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Dependence Among Order Statistics for Time-transformed Exponential Models

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Abstract

Let $(X_1, \ldots, X_n)$ be a random vector distributed according to *a time-transformed exponential model*. This is a special class of exchangeable models, which, in particular, includes multivariate distributions with Schur-constant survival functions and with identical marginals. Let for $1 \leq i \leq n$, $X_{i:n}$ denote the corresponding ith order statistic. We consider the problem of comparing the strength of dependence between any pair of X_i's with that of the corresponding order statistics. It is proved that for $m = 2, \ldots, n$, the dependence of $X_{2:m}$ on $X_{1:m}$ is more than that of X_2 on X_1 according to more stochastic increasingness (positive monotone regression) order, which in turn implies that $(X_{1:m}, X_{2:m})$ is more concordant than (X_1, X_2). It will be interesting to examine whether these results can be extended to other exchangeable models.

Key Words: Concordance order, More stochastic increasing order, Archimedean copulas, Exchangeable random variables, Kendall's tau.

1 Introduction

Let $\{X_1, \ldots, X_n\}$ be a set of n random variables and let for $1 \le i \le n$, $X_{i:n}$ denote the corresponding ith order statistic. The problem of dependence among order statistics has been well studied in the literature when the random variables X_i's are independent and identically distributed. It is well known that in this case, any pair of order statistics are positively dependent according to likelihood ratio dependence, a very strong notion of positive dependence. *Averous, Genest and Kochar* (2005) studied the problem of comparing the relative degree of dependence among two pairs of order statistics based on independent and identically distributed continuous random variables. Besides other results, they proved that for any $1 \le i < j \le n$, the dependence of $X_{j:n}$ on $X_{i:n}$ decreases in the sense of more stochastic increasingness (also known as more monotone regression dependence) as i and j draw further apart. *Genest, Kochar and Xu* (2009) and *Kochar* (2022 a) extended some of these results to the case when the parent observations are independent with proportional hazard rates. See *Boland et al.* (1996) and Chapter 8 of *Kochar* (2022 b) for more details and other results on this topic.

However, this problem of dependence among order statistics has not been fully studied when the X_i's are dependent. It is well known that when the X_i's are associated, so are the order statistics based on them (cf. *Barlow and Proschan*, 1981) which implies in turn that $cov(X_{i:n}, X_{j:n}) \ge 0$ for any $1 \le i \le j \le n$. One might think that the order statistics are *a*lways positively dependent. However, this is not always true as shown in *Boland et al.* (1996). Further in this paper, towards the end of their paper, they posed the question to explore the conditions on the joint distribution of the X_i's, under which the order statistics are positively associated.

Another question of interest is to examine whether the degree of dependence among order statistics is more than that exist between the parent observations. This is certainly true when the observations are independent. We investigate this second problem in this paper. *Navarro and Balakrishnan* (2012) obtained expressions for the Pearson's coefficient of correlation, Kendall's τ and Spearman's ρ coefficient between the first two order statistics for some exchangeable bivariate distributions. *Kochar and Xu* (2013) proved that in the case of exchangeable bivariate Pareto distribution, the dependence of the second order statistic on the first order statistic is more than that of the parent observations according to more stochastic increasingness order. This problem has also been briefly discussed in Chapter 8 of *Kochar* (2022 b).

While the general solution is still elusive, we analyze the case when the joint distribution of the observations has a Schur-constant survival function or, more generally, when the case of a *time-transformed exponential model* is considered. Besides allowing for clear solutions, these models are of interest in view of some specific aspects emerging in the analysis of dependence properties for Archimedean copulas. It is proved that, for $m = 2, \ldots, n$, the dependence of $X_{2:m}$ on $X_{1:m}$ is more than that of X_2 on X_1 according to more stochastic increasingness order. As the concept of more stochastic increasingness order as defined in the next section is copula based, it follows that $\kappa(X_1, X_i) \leq \kappa(X_{1:m}, X_{2:m})$, for $i = 1, \ldots, m$ and for any margin-free measure of concordance κ satisfying the axioms of *Scarsini* (1984), e.g., Kendall's tau or Spearman's rho. We also find conditions under which different types of stochastic dependence relations between $X_{2:m}$ and $X_{1:m}$ hold. It will be interesting to examine whether these results can be extended to other exchangeable models.

The plan of the paper is as follows. In Section 2, we review some concepts of positive dependence and dependence orders. In Section 3, some basic definitions and properties of the multivariate distributions of the type time-transformed exponential model are recalled. A special case of the latter class is described by the condition that the joint survival function is Schur-constant. For this case, we give some basic results that will be used for the analysis of the general case of time-transformed exponential model. The main results of this paper are presented in the last section.

2 Some dependence concepts and dependence orders

In the literature there exist several notions of monotone dependence between random variables. Researchers have also developed the corresponding dependence (partial) orderings which compare the degree of (monotone) dependence within the components of different random vectors of the same length. For details, see the pioneering paper of *Lehmann*(1966) and Chapter 5 of *Barlow and Proschan* (1981) for different notions of positive dependence, and that of *Kimeldorf and Sampson* (1989) for a unified presentation of families, orderings and measures of monotone dependence. Other details about these concepts may be found in the Chapter 2 of *Joe* (1997), Chapter 5 of *Nelsen* (1999) and Chapters 5 and 8 of *Kochar* (2022 b). See also *Foschi and Spizzichino* (2013).

In this section we first review some of the notions of monotone dependence for a bivariate vector (X, Y) with joint cdf $H(x, y)$, joint survival function $\overline{H}$, and with marginal cdf's F and G, respectively. Remind that the joint survival function of (X, Y) is defined by

$$\overline{H}(x, y) = P[X > x, Y > y] = 1 - F(x) - G(y) + H(x, y) \tag{2.1}$$

In the case when the distributions F and G are absolutely continuous with unique inverses, F^{-1} and G^{-1}, the *connecting copula* associated with H is defined as

$$C(u, v) = H(F^{-1}(u), G^{-1}(v)), \quad (u, v) \in (0, 1)^2.$$

In other words, C is the distribution of the pair $(U, V) \equiv (F(X), G(Y))$ whose margins are uniform on the interval $(0, 1)$. The *survival copula* is defined by

$$\widehat{C}\,(u, v) = \overline{H}\left(\overline{F}^{-1}(u), \overline{G}^{-1}(v)\right)$$

See, for example, Chapter 1 of *Nelsen* (1999) for details.

Perhaps the most widely used and understood notion of positive dependence is that of positive quadrant dependence as defined below.

Definition 2.1 *Let (X, Y) be a bivariate random vector with joint distribution function H. X and Y are said to be positively quadrant dependent (**PQD**) if*

$$P[X \leq x, Y \leq y] \geq P[X \leq x]\, P[Y \leq y] \ \textit{for all } (x, y) \in \mathbb{R}^2,$$

or equivalently if

$$P[X \leq F^{-1}(u), Y \leq G^{-1}(v)] \geq P[X \leq F^{-1}(u)]\, P[Y \leq G^{-1}(v)] \ \textit{for all } (u, v) \in [0, 1]^2,$$

*in case the random variables are continuous with unique inverses. That is, (X, Y) are **PQD** if and only if*

$$C(u, v) \geq uv \ \textit{for all } (u, v) \in [0, 1]^2.$$

Notice that $C(u, v) \geq u \cdot v$ if and only if $\widehat{C}(u, v) \geq u \cdot v$.

A well known partial order to compare dependence between two pairs of random variables is that of *more positive quadrant dependence* order as defined below.

Definition 2.2 *(X_2, Y_2) is said to be more positive quadrant dependent than (X_1, Y_1), denoted by $(X_1, Y_1) \prec_{\mathrm{PQD}} (X_2, Y_2)$, if and only if,*

$$C_1(u, v) \leq C_2(u, v) \ \textit{for all } u, v \in (0, 1), \tag{2.2}$$

or equivalently if

$$\widehat{C}_1(u, v) \leq \widehat{C}_2(u, v)$$

where $\widehat{C}_1, \widehat{C}_2$ are the survival copulas of $(X_i, Y_i), i = 1, 2$, respectively.

In the literature the *more PQD* order is also known as the *more concordance* order (cf. *Joe,* 1997 pp 36). It is also well known that $(X_1, Y_1) \prec_{\mathrm{PQD}} (X_2, Y_2) \Rightarrow \kappa(X_1, Y_1) \leq \kappa(X_2, Y_2)$, where $\kappa(S, T)$ represents Spearman's rho, Kendall's tau, Gini's coefficient, or indeed any other copula-based measure of concordance satisfying the axioms of *Scarsini* (1984).

Lehmann (1966) in his seminal work introduced the notion of monotone regression dependence (MRD) which is also known in the literature as *stochastic increasingness* (SI).

Definition 2.3 *For a bivariate random vector (X, Y), Y is said to be stochastically increasing (SI) in X if for all $(x, x')s \in I\!R^2$,*

$$x < x' \Rightarrow P(Y \geq y|\ X = x') \geq P(Y \geq y|\ X = x),\ \textit{for all}\ y \in I\!R. \tag{2.3}$$

If we denote by $\overline{H}_x$ the survival function of the conditional distribution of Y given $X = x$, then (2.3) can be rewritten as

$$x < x' \Rightarrow \overline{H}_{x'} \circ \overline{H}_x^{-1}(u) \geq u, \quad \text{for } 0 \leq u \leq 1. \tag{2.4}$$

Note that in case X and Y are independent, $\overline{H}_{x'} \circ \overline{H}_x^{-1}(u) = u$, for $0 \leq u \leq 1$ and for all (x, x'). The SI property is a very strong notion of positive dependence and many of the other notions of positive dependence follow from it. In particular it implies *association* (and hence *positive correlation*) between X and Y. Also note that SI property, in general, is not symmetric in X and Y; however it obviously is symmetric in the case of exchangeability.

Denoting by $\xi_p = {F_X}^{-1}(p)$ the p-th quantile of the marginal distribution of X, we see that (2.4) will hold if and only if for all $0 \leq u \leq 1$,

$$0 \leq p < q \leq 1 \Rightarrow \overline{H}_{\xi_q} \circ \overline{H}_{\xi_p}^{-1}(u) \geq u. \tag{2.5}$$

Suppose we have two pairs of continuous random variables (X_i, Y_i) with joint cumulative distribution functions H_i and marginals F_i and G_i for $i = 1, 2$. We would like to compare these two pairs according to the strength of stochastic increasingness (monotone regression dependence) between them.

Definition 2.4 *Y_2 is said to be more stochastically increasing in X_2 than Y_1 is in X_1, denoted by $(Y_1|\, X_1) \prec_{\mathrm{SI}} (Y_2|\, X_2)$ or $H_1 \prec_{\mathrm{SI}} H_2$, if*

$$0 < p \le q < 1 \Longrightarrow \overline{H}_{2,\xi_{2q}} \circ \overline{H}^{-1}_{2,\xi_{2p}}(u) \ge \overline{H}_{1,\xi_{1q}} \circ \overline{H}^{-1}_{1,\xi_{1p}}(u), \tag{2.6}$$

for all $u \in (0,1)$, where for $i = 1, 2$, $\overline{H}_{i,s}$ denotes the conditional survival function of Y_i given $X_i = s$, and $\xi_{ip} = F_i^{-1}(p)$ stands for the pth quantile of the marginal distribution of X_i.

Note that (2.6) implies that if Y_1 is SI in X_1, then so is Y_2 in X_2 and conversely if Y_2 is stochastically decreasing in X_2, then so will be Y_1 in X_1.

Remark 2.1 *It can be checked that the definition of "more SI" as given above is equivalent to the one given by* Capéraà and Genest *(1990) as* ***applied to their copulas****. This is also equivalent to the one given by* Avérous, Genest and Kochar *(2005) who define "more SI" in terms of conditional distribution functions instead of conditional survival functions as defined above.*

Definition 2.5 *For a bivariate random vector (X, Y), Y is said to be right tail increasing (RTI) in X if for all $(x, x')s \in I\!R^2$,*

$$x < x' \Rightarrow P(Y \ge y|\ X \ge x') \ge P(Y \ge y|\ X \ge x), \text{ for all } y \in I\!R. \tag{2.7}$$

By conditioning on the quantiles in the definition of more RTI as proposed by *Avérous and Dortet-Bernadet* (2000), *Dolati, Genest and Kochar* (2008) proposed a weaker dependence order for comparing two bivariate random vectors based on RTI considerations .

Definition 2.6 *Y_2 is said to be more* right-tail increasing *(RTI) in X_2 than Y_1 is in X_1, denoted by $(Y_1|X_1) \preceq_{RTI} (Y_2|X_2)$, if and only if, for $0 \le u \le 1$,*

$$0 < p < q < 1 \Rightarrow H^*_{2,\,\xi_{2q}} \circ H^{*\,-1}_{2,\,\xi_{2p}}(u) \le H^*_{1,\,\xi_{1q}} \circ H^{*\,-1}_{1,\,\xi_{1p}}(u), \tag{2.8}$$

where $\xi_{ip} = F_i^{-1}(p)$ stands for the pth quantile of the marginal distribution of X_i, and $H^*_{i,\,s}$ denotes the conditional distribution of Y_i given $X_i > s$, for $i = 1, 2$.

Likewise, Y is said to be left tail decreasing (LTD) in X if for all $(x, x')s \in I\!R^2$,

$$x < x' \Rightarrow P(Y \leq y|\ X \leq x') \leq P(Y \leq y|\ X \leq x),\ \text{for all } y \in I\!R. \tag{2.9}$$

Avérous and Dortet-Bernadet (2000) analogously defined the concept of *more LTD* (after conditioning on the quantiles instead) and noted the following following chains of implications

$$(Y_1|X_1) \preceq_{\text{SI}} (Y_2|X_2) \Rightarrow (Y_1|X_1) \preceq_{\text{RTI}} (Y_2|X_2) \Rightarrow (X_1, Y_1) \preceq_{\text{PQD}} (X_2, Y_2),$$

$$(Y_1|X_1) \preceq_{\text{SI}} (Y_2|X_2) \Rightarrow (Y_1|X_1) \preceq_{\text{LTD}} (Y_2|X_2) \Rightarrow (X_1, Y_1) \preceq_{\text{PQD}} (X_2, Y_2).$$

An interesting feature of more SI, more RTI and more LTD orders as defined in this section is that, though they are copula based, one does not need the expressions for the copulas in explicit forms.

3 Schur-constant models, Archimedean copulas and related dependence properties

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be an n-dimensional random vector with absolutely continuous joint distribution. We fix attention on the case when $X_1, X_2, \ldots, X_n$ are non-negative random variables and describe their joint distribution in terms of their joint survival function

$$S(x_1, \ldots, x_n) = P[\mathbf{X} > \mathbf{x}] = P[X_1 > x_1, \ldots, X_n > x_n].$$

In particular, we start by considering the special case of Schur-constant survival function:

$$S(x_1, \ldots, x_n) = \overline{G}(x_1 + \cdots + x_n), \tag{3.10}$$

for any $\mathbf{x} = (x_1, \ldots, x_n)$ in $[0, \infty)^n$ and for an appropriate univariate survival function $\overline{G}$ over $[0, \infty)$. First of all, we assume that $\overline{G}$ is strictly decreasing all over the interval

$[0,\infty)$. Moreover, we assume that $\overline{G}$ is n times differentiable and *n-monotonic:*

$$g^{(m)}(x) = (-1)^m \frac{d^m}{dx_m} \overline{G}(x) > 0, m = 1, ..., n.$$

Then the joint probability density of $\mathbf{X}$ exists and is given by

$$f(x_1, ..., x_n) = g^{(n)}(\textstyle\sum_{j=1}^{n} x_j).$$

In particular we will use the symbol $g = g^{(1)}$ for the probability density function of $\overline{G}$. For any $m = 2, ..., n$, $\overline{G}$ and g respectively are the common one-dimensional survival function and the common density function of the random variables $X_1 ..., X_m$. Schur-constant survival functions and some of their basic properties were initially discussed, in particular, in *Barlow and Mendel* (1993), *Caramellino and Spizzichino* (1994, 1996), *Spizzichino* (2001), *Bassan and Spizzichino* (2005) and *Nelsen* (2005).

It is obvious that if $\mathbf{X}$ has a Schur-constant joint survival function, then the components of $\mathbf{X}$ are exchangeable and all lower dimensional marginal joint survival functions are also Schur-constant.

There is a strict relation between Schur-constant survival models and *Archimedean copulas* (see e.g. *Nelsen* (2005) and *Durante and Sempi* (2016)). In fact the survival copula $\widehat{C}_{\overline{G}}$ of the model in (3.10) is the Archimedean copula with the inverse of $\overline{G}$ as a generator. This property is immediately checked by observing that, by the very definition of survival copula, one can write:

$$\begin{aligned} \widehat{C}_{\overline{G}}(u_1, ..., u_n) : &= S(\overline{G}^{-1}(u_1), \ldots, \overline{G}^{-1}(u_n)) \\ &= \overline{G}(\overline{G}^{-1}(u_1) + \cdots + \overline{G}^{-1}(u_n)). \end{aligned} \tag{3.11}$$

Note that the survival copula (and hence the stochastic dependence properties thereof) is uniquely determined by the univariate survival function $\overline{G}$ which is an immediate consequence of the fact that the joint probability distribution is fully determined as soon as the univariate survival function $\overline{G}$ is specified. In particular, under the assumption of Schur-constant model, the components of $\mathbf{X}$ are independent if and only if they are exponentially distributed.

One main reason of interest for Schur-constant models dwells in the following property: for any $i \neq j$ in $\{1, 2, \dots, n\}$, and $\mathbf{x} \in [0, \infty)^n$, and any $t \geq 0$,

$$P[X_i - x_i > t | \mathbf{X} > \mathbf{x}] = \frac{\overline{G}(x_1 + \cdots + x_n + t)}{\overline{G}(x_1 + \cdots + x_n)} = P[X_j - x_j > t | \mathbf{X} > \mathbf{x}], \qquad (3.12)$$

that is, the residual lifetimes of $X_i - x_i$ and $X_j - x_j$ of two components of two different ages, x_i and x_j, respectively, have the same conditional distributions, conditional on the observed survival data $(\mathbf{X} > \mathbf{x})$. More generally, conditional on $(\mathbf{X} > \mathbf{x})$, the joint survival function of all the residual lifetimes $X_i - x_i$ (for $i = 1, ..., n$) is still exchangeable and Schur-constant. In fact we can write

$$P[\mathbf{X} - \mathbf{x} > \boldsymbol{t} | \mathbf{X} > \mathbf{x}] = \frac{\overline{G}(x_1 + \cdots + x_n + t_1 + ... + t_n)}{\overline{G}(x_1 + \cdots + x_n)}.$$

This is thus one way to extend the no-aging concept and the memory-less property of the univariate exponential distribution to the multivariate case. See the above references for more details about this aspect of Schur-constant survival functions.

Two important families of bivariate distributions with Schur-constant survival functions are:

1. Bivariate Pareto distribution with survival function,

$$S_1(x, y) = (1 + ax + ay)^{-\theta}, \qquad (3.13)$$

   where $x, y \geq 0$ and $a > 0$ and $\theta > 2$ are called the scale and the shape parameters. The corresponding copula is

$$C_1(u, v) = (u^{-1/\theta} + v^{-1/\theta} - 1)^{-\theta}. \qquad (3.14)$$

2. Bivariate Weibull distribution with survival function,

$$S(x, y) = exp[-(x + y)^{\theta}],$$

   where $x, y \geq 0$ and $\theta \in (0, 1]$.

Here $\overline{G}(x) = \exp\{-x^\theta\}$ and, for $z \in (0,1)$, $\overline{G}^{-1}(z) = (-\log z)^{\frac{1}{\theta}}$, so that

$$\begin{aligned} \widehat{C}(u,v) &= S(\overline{G}^{-1}(u), \overline{G}^{-1}(v)) \\ &= \exp\left\{-\left[(-\log u)^{\frac{1}{\theta}} + (-\log v)^{\frac{1}{\theta}}\right]^\theta\right\} \\ &= \overline{G}\left(\overline{G}^{-1}(u) + \overline{G}^{-1}(v)\right). \end{aligned}$$

REMARK : The condition $\theta \in (0,1]$ in the Weibull model is required in order to satisfy the requirement that $\overline{G}$ is convex.

Some important properties of Schur-constant survival functions are summarized in the next theorem.

THEOREM **3.1** *Let* $\mathbf{X}$ *be a random vector with a Schur-constant survival function given by* (3.10)*. Then for* $m = 2, ..., n-1$*,*

(a)

$$P\left(X_2 > x_2, ..., X_m > x_m | X_1 = x\right) = \frac{g(x + \sum_{j=2}^m x_j)}{g(x)}. \tag{3.15}$$

(b) For $t > x$*, one has*

$$P\left(X_{2:m} > t | X_{1:m} = x\right) = \frac{g(x + (m-1)t)}{g(mx)}. \tag{3.16}$$

(c)

$$P\left(X_2 > x_2, ..., X_m > x_m | X_1 > x\right) = \frac{\overline{G}(x + \sum_{j=2}^m x_j)}{\overline{G}\left(x\right)}. \tag{3.17}$$

Proof

(a)

$$\begin{aligned} &P\left(x \le X_1 \le x + \Delta x, X_2 > x_2, ..., X_m > x_m\right) \qquad (3.18) \\ &= \int_x^{x+\Delta x} \int_{x_2}^\infty ... \int_{x_m}^\infty f^{(m)}\left(\xi_1, ..., \xi_m\right) d\xi_1 ... d\xi_m \\ &= \int_x^{x+\Delta x} \int_{x_2}^\infty ... \int_{x_m}^\infty g^{(m)}(\xi_1 + \sum_{j=2}^m \xi_j) d\xi_1 ... d\xi_m. \end{aligned}$$

By progressively integrating with respect to the variables $\xi_m, \xi_{m-1}, ..., \xi_2$, we arrive to the identity

$$\int_x^{x+\Delta x} \int_{x_2}^{\infty} ... \int_{x_m}^{\infty} g^{(m)}(\xi_1 + \sum_{j=2}^m \xi_j) d\xi_1 ... d\xi_m = \int_x^{x+\Delta x} g(\xi_1 + \sum_{j=2}^m x_j) d\xi_1.$$

Whence (3.15) follows immediately.

(b) By using the exchangeability of $\mathbf{X}$, it follows that

$$\begin{aligned} P\left(X_{2:m} > t | X_{1:m} = x\right) &= P\left(X_2 > t, ..., X_m > t | X_1 = x, X_2 > x, ..., X_m > x\right) \\ &= \frac{P\left(X_2 > t, ..., X_m > t | X_1 = x\right)}{P\left(X_2 > x, ..., X_m > x | X_1 = x\right)}. \end{aligned}$$

Thus the required result is obtained by rewriting in terms of formula (3.15) both the numerator and the denominator in the above r.h.s.:

$$P\left(X_{2:m} > t | X_{1:m} = x\right) = \frac{g\left(x + (m-1)t\right)}{g(x)} \cdot \frac{g\left(x\right)}{g\left(x + (m-1)x\right)}.$$

(c) This is an immediate consequence of (3.10).

Remark 3.1 *By taking into account that we start from the condition that*

$$\int_0^{\infty} \int_0^{\infty} ... \int_0^{\infty} g^{(n)}(\sum_{j=1}^n \xi_j) d\xi_1 ... d\xi_n = 1,$$

we notice that no convergence problem can arise along the integrations in the above formula (3.18).

Now, in the rest of this section, we concentrate our attention on the simple case $n = 2$. Let us then consider two non-negative random variables X_1 and X_2 with joint survival function

$$S(x_1, x_2) = \overline{G}\left(x_1 + x_2\right).$$

X_1 and X_2 are then identically distributed with marginal survival function $\overline{G}$ and Archimedean survival copula given by

$$\widehat{C}_{\overline{G}}(u, v) = \overline{G}\left(\overline{G}^{-1}(u) + \overline{G}^{-1}(v)\right), u, v \in [0, 1] \times [0, 1]. \tag{3.19}$$

By specializing the claims on Theorem 3.1, we first of all obtain the two simple formulas that, in such a case, can be respectively given for the conditional probabilities $P[X_2 > y|X_1{>}x]$ and $P[X_2 > y|X_1{=}x]$:

$$P[X_2 > y|X_1{>}x] = \frac{\overline{G}(x+y)}{\overline{G}(x)}, \tag{3.20}$$

$$P[X_2 > y|X_1{=}x] = \frac{g(x+y)}{g(x)}. \tag{3.21}$$

These formulas can in particular be applied to show that the positive dependence properties of positive quadrant dependence (PQD), right tail increasing (RTI), and stochastically increasing (SI) are respectively characterized by simple properties of *negative ageing* for the univariate distribution of X_1, X_2. In fact, the following result can be easily obtained in terms of the above formulas.

Theorem **3.2** *Let X_1, X_2 be jointly distributed according to a Schur–constant survival model, characterized by the univariate survival function $\overline{G}$. Then*

(a) (X_1, X_2) *PQD* $\Leftrightarrow \overline{G}$ *NBU*

(b) (X_1, X_2) *RTI* $\Leftrightarrow \overline{G}$ *DFR*

(c) (X_1, X_2) *SI* $\Leftrightarrow g$ *log-convex.*

See *Avérous and Dortet-Bernadet(2004)*, *Caramellino and Spizzichino* (1994, 1996), *Spizzichino* (2001), *Bassan and Spizzichino* (2005), and *Nelsen* (2005) for details.

It is not surprising that positive dependence properties of (X_1, X_2) are related to conditions on the survival function $\overline{G}$, since there is a one-to-one correspondence between $\overline{G}$ and the survival copula, $\widehat{C}_{\overline{G}}$. It is remarkable however that the conditions on $\overline{G}$, involved in such correspondence, have precisely the form of negative ageing of $\overline{G}$. In this vein, one can see that X_1, X_2 are independent if and only if they are exponentially distributed.

Let now $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing function with $\varphi(0) = 0, \varphi(\infty) = \infty$ and consider the non-negative random variables $X_1' = \varphi(X_1)$ and $X_2' = \varphi(X_2)$.

Obviously, also X_1' and X_2' are exchangeable but (X_1', X_2') is not Schur-constant. The joint survival function is given by

$$\begin{aligned} S'(x_1, x_2) : &= P[X_1' > x_1, X_2' > x_2] \\ &= P[\varphi(X_1) > x_1, \varphi(X_2) > x_2] \\ &= S\left(\varphi^{-1}(x_1), \varphi^{-1}(x_2)\right) \\ &= \overline{G}(\varphi^{-1}(x_1) + \varphi^{-1}(x_2)). \end{aligned} \tag{3.22}$$

X_1' and X_2' are identically distributed with survival function $\overline{G}(\varphi^{-1}(x))$ and their survival copula is still $\widehat{C}_{\overline{G}}$.

The term Time Transformed Exponential model has been used in some papers to designate the survival model in (3.22) (see e.g. *Spizzichino* (2001) and references cited therein). In conclusion, a bivariate Time Transformed Exponential model is a survival model characterized by the following two conditions: identical univariate distributions and Archimedean survival copula as in (3.19).

Consider now a bivariate (Time-Transformed Exponential) model with survival function given by

$$S(x_1, x_2) = W\left(R(x_1) + R(x_2)\right), \tag{3.23}$$

with $W : [0, \infty) \to [0, 1]$ strictly decreasing and $R : [0, \infty) \to [0, \infty)$ strictly increasing. The corresponding univariate survival function $\overline{H}(x)$ is given by $\overline{H}(x) = W\left(R(x)\right)$. So that one has, for $z \in (0, 1)$,

$$\overline{H}^{-1}(z) = R^{-1}(W^{-1}(z))$$

and the survival copula is then

$$\widehat{C}_W(u, v) = W\left(W^{-1}(u) + W^{-1}(v)\right). \tag{3.24}$$

One can thus immediately obtain the following result.

Proposition 3.1 *The Time Transformed Exponential model (3.23) shares the same survival copula with the Schur-constant model with univariate survival function W.*

Let us now take into account that the properties PQD, RTI, SI hold for a bivariate random vector (X_1, X_2) if and only if they respectively hold for the survival copula of (X_1, X_2). As a first consequence of the above Proposition one can retrieve the following result by *Avérous and Dortet-Bernadet* (2005). See also the results by *Mosler and Scarsini* (2005).

THEOREM **3.3** *For an Archimedean copula K_W as in (3.24), the following equivalences hold:*

(a) $\widehat{C}_W$ *PQD* $\Leftrightarrow$ *W NBU*

(b) $\widehat{C}_W$ *RTI* $\Leftrightarrow$ *W DFR*

(c) $\widehat{C}_W$ *SI* $\Leftrightarrow w(x) = -W'(x)$ *log-convex.*

More generally we can conclude with the following principle concerning copula-based properties of dependence for bivariate survival models:

An arbitrary copula-based property of dependence holds for a Time-Transformed Exponential model in (3.23) if and only if it holds for the Schur-constant model with univariate survival function W.

This principle have inspired the developments that will be presented in the next Section and that are initially formulated with reference to Schur-constant models.

4 Positive dependence properties between order statistics for bivariate TTE models

In the previous sections we have reviewed the positive dependence properties of NBU, LTD, RTI, SI and related characterizations in the case of Schur-constant models and, slightly more generally, in the case of Time-transformed exponential models.

This section is devoted to describing sufficient conditions and characterizations of the same dependence properties for corresponding cases of pairs of order statistics.

As it has been recalled above a number of different papers almost simultaneously, but from different viewpoints, had pointed out the strict connection between positive dependence properties and negative ageing properties for exchangeable pairs with Archimedean copula. One path to obtain such results hinges on the identities (3.20) and (3.21). The latter identities are also useful for the purposes of this section.

We will essentially concentrate attention on a pair of lifetimes (X_1, X_2) following a Schur-constant model, where the joint survival function has then the form

$$S(x_1, x_2) = \overline{G}(x_1 + x_2), \tag{4.25}$$

with $\overline{G}$ decreasing and convex. We will also assume that $\overline{G}$ is differentiable two times and use the notation

$$g(x) = -\overline{G}'(x), \gamma(x) = -g'(x),$$

so that $g(x)$ is the marginal density of X_1, X_2, and the joint density of (X_1, X_2) exists and has the form

$$s(x_1, x_2) = \gamma(x_1 + x_2).$$

Towards the end, the obtained results will be reformulated for the Time-transformed exponential case by simply applying Proposition 3.1.

Let us start by preliminarily recalling attention on the following properties and simple results, concerning the pair $(X_{1:2}, X_{2:2})$.

$$\overline{F}_{1,2:2}(s,t) = P(X_{1:2} > s, X_{2:2} > t) = \begin{cases} 2\overline{G}(s+t) - \overline{G}(2t) & \text{for } s < t \\ \overline{G}(2s) & \text{for } s > t \end{cases},$$

$$\overline{F}_{1:2}(s) = P(X_{1:2} > s) = \overline{G}(2s), \ \overline{F}_{2:2}(t) = P(X_{2:2} > t) = 2\overline{G}(t) - \overline{G}(2t).$$

Thus, denoting $\widehat{L}_{[s]}(t) = P(X_{2:2} > t | X_{1:2} > s)$, for $t > s$ one has

$$\widehat{L}_{[s]}(t) = \frac{2\overline{G}(s+t) - \overline{G}(2t)}{\overline{G}(2s)}. \tag{4.26}$$

We also remind that for the case $n = 2$, Theorem 3.1 in particular gives for $t > s$

$$\widehat{H}_{[s]}(t) = P(X_{2:2} > t | X_{1:2} = s) = \frac{g(s+t)}{g(2s)}. \tag{4.27}$$

The survival copula $\widehat{C}(u, v)$ of $(X_{1:2}, X_{2:2})$ is given by

$$\widehat{C}(u,v) = \overline{F}_{1,2:2}(\overline{F}_{1:2}^{-1}(u), \overline{F}_{2:2}^{-1}(v))$$

$$= \begin{cases} 2\overline{G}(\overline{F}_{1:2}^{-1}(u) + \overline{F}_{2:2}^{-1}(v)) - \overline{G}(2\overline{F}_{2:2}^{-1}(v)) & \text{for } \overline{F}_{1:2}^{-1}(u) < \overline{F}_{2:2}^{-1}(v) \\ u & \text{for } \overline{F}_{1:2}^{-1}(u) > \overline{F}_{2:2}^{-1}(v) \end{cases}. \tag{4.28}$$

In the following subsections, we separately analyze the different dependence properties. We notice that, for the Schur-constant model of the parent variables, the two properties of RTI and LTD of (X_1, X_2) are equivalent (they are both equivalent to the DFR property of $\overline{G}$). Concerning with the ensuing analysis of $(X_{1:2}, X_{2:2})$, we concentrate on RTI and omit the analysis of the LTD property.

4.1 SI Property

As a main goal of the paper, we first aim to show that, for a pair (X_1, X_2) following a bivariate Schur-costant model, one has that $(X_{2:m}|X_{1:m})$ is more SI than $(X_2|X_1)$. A more general and related result, concerning with a random vector $\mathbf{X}$ following a n-dimensional Schur-costant model ($n > 2$), will be presented at the end of this subsection.

We will need the following notation.

For $z \in (0,1)$, let $\xi_z = \overline{G}^{-1}(z)$, $\zeta_z = \overline{G}_{(1)}^{-1}(z) = \frac{1}{2}\overline{G}^{-1}(z)$.

Recalling that,

$$\overline{H}_{[s]}(t) = P(X_2 > t | X_1 = s) = \frac{g\,(s+t)}{g\,(s)},$$

we then have for $s < t$,

$$\overline{H}_{[\xi_z]}(t) = \frac{g(\overline{G}^{-1}(z) + t)}{g(\overline{G}^{-1}(z))}. \tag{4.29}$$

Using (4.27), we have for $\zeta_z < t$,

$$\widehat{H}_{[\zeta_z]}(t) = P(X_{2:2} > t | X_{1:2} = \zeta_z) = \frac{g(\frac{1}{2}\overline{G}^{-1}(z) + t)}{g(\overline{G}^{-1}(z))}. \tag{4.30}$$

THEOREM **4.1** *Let (X_1, X_2) be a random vector with a Schur-constant survival function given by (4.25). Then*

$$(X_2|X_1) \prec_{\mathrm{SI}} (X_{2:2}|X_{1:2}) \tag{4.31}$$

Proof From the identity (4.29) we can obtain that

$$\overline{H}_{[\xi_p]}^{-1}(u) = g^{-1}\left(u \cdot g\left(\overline{G}^{-1}(p)\right)\right) - \overline{G}^{-1}(p),$$

by reminding that g is invertible since it is strictly decreasing.

From this we can in turn write

$$\begin{aligned} K_{p,q}(u) &= \overline{H}_{[\xi_q]}\left(\overline{H}_{[\xi_p]}^{-1}(u)\right) \\ &= \overline{H}_{[\xi_{\overline{q}}]}\left(g^{-1}\left(u \cdot g\left(\overline{G}^{-1}(\overline{p})\right)\right) - \overline{G}^{-1}(\overline{p})\right) \\ &= \frac{g\left(\overline{G}^{-1}(\overline{q}) - \overline{G}^{-1}(\overline{p}) + g^{-1}\left(u \cdot g\left(\overline{G}^{-1}(\overline{p})\right)\right)\right)}{g\left(\overline{G}^{-1}(\overline{q})\right)} \end{aligned} \tag{4.32}$$

We now consider the identity (4.30), whence we obtain

$$\widehat{H}_{[\zeta_p]}^{-1}(u) = \left[g^{-1}\left(u \cdot g(\overline{G}^{-1}(p))\right) - \frac{1}{2}\overline{G}^{-1}(p)\right]$$

and

$$\begin{aligned} \widehat{K}_{p,q}(u) &= \widehat{H}_{[\zeta_q]}\left(\widehat{H}_{[\zeta_p]}^{-1}(u)\right) \\ &= \frac{g\left(\frac{1}{2}\overline{G}^{-1}(q) + g^{-1}\left(u \cdot g(\overline{G}^{-1}(p))\right) - \frac{1}{m}\overline{G}^{-1}(p)\right)}{g(\overline{G}^{-1}(q))} \\ &= \frac{g\left(\frac{1}{2}\{\overline{G}^{-1}(q) - \overline{G}^{-1}(p)\} + g^{-1}\left(u \cdot g\left(\overline{G}^{-1}(p)\right)\right)\right)}{g\left(\overline{G}^{-1}(q)\right)}. \end{aligned} \tag{4.33}$$

We are now ready to conclude our proof.

Since $g, \overline{G}^{-1}$ are decreasing and $\left(\overline{G}^{-1}(q) - \overline{G}^{-1}(p)\right)$ is non-negative for $0 \leq q < p \leq 1$, it follows that for $0 \leq q < p \leq 1$ and $u \in (0,1)$,

$$K_{p,q}(u) \leq \widehat{K}_{p,q}(u)$$

proving thereby the thesis $(X_2|X_1) \prec_{\mathrm{SI}} (X_{2:m}|X_{1:m})$ in view of the definition of $\prec_{\mathrm{SI}}$ as recalled in Section 2. ■

Also recalling from Sections 2 and 3 that the condition $(X_2|X_1)$ SI is equivalent to g being log-convex and that it means $K_{p,q}(u) \geq u$ (for $0 \leq q < p \leq 1, u \in (0,1)$), we get the following sufficient condition for $(X_{2:2}|X_{1:2})$ SI

COROLLARY 4.1 *If g is log-convex then $X_{2:2}$ is stochastically increasing in $X_{1:2}$.*

We point out however that also a sufficient and necessary condition for $(X_{2:2}|X_{1:2})$ SI is easily obtained. In view of the identity (4.27), we have in fact the following characterization

PROPOSITION 4.1 *The condition $(X_{2:2}|X_{1:2})$ SI holds if and only if the function $\frac{g(s+t)}{g(2s)}$ is increasing in s, for any $t > s$.*

REMARK 4.1 *In view of Proposition 4.1, we notice that Corollary 4.1 may also be alternatively obtained by using the following argument: as it can be checked, the condition $\frac{g(s+t)}{g(2s)}$ increasing in s is actually implied by the one that $\frac{g(s+t)}{g(s)}$ is increasing in s. On the other hand, for what concerns the claim in Proposition 4.1, the condition $(X_{2:2}|X_{1:2})$ SI can also be expressed by the inequality $\widehat{K}_{p,q}(u) \geq u$, for $u \in (0,1)$. It can also be seen that this inequality is also equivalent to $\frac{g(s+t)}{g(2s)}$ being increasing in s.*

Along the same lines as followed in the proof of Theorem 4.1, one can obtain the following result about a random vector $(X_1, X_2, ..., X_n)$ with $n > 2$ and with a Schur-constant survival function.

THEOREM **4.2** *Let* $\mathbf{X}$ *be a random vector with a Schur-constant survival function given by* (3.10). *Then for* $m = 2, ..., n-1$,

(a)

$$(X_2 | X_1) \prec_{\mathrm{SI}} (X_{2:m} | X_{1:m}) \,, \tag{4.34}$$

(b)

$$(X_{2:m} | X_{1:m}) \prec_{\mathrm{SI}} (X_{2:m+1} | X_{1:m+1}) \,. \tag{4.35}$$

Proof

For the pair (X_1, X_2), the identity

$$K_{p,q}(u) = \frac{g\left(\overline{G}^{-1}(q) - \overline{G}^{-1}(p) + g^{-1}\left(u \cdot g\left(\overline{G}^{-1}(p)\right)\right)\right)}{g\left(\overline{G}^{-1}(q)\right)}$$

has been shown in the proof of Theorem 4.1. In place of the pair $(X_{1:2}, X_{2:2})$, that has been moreover considered therein, we here consider the pairs $(X_{1:m}, X_{2:m})$. In this respect we now set

$$\overline{F}_{1:m}(x) = P(X_{1:m} > x) = \overline{G}(mx),$$

whence

$$\zeta_{m,z} = \overline{F}_{1:m}^{-1}(z) = \frac{1}{m}\overline{G}^{-1}(z) \tag{4.36}$$

From (3.16) and (4.36), we get,

$$\widehat{H}_{m,[\zeta_{m,z}]}(t) = P(X_{2:m} > t | X_{1:m} = \zeta_{m,z}) = \frac{g(\frac{1}{m}\overline{G}^{-1}(z) + (m-1)\,t)}{g(\overline{G}^{-1}(z))}.$$

and

$$\left(\widehat{H}_{m,[\zeta_{m,z}]}\right)^{-1}(u) = \frac{1}{m-1}\left[g^{-1}\left(u \cdot g(\overline{G}^{-1}(z))\right) - \frac{1}{m}\overline{G}^{-1}(z)\right].$$

Whence

$$\begin{aligned}
\widehat{K}_{p,q}^{(m)}(u) &= \widehat{H}_{m,[\zeta_{m,q}]}\left(\left(\widehat{H}_{m,[\zeta_{m,p}]}\right)^{-1}(u)\right) \\
&= \frac{g\left(\frac{1}{m}\overline{G}^{-1}(q) + g^{-1}\left(u \cdot g(\overline{G}^{-1}(p))\right) - \frac{1}{m}\overline{G}^{-1}(p)\right)}{g(\overline{G}^{-1}(q))} \\
&= \frac{g\left(\frac{1}{m}\{\overline{G}^{-1}(q) - \overline{G}^{-1}(p)\} + g^{-1}\left(u \cdot g\left(\overline{G}^{-1}(p)\right)\right)\right)}{g\left(\overline{G}^{-1}(q)\right)}.
\end{aligned} \tag{4.37}$$

Since $\overline{G}^{-1}(q) - \overline{G}^{-1}(p)$ is non-negative for $0 \le q < p \le 1$, one has

$$\frac{1}{m}\{\overline{G}^{-1}(q) - \overline{G}^{-1}(p)\} \le \{\overline{G}^{-1}(q) - \overline{G}^{-1}(p)\}$$

and

$$\frac{1}{m+1}\{\overline{G}^{-1}(q) - \overline{G}^{-1}(p)\} \le \frac{1}{m}\{\overline{G}^{-1}(q) - \overline{G}^{-1}(p)\}.$$

Using the fact that g is decreasing, we obtain

$$K_{p,q}(u) \le \widehat{K}_{p,q}^{(m)}(u)$$

$$\widehat{K}_{p,q}^{(m)}(u) \le \widehat{K}_{p,q}^{(m+1)}(u),$$

for $0 \le q < p \le 1$, $u \in (0,1)$, $m = 2, ..., n-1$, proving thereby (4.34) and (4.35), respectively. ■

4.2 RTI Property

Analogously to the above subsection here we will examine, for a bivariate Schur-constant model characterized by a survival function $\overline{G}$, the following three questions concerning with the RTI property for the pair $(X_{1:2}, X_{2:2})$:

(i) Does the order relation

$$(X_2|X_1) \prec_{\mathrm{RTI}} (X_{2:2}|X_{1:2}) \quad \text{hold?} \tag{4.38}$$

(ii) Does $(X_2|X_1)$ RTI imply $(X_{2:2}|X_{1:2})$ RTI?

(iii) What is a necessary and sufficient condition on the survival function $\overline{G}$ for the RTI property of $(X_{2:2}|X_{1:2})$?

Let us introduce some general notation. For $p, q \in (0,1)$ and a pair of lifetimes $\mathbf{X} = (X_1, X_2)$ with $\overline{F}_1(s) = P(X_1 > s)$ set

$$L_{[s]}^{(\mathbf{X})}(t) = P(X_2 > t | X_1 > s),$$

$$\mathcal{R}_{p,q}^{(\mathbf{X})}(u) = L_{[\overline{F}_1^{-1}(q)]}^{(\mathbf{X})}\left(\left(L_{[\overline{F}_1^{-1}(p)]}^{(\mathbf{X})}\right)^{-1}(u)\right).$$

By reformulating under such notation the definition of the More RTI ordering, as given in *Dolati, Genest and Kochar* (2006) and recalled above as Definition 2.6, for two different pairs $\mathbf{X}', \mathbf{X}''$ we can write

$$\mathbf{X}' \preceq_{RTI} \mathbf{X}'' \Leftrightarrow \mathcal{R}_{p,q}^{(\mathbf{X}')}(u) \leq \mathcal{R}_{p,q}^{(\mathbf{X}'')}(u), \text{for } q < p.$$

Since, for two independent lifetimes Z_1, Z_2 it is $\mathcal{R}_{p,q}^{(\mathbf{Z})}(u) = u$, it also follows that, for $q < p$, $u \in (0,1)$,

$$\mathbf{X}\ RTI \Leftrightarrow \mathcal{R}_{p,q}^{(\mathbf{X})}(u) \geq u. \tag{4.39}$$

By combining the above Theorem 4.1 with the chains of implications

$$SI \Rightarrow RTI \Rightarrow PQD,$$

as proved in *Averous and Dortet-Bernadette* (2000), one immediately obtains the answer to the above question (i).

Corollary 4.2 *Let (X_1, X_2) be a random vector with a Schur-constant survival function given by (4.25). Then*

$$(X_2|X_1) \prec_{\text{RTI}} (X_{2:2}|X_{1:2}). \tag{4.40}$$

By recalling from Section 3 that the condition $(X_2|X_1)$ RTI is equivalent to $\overline{G}$ being DFR and that it means $\mathcal{R}_{p,q}^{(\mathbf{X})}(u) \geq u$ (for $0 \leq q < p \leq 1, u \in (0,1)$), we get the following sufficient condition for $(X_{2:2}|X_{1:2})$ to be RTI.

Corollary 4.3 *If $\overline{G}$ is DFR then the condition $(X_{2:2}|X_{1:2})$ RTI holds.*

Also in this case, however, we can easily write down necessary and sufficient condition for the property $(X_{2:2}|X_{1:2})$ RTI to hold.

Recalling in fact the formula (4.26) one can then obtain

Proposition 4.2 *The condition $(X_{2:2}|X_{1:2})$ RTI holds if and only if, for any t, the function $\widehat{L}_{[s]}(t) = \frac{2\overline{G}(s+t)-\overline{G}(2t)}{\overline{G}(2s)}$ is increasing for $s \in (0,t]$.*

Remark 4.2 *Notice that the implication $\overline{G}$ DFR $\Rightarrow$ $(X_{2:2}|X_{1:2})$ RTI can also be proved directly, without using the above Theorem 4.1 and the general implication $SI \Rightarrow RTI$. In fact, by computing the derivative of the function $\widehat{L}_{[s]}(t)$ and after some manipulations, one can show that the condition that $\widehat{L}_{[s]}(t)$ is increasing is verified when $\overline{G}$ is DFR.*

4.3 PQD Property

Also in the present subsection we examine, for a bivariate Schur-constant model characterized by a survival function $\overline{G}$, questions that are analogous to(i), (ii), (iii) above. Now we are concerned with the PQD property and consider

(i') Does the order relation

$$(X_1, X_2) \prec_{\mathrm{PQD}} (X_{1:2}, X_{2:2}) \quad \text{hold?} \tag{4.41}$$

(ii') Does (X_1, X_2) PQD imply $(X_{1:2}, X_{2:2})$ PQD?

(iii') What is a necessary and sufficient condition on the survival function $\overline{G}$ for the PQD property of $(X_{1:2}, X_{2:2})$?

By combining Theorem 4.1 with the implication $SI \Rightarrow PQD$, one immediately obtains answer to the question (i').

Corollary 4.4 *Let (X_1, X_2) be a random vector with a Schur-constant survival function given by (4.25). Then*

$$(X_1, X_2) \prec_{\mathrm{PQD}} (X_{1:2}, X_{2:2}) \tag{4.42}$$

and as a result,

$$\kappa(X_1, X_2) \leq \kappa(X_{1:m}, X_{2:2}) \tag{4.43}$$

where $\kappa(S,T)$ represents Spearman's rho, Kendall's tau, Gini's coefficient, or indeed any other copula-based measure of concordance satisfying the axioms of Scarsini (1984).

By recalling from Section 3 that the condition (X_1, X_2) PQD is equivalent to $\overline{G}$ being NBU, we get the following sufficient condition.

Corollary 4.5 *If $\overline{G}$ is NBU then the PQD property of $(X_{1:2}, X_{2:2})$ holds.*

Also this time, however, we can easily write down a necessary and sufficient condition for the property $(X_{2:2}|X_{1:2})$ PQD to hold.

Recalling in fact the formula (4.28), and that being PQD means that the survival copula is greater then the copula of independence $\Pi(u,v) = u \cdot v$, one can then obtain

Proposition 4.3 *The condition $(X_{1:2}, X_{2:2})$ PQD holds if and only if*

$$2\overline{G}(\frac{1}{2}\overline{G}^{-1}(u) + \overline{F}_{2:2}^{-1}(v)) - \overline{G}(2\overline{F}_{2:2}^{-1}(v)) \geq u \cdot v$$

for $\overline{F}_{1:2}^{-1}(u) = \frac{1}{2}\overline{G}^{-1}(u) < \overline{F}_{2:2}^{-1}(v)$.

A different characterization of PQD for $(X_{1:2}, X_{2:2})$, which can of course be obtained by directly applying the definition of PQD, becomes:

$$2\overline{G}(s+t) - \overline{G}(2t) \geq \overline{G}(2s) \cdot \left(2\overline{G}(t) - \overline{G}(2t)\right),$$

for $s < t$.

4.4 Conclusions concerning dependence properties of order statistics for bivariate Time-transformed exponential models.

We here conclude by summarizing implications of the results that have been obtained in the previous subsections. Actually, we will reformulate those results for the case of a pair of lifetimes X_1, X_2, jointly distributed according to a Time-transformed exponential model, characterized by a characterized by a survival function of the form $S(x_1, x_2) = W\left(R(x_1) + R(x_2)\right)$, as considered in (3.19). The function W is assumed to be two-times differentiable, strictly decreasing, and convex, whereas $R(x)$ is assumed to be increasing. Here, we set the notation

$$w(x) = \frac{d}{dx} W(x).$$

As it has been observed in Section 3 we remind that, in such a case, X_1 and X_2 are identically distributed with a marginal survival function $P\left(X_1 > x\right) = W\left(R(x)\right)$ and their survival copula is the Archimedean copula $\widehat{C}(u,v) = W\left(W^{-1}(u) + W^{-1}(v)\right)$.

In what follows we can take into account such a form for $\widehat{C}(u,v)$ and the circumstance that the dependence properties of PQD, RTI, and SI are copula-based. Also recalling the previous Proposition 3.1, we can thus conclude that one can extend to the pair (X_1, X_2) all the above dependence-type results valid for a Schur-constant model characterized by a marginal survival function $\overline{G}(x) = W(x)$.

More explicitly we can list the following claims concerning with the pair of the order statistics,

$$X_{1:2} = \min\left(X_1, X_2\right), X_{2:2} = \max\left(X_1, X_2\right).$$

1. $$\left(X_{1:2}, X_{2:2}\right) \succeq_{SI} \left(X_1, X_2\right), \left(X_{1:2}, X_{2:2}\right) \succeq_{RTI} \left(X_1, X_2\right), \left(X_{1:2}, X_{2:2}\right) \succeq_{PQD} \left(X_1, X_2\right).$$

2. If $w(x)$ is log-convex then $(X_{1:2}, X_{2:2})$ is SI

3. If $W(x)$ is DFR then $(X_{1:2}, X_{2:2})$ is RTI

4. If $W(x)$ is NWU then $(X_{1:2}, X_{2:2})$ is PQD

5. $(X_{1:2}, X_{2:2})$ is SI if and only if the function

$$\frac{w(s+t)}{w(2s)}$$

   is increasing for $s \in (0, t]$

6. $(X_{1:2}, X_{2:2})$ is RTI if and only if the function

$$\frac{2W(s+t) - W(2t)}{W(2s)}$$

   is increasing for $s \in (0, t]$.

7. $(X_{1:2}, X_{2:2})$ is PQD if and only if

$$2W(s+t) - W(2t) \geq W(2s) \cdot (2W(t) - W(2t)) \, .$$

   for $s < t$.

The claim presented in the above item 4. is equivalent to saying that, for bivariate Time-transformed exponential models, the PQD property of the pair of the order statistics is indeed implied by the same property for the parent variables.

Huang et al (2013) studied a related problem. They proved that if $(X_1, Y_1), \ldots, (X_n, Y_n)$ is a random sample from a bivariate distribution which is PQD, then so is the joint cdf of $(X_{i:n}, Y_{j:n})$, where $X_{1:n} \leq \cdots \leq X_{n:n}$ and $Y_{1:n} \leq \cdots \leq Y_{n:n}$ are the orders statistics of $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, respectively and $1 \leq i < j \leq n$. However this result is different from ours in that it is based on a completely different construction of pairs of order statistics.

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