\omega_{3,j}^A \omega_{3,j}^C}$, we invoke the inequality given in the footnote 3. Then, the right-hand side of Eq. (25) reduces to

$$\sum_{j=1}^{3} \left(\prod_{k=A,C} \omega_{3,j}^k\right)^{\frac{1}{2}} \leqslant \prod_{k=A,C} \left(\sum_{j=1}^{3} \omega_{3,j}^k\right)^{\frac{1}{2}}. \tag{27}$$

Further, by applying the convex inequality,[4] the quantity $\sum_{j=1}^{3} \omega_{3,j}^k$ can be written as

$$\sum_{j=1}^{3} (\omega_{3,j}^k) \leqslant \sqrt{3 \sum_{j=1}^{3} (\omega_{3,j}^k)^2}. \tag{28}$$

[4]From the Jensen's inequality given by $f(\sum_{k=1}^{t} r_k x_k) \leqslant \sum_{k=1}^{t} r_k f(x_k)$ where $\sum_{k=1}^{t} r_k = 1$, the following inequality can be derived:

$$\sum_{k=1}^{t} \omega_k \leqslant \sqrt{t \sum_{k=1}^{t} \omega_k^2}.$$

Then, by combining Eqs. (27) and (28), from Eq. (25) we obtain

$$(\mathcal{B}_3)_Q^{\text{opt}} = \max \left[\prod_{k=A,C} \left(3 \sum_{j=1}^{3} (\omega_{3,j}^k)^2\right)\right]^{\frac{1}{4}}, \tag{29}$$

where each $(\omega_{3,j}^k)^2$ is evaluated from Eq. (24) as follows:

$$\begin{aligned}
(\omega_{3,1}^A)^2 &= \langle\psi|\,(4 + \{A_{3,1}, (A_{3,2} + A_{3,3} - A_{3,4})\} \\
&\quad + \{A_{3,2}, (A_{3,3} - A_{3,4})\} - \{A_{3,3}, A_{3,4}\})\,|\psi\rangle\,, \\
(\omega_{3,2}^A)^2 &= \langle\psi|\,(4 + \{A_{3,1}, (A_{3,2} - A_{3,3} + A_{3,4})\} \\
&\quad + \{A_{3,2}, (-A_{3,3} + A_{3,4})\} - \{A_{3,3}, A_{3,4}\})\,|\psi\rangle\,, \\
(\omega_{3,3}^A)^2 &= \langle\psi|\,(4 + \{A_{3,1}, (-A_{3,2} + A_{3,3} + A_{3,4})\} \\
&\quad - \{A_{3,2}, (A_{3,3} + A_{3,4})\} + \{A_{3,3}, A_{3,4}\})\,|\psi\rangle\,,
\end{aligned} \tag{30}$$

$$\begin{aligned}
(\omega_{3,1}^C)^2 &= \langle\psi|\,(2 + \{C_{3,1}, C_{3,2}\})\,|\psi\rangle\,, \\
(\omega_{3,2}^C)^2 &= \langle\psi|\,(2 + \{C_{3,2}, C_{3,3}\})\,|\psi\rangle\,, \\
(\omega_{3,3}^C)^2 &= \langle\psi|\,(2 - \{C_{3,1}, C_{3,3}\})\,|\psi\rangle\,.
\end{aligned} \tag{31}$$

Now, in the following we calculate $\sum_{j=1}^{3}(\omega_{3,j}^A)^2$ from Eq. (30) and $\sum_{j=1}^{3}(\omega_{3,j}^C)^2$ from Eq. (31) separately.

1. Evaluation of $\sum_{j=1}^{3}(\omega_{3,j}^A)^2$

$$\begin{aligned}
\sum_{j=1}^{3} (\omega_{3,j}^A)^2 &= \langle\psi|\,(12 + \{A_{3,1}, (A_{3,2} - A_{3,3} + A_{3,4})\} \\
&\quad - \{A_{3,2}, (A_{3,3} + A_{3,4})\} - \{A_{3,3}, A_{3,4}\})\,|\psi\rangle \\
&= \langle\psi|\,(12 + \Delta_3)\,|\psi\rangle\,,
\end{aligned} \tag{32}$$

where $\Delta_3 = \{A_{3,1}, (A_{3,2} - A_{3,3} + A_{3,4})\} - \{A_{3,2}, (A_{3,3} + A_{3,4})\} - \{A_{3,3}, A_{3,4}\}$. Without loss of generality we can always write $|\psi'\rangle = (A_{3,1} + A_{3,2} + A_{3,3} - A_{3,4})\,|\psi\rangle$ such that $|\psi\rangle \neq 0$. Therefore, $\langle\psi'|\psi'\rangle = \langle\psi|\,(4 - \Delta_3)\,|\psi\rangle$ implies $\langle\Delta_3\rangle = 4 - \langle\psi'|\psi'\rangle$. Then it immediately follows that $\langle\Delta_3\rangle_{\max}$ is obtained *iff* $\langle\psi'|\psi'\rangle = 0$. Since $|\psi\rangle \neq 0$, then the following relation must satisfy

$$A_{3,1} - A_{3,2} - A_{3,3} - A_{3,4} = 0. \tag{33}$$

Hence, in order to obtain the quantum optimal value of $(\mathcal{B}_3)_Q^{\text{opt}}$, observables of Alice must satisfy the linear condition given by Eq. (33). Therefore, $\langle\Delta_3\rangle_{\max} = 4$ leads to

$$\sum_{j=1}^{3} (\omega_{3,j}^A)^2 = \langle\psi|\,(12 + \Delta_3)\,|\psi\rangle \leqslant 16. \tag{34}$$

2. Evaluation of $\sum_{j=1}^{3}(\omega_{3,j}^C)^2$

$$\begin{aligned}
\sum_{j=1}^{3} (\omega_{3,j}^C)^2 &= \langle\psi|\,(6 + \{C_{3,2}, (C_{3,1} + C_{3,3})\} - \{C_{3,1}, C_{3,3}\})\,|\psi\rangle \\
&= \langle\psi|\,(6 + 3\mathbb{I} - (C_{3,1} - C_{3,2} + C_{3,3})^2)\,|\psi\rangle \leqslant 9.
\end{aligned} \tag{35}$$

Equation (35) is maximized when

$$C_{3,1} - C_{3,2} + C_{3,3} = 0. \tag{36}$$

Finally, we obtain the optimal quantum value from Eqs. (25), (34), and (35) as

$$(\mathcal{B}_3)_Q^{\text{opt}} = 6. \tag{37}$$

It is important to remark here that the optimal quantum value $(\mathcal{B}_3)_Q^{\text{opt}} = 6$ is evaluated without specifying the dimension of both the system and observables. The optimal value fixes the states, and the observables are the following.

B. The state and observables for the optimal quantum violation of $\mathcal{B}_3$

We further obtain relationships between the observables of all the parties for achieving the optimal quantum violation. Such relationships are given in Eqs. (33) and (36), which in turn provide the relationship between the observables in terms of the anticommuting relations:

$$\{A_{3,1}, A_{3,2}\} = \{A_{3,1}, A_{3,3}\} = \{A_{3,1}, A_{3,4}\} = \tfrac{2}{3}\mathbb{I}_d,$$
$$\{A_{3,2}, A_{3,3}\} = \{A_{3,2}, A_{3,4}\} = \{A_{3,3}, A_{3,4}\} = -\tfrac{2}{3}\mathbb{I}_d,$$
$$\{C_{3,1}, C_{3,2}\} = \{C_{3,2}, C_{3,3}\} = -\{C_{3,1}, C_{3,3}\} = \mathbb{I}_d. \tag{38}$$

By using the above relations between the observables given by Eq. (38) on the observables, one can always construct a set of observables for Alice and Charlie in the Hilbert space dimension $\mathcal{H}^d, \ \forall\, d \geqslant 2$.

Next, we recall the optimization condition obtained in the SOS method from Eq. (26) to find the constraints on Bob's observable. The specific condition $M_{3,j}\,|\psi\rangle = 0\,,\ \forall\ j \in \{1, 2, 3\}$, implies

$$B_{3,j} = \frac{\tilde{\mathcal{A}}_{3,j}}{\omega^A_{3,j}} \otimes \frac{\tilde{C}_{3,j}}{\omega^C_{3,j}}. \tag{39}$$

In the following, we then explicitly construct a set of observables for the Hilbert space dimension $\mathcal{H}^2$:

$$C_{3,1} = \sigma_z, \quad C_{3,2} = \left(\frac{\sqrt{3}}{2}\sigma_x + \frac{\sigma_z}{2}\right), \quad C_{3,3} = \left(\frac{\sqrt{3}}{2}\sigma_x - \frac{\sigma_z}{2}\right);$$
$$A_{3,1} = \frac{\sigma_x + \sigma_y + \sigma_z}{\sqrt{3}}, \qquad A_{3,2} = \frac{\sigma_x + \sigma_y - \sigma_z}{\sqrt{3}},$$
$$A_{3,3} = \frac{\sigma_x - \sigma_y + \sigma_z}{\sqrt{3}}, \qquad A_{3,4} = \frac{-\sigma_x + \sigma_y + \sigma_z}{\sqrt{3}}. \tag{40}$$

Note that employing the above-mentioned observables, we find that the quantum optimal value $(\mathcal{B}_3)_Q^{\text{opt}} = 6$ is achieved when two maximally entangled two-qubit states are shared between Alice-Bob and Bob-Charlie.

IV. ASYMMETRIC BILOCAL NETWORK: SCENARIO II

Here we consider (see Fig. 3) the central party Bob performs equal number of measurements (four) as Alice in contrast to the preceding scenario discussed in Sec. III where the number of measurements for Bob and Charlie was considered to be equal. In this scenario, let us consider the nonlinear bilocal inequality of the form

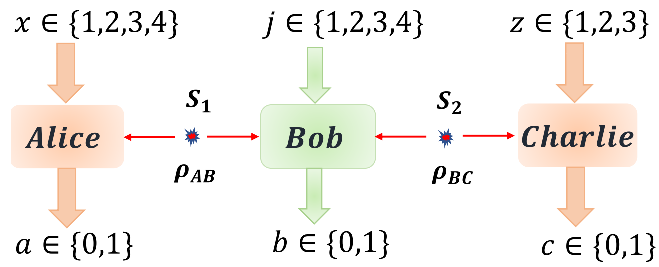

FIG. 3. Asymmetric bilocal network scenario featuring two edge parties (Alice and Charlie) and a central party Bob. The independent sources S_1 and S_2 emit physical systems for Alice-Bob and Charlie-Bob, respectively.

$$\mathcal{B}'_3 = \sum_{j=1}^{4} \sqrt{\left|\langle \tilde{\mathcal{A}}'_{3,j} B_{3,j} \tilde{C}'_{3,j}\rangle\right|} \leqslant (\mathcal{B}'_3)_{bl}, \tag{41}$$

where $(\mathcal{B}'_3)_{bl}$ is the bilocal bound of $\mathcal{B}'_3$ and $\tilde{\mathcal{A}}'_{3,j} = A_{3,j} + A_{3,j+1}$ satisfying $A_{3,4} = -A_{3,1}$; $\tilde{C}'_{3,1} = C_{3,1} + C_{3,2} + C_{3,3}$; $\tilde{C}'_{3,2} = C_{3,1} + C_{3,2} - C_{3,3}$; $\tilde{C}'_{3,3} = C_{3,1} - C_{3,2} + C_{3,3}$; $\tilde{C}'_{3,4} = -C_{3,1} + C_{3,2} + C_{3,3}$. We find (see Appendix A 1) that the bilocal bound in this scenario is $(\mathcal{B}'_3)_{bl} = 6$. It can be shown that there are suitable states and observables for which quantum correlations violate the bilocal bound. In the following, we evaluate the quantum optimal bound of $\mathcal{B}'_3$.

The optimal quantum bound of $\mathcal{B}'_3$ without assuming the dimension of the system is derived by suitably invoking the SOS approach. The explicit construction of the SOS method is provided in Appendix A 2. The optimal quantum bound is found to be

$$(\mathcal{B}'_3)_Q^{\text{opt}} = 4[3(2 + \sqrt{2})]^{\frac{1}{4}} \approx 7.16. \tag{42}$$

It is to be noted here that the obtained optimal quantum bound $((\mathcal{B}'_3)_Q^{\text{opt}} \approx 7.16)$ in this scenario is greater than that obtained $((\mathcal{B}_3)_Q^{\text{opt}} = 6)$ in the earlier scenario I. Thus, the bilocal as well as the quantum optimal bound depends on how the asymmetry is invoked in the bilocal scenario. The corresponding state and observables for which the quantum optimal bound is achieved are given in Appendix A 3.

V. ASYMMETRIC TRILOCAL NETWORK: SCENARIO I

Here we introduce a variant of asymmetric trilocal network scenarios in which one of the edge party Charlie performs one of three dichotomic measurements and rest of the edge parties Alice and Diana as well as the central party Bob perform four dichotomic measurements (see Fig. 4). In this scenario, let us introduce the nonlinear trilocal inequality of the form

$$\mathcal{T}_3 = \sum_{j=1}^{4} |\mathcal{J}_{3,j}|^{\frac{1}{3}} \leqslant (\mathcal{T}_3)_{tl}, \tag{43}$$

where $(\mathcal{T}_3)_{tl}$ is the local bound of $\mathcal{T}_3$ and the quantity $\mathcal{J}_{3,j} = \langle \tilde{\mathscr{A}}_{3,j} B_{3,j} \tilde{\mathscr{C}}_{3,j} \tilde{\mathscr{D}}_{3,j}\rangle$. We define the quantities $\tilde{\mathscr{A}}_{3,j}$, $\tilde{\mathscr{C}}_{3,j}$, and

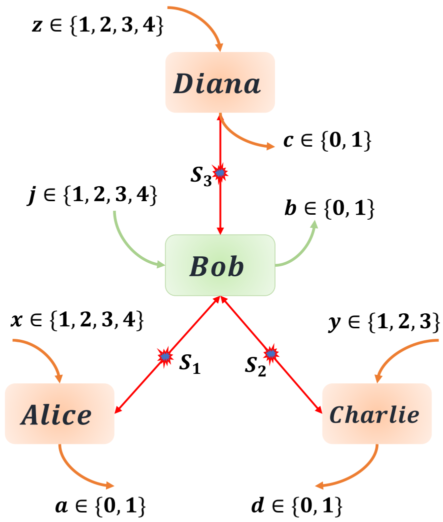

FIG. 4. Asymmetric trilocal network scenario featuring three edge parties (Alice, Charlie, and Diana) and a central party Bob. The independent sources S_1, S_2, and S_3 emit physical systems for Alice-Bob, Charlie-Bob, and Diana-Bob, respectively.

$\tilde{\mathscr{D}}_{3,j}$ as follows:

$$\begin{aligned}
&\tilde{\mathscr{A}}_{3,j} = A_{3,j} + A_{3,j+1} \quad \text{with} \quad A_{3,5} = -A_{3,1},\\
&\tilde{\mathscr{C}}_{3,1} = C_{3,1} + C_{3,2} + C_{3,3}, \quad \tilde{\mathscr{C}}_{3,2} = C_{3,1} + C_{3,2} - C_{3,3},\\
&\tilde{\mathscr{C}}_{3,3} = C_{3,1} - C_{3,2} + C_{3,3}, \quad \tilde{\mathscr{C}}_{3,4} = -C_{3,1} + C_{3,2} + C_{3,3},\\
&\tilde{\mathscr{D}}_{3,1} = D_{3,1} + D_{3,2} + D_{3,3} + D_{3,4},\\
&\tilde{\mathscr{D}}_{3,2} = D_{3,1} + D_{3,2} + D_{3,3} - D_{3,4},\\
&\tilde{\mathscr{D}}_{3,3} = D_{3,1} + D_{3,2} - D_{3,3} - D_{3,4},\\
&\tilde{\mathscr{D}}_{3,4} = D_{3,1} - D_{3,2} - D_{3,3} - D_{3,4}.
\end{aligned} \tag{44}$$

Now, invoking the reproducibility condition given by Eq. (2) it follows that

$$\begin{aligned}
\mathcal{J}_{3,1} = &\iiint d\lambda_1 d\lambda_2 d\lambda_3 \; \mu(\lambda_1)\mu(\lambda_2)\mu(\lambda_3)\\
&\times \left[\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1}\right]\\
&\times \left[\langle B_{3,1}\rangle_{\lambda_1,\lambda_2,\lambda_3}\right]\left[\langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2} + \langle C_{3,3}\rangle_{\lambda_2}\right]\\
&\times \left[\langle D_{3,1}\rangle_{\lambda_3} + \langle D_{3,2}\rangle_{\lambda_3} + \langle D_{3,3}\rangle_{\lambda_3} + \langle D_{3,4}\rangle_{\lambda_3}\right].
\end{aligned} \tag{45}$$

Since $|\langle B_{3,1}\rangle_{\lambda_1,\lambda_2,\lambda_3}| \leqslant 1$, we obtain

$$\begin{aligned}
|\mathcal{J}_{3,1}| = &\iiint d\lambda_1 d\lambda_2 d\lambda_3 \; \mu(\lambda_1)\mu(\lambda_2)\mu(\lambda_3)\\
&\times \left|\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1}\right|\\
&\times \left|\langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2} + \langle C_{3,3}\rangle_{\lambda_2}\right|\\
&\times \left|\langle D_{3,1}\rangle_{\lambda_3} + \langle D_{3,2}\rangle_{\lambda_3} + \langle D_{3,3} + \langle D_{3,4}\rangle_{\lambda_3}\right|.
\end{aligned} \tag{46}$$

The terms $|\mathcal{J}_{3,2}|$, $|\mathcal{J}_{3,3}|$, and $|\mathcal{J}_{3,4}|$ can also be written in a similar manner as Eq. (46). Then, we obtain

$$(\mathcal{T}_3)_{tl} \leqslant \left(\iiint d\lambda_1 d\lambda_2 d\lambda_3 \; \mu(\lambda_1)\mu(\lambda_2)\mu(\lambda_3)\; \eta_1\eta_2\eta_3\right)^{\frac{1}{3}}, \tag{47}$$

where $\eta_1 = [|\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1}| + |\langle A_{3,2}\rangle_{\lambda_1} + \langle A_{3,3}\rangle_{\lambda_1}| + |\langle A_{3,3}\rangle_{\lambda_1} + \langle A_{3,4}\rangle_{\lambda_1}| + |\langle A_{3,4}\rangle_{\lambda_1} - \langle A_{3,1}\rangle_{\lambda_1}|]$, $\eta_2 = [|\langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2} + \langle C_{3,3}\rangle_{\lambda_2}| + |\langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2} - \langle C_{3,3}\rangle_{\lambda_2}| + |\langle C_{3,1}\rangle_{\lambda_2} - \langle C_{3,2}\rangle_{\lambda_2} + \langle C_{3,3}\rangle_{\lambda_2}| + | - \langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2} + \langle C_{3,3}\rangle_{\lambda_2}|]$, and $\eta_3 = [|\langle D_{3,1}\rangle_{\lambda_3} + \langle D_{3,2}\rangle_{\lambda_3} + \langle D_{3,3}\rangle_{\lambda_3} + \langle D_{3,4}\rangle_{\lambda_3}| + |\langle D_{3,1}\rangle_{\lambda_3} + \langle D_{3,2}\rangle_{\lambda_3} + \langle D_{3,3}\rangle_{\lambda_3} - \langle D_{3,4}\rangle_{\lambda_3}| + |\langle D_{3,1}\rangle_{\lambda_3} + \langle D_{3,2}\rangle_{\lambda_3} - \langle D_{3,3}\rangle_{\lambda_3} - \langle D_{3,4}\rangle_{\lambda_3}| + |\langle D_{3,1}\rangle_{\lambda_3} - \langle D_{3,2}\rangle_{\lambda_3} - \langle D_{3,3}\rangle_{\lambda_3} - \langle D_{3,4}\rangle_{\lambda_3}|]$.

Since all the observables are dichotomic with eigenvalues ± 1, it is straightforward to derive that $\eta_1 \leqslant 6$, $\eta_2 \leqslant 6$, and $\eta_3 \leqslant 8$. Therefore, from Eq. (47), integrating over λ_1, λ_2, and λ_3 we get

$$(\mathcal{T}_3)_{tl} \leqslant 2(6)^{\frac{2}{3}} \approx 6.60, \tag{48}$$

i.e., the trilocal bound $(\mathcal{T}_3)_{tl} = 2(6)^{\frac{2}{3}} \approx 6.60$. There are suitable states and observables for which the trilocal bound can be violated in quantum theory. Now, in the following, we evaluate the quantum optimal bound of $\mathcal{T}_3$.

A. Optimal quantum bound of the asymmetric trilocality inequality for scenario I

Here, by invoking the SOS approach, we evaluate the optimal quantum bound of $\mathcal{T}_3$ without assuming the dimension of the system. We first show that there exists a positive-semidefinite operator Γ_3 satisfying $\langle\Gamma_3\rangle = \zeta_3 - (\mathcal{T}_3)_Q$. The existence of such operator can be proved by considering a set of operators $L_{3,j}$, $\forall\, j \in \{1, 2, 3, 4\}$, such that

$$\langle\Gamma_3\rangle = \sum_{j=1}^{4} \frac{(\omega_{3,j})^{\frac{1}{3}}}{2} |L_{3,j}\,|\psi\rangle\,|^2, \tag{49}$$

where $\omega_{3,j} \geqslant 0$ and $\omega_{3,j} = \omega^A_{3,j}\omega^C_{3,j}\omega^D_{3,j}$. We choose $L_{3,j}$ and $\omega_{3,j}$ as

$$|L_{3,j}\,|\psi\rangle\,| = \left|\left(\frac{\tilde{\mathscr{A}}_{3,j}}{\omega^A_{3,j}} \otimes \frac{\tilde{\mathscr{C}}_{3,j}}{\omega^C_{3,j}} \otimes \frac{\tilde{\mathscr{D}}_{3,j}}{\omega^D_{3,j}}\right)|\psi\rangle\right|^{\frac{1}{3}} - |B_{3,j}\,|\psi\rangle\,|^{\frac{1}{3}}, \tag{50}$$

$$\begin{aligned}
&\omega^A_{3,j} = ||\tilde{\mathscr{A}}_{3,j}\,|\psi\rangle\,||_2, \quad \omega^C_{3,j} = ||\tilde{\mathscr{C}}_{3,j}\,|\psi\rangle\,||_2,\\
&\omega^D_{3,j} = ||\tilde{\mathscr{D}}_{3,j}\,|\psi\rangle\,||_2.
\end{aligned} \tag{51}$$

Now, putting $|L_{3,j}\,|\psi\rangle\,|$ from Eq. (50) into Eq. (49), after a simple algebraic evaluation, we obtain $\langle\Gamma_3\rangle = \sum_{j=1}^{3}(\omega_{3,j})^{\frac{1}{3}} - (\mathcal{T}_3)_Q$. Then, it follows that the quantum optimal value corresponds $\langle\Gamma_3\rangle = 0$. Therefore,

$$(\mathcal{T}_3)_Q^{\text{opt}} = \sum_{j=1}^{4} \left(\omega^A_{3,j}\omega^C_{3,j}\omega^D_{3,j}\right)^{\frac{1}{3}}. \tag{52}$$

Such optimal quantum value will occur under the following optimization condition:

$$L_{3,j}\;|\psi\rangle = 0, \quad \forall\, j \in \{1, 2, 3, 4\}. \tag{53}$$

Hence, from Eq. (52) and by using the inequalities given in footnotes 3 and 4, we obtain the quantum optimal value as follows:

$$(\mathcal{T}_3)_Q^{\text{opt}} = 2\left[\prod_{k=A,C,D}\left(\max\sum_{j=1}^{4}\left(\omega_{3,j}^{k}\right)^2\right)\right]^{\frac{1}{6}}. \tag{54}$$

Note that the quantities $\sum_{j=1}^{3}(\omega_{3,j}^{A})^2 \leqslant 4(2+\sqrt{2})$ and $\sum_{j=1}^{3}(\omega_{3,j}^{C})^2 \leqslant 12$ have already derived in Appendixes (A 2 a) and (A 2 b). Now, we evaluate the quantity $\sum_{j=1}^{4}(\omega_{3,j}^{D})^2$ from Eq. (51) as follows:

$$\sum_{j=1}^{4}\left(\omega_{3,j}^{D}\right)^2 = \langle\psi|\,(16 + 2(\{D_{3,1}, (D_{3,2} - D_{3,4})\} + \{D_{3,3}, (D_{3,2} + D_{3,4})\})\,|\psi\rangle\,. \tag{55}$$

Since the quantities $D_{3,1}$ and $D_{3,3}$ appeared independently with $\{D_{3,1}, (D_{3,2} - D_{3,4})\}$ and $\{D_{3,3}, (D_{3,2} + D_{3,4})\}$, respectively, without loss of generality, we chose $D_{3,1} = (D_{3,2} - D_{3,4})/\nu_1$ and $D_{3,3} = (D_{3,2} + D_{3,4})/\nu_1$, where $\nu_1 = ||(D_{3,2} - D_{3,4})\,|\psi\rangle\,||_2$ and $\nu_2 = ||(D_{3,2} + D_{3,4})\,|\psi\rangle\,||_2$. Therefore, Eq. (55) reduces to

$$\sum_{j=1}^{4}\left(\omega_{4,j}^{D}\right)^2 = 16 + 4[\sqrt{2 - \langle\{D_{3,2}, D_{3,4}\}\rangle} + \sqrt{2 - \langle\{D_{3,2}, D_{3,4}\}\rangle}]. \tag{56}$$

The maximum value of $\sum_{j=1}^{4}(\omega_{4,j}^{D})^2 = 8(2+\sqrt{2})$ is then achieved when $\{D_{3,2}, D_{3,4}\} = 0$ which automatically implies $\nu_1 = \nu_2 = \sqrt{2}$. Therefore, for the optimal quantum violation, the linear constraints on Diana's observables are given as

$$D_{3,4} - \sqrt{2}\,D_{3,1} - D_{3,2} = D_{3,2} + D_{3,4} - \sqrt{2}\,D_{3,3} = 0. \tag{57}$$

Therefore, we obtain the quantum optimal value as given by

$$(\mathcal{T}_3)_Q^{\text{opt}} = 4\,[2\sqrt{3}(1+\sqrt{2})]^{\frac{1}{3}} \approx 8.12. \tag{58}$$

It is important to remark here that the optimal quantum value $(\mathcal{T}_3)_Q^{\text{opt}} = 4\;[2\sqrt{3}(1+\sqrt{2})]^{\frac{1}{3}} \approx 8.12$ is evaluated without specifying the dimension of both the system and observables. The states, and the observables for which the optimal value will be achieved, are given in the following.

B. The state and observables for the optimal quantum violation of $(\mathcal{T})_3$

We further obtain relationships between the observables of all the parties in terms of the anticommuting relations for achieving the optimal quantum violation. The anticommutation relations for Alice's and Charlie's observables are evaluated from Eqs. (A11) and (A13). The anticommutation relations for Diana's observables are evaluated from Eq. (57). All the relations are given as follows:

$$\begin{aligned}
\{A_{3,1}, A_{3,2}\} &= \{A_{3,2}, A_{3,3}\} = \{A_{3,3}, A_{3,4}\}\\
&= -\{A_{3,1}, A_{3,4}\} = \sqrt{2}\,\mathbb{I}_d,\\
\{A_{3,1}, A_{3,3}\} &= \{A_{3,2}, A_{3,4}\} = 0;\\
\{D_{3,1}, D_{3,3}\} &= \{D_{3,2}, D_{3,4}\} = 0,\\
\{D_{3,1}, D_{3,2}\} &= \{D_{3,2}, D_{3,3}\} = \{D_{3,3}, D_{3,4}\}\\
&= -\{D_{3,1}, D_{3,4}\} = \sqrt{2}\,\mathbb{I}_d,\\
\{C_{3,1}, C_{3,2}\} &= \{C_{3,2}, C_{3,3}\} = -\{C_{3,1}, C_{3,3}\} = \mathbb{I}_d.
\end{aligned} \tag{59}$$

By using the above relations between the observables given by Eq. (59) on the observables, one can always construct a set of observables for Alice and Charlie in the Hilbert space dimension $\mathcal{H}^d$, $\forall\, d \geqslant 2$.

Next, we recall the optimization condition obtained in the SOS method from Eq. (53) to find the constraints on Bob's observable. The specific condition $L_{3,j}\,|\psi\rangle = 0$, $\forall\, j \in \{1, 2, 3, 4\}$, implies the following:

$$B_{3,j} = \frac{\tilde{\mathscr{A}}_{3,j}}{\omega_{3,j}^{A}} \otimes \frac{\tilde{\mathscr{C}}_{3,j}}{\omega_{3,j}^{C}} \otimes \frac{\tilde{\mathscr{D}}_{3,j}}{\omega_{3,j}^{C}}. \tag{60}$$

We explicitly construct a set of observables of Alice, Charlie, and Diana for the Hilbert space dimension $\mathcal{H}^2$ as follows:

$$\begin{aligned}
&A_{3,1} = r\,\sigma_x + \sqrt{1-r^2}\sigma_z, \quad A_{3,2} = t\,\sigma_x + \sqrt{1-t^2}\sigma_z,\\
&A_{3,3} = t\,\sigma_x - \sqrt{1-t^2}\,\sigma_z, \quad A_{3,4} = r\,\sigma_x - \sqrt{1-r^2}\,\sigma_z,\\
&D_{3,1} = -t\,\sigma_x + \sqrt{1-t^2}\sigma_z, \quad D_{3,2} = -t\,\sigma_x - \sqrt{1-t^2}\sigma_z,\\
&D_{3,3} = -r\,\sigma_x - \sqrt{1-r^2}\,\sigma_z, \quad D_{3,4} = r\,\sigma_x - \sqrt{1-r^2}\,\sigma_z,\\
&C_{3,1} = \sigma_x, \quad C_{3,2} = \sigma_y, \quad C_{3,3} = \sigma_z,
\end{aligned} \tag{61}$$

where $r = \frac{1}{2}\sqrt{2-\sqrt{2}}$ and $t = \frac{1}{2}\sqrt{2+\sqrt{2}}$. Note that employing the above-mentioned observables, we find that the quantum optimal value $(\mathcal{T}_3)_Q^{\text{opt}} \approx 7.23$ is achieved when three maximally entangled two-qubit states are shared between Alice-Bob, Charlie-Bob, and Diana-Bob.

VI. ASYMMETRIC TRILOCAL NETWORK: SCENARIO-II

Here we present another asymmetric trilocal network scenario in which one of the edge party Alice performs one of four dichotomic measurements and rest of the edge parties Charlie and Diana as well as the central party Bob perform three dichotomic measurements (see Fig. 5). In this scenario, let us consider the nonlinear trilocal inequality of the form

$$\mathcal{T}_3' = \sum_{j=1}^{3}\left|\langle\tilde{\mathscr{A}}_{3,j}' B_{3,j}\tilde{\mathscr{C}}_{3,j}'\tilde{\mathscr{D}}_{3,j}'\rangle\right|^{\frac{1}{3}} \;\leqslant\; (\mathcal{T}_3')_{tl}, \tag{62}$$

where $(\mathcal{T}_3')_{tl}$ is the local bound of $\mathcal{T}_3'$ and the quantities $\tilde{\mathscr{A}}_{3,1}' = A_{3,1} + A_{3,2} + A_{3,3} - A_{3,4}$; $\tilde{\mathscr{D}}_{3,1}' = D_{3,1} + D_{3,2} + D_{3,3}$; $\tilde{\mathscr{A}}_{3,2}' = A_{3,1} + A_{3,2} - A_{3,3} + A_{3,4}$; $\tilde{\mathscr{D}}_{3,2}' = D_{3,1} - D_{3,2} + D_{3,3}$; $\tilde{\mathscr{A}}_{3,3}' = A_{3,1} - A_{3,2} + A_{3,3} + A_{3,4}$; $\tilde{\mathscr{D}}_{3,3}' = D_{3,1} - D_{3,2} - D_{3,3}$; $\tilde{\mathscr{C}}_{3,1}' = C_{3,j} + C_{3,j+1}$ satisfying $C_{3,4} = -C_{3,1}$. We find (see Appendix B 1) that the trilocal bound in this scenario is given by

$$(\mathcal{T}_3')_{tl} = 2(15)^{\frac{1}{3}} \approx 4.93. \tag{63}$$

It can be shown that there are suitable states and observables for which quantum correlations violate the trilocal bound. In the following, we evaluate the quantum optimal bound of $\mathcal{T}_3'$.

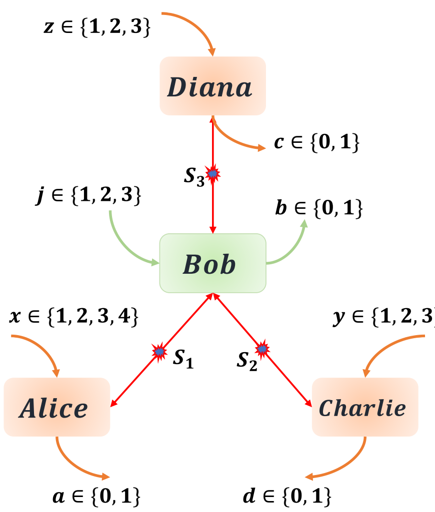

FIG. 5. Asymmetric trilocal network scenario featuring three edge parties (Alice, Charlie, and Diana) and a central party Bob. The independent sources S_1, S_2, and S_3 emit physical systems for Alice-Bob, Charlie-Bob, and Diana-Bob, respectively.

The optimal quantum bound of $\mathcal{T}_3'$ without assuming the dimension of the system is derived by suitably invoking the SOS approach as discussed earlier. The explicit construction of the SOS method is provided in Appendix B 2. The optimal quantum bound is found to be

$$(\mathcal{T}_3')_Q^{\text{opt}} = 6. \tag{64}$$

It is to be noted here that the obtained optimal quantum bound ($(\mathcal{T}_3')_Q^{\text{opt}} = 6$) in this scenario is lesser than that obtained ($(\mathcal{T}_3)_Q^{\text{opt}} \approx 7.23$) in the earlier scenario I. Thus, the trilocal as well as the quantum optimal bound depend on how the asymmetry invoked in the trilocal scenario.

The corresponding state and observables for which the quantum optimal bound is achieved are given in Appendix B 3.

VII. ASYMMETRIC NETWORK FOR ARBITRARY *n*

Let us now provide a sketch of the result in an asymmetric bilocal network scenario for arbitrary n. Alice performs one of $m_A = 2^{n-1}$ dichotomic measurements and both the central party (Bob) and Charlie perform one of $m_B = n$ and $m_c = n$ dichotomic measurements, respectively. In this scenario, we consider the nonlinear asymmetric bilocality inequality for arbitrary input n of the form

$$\mathcal{B}_n = \sum_{j=1}^{n} \sqrt{|\mathcal{I}_{n,j}|} \leqslant (\mathcal{B}_n)_{bl}, \tag{65}$$

where $(\mathcal{B}_n)_{bl}$ is the bilocal bound of $\mathcal{B}_n$ and $\mathcal{I}_{n,j} = \langle \tilde{\mathcal{A}}_{n,j} \otimes B_{n,j} \otimes \tilde{C}_{n,j}\rangle$ with $\tilde{\mathcal{A}}_{n,j}$ and $\tilde{C}_{n,j}$ are unnormalized observables given as follows:

$$\tilde{\mathcal{A}}_{n,j} = \sum_{x=1}^{2^{n-1}} (-1)^{y_j^x} A_{n,x}; \quad \tilde{C}_{n,j} = \sum_{z=1}^{n} (C_{n,z} + C_{n,z+1}) \tag{66}$$

with $C_{n,n+1} = -C_{n,1}$ and $y_j^x \in \{0,1\}$, $\forall\ j \in [n]$. For our purpose, by using the encoding scheme which was earlier introduced in the context of random access codes (RACs) [43–47] protocol, we fix the values of y_j^x in the following way. Let us consider a random variable $y^\alpha \in \{0,1\}^n$ with $\alpha \in \{1, 2 \dots 2^n\}$. Each element of the bit string can be written as $y^\alpha = y^\alpha_{j=1} y^\alpha_{j=2} y^\alpha_{j=3} \dots y^\alpha_{j=n}$. For example, if $y^\alpha = 011\dots00$, then $y^\alpha_{j=1} = 0$, $y^\alpha_{j=2} = 1$, $y^\alpha_{j=3} = 1$ and so on. We denote the n-bit binary strings as y^x. Here we consider the bit strings such that for any two x and x', $y^x \oplus_2 y^{x'} = 11\dots1$. Clearly, we have $x \in \{1, 2 \dots 2^{n-1}\}$ constituting the inputs for Alice. If $x = 1$, we get all the first bit of each bit string y_j for every $j \in \{1, 2 \dots n\}$.

Now it follows from the reproducibility condition given by Eq. (2) that

$$\mathcal{I}_{n,1} = \iint \mu(\lambda_1)\mu(\lambda_2) d\lambda_1 d\lambda_2 \langle \tilde{\mathcal{A}}_{n,1}\rangle_{\lambda_1} \langle B_{n,1}\rangle_{\lambda_1,\lambda_2} \langle \tilde{C}_{n,1}\rangle_{\lambda_1}. \tag{67}$$

Since $|\langle B_{n,1}\rangle_{\lambda_1,\lambda_2}| \leqslant 1$, we obtain from Eq. (67)

$$|\mathcal{I}_{n,j}| \leqslant \iint \mu(\lambda_1)\mu(\lambda_2) d\lambda_1 d\lambda_2 \left|\langle \tilde{\mathcal{A}}_{n,j}\rangle_{\lambda_1}\right| \left|\langle \tilde{C}_{n,j}\rangle_{\lambda_2}\right|, \quad \forall\ j \in \{1, 2, \dots, n\}. \tag{68}$$

By putting the values of Eq. (68) in (65) and, then by applying the property of the inequality given in footnote 3, we obtain

$$(\mathcal{B}_n)_{bl} \leqslant \sqrt{\int d\lambda_1 \mu(\lambda_1) \left|\langle \tilde{\mathcal{A}}_{n,j}\rangle_{\lambda_1}\right|} \sqrt{\int d\lambda_2 \mu(\lambda_2) \left|\langle \tilde{C}_{n,j}\rangle_{\lambda_2}\right|}. \tag{69}$$

Note that all the observables $A_{n,x}$ and $C_{n,z}$ are dichotomic with eigenvalues ± 1. In [44,45] it was derived that the value of $|\langle \tilde{\mathcal{A}}_n\rangle_{\lambda_1}| = n \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$ and $|\langle \tilde{C}_n\rangle_{\lambda_2}| = (2n-2)$. Hence, the bilocal bound is given by

$$(\mathcal{B}_n)_{bl} = \sqrt{2n(n-1)\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}}, \tag{70}$$

where $\lfloor x \rfloor$ denotes the largest integer smaller or equal to x.

We derive the optimal quantum value of $(\mathcal{B}_n)_Q$ of the bilocality inequality proposed in Eq. (65) by again invoking the SOS approach. We consider a positive-semidefinite operator $\langle \gamma_n \rangle \geqslant 0$, that can be expressed as $\langle \gamma_n \rangle = \beta_n \mathbb{I} - (\mathcal{B}_n)_Q$, where β_n is the optimal value that can be obtained when $\langle \gamma_n \rangle$ is equal to zero. This can be proved by considering a set of positive operators $M_{n,j}$ which are polynomial functions of $A_{n,x}$, $C_{n,z}$, $B_{n,j}$ such that

$$\langle \gamma_n \rangle = \sum_{j=1}^{n} \frac{(\omega_{n,j})^{\frac{1}{2}}}{2} \langle \psi | (M_{n,j})^\dagger (M_{n,j}) | \psi \rangle, \tag{71}$$

where $\omega_{n,j}$ are suitable positive numbers and $\omega_{n,j} = (\omega^A_{n,j})\, (\omega^C_{n,j})$. The optimal quantum value of $(\mathcal{B}_n)_Q$ is obtained if $\langle \gamma_n \rangle = 0$, implying that $M_{n,j}|\psi\rangle = 0$. We choose a suitable set of positive operators $M_{n,j}$ (with $j \in [n]$), such that

$$M_{n,j}|\psi\rangle = \sqrt{\left|(\mathcal{A}_{n,j} \otimes C_{n,j})|\psi\rangle\right|} - \sqrt{\left|B_{n,j}|\psi\rangle\right|}, \tag{72}$$

where $\mathcal{A}_{n,j} = \frac{\tilde{\mathcal{A}}_{n,j}}{\omega^A_{n,j}}$ and $C_{n,j} = \frac{\tilde{C}_{n,j}}{\omega^C_{n,j}}$ with $\omega^A_{n,j} = ||\tilde{\mathcal{A}}_{n,j}|\psi\rangle||_2$ and $\omega^C_{n,j} = ||\tilde{C}_{n,j}|\psi\rangle||_2$. By inserting Eq. (72) in (71), we obtain $\langle\gamma_n\rangle = -(\mathcal{B}_n)_Q + \sum_{j=1}^n(\omega_{n,j})^{\frac{1}{2}}$. The optimal value of $(\mathcal{B}_n)_Q$ is obtained if $\langle\gamma_n\rangle = 0$. Therefore, the optimal value is given as follows:

$$(\mathcal{B}_n)_Q^{\text{opt}} = \max\left(\sum_{j=1}^n(\omega_{n,j})^{\frac{1}{2}}\right). \tag{73}$$

Now, by using the inequality given in footnote 3 along with the convex inequality given in the footnote 4, we obtain

$$(\mathcal{B}_n)_Q^{\text{opt}} = \max\left[\prod_{k=A,C}\left(n\sum_{j=1}^n\left(\omega^k_{n,j}\right)^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{74}$$

We can always evaluate each term $(\omega^A_{n,j})^2$ and $(\omega^C_{n,j})^2$ by following the procedure discussed for the $n = 3$ case. However, such evaluation will take rigorous algebraic treatment which we are skipping here. Thus, instead of following the same path, in the following, we present an argument that leads to the optimal quantum value of $(\mathcal{B}_n)_Q$.

To begin with, let us revisit the expression $\mathcal{B}_n$ given by Eq. (65). Since all $\mathcal{I}_{n,j}$ are real numbers and positive by construction, we can always invoke the convex inequality (as mentioned in footnote 4) to obtain

$$(\mathcal{B}_n)_Q \leqslant \left(n\sum_{j=1}^n\mathcal{I}_{n,j}\right)^{\frac{1}{2}}. \tag{75}$$

Then the quantity $\sum_{j=1}^n\mathcal{I}_{n,j}$ is evaluated from Eq. (16):

$$\sum_{j=1}^n\mathcal{I}_{n,j} = \left\langle\sum_{j=1}^n(\tilde{\mathcal{A}}_{n,j}\otimes B_{n,j}\otimes\tilde{C}_{n,j})\right\rangle. \tag{76}$$

Now, by noting the optimization condition $M_{n,j}|\psi\rangle = 0$ form the SOS method, we write

$$B_{n,j} = \mathcal{A}_{n,j}\otimes C_{n,j}, \quad \forall\ j\in\{1,2,\dots,n\}. \tag{77}$$

Hence, from Eq. (77), we conclude that for achieving the quantum optimal value, it is sufficient to assume that Bob measures his system on a product basis. Thus, Bob's observables are given by $B_{n,j} = B^A_{n,j}\otimes B^C_{n,j}$, where $B^A_{n,j}$ and $B^C_{n,j}$ are normalized observables. The ability to express Bob's observables in such a way buoyed up the fact that for the optimal value, the quantity $\sum_{j=1}^n\mathcal{I}_{n,j}$ can be expressed as

$$\max\sum_{j=1}^n\mathcal{I}_{n,j} = (\mathcal{S}_n)_Q^{\text{opt}}\,(\mathcal{L}_n)_Q^{\text{opt}}, \tag{78}$$

where $\mathcal{S}_n$ is the Bell functional proposed in [47] and $\mathcal{L}_n$ is the n-settings Chain-Bell functional [48]. Note that the optimal quantum values of $\mathcal{S}_n$ and $\mathcal{L}_n$ have already been derived as $(\mathcal{S})_Q^{\text{opt}} = 2^{n-1}\sqrt{n}$ and $(\mathcal{L}_n)_Q^{\text{opt}} = 2n\cos\frac{\pi}{2n}$, respectively [47,49]. By combining these results, it is straightforward to obtain the quantum optimal bound of $(\mathcal{B}_n)_Q$:

$$(\mathcal{B}_n)_Q^{\text{opt}} = \left(2^n\,n^{\frac{3}{2}}\,\cos\frac{\pi}{2n}\right)^{\frac{1}{2}} \tag{79}$$

which violates the local bound $(\mathcal{B}_n)$ for any arbitrary n.

The state and observables for the optimal quantum violation of $(\mathcal{B})_n$

Note that the optimal quantum value $(\mathcal{B}_n)_Q^{\text{opt}}$ is evaluated without specifying the dimension of both the system and observables. Importantly, from the argument presented in the preceding section, the optimal value is achieved when both the quantity $\mathcal{S}_n$ and $\mathcal{L}_n$ are optimized simultaneously. It has earlier been shown [47] that the optimal of $\mathcal{B}_n$ implies the observables $B^A_{n,j}$ are mutually anticommuting, i.e., $\{B^A_{n,j}, B^A_{n,i}\} = 0,\ \forall\ i, j$.

Thus, it is crucial to remark that to achieve the optimal quantum violation of the asymmetric bilocality inequality, there should be n mutually anticommuting operators. Further it follows that $\mathcal{A}_{n,j}\in\mathcal{H}^d_A$ with necessarily $(d_A)_{\min} = 2^{\lfloor n/2\rfloor}$. Thus, Bob's observables $B_{n,j} = \mathcal{A}_{n,j}\otimes C_{n,j}$ must belong to $\mathcal{H}^d_B$ with necessarily $(d_B)_{\min} > 2^{\lfloor n/2\rfloor}$. We can also conclude that the optimal value cannot be achieved if Alice-Bob and Bob-Charlie share a single copy of the maximally entangled two-qubit state.

Hence, taking a cue from the preceding discussions, we find that if there should be *at least* $N = \lfloor n/2\rfloor$ copies of maximally entangled two-qubit states between Alice-Bob and *at least* a single copy of maximally entangled two-qubit state between Bob-Charlie, then the quantum optimal value $(\mathcal{B}_n)_Q^{\text{opt}}$ will achieve for the observables in total dimension $d_{\min} = 2^N\times 2^{N+1}\times 2 = 4^{N+1}$.

Now, following a similar argument, it is straightforward to obtain the bilocal as well as the quantum optimal bound of asymmetric bilocality inequality for scenario 2 where Alice and Bob perform 2^{n-1} number of measurements and Charlie performs n number of measurements. In this scenario, the bilocal bound of the n-settings asymmetric bilocal inequality is given by

$$\mathcal{B}'_n = \sum_{j=1}^{2^{n-1}}\sqrt{|\langle\tilde{\mathcal{A}}'_{n,j}\otimes B_{n,j}\otimes\tilde{C}'_{n,j}\rangle|} \leqslant \sqrt{2n(2^{n-1}-1)\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}}, \tag{80}$$

where $\tilde{\mathcal{A}}_{n,j} = \sum_{x=1}^{2^{n-1}}(C_{n,z}+C_{n,z+1})$ with $A_{n,n+1} = -A_{n,1}$ and $\tilde{C}_{n,j} = \sum_{z=1}^n(-1)^{y^z_j}C_{n,z}$. The optimal quantum bound in this case is given by

$$(\mathcal{B}_n)_Q^{\text{opt}} = \left(2^{2n-1}\sqrt{n}\,\cos\frac{\pi}{2^n}\right)^{\frac{1}{2}}. \tag{81}$$

Next, for the asymmetric trilocal network scenario I, Alice, Bob, and Diana perform 2^{n-1} measurements and Charlie performs n measurements. In this scenario, the trilocal bound is given by

$$(\mathcal{T}_n)_{tl} = \left[2n\,(2^{n-1}-1)\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}\left\lfloor 2^{2n-3}+\frac{1}{2}\right\rfloor\right]^{\frac{1}{3}}. \tag{82}$$

The optimal quantum value can be written in the form $(\mathcal{T}_n)_{tl}^{\text{opt}} = (\mathcal{L}_{2^{n-1}})_Q^{\text{opt}} \ (\mathcal{G}_{2^{n-1}})_Q^{\text{opt}} \ (\mathcal{S}_n)_Q^{\text{opt}}$. The Bell functional $\mathcal{G}_n$ is the family of n-settings Bell inequalities proposed in [50] with the optimal quantum value given by $(\mathcal{G}_n)_Q^{\text{opt}} = 2n \cos\frac{\pi}{2n}/\sin\frac{\pi}{n}$. Thus, the corresponding optimal quantum bound of $(\mathcal{T}_n)_{tl}$ is given by

$$(\mathcal{T}_n)_Q^{\text{opt}} = 2^{n-1}\left(2\sqrt{n}\cot\frac{\pi}{2^n}\right)^{\frac{1}{3}}. \tag{83}$$

Now, for the asymmetric trilocal network scenario II, the respective trilocal and optimal quantum bound is given as follows:

$$(\mathcal{T}'_n)_{tl} = \left[2n(n-1)\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}\left\lfloor\frac{n^2+1}{2}\right\rfloor\right]^{\frac{1}{3}}, \tag{84}$$

$$(\mathcal{T}'_n)_Q^{\text{opt}} = \left(2^n n^{\frac{5}{2}}\cot\frac{\pi}{2n}\right)^{\frac{1}{3}}. \tag{85}$$

The optimal quantum values $(\mathcal{T}_n)_Q^{\text{opt}} > (\mathcal{T}_n)_{tl}$ and $(\mathcal{T}'_n)_Q^{\text{opt}} > (\mathcal{T}'_n)_{tl}$ for any value of n, thereby demonstrating the nonlocality in trilocal network featuring arbitrary inputs.

VIII. RESISTANCE TO WHITE NOISE

A. Resistance to white noise for bilocality scenarios I and II

Let us assume that each of the independent sources S_1 and S_2 does not produce maximally entangled two-qubit state, but a mixture of maximally entangled state with a white noise, known as Werner state [51]. Let the two sources produce such Werner states with different noise parameters v_1 and v_2. The Werner states between Alice and Bob are $\rho_{AB}^w(v_1)$ and for Bob and Charlie are $\rho_{BC}^w(v_2)$, given by $\rho_{AB}^w(v_k) = v_k\,|\psi\rangle\langle\psi| + (1-v_k)\frac{\mathbb{I}}{4}$ with $k\in\{1,2\}$ and $|\psi\rangle\langle\psi|$ is a maximally entangled two-qubit state and the joint tripartite physical system is given by $\rho_{ABC}^w(v_1,v_2) = \rho_{AB}^w(v_1)\otimes\rho_{BC}^w(v_2)$.

For convenience, we first evaluate the robustness of the asymmetric bilocality scenario I for arbitrary n. Since the optimal quantum violation of $\mathcal{B}_n$ is achieved when $\lfloor n/2\rfloor$ copies of maximally entangled two-qubit states shared between Alice-Bob and a single copy of maximally entangled two-qubit state is shared between Bob-Charlie. Thus, we take the Werner states of the form $\mathcal{W} = \rho_{AB}(v_1)^{\otimes N}\otimes\rho_{BC}(v_2)$ where we take $N = \lfloor n/2\rfloor$. Then, by invoking the conditions on the observables given in Sec. VI, we obtain

$$(\mathcal{B}_n)_Q^{\mathcal{W}} = \left(v_1^N v_2\right)^{\frac{1}{2}}\left(2^n\, n^{\frac{3}{2}}\cos\frac{\pi}{2n}\right)^{\frac{1}{2}}. \tag{86}$$

Therefore, in this noisy case if we take all the noise parameter as the same (v), the quantum violation will be achieved when

$$v^{N+1} > 2^{1-n}\sqrt{n}\left(1-\frac{1}{n}\right)\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}\sec\frac{\pi}{2n}. \tag{87}$$

Note that $n = 2, 3$ correspond $N = 1$. For $n = 2$ the critical noise parameter for each of the Werner states is $v_c = 1/\sqrt{2}$. This is exactly to be expected because this is the critical noise parameter for the Werner state for violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality. On the other hand, for $n = 3$ the critical noise parameter $v_c = \sqrt{2/3}\approx 0.82$ which is greater that $1/\sqrt{2}$. So, no advantage has been found over the standard Bell nonlocality or bilocality scenario for demonstrating the quantum nature of the noisy maximally entangled bipartite states.

In the asymmetric bilocality scenario II, for demonstrating nonlocality, the critical noise parameter is found to be

$$v^{N+1} > \sqrt{n}\,2^{1-n}\,(1-2^{1-n})\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}\sec\left(\frac{\pi}{2^n}\right). \tag{88}$$

For $n = 3$, the critical parameter in scenario II is given by $v_c = \frac{3}{4}\sqrt{3-\frac{3}{\sqrt{2}}}\approx 0.84$. Thus, in the asymmetric bilocality scenario, scenario I is more robust against the white noise than the scenario II.

B. Resistance to white noise for trilocality scenarios I and II

In the trilocality scenario, all the independent sources produce Werner state with noise parameters v_1, v_2, and v_3. The joint four-partite physical system is given by $\rho_{ABCD}^w(v_1,v_2,v_3) = \rho_{AB}^w(v_1)\otimes\rho_{BC}^w(v_2)\otimes\rho_{DB}^w(v_3)$.

In the asymmetric trilocality scenario I, the critical noise parameter for demonstrating nonlocality is given by

$$v^{N+2} > \sqrt{n}\,2^{2(1-n)}\,(1-2^{1-n})\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}\times\left\lfloor 2^{2n-3}+\frac{1}{2}\right\rfloor\tan\left(\frac{\pi}{2^n}\right). \tag{89}$$

For $n = 3$ critical noise parameter per Werner state is $v_c = \frac{\sqrt{3}}{\sqrt[3]{2(\sqrt{2}+2)}}\approx 0.92$.

On the other hand, in the asymmetric trilocality scenario II, the critical noise parameter is given by

$$v^{N+2} > 2^{1-n}\frac{1}{\sqrt{n}}\left(1-\frac{1}{n}\right)\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}\left\lfloor\frac{n^2+1}{2}\right\rfloor\tan\left(\frac{\pi}{2n}\right). \tag{90}$$

For $n = 3$, the critical noise parameter per Werner state is $v_c = (\frac{5}{9})^{\frac{1}{3}}\approx 0.82$. Thus, in the asymmetric trilocality cases, the scenario II is more robust to white noise than the scenario I.

IX. SUMMARY AND DISCUSSION

In this work, we have explored the quantum nonlocality in arbitrary n input asymmetric bilocal as well as trilocal network scenario. The asymmetric bilocal scenario proposed here features two edge parties Alice and Charlie who perform 2^{n-1} and n number of measurements, respectively. We derive two families of bilocality inequalities specifically designed for the asymmetric scenario when the central party Bob measures n (Fig. 2) and 2^{n-1} number of measurements.

Furthermore, we have extended the asymmetric network scenario into the trilocal network. In particular, we have introduced two variants of asymmetric trilocal network: (i) when one edge party, Alice, performs 2^{n-1} measurements, the other two edge parties, Charlie and Diana, perform n measurements each and the central party Bob performs 2^{n-1} measurements. (ii) When one edge party, Charlie, performs n measurements, the other two edge parties, Alice and Diana, perform 2^{n-1}

measurements each and the central party Bob performs n measurements.

In Secs. III and IV, the detailed analytical treatments of the bilocal bounds for the proposed asymmetric bilocal scenarios have been provided. It has been found that the bilocal bounds depend on how the asymmetricity is invoked within the bilocal scenario. In particular, the bilocal bounds have been found to be 4.89 and 6 for scenarios I and II, respectively. Subsequently, by invoking the SOS technique, the DI optimal quantum bound for the asymmetric bilocality scenario I ($(\mathcal{B}_3)_Q^{\text{opt}} = 6$) is found to be less than that obtained ($(\mathcal{B}_3')_Q^{\text{opt}} = 4[3(2+\sqrt{2})]^{\frac{1}{4}} \approx 7.16$) in the scenario II (Sec. III A). In both the optimization processes corresponding to scenarios I and II, we obtain the relational constraints on the observables along with shared states required for Alice-Bob and Bob-Charlie (Sec. III B and Appendix A 3). The constraints on the observables of all the parties in terms of the anticommuting relations have also been provided in Secs. III and V.

For the asymmetric trilocality scenarios, the trilocal bounds evaluated to be approximately 6.60 and 4.93 for the scenarios I and II, respectively (Secs. V and VI). Consequently, using the SOS method, while for the scenario I, the optimal quantum bound is found to be $(\mathcal{T}_3)_Q^{\text{opt}} = 4(2\sqrt{3}+\sqrt{6})^{\frac{1}{3}} \approx 7.23$ for scenario II, it is found to be $(\mathcal{T}_3')_Q^{\text{opt}} = 6$. In this process, we have also obtained the constraint relations of all the parties' observables in terms of the anticommutation relations (Sec. V B and Appendix B 3). We have demonstrated that the optimal quantum bound will be achieved if all the edge parties share three maximally entangled two-qubit states with the central party Bob.

Moreover, we have generalised our results for arbitrary n. For this purpose, using the SOS method, we first established that it is sufficient for Bob to measure in the product basis in order to obtain the optimal quantum violations in both the bilocality and trilocality scenarios. Then, an interesting algebraic manipulation reduces the nonlinear bilocal inequalities into a product of two standard Bell inequalities (Sec. VII). Importantly, such reduction in terms of the product of two standard Bell inequalities will only be possible at the optimal condition. It is crucial to note here that the rigorous algebraic manipulation for the case of $n = 3$ buoyed up such deep-seated understanding of the quantum optimal bound, which then provides the necessary intuition for such a simple proof for the case of arbitrary n. Then, following the similar argument of the bilocality scenarios, the optimal quantum bounds of asymmetric trilocality inequalities have also been evaluated.

Finally, for both the asymmetric bilocal and trilocal scenarios, we have demonstrated the robustness of the quantum violations of the proposed inequalities in the presence of white noise. We found that the proposed asymmetric inequalities are most robust to white noise in the simplest bilocal scenario with two measurement settings for each party. In this case, two Werner states exhibit nonlocality if $v_1 v_2 > \frac{1}{2}$. Although, unfortunately, our proposed inequality becomes less robust with increasing number of measurement settings or with increasing party (or, source), it indeed possesses some independent interests.

We conclude by raising some open questions which can be studied in the future.

(i) Since the optimal quantum violations of the asymmetric bilocality inequalities cannot always be achieved with the single copy of maximally entangled two-qubit state. In fact, for the asymmetric bilocality scenario I, while Alice-Bob needs to share at least of $\lfloor n/2 \rfloor$ copies of maximally entangled two-qubit state, Bob-Charlie need to share at least one copy of it. Similar results can be proved for other scenarios also. Thus, in the network scenario, our proposed inequality has the potential to be used as a dimension witness. Construction of such proof may lead to a wide variety of interesting results in the field of self-testing of many copies of maximally entangled two-qubit states, cryptographic applications, or randomness generation protocols.

(ii) A straightforward extension of our proposed bilocality inequality would be to invoke the four-outcome measurement scenario for Bob. In this regard, for the symmetric bilocal scenario, different bilocal inequalities have been tailored to the four-outcome scenario in the context of both the Bell state measurement and the elegant joint measurement scheme [35] for Bob. Although the inequality with Bell state measurement does not provide any advantage over the usual bilocal scenario, the inequality involving elegant joint measurement is more advantageous in presence of the noise than the earlier bilocal or standard Bell inequalities. Thus, an extensive study of such a scenario in the asymmetric case may lead to interesting findings.

(iii) Of course, one can always generalize our asymmetric bilocal scenario by going beyond the four-party three-independent sources into a multiparty multisource scenario. From our evaluation of the robustness to white noise, the asymmetric bilocal scenario does not provide any advantage over the noise tolerance over the standard Bell scenario. However, with increasing the number of parties and sources, one can introduce the asymmetry in many ways, which may lead to a multiparty n-locality inequality that may provide such advantages.

In sum, the essence of this work lies in constructing a family of asymmetric bilocality as well as trilocality inequalities and evaluating their quantum optimal bounds, importantly, by not specifying the dimension of the system or the dichotomic observables. Our work has the potential to open up interesting avenues for future research such as self-testing of many copies of entangled states, sequential sharing of quantum correlations, unbounded generation of randomness, and secret key sharing in one-to-many scenarios that calls for further study.

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APPENDIX A: DETAILED CALCULATION FOR THE ASYMMETRIC BILOCALITY NETWORK SCENARIO II

1. Bilocal bound for the asymmetric bilocality network scenario II

Note the following:

$$\mathcal{B}_3' = \sum_{j=1}^{4} \sqrt{\left|\langle \tilde{\mathcal{A}}_{3,j}' B_{3,j} \tilde{C}_{3,j}' \rangle\right|} \leqslant (\mathcal{B}_3')_{bl}, \tag{A1}$$

where the quantities $\tilde{\mathcal{A}}_{3,j}'$ and $\tilde{C}_{3,j}'$ are defined in Sec. IV. Now, invoking the reproducibility condition given by Eq. (2) and since $|\langle B_{3,1}\rangle_{\lambda_1,\lambda_2}| \leqslant 1$, we obtain

$$|\langle \tilde{\mathcal{A}}_{3,1}' B_{3,1} \tilde{C}_{3,1}' \rangle| \leqslant \iint d\lambda_1\, d\lambda_2\, \mu(\lambda_1)\, \mu(\lambda_2) \big|\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1}\big| \big|\langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2} + \langle C_{3,3}\rangle_{\lambda_2}\big|. \tag{A2}$$

The other terms can also be written in a similar manner as Eq. (A2). Thus, from Eq. (A1), we obtain

$$(\mathcal{B}_3')_{bl} \leqslant \left(\iint d\lambda_1\, d\lambda_2\, \mu(\lambda_1)\, \mu(\lambda_2)\, \delta_1'\, \delta_2' \right)^{\frac{1}{2}}, \tag{A3}$$

where $\delta_1' = [|\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1}| + |\langle A_{3,2}\rangle_{\lambda_1} + \langle A_{3,3}\rangle_{\lambda_1}| + |\langle A_{3,3}\rangle_{\lambda_1} + \langle A_{3,4}| + |\langle A_{3,4}\rangle_{\lambda_1} - \langle A_{3,1}\rangle_{\lambda_1}|]$ and $\delta_2' = [|\langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2} + \langle C_{3,3}\rangle_{\lambda_2}| + |\langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2} - \langle C_{3,3}\rangle_{\lambda_2}| + |\langle C_{3,1}\rangle_{\lambda_2} - \langle C_{3,2}\rangle_{\lambda_2} + \langle C_{3,3}\rangle_{\lambda_2}| + | - \langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2} + \langle C_{3,3}\rangle_{\lambda_2}|]$.

Since all the observables are dichotomic with eigenvalues ±1, it is straightforward to derive that $\delta_1' \leqslant 6$ and $\delta_2' \leqslant 6$. Therefore, from Eq. (A3), integrating over λ_1 and λ_2 we obtain

$$(\mathcal{B}_3')_{bl} \leqslant 6. \tag{A4}$$

2. Optimal quantum bound of the asymmetric bilocality inequality for scenario II

Let us consider a suitable positive-semidefinite operator γ_3' satisfying $\langle\gamma_3'\rangle = \beta_3' - (\mathcal{B}_3')_Q$. The existence of such operator is constructed by considering a set of operators $M_{3,j}'$, $\forall\, j \in \{1, 2, 3, 4\}$, such that

$$\langle\gamma_3'\rangle = \sum_{j=1}^{4} \frac{\sqrt{\omega_{3,j}}}{2} |M_{3,j}' |\psi\rangle |^2, \tag{A5}$$

where $\omega_{3,j} \geqslant 0$ and $\omega_{3,j} = \omega_{3,j}^A \cdot \omega_{3,j}^C$. We choose $M_{3,j}'$ and the quantity $\omega_{3,j}$ as

$$\big|M_{3,j}' |\psi\rangle \big| = \sqrt{\left|\left(\frac{\tilde{\mathcal{A}}_{3,j}'}{\omega_{3,j}^A} \otimes \frac{\tilde{C}_{3,j}'}{\omega_{3,j}^C}\right) |\psi\rangle\right|} - \sqrt{|B_{3,j} |\psi\rangle |}, \quad \forall\, j \in \{1, 2, 3, 4\} \tag{A6}$$

$$\omega_{3,j}^A = ||\tilde{\mathcal{A}}_{3,j}' |\psi\rangle ||_2; \quad \omega_{3,j}^C = ||\tilde{C}_{3,j}' |\psi\rangle ||_2. \tag{A7}$$

Putting $|M_{3,j}' |\psi\rangle |$ and $\omega_{3,j}^k$ from Eq. (A6) into (A5) and, by using the inequalities given in footnotes 3 and 4, we obtain the quantum optimal value as follows:

$$(\mathcal{B}_3')_Q^{\text{opt}} = \max \left[\prod_{k=A,C} \left(4 \sum_{j=1}^{4} \left(\omega_{3,j}^k\right)^2 \right) \right]^{\frac{1}{4}} \quad \text{with the optimality condition } M_{3,j}'\ |\psi\rangle = 0, \ \ \forall\, j \in \{1, 2, 3, 4\}. \tag{A8}$$

In the following we evaluate $\sum_{j=1}^{4}(\omega_{3,j}^A)^2$ and $\sum_{j=1}^{4}(\omega_{3,j}^C)^2$ from Eq. (A7) separately.

a. Evaluation of $\sum_{j=1}^{4}(\omega_{3,j}^A)^2$

From Eq. (A7) we obtain

$$\sum_{j=1}^{4} \left(\omega_{4,j}^A\right)^2 = \langle\psi| (8 + \{A_{4,2}, (A_{4,1} + A_{4,3})\} + \{A_{4,4}, (A_{4,3} - A_{4,1})\}) |\psi\rangle\,. \tag{A9}$$

Note that in Eq. (A9) the quantities $A_{4,2}$ and $A_{4,4}$ appeared independently with $\{A_{4,2}, (A_{4,1} + A_{4,3})\}$ and $\{A_{4,4}, (A_{4,3} - A_{4,1})\}$, respectively. Thus, we can always define $A_{4,2}$ and $A_{4,4}$ independently. Hence, without loss of generality, we chose $A_{4,2} = (A_{4,3} + A_{4,1})/\nu_1$ and $A_{4,4} = (A_{4,3} - A_{4,1})/\nu_2$, where $\nu_1 = ||\ (A_{4,3} + A_{4,1})\ ||_2$ and $\nu_2 = ||\ (A_{4,3} - A_{4,1})\ ||_2$. Therefore, Eq. (A9) reduces to

$$\sum_{j=1}^{4} \left(\omega_{4,j}^A\right)^2 = 8 + 2[\sqrt{4 + 2\sqrt{4 - \langle\{A_{4,1}, A_{4,3}\}\rangle}}] \leqslant 4(2 + \sqrt{2}). \tag{A10}$$

The maximum value of $\sum_{j=1}^{4} \omega_{4,j}^{A})^2$ is then achieved when $\{A_{4,1}, A_{4,3}\} = 0$ which automatically implies $\nu_1 = \nu_2 = \sqrt{2}$. Therefore, for the optimal quantum violation, the linear constraints on Alice's observables are given as

$$A_{4,1} - \sqrt{2}A_{4,2} + A_{4,3} = 0, \quad -A_{4,1} + A_{4,3} - \sqrt{2}A_{4,4} = 0. \tag{A11}$$

b. Evaluation of $\sum_{j=1}^{3}(\omega_{3,j}^{C})^2$

By Eq. (A7), a straightforward calculation leads to

$$\sum_{j=1}^{4} \left(\omega_{3,j}^{C}\right)^2 \leqslant 12 \tag{A12}$$

with the constraints on Charlie's observables given by

$$\{C_{3,1}, C_{3,2}\} = \{C_{3,2}, C_{3,3}\} = \{C_{3,1}, C_{3,3}\} = 0. \tag{A13}$$

Finally, we obtain the optimal quantum value from Eqs. (A8), (A10), and (A12) as

$$(\mathcal{B}_3')_Q^{\text{opt}} = 4[3(2+\sqrt{2})]^{\frac{1}{4}}. \tag{A14}$$

3. The state and observables for the optimal quantum violation of $(\mathcal{B}')_3$

We further obtain relationships between the observables of all the parties for achieving the optimal quantum violation in terms of the anticommuting relations. The anticommutation relations for Charlie's observables are already given in Eq. (36). From Eq. (33), we obtain the anticommuting relations for Alice's observables as

$$\{A_{3,1}, A_{3,2}\} = \{A_{3,2}, A_{3,3}\} = \{A_{3,3}, A_{3,4}\} = -\{A_{3,1}, A_{3,4}\} = \sqrt{2}\,\mathbb{I}_2; \quad \{A_{3,1}, A_{3,3}\} = \{A_{3,2}, A_{3,4}\} = 0. \tag{A15}$$

By using the above relations between the observables given by Eqs. (A13) and (A15) on the observables, one can always construct a set of observables for Alice and Charlie in the Hilbert space dimension $\mathcal{H}^d, \ \forall\, d \geqslant 2$.

Next, we recall the optimization condition obtained in the SOS method from Eq. (A8) to find the constraints on Bob's observable. The specific condition $M'_{3,j}|\psi\rangle = 0, \ \forall\, j \in \{1, 2, 3\}$, implies the following:

$$B_{3,j} = \frac{\tilde{\mathcal{A}}'_{3,j}}{\omega_{3,j}^{A}} \otimes \frac{\tilde{C}'_{3,j}}{\omega_{3,j}^{C}}, \quad \forall\, j \in \{1, 2, 3, 4\}. \tag{A16}$$

We explicitly construct a set of observables for the Hilbert space dimension $\mathcal{H}^2$ as follows:

$$A_{3,1} = r\sigma_x + \sqrt{1-r^2}\,\sigma_z, \quad A_{3,2} = t\sigma_x + \sqrt{1-t^2}\,\sigma_z, \quad A_{3,3} = t\,\sigma_x - \sqrt{1-t^2}\,\sigma_z, \quad A_{3,4} = r\,\sigma_x - \sqrt{1-r^2}\,\sigma_z,$$
$$C_{3,1} = \sigma_x, \quad C_{3,2} = \sigma_y, \quad C_{3,3} = \sigma_z\left[\text{where } r = \frac{1}{2}\sqrt{2-\sqrt{2}} \text{ and } t = \frac{1}{2}\sqrt{2+\sqrt{2}}\right]. \tag{A17}$$

Bob's observables can be constructed from Eq. (A16). Note that employing the above-mentioned observables, we find that the quantum optimal value $(\mathcal{B}_3')_Q^{\text{opt}} = 6$ is achieved when two maximally entangled two-qubit states are shared between Alice-Bob and Bob-Charlie.

APPENDIX B: DETAILED CALCULATION FOR THE ASYMMETRIC TRILCALITY NETWORK SCENARIO -II

1. Trilocal bound for the asymmetric trilocality network scenario II

$$\mathcal{T}_3' = \sum_{j=1}^{3} |\mathcal{J}_{3,j}'|^{\frac{1}{3}} \leqslant (\mathcal{T}_3')_{tl} \quad \text{with} \quad \mathcal{J}_{3,j}' = \langle \tilde{\mathscr{A}}_{3,j}' B_{3,j} \tilde{\mathscr{C}}_{3,j}' \tilde{\mathscr{D}}_{3,j}' \rangle, \tag{B1}$$

where the quantities $\tilde{\mathscr{A}}_{3,j}'$, $\tilde{\mathscr{C}}_{3,j}'$, and $\tilde{\mathscr{D}}_{3,j}'$ are defined in Sec. VI of the main text. Now, invoking the reproducibility condition given by Eq. (2) and since $|\langle B_{3,1}\rangle_{\lambda_1,\lambda_2,\lambda_3}| \leqslant 1$, we obtain

$$\mathcal{J}_{3,1}' = \iiint d\lambda_1 d\lambda_2 d\lambda_3\, \mu(\lambda_1)\mu(\lambda_2)\mu(\lambda_3)\big|\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1} + \langle A_{3,3}\rangle_{\lambda_1} - \langle A_{3,4}\rangle_{\lambda_1}\big|\big|\langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2}\big|\big|\langle D_{3,1}\rangle_{\lambda_3}$$
$$+ \langle D_{3,2}\rangle_{\lambda_3} + \langle D_{3,3}\rangle_{\lambda_3}\big|. \tag{B2}$$

The terms $|\mathcal{J}'_{3,2}|$, and $|\mathcal{J}'_{3,3}|$ given by Eq. (B1) can also be written in a similar manner as Eq. (B2). Then, we obtain the following:

$$(\mathcal{T}'_3)_{bl} \leqslant \left(\iiint d\lambda_1 d\lambda_2 d\lambda_3\ \mu(\lambda_1)\mu(\lambda_2)\mu(\lambda_3)\ \eta'_1\eta'_2\eta'_3\right)^{\frac{1}{3}}, \tag{B3}$$

where $\eta'_1 = [|\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1} + \langle A_{3,3}\rangle_{\lambda_1} - \langle A_{3,4}\rangle_{\lambda_1}| + |\langle A_{3,1}\rangle_{\lambda_1} + \langle A_{3,2}\rangle_{\lambda_1} - \langle A_{3,3}\rangle_{\lambda_1} + \langle A_{3,4}\rangle_{\lambda_1}| + |\langle A_{3,1}\rangle_{\lambda_1} - \langle A_{3,2}\rangle_{\lambda_1} + \langle A_{3,3}\rangle_{\lambda_1} + \langle A_{3,4}\rangle_{\lambda_1}|]$, $\eta'_2 = [|\langle C_{3,1}\rangle_{\lambda_2} + \langle C_{3,2}\rangle_{\lambda_2}| + |\langle C_{3,2}\rangle_{\lambda_2} + \langle C_{3,3}\rangle_{\lambda_2}| + |\langle C_{3,3}\rangle_{\lambda_2} - \langle C_{3,1}\rangle_{\lambda_2}|]$, and $\eta'_3 = [|\langle D_{3,1}\rangle_{\lambda_3} + \langle D_{3,2}\rangle_{\lambda_3} + \langle D_{3,3}\rangle_{\lambda_3}| + |\langle D_{3,1}\rangle_{\lambda_3} + \langle D_{3,2}\rangle_{\lambda_3} - \langle D_{3,3}\rangle_{\lambda_3}| + |\langle D_{3,1}\rangle_{\lambda_3} - \langle D_{3,2}\rangle_{\lambda_3} - \langle D_{3,3}\rangle_{\lambda_3}|]$. Since all the observables are dichotomic with eigenvalues ± 1, it is straightforward to derive that $\eta'_1 \leqslant 6$, $\eta'_2 \leqslant 4$, and $\eta'_3 \leqslant 5$. Therefore, from Eq. (B3), integrating over λ_1 and λ_2 we obtain

$$(\mathcal{T}_3)_{bl} \leqslant 2(15)^{\frac{1}{3}} \approx 4.93. \tag{B4}$$

2. Optimal quantum bound of the asymmetric trilocality inequality for scenario II

To derive the optimal quantum bound of $\mathcal{T}'_3$ without assuming the dimension of the system, we again invoke the SOS approach discussed in the preceding Sec. II. Following the similar argument presented earlier in Sec. II, we first show that there exists a positive-semidefinite operator $\langle\Gamma'_3\rangle = \zeta'_3 - (\mathcal{T}'_3)_Q$. The existence of such operator can be proved by considering a set of operators $L'_{3,j},\ \forall\ j \in \{1, 2, 3\}$, such that

$$\langle\Gamma'_3\rangle = \sum_{j=1}^{4} \frac{(\omega_{3,j})^{\frac{1}{3}}}{2} |L'_{3,j}\,|\psi\rangle\,|^2, \tag{B5}$$

where $\omega_{3,j} \geqslant 0$ and $\omega_{3,j} = \omega^A_{3,j}\omega^C_{3,j}\omega^D_{3,j}$. We choose $L'_{3,j}$ and the quantity $\omega_{3,j}$ as

$$|L'_{3,j}\,|\psi\rangle\,| = \left|\left(\frac{\tilde{\mathscr{A}}'_{3,j}}{\omega^A_{3,j}} \otimes \frac{\tilde{\mathscr{C}}'_{3,j}}{\omega^C_{3,j}} \otimes \frac{\tilde{\mathscr{D}}'_{3,j}}{\omega^C_{3,j}}\right)|\psi\rangle\right|^{\frac{1}{3}} - |B_{3,j}\,|\psi\rangle\,|^{\frac{1}{3}}, \quad \forall\ j \in \{1, 2, 3\} \tag{B6}$$

$$\omega^A_{3,j} = ||\tilde{\mathscr{A}}'_{3,j}\,|\psi\rangle\,||_2; \quad \omega^C_{3,j} = ||\tilde{\mathscr{C}}'_{3,j}\,|\psi\rangle\,||_2; \quad \omega^D_{3,j} = ||\tilde{\mathscr{D}}'_{3,j}\,|\psi\rangle\,||_2, \tag{B7}$$

where $||\cdot||_2$ denotes the Frobenious norm given by $||\ \mathcal{O}\ ||_2 = \sqrt{\langle\psi|\,\mathcal{O}^\dagger\mathcal{O}\,|\psi\rangle}$.

Putting $L'_{3,j}$ and $\omega_{3,j}$ from Eqs. (B6) and (B7) into Eq. (B5), and by using the inequalities given in footnotes 3 and 4, we obtain the quantum optimal value as follows:

$$(\mathcal{T}'_3)^{\text{opt}}_Q = \max\left[\prod_{k=A,C,D}\left(3\sum_{j=1}^{3}\left(\omega^k_{3,j}\right)^2\right)\right]^{\frac{1}{6}} \quad \text{with the optimality condition } L'_{3,j}\ |\psi\rangle = 0, \quad \forall\ j \in \{1, 2, 3, 4\}. \tag{B8}$$

Note that from Eqs. (34) and (35), $\max\sum_{j=1}^{3}(\omega^A_{3,j})^2 = 16$ and $\max\sum_{j=1}^{3}(\omega^C_{3,j})^2 = 9$, respectively. We evaluate $\sum_{j=1}^{3}(\omega^D_{3,j})^2$ as follows:

$$\sum_{j=1}^{3}\left(\omega^D_{3,j}\right)^2 = \langle\psi|\,(9 + \{D_{3,1}, (D_{3,2} - D_{3,3})\} + \{D_{3,2}, D_{3,3}\})\,|\psi\rangle = \langle\psi|\,(9 + 3\mathbb{I} - (D_{3,1} - D_{3,2} + D_{3,3})^2)\,|\psi\rangle \leqslant 12. \tag{B9}$$

Equation (B9) provides maximum value when

$$D_{3,1} - D_{3,2} + D_{3,3} = 0. \tag{B10}$$

By placing the value of $\sum_{j=1}^{3}(\omega^A_{3,j})^2$, $\sum_{j=1}^{3}(\omega^C_{3,j})^2$, and $\sum_{j=1}^{3}(\omega^D_{3,j})^2$ in Eq. (B8) we obtain the optimal quantum bound as

$$(\mathcal{T}'_3)^{\text{opt}}_Q = 6. \tag{B11}$$

It is important to remark here that the optimal quantum value $(\mathcal{T}_3)^{\text{opt}}_Q = 6$ is evaluated without specifying the dimension of both the system and observables. The optimal value fixes the states, and the observables are the following.

3. The state and observables for the optimal quantum violation of $\mathcal{T}'_3$

We further obtain relationships between the observables of all the parties for achieving the optimal quantum violation. It follows from the earlier derived results [Eqs. (33) and (36)] and Eq. (B10) the following anticommuting relations of the observables for all the parties:

$$\{A_{3,1}, A_{3,2}\} = \{A_{3,1}, A_{3,3}\} = \{A_{3,1}, A_{3,4}\} = \tfrac{2}{3}\mathbb{I}_d; \quad \{A_{3,2}, A_{3,3}\} = \{A_{3,2}, A_{3,4}\} = \{A_{3,3}, A_{3,4}\} = -\tfrac{2}{3}\mathbb{I}_d, \tag{B12}$$

$$\{C_{3,1}, C_{3,2}\} = \{C_{3,2}, C_{3,3}\} = -\{C_{3,1}, C_{3,3}\} = \{D_{3,1}, C_{3,2}\} = \{D_{3,2}, D_{3,3}\} = -\{D_{3,1}, D_{3,3}\} = \mathbb{I}_d. \tag{B13}$$

By using the above relations between the observables given by Eqs. (B12) and (B13) on the observables, one can always construct a set of observables for Alice and Charlie in the Hilbert space dimension $\mathcal{H}^d, \ \forall\, d \geqslant 2$.

Next, we recall the optimization condition obtained in the SOS method from Eq. (B8) to find the constraints on Bob's observable. The specific condition $L'_{3,j}\,|\psi\rangle = 0, \ \ \forall\, j \in \{1, 2, 3\}$, implies the following:

$$B_{3,j} = \frac{\tilde{\mathscr{A}}'_{3,j}}{\omega^A_{3,j}} \otimes \frac{\tilde{\mathscr{C}}'_{3,j}}{\omega^C_{3,j}} \otimes \frac{\tilde{\mathscr{D}}'_{3,j}}{\omega^C_{3,j}}. \tag{B14}$$

We explicitly construct a set of observables for the Hilbert space dimension $\mathcal{H}^2$ as follows:

$$A_{3,1} = \frac{\sigma_x + \sigma_y + \sigma_z}{\sqrt{3}}, \quad A_{3,2} = \frac{\sigma_x + \sigma_y - \sigma_z}{\sqrt{3}}, \quad A_{3,3} = \frac{\sigma_x - \sigma_y + \sigma_z}{\sqrt{3}}, \quad A_{3,4} = \frac{-\sigma_x + \sigma_y + \sigma_z}{\sqrt{3}}; \quad C_{3,1} = \sigma_z,$$
$$C_{3,2} = \left(\frac{\sqrt{3}}{2}\sigma_x + \frac{\sigma_z}{2}\right), \quad C_{3,3} = \left(\frac{\sqrt{3}}{2}\sigma_x - \frac{\sigma_z}{2}\right); \quad D_{3,3} = -\sigma_z, \quad D_{3,1} = \left(\frac{-\sqrt{3}}{2}\sigma_x + \frac{\sigma_z}{2}\right), \quad D_{3,2} = \left(-\frac{\sqrt{3}}{2}\sigma_x - \frac{\sigma_z}{2}\right). \tag{B15}$$

Note that Bob's observables can be constructed from Eq. (B14). Now, employing the above-mentioned observables, we find that the quantum optimal value $(\mathcal{T}'_3)^{\text{opt}}_Q = 6$ is achieved when three maximally entangled two-qubit states are shared between Alice-Bob, Charlie-Bob, and Diana-Bob.

[1] A. Tavakoli, A. Pozas-Kerstjens, M.-X. Luo, and M.-O. Renou, Bell nonlocality in networks, Rep. Pro. Phys. **85**, 056001 (2022).

[2] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. **86**, 419 (2014).

[3] C. Branciard, N. Gisin, and S. Pironio, Characterizing the Nonlocal Correlations Created via Entanglement Swapping, Phys. Rev. Lett. **104**, 170401 (2010).

[4] T. Fritz, Beyond bell's theorem: Correlation scenarios, New J. Phys. **14**, 103001 (2012).

[5] C. Branciard, D. Rosset, N. Gisin, and S. Pironio, Bilocal versus nonbilocal correlations in entanglement-swapping experiments, Phys. Rev. A **85**032119 (2012).

[6] A. Tavakoli, P. Skrzypczyk, D. Cavalcanti, and A. Acín, Nonlocal correlations in the star-network configuration, Phys. Rev. A **90**, 062109 (2014).

[7] F. Andreoli, G. Carvacho, L. Santodonato, M. Bentivegna, R. Chaves, and F. Sciarrino, Experimental bilocality violation without shared reference frames, Phys. Rev. A **95**, 062315 (2017).

[8] A. Tavakoli, Quantum correlations in connected multipartite bell experiments, J. Phys. A: Math. Theor. **49**145304 (2016).

[9] D. Rosset, C. Branciard, T. J. Barnea, G. Pütz, N. Brunner, and N. Gisin, Nonlinear Bell Inequalities Tailored for Quantum Networks, Phys. Rev. Lett. **116**, 010403 (2016).

[10] R. Chaves, Polynomial Bell Inequalities, Phys. Rev. Lett. **116**, 010402 (2016).

[11] A. Tavakoli, M. O. Renou, N. Gisin, and N. Brunner, Correlations in star networks: From bell inequalities to network inequalities, New J. Phys. **19**, 073003 (2017).

[12] N. Gisin, Q. Mei, A. Tavakoli, M. O. Renou, and N. Brunner, All entangled pure quantum states violate the bilocality inequality, Phys. Rev. A **96**, 020304(R) (2017).

[13] N. Gisin, The elegant joint quantum measurement and some conjectures about n-locality in the triangle and other configurations, arXiv:1708.05556.

[14] M. X. Luo, Computationally Efficient Nonlinear Bell Inequalities for Quantum Networks, Phys. Rev. Lett. **120**, 140402 (2018).

[15] A. Pozas-Kerstjens, R. Rabelo, Ł. Rudnicki, R. Chaves, D. Cavalcanti, M. Navascués, and A. Acín, Bounding the Sets of Classical and Quantum Correlations in Networks, Phys. Rev. Lett. **123**, 140503 (2019).

[16] C. M. Lee, Device-independent certification of non-classical joint measurements via causal models, npj Quantum Inf. **5**, 34 (2019).

[17] T. Kriváchy, Y. Cai, D. Cavalcanti, A. Tavakoli, N. Gisin, and N. Brunner, A neural network oracle for quantum nonlocality problems in networks, npj Quantum Inf. **6**, 70 (2020).

[18] I. Šupić, J. D. Bancal, and N. Brunner, Quantum Nonlocality in Networks Can Be Demonstrated with an Arbitrarily Small Level of Independence between the Sources, Phys. Rev. Lett. **125**, 240403 (2020).

[19] A. Kundu, M. K. Molla, I. Chattopadhyay, and D. Sarkar, Maximal qubit violation of n -local inequalities in a quantum network, Phys. Rev. A **102**, 052222 (2020).

[20] X. Coiteux-Roy, E. Wolfe, and M. O. Renou, No Bipartite-Nonlocal Causal Theory Can Explain Nature's Correlations, Phys. Rev. Lett. **127**, 200401 (2021).

[21] P. Contreras-Tejada, C. Palazuelos, and J. I. D. Vicente, Genuine Multipartite Nonlocality Is Intrinsic to Quantum Networks, Phys. Rev. Lett. **126**, 040501 (2021).

[22] B. D. M. Jones, I. Šupić, R. Uola, N. Brunner, and P. Skrzypczyk, Network Quantum Steering, Phys. Rev. Lett. **127**, 170405 (2021).

[23] M. O. Renou and S. Beigi, Nonlocality for Generic Networks, Phys. Rev. Lett. **128**, 060401 (2022).

[24] M. O. Renou, E. Bäumer, S. Boreiri, N. Brunner, N. Gisin, and S. Beigi, Genuine Quantum Nonlocality in the Triangle Network, Phys. Rev. Lett. **123**, 140401 (2019).

[25] J. D. Bancal and N. Gisin, Nonlocal boxes for networks, Phys. Rev. A **104**, 052212 (2021).

[26] A. Sen(De), U. Sen, Č. Brukner, V. Bužek, and M. Żukowski, Entanglement swapping of noisy states: A kind of superadditivity in nonclassicality, Phys. Rev. A **72**, 042310 (2005).
[27] D. Cavalcanti, M. L. Almeida, V. Scarani, and A. Acín, Quantum networks reveal quantum nonlocality, Nat. Commun. **2**, 184 (2011).
[28] G. Carvacho, F. Andreoli, L. Santodonato, M. Bentivegna, R. Chaves, and F. Sciarrino, Experimental violation of local causality in a quantum network, Nat. Commun. **8**, 14775 (2017).
[29] D. J. Saunders, A. J. Bennet, C. Branciard, and G. J. Pryde, Experimental demonstration of nonbilocal quantum correlations, Sci. Adv. **3**, e1602743 (2017).
[30] Q.-C. Sun, Y.-F. Jiang, B. Bai, W. Zhang, H. Li, X. Jiang, J. Zhang, L. You, X. Chen, Z. Wang, Q. Zhang, J. Fan, and J.-W. Pan, Experimental demonstration of non-bilocality with truly independent sources and strict locality constraints, Nat. Photonics **13**, 687 (2019).
[31] D. Poderini, I. Agresti, G. Marchese, E. Polino, T. Giordani, A. Suprano, M. Valeri, G. Milani, N. Spagnolo, G. Carvacho, R. Chaves, and F. Sciarrino, Experimental violation of n-locality in a star quantum network, Nat. Commun. **11**, 2467 (2020).
[32] E. Bäumer, N. Gisin, and A. Tavakoli, Demonstrating the power of quantum computers, certification of highly entangled measurements and scalable quantum nonlocality, npj Quantum Inf. **7**, 117 (2021).
[33] C.-X. Huang, X.-M. Hu, Y. Guo, C. Zhang, B.-H. Liu, Y.-F. Huang, C.-F. Li, G.-C. Guo, N. Gisin, C. Branciard, and A. Tavakoli, Entanglement Swapping and Quantum Correlations via Symmetric Joint Measurements, Phys. Rev. Lett. **129**, 030502 (2022).
[34] E. Hakansson, A. Piveteau, S. Muhammad, and M. Bourennane, Experimental demonstration of full network nonlocality in the bilocal scenario, arXiv:2201.06361.
[35] A. Tavakoli, N. Gisin, and C. Branciard, Bilocal Bell Inequalities Violated by the Quantum Elegant Joint Measurement, Phys. Rev. Lett. **126**, 220401 (2021).
[36] A. Pozas-Kerstjens, N. Gisin, and A. Tavakoli, Full Network Nonlocality, Phys. Rev. Lett. **128**, 010403 (2022).
[37] I. Agresti, B. Polacchi, D. Poderini, E. Polino, A. Suprano, I. Šupić, J. Bowles, G. Carvacho, D. Cavalcanti, and F. Sciarrino, Experimental robust self-testing of the state generated by a quantum network, PRX Quantum **2**, 020346 (2021).
[38] M. O. Renou, D. Trillo, M. Weilenmann, T. P. Le, A. Tavakoli, N. Gisin, A. Acín, and M. Navascués, Quantum theory based on real numbers can be experimentally falsified, Nature (London) **600**, 625 (2021).
[39] I. Šupi c, J. Bowles, M.-O. Renou, A. Acín, and M. J. Hoban, Quantum networks self-test all entangled states, arXiv:2201.05032.
[40] I. Šupić, J. D. Bancal, Y. Cai, and N. Brunner, Genuine network quantum nonlocality and self-testing, Phys. Rev. A **105**, 022206 (2022).
[41] Z. D. Li, Y. L. Mao, M. Weilenmann, A. Tavakoli, H. Chen, L. Feng, S. J. Yang, M. O. Renou, D. Trillo, T. P. Le, N. Gisin, A. Acín, M. Navascués, Z. Wang, and J. Fan, Testing Real Quantum Theory in an Optical Quantum Network, Phys. Rev. Lett. **128**, 040402 (2022).
[42] M.-C. Chen, C. Wang, F.-M. Liu, J.-W. Wang, C. Ying, Z.-X. Shang, Y. Wu, M. Gong, H. Deng, F.-T. Liang, Q. Zhang, C.-Z. Peng, X. Zhu, A. Cabello, C.-Y. Lu, and J.-W. Pan, Ruling Out Real-Valued Standard Formalism of Quantum Theory, Phys. Rev. Lett. **128**, 040403 (2022).
[43] A. K. Pan and S. S. Mahato, Device-independent certification of the hilbert-space dimension using a family of bell expressions, Phys. Rev. A **102**, 052221 (2020).
[44] S. Munshi, R. Kumar, and A. K. Pan, Generalized n -locality inequalities in a star-network configuration and their optimal quantum violations, Phys. Rev. A **104**, 042217 (2021).
[45] S. Munshi and A. K. Pan, Characterizing nonlocal correlations through various n-locality inequalities in a quantum network, Phys. Rev. A **105**, 032216 (2022).
[46] A. Ambainis, D. Leung, L. Mancinska, and M. Ozols, Quantum random access codes with shared randomness, arXiv:0810.2937.
[47] S. Ghorai and A. K. Pan, Optimal quantum preparation contextuality in an n -bit parity-oblivious multiplexing task, Phys. Rev. A **98**, 032110 (2018).
[48] S. L. Braunstein and C. M. Caves, Chained bell inequalities, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Springer, Dordrecht, 1989), pp. 27–36.
[49] I. Šupić, R. Augusiak, A. Salavrakos, and A. Acín, Self-testing protocols based on the chained bell inequalities, New J. Phys. **18**, 035013 (2016).
[50] N. Gisin, Bell inequality for arbitrary many settings of the analyzers, Phys. Lett. A **260**, 1 (1999).
[51] R. F. Werner, Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable model, Phys. Rev. A **40**, 4277 (1989).