UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

# Invariant Solutions of the Black Scholes Equation

by

Makoba Melidah Kholofelo Mothiba

12044149

Supervisors: Dr Gaza Maluleke and Dr Mokhwetha Mabula

Submitted in fulfillment of the requirements for the degree

Magister Scientiae

in the Department of Mathematics and Applied Mathematics,

in the Faculty of Natural and Agricultural Sciences.

University of Pretoria

June 2022

# Declaration

I, Makoba Mothiba declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work has not previously been submitted by me for a degree at this or any other tertiary institution.

**Signature:** ______________________________

**Date:** June 2022

# Abstract

In this study, we discuss derivatives, Lie symmetries and invariant solutions of the Black Scholes equation. We combine the Lie group methods with the Adomian decomposition method to solve the Black and Scholes equation via the heat equation.

We further discuss several examples to illustrate the theory in this study.

# Acknowledgment

First and foremost, I want to thank God for giving me the strength to complete my research. I would like to convey my heartfelt appreciation to my supervisors, Dr. Mokhwetha Mabula and Dr. Gaza Maluleke, for their patience and ongoing support of my research. I am overwhelmed with humility and gratitude for the invaluable guidance provided throughout the research. I will forever be grateful.

I would like to express my sincere gratitude to the Department of Mathematics and Applied Mathematics at the University of Pretoria for providing me with the opportunity to study there and for enabling me to take a sick leave when I became ill. I would also like to thank Upstream Training Trust for their financial assistance during my research. They gave me an opportunity that I will remember for the rest of my life.

I thank Mr Sebole, Mr Mazwi, Prof Dimas, and Dr Mabitsela for reading my work and providing constructive feedback. Many thanks to Tshwane University of Technology for giving me the opportunity to teach and for providing me with a skill that will benefit my career.

My heartfelt gratitude goes to my family, especially my mother and sisters, who supported and continues to support me in everything I do.

Finally, many thanks to everyone who has directly and indirectly helped me complete the research work.

# Contents

# 1 Introduction

In the corporate business world, finance is one of the most rapidly changing and expanding industries. The valuation of options contracts has been a topic of interest to researchers. There are various types of mathematical models for pricing different kinds of options. Paul Samuelson wrote an unpublished paper titled "Brownian motion in the stock market", in 1955. Many other mathematicians like Sprenkle, Ayes, James Boness [6] and Chen etc. worked on the valuation of options and developed valuation formulas of the general form but their formulas were not complete. In the year 1973, F. Black and M. Scholes developed the original option pricing formula in the paper titled "The pricing of options and corporate liabilities" [5]. In the same year Black and Scholes transformed the option pricing problem into a new partial differential equation with variable coefficients.

Partial differential equations (PDEs) play an important role in mathematics and physics, PDEs falls under both abstract mathematics, in particular operator theory and also under applied sciences, for example in finance. In this study, we focus our attention on mathematics of finance, we consider a particular type of a PDE called the Black Scholes equation. The main idea of Black Scholes equation is to construct a risk-less portfolio taking positions in cash (bonds), options and the underlying stock. The Black Scholes equation is also utilized in the calculation of the theoretical price of the European style option (put or call option). This does not consider any dividends paid during the life of the option. It is only considered at the time of expiration. However, this equation can be modified to take into account the effects of dividends paid during the option's lifetime by determining the ex-dividend date value of the underlying stock. There are various ways of deriving this equation, using various types of mathematics and different levels of complexity. Some of the derivations that have been published in literature include the Capital Asset Pricing Model (CAPM), which was originally due to Cox and Rubinstein (1985). We also have the Martingale approach and the Nume-raire approach, among other derivations. Recently, S.Grandville derived the Black Schole equation from basic principle he assumed no knowledge of stochastic calculus in [20].

It is known that probability theory, Lebesgue's integration and Ito calculus are the main ingredients for Black Scholes equation and these rely on set theory analysis and an axiomatic approach to mathematics. The other equation that will be discussed and used in this study is the Stochastic Differential Equations (SDEs). These equations are very important in the modelling of evolution, finance, biology and oceanography. The SDEs can be defined as a differential equation where one or more of the terms is a stochastic process or is a differential equation whose coefficients are random numbers or random functions of independent variables [34]. The SDEs contains a variable which is referred to as a derivative of Brownian Motion. The theory and the study of SDEs was first done in the 1940s. In particular, K. Ito introduced this theory to study and describe motion due to random events [27]. Thereafter, many developments of this theory followed, and recently, M.C Lopez-Diaz and M. Lopez-Diaz [9] studied SDEs in an ordered theoretic setting. In their case, they considered a partially ordered set (also called a poset). A partially ordered set $P$ is a non-empty set equipped with a binary relation $\leq$ satisfying the following properties,

1. If $x \in P$, then $x \leq x$ in $P$ (Reflexive property).

2. If $x, y \in P$, $x \leq y \in P$ and $y \leq x \in P$, then $x = y$ (Anti-symmetric property).

3. If $x, y, z \in P$, $x \leq y \in P$ and $y \leq z \in P$, then $x \leq z \in P$ (Transitive Property).

That is, (P, $\leq$) is a partially ordered set. The theoretical results in [9], can be applied to the comparison of Maritime areas with respect to chemical component of sea weeds. They also showed that their results to the search of Maritime areas with values of chemical components can be applied in Maritime fields. This indicates the importance of the theory of stochastic differential equations in both abstract and applied settings.
One of the most studied stochastic partial differential equation is the stochastic heat equation and is represented by

$$\frac{\partial u}{\partial t} = \Delta u + \psi$$

where $\psi$ is white noise (chaotic behaviour of a solution as $t \to \infty$) in both space and time and $\Delta$ is the Laplacian. In this study, we will be focusing on the one dimensional heat equation, given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

The heat equation has been well studied in literature. The first person to develop and solve the heat equation was Joseph Fourier in 1822. This equation has many application in various branches in the scientific field, for example in the field of financial mathematics it is used to solve the Black Scholes partial differential equation. D. V Widdder (1976) [35] list a few methods of generating or producing the solution of the heat equation and because of the great advancement in computing, numerical solutions of the heat equation have also been derived (see Anis Zafirah Azmi [3]).

In our discussion of the invariant solutions of the Black Scholes equation, we use the notion of symmetries. This notion play an important role in solving differential equations [31]. We discuss both the classical Lie point symmetries and the non-classical symmetries. The notion can be applied in solving some problems of fluid draining, epidemiology of AIDS and meteorology. It is known that for problems that gives rise to a number of three dimensional non-linear PDEs of Black Scholes type, it is best to reduce such problems by finding the symmetry of the equation using Lie Group analysis [7]. In the study of differential equations using symmetries, Nucci [31] showed that the iterations of the non-classical symmetries method yields new non-linear equations, which inherits the Lie symmetry algebra of the given equation.

We organise our work as follows:
In Chapter 1, the preliminary results that are needed in our discussion and some basic assumptions are defined.
In Chapter 2, we present assumptions that must be considered before deriving the Black Scholes equations. We discuss two different derivations of the Black Scholes equation and consider some examples, forward contract, Perpetual and Tradeable derivatives.

In Chapter 3, we discuss a method of solving nonlinear partial differential equations, the Adomian Decomposition Method (ADM). We consider examples to demonstrate the method.
In Chapter 4, we present symmetry analysis of the Black Scholes equation, commutators are computed for the symmetries of the Black Scholes equation and symmetry transformation are constructed using the solution of the Lie equation and exponentiation.
In chapter 5, the Black Scholes is transformed into the heat equation. Lie symmetries of the heat equation are calculated, and the Adomian decomposition method is used to solve the Heat equation.
In chapter 6 we calculate the invariant solutions of the heat equation and Black Scholes equation.
In chapter 7, we give a summary of the dissertation and we discuss the future work of this study.

## 1.1 Preliminaries

### Definitions

In this chapter, we recall some notions that will be used in our study.

1. **Option** ([33, *Definition 2.1*])
   An option is a security that gives its holder the right to buy and sell an asset, within a specified time frame, subject to certain conditions.
   There are two types of options, we have the "call option", which is an option that allows its holder the right to buy the underlying asset at a strike price at some future time T, and the "put option" is another type of an option that allows its holder to sell the underlying asset at a strike price at some future time T.

2. **European Option** ([33, *Definition 2.2.1*])
   An option which cannot be exercised until the expiration date is called an European option.

3. **American Option** ([33, *Definition 2.2.2*])
   An American option is an option which can be exercised at any time up to and including the expiration date.

4. **Strike Price** ([33, *Definition 2.5*])
   A price that is determined in advance for an underlying asset is called a strike price.

5. **Forward Contract** ([23, *Chapter 1,Section 1.3*]) It is an agreement between two parties to buy or sell an asset at a certain future time for a certain price.

6. **Future Contract** ([23, *Chapter 1,Section 1.4*]) It is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price.

7. **Volatility** ([23, *Chapter 15,Section 15.4*])
   The volatility of the underlying asset is the measure of uncertainty about the returns provided by the underlying asset over time.

8. **Hedge** ([33, *Definition 2.6*])
   A transaction that reduce the risk of an investment.

9. **Portfolio** ([33, *Definition 2.7*])
   An investment institution or company's collection of financial assets, such as stocks, bonds, and cash equivalents.

10. **Delta-hedging** ([10, *Investopedia*])
    Delta-hedging is an option's strategy that aims to reduce, or hedge, the risk associated with price movements in the underlying asset, by offsetting long and short positions. The value of the delta hedge portfolio is

$$\Pi = u - \Delta x \quad \text{where} \quad \Delta = \frac{\partial u}{\partial x}.$$

    This portfolio is made up of one position worth $u$ and $\Delta$ units of the underlying asset worth $x$.

11. **Wiener Process (or Brownian Motion)**([23], *Chapter 14.2*)
    A continuous-time stochastic process with a variable $z$ follows a Wiener process if it has the properties listed below,

    (a) The change $dz$ for a very short amount of time $dt$ is $dz = \epsilon\sqrt{dt}$ where $\epsilon$ has a standardized Normal distribution $\phi(0, 1)$ with mean zero and standard deviation of one.

    (b) The values of $dz$ for any two different short intervals of time, $dt$, are independent.

12. **Arithmetic Brownian Motion** (ABM) ( [29])
    An arithmetic Brownian motion is a Brownian motion with drift that is modelled by a stochastic differential equation of the form.

$$dx = \mu dt + \sigma dz,$$

    where $\mu$ is called the drift and $\sigma$ is called the volatility.

13. **Stochastic Differential Equation** ([33], *Definition 2.10*)
    Let $(\Omega, F, P)$ be a probability space and let $x(t), t \in R_+$ be a stochastic process $x : \Omega \times R_+ \to R$. Assume that $a(x, t) : \Omega \times R \times R_+ \to R$ and $b(x, t) : \Omega \times R \times R_+ \to R$ are stochastic-ally integrable functions of $t \in R_+$. Then the equation

$$dx = a(x, t)dt + b(x, t)dz \tag{1.1.1}$$

    is called Stochastic differential equation. The symbolic notation of the stochastic integral equation of (1.1.1) is,

$$x(t) = x(0) + \int_0^t a(x(s), s)ds + \int_0^t b(x(s), s)dz \tag{1.1.2}$$

    where $a(x, t)$ and $b(x, t)$ are referred to as the drift term and the diffusion term, respectively.

14. **Geometric Brownian motion** (GBM) ([33, *Definition 2.8*])
The Geometric Brownian motion describes a continuous-time stochastic process where the logarithm of a randomly fluctuating quantity follows a Brownian motion.

$$dx = A(x,t)xdt + B(x,t)xdz$$

where $A$ represents the drift and $B$ represents the volatility.

15. **Ito's process** ([33, *Definition 2.11*])
A stochastic process $x$ satisfying equation

$$dx = A(x,t)dt + B(x,t)dz$$

is considered to be an Ito's process.

16. **Ito's Lemma** ([8, Appendix 10A])
Consider a continuous and differentiable function $u$ of variable $x$. If $dx$ is a small change in $x$ and $du$ is the resulting small change in $u$, then

$$du \approx \frac{du}{dx}dx. \tag{1.1.3}$$

In other words, $du$ is roughly equivalent to the rate of change of $u$ in relation to $x$ multiplied by $dx$. If more precision is required, a Taylor series expansion of $du$ can be used:

$$du = \frac{du}{dx}dx + \frac{1}{2}\frac{d^2u}{dx^2}dx^2 + \frac{1}{6}\frac{d^3u}{dx^3} + \ldots \tag{1.1.4}$$

For a continuous and differentiable function $u$ of two variables, $x$ and $t$, the result is similar to equation (1.1.3) is

$$du \approx \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial t}dt \tag{1.1.5}$$

and the Taylor series expansion of $du$ is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial t}dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}dx^2 + \frac{\partial^2 u}{\partial x \partial t}dxdt + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}dt^2 + \ldots \tag{1.1.6}$$

If the limit of $dxdt$ and $dt^2$ tend to zero, equation (1.1.6) yields

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial t}dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}dx^2. \tag{1.1.7}$$

Now suppose that the value of a variable $x$ follows the Ito's process

$$dx = \mu dt + \sigma dz,$$

where $dz$ is a Wiener process. The variable $x$ has a drift rate or expected return of $\mu$ and has a variance of $\sigma^2$. Substituting $dx$ in equation (1.1.7) yeilds,

$$du = \frac{\partial u}{\partial x}\Big(\mu dt + \sigma dz\Big) + \frac{\partial u}{\partial t}dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}dx^2. \tag{1.1.8}$$

Expanding $dx^2$ gives $\mu^2 dt^2 + \mu\sigma dtdz + \mu\sigma dtdz + \sigma^2 dt$ and using Ito's multiplication rule,

| × | dz | dt |
|---|---|---|
| dz | dt | 0 |
| dt | 0 | 0 |

Equation (1.1.8) reduces to

$$du = \frac{\partial u}{\partial x}\Big(\mu dt + \sigma dz\Big) + \frac{\partial u}{\partial t}dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2 dt. \tag{1.1.9}$$

Simplifying equation (10), we obtain

$$du = \Big(\frac{\partial u}{\partial x}\mu + \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2\Big)dt + \frac{\partial u}{\partial x}\sigma dz.$$

This is Ito's Lemma.

**Example (Stock prices)** [15, ***Chapter 5, Example 3***]

Let $S(t)$ represent the stock price at time $t$. We model the evolution of $S(t)$ in time by supposing that $\frac{dS(t)}{S(t)}$ evolves according to the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW$$

where $dW$ is Wiener process or Brownian motion, $\mu > 0$ (the drift coefficient) and $\sigma$ (the volatility) are constants. One needs to realise that "$\mu$" is usually used to model deterministic trends and "$\sigma$" is used to model a set of random events occurring during this motion. The stochastic differential equation $dS(t) = \mu S(t)dt + \sigma S(t)dW$ has the following analytic solution,

$$S(t) = s_0 exp^{\sigma W(t) + (\mu - \frac{\sigma^2}{2})t}. \tag{1.1.10}$$

To give details on how the analytical solution was found, we first apply the Ito's formula to Ito's lemma to get

$$\begin{aligned} d(ln(S(t))) &= (ln(S(t)))' dS(t) + \frac{1}{2}(ln(S(t)))'' dS(t)dS(t) \\ &= \frac{dS(t)}{S(t)} - \frac{1}{2}\frac{1}{S(t)^2}dS(t)dS(t) \\ &= \mu dt + \sigma dW - \frac{1}{2}\frac{1}{S(t)^2}dS(t)dS(t). \end{aligned} \tag{1.1.11}$$

But,

$$dS(t)dS(t) = \sigma^2 S(t)^2 dW^2 + 2\sigma S(t)^2 \mu dW dt + \mu^2 S(t)^2 dt^2$$

and from the ito's multiplication table $dW^2 = dt$, $dWdt = 0$ and $dt^2 = 0$, Hence,

$$dS(t)dS(t) = \sigma^2 S(t)^2 dt.$$

Thus, equation (1.1.11) becomes,

$$d(ln(S(t)) = (\mu - \frac{\sigma^2}{2})dt + \sigma dt$$

So the analytical solution is,

$$S(t) = s_0 \exp^{\sigma W(t) + (\mu - \frac{\sigma^2}{2})t}.$$

□

# 2 Black Scholes Equation

In this chapter, we discuss some derivations of the Black Scholes, but we first recall some assumptions needed to obtain the derivations.

**Assumptions** ([23], *Chapter 15.5*)
We assume the following,

1. The stock price ($x$) follows a stochastic process.
2. During the life of the derivative, there are no dividends and transaction expenses or taxes. (We follow a European style option).
3. There are no risk-less arbitrage opportunities.
4. The trading of securities is continuous.
5. The interest rate remains constant.
6. The stock returns follows a normal distribution, hence volatility remains constant overtime.

## 2.1 Derivation of the Black Scholes Equation

In this section, we consider the derivation of the Black Scholes equation. The Black Sholes model assumes that the percentage changes in the stock price over a short period of time are normally distributed, where else changes in the stock price at a future time follows a log-normal distribution (see Figure 1 in the next page) and a variable with a log-normal distribution can have any value between zero and infinity [23, *Chapter 15.11*]. This motivates us to consider the derivation of the Black Scholes equation via Geometric Brownian Motion (GBM). However, this does not rule out the possibility of using Arithmetic Brownian Motion (ABM) to derive the Black Scholes equation. In fact, Marek, in [30], used the ABM to derive the Black Scholes equation, with the resulting equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}B^2 + \frac{\partial u}{\partial x}Cx - Du = 0.$$

In GBM,

$$dx = Axdt + Bxdz \tag{2.1.1}$$

where both $A$ and $B$ are proportional to the underlying asset $x$. The asset follows a log-normal random walk. A process like $dx$ is a model usually used to model the price of a stock and the variable $z$ follows a Wiener process, with properties as in section (1.1). If we take the squares of the GBM, we obtain,

$$(dx)^2 = (Axdt + Bxdz)^2$$

which can be written as,

$$dx^2 = A^2x^2dt^2 + B^2x^2dz^2 + 2BAx^2dtdz \tag{2.1.2}$$

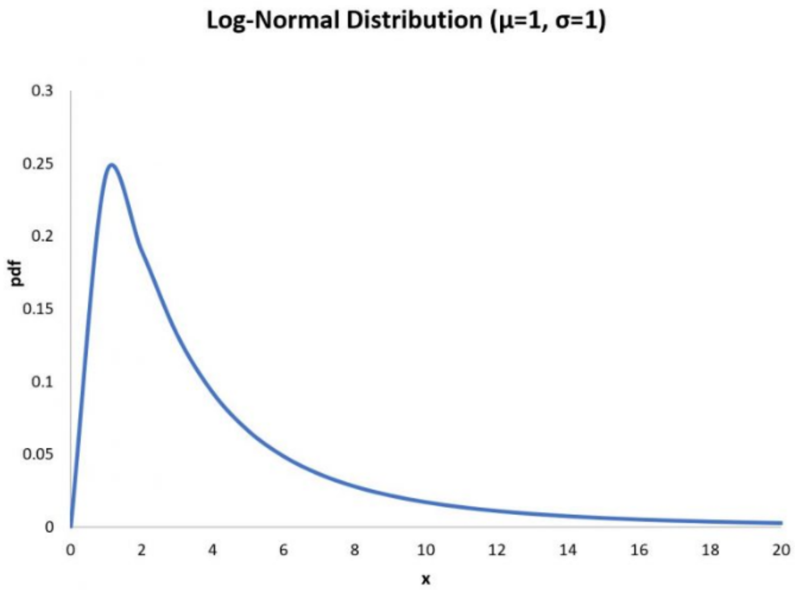


Figure 1: Log-normal distribution

from stochastic calculus we know that $dz^2 \approx dt$, $dt^2 \approx 0$ and $dzdt \approx 0$. Therefore, equation (2.1.2) reduces to,

$$dx^2 = B^2 x^2 dt, \tag{2.1.3}$$

Now suppose that $u(x,t)$ is the price of a call option. From Ito's lemma,

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial t}dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}dx^2 + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}dt^2 + \frac{\partial^2 u}{dtdx}dtdx \\ &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial t}dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}dx^2 \end{aligned} \tag{2.1.4}$$

and by substituting the GBM, we get

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial t}dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}B^2 x^2 dt. \tag{2.1.5}$$

The Wiener process (2.1.1) is the source of uncertain (risk) in the Ito equation (2.1.5). To eliminate it, we consider a portfolio $\Pi$ of stock and derivative such

$$\Pi = u - \Delta x, \quad \text{where} \quad \Delta = \frac{\partial u}{\partial x}. \tag{2.1.6}$$

The holder of this portfolio is short an amount $\Delta = \frac{\partial u}{\partial x}$ of shares and long one derivative (call option). We want to know the change in value of the portfolios, we then take the derivative of the portfolio to get

$$\begin{aligned} d\Pi &= du - \Delta dx \\ d\Pi &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial t}dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}B^2 x^2 dt - \Delta dx \end{aligned} \tag{2.1.7}$$

where,

$$\Delta dx = \frac{\partial u}{\partial x} dx,$$

so we obtain,

$$d\Pi = \frac{\partial u}{\partial t} dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} B^2 x^2 dt. \tag{2.1.8}$$

Since there is no longer a source of randomness ($z$) in equation (2.1.8) then the $\Pi$ will earn a risk-free rate, thus $d\Pi = r\Pi dt$. Hence,

$$r(u - \Delta x)dt = \frac{\partial u}{\partial t} dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} B^2 x^2 dt. \tag{2.1.9}$$

Therefore, equation (2.1.9) can be re-written as:

$$Du(x,t) - \frac{\partial u(x,t)}{\partial x} Cx = \frac{\partial u(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} B^2 x^2, \tag{2.1.10}$$

where $B$ is the standard deviation, $C$ and $D$ are risk-free interest rate and $B, C, D$ are constants, $x$ is the current value of the underlying asset (Stock Price) and $t$ represent time, so equation (2.1.10) can then be re-written as,

$$\frac{\partial u(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} B^2 x^2 + \frac{\partial u(x,t)}{\partial x} Cx - Du(x,t) = 0. \tag{2.1.11}$$

We will now discuss Grandville Sewell's derivation of the Black Scholes model from basic principle. In this derivation, no knowledge of stochastic calculus is assummed.

## 2.2 Derivation of the Black Scholes from basic principle

In this section, we consider another derivation of the Black Scholes equation that was recently derived by Granville Sewell in [20].
Now, first let $K$ be the strike price and $S$ the price at expiration date. Then the payoff of the call option value is determined by piecewise function,

$$P(S) = \begin{cases} S - K, & if \quad S > K \\ 0, & if \quad S < K. \end{cases}$$

This can be written as,

$$P(S) = max\{0, S - K\}. \tag{2.2.1}$$

Equation (2.2.1) implies that if $S < K$, then $P(S) = 0$. Hence, there is no need to exercise the option, since there is no profit to gain. If $K < S$, then $P(S) = K - S$. Hence, the option can be exercised. Note that if the stock price is zero at some time t, the option will be given by $P(0) = max\{0, -K\} = 0$, when the stock price increases to infinity, the option can be exercised in a case where $K < S$, the value of an option is given by $u(S,t) = S - Ke^{-r(T-t)}$, where t represents time and $r$ the risk-free interest rate.

The log normal distribution in the logarithm of price variable $z = ln(S) - \alpha$ is

$$p(t,z) = \frac{1}{\sqrt{2\pi\sigma_1^2(T-t)}} \exp\left[\frac{-z^2}{2\sigma_1^2(T-t)}\right], \tag{2.2.2}$$

where $z = ln(S) - \alpha = ln(S) - ln(s) + (\frac{1}{2}\sigma_1^2 - r)(T-t)$, and $s$ represents the current price of the asset and $S$ the final price of the asset, then current value of the option is

$$u(s,t) = e^{-r(T-t)} \int_0^\infty \frac{P(S)}{S} p(t,z) dS. \tag{2.2.3}$$

Now,

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial}{\partial t}\Big(ln(S) - ln(s) + (\frac{1}{2}\sigma_1^2 - r)(T-t)\Big) \\ &= r - \frac{1}{2}\sigma_1^2 \end{aligned} \tag{2.2.4}$$

and

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial}{\partial s}\Big(ln(S) - ln(s) + (\frac{1}{2}\sigma_1^2 - r)(T-t)\Big) \\ &= -\frac{1}{s}. \end{aligned} \tag{2.2.5}$$

The log normal distribution will be at its peak when the original price is the same as the final price, that is, $S = s$. When $s$ is not equal to $S$, then equation (2.2.3) will not be at its peak. Moreover, $z \to \infty$ when $s \to \infty$ and $z \to -\infty$ when $s \to 0$, so considering equation (2.2.3) and differentiating it with respect to $t$, in order to derive the Black Scholes equation we get,

$$\begin{aligned} u_t &= e^{-r(T-t)} r \int_0^\infty \frac{P(S)}{S} p(t,z) dS + e^{-r(T-t)} \int_0^\infty \frac{P(S)}{S}\Big[p_t + \frac{\partial z}{\partial t} p_z\Big] dS \\ u_t &= ru(s,t) + e^{-r(T-t)} \int_0^\infty \frac{P(S)}{S}\Big[p_t + (r - \frac{1}{2}\sigma_1^2) p_z\Big] dS. \end{aligned} \tag{2.2.6}$$

Now, when we differentiate equation (2.2.3) again with respect to $s$, we obtain,

$$\begin{aligned} u_s &= e^{-r(T-t)} \int_0^\infty \frac{P(S)}{S}\Big[p_s(t,z)(0) + p_z \frac{\partial z}{\partial s}\Big] dS \\ su_s &= -e^{-r(T-t)} \int_0^\infty \frac{P(S)}{S} p_z dS \\ s(su_s)_s &= e^{-r(T-t)} \int_0^\infty \frac{P(S)}{S} p_{zz} dS. \end{aligned} \tag{2.2.7}$$

We shall now multiply the last equation of equation (2.2.7) with $\sigma_1^2/2$ so that we can be able to combine the second equation of equation (2.2.6) and the last equation of equation (2.2.7) to get

$$u_t + \frac{1}{2}\sigma_1^2 s(su_s)_s = ru + e^{-r(T-t)} \int_0^\infty \frac{P(S)}{S}\Big[p_t + (r - \frac{1}{2}\sigma_1^2)p_z + \frac{1}{2}\sigma_1^2 p_{zz}\Big] dS. \tag{2.2.8}$$

Hence,

$$u_t + \frac{1}{2}\sigma_1^2 s(su_s)_s = ru + (r - \frac{1}{2}\sigma_1^2)e^{-r(T-t)} \int_0^\infty \frac{P(S)}{S} p_z dS, \tag{2.2.9}$$

and using equation (2.2.7) we get

$$u_t + \frac{1}{2}\sigma_1^2 s(su_s)_s = ru - (r - \frac{1}{2}\sigma_1^2)su_s, \tag{2.2.10}$$

since $(su_s)_s = su_{ss} + u_s$. Expanding $u$ we get the following

$$u_t + \frac{1}{2}\sigma_1^2 s^2 u_{ss} + \frac{1}{2}\sigma_1^2 su_s = ru - rsu_s + \frac{1}{2}\sigma_1^2 su_s, \tag{2.2.11}$$

which can simply be written as,

$$u_t + \frac{1}{2}\sigma_1^2 s^2 u_{ss} + rsu_s - ru = 0. \tag{2.2.12}$$

Equation (2.2.12) is known as the Black Scholes equation. If we consider the assumptions under section (2.2) then equation (2.2.12) will be the same as equation (2.1.11) provided $s = x, \quad r = C$ and $\sigma = B$.

We discuss an example of a forward contract on a non-dividend paying stock. This derivative depends on the stock. First recall from [23], a general result, applicable to all long forward contracts, is

$$u = (F_0 - K)e^{-rT} \quad \text{where } F_0 \text{ is the expected return on the stock.} \tag{2.2.13}$$

This result is applicable to the both contracts on investment assets and those on consumption assets. It is also known that when considering a forward contract on an investment asset and price $S_0$ that provides no income, we have the following relationship between $F_0$ and $S_0$,

$$F_0 = S_0 e^{rT}. \tag{2.2.14}$$

For the value of a forward contract on an investment asset that provides no income, we simply substitute equation (2.2.14) into equation (2.2.13) to obtain,

$$u = S_0 - Ke^{-rT}. \tag{2.2.15}$$

We are now in a position to consider an example. We want to illustrate the link between the Black schole equation to fare price and arbitrage free prices of derivatives.

**Example 2.2.1** ([23], ***Example 15.5***)

We consider a forward contract on a non-dividend paying stock. We will use the equation

$$u = S - Ke^{-r(T-t)}, \tag{2.2.16}$$

for the forward contract $u$, at time $t$, given in terms of the stock price $S$, and $K$ as the delivery price. Now,

$$\frac{\partial u}{\partial t} = -rKe^{-r(T-t)}, \qquad \frac{\partial u}{\partial S} = 1, \qquad \frac{\partial^2 u}{\partial S} = 0, \tag{2.2.17}$$

so if we substitute into the Black Scholes equation.

$$u_t + \frac{1}{2}\sigma_1^2 S^2 u_{SS} + rSu_S - ru = 0. \tag{2.2.18}$$

We obtain,

$$\begin{aligned} -rKe^{-r(T-t)} + \frac{1}{2}\sigma_1^2 s^2(0) + rS(1) - ru = 0 \\ -rKe^{-r(T-t)} + rS(1) - ru = 0. \end{aligned} \tag{2.2.19}$$

Therefore the Black Scholes equation is satisfied.

## 2.3 A Perpetual Derivative

A derivative is a type of a financial contract whose value is dependent on an underlying asset, group of assets, or benchmark[16]. A perpetual contract is a sort of derivative that allows you to simply bet on an asset's price. Consider a perpetual derivative that pays off a fixed amount $Q$ when the stock price equals $H$ for the first time. This means that the value of the stock in question does not depend on time, that is

$$u_t + \frac{1}{2}\sigma_1^2 S^2 u_{SS} + rSu_S - ru = 0 \tag{2.3.1}$$

becomes the following ordinary differential equation

$$\frac{1}{2}\sigma_1^2 S^2 u_{SS} + rSu_S - ru = 0, \quad \text{since} \quad u_t = 0. \tag{2.3.2}$$

Lets assume that $S < H$, where $S$ is the stock price at time $T$ and $H$ is the stock price at $t = 0$. The boundary conditions for the derivative will be

$$u = \begin{cases} 0, & S = 0 \\ Q, & S = H \end{cases}$$

or

$$u = max\{H - S, 0\} \quad \text{when} \quad t = T.$$

We now find the value of the derivative that satisfy the boundary conditions as well as equation (2.3.1). So, the value of the derivative can be $u = \frac{QS}{H}$, since if $S = 0$, $u = \frac{(0 \times Q)}{H} = 0$ and if $S = H$, $u = \frac{(QH)}{H} = Q$ or $u = \frac{(QS)}{S} = Q$.
Now, if we assume $S > H$, the boundary conditions will be given by

$$u = \begin{cases} 0, & S \to \infty \\ Q, & S = H \end{cases}$$

and a function $u$ that satisfies this boundary conditions is,

$$u = Q\left(\frac{S}{H}\right)^{-\alpha} \tag{2.3.3}$$

where $\alpha$ is positive and the differential equation (2.3.1) is satisfied when

$$-r\alpha + \frac{1}{2}\sigma^2\alpha(\alpha+1) - r = 0. \tag{2.3.4}$$

Hence, the value of the derivative is

$$u = Q\Big(\frac{S}{H}\Big)^{-2r/\sigma^2}. \tag{2.3.5}$$

**Example 2.3.1( [23], *Problem 15.23*)**

We consider equation (2.3.5) to determine the value of a perpetual American put option on a non-dividend paying stock with strike price $K$ if it is exercised.
Solution:
If the perpetual American put option is exercised when $S = H$, it yields a payoff of $(K - H)$ and then we obtain its value by setting $\alpha = K - H$ in equation (2.3.5) as,

$$\begin{aligned} u &= (K-H)\Big(\frac{S}{H}\Big)^{-2r/\sigma^2} \\ &= (K-H)\Big(\frac{H}{S}\Big)^{2r/\sigma^2}. \end{aligned} \tag{2.3.6}$$

Now,

$$\begin{aligned} \frac{du}{dH} &= \Big(\frac{H}{S}\Big)^{2r/\sigma^2} + \Big(\frac{K-H}{S}\Big)\Big(\frac{2r}{\sigma^2}\Big)\Big(\frac{H}{S}\Big)^{\frac{2r}{\sigma^2}-1} \\ &= \Big(\frac{H}{S}\Big)^{\frac{2r}{\sigma^2}}\Big(-1 + \frac{2r(K-H)}{H\sigma^2}\Big) \\ \frac{d^2u}{dH^2} &= \frac{-2rk}{H^2\sigma^2}\Big(\frac{H}{S}\Big)^{2r/\sigma^2} + \Big(-1 + \frac{2r(K-H)}{H\sigma^2}\Big)\frac{2r}{\sigma^2 s}\Big(\frac{H}{S}\Big)^{\frac{2r}{r^2}-1} \end{aligned} \tag{2.3.7}$$

So $\frac{du}{dH}$ is zero when, $H = \frac{2rk}{(2r+\sigma^2)}$ and the value of the perpetual American put option is maximised if it is exercised when $S$ equals the value of $H$. Hence, the value of the perpetual American put option is given by

$$\Big(K - H\Big)\Big(\frac{S}{H}\Big)^{\frac{r}{\sigma^2}} \quad \text{when} \quad H = \frac{2rK}{\sigma^2 + 2r}. \tag{2.3.8}$$

## 2.4 Prices of Tradeable Derivatives

**Theorem 1.** Any function $u(S,t)$ that is a solution of the differential equation

$$u_t + \frac{1}{2}\sigma_1^2 S^2 u_{SS} + rSu_S - ru = 0 \tag{2.4.1}$$

is the theoretical price of a derivative that could be traded. If a derivative with that price existed, it would not create any arbitrage opportunities. Conversely, if a function $u(S,t)$ does not satisfy the differential equation (2.4.1), it cannot be the price of a derivative without creating arbitrage opportunities for traders.

We consider, the demonstration of the theorem above, with the following examples.

**Example 2.4.1**

We consider the function $u(S,t) = e^S$, and show that it does not satisfy equation (2.4.1). If $u(S,t) = e^S$, then

$$\begin{aligned} \frac{\partial u}{\partial t} &= 0, \\ \frac{\partial u}{\partial S} &= e^S, \\ \frac{\partial^2 u}{\partial S^2} &= e^S. \end{aligned} \tag{2.4.2}$$

We now substitute equation (2.4.2) into equation (2.4.1) to get,

$$rS(e^S) + \frac{1}{2}\sigma^2(e^S) = ru. \tag{2.4.3}$$

Equation (2.4.3) does not satisfy the differential equation (2.4.1) because when we derive $u(S,t)$ we do not obtain all the terms in equation(2.4.1). This means the function $u(S,t) = e^S$, cannot be one of the price of a derivative dependent on the stock price. If an instrument whose price was always $e^S$ existed, there would be an arbitrage opportunity.

**Example 2.4.2**

We consider the function

$$u(S,t) = \frac{e^{(\sigma^2-2r)(T-t)}}{S} \tag{2.4.4}$$

and show that it does not satisfy equation (2.4.1).
If $u(S,t) = (e^{(\sigma^2-2r)(T-t)})/S$ then

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{e^{T\sigma^2-t\sigma^2-2Tr+2tr}(-\sigma^2+2r)}{S}, \\ \frac{\partial u}{\partial S} &= -\frac{e^{(\sigma^2-2r)(T-t)}}{S^2}, \\ \frac{\partial^2 u}{\partial S^2} &= 2\frac{e^{(\sigma^2-2r)(T-t)}}{S^3}. \end{aligned} \tag{2.4.5}$$

Substituting in equation (2.4.1) , we obtain

$$\frac{e^{T\sigma^2-t\sigma^2-2Tr+2tr}(-\sigma^2+2r)}{S} - r\frac{e^{(\sigma^2-2r)(T-t)}}{S} + \frac{1}{2}\sigma^2S^22\frac{e^{(\sigma^2-2r)(T-t)}}{S} = ru. \tag{2.4.6}$$

Equation (2.4.4) does not satisfy equation (2.4.1), so in theory, the function $u(S,t) = \frac{(e^{(T-t)})}{S}$ is a price of a trade-able security.

**Example 2.4.3 ( [23], *Problem 15.12*)**

We consider a derivative that pays off $S_T^n$ at future time $T$, where $S_t$ is the stock price at the current time. When the stock pays no dividends and its price follows a geometric Brownian motion, it can be shown that its price at time $t$ where $(t < T)$ has the form $h(t, T)S^n$, where $S$ is the stock price at time $t$ and $h$ is a function of $t$ and $T$.
We show the following,

1. We derive a differential equation satisfied by $h(t, T)$.
2. We find the boundary condition for the differential equation $h(t, T)$ using the Black Scholes equation.
3. Show that $h(t, T) = e^{[0.5\sigma n(n-1)+r(n-1)]}$ where $r$ is the risk-free interest rate and $\sigma$ is the stock price volatility.

Solution:
If $G(S, t) = h(t, T)S^n$, then

$$\begin{aligned} \frac{\partial G}{\partial t} &= h_t S^n \\ \frac{\partial G}{\partial S} &= hnS^{n-1} \\ \frac{\partial^2 G}{\partial S^2} &= hn(n-1)S^{n-2}, \end{aligned} \tag{2.4.7}$$

where, $h_t = \dfrac{\partial h}{\partial t}$. Substituting into equation (2.4.1) we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh \tag{2.4.8}$$

The derivative is worth $S^n$ when $t = T$. The boundary condition for this differential equation is therefore $h(T, T) = 1$. The equation

$$h(t, T) = e^{(0.5\sigma^2 n(n-1) - r(n-1))(T-t)} \tag{2.4.9}$$

satisfies the boundary condition since it reduces to $h = 1$ when $t = T$.

# 3 Adomian Decomposition Method

In this chapter, we describe the method of solving nonlinear partial differential equations involving two variables, namely, the Adomian Decomposition Method(ADM). The computational efficiency and the type of solution of the proposed method will be discussed and analysed with some examples. We will use the Adomian decomposition method later in our study to solve the heat equation after transforming the Black Schole equation to the heat equation.

## 3.1 Description of the Method

### 3.1.1 The Adomian Decomposition Method

Consider a nonlinear partial differential equation of the form [19, *Section 3.1*]

$$u_x(x,y) + u_y(x,y) + R(u(x,y)) + N(u(x,y)) = 0, \tag{3.1.1}$$

which can be written as,

$$L_x u(x,y) + L_y u(x,y) + R(u(x,y)) + N(u(x,y)) = 0, \tag{3.1.2}$$

where,

1. $L_x = \frac{\partial^n}{\partial x^n}$ for $n = 1, 2, 3, ...$ is the highest order in $x$ and the inverse of $L_x$ is given by $L_x^{-1} = \int \cdots \int (.) dx_1 \cdots dx_n$.
2. $L_y = \frac{\partial^n}{\partial y^n}$ for $n = 1, 2, 3, ...$ is the highest order in $y$ and the inverse of $L_y$ is given by $L_y^{-1} = \int \cdots \int (.) dx_1 \cdots dx_n$.
3. $R(u(x,y))$ represents lower order terms in $x$ and $y$.
4. $N(u(x,y))$ represents nonlinear terms in $x$ and $y$.

The solutions to $u(x,y)$ from the operators $L_x$ and $L_y$ are called partial solutions, because either $L_x$ or $L_y$ can be used to get the solution. One can either choose to use the $L_x$ or the $L_y$ operator at a time. The decision on which operator to use is based on the following reasons.

1. Which one minimises the size of the computation? (Check the terms that are simple to evaluate when applying the $L_x$ or the $L_y$ operator)
2. Which one has the best conditions (for example the coefficients of the differential equation you are given) to evaluate the solution's components more quickly?

Suppose that $L_x$ meets these two conditions ($L_y$ operator can still be used, even if it does not meet the conditions, these conditions are just there to try to simplify computations), then we write $L_x$ as the subject of the formula of equation (3.1.2),

$$L_x u(x,y) = -L_y u(x,y) - R(u(x,y)) - N(u(x,y)) \tag{3.1.3}$$

and $L_x$ is invertible, that is, $L_x^{-1}$ exists (when dealing with Adomian decomposition method, it is always assumed that the operator $L_x$ and the operator $L_y$ are invertible) and its integral operator is a definite operator defined by, $L_x^{-1} = \int_0^x (.)dx$. Apply the inverse operator to equation (3.1.3) to get

$$L_x^{-1} L_x u(x,y) = u(0,y) - L_x^{-1} L_y u(x,y) - L_x^{-1} R(u(x,y)) - L_x^{-1} N(u(x,y)), \tag{3.1.4}$$

where $u(0,y)$ is a constant of integration. Re-writing equation (3.1.4) we get,

$$u(x,y) = u(0,y) - L_x^{-1} L_y u(x,y) - L_x^{-1} R(u(x,y)) - L_x^{-1} N(u(x,y)). \tag{3.1.5}$$

If $L_x^{-1}$ does not exists, the Adomian decomposition method becomes redundant. Now lets carry on with the invertible operator $L_x$. The solution $u(x,y)$ of equation (3.1.5) can be presented as an infinite series

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y). \tag{3.1.6}$$

Applying equation (3.1.6) to equation (3.1.5) we get,

$$\sum_{n=0}^{\infty} u_n(x,y) = u(0,y) - L_x^{-1} L_y \Big( \sum_{n=0}^{\infty} u_n(x,y) \Big) - L_x^{-1} \Big( R(u(x,y)) \sum_{n=0}^{\infty} u_n(x,y) \Big) - L_x^{-1} \Big( \sum_{n=0}^{\infty} A_n \Big) \tag{3.1.7}$$

where

$$N(u(x,y)) = \sum_{n=0}^{\infty} A_n$$

where $A_n$'s are known as the Adomian polynomials and they depend on $u_0, u_1, ..., u_n$. The Adomian polynomials are given by the following formula

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} N \Big[ \sum_{j=0}^{n} \lambda^j u_j \Big] |_{\lambda=0} \qquad n = 0, 1, 2, 3, ...$$

The first few polynomials are defined as follows,

$$\begin{aligned}
n = 0 \quad A_0 &= \frac{1}{0!} \frac{\partial^0}{\partial \lambda^0} N \Big[ \sum_{j=0}^{0} \lambda^j u_j \Big] |_{\lambda=0} = \frac{1}{0!} N(\lambda^0 u_0) = N(u_0) \\
n = 1 \quad A_1 &= \frac{1}{1!} \frac{\partial^1}{\partial \lambda^1} N \Big[ \sum_{j=0}^{1} \lambda^j u_j \Big] |_{\lambda=0} = \frac{\partial^1}{\partial \lambda^1} N(u_0 + \lambda u_1) = N'(u_0 + \lambda u_1)(0 + u_1)|_{\lambda=0} = N'(u_0)u_1, \\
n = 2 \quad A_2 &= \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} N \Big[ \sum_{j=0}^{2} \lambda^j u_j \Big] |_{\lambda=0} = \frac{1}{2!} \frac{\partial^2}{\partial \lambda^2} N(\lambda^0 u_0 + \lambda^1 u_1 + \lambda^2 u_2)|_{\lambda=0} = u_2 N'(u_0) + \frac{u_1^2}{2!} N''(u_0), \\
n = 3 \quad A_3 &= u_3 N'(u_0') + u_1 u_2 N''(u_0) + \frac{u_1^3}{3!} N'''(u_0). \\
&\vdots
\end{aligned} \tag{3.1.8}$$

The above equations (3.1.8), were derived by Adomian [1]. Consider (3.1.7) and note that it is possible to write the components $u_n(x,y)$, $n \geq 0$ of the solution $u(x,y)$ iteratively by

$$\begin{aligned} u_0 &= u(0,y) \\ u_{k+1} &= -L_x^{-1}L_y u_k(x,y) - L_x^{-1}R(u_k(x,y)) - L_x^{-1}N(u_k(x,y)). \end{aligned} \tag{3.1.9}$$

Using the Adomian polynomials $A_n$ of the nonlinear term $N(u(x,y))$, the components $u_n(x,y)$ are given as,

$$\begin{aligned} u_0(x,y) &= u(0,y) \\ u_1(x,y) &= -L_x^{-1}L_y u_0(x,y) - L_x^{-1}R(u_0(x,y)) - L_x^{-1}A_0 \\ u_2(x,y) &= -L_x^{-1}L_y u_1(x,y) - L_x^{-1}R(u_1(x,y)) - L_x^{-1}A_1 \\ &\vdots \end{aligned} \tag{3.1.10}$$

As a result, the solution as a series form is as follows,

$$u_0+u_1+u_2+... = u(0,y) - L_x^{-1}L_y\sum_{n=0}^{\infty} u_n(x,y) - L_x^{-1}R\sum_{n=0}^{\infty} u_n(x,y) - L_x^{-1}\sum_{n=0}^{\infty} A_n. \tag{3.1.11}$$

The solution of $u$ is found as a series that converges rapidly to an accurate solution, that means when we evaluate the terms we get the exact solution in a very short period of time. ( [22, *Section 2*] explain these claim in details). [12] proposed a hypothesis that led us to a theorem that proves that indeed the Adomian decomposition method converges, and it is absolutely convergent.

### 3.1.2 Theorem of convergence [12, *Section 2*]

Let us consider the following nonlinear equation

$$u = N(u) + u_0 \tag{3.1.12}$$

where $N$ and $u_0$ represent an operator and a function given in a suitable space, respectively.

**Theorem 2.** [12, *Section 2*]
Assume that

1. A series of functions $(u_i)$ can be used to express the solution of equation (3.1.12), where the series is assumed to be absolutely convergent, that is, $\sum |u_i| < \infty$.

2. The nonlinear term $N(u)$ in equation (3.1.12) can be developed on the entire series, with a convergent radius equal to $\infty$, that is

$$\begin{aligned} N(u) &= \sum_{n=0}^{\infty} N_0^n \frac{u^n}{n!} \\ N(u) &= N_0^0\frac{u^0}{0!} + N_0^1\frac{u^1}{1!} + N_0^2\frac{u^2}{2!} + ... \end{aligned} \tag{3.1.13}$$

with $|u| < \infty$.

Then the solution of equation (3.1.12) is the series $u_n = \sum_{i=0}^{n} u_i$, when $u$ satisfy equation (3.1.10).

*Proof.* Assumption (2) above assures us that the series $\sum N_0^n \frac{u^n}{n!}$ converges for any $u$. Since $u = \sum_{i=0}^{\infty} u_i$ is absolutely convergent, the equation (3.1.13) can further be expressed as

$$N(u) = \sum_{n=0}^{\infty} N_0^n \frac{(\sum_{i=0}^{\infty} u_i)^n}{n!} \tag{3.1.14}$$

and $u^n$ in equation (3.1.13) is now defined as follows,

$$u^n = (\sum_{i=0}^{\infty} u_i)^n = (u_0 + u_1 + u_2 ...)^n = \sum_{q=0}^{\infty} A_{nq}(u_0, .., u_q).$$

due to $u = \sum_{i=0}^{\infty} u_i$ being absolutely convergent, we then have $\sum_{i=0}^{\infty} |u_i| = U < \infty$ or $\sum_{q=0}^{\infty} |A_{nq}| \leq U^n < \infty$ as a result. ($A_{nq}$ depends only on $u_0 + u_1 + u_2 ....$). We say equation (3.1.14) is also, absolutely convergent because

$$\begin{aligned} N(u) &= \sum_{n=0}^{\infty} \left[ \frac{N_0^n}{n!} \sum_{q=0}^{\infty} A_{nq}(u_0, ..., u_q) \right] \\ &= \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \frac{N_0^n}{n!} A_{nq}(u_0, ..., u_q) \end{aligned} \tag{3.1.15}$$

so taking the absolute value of $N(u)$, we get that

$$|N(u)| \leq \sum_{n=0}^{\infty} \left| \frac{N_0^n}{n!} \right| U^n$$

where $\sum_{n=0}^{\infty} \left| \frac{N_0^n}{n!} \right| U^n$ converges as a result of assumption (2). Hence, it indicates that the equation (3.1.15) is absolutely convergent. Now, taking into account that $u(x, y) = \sum_{n=0}^{\infty} u_n(x, y)$ and

$$\begin{aligned} A_0 &= N(u_0) \\ A_1 &= u_1 N'(u_0), \\ A_2 &= u_2 N'(u_0) + \frac{u_1^2}{2!} N''(u_0), \\ A_3 &= u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{u_1^3}{3!} N'''(u_0). \\ &\vdots \end{aligned} \tag{3.1.16}$$

then equation (3.1.12) becomes,

$$\sum_{i=0}^{\infty} u_i = \sum_{n=0}^{\infty} A_n = u_0$$

This completes the proof. □

## 3.2 Application of the Method

In this section, we discuss some applications of Adomian decomposition method. We first consider an example of a partial differential equation without initial conditions, and show that in this case we get a series of solutions.

**Example 1.** [25, *Exercise 4.4.2*] Solve the following nonlinear equation by Adomian Decomposition Method

$$F = pq + xp + yq - u = 0. \tag{3.2.1}$$

**Solution:** Note that equation (3.2.1) is the same as

$$u_x u_y + xu_x + yu_y - u = 0 \tag{3.2.2}$$

since $p = \partial u/\partial x$ and $q = \partial u/\partial y$, Writing equation (3.2.2) in an operator form

$$L_x u(x,y)L_y u(x,y) + xL_x u(x,y) + yL_y u(x,y) - u(x,y) = 0 \tag{3.2.3}$$

where $L_x = \partial/\partial x$ and $L_y = \partial/\partial y$. Then

$$\begin{aligned} xL_x u(x,y) &= -L_x u(x,y)L_y u(x,y) - yL_y u(x,y) + u(x,y) \\ L_x u(x,y) &= -\frac{1}{x}L_x u(x,y)L_y u(x,y) - \frac{y}{x}L_y u(x,y) + \frac{1}{x}u(x,y) \end{aligned} \tag{3.2.4}$$

Applying the inverse operator of $L_x^{-1} = \int_0^x (.)dx$ to equation (3.2.4), we get

$$\begin{aligned} L_x^{-1}L_x u(x,y) &= -\frac{1}{x}L_x^{-1}L_x u(x,y)L_y u(x,y) - \frac{y}{x}L_x^{-1}L_y u(x,y) + \frac{1}{x}L_x^{-1}u(x,y) \\ u(x,y) &= u(0,y) - \frac{1}{x}u(x,y)L_y u(x,y) - \frac{y}{x}L_x^{-1}L_y u(x,y) + \frac{1}{x}L_x^{-1}u(x,y) \\ u(x,y) &= u(0,y) - \frac{1}{2x}L_y u(x,y)^2 - \frac{y}{x}L_x^{-1}L_y u(x,y) + \frac{1}{x}L_x^{-1}u(x,y) \end{aligned} \tag{3.2.5}$$

since the Adomian solution $u(x,y)$ has a series form

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$$

equation (3.2.5) becomes,

$$\sum_{n=0}^{\infty} u_n(x,y) = u(0,y) - \frac{1}{2x}L_y \sum_{n=0}^{\infty} A_n - \frac{y}{x}L_x^{-1}L_y \sum_{n=0}^{\infty} u_n(x,y) + L_x^{-1}\frac{1}{x}\sum_{n=0}^{\infty} u_n(x,y) \tag{3.2.6}$$

where $A_n$ is the Adomian Polynomial and is evaluated by equation (3.1.8), so our solution is given as

$$\begin{aligned} u_0 + u_1 + u_2 + ... = u(0,y) - \frac{1}{2x}L_y(A_0 + A_1 + A_2 + A_3 + ...) - \frac{y}{x}L_x^{-1}L_y(u_0 + u_1 \\ + u_2 + ...) + L_x^{-1}\frac{1}{x}(u_0 + u_1 + u_2 + ...). \end{aligned} \tag{3.2.7}$$

## Application of the methods for a nonlinear first order initial value problem

In this section we consider a nonlinear partial differential equation with suitable initial condition, namely the Inviscid Burgers equation.

$$u_y + uu_x = 0, \tag{3.2.8}$$

with initial condition

$$u(x,0) = x.$$

In the case where initial conditions are provided, the Adomian decomposition method provides exact solution.

**Example 2.**( [13], ***Exercise 3.2***) We solve the above mentioned equation (3.2.8) using the Adomian Decomposition Method.

**Solution:** The nonlinear partial differential equation (3.2.8) in an operator form is given as

$$L_y u(x,y) + uL_x u(x,y) = 0 \tag{3.2.9}$$

where $L_x = \frac{\partial}{\partial x}$ and $L_y = \frac{\partial}{\partial y}$. Then

$$L_y u(x,y) = -u(x,y)L_x u(x,y) = -\frac{1}{2}L_x u(x,y)^2 \tag{3.2.10}$$

when we apply the inverse operator $L_y^{-1} = \int_0^y (.)dy$ to equation (3.2.10), we get

$$\begin{aligned} L_y^{-1}L_y u(x,y) &= -\frac{1}{2}L_y^{-1}L_x u(x,y)^2 \\ u(x,y) &= u(x,0) - \frac{1}{2}L_y^{-1}L_x u(x,y)^2, \end{aligned} \tag{3.2.11}$$

Using the initial condition we get that

$$u(x,y) = x - \frac{1}{2}L_y^{-1}L_x u(x,y)^2 \tag{3.2.12}$$

since the Adomian solution has a series form

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$$

equation (3.2.12) becomes

$$\sum_{n=0}^{\infty} u_n(x,y) = x - \frac{1}{2}L_y^{-1}L_x \sum_{n=0}^{\infty} A_n, \tag{3.2.13}$$

where $\sum_{n=0}^{\infty} A_n = u(x,y)^2$. Equation (3.2.13) can be written as

$$u_0 + u_1 + u_2 + .... = x - \frac{1}{2}L_y^{-1}L_x(A_0 + A_1 + A_2 + ...). \tag{3.2.14}$$

Evaluating each component of the series solution using equation (3.1.8) and (3.1.10), we have

$$\begin{aligned} u_0(x,y) &= x \\ u_1(x,y) &= -\frac{1}{2}L_y^{-1}L_xA_0 = -xy \\ u_2(x,y) &= -\frac{1}{2}L_y^{-1}L_xA_1 = xy^2 \\ u_3(x,y) &= -\frac{1}{2}L_y^{-1}L_xA_1 = -xy^3 \\ &\vdots \end{aligned} \tag{3.2.15}$$

Thus,

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) = u_0 + u_1 + u_2 + ... = x - xy + xy^2 - xy^3 + ..., \tag{3.2.16}$$

The solution of equation (3.2.8) is

$$u(x,y) = x(1+y)^{-1} = \frac{x}{y+1}. \tag{3.2.17}$$

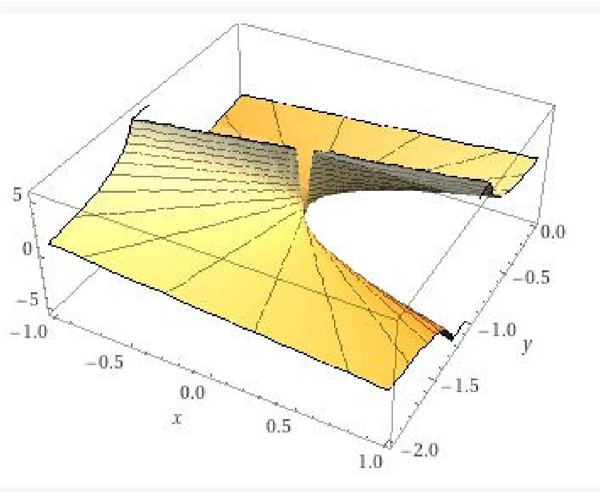


Figure 2: This graph represent the solution to equation (3.2.8), u(x,y)=x/y+1

.

## Application of the method to first-order ordinary differential equation with one independent variable

In this section we discuss the solution of a first-order ordinary differential equation using Adomian decomposition method.

**Example 3.**
Solve the following differential equation using the Adomian.

$$\frac{dy(x)}{dx} = 1 - 2\Big[y(x)\Big]^2, \quad y(0) = 0, \quad 0 \le x \le 1. \tag{3.2.18}$$

**Solution:** Writing the equation (3.2.18) in operator form we get,

$$L_x y = 1 - 2\Big[y(x)\Big]^2. \tag{3.2.19}$$

Applying the inverse operator $L_x^{-1} = \int_0^x (.)dx$ to equation (3.2.18) yields,

$$y(x) = y_0(x) + x - 2L_x^{-1}\Big[y(x)\Big]^2. \tag{3.2.20}$$

Since the Adomian solution has a series form, then

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

$$\sum_{n=0}^{\infty} y_n(x) = y_0(x) + x - 2L^{-1}\sum_{n=0}^{\infty} A_n \tag{3.2.21}$$

where $\sum_{n=0}^{\infty} A_n = \Big[y(x)\Big]^2$, and $A_n$ are the Adomian polynomial. Expanding equation (3.2.21) , we have,

$$y_0(x) + y_1(x) + y_2(x) + ... = x - 2L^{-1}\Big[A_0 + A_1 + A_2 + ...\Big]. \tag{3.2.22}$$

Evaluating each component $y(x)$ and using the following

$$\begin{aligned} A_0 &= N(u_0) \\ A_1 &= u_1 N'(u_0), \\ A_2 &= u_2 N'(u_0) + \frac{u_1^2}{2!}N''(u_0), \\ A_3 &= u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{u_1^3}{3!}N'''(u_0). \\ &\vdots \end{aligned} \tag{3.2.23}$$

yeilds,

$$\begin{aligned} y_0(x) &= 0 \\ y_1(x) &= x - 2L^{-1}\tilde{A}_0 = x - 2L^{-1}(y_0)^2 = x \\ y_2(x) &= x - 2L^{-1}A_1 = x - 2L^{-1}(2y_0y_1) = x \\ y_3(x) &= x - 2L^{-1}A_2 = x - 2L^{-1}(2y_0y_2 + y_1^2) = x - \frac{2}{3}x^3 \\ y_4(x) &= x - 2L^{-1}A_3 = x - 2L^{-1}(2y_0y_3 + 2y_1y_2) = x - \frac{4}{3}x^3. \end{aligned} \tag{3.2.24}$$

Therefore the particular solution is,

$$y_x = \sum_{n=0}^{\infty} y_n(x) = y_0 + y_1 + y_3 + ... = 0 + x + x + (x - \frac{2}{3}x^3) + (x - \frac{4}{3}x^3) + ... \tag{3.2.25}$$

## Application of the method to second-order partial differential equation.

In this section we discuss the solution of a second order partial differential equation using Adomian decomposition method. Consider the following second order partial differential equation

$$\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} - u(1-u). \tag{3.2.26}$$

**Example 4.** Lets consider equation (3.2.26) in this form

$$F = -u_y + u_{xx} - u + u^2 = 0 \tag{3.2.27}$$

**Solution:** Equation (3.2.26) in operator form is given as

$$L_x u(x,y) = L_y u(x,y) + R(u(x,y)) - N(u(x,y)^2) \tag{3.2.28}$$

where $L_x = \frac{\partial^2}{\partial x^2}$ and $L_y = \frac{\partial}{\partial y}$. Applying the inverse operator $L_x^{-1} = \int_0^x \int_0^x (.)dxdx$ to equation (3.2.28), we get

$$\begin{aligned} L_x^{-1} L_x u(x,y) &= L_x^{-1} L_y u(x,y) + L_x^{-1}(u(x,y)) - L_x^{-1} u(x,y)^2 \\ u(x,y) &= u(0,y) + u(0,y)x + L_x^{-1} L_y u(x,y) + L_x^{-1}(u(x,y)) - L_x^{-1} u(x,y)^2 \end{aligned} \tag{3.2.29}$$

since the Adomian solution has a series form

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y).$$

Equation (3.2.29) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,y) = u(0,y) + u(0,y)x + L_x^{-1} L_y \sum_{n=0}^{\infty} u_n(x,y) + L_x^{-1}(\sum_{n=0}^{\infty} u_n(x,y)) \\ - L_x^{-1}(\sum_{n=0}^{\infty} A_n) \end{aligned} \tag{3.2.30}$$

where $A_n = u(x,y)^2$ is the Adomian polynomial. Hence, the solution is

$$\begin{aligned} u_0 + u_1 + u_2 + ... = u(0,y) + u(0,y)x + L_x^{-1} L_y (u_0 + u_1 + u_2 + ...) \\ + L_x^{-1}(u_0 + u_1 + u_2 + ...) - L_x^{-1}(A_0 + A_1 + A_2 + ...). \end{aligned} \tag{3.2.31}$$

# 4 Symmetries of the Black Scholes equation

## 4.1 Lie symmetries

The study of Lie symmetries is a branch within the Lie group theory. The Lie Group theory is a mathematical theory developed in the nineteenth century by Sophus Lie to study the solutions and group symmetries of differential equations. We will start by defining certain terms that will be useful later in this section.

**Definition 4.1: Symmetry** ( *[4], Section 1*)
A symmetry of a differential equation is a transformation that transforms any solution of the differential equation to another solution.

We now discuss concept of symmetries. Symmetries in mathematics refers to any object that is invariant under various transformations, scaling, rotations and reflection.We have discrete and continuous symmetries. A basic equilateral triangle, is an example of an object with discrete symmetries, since they do not depend upon continuous parameters [21]. Consider an equilateral triangle with vertices $A, B$ and $C$, after rotations of $\frac{2\pi}{3}$, $\frac{2\pi}{3}$ and $2\pi$ about its center and some reflection through any one of the bisection axis, you will realise that such transformation leaves the triangle unchanged or "invariant" and we can conclude it is invariant under such transformation.
Another example of an object with continuous symmetries is a unit circle. A unit circle will remain invariant if it is rotated by any radians measure about its origin.

**Definition 4.2 : Lie point symmetry**
A Lie point symmetry is distinguished by an infinitesimal transformation that renders the specified differential equation invariant under the transformation of all independent and dependent variables.

**Definition 4.3: Commutator***( [25], Section 7.3.1)*
The commutator of any two operators $X_i$ and $X_j$ is the differential operator $[X_i, X_j]$ of the first order defined by

$$[X_i, X_j] = X_iX_j - X_jX_i,$$

or in the following equivalent form

$$[X_i, X_j] = \sum_{a=1}^{n} \Big( X_i(\xi_i^a) - X_j(\xi_j^a) \Big) \frac{\partial}{\partial x^a}.$$

It follows from the above definition that the commutator is bi-linear:

$$[c_1X_1 + c_2X_2, X] = c_1[X_1, X] + c_2[X_2, X],$$

$$[X, c_1X_1 + c_2X_2] = c_1[X, X_1] + c_2[X, X_2]$$

skew-symmetric:

$$[X_1, X_2] = -[X_2, X_1],$$

and also satisfies the Jacobi identity:

$$[X_1,[X_2,X_3]]+[X_2,[X_3,X_1]]+[X_3,[X_1,X_2]]=0.$$

**Definition 4.4: Lie Algebra of operators**. *( [25], Definition 7.3.1 )*
The Lie algebra is a vector space $L$ of operators $X=\xi^i(x)\dfrac{\partial}{\partial x^i}$ with the following property. If the operators

$$X_1=\xi_1^i(x)\frac{\partial}{\partial x^i},\qquad X_2=\xi_2^i\frac{\partial}{\partial x^i},$$

are elements of $L$, then their commutator

$$[X_1,X_2]\equiv X_1X_2-X_2X_1=(X_1(\xi_1^i)-X_2(\xi_2^i))\frac{\partial}{\partial x^i}$$

is also an element of L.

**Definition 4.5: Basis of the vector space***( [24], Section 1.1)*
Let $L_r$ be a finite dimension Lie algebra and suppose that $X_\alpha=\xi_\alpha^i(x)\frac{\partial}{\partial x^i}$ for $\alpha=1,..,r$ be a basis of a vector space $L_r$. In particular $[X_\alpha,X_\beta]\in L$, hence $[X_\alpha,X_\beta]=C^r_{\alpha\beta}X_r$ for $\alpha,\beta=1,...r$. The constant coefficients $C^r_{\alpha\beta}$ are called structure constants of the algebra $L_r$.

**Definition 4.6: A Local Group***( [25], Section 7.1.2)*
A set G of transformations $T_a\quad(\bar{x}=f(x,a))$ in $\mathbb{R}^2$ given by $\bar{x}^i=f^i(x,a),\quad i=1,...,n.$ is called a one parameter local group if there exists a sub-interval $U'\subset U$ containing $a_0$ such that the function $f^i(x,a)$ satisfy the composition rule

$$f^i(f(x,a),b)=f^i(x,c),\qquad i,...,n,$$

for all values $a,b,c\in U'$.

## 4.2 Construction of symmetries

In this section, we discuss the symmetries of partial differential equations. Since our aim is to identify the symmetries of the one-dimensional Black Scholes equation, we limit our research to second-order equation with $t$ and $x$ as independent variables and $u$ as the dependent variable. Consider a partial differential equation of second order

$$F(t,x,u,u_t,u_x,u_{tt},u_{xx},u_{tx})=0.$$

The infinitesimal generator of the one-parameter group of transformation is defined as follows,

$$X=\xi^1(x,t,u)\frac{\partial}{\partial x}+\xi^2(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u}.\tag{4.2.1}$$

Then we denote the first prolongation of (4.2.1) as $X^{[1]}$:

$$X^{[1]}=\xi^1(x,t,u)\frac{\partial}{\partial x}+\xi^2(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u}+\zeta_1\frac{\partial}{\partial u_x}+\zeta_2\frac{\partial}{\partial u_t}\tag{4.2.2}$$

and the second prolongation is denoted as $X^{[2]}$ and defined below as:

$$X^{[2]} = X^{[1]} + \zeta_{11}\frac{\partial}{\partial u_{xx}} + \zeta_{12}\frac{\partial}{\partial u_{xt}} + \zeta_{22}\frac{\partial}{\partial u_{tt}} \tag{4.2.3}$$

where $\zeta_1, \zeta_2, \zeta_{11}, \zeta_{12}, \zeta_{22}$ are given by

$$\begin{aligned} \zeta_1 &= D_x(\eta) - u_x D_x(\xi^1) - u_t D_x(\xi^2), \\ \zeta_2 &= D_t(\eta) - u_x D_y(\xi^1) - u_t D_t(\xi^2), \\ \zeta_{11} &= D_x(\zeta_1) - u_{xx} D_x(\xi^1) - u_{xt} D_x(\xi^2), \\ \zeta_{12} &= D_t(\zeta_1) - u_{xx} D_t(\xi^1) - u_{xt} D_t(\xi^2), \\ \zeta_{22} &= D_t(\zeta_2) - u_{xt} D_y(\xi^1) - u_{tt} D_t(\xi^2), \end{aligned} \tag{4.2.4}$$

where $D$ denotes the total derivatives of $x$ and $t$, represented as $D_x$ and $D_t$ *([25],p.217)*, that is,

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + ..., \\ D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + ... \end{aligned} \tag{4.2.5}$$

substituting equation (4.2.5) into equations(4.2.4) (also known as the prolongation formula's), one obtains the following:

$$\begin{aligned} \zeta_1 &= \eta_x + u_x\eta_u - u_x\xi^1_x - (u_x)^2\xi^1_u - u_t\xi^2_x - u_x u_t\xi^2_u, \\ \zeta_2 &= \eta_t + u_t\eta_u - u_x\xi^1_t - (u_t)^2\xi^2_u - u_t\xi^2_t - u_x u_t\xi^1_u, \\ \zeta_{11} &= \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + (u_x)^2\eta_{uu} - 2u_{xx}\xi^1_x - u_x\xi^1_{xx} - 2(u_x)^2\xi^1_{xu} \\ &\quad - 3u_x u_{xx}\xi^1_u - (u_x)^3\xi^1_{uu} - 2u_{xt}\xi^2_x - u_t\xi^2_{xx} - 2u_x u_t\xi^2_{xu} - (u_t u_{xx} \\ &\quad + 2u_x u_{xt})\xi^2_u - (u_x)^2 u_y\xi^2_{uu}, \\ \zeta_{12} &= \eta_{xt} + u_t\eta_{xu} + u_x\eta_{tu} + u_{tx}\eta_u + u_x u_t\eta_{uu} - u_{xt}(\xi^1_x + \xi^2_t) - u_x\xi^1_{xt} - u_{xx}\xi^1_t \\ &\quad - u_x u_t(\xi^1_{xu} + \xi^2_{tu}) - (u_x)^2\xi^1_{tu} - (2u_x u_{xt} + u_t u_{xx})\xi^1_u - (u_x)^2 u_t\xi^1_{uu} - u_t\xi^2_{xt} \\ &\quad - u_{tt}\xi^2_x - (u_t)^2\xi^2_{xu} - (2u_t u_{xt} + u_x u_{tt})\xi^2_u - u_x(u_t)^2\xi^2_{uu}, \\ \zeta_{22} &= \eta_{tt} + 2u_t\eta_{tu} + u_{tt}\eta_u + (u_t)^2\eta_{uu} - 2u_{tt}\xi^2_t - u_t\xi^2_{tt} - 2(u_t)^2\xi^2_{tu} \\ &\quad - 3u_t u_{tt}\xi^2_u - (u_t)^3\xi^2_{uu} - 2u_{xt}\xi^1_t - u_x\xi^1_{tt} - 2u_x u_t\xi^1_{tu} - (u_x u_{tt} \\ &\quad + 2u_t u_{xt})\xi^1_u - (u_t)^2 u_x\xi^1_{uu}. \end{aligned} \tag{4.2.6}$$

**Determining equation**

**Definition 4.7: Determining equation**
A determining equation is a linear system of partial differential equation with unknown $\xi$ and $\eta$ with variables $x$ and $t$.

## 4.3 Symmetries analysis of the Black Scholes equation

We now compute the Lie symmetries of the Black Scholes equation. Consider the Black Scholes equation in the following form:

$$u_t + \frac{1}{2}B^2x^2u_{xx} + Cxu_x - Du = 0, \tag{4.3.1}$$

where $B, C$ and $D$ are constants. The infinitesimal generator of the Black Scholes equation is as follows,

$$X = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,u)\frac{\partial}{\partial t} + \eta(t,x,u)\frac{\partial}{\partial u}$$

then the determining equation is given below as

$$\zeta_2 + \frac{1}{2}B^2x^2\zeta_{11} + \frac{1}{2}B^2x\xi^1u_{xx} + Cx\zeta_1 + C\xi^1u_x - \eta C = 0. \tag{4.3.2}$$

Substituting $\zeta_1, \zeta_{11}$ and $\zeta_2$ as defined in equation (4.3.2), we get the following determining equations:

$$\begin{aligned}
&\Big[\eta_t + u_t\eta_u - u_x\xi^1_t - (u_t)^2\xi^2_u - u_t\xi^2_t - u_xu_t\xi^1_u\Big] + \frac{1}{2}B^2x^2\Big[\eta_{xx} + 2u_x\eta_{xu} \\
&+ u_{xx}\eta_u + (u_x)^2\eta_{uu} - 2u_{xx}\xi^1_x - u_x\xi^1_{xx} - 2(u_x)^2\xi^1_{xu} - 3u_xu_{xx}\xi^1_u \\
&- (u_x)^3\xi^1_{uu} - 2u_{xt}\xi^2_x - u_t\xi^2_{xx} - 2u_xu_t\xi^2_{xu} - (u_tu_{xx} + 2u_xu_{xt})\xi^2_u \\
&- (u_x)^2u_y\xi^2_{uu}\Big] + \frac{1}{2}B^2x\xi^1u_{xx} + Cx\Big[\eta_x + u_x\eta_uu_x\xi^1_x - (u_x)^2\xi^1_u - u_t\xi^2_x \\
&- u_xu_y\xi^2_u\Big] + C\xi^1u_x - \eta C = 0.
\end{aligned} \tag{4.3.3}$$

The following are simplified determining equations derived from equation (4.3.3),

1. $B\xi^2_x = 0, \quad B\xi^2_u = 0, \quad B\xi^2_{uu} = 0, \quad B(-\xi^1_{uu} + Cx\xi^2_{uu}) = 0,$
2. $B(-2\xi^1_u + x(2(B^2+C)\xi^2_u + B^2x\xi^2_{xu})) = 0,$
3. $B(2(C-D)\xi^2_u + \eta_{uu} - Du\xi^2_{uu} - 2\xi^1_{xu} + 2Cx\xi^2_{xu} = 0,$
4. $B(4\xi^1 + x(2Du\xi^2_u + 2\xi^2_t - 4\xi^1_x + 4B^2x\xi^2_x + 6Cx\xi^2_x + Bx^2\xi^2_{xx})) = 0,$
5. $-2D\eta + 2D\eta_u - 2D^2u^2\xi^2_u + 2\eta_t - 2Du\xi^2_t + 2Dx\eta_x - 2CDux\xi^2_x + B^2x^2\eta_{xx}$ $-B^2Dux^2\xi^2_{xx} = 0,$
6. $2C\xi^1 - 2Du\xi^1_u + 2CDux\xi^2_u - 2\xi^1_t + 2Cx\xi^2_t - 2Cx\xi^1_x + 2B^2Cx^2\xi^2_x + 2C^2x^2\xi^2_x$ $- 2B^2Dx^2\xi^2_x + 2B^2x^2\eta_{xu} - 2B^2Dux^2\xi^2_{xu} - B^2x^2\xi^1_{xx} + B^2Cx^3\xi^2_{xx} = 0.$

$$. \tag{4.3.4}$$

We solve the determining equations with "B", "C" and "D" being non-zero. To obtain the coefficients of the infinitesimal generator of the Black Scholes equations equation;

$$\begin{aligned}
&\xi^1(x,t,u) = x(c_4+tc_5)+\frac{1}{2}x(c_2+2tc_3)lnx,\\
&\xi^2(x,t,u) = c_1+t(c_2+tc_3),\\
&\eta(x,t,u) = \frac{1}{8B^2}u(B^4t(c_2+tc_3)+4Ct(C(c_2+tc_3)-2c_5)-4B^2(tc_3+Ct(c_2\\
&+tc_3)-2Dt(c_2+tc_3)-tc_5-2c_6)+2(B^2(c_2+2tc_3)-2C(c_2+2tc_3)+4c_5)\\
&lnx+4c_3lnx^2)+\mathcal{F}_1(x,t)),
\end{aligned} \tag{4.3.5}$$

under the following constraint

$$-2D\mathcal{F}_1(x,t)+2\mathcal{F}_{1,t}+2Cx\mathcal{F}_{1,x}+B^2x^2\mathcal{F}_{1,xx}=0,$$

with $c_1, c_2, c_3, c_4, c_5$ and $c_6$ as constants. We consider the following symmetries of equation (4.3.1) obtained using Symbolic package [14];

$$\begin{aligned}
&X_1=\frac{\partial}{\partial t}, \qquad X_2 = x\frac{\partial}{\partial x},\\
&X_3 = u((B^4+4C^2-4B^2(C-2D))t+2(B^2-2C)\ln x)\frac{\partial}{\partial u}+4B^2x\ln x\frac{\partial}{\partial x}+8B^2t\frac{\partial}{\partial t},\\
&X_4 = u((B^2-2C)t+2\ln x)\frac{\partial}{\partial u}+2B^2tx\frac{\partial}{\partial x},\\
&X_5 = u(t(B^4t+4C^2t-4B^2(1+Ct-2Dt))+4(B^2-2C)t\ln x+4(\ln x)^2)\frac{\partial}{\partial u}\\
&\quad +8B^2t^2\frac{\partial}{\partial t}+8B^2tx\ln x\frac{\partial}{\partial x},\\
&X_6 = u\frac{\partial}{\partial u}, \qquad X_{\mathcal{F}_\infty} = \mathcal{F}_1.
\end{aligned} \tag{4.3.6}$$

We now consider a known basis of the same Lie Algebra of the Black Scholes symmetries as derived in ( *[17], Section 3.1*),

$$\begin{aligned}
&Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = x\frac{\partial}{\partial x},\\
&Y_3 = 2t\frac{\partial}{\partial t}+(\ln x+\mathcal{D}t)x\frac{\partial}{\partial x}+2Dtu\frac{\partial}{\partial u},\\
&Y_4 = B^2tx\frac{\partial}{\partial x}+(\ln x-\mathcal{D}t)u\frac{\partial}{\partial u},\\
&Y_5 = 2B^2t^2\frac{\partial}{\partial t}+2B^2tx\ln x\frac{\partial}{\partial x}+((\ln x-\mathcal{D}t)^2+2B^2Ct^2-B^2t)u\frac{\partial}{\partial u},\\
&Y_6 = u\frac{\partial}{\partial u}, \quad Y_\phi = \phi(t,x)\frac{\partial}{\partial u}.
\end{aligned} \tag{4.3.7}$$

where $\mathcal{D} = C - \frac{(B^2)}{2}$. We note from our discussion that the basis for Lie algebra of symmetries of Black Scholes are unique. However, each symmetry of the basis can be written as a linear combination of the other basis which means that they span the same Lie algebra.

$$\begin{aligned}
Y_1 &= X_1\\
Y_2 &= X_2\\
Y_3 &= \frac{1}{4B^2}X_3 + \frac{\mathcal{D}}{2B^2}\\
Y_4 &= \frac{B^2}{2}X_4\\
Y_5 &= \frac{B^2}{4}X_5\\
Y_6 &= X_6\\
Y_\phi &= \mathcal{F}_\infty.
\end{aligned}$$

## 4.4 Commutators

We have already seen that one set of symmetries can be written as a linear combination of the other, which means they span the same Lie algebra. We compute the commutators of the set of operators (4.3.6) and (4.3.7) to further show that the symmetries produced by the Symbolic package and those derived in the article [17] span the same Lie algebra.

We first consider the commutators of the operators (4.3.7),

1. The commutator between $Y_1$ and $Y_1$ is;

$$\begin{aligned}
[Y_1, Y_1] &= Y_1Y_1 - YY_1\\
&= \Big(\frac{\partial}{\partial t}\frac{\partial}{\partial t}\Big) - \Big(\frac{\partial}{\partial t}\frac{\partial}{\partial t}\Big)\\
&= 0.
\end{aligned}$$

2. The commutator between $Y_1$ and $Y_2$ is;

$$\begin{aligned}
[Y_1, Y_2] &= Y_1Y_2 - Y_2Y_1\\
&= \Big(\frac{\partial}{\partial t}x\frac{\partial}{\partial x}\Big) - \Big(x\frac{\partial}{\partial x}\frac{\partial}{\partial t}\Big)\\
&= 0.
\end{aligned}$$

3. The commutator between $Y_1$ and $Y_3$ is;

$$\begin{aligned}
[Y_1, Y_3] &= Y_1Y_3 - Y_3Y_1\\
&= \Big(\frac{\partial}{\partial t}\Big(2t\frac{\partial}{\partial t} + (\ln Y + \mathcal{D}t)x\frac{\partial}{\partial x} + 2\mathcal{D}tu\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x}\\
&\quad + 2\mathcal{D}tu\frac{\partial}{\partial u}\Big)\frac{\partial}{\partial t}\Big)\\
&= 2\frac{\partial}{\partial t} + \mathcal{D}x\frac{\partial}{\partial x} + 2\mathcal{D}u\frac{\partial}{\partial u}\\
&= 2Y_1 + \mathcal{D}Y_2 + 2\mathcal{D}Y_6.
\end{aligned}$$

4. The commutator between $Y_1$ and $Y_4$ is;

$$
\begin{aligned}
[Y_1, Y_4] &= Y_1Y_4 - Y_4Y_1 \\
&= \Big(\frac{\partial}{\partial t}\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\frac{\partial}{\partial t}\Big) \\
&= B^2x\frac{\partial}{\partial x} - \mathcal{D}u\frac{\partial}{\partial u} \\
&= B^2Y_2 - \mathcal{D}Y_6.
\end{aligned}
$$

5. The commutator between $Y_1$ and $Y_5$ is;

$$
\begin{aligned}
[Y_1, Y_5] &= Y_1Y_5 - Y_5Y_1 \\
&= \Big(\frac{\partial}{\partial t}\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Ct^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big) \\
&- \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Ct^2 - B^2t)u\frac{\partial}{\partial u}\Big)\frac{\partial}{\partial t}\Big) \\
&= 4B^2t\frac{\partial}{\partial t} + 2B^2x\ln x\frac{\partial}{\partial x} + 2(\ln x - \mathcal{D}t)(-\mathcal{D})u\frac{\partial}{\partial u} + (4B^2Ct)u\frac{\partial}{\partial u} - B^2u\frac{\partial}{\partial u} \\
&= 2B^2\Big(2t\frac{\partial}{\partial t} + x\ln x\frac{\partial}{\partial x} + \mathcal{D}tx\frac{\partial}{\partial x} + 2Ctu\frac{\partial}{\partial u}\Big) - 2\mathcal{D}\Big(\Big(\ln x - \mathcal{D}t\Big)u\frac{\partial}{\partial u} \\
&+ B^2tx\frac{\partial}{\partial x}\Big) - 2B^2u\frac{\partial}{\partial u} \\
&= 2B^2Y_3 - 2\mathcal{D}Y_4 - 2B^2Y_6.
\end{aligned}
$$

6. The commutator between $Y_1$ and $Y_6$ is;

$$
\begin{aligned}
[Y_1, Y_6] &= Y_1Y_6 - Y_6Y_1 \\
&= (\frac{\partial}{\partial t}u\frac{\partial}{\partial u}) - (u\frac{\partial}{\partial u}\frac{\partial}{\partial t}) \\
&= 0.
\end{aligned}
$$

7. The commutator between $Y_1$ and $Y_7$ is;

$$
\begin{aligned}
[Y_1, Y_7] &= Y_1Y_7 - Y_7Y_1 \\
&= \Big(\frac{\partial}{\partial t}\phi(t,x)\frac{\partial}{\partial u}\Big) - \Big(\phi(t,x)\frac{\partial}{\partial u}\frac{\partial}{\partial t}\Big) \\
&= \phi(t,x)_t - 0 \\
&= \phi(t,x)_t.
\end{aligned}
$$

8. The commutator between $Y_2$ and $Y_1$ is;

$$
\begin{aligned}
[Y_2, Y_1] &= Y_2Y_1 - Y_1Y_2 \\
&= (x\frac{\partial}{\partial x}\frac{\partial}{\partial t}) - (\frac{\partial}{\partial t}\frac{\partial}{\partial x}) \\
&= 0.
\end{aligned}
$$

9. The commutator between $Y_2$ and $Y_2$ is;

$$\begin{aligned}[Y_2, Y_2] &= Y_2Y_2 - Y_2Y_2\\ &= (Y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}) - (x\frac{\partial}{\partial x}x\frac{\partial}{\partial x})\\ &= 0.\end{aligned}$$

10. The commutator between $Y_2$ and $Y_3$ is;

$$\begin{aligned}[Y_2, Y_3] &= Y_2Y_3 - Y_3Y_2\\ &= \left(x\frac{\partial}{\partial x}\left(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\right)\right) - \left(\left(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x}\right.\right.\\ &\left.\left.+ 2Dtu\frac{\partial}{\partial u}\right)x\frac{\partial}{\partial x}\right)\\ &= 0 + x(\ln x + 1)\frac{\partial}{\partial x} + x\mathcal{D}t\frac{\partial}{\partial x} + 0 - 0 - (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} - 0\\ &= x\ln x\frac{\partial}{\partial x} + x\frac{\partial}{\partial x} + x\mathcal{D}t\frac{\partial}{\partial x} - x\ln x\frac{\partial}{\partial x} - x\mathcal{D}t\frac{\partial}{\partial x}\\ &= x\frac{\partial}{\partial x} = Y_2.\end{aligned}$$

11. The commutator between $Y_2$ and $Y_4$ is;

$$\begin{aligned}[Y_2, Y_4] &= Y_2Y_4 - Y_4Y_2\\ &= \left(x\frac{\partial}{\partial x}\left(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\right)\right) - \left(\left(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\right)x\frac{\partial}{\partial x}\right)\\ &= xB^2t\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} - 0 - B^2tx\frac{\partial}{\partial x} - 0\\ &= u\frac{\partial}{\partial u} = Y_6.\end{aligned}$$

12. The commutator between $Y_2$ and $Y_5$ is;

$$\begin{aligned}[Y_2, Y_5] &= Y_2Y_5 - Y_5Y_2\\ &= \left(x\frac{\partial}{\partial x}\left(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\right)\right)\\ &- \left(\left(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\right)x\frac{\partial}{\partial x}\right)\\ &= x2B^2t\ln x\frac{\partial}{\partial x} + x2B^2t\frac{\partial}{\partial x} + 2(\ln x - \mathcal{D}t)u\frac{\partial}{\partial u} - 2B^2tx\ln x\frac{\partial}{\partial x}\\ &= x2B^2t\frac{\partial}{\partial x} + 2(\ln x - \mathcal{D}t)u\frac{\partial}{\partial u} = 2(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u})\\ &= 2Y_4.\end{aligned}$$

13. The commutator between $Y_2$ and $Y_6$ is;

$$\begin{aligned}[Y_2, Y_6] &= Y_2Y_6 - Y_6Y_2\\ &= \left(x\frac{\partial}{\partial x}u\frac{\partial}{\partial u}\right) - (u\frac{\partial}{\partial u}x\frac{\partial}{\partial x})\\ &= 0.\end{aligned}$$

14. The commutator between $Y_2$ and $Y_7$ is;

$$\begin{aligned}[Y_2, Y_7] &= Y_2Y_7 - Y_7Y_2 \\ &= (x\frac{\partial}{\partial x}\phi(t,x)\frac{\partial}{\partial u}) - (\phi(t,x)\frac{\partial}{\partial u}x\frac{\partial}{\partial x}) \\ &= x\phi(t,x)_x\frac{\partial}{\partial u}.\end{aligned}$$

15. The commutator between $Y_3$ and $Y_1$ is;

$$\begin{aligned}[Y_3, Y_1] &= Y_3Y_1 - Y_1Y_3 \\ &= \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\frac{\partial}{\partial t}\Big) - \Big(\frac{\partial}{\partial t}\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t) \\ &x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big) \\ &= -2\frac{\partial}{\partial t} - \mathcal{D}x\frac{\partial}{\partial x} - 2Du\frac{\partial}{\partial u} \\ &= -2Y_1 - \mathcal{D}Y_2 - 2DY_6.\end{aligned}$$

16. The commutator between $Y_3$ and $Y_2$ is;

$$\begin{aligned}[Y_3, Y_2] &= Y_3Y_2 - Y_2Y_3 \\ &= \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)x\frac{\partial}{\partial x}\Big) - \Big(x\frac{\partial}{\partial x}\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t) \\ &x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big) \\ &= 0 + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 0 - 0 - x\ln x\frac{\partial}{\partial x} - x\frac{\partial}{\partial x} - x\mathcal{D}t\frac{\partial}{\partial x} - 0 \\ &= -Y_2.\end{aligned}$$

17. The commutator between $Y_3$ and $Y_3$ is;

$$\begin{aligned}[Y_3, Y_3] &= Y_3Y_3 - Y_3Y_3 \\ &= \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big) \\ &- \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big) \\ &= 0.\end{aligned}$$

18. The commutator between $Y_3$ and $Y_4$ is;

$$
\begin{aligned}
[Y_3, Y_4] &= Y_3Y_4 - Y_4Y_3 \\
&= \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big) \\
&- \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big) \\
&= 2tB^2x\frac{\partial}{\partial x} - 2t\mathcal{D}u\frac{\partial}{\partial u} + (\ln x + \mathcal{D}t)xB^2t\frac{\partial}{\partial x} + (\ln x + \mathcal{D}t)u\frac{\partial}{\partial u} + 0 \\
&+ 2Dtu(\ln x - \mathcal{D}t)\frac{\partial}{\partial u} - B^2tx\ln x\frac{\partial}{\partial x} - B^2tx\frac{\partial}{\partial x} - B^2tx\mathcal{D}t\frac{\partial}{\partial x} - \\
&- (\ln x - \mathcal{D}t)u2Dt\frac{\partial}{\partial u} \\
&= B^2tx\frac{\partial}{\partial x} - 2t\mathcal{D}u\frac{\partial}{\partial u} + (\ln x + \mathcal{D}t)u\frac{\partial}{\partial u} \\
&= B^2tx\frac{\partial}{\partial x} + \ln xu\frac{\partial}{\partial u} - \mathcal{D}tu\frac{\partial}{\partial u} = Y_4.
\end{aligned}
$$

19. The commutator between $Y_3$ and $Y_5$ is;

$$
\begin{aligned}
[Y_3, Y_5] &= Y_3Y_5 - Y_5Y_3 \\
&= \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 \\
&+ 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 \\
&- B^2t)u\frac{\partial}{\partial u}\Big)\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big) \\
&= 8t^2B^2\frac{\partial}{\partial t} + 4tB^2x\ln x\frac{\partial}{\partial x} - 4\mathcal{D}t\ln xu\frac{\partial}{\partial u} + 4(\mathcal{D}t)^2u\frac{\partial}{\partial u} + 8t^2B^2Du\frac{\partial}{\partial u} - 2tB^2u\frac{\partial}{\partial u} \\
&+ 2B^2tx(\ln x)^2\frac{\partial}{\partial x} + 2B^2t^2x\ln x\mathcal{D} + 2B^2t(\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2((\ln x)^2 - (\mathcal{D}t)^2)u\frac{\partial}{\partial u} \\
&+ 2Dtu((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\frac{\partial}{\partial u} - 4B^2t^2\frac{\partial}{\partial t} - 2B^2t^2\mathcal{D}x\ln x\frac{\partial}{\partial x} \\
&- 8B^2t^2Du\frac{\partial}{\partial u} - 2B^2tx(\ln x)^2\frac{\partial}{\partial x} - 2B^2tx\ln x\frac{\partial}{\partial x} - 2B^2t^2x\ln x\mathcal{D}\frac{\partial}{\partial x} \\
&- 2Dtu((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\frac{\partial}{\partial u} \\
&= 4B^2t^2\frac{\partial}{\partial t} + 4B^2tx\ln x\frac{\partial}{\partial x} + \Big(2(\ln x)^2 - 4\mathcal{D}t\ln x + 2(\mathcal{D}t)^2\Big)u\frac{\partial}{\partial u} \\
&= 2Y_5.
\end{aligned}
$$

20. The commutator between $Y_3$ and $Y_6$ is;

$$\begin{aligned}[Y_3, Y_6] &= Y_3Y_6 - Y_6Y_3\\ &= \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big(u\frac{\partial}{\partial u}\Big)\Big)\\ &- \Big(u\frac{\partial}{\partial u}\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big)\\ &= 0 + 0 + 2Ctu\frac{\partial}{\partial u} - 0 - 0 - 0 - 2Dtu\frac{\partial}{\partial u}\\ &= 0.\end{aligned}$$

21. The commutator between $Y_3$ and $Y_7$ is;

$$\begin{aligned}[Y_3, Y_7] &= Y_3Y_7 - Y_7Y_3\\ &= \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\phi(t,x)\frac{\partial}{\partial u}\Big) - \Big(\phi(t,x)\frac{\partial}{\partial u}\Big(2t\frac{\partial}{\partial t}\\ &+ (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big)\\ &= 2t\phi(t,x)_t\frac{\partial}{\partial u} + (\ln x + \mathcal{D}t)x\phi(t,x)_x\frac{\partial}{\partial u} - 2\phi(t,x)Dt\frac{\partial}{\partial u}.\end{aligned}$$

22. The commutator between $Y_4$ and $Y_1$ is;

$$\begin{aligned}[Y_4, Y_1] &= Y_4Y_1 - Y_1Y_4\\ &= \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\frac{\partial}{\partial t}\Big) - \Big(\frac{\partial}{\partial t}\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big)\\ &= 0 + 0 - B^2x\frac{\partial}{\partial x} + 0 - \mathcal{D}u\frac{\partial}{\partial u}\\ &= -B^2Y_2 - \mathcal{D}Y_6.\end{aligned}$$

23. The commutator between $Y_4$ and $Y_2$ is;

$$\begin{aligned}[Y_4, Y_2] &= Y_4Y_2 - Y_2Y_4\\ &= \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)x\frac{\partial}{\partial x}\Big) - \Big(x\frac{\partial}{\partial x}\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big)\\ &= B^2tx\frac{\partial}{\partial x} + 0 - xB^2t\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}\\ &= -u\frac{\partial}{\partial u} = -Y_6.\end{aligned}$$

24. The commutator between $Y_4$ and $Y_3$ is;

$$\begin{aligned}
[Y_4, Y_3] &= Y_4Y_3 - Y_3Y_4 \\
&= \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big) \\
&- \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big) \\
&= 0 + B^2tx\ln x\frac{\partial}{\partial x} + B^2tx\frac{\partial}{\partial x} + B^2tx\mathcal{D}t\frac{\partial}{\partial x} + 0 + 0 + 0 \\
&+ (\ln x - \mathcal{D}t)u2Dt\frac{\partial}{\partial u} - 2tB^2x\frac{\partial}{\partial x} - 2t\mathcal{D}u\frac{\partial}{\partial u} - (\ln x + \mathcal{D}t)xB^2t\frac{\partial}{\partial x} \\
&- (\ln x + \mathcal{D}t)u\frac{\partial}{\partial u} - 0 - 2Dtu(\ln x - \mathcal{D}t)\frac{\partial}{\partial u} \\
&= -Y_4.
\end{aligned}$$

25. The commutator between $Y_4$ and $Y_4$ is;

$$\begin{aligned}
[Y_4, Y_4] &= Y_4Y_4 - Y_4Y_4 \\
&= \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big) \\
&- \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big) \\
&= 0.
\end{aligned}$$

26. The commutator between $Y_4$ and $Y_5$ is;

$$\begin{aligned}
[Y_4, Y_5] &= Y_4Y_5 - Y_5Y_4 \\
&= \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x \\
&- \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x \\
&- \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big) \\
&= 2B^4t^2x\ln x\frac{\partial}{\partial x} + 2B^4t^2x\frac{\partial}{\partial x} + 2B^2tu(\ln x - \mathcal{D}t)\frac{\partial}{\partial u} + (\ln x - \mathcal{D}t)u((\ln x - \mathcal{D}t)^2 \\
&+ 2B^2Dt^2 - B^2t)\frac{\partial}{\partial u} - 2B^4t^2x\frac{\partial}{\partial x} + 2B^2tu(\ln x - \mathcal{D})\frac{\partial}{\partial u} - 2B^4t^2x\ln x\frac{\partial}{\partial x} \\
&- ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)(\ln x - \mathcal{D}t)u\frac{\partial}{\partial u} \\
&= 0
\end{aligned}$$

27. The commutator between $Y_4$ and $Y_6$ is;

$$\begin{aligned}
[Y_4, Y_6] &= Y_4Y_6 - Y_6Y_4 \\
&= \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)u\frac{\partial}{\partial u}\Big) - \Big(u\frac{\partial}{\partial u}\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big) \\
&= 0 + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u} - u(\ln x - \mathcal{D}t)\frac{\partial}{\partial u} \\
&= 0
\end{aligned}$$

28. The commutator between $Y_4$ and $Y_7$ is;

$$\begin{aligned}[Y_4, Y_7] &= Y_4Y_7 - Y_7Y_4 \\ &= \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\phi(t,x)\frac{\partial}{\partial u}\Big) - \Big(\phi(t,x)\frac{\partial}{\partial u}\Big(B^2tx\frac{\partial}{\partial x} \\ &+ (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big) \\ &= B^2tx\phi(t,x)_x\frac{\partial}{\partial u} + 0 - 0 - \phi(t,x)(\ln x - \mathcal{D}t)\frac{\partial}{\partial u}\end{aligned}$$

29. The commutator between $Y_5$ and $Y_1$ is;

$$\begin{aligned}[Y_5, Y_1] &= Y_5Y_1 - Y_1Y_5 \\ &= \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\frac{\partial}{\partial t}\Big) \\ &- \Big(\frac{\partial}{\partial t}\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big) \\ &= -4B^2t\frac{\partial}{\partial t} - 2B^2x\ln x\frac{\partial}{\partial x} + 2\mathcal{D}\Big(\ln x - \mathcal{D}t\Big)u\frac{\partial}{\partial u} - 4B^2Dtu\frac{\partial}{\partial u} + B^2u\frac{\partial}{\partial u} \\ &= -2B^2\Big(2t\frac{\partial}{\partial t} + x\ln x\frac{\partial}{\partial x} + \mathcal{D}tx\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big) + 2\mathcal{D}\Big(\Big(\ln x - \mathcal{D}t\Big)u\frac{\partial}{\partial u} \\ &+ B^2tx\frac{\partial}{\partial x}\Big) + 2B^2u\frac{\partial}{\partial u} \\ &= -2B^2Y_3 + 2\mathcal{D}Y_4 + B^2Y_6\end{aligned}$$

30. The commutator between $Y_5$ and $Y_2$ is;

$$\begin{aligned}[Y_5, Y_2] &= Y_5Y_2 - Y_2Y_5 \\ &= \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big(x\frac{\partial}{\partial x}\Big)\Big) \\ &- \Big(\Big(x\frac{\partial}{\partial x}\Big)\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big) \\ &= -2Y_4\end{aligned}$$

31. The commutator between $Y_5$ and $Y_3$ is;

$$\begin{aligned}[Y_5, Y_3] &= Y_5Y_3 - Y_3Y_5 \\ &= \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big) \\ &\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big) \\ &\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big) \\ &= -4B^2t^2\frac{\partial}{\partial t} - 4B^2tx\ln x\frac{\partial}{\partial x} - \Big(2(\ln x)^2 - 4\mathcal{D}t\ln x + 2(\mathcal{D}t)^2\Big)u\frac{\partial}{\partial u} \\ &= -2Y_5\end{aligned}$$

32. The commutator between $Y_5$ and $Y_4$ is;

$$
\begin{aligned}
[Y_5, Y_4] &= Y_5Y_4 - Y_4Y_5 \\
&= \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big) \\
&\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big) \\
&\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big) \\
&= -2B^4t^2x\ln x\frac{\partial}{\partial x} - 2B^4t^2x\frac{\partial}{\partial x} - 2B^2tu(\ln x - \mathcal{D}t)\frac{\partial}{\partial u} - (\ln x - \mathcal{D}t) \\
&u((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)\frac{\partial}{\partial u} + 2B^4t^2x\frac{\partial}{\partial x} + 2B^2tu(\ln x - \mathcal{D}t)\frac{\partial}{\partial u} \\
&+ 2B^4t^2x\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)(\ln x - \mathcal{D}t)u\frac{\partial}{\partial u} \\
&= 0
\end{aligned}
$$

33. The commutator between $Y_5$ and $Y_5$ is;

$$
\begin{aligned}
[Y_5, Y_5] &= Y_5Y_5 - Y_5Y_5 \\
&= \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big) \\
&\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big) \\
&- \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big) \\
&\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big) \\
&= 0.
\end{aligned}
$$

34. The commutator between $Y_5$ and $Y_6$ is;

$$
\begin{aligned}
[Y_5, Y_6] &= Y_5Y_6 - Y_6Y_5 \\
&= \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\frac{\partial}{\partial t}\Big) \\
&- \Big(\frac{\partial}{\partial t}\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big) \\
&= 0.
\end{aligned}
$$

35. The commutator between $Y_5$ and $Y_7$ is;

$$\begin{aligned}
[Y_5, Y_7] &= Y_5Y_7 - Y_7Y_5 \\
&= \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big) \\
&\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 \\
&+ 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big) \\
&= -\phi(t,x)((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)\frac{\partial}{\partial u} + 2B^2t^2\phi(t,x)_t\frac{\partial}{\partial u} \\
&+ 2B^2tx\ln x\phi(t,x)_x\frac{\partial}{\partial u}.
\end{aligned}$$

36. The commutator between $Y_6$ and $Y_1$ is;

$$\begin{aligned}
[Y_6, Y_1] &= Y_6Y_1 - Y_1Y_6 \\
&= \Big(\Big(u\frac{\partial}{\partial u}\Big)\Big(\frac{\partial}{\partial t}\Big) - \Big(\Big(\frac{\partial}{\partial t}\Big)\Big(u\frac{\partial}{\partial u}\Big) \\
&= 0.
\end{aligned}$$

37. The commutator between $Y_6$ and $Y_2$ is;

$$\begin{aligned}
[Y_6, Y_2] &= Y_6Y_2 - Y_2Y_6 \\
&= \Big(\Big(u\frac{\partial}{\partial u}\Big)\Big(x\frac{\partial}{\partial x}\Big) - \Big(\Big(x\frac{\partial}{\partial x}\Big)\Big(u\frac{\partial}{\partial u}\Big) \\
&= 0.
\end{aligned}$$

38. The commutator between $Y_6$ and $Y_3$ is;

$$\begin{aligned}
[Y_6, Y_3] &= Y_6Y_3 - Y_3Y_6 \\
&= \Big(\Big(u\frac{\partial}{\partial u}\Big)\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big) \\
&- \Big(\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big(u\frac{\partial}{\partial u}\Big) \\
&= 0 + 0 + u2Ct\frac{\partial}{\partial u} - 0 - 0 - 2Dtu\frac{\partial}{\partial u} \\
&= 0.
\end{aligned}$$

39. The commutator between $Y_6$ and $Y_4$ is;

$$\begin{aligned}
[Y_6, Y_4] &= Y_6Y_4 - Y_4Y_6 \\
&= \Big(\Big(u\frac{\partial}{\partial u}\Big)\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big) \\
&- \Big(\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big(u\frac{\partial}{\partial u}\Big) \\
&= 0 + u(\ln x - \mathcal{D}t)\frac{\partial}{\partial u} - 0 - (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u} \\
&= 0.
\end{aligned}$$

40. The commutator between $Y_6$ and $Y_5$ is;

$$
\begin{aligned}
[Y_6, Y_5] &= Y_6Y_5 - Y_5Y_6 \\
&= \Big(\Big(u\frac{\partial}{\partial u}\Big)\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big) \\
&- \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)u\frac{\partial}{\partial u}\Big)\Big(u\frac{\partial}{\partial u}\Big) \\
&= u((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)\frac{\partial}{\partial u} - ((\ln x - \mathcal{D}t)^2 \\
&+ 2B^2Ct^2 - B^2t)u\frac{\partial}{\partial u} \\
&= 0.
\end{aligned}
$$

41. The commutator between $Y_6$ and $Y_6$ is;

$$
\begin{aligned}
[Y_6, Y_6] &= Y_6Y_6 - Y_6Y_6 \\
&= \Big(\Big(u\frac{\partial}{\partial u}\Big)\Big(u\frac{\partial}{\partial u}\Big) - \Big(\Big(u\frac{\partial}{\partial u}\Big)\Big(u\frac{\partial}{\partial u}\Big)\Big) \\
&= 0.
\end{aligned}
$$

42. The commutator between $Y_6$ and $Y_7$ is;

$$
\begin{aligned}
[Y_6, Y_7] &= Y_6Y_7 - Y_7Y_6 \\
&= \Big(\Big(u\frac{\partial}{\partial u}\Big)\Big(\phi(t,x)\frac{\partial}{\partial u}\Big) - \Big(\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big(u\frac{\partial}{\partial u}\Big) \\
&= 0 - \phi(t,x)\frac{\partial}{\partial u} = -Y_7.
\end{aligned}
$$

43. The commutator between $Y_7$ and $Y_1$ is;

$$
\begin{aligned}
[Y_7, Y_1] &= Y_7Y_1 - Y_1Y_7 \\
&= \Big(\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big(\frac{\partial}{\partial t}\Big)\Big) - \Big(\Big(\frac{\partial}{\partial t}\Big)\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big) \\
&= -\phi(t,x)_t\frac{\partial}{\partial u}.
\end{aligned}
$$

44. The commutator between $Y_7$ and $Y_2$ is;

$$
\begin{aligned}
[Y_7, Y_2] &= Y_7Y_2 - Y_2Y_7 \\
&= \Big(\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big(x\frac{\partial}{\partial x}\Big)\Big) - \Big(\Big(x\frac{\partial}{\partial x}\Big)\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big) \\
&= -x\phi(t,x)_x\frac{\partial}{\partial u}.
\end{aligned}
$$

45. The commutator between $Y_7$ and $Y_3$ is;

$$\begin{aligned}[Y_7, Y_3] &= Y_7Y_3 - Y_3Y_7 \\ &= \Big(\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big(2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Dtu\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(2t\frac{\partial}{\partial t} \\ &+ (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2dtu\frac{\partial}{\partial u}\Big)\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big) \\ &= \phi(t,x)2Dt\frac{\partial}{\partial u} - 2t\phi(t,x)_t\frac{\partial}{\partial u} - (\ln x + \mathcal{D}t)x\phi(t,x)_x\frac{\partial}{\partial u}.\end{aligned}$$

46. The commutator between $Y_7$ and $Y_4$ is;

$$\begin{aligned}[Y_7, Y_4] &= Y_7Y_4 - Y_4Y_7 \\ &= \Big(\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big(B^2tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(B^2tx\frac{\partial}{\partial x} \\ &+ (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}\Big)\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big) \\ &= \phi(t,x)(\ln x - \mathcal{D}t)\frac{\partial}{\partial u} - B^2tx\phi(t,x)_x\frac{\partial}{\partial u}.\end{aligned}$$

47. The commutator between $Y_7$ and $Y_5$ is;

$$\begin{aligned}[Y_7, Y_5] &= Y_7Y_5 - Y_5Y_7 \\ &= \Big(\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 \\ &- B^2t)u\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(2B^2t^2\frac{\partial}{\partial t} + 2B^2tx\ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 \\ &- B^2t)u\frac{\partial}{\partial u}\Big)\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big) \\ &= \phi(t,x)((\ln x - \mathcal{D}t)^2 + 2B^2Dt^2 - B^2t)\frac{\partial}{\partial u} - 2B^2t^2\phi(t,x)_t\frac{\partial}{\partial u} \\ &- 2B^2tx\ln x\phi(t,x)_x\frac{\partial}{\partial u}.\end{aligned}$$

48. The commutator between $Y_7$ and $Y_6$ is;

$$\begin{aligned}[Y_7, Y_6] &= Y_7Y_6 - Y_6Y_7 \\ &= \Big(\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big(u\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(u\frac{\partial}{\partial u}\Big)\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big) \\ &= \phi(t,x)\frac{\partial}{\partial u} = Y_7.\end{aligned}$$

49. The commutator between $Y_7$ and $Y_7$ is;

$$\begin{aligned}[Y_7, Y_7] &= Y_7Y_7 - Y_7Y_7 \\ &= \Big(\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big) - \Big(\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big(\phi(t,x)\frac{\partial}{\partial u}\Big)\Big) \\ &= 0.\end{aligned}$$

We present the commutators of the basis of the symmetries of Black Scholes equation, with two tables below, Table 1 and Table 2.

+

Table 1: Commutator table corresponding to operators (4.3.7)

| | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ | $Y_6$ | $Y_7$ |
|---|---|---|---|---|---|---|---|
| $Y_1$ | 0 | 0 | $2Y_1 + \mathcal{D}Y_2$ $+2DY_6$ | $B^2Y_2 - DY_6$ | $2B^2Y_3 - 2\mathcal{D}Y_4$ $-B^2Y_6$ | 0 | $\phi(t,x)_t\frac{\partial}{\partial u}$ |
| $Y_2$ | 0 | 0 | $Y_2$ | $Y_6$ | $2Y_4$ | 0 | $x\phi(t,x)_x\frac{\partial}{\partial u}$ |
| $Y_3$ | $-2Y_1 - \mathcal{D}Y_2$ $-2DY_6$ | $-Y_2$ | 0 | $Y_4$ | $2Y_5$ | 0 | $2t\phi(t,x)_t\frac{\partial}{\partial u}$ $+(\ln x + \mathcal{D}t)x$ $\phi(t,x)_x\frac{\partial}{\partial u} - 2\phi(t,x)$ $Ct\frac{\partial}{\partial u}$ |
| $Y_4$ | $-B^2Y_2 - \mathcal{D}Y_6$ | $-Y_6$ | $-Y_4$ | 0 | 0 | 0 | $B^2tx\phi(t,x)_x\frac{\partial}{\partial u}$ $-\phi(t,x)(\ln x$ $-\mathcal{D}t)\frac{\partial}{\partial u}$ |
| $Y_5$ | $-2B^2Y_3 + 2\mathcal{D}Y_4$ $+B^2Y_6$ | $-2Y_4$ | $-2Y_5$ | 0 | 0 | 0 | $-\phi(t,x)((\ln x-$ $\mathcal{D}t)^2 + 2B^2Ct^2$ $-A^2t)\frac{\partial}{\partial u} + 2B^2t^2$ $\phi(t,x)_t\frac{\partial}{\partial u}$ $+2B^2tx\ln x\phi(t,x)_x$ $\frac{\partial}{\partial u}$ |
| $Y_6$ | 0 | 0 | 0 | 0 | 0 | 0 | $-Y_7$ |
| $Y_7$ | $-\phi(t,x)_t\frac{\partial}{\partial u}$ | $-x\phi(t,x)_x\frac{\partial}{\partial u}$ | $\phi(t,x)2Dt\frac{\partial}{\partial u}$ $-2t\phi(t,x)_t\frac{\partial}{\partial u}$ $-(\ln x + \mathcal{D}t)$ $x\phi(t,x)_x\frac{\partial}{\partial u}$ | $\phi(t,x)(\ln x-$ $\mathcal{D}t)\frac{\partial}{\partial u}$ $-B^2tx\phi(t,x)_x\frac{\partial}{\partial u}$ | $\phi(t,x)((\ln x - \mathcal{D}t)^2$ $+2B^2Dt^2 - B^2t)\frac{\partial}{\partial u}$ $-2B^2t^2\phi(t,x)_t\frac{\partial}{\partial u}$ $-2B^2tx\ln x\phi(t,x)_x\frac{\partial}{\partial u}$ | $Y_7$ | 0 |

Table 2: Commutator table corresponding to operators (4.3.6)

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ |
|---|---|---|---|---|---|---|---|
| $X_1$ | 0 | 0 | $8B^2X_1+\Big(B^4 +4C^2-4B^2\Big(C -2D\Big)\Big)X_6$ | $2B^2X_2+\Big(B^2 -2C\Big)X_6$ | $2X_3 -4B^2X_6$ | 0 | $\mathcal{F}_{1,t}\frac{\partial}{\partial u}$ |
| $X_2$ | 0 | 0 | $4B^2X_2+2\Big(B^2 -2C\Big)X_6$ | $2X_6$ | $4X_4$ | 0 | $x\mathcal{F}_{1,x}\frac{\partial}{\partial u}$ |
| $X_3$ | $-8B^2X_1-\Big(B^4 +4C^2-4B^2 \Big(C-2D\Big)\Big)X_6$ | $-4B^2X_2-2 \Big(B^2-2C\Big)X_6$ | 0 | $4B^2X_4$ | $8B^2X_5$ | 0 | $\Big(\mathcal{F}_1\Big(-4C^2t+B^2 \Big(-1+4C-8D\Big)t -2\Big(B^2-2C\Big)\ln x\Big) +4B^2\Big(x\ln x\mathcal{F}_{1,x} +2t\mathcal{F}_{1,t}\Big)\Big)\frac{\partial}{\partial u}$ |
| $X_4$ | $-2B^2X_2- \Big(B^2-2C\Big)X_6$ | $-2X_6$ | $-4B^2X_4$ | 0 | 0 | 0 | $-\mathcal{F}_1\Big(\Big(B^2- 2C\Big)t+2\ln x\Big) +2B^2tx\mathcal{F}_{1,x}$ |
| $X_5$ | $-2X_3+4B^2X_6$ | $-4X_4$ | $-8B^2X_5$ | 0 | 0 | 0 | $\Big(-\mathcal{F}_1\Big(t\Big(B^4t 4C^2t-4B^2\Big(1+Ct -2Dt\Big)\Big)+4\Big(B^2-2C\Big) t\ln x+4\ln x^2\Big) +8B^2t\Big(x\ln x\mathcal{F}_{1,x}+t\mathcal{F}_{1,t}\Big)\Big)$ |
| $X_6$ | 0 | 0 | 0 | 0 | 0 | 0 | $-X_7$ |
| $X_7$ | $-\mathcal{F}_{1,t}\frac{\partial}{\partial u}$ | $-x\mathcal{F}_{1,x}\frac{\partial}{\partial u}$ | $\Big(\mathcal{F}_1\Big(4C^2t+B^2 Big(1-4C+8D\Big) t+2\Big(B^2-2C\Big) \ln x\Big)-4B^2\Big(x\ln x\mathcal{F}_{1,x} +2t\mathcal{F}_{1,t}\Big)\Big)\frac{\partial}{\partial u}$ | $\mathcal{F}_1\Big(\Big(B^2-2C\Big)t +2\ln x\Big)- 2B^2tx\mathcal{F}_{1,x}\frac{\partial}{\partial u}$ | $\Big(\mathcal{F}_1\Big(t\Big(B^4t+4C^2t -4B^2\Big(1+Ct-2Dt\Big)\Big) +4\Big(B^2-2C\Big) t\ln x+\ln x^2\Big) -8B^2t\Big(x\ln x\mathcal{F}_{1,x} +t\mathcal{F}_{1,t}\Big)\Big)\frac{\partial}{\partial u}$ | $X_7$ | 0 |

## 4.5 Symmetry transformations

In this section we use the solution of the Lie equations and Exponentiation to construct symmetry transformations of the infinitesimal generators (4.3.7).

### 4.5.1 Lie Equations

We discuss the theorem for solving the Lie equations.

**Theorem 3.** *( [25],Section 7.1.5)*
Let G be a local group and let $\bar{x}^i \approx x^i + a\xi^i(x), \quad i = 1, .., n$ be the infinitesimal transformation of the group G. The system of first-order ordinary differential equations (known as Lie equations) is then solved using the function $\bar{x}^i = f^i(x, a)$.

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}), i = 1, ..n \tag{4.5.1}$$

with the initial condition $\bar{x}^i|_{a=0} = x^i$, that is,

$$\frac{d\bar{x}^i}{da} = \xi(\bar{x}), \qquad \bar{x}|_{a=0} = x. \tag{4.5.2}$$

The corresponding transformations to generators (4.3.7) are:

1. For $Y_1 = \dfrac{\partial}{\partial t}$, we solve,

$$\frac{d\bar{t}}{da_1} = 1, \text{ with initial condition } \quad \bar{t}|_{a_1=0} = t, \tag{1}$$

$$\frac{d\bar{x}}{da_1} = 0, \text{ with initial condition } \quad \bar{x}|_{a_1=0} = x, \tag{2}$$

$$\frac{d\bar{u}}{da_1} = 0, \text{ with initial condition } \quad \bar{u}|_{a_1=0} = u. \tag{3}$$

   From equation (1), $\bar{t} = a_1 + c_1$, then we substitute the initial condition $\bar{t}|_{a_1=0} = t$, $t = 0 + c_1$ thus, $\bar{t} = a_1 + t$.
   From equation (2), $\bar{x} = c_2$, substituting the initial condition $\bar{x}|_{a_1=0} = x$, $x = c_2$ then, $\bar{x} = x$.
   From equation (3), $\bar{u} = c_3$, substituting the initial condition $\bar{u}|_{a_1=0} = u$, $u = c_3$, then, $\bar{u} = u$.

2. For $Y_2 = x\dfrac{\partial}{\partial x}$, we solve,

$$\frac{d\bar{t}}{da_2} = 0, \text{ with initial condition } \quad \bar{t}|_{a_2=0} = t, \tag{4}$$

$$\frac{d\bar{x}}{da_2} = \bar{x}, \text{ with initial condition } \quad \bar{x}|_{a_2=0} = x, \tag{5}$$

$$\frac{d\bar{u}}{da_2} = 0, \text{ with initial condition } \quad \bar{u}|_{a_2=0} = u. \tag{6}$$

From equation (4), $\bar{t} = c_4$, we then substitute the initial condition $\bar{t}|_{a_2=0}= t$, $t = c_4$ then, $\bar{t} = t$.
From equation (5), $\bar{x} = c_5 \exp\{a_2\}$, substituting the initial condition $\bar{x}|_{a_2=0}= x$, $x = c_5$ then, $\bar{x} = x \exp\{a_2\}$ .
From equation (6), $\bar{u} = c_6$, substituting the initial condition $\bar{u}|_{a_2=0}= u$, $c_6 = u$ then, $\bar{u} = u$.

3. For $Y_3 = 2t\dfrac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\dfrac{\partial}{\partial x} + 2Dtu\dfrac{\partial}{\partial u}$, we solve,

$$\frac{d\bar{t}}{da_3} = 2\bar{t}, \text{ with initial condition } \quad \bar{t}|_{a_3=0}= t, \tag{7}$$

$$\frac{d\bar{x}}{da_3} = \bar{x}\ln\bar{x} + \mathcal{D}\bar{t}\bar{x}, \text{ with initial condition } \quad \bar{x}|_{a_3=0}= x, \tag{8}$$

$$\frac{d\bar{u}}{da_3} = 2D\bar{t}\bar{u}, \text{ with initial condition } \quad \bar{u}|_{a_3=0}= u. \tag{9}$$

From equation (7), $\ln\bar{t} = 2a_3 + c_7, \bar{t} = c_7 \exp\{2a_3\}$, we then substitute the initial condition $\bar{t}|_{a_3=0}= t$, $t = c_7$ then, $\bar{t} = t \exp\{2a_3\}$.
From equation (8) and $\bar{t} = t\exp\{2a_3\}$ we let

$$v = \ln\bar{x} + \mathcal{D}t\exp\{2a_3\}$$
$$dv = \frac{1}{\bar{x}}d\bar{x},$$

hence, $\ln v = a_3 + c_8$,$v = c_8 \exp\{a_3\}$, and substituting back $v$ we have $\ln\bar{x}$ $+ \mathcal{D}t\exp\{2a_3\} = c_8\exp\{a_3\}$ substituting the initial condition $\bar{x}|_{a_3=0}= x$, $\ln x + \mathcal{D}t = c_8$ then,

$$\bar{x} = \exp\{(\ln x + \mathcal{D}t)\exp\{a_3\} - \mathcal{D}t\exp\{2a_3\}\}$$

From equation (9) and $\bar{t} = t\exp\{2a_3\}$ , we let

$$v = 2a_3$$
$$dv = 2da_3.$$

Hence $\bar{u} = c_9\exp\{(Dt\exp\{2a_3\}\})$, substituting the initial condition $\bar{u}|_{a_3=0}= u$, $u\exp\{-Dt\} = c_9$ then,

$$\bar{u} = u\exp\{Dt(\exp\{2a_3\} - 1)\}.$$

4. For $Y_4 = B^2tx\dfrac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\dfrac{\partial}{\partial u}$, we solve,

$$\frac{d\bar{t}}{da_4} = 0, \text{ with initial condition } \quad \bar{t}|_{a_4=0}= t, \tag{10}$$

$$\frac{d\bar{x}}{da_4} = B^2\bar{t}\bar{x}, \text{ with initial condition } \quad \bar{x}|_{a_4=0}= x, \tag{11}$$

$$\frac{d\bar{u}}{da_4} = (\ln\bar{x} - \mathcal{D}\bar{t})\bar{u}, \text{ with initial condition } \quad \bar{u}|_{a_4=0}= u. \tag{12}$$

From equation (10), $\bar{t} = c_{10}$, we then substitute the initial condition $\bar{t}|_{a_4=0} = t$, $t = c_{10}$ then, $\bar{t} = t$.
From equation (11) and $\bar{t} = t$, $\ln \bar{x} = B^2 t a_4 + c_{11}$, $\bar{x} = c_{11} \exp\{(B^2 t a_4)\}$, substituting the initial condition $\bar{x}|_{a_4=0} = x$, $x = c_{11}$ then,

$$\bar{x} = x \exp\big\{(B^2 t a_4)\big\},$$

From equation (12) and $\bar{t} = t$ and $\bar{x} = x \exp\{(B^2 t a_4)\}$, $\ln \bar{u} = \ln x a_4 + B^2 t a_4^2 - \mathcal{D} t a_4 + c_{12}$, $\bar{u} = c_{12} \exp\{\ln x a_4 + B^2 t a_4^2 - \mathcal{D} t a_4\}$, substituting the initial condition $\bar{u}|_{a_4=0} = u$, $u = c_{12}$ then,

$$\bar{u} = u \exp\big\{(\ln x a_4 + B^2 t a_4^2 - \mathcal{D} t a_4)\big\}.$$

5. For $Y_5 = 2B^2 t^2 \dfrac{\partial}{\partial t} + 2B^2 t x \ln x \dfrac{\partial}{\partial x} + ((\ln x - \mathcal{D})^2 + 2B^2 D t^2 - B^2 t) u \dfrac{\partial}{\partial u}$, we solve,

$$\frac{d\bar{t}}{da_5} = 2B^2 \bar{t}^2, \text{ with initial condition } \quad \bar{t}|_{a_5=0} = t, \tag{13}$$

$$\frac{d\bar{x}}{da_5} = 2B^2 \bar{t} \bar{x} \ln \bar{x}, \text{ with initial condition } \quad \bar{x}|_{a_5=0} = x, \tag{14}$$

$$\frac{d\bar{u}}{da_5} = ((\ln \bar{x} - \mathcal{D})^2 + 2B^2 D \bar{t}^2 - B^2 \bar{t})\bar{u}, \text{ with initial condition } \quad \bar{u}|_{a_5=0} = u. \tag{15}$$

From equation (13), $\bar{t}^{-1} = -2B^2 a_5 - c_{13}$, $\bar{t} = \dfrac{1}{-2B^2 a_5 - c_{13}}$, we then substitute the initial condition $\bar{t}|_{a_1=0} = t$, $t = \dfrac{1}{-c_{13}}$ then,

$$\bar{t} = \frac{t}{-2B^2 a_5 t + 1}.$$

From equation (14) and $\bar{t} = \dfrac{t}{-2B^2 a_5 t + 1}$, we let

$$v = \ln \bar{x}, \; u = -2B^2 t a_5 + 1,$$

$$dv = \frac{1}{\bar{x}} d\bar{x}, \; du = -2B^2 t da_5,$$

then $\ln \bar{x} = c_{14}\Big(\dfrac{1}{-2B^2 t a_5 + 1}\Big)$, $\bar{x} = \exp\Big\{c_{14} \dfrac{1}{-2B^2 t a_5 + 1}\Big\}$, substituting the initial condition $\bar{x}|_{a_5=0} = x$, $\ln x = c_{14}$, then,

$$\bar{x} = x^{\left(\frac{1}{-2B^2 t a_5 + 1}\right)}.$$

From equation (15) and $\bar{t} = \dfrac{t}{-2B^2 a_5 t + 1}$ and $\bar{x} = x^{\left(\frac{1}{-2B^2 t a_5 + 1}\right)}$,

$$\ln \bar{u} = \frac{-\ln x^2}{-2B^2 t(-2B^2 t a_5 + 1)} - \frac{2\mathcal{D} \ln x (\ln(-2B^2 t a_5 + 1))}{-2B^2 t} + \mathcal{D}^2 a_5 - \frac{Dt}{2B^2 t a_5 - 1}$$
$$- \frac{\ln -2B^2 t a_5 + 1}{-2} + c_{15}.$$

Thus,

$$\bar{u} = c_{15}\exp\left\{\frac{-\ln x^2}{-2B^2t(-2B^2ta_5+1)} - \frac{2\mathcal{D}\ln x(\ln(-2B^2ta_5+1))}{-2B^2t} + \mathcal{D}^2a_5\right\}$$
$$\times\exp\left\{-\frac{Dt}{2B^2ta_5-1} - \frac{\ln -2B^2ta_5+1}{-2}\right\}$$

substituting the initial condition $\bar{u}|_{a_5=0}= u$,

$$u = c_{15}\exp\left\{\frac{-\ln x^2}{-2B^2t} - Dt\right\},\ c_{15} = u\exp\left\{-\left(\frac{-\ln x^2}{-2B^2t} - Dt\right)\right\},$$

then

$$\bar{u} = u\exp\left\{-\left(\frac{-\ln x^2}{-2B^2t} - Dt\right)\right\}\exp\left\{\frac{-\ln x^2}{-2B^2t(-2B^2ta_5+1)} - \frac{2\mathcal{D}\ln x(\ln(-2B^2ta_5+1))}{-2B^2t}\right\}$$
$$\times\exp\left\{+\mathcal{D}^2a_5 - \frac{Dt}{2B^2ta_5-1} - \frac{\ln -2B^2ta_5+1}{-2}\right\} \tag{4.5.3}$$

6. For $Y_6 = u\dfrac{\partial}{\partial u}$, we solve,

$$\frac{d\bar{t}}{da_6} = 0,\ \text{with initial condition}\quad \bar{t}|_{a_6=0}= t, \tag{16}$$

$$\frac{d\bar{x}}{da_6} = 0,\ \text{with initial condition}\quad \bar{x}|_{a_6=0}= x, \tag{17}$$

$$\frac{d\bar{u}}{da_6} = \bar{u},\ \text{with initial condition}\quad \bar{u}|_{a_6=0}= u. \tag{18}$$

From equation (16), $\bar{t} = c_{16}$, we then substitute the initial condition $\bar{t}|_{a_6=0}= t, t = c_{16}$ then, $\bar{t} = t$.
From equation (17), $\bar{x} = c_{17}$, substituting the initial condition $\bar{x}|_{a_6=0}= x$, $x = c_{17}$ then, $\bar{x} = x$.
From equation (18), $\bar{u} = c_{18}\exp\{a_6\}$, substituting the initial condition $\bar{u}|_{a_6=0}= u$, $u = c_{18}$ then, $\bar{u} = u\exp\{a_6\}$.

7. For $Y_\phi = \phi(t,x)\dfrac{\partial}{\partial u}$, we solve,

$$\frac{d\bar{t}}{da_7} = 0,\ \text{with initial condition}\quad \bar{t}|_{a_\phi=0}= t, \tag{19}$$

$$\frac{d\bar{x}}{da_7} = 0,\ \text{with initial condition}\quad \bar{x}|_{a_\phi=0}= x, \tag{20}$$

$$\frac{d\bar{u}}{da_7} = \phi(\bar{t}, \bar{x}), \text{ with initial condition } \quad \bar{u}|_{a_\phi=0} = u \tag{21}$$

From equation (19), $\bar{t} = c_{19}$, we then substitute the initial condition $\bar{t}|_{a_1=0} = t, t = c_{19}$ then, $\bar{t} = t$.
From equation (20), $\bar{x} = c_{20}$, substituting the initial condition $\bar{x}|_{a_7=0} = x$, $x = c_{20}$ then, $\bar{x} = x$.
From equation (21) and $\bar{x} = x$ and $\bar{t} = t$, $\bar{u} = \phi(t, x)a + c_{21}$, substituting the initial condition $\bar{u}|_{a_7=0} = u, u = c_{21}$ then, $\bar{u} = \phi(t, x)a_7 + u$.

### 4.5.2 The Exponential Map

Given a generator $Y = \xi^i(x)\frac{\partial}{\partial x^i}$,one can obtain the group transformation in the form of an infinite series by using the exponential map

$$\bar{x}^i = \exp\{aY\}(x^i), \quad i = 1, ..., n \tag{4.5.4}$$

where

$$\exp\{aY\} = 1 + \frac{a}{1!}Y + \frac{a^2}{2!}Y^2 + \frac{a^3}{3!}Y^3 + ... + \frac{a^s u}{s!}Y^s + ...$$

Consider the generator $Y_1 = \frac{\partial}{\partial t}$, the corresponding group transformation using exponential map is given by

$$\bar{t} = \exp\{a_1 Y\}(t), \ \bar{x} = \exp\{a_1 Y\}(x) \ \bar{u} = \exp\{a_1 Y\}(u) \tag{4.5.5}$$

so we now calculate $Y^s(t)$ for $s = 1, 2, ...,$ $Y(t) = 1, \quad Y^2(t) = Y(Y(t))$
$= Y(1) = 0, \quad Y^3(0) = 0 \quad \cdots Y^s(0) = 0$
substituting the above expressions into the exponential map formula

$$\exp\{a_1 Y\}(t) = t + a_1(1) + a_1^2(0) + \cdots + a_1^s(0) + \cdots = t + a_1 \tag{4.5.6}$$

calculating $Y^s(x)$ and $Y^s(u)$ for $s = 1, 2, ...$ we get $\bar{x} = \exp\{a_1 Y\}(x) = x + 0 + ...$ and $\bar{u} = \exp\{aY\}(u) = u + 0 + ...$, then,

$$Y_1 : \ \bar{t} = t + a_1, \ \bar{x} = x, \ \bar{u} = u$$

For the generator $Y_2 = x\frac{\partial}{\partial x}$, the group transformation using exponential map is $\bar{x} = \exp\{a_2 Y\}(x)$. We compute $Y^s(x)$ for $s = 1, ...$ as follows

$$Y^1(x) = x, \quad Y^2(x) = Y(Y(x)) = Y(x) = x, \quad Y^3(x) = x, .., Y^s(x) = x, ..$$

Substituting all the expressions into the exponential map formula

$$\begin{aligned}
\exp\{a_2 X\}(x) &= x + \frac{a_2 x}{1!} + \frac{a_2^2 x}{2!} + \frac{a_2^3 x}{3!} + ... + \frac{a_2^s x}{s!} + ... \\
&= x(1 + \frac{a_2}{1!} + \frac{a_2^2}{2!} + \frac{a_2^3}{3!} + ... + \frac{a_2^s}{s!} + ...) \\
&= x \exp\{a_2\}
\end{aligned}$$

and now, the group transformation are given as

$$Y_2: \ \bar{x} = x\exp\{a_2\}, \ \bar{t} = t \ \bar{u} = u$$

Similarly ,we get

$$\begin{aligned}
&Y_3: \bar{x} = \exp\{\exp\{2a_3\}\mathcal{D}t + \exp\{a_3(-\mathcal{D}t + \ln x)\}\}, \quad \bar{t} = \exp\{2a_3\}t, \\
&\quad \bar{u} = u\exp\{-\mathcal{D}t + \mathcal{D}\exp\{2a_3 t\}\}; \\
&Y_4: \bar{x} = x\exp\Big\{B^2 t a_4\Big\}, \ \bar{t} = t, \ \bar{u} = u\exp\Bigg\{\Bigg(-\mathcal{D}t a_4 - \frac{(\ln x)^2}{2B^2 t} + \frac{(\ln x\exp\{B^2 t a_4\})^2}{2B^2 t}\Bigg)\Bigg\}; \\
&Y_5: \bar{x} = x^{\left(\frac{1}{-2B^2 t a_5 + 1}\right)}, \ \bar{t} = \frac{t}{-2B^2 a_5 t + 1}, \ \bar{u} = u\exp\Bigg(\ln(1 - 2B^2 t a_5)^{0.5} \\
&\quad + \frac{(\ln x)^2 a_5 - 2\ln x\mathcal{D}t a_5 + \mathcal{D}^2 t^2 a_5}{1 - 2B^2 t a_5} + \frac{2B^2 D t^2 a_5}{1 - 2B^2 t a_5}\Bigg); \\
&Y_6: \bar{x} = x, \ \bar{t} = t, \ \bar{u} = \exp\{a_6\}u; \\
&Y_\Phi: \bar{x} = x, \ \bar{t} = t, \ \bar{u} = u + a\phi(t, x).
\end{aligned}$$

The exponential map and the solution of Lie equations yields the same transformations of the generators.

# 5 Transformation of the Black Scholes into Heat equation.

## 5.1 Symmetry Analysis of the Heat equation

In this section we discuss the Lie symmetries of the Heat equation, which is represented by the following equation

$$u_t = u_{xx}. \tag{5.1.1}$$

The invariance condition also referred as the infinitesimal generator is given by:

$$X = \xi^1(x,t,u)\frac{\partial}{\partial t} + \xi^2(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u), \tag{5.1.2}$$

the coeficient $\xi$ and $\eta$ that depends on $t, x$ and $u$ will be found from following the determining equation,

$$X(u_t - u_{xx})|_{u_t=u_{xx}} = \zeta_1 - \zeta_{22} = 0.$$

We expand the above equation by substituting the prolongation formulas (see equation 4.2.4) we get:

$$\begin{aligned} &\eta_x + u_x\eta_u - u_x\xi^1_x - (u_x)^2\xi^1_u - u_t\xi^2_x - u_xu_t\xi^2_u - (\eta_{tt} + 2u_t\eta_{tu} + u_{tt}\eta_u + (u_t)^2\eta_{uu} \\ &\quad - 2u_{tt}\xi^2_t - u_t\xi^2_{tt} - 2(u_t)^2\xi^2_{tu} - 3u_tu_{tt}\xi^2_u - (u_t)^3\xi^2_{uu} - 2u_{xt}\xi^1_t - u_x\xi^1_{tt} - 2u_xu_t\xi^1_{tu} \\ &\quad - (u_xu_{tt} + 2u_tu_{xt})\xi^1_u - (u_t)^2u_x\xi^1_{uu}) = 0. \end{aligned} \tag{5.1.3}$$

Thereafter, we find the following basis of the Lie algebra of symmetries of the Heat equation.

$$\begin{aligned} &P_1 = \frac{\partial}{\partial x}, \quad P_2 = \frac{\partial}{\partial t} \quad P_3 = u\frac{\partial}{\partial u} \\ &P_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \\ &P_5 = 2t\frac{\partial}{\partial x} - ux\frac{\partial}{\partial u} \\ &P_6 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - 2tu\frac{\partial}{\partial u} - x^2u\frac{\partial}{\partial u} \\ &P_{\mathcal{F}} = \mathcal{F}_1(t,x)\frac{\partial}{\partial u} \quad \text{given that} \quad \mathcal{F}_{1,xx} - \mathcal{F}_{1,t} = 0 \quad \mathcal{F}_1(t,x)\frac{\partial}{\partial u} \end{aligned}$$

with $\mathcal{F}_1$ as an arbitrary solution of the heat equation.

## 5.2 Transforming of the Black Scholes equation to the Heat equation

Any parabolic equations that admits the symmetry group of their highest order can be reduced to the heat equation, [26]. We present the reduction the Black Scholes problem to the heat equation since it is also a parabolic equation.

Consider the Black Scoles equation in the following form,

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 u}{\partial x^2}+rx\frac{\partial u}{\partial x}-ru=0, \tag{5.2.1}$$

$$\text{with} \quad 0\leq x\leq\infty, \quad 0\leq t\leq T$$

where $x$ is the value of the underlying asset, $\sigma$ represent the volatility (we assume a constant volatility through out the transformation) and r is a risk-less interest rate. The boundary condition of $u$ (the price of a call option) is given by,

$$u(x,T)=f(x)=max(x-K,0), \tag{5.2.2}$$

where $K$ is the strike price of the call option. We consider the following transformations of the BS to the heat equation,

$$x=e^y, \quad t=T-\frac{2\tau}{\sigma^2}, \tag{5.2.3}$$

$$u(x,t)=v(y,\tau) \quad \text{with} \quad y=\ln x \quad \text{and} \quad \tau=\frac{\sigma^2}{2}(T-t).$$

The corresponding derivatives of equation (5.2.1) with respect to the new variables

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial\tau}\frac{\partial\tau}{\partial t}=-\frac{\sigma^2}{2}\frac{\partial v}{\partial\tau} \tag{5.2.4}$$

and

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial v}{\partial y}\frac{\partial y}{\partial x}=\frac{1}{x}\frac{\partial v}{\partial y}\\ \frac{\partial^2 u}{\partial x^2}&=\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x}\Big)\\ &=\frac{\partial}{\partial x}\Big(\frac{1}{x}\frac{\partial v}{\partial y}\Big)\\ &=-\frac{1}{x^2}\frac{\partial v}{\partial y}+\frac{1}{x}\frac{\partial}{\partial x}\frac{\partial y}{\partial x}\frac{\partial v}{\partial y}\\ &=-\frac{1}{x^2}\frac{\partial v}{\partial y}+\frac{1}{x^2}\frac{\partial}{\partial y}\frac{\partial v}{\partial y}\\ &=-\frac{1}{x^2}\frac{\partial v}{\partial y}+\frac{1}{x^2}\frac{\partial^2}{\partial y^2}\end{aligned} \tag{5.2.5}$$

substituting equation (5.2.4) and (5.2.5) into (5.2.1)

$$\begin{aligned}-\frac{\sigma^2}{2}\frac{\partial v}{\partial\tau}+\frac{1}{2}\sigma^2x^2\Big(-\frac{1}{x^2}\frac{\partial v}{\partial y}+\frac{1}{x^2}\frac{\partial^2}{\partial y^2}\Big)+rx\Big(\frac{1}{x}\frac{\partial v}{\partial y}\Big)-rv&=0\\ -\frac{\sigma^2}{2}\frac{\partial v}{\partial\tau}+\frac{\sigma^2}{2}\Big(-\frac{\partial v}{\partial y}+\frac{\partial^2}{\partial y^2}\Big)+r\frac{\partial v}{\partial y}-rv&=0\\ -\frac{\sigma^2}{2}\frac{\partial v}{\partial\tau}-\frac{\sigma^2}{2}\frac{\partial v}{\partial y}+\frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}+r\frac{\partial v}{\partial y}-rv&=0\\ -\frac{\sigma^2}{2}\frac{\partial v}{\partial\tau}+\Big(-\frac{\sigma^2}{2}+r\Big)\frac{\partial v}{\partial y}+\frac{\sigma^2}{2}\frac{\partial^2 v}{\partial y^2}-rv&=0\end{aligned} \tag{5.2.6}$$

rearranging the last equation

$$\frac{\partial v}{\partial \tau} = \frac{2}{\sigma^2}\Big( - \frac{\sigma^2}{2} + r\Big)\frac{\partial v}{\partial y} + \frac{2}{\sigma^2}\frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2} - \frac{2}{\sigma^2} rv$$

simplifying,

$$\frac{\partial v}{\partial \tau} = \Big( - 1 + \frac{2r}{\sigma^2}\Big)\frac{\partial v}{\partial y} + \frac{\partial^2}{\partial y^2} - \frac{2r}{\sigma^2} u \tag{5.2.7}$$

setting $\kappa = \dfrac{2r}{\sigma^2}$ to simplify calculation and letting $t = \tau$ , then (5.2.1) becomes

$$\frac{\partial v}{\partial \tau} = \Big( - 1 + \kappa\Big)\frac{\partial v}{\partial y} + \frac{\partial^2}{\partial y^2} - \kappa u \tag{5.2.8}$$

$$\text{with} \quad - \infty \leq y \leq \infty \quad 0 \leq t \leq \frac{\sigma^2}{2} T$$

and (5.2.2) becomes

$$v(y, 0) = u(e^y, T) = f(e^x) = max(e^x - K, 0).$$

In order to eliminate $\Big( - 1 + \dfrac{2r}{\sigma^2}\Big)\dfrac{\partial v}{\partial y}$ and $\kappa u$ in (5.2.8), we need to transform one more variable

$$v(y, t) = \exp\{\alpha y + \beta t\}V(y, t)$$

where $\alpha$ and $\beta$ are arbitrary constants. Computing the partial derivatives of $V$ with respect to $y$ and $t$, we get

$$\frac{\partial v}{\partial t} = \exp\{\alpha y + \beta t\}\beta V + \exp\{\alpha y + \beta t\}\frac{\partial V}{\partial t} \tag{5.2.9}$$

$$\frac{\partial v}{\partial y} = \exp\{\alpha y + \beta t\}\alpha V + \exp\{\alpha y + \beta t\}\frac{\partial V}{\partial t}$$

$$\begin{aligned}
\frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial y}\Big( \exp\{\alpha y + \beta t\}\alpha V + \exp\{\alpha y + \beta t\}\frac{\partial V}{\partial t}\Big) \\
&= \exp\{\alpha y + \beta t\}\alpha^2 V + \exp\{\alpha y + \beta t\}\alpha\frac{\partial V}{\partial y} + \exp\{\alpha y + \beta t\}\alpha\frac{\partial V}{\partial y} + \exp\{\alpha y + \beta t\}\frac{\partial^2 V}{\partial y^2} \\
&= \exp\{\alpha y + \beta t\}\alpha^2 V + 2\alpha \exp\{\alpha y + \beta t\}\frac{\partial V}{\partial y} + \exp\{\alpha y + \beta t\}\frac{\partial^2 V}{\partial y^2}
\end{aligned} \tag{5.2.10}$$

substituting (5.2.9) and (5.2.10) back into equation (5.2.8)

$$\begin{aligned}
\exp\{\alpha y + \beta t\}\beta V + \exp\{\alpha y + \beta t\}\frac{\partial V}{\partial t} = \Big( - 1 + \kappa\Big)&\Big( \exp\{\alpha y + \beta t\}\alpha V + \exp\{\alpha y + \beta t\}\frac{\partial V}{\partial y}\Big) \\
&+ \exp\{\alpha y + \beta t\}\alpha^2 V + 2\alpha \exp\{\alpha y + \beta t\}\frac{\partial V}{\partial y} \\
&+ \exp\{\alpha y + \beta t\}\frac{\partial^2 V}{\partial y^2} - \kappa \exp\{\alpha y + \beta t\}V
\end{aligned}$$

simplifying and dividing both sides by $\exp\{\alpha y + \beta t\}$ we get,

$$\begin{aligned} \beta V + \frac{\partial V}{\partial t} &= -\alpha V - \frac{\partial V}{\partial y} + \kappa\alpha V + \kappa\frac{\partial V}{\partial y} + \alpha^2 V + 2\alpha\frac{\partial V}{\partial y} + \frac{\partial^2 V}{\partial y^2} - \kappa V \\ \frac{\partial V}{\partial t} &= \Big( -1 + \kappa + 2\alpha\Big)\frac{\partial V}{\partial y} + \frac{\partial^2 V}{\partial y^2} + \Big( -\alpha + \kappa\alpha + \alpha^2 - \kappa - \beta\Big) \end{aligned} \tag{5.2.11}$$

since $\alpha$ and $\beta$ are arbitrary constants, we set

$$\alpha = \frac{1-\kappa}{2}$$

$$\beta = -\alpha + \kappa\alpha + \alpha^2 - \kappa = \frac{-\kappa^2 - 2\kappa - 1}{4} = -\frac{-(\kappa+1)^2}{4}$$

substituting back into equation (5.2.11), the resulting equation is a heat equation

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial y^2} \tag{5.2.12}$$

$$\text{with} \quad y > 0 \quad \text{and} \quad 0 \leq t \leq \frac{\sigma^2}{2}T.$$

In this chapter and chapter 4, we have determined the Lie symmetries of the Heat equation and the Black Scholes equation. In the following chapter we will find the invariant solutions corresponding to the Lie symmetries that where found.

# 6 Invariant solutions

In this sections we discuss the invariant solutions of the symmetry generators of the Black Scholes equation and the Heat equation. Invariant solutions are exact solutions that are invariant under a subgroup of the full symmetry group.
**Definition 6.1**( [25], *Chapter 9.4*)
Let a system S of differential equations

$$F_\sigma(x, u, u_{(1)}, ..., u_{(k)}) = 0,\ \sigma = 1, ..., s, \tag{1}$$

where the order $k$ refers to the highest derivatives appearing (1), admit a group G, and let H be a subgroup of G. A solution of equation (1)

$$u^\alpha = h^\alpha(x),\ \alpha = 1, ..., m, \tag{2}$$

is called an invariant solution of the system $S$ if (2) is an invariant manifold for $H$.

We start by demonstrating how to get invariant solutions using the heat operators as in section (5.1).

## 6.1 Invariant solutions of the Heat question

We recall the symmetries of the heat equation below

$$\begin{aligned}
&P_1 = \frac{\partial}{\partial x}, \quad P_2 = \frac{\partial}{\partial t}, \quad P_3 = u\frac{\partial}{\partial u}\\
&P_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\\
&P_5 = 2t\frac{\partial}{\partial x} - ux\frac{\partial}{\partial u}\\
&P_6 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - 2tu\frac{\partial}{\partial u} - x^2u\frac{\partial}{\partial u}\\
&P_{\mathcal{F}} = \mathcal{F}_1[t, x]\frac{\partial}{\partial u} \quad \text{given that} \quad \mathcal{F}_{1,xx} - \mathcal{F}_{1,t} = 0\mathcal{F}_1[t, x]\frac{\partial}{\partial u}
\end{aligned}$$

with $\mathcal{F}_1$ as an arbitrary solution of the heat equation.

1. The invariant solutions for the symmetry generator $P_1 = \dfrac{\partial}{\partial x}$ is as follows
   The characteristic equation is given by

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}$$

   which can be reduced into two equations as shown below

$$1. \quad \frac{dx}{1} = \frac{dt}{0}, \quad 2. \quad \frac{dx}{1} = \frac{du}{0}.$$

We solve equation (1), $t = M_1$ where $M_1$ is a constant of integration, and the second equation (2), $u = M_2$. Thus, the invariant solution is given by

$$M_2 = \phi(M_1), \text{ that is, } u = \phi(t) \tag{6.1.1}$$

when we compute the derivatives of equation (6.1.1) with respect to $t$ and $x$

$$u_t = \phi'(t), \; u_x = 0 \; u_{xx} = 0.$$

Now we substitute these equations into the heat equation (5.3.1), to obtain

$$\phi'(t) = 0 \longrightarrow \frac{d\phi}{dt} = 0. \tag{6.1.2}$$

We now integrate equation (6.1.2) , $\phi = C_1$. Thus, the invariant solution of $P_1$ is

$$u = C_1, \text{ where } C \text{ is a constant.} \tag{6.1.3}$$

2. We compute the invariant solution of the symmetry generator $P_2 = \dfrac{\partial}{\partial t}$ to obtain the characteristic equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}.$$

Therefore, we obtain, $x = M_1$ and $u = M_2$ and expressing $M_2$ in terms of $M_1$, we obtain

$$M_2 = \phi(M_1) = \phi(x) \longrightarrow u = \phi(x). \tag{6.1.4}$$

Taking the derivatives of $u$ with respect to $t$ and $x$ , $u_t = 0$ , $u_x = \phi'(x)$ and $u_{xx} = \phi''(x)$. Substituting back into the heat equation (5.1.1),

$$\phi''(x) = 0. \tag{6.1.5}$$

Upon integration, $\phi = C_1 x + C_2$, so the invariant solution is

$$u = C_1 x + C_2 \tag{6.1.6}$$

were $C_1, C_2$ are constants.

3. For the symmetry generator $P_3 = 2t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x}$, the characteristic equation of this generator is,

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{du}{0}.$$

Then we separately solve

$$1. \quad \frac{dt}{2t} = \frac{dx}{x}, \qquad 2. \quad \frac{dx}{x} = \frac{du}{0}.$$

We considering (1) and integrate

$$\frac{1}{2}\ln t + M_1 = \ln x,$$

which can be written as

$$x = M_1\sqrt{t}. \tag{6.1.7}$$

We consider equation (2) and integrate to obtain, $u = M_2$. Designating $M_2$ as a function of $M_1$

$$M_2 = \phi(M_1) \longrightarrow M_2 = \phi(\frac{x}{\sqrt{t}}).$$

Then $u = \phi(\frac{x}{\sqrt{t}})$. Taking the derivatives of $u$ with respect to $x$ and $t$,

$$\begin{aligned} u_t &= \phi'(\frac{x}{\sqrt{t}})\Big(-\frac{x}{2t^{3/2}}\Big) \\ u_x &= \phi'(\frac{x}{\sqrt{t}})\Big(\frac{1}{\sqrt{t}}\Big) \\ u_{xx} &= \phi''(\frac{x}{\sqrt{t}})\Big(\frac{1}{t}\Big). \end{aligned} \tag{6.1.8}$$

Now we substitute the derivatives back into the heat equation (5.1.1)

$$\phi'(\frac{x}{\sqrt{t}})\Big(-\frac{x}{2t^{3/2}}\Big) = \phi''(\frac{x}{\sqrt{t}})\Big(\frac{1}{t}\Big) \longrightarrow \phi'' + \frac{x}{2\sqrt{t}}\phi' = 0, \tag{6.1.9}$$

this is a second order homogeneous differential equation. Now let $\phi' = y(M_1)$, then

$$y' + \frac{x}{2\sqrt{t}}y = 0 \longrightarrow \frac{dy}{y} = \frac{x}{2\sqrt{t}}dM_1, \tag{6.1.10}$$

which yields the following derivation,

$$\ln y = \frac{M_1 x}{2\sqrt{t}} + K_1 \longrightarrow y = K_1 \exp\Big(\frac{M_1 x}{2\sqrt{t}}\Big). \tag{6.1.11}$$

We substituting back $\phi'$, to obtain $\phi' = K_1 \exp\Big(\frac{M_1 x}{2\sqrt{t}}\Big)$, thus

$$\frac{d\phi}{dM_1} = K_1 \exp\Big(\frac{M_1 x}{2\sqrt{t}}\Big) \longrightarrow d\phi = K_1 \exp\Big(\frac{M_1 x}{2\sqrt{t}}\Big)dM_1. \tag{6.1.12}$$

We solve again

$$\phi = K_1\frac{2\sqrt{t}}{x}\exp\Big(\frac{M_1 x}{2\sqrt{t}}\Big) + K_2. \tag{6.1.13}$$

We obtain, the invariant solution as

$$u = K_1\frac{2\sqrt{t}}{x}\exp\Big(\frac{M_1 x}{2\sqrt{t}}\Big) + K_2. \tag{6.1.14}$$

4. We consider invariant solution of the symmetry generator $P_5 = 2t\frac{\partial}{\partial x} - ux\frac{\partial}{\partial u}$
We compute the characteristic equation,

$$\frac{dt}{0} = \frac{dx}{2t} = \frac{du}{-ux}$$

and we separately solve

$$1.\quad \frac{dt}{0}=\frac{dx}{2t},\qquad 2.\quad \frac{dx}{2t}=\frac{du}{-ux}.$$

We obtain $t=M_1$ from equation (1) and equation (2) yields

$$-\frac{x^2}{4t}-M_2=\ln u \longrightarrow u=M_2\exp\left(-\frac{x^2}{4t}\right). \tag{6.1.15}$$

Now we write $M_2$ as a function of $M_1$, $M_2=\phi(t)$, thus equation (6.1.15) is written as,

$$u=\phi(t)\exp\left(-\frac{x^2}{4t}\right). \tag{6.1.16}$$

Then we differentiate $u$ with respect to $t$ and $u$ to obtain

$$\begin{aligned} u_t &= \phi'(t)\exp\left(-\frac{x^2}{4t}\right)+\phi(t)\exp\left(-\frac{x^2}{4t}\right)\left(\frac{x^2}{4t^2}\right)\\ u_x &= \phi(t)\exp\left(-\frac{x^2}{4t}\right)\left(-\frac{2x}{4t}\right)\\ u_{xx} &= \phi(t)\exp\left(-\frac{x^2}{4t}\right)\left(\frac{4x^2}{16t^2}\right)+\phi(t)\exp\left(-\frac{x^2}{4t}\right)\left(-\frac{1}{2t}\right)\end{aligned} \tag{6.1.17}$$

we then rewrite equation (5.1.1) in terms of $u_t$ and $u_{xx}$

$$\begin{aligned}\phi'(t)\exp\left(-\frac{x^2}{4t}\right) &= \phi(t)\exp\left(-\frac{x^2}{4t}\right)\frac{4x^2}{16t^2}-\phi(t)\exp\left(-\frac{x^2}{4t}\right)\frac{1}{2t}\\ &\quad +\phi(t)\exp\left(-\frac{x^2}{4t}\right)\frac{x^2}{4t^2}.\end{aligned} \tag{6.1.18}$$

Thereafter we divide both side of the equation by $\exp\left(-\frac{x^2}{4t}\right)$,

$$\phi'(t)+\phi(t)\frac{x^2}{4t^2}=\phi(t)\frac{4x^2}{16t^2}-\phi(t)\cdot\frac{1}{2t} \tag{6.1.19}$$

which becomes,

$$\begin{aligned}\frac{\phi'}{\phi} &= \left(\frac{4x^2}{16t^2}-\frac{1}{2t}-\frac{x^2}{4t^2}\right)\\ \frac{d\phi}{\phi} &= \left(-\frac{1}{2t}\right)dt.\end{aligned} \tag{6.1.20}$$

We then integrate and obtain $\phi=\frac{C}{\sqrt{t}}$, were $C$ is an integration constant. Hence, the invariant solution of $P_5$ is

$$u=\frac{C}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right). \tag{6.1.21}$$

5. For the symmetry generator $P_6 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - 2tu\frac{\partial}{\partial u} - x^2u\frac{\partial}{\partial u}$ the characteristic equation is,

$$\frac{dt}{t^2} = \frac{dx}{tx} = \frac{du}{(-\frac{1}{2}tu - \frac{1}{4}x^2u)}$$

which can be written separately as

$$1. \quad \frac{dt}{t^2} = \frac{dx}{tx}, \qquad 2. \quad \frac{dx}{tx} = \frac{du}{(-\frac{1}{2}t - \frac{1}{4}x^2)u}.$$

Solving equation (1) we obtain,

$$M_1 = \frac{x}{t}$$

and solving equation (2) we obtain,

$$\Big(-\frac{1}{2x} - \frac{x}{4t}\Big)dx = \frac{du}{u}.$$

We then integrate to get $u = M_2 \exp\Big(-\frac{\ln x}{2} - \frac{x^2}{8t}\Big)$, then write $M_2 = \phi(M_1) = \phi(\frac{x}{t})$, hence

$$u = \phi(\frac{x}{t}) \exp\Big(-\frac{\ln x}{2} - \frac{x^2}{8t}\Big) = \phi(\frac{x}{t}) \cdot \frac{1}{\sqrt{x}} \cdot \exp\Big(-\frac{x^2}{8t}\Big)$$

We differentiate "u" with respect to $x$ and $t$,

$$\begin{aligned}
u_t &= \phi'\Big(-\frac{\sqrt{x}}{t^2}\Big)\exp\Big(-\frac{x^2}{8t}\Big) + \phi\Big(\frac{x^{3/2}}{8t^2}\Big)\exp\Big(-\frac{x^2}{8t}\Big)\\
u_x &= \phi'\Big(\frac{1}{t\sqrt{x}}\Big)\exp\Big(-\frac{x^2}{8t}\Big) - \phi\Big(\frac{\sqrt{x}}{4t}\Big)\exp\Big(-\frac{x^2}{8t}\Big) - \phi\Big(\frac{1}{2x^{3/2}}\Big)\exp\Big(-\frac{x^2}{8t}\Big)\\
u_{xx} &= \phi''\Big(\frac{1}{t^2\sqrt{x}}\Big)\exp\Big(-\frac{x^2}{8t}\Big) + \phi'\Big(\Big(-\frac{t}{x^{3/2}}\Big)\exp\Big(-\frac{x^2}{8t}\Big) - \Big(\frac{\sqrt{x}}{2}\Big)\\
&\exp\Big(-\frac{x^2}{8t}\Big)\Big) + \phi\Big(\Big(\frac{3}{4x^{5/2}}\Big)\exp\Big(-\frac{x^2}{8t}\Big) + \frac{1}{4t\sqrt{x}\cdot\exp(-\frac{x^2}{8t})} - \Big(\frac{\sqrt{x}}{4t}\Big)\\
&\exp\Big(-\frac{x^2}{8t}\Big) + \Big(\frac{x^{5/2}}{16t^2}\Big)\exp\Big(-\frac{x^2}{8t}\Big)\Big).
\end{aligned} \tag{6.1.22}$$

Now we re-write equation (5.1.1) as,

$$\phi''(16x^2) + \phi'(-16tx + 8x^3) + \phi(12t^2 - x^4) = 0. \tag{6.1.23}$$

## 6.2 Solution of (5.2.12) via the Adomian Decomposition Method

We consider

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial y^2}, \quad V(0,y) = V_0(y) \tag{6.2.1}$$

$$\text{with} \quad y > 0 \quad \text{and} \quad 0 \le t \le \frac{\sigma^2}{2}T.$$

The operator form is as follows,

$$L_yV(t,y) - L_tV(x,y) = 0 \tag{6.2.2}$$

where $L_y = \frac{\partial^2}{\partial y^2}$ and $L_t = \frac{\partial}{\partial t}$ and the inverse operator $L_y^{-1} = \int_0^y \int_0^y (.)dydy$, applying the inverse operator we get,

$$L_y^{-1}L_yV(t,y) = V_0(y) + L_y^{-1}L_tV(x,y) \tag{6.2.3}$$

simplifying we obtain

$$V(t,y) = V_0(y) + L_y^{-1}L_tV(x,y). \tag{6.2.4}$$

The Adomian solution has a series form

$$V(x,y) = \sum_{n=0}^{\infty} V_n(x,y)$$

so, equation (6.2.4) becomes

$$\sum_{n=0}^{\infty} V_n(x,y) = V_0(y) + L_y^{-1}L_t \sum_{n=0}^{\infty} V_n(x,y) \tag{6.2.5}$$

thus equation (6.2.5) is written as

$$V_0 + V_1 + V_2 + ... = V_0(y) + L_y^{-1}L_t(V_0 + V_1 + V_2 + ...) \tag{6.2.6}$$

which is a sequence of solutions of (6.2.1). The infinite series (6.2.6) will also be a solution of the heat equation, under appropriate initial or boundary conditions.

## 6.3 Invariant solutions of the Black Scholes equation

We consider the following symmetries of the Black Scholes equation.

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \qquad X_2 = x\frac{\partial}{\partial x}, \\
&X_3 = u((B^4 + 4C^2 - 4B^2(C - 2D))t + 2(B^2 - 2C)\ln x)\frac{\partial}{\partial u} + 4B^2x\ln x\frac{\partial}{\partial x} + 8B^2t\frac{\partial}{\partial t}, \\
&X_4 = u((B^2 - 2C)t + 2\ln x)\frac{\partial}{\partial u} + 2B^2tx\frac{\partial}{\partial x}, \\
&X_5 = u(t(B^4t + 4C^2t - 4B^2(1 + Ct - 2Dt)) + 4(B^2 - 2C)t\ln x + 4(\ln x)^2)\frac{\partial}{\partial u} \\
&\quad + 8B^2t^2\frac{\partial}{\partial t} + 8B^2tx\ln x\frac{\partial}{\partial x}, \\
&X_6 = u\frac{\partial}{\partial u}, \qquad X_{\mathcal{F}_\infty} = \mathcal{F}_1.
\end{aligned}$$

1. For the symmetry generator $X_1 = \dfrac{\partial}{\partial t}$ the characteristic equations are as follows

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}.$$

We separate the above equations into two linear equations

$$1.\quad \frac{dt}{1} = \frac{dx}{0}, \quad 2.\quad \frac{dt}{1} = \frac{du}{0}.$$

We solve equation (1), $x = M_1$ where $M_1$ is a constant of integration,and solving (2) yeilds $u = M_2$. Hence, the invariant solution is given by

$$M_2 = \phi(M_1), \text{ that is, } u = \phi(x). \tag{6.3.1}$$

Then now we take the derivatives of equation (6.3.1) with respect to $t$ and $x$

$$u_t = 0,\ u_x = \phi^{'}(x),\ u_{xx} = \phi^{''}(x).$$

By substituting into the Black Scholes equation (4.3.1), we get,

$$\frac{1}{2}B^2x^2\phi^{''}(x) + Cx\phi^{'}(x) - D\phi(x) = 0. \tag{6.3.2}$$

The equation (6.3.2) is known as Cauchy-Euler equation of order 2 and its solution is

$$\begin{aligned}\phi(x) = c_1 x^{\left(\frac{(-i\sqrt{-16-\frac{2B^2}{D}+\frac{8C}{D}-\frac{8C^2}{B^2D}}+\frac{\sqrt{2}B}{\sqrt{D}}-\frac{2\sqrt{2}C}{B\sqrt{D}})\sqrt{D}}{2\sqrt{2}B}\right)}\\ + c_2 x^{\left(\frac{(-i\sqrt{-16-\frac{2B^2}{D}+\frac{8C}{D}-\frac{8C^2}{B^2D}}+\frac{\sqrt{2}B}{\sqrt{D}}-\frac{2\sqrt{2}C}{B\sqrt{D}})\sqrt{D}}{2\sqrt{2}B}\right)}\end{aligned}$$

where $c_1$ and $c_2$ are arbitrary constants. Thus, the invariant solution of the Black Scholes equation under $X_1$ is

$$\begin{aligned}u(t,x) = c_1 x^{\left(\frac{(-i\sqrt{-16-\frac{2B^2}{D}+\frac{8C}{D}-\frac{8C^2}{B^2D}}+\frac{\sqrt{2}B}{\sqrt{D}}-\frac{2\sqrt{2}C}{B\sqrt{D}})\sqrt{D}}{2\sqrt{2}B}\right)}\\ + c_2 x^{\left(\frac{(-i\sqrt{-16-\frac{2B^2}{D}+\frac{8C}{D}-\frac{8C^2}{B^2D}}+\frac{\sqrt{2}B}{\sqrt{D}}-\frac{2\sqrt{2}C}{B\sqrt{D}})\sqrt{D}}{2\sqrt{2}B}\right)}.\end{aligned}$$

2. For the symmetry generator $X_2 = x\frac{\partial}{\partial x}$ the characteristic equations are

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{0},$$

then we separately solve

$$1.\quad \frac{dt}{0} = \frac{dx}{x}, \quad 2.\quad \frac{dx}{0} = \frac{du}{0}.$$

We consider (1) and integrate to get $t = M_1$. We also consider equation (2) and solve $u = M_2$. Then the invariant solution is

$$M_2 = \phi(M_1), \text{ that is, } u = \phi(t). \tag{6.3.3}$$

Then now we take the derivatives of equation (6.3.3) with respect to $t$ and $x$

$$u_t = \phi^{'}(t),\ u_x = 0,\ u_{xx} = 0.$$

We now substitute into the Black Scholes equation (4.3.1) to obtain

$$\phi^{'}(t) - D\phi(t) = 0, \tag{6.3.4}$$

then

$$\frac{d\phi(t)}{dt} = D\phi(t), \longrightarrow \frac{d\phi(t)}{\phi(t)} = Ddt,$$

upon integration

$$\ln\phi = Dt + c_1, \text{ then } \phi(t) = c_1 e^{Dt}.$$

So the invariant solution under $X_2$ is

$$u = c_1 e^{Dt}. \tag{6.3.5}$$

3. For the symmetry generator

$$X_3 = u((B^4 + 4C^2 - 4B^2(C - 2D))t + 2(B^2 - 2C)\ln x)\frac{\partial}{\partial u} + 4B^2 x \ln x \frac{\partial}{\partial x} + 8B^2 t \frac{\partial}{\partial t} \tag{6.3.6}$$

the characteristic equations are,

$$\frac{dt}{8B^2t} = \frac{dx}{4B^2x\ln x} = \frac{du}{u(B^4t + 4C^2t - 4CB^2t + 8B^2tD + 2B^2\ln x - 4C\ln x)}.$$

We start by solving

$$\frac{dt}{8B^2t} = \frac{dx}{4B^2x\ln x} \longrightarrow \frac{dt}{2t} = \frac{dx}{x\ln x}, \tag{6.3.7}$$

before we can integrate, we make the following substitution

$$v = \ln x,\ dv = \frac{1}{x}dx,$$

thus equation (6.3.7) is written as,

$$\frac{dt}{2t} = \frac{dv}{v}.$$

We now integrate the above equation to get $M_1 t^{1/2} = v$, and finally we have

$$M_1 t^{1/2} = \ln x \longrightarrow M_1 = \frac{\ln x}{t^{1/2}}.$$

Then now we consider the second pair of equation from the characteristics equation we computed earlier

$$\frac{dt}{8B^2t} = \frac{du}{u(B^4t + 4C^2t - 4CB^2t + 8B^2tD + 2B^2\ln x - 4C\ln x)}. \tag{6.3.8}$$

This is then written as,

$$\begin{aligned}(B^4t + 4C^2t - 4CB^2t + 8B^2tD + 2B^2\ln x - 4C\ln x)dt/8B^2t &= \frac{du}{u}\\ \Big(\frac{B^2}{8} + \frac{C^2}{2B^2} - \frac{C}{2} + D + \frac{\ln x}{4t} - \frac{c\ln x}{2B^2t}\Big)dt &= \frac{du}{u}.\end{aligned} \tag{6.3.9}$$

We now integrate equation (6.3.9) ,

$$\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{Ct}{2} + Dt + \frac{\ln x \ln t}{4} - \frac{c\ln x\ln t}{2B^2} + M_2 = \ln u.$$

We solve for $u$,

$$u = M_2 \exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{Ct}{2} + Dt\Big)\Big(t^{\frac{\ln x}{4} - \frac{c\ln x}{2B^2}}\Big). \tag{6.3.10}$$

Now we write $M_2$ as a function of $M_1$ ,

$$M_2 = \phi(M_1) = \phi(\frac{\ln x}{\sqrt{t}}), \tag{6.3.11}$$

thus equation (6.3.10) can be written as,

$$u = \phi\Big(\frac{\ln x}{\sqrt{t}}\Big)\exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{Ct}{2} + Dt + \frac{\ln x\ln t}{4} - \frac{C\ln x\ln t}{2B^2}\Big). \tag{6.3.12}$$

We now differentiate equation (6.3.12) with respect to $t$ and $x$

$$\begin{aligned}u_t = {}& \phi'\Big(\frac{\ln x}{\sqrt{t}}\Big)\Big(\frac{-\ln x}{2t^{3/2}}\Big)\exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{Ct}{2} + Dt + \frac{\ln x\ln t}{4}\\ & - \frac{C\ln x\ln t}{2B^2}\Big) + \phi\exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{Ct}{2} + Dt + \frac{\ln x\ln t}{4}\\ & - \frac{C\ln x\ln t}{2B^2}\Big)\Big(\frac{B^2}{8} + \frac{c^2}{2B^2} - \frac{C}{2} + D + \frac{\ln x}{4t} - \frac{c\ln x}{2B^2t}\Big),\end{aligned} \tag{6.3.13}$$

$$\begin{aligned}u_x = {}& \phi'\Big(\frac{\ln x}{\sqrt{t}}\Big)\Big(\frac{1}{x\sqrt{t}}\Big)\exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{Ct}{2} + Dt + \frac{\ln x\ln t}{4}\\ & - \frac{C\ln x\ln t}{2B^2}\Big) + \phi\exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{Ct}{2} + Dt + \frac{\ln x\ln t}{4}\\ & - \frac{C\ln x\ln t}{2B^2}\Big)\Big(\frac{\ln t}{4x} - \frac{C\ln t}{2B^2x}\Big),\end{aligned} \tag{6.3.14}$$

$$
\begin{aligned}
u_{xx} = {} & \phi''\Big(\frac{\ln x}{\sqrt{t}}\Big)\Big(\frac{1}{tx^2}\Big)\exp\Big(\frac{B^2t}{8}+\frac{C^2t}{2B^2}-\frac{Ct}{2}+Dt+\frac{\ln x\ln t}{4} \\
& -\frac{C\ln x\ln t}{2B^2}\Big)+\phi'(\frac{\ln x}{\sqrt{t}})\Big(\frac{-1}{x^2\sqrt{t}}\Big)\exp\Big(\frac{B^2t}{8}+\frac{C^2t}{2B^2}-\frac{Ct}{2}+Dt \\
& +\frac{\ln x\ln t}{4}-\frac{C\ln x\ln t}{2B^2}\Big)+\phi'(\frac{\ln x}{\sqrt{t}})\Big(\frac{2}{x\sqrt{t}}\Big)\exp\Big(\frac{B^2t}{8}+\frac{C^2t}{2B^2} \\
& -\frac{Ct}{2}+Dt+\frac{\ln x\ln t}{4}-\frac{C\ln x\ln t}{2B^2}\Big)\Big(\frac{\ln t}{4x}-\frac{C\ln t}{2B^2x}\Big)+\phi\exp\Big(\frac{B^2t}{8} \\
& +\frac{C^2t}{2B^2}-\frac{Ct}{2}+Dt+\frac{\ln x\ln t}{4}-\frac{C\ln x\ln t}{2B^2}\Big)\Big(\frac{\ln t}{4x}-\frac{C\ln t}{2B^2x}\Big)^2+\phi \\
& \exp\Big(\frac{B^2t}{8}+\frac{C^2t}{2B^2}-\frac{Ct}{2}+Dt+\frac{\ln x\ln t}{4}-\frac{C\ln x\ln t}{2B^2}\Big)\Big(\frac{-\ln t}{4x^2} \\
& +\frac{C\ln t}{2B^2x^2}\Big).
\end{aligned}
\tag{6.3.15}
$$

We substitute back into the Black Scholes equation (4.3.1) to obtain,

$$
\begin{aligned}
& \phi''\Big(16B^4\sqrt{t}\Big)+\phi'\Big(8B^4t\ln t-16B^4t+32B^2Ct-16B^2\ln x+16Ct\ln t\Big) \\
& +\phi\sqrt{t}\Big(4B^4t-16B^2Ct+16C^2t+16C\ln x-4B^4t\ln t+8B^2Ct\ln t \\
& +B^4t(\ln t)^2+8B^2\ln x+4Ctx\ln t(4C+B^2\ln t)+8Ctx(\ln t)^2 \\
& -8B^2Ct\ln t\Big)=0.
\end{aligned}
\tag{6.3.16}
$$

4. For the symmetry generator

$$
X_4 = u((B^2-2C)t+2\ln x)\frac{\partial}{\partial u}+2B^2tx\frac{\partial}{\partial x}, \tag{6.3.17}
$$

the characteristics equation are;

$$
\frac{dt}{0}=\frac{dx}{2B^2tx}=\frac{du}{u((B^2-2C)t+2\ln x)}.
$$

We separately solve

1. $\dfrac{dt}{0}=\dfrac{dx}{2B^2tx}$ and
2. $\dfrac{dx}{2B^2tx}=\dfrac{du}{u((B^2-2C)t+2\ln x)}$.

Upon integration, equation (1), becomes $t = M_1$, where $M_1$ is an integrating constant. Similarly, we solve for equation (2) ,

$$
\begin{aligned}
\Big((B^2-2C)t+\frac{2\ln x}{2B^2tx}\Big)dx &= \frac{du}{u}, \\
\Big(\frac{B^2t}{2B^2tx}-\frac{2Ct}{2B^2tx}+\frac{2\ln x}{2B^2tx}\Big)dx &= \frac{du}{u},
\end{aligned}
$$

this can written as,

$$\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t} + M_2 = \ln u.$$

We solve again to obtain,

$$M_2 \exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big) = u \text{ or } M_2 = u\exp\Big(-\frac{\ln x}{2} + \frac{C\ln x}{B^2} - \frac{\ln x^2}{2B^2t}\Big).$$

Hence $M_2 = \phi(M_1) = \phi(t)$, so the invariant solution is

$$u = \phi(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big). \tag{6.3.18}$$

We now differentiate $u$ with respect to $t$ and $x$,

$$u_t = \phi'(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big) + \phi(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big)\Big(-\frac{\ln x^2}{2B^2t^2}\Big),$$

$$u_x = \phi(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big)\Big(\frac{1}{2x} - \frac{C}{B^2x} + \frac{\ln x}{B^2tx}\Big),$$

$$u_{xx} = \phi(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big)\Big(\frac{1}{2x} - \frac{C}{B^2x} + \frac{\ln x}{B^2tx}\Big)^2 + \phi(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big)\Big(-\frac{1}{2x^2} + \frac{C}{B^2x^2} + \frac{(1-\ln x)}{B^2tx^2}\Big),$$

then we substitute back into the Black scholes equation (4.3.1) to get

$$\phi'(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big) + \phi(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big)\Big(-\frac{\ln x^2}{2B^2t^2}\Big) - \frac{1}{2}B^2x^2\Big[\phi(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big)\Big(\frac{1}{2x} - \frac{C}{B^2x} + \frac{\ln x}{B^2tx}\Big)^2 + \phi(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big)\Big(-\frac{1}{2x^2} + \frac{C}{B^2x^2} + \frac{(1-\ln x)}{B^2tx^2}\Big)\Big] - CxBigg(\phi(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big)\Big(\frac{1}{2x} - \frac{C}{B^2x} + \frac{\ln x}{B^2tx}\Big)\Big) + D\Big(\phi(t)\exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t}\Big)\Big) = 0.$$

We then simplify to obtain the following

$$\begin{aligned}
\frac{\phi'(t)}{\phi(t)} &= \Big(-\frac{\ln x^2}{2B^2t^2}\Big) - \frac{1}{2}B^2x^2\Big[\Big(\frac{1}{2x} - \frac{C}{B^2x} + \frac{\ln x}{B^2tx}\Big)^2 + \Big(-\frac{1}{2x^2} + \frac{C}{B^2x^2} \\
&+ \frac{(1-\ln x)}{B^2tx^2}\Big)\Big] - Cx\Big(\frac{1}{2x} - \frac{C}{B^2x} + \frac{\ln x}{B^2tx}\Big) + D \\
&= \Big(-\frac{\ln x^2}{2B^2t^2}\Big) - \frac{1}{2}B^2x^2\Big[\Big(\frac{1}{4x^2} - \frac{C}{2B^2x^2} + \frac{\ln x}{2B^2tx^2} - \frac{C}{2B^2x^2} \\
&+ \frac{C^2}{B^4x^2} - \frac{C\ln x}{B^4tx^2} + \frac{\ln x}{2B^2tx^2} - \frac{C\ln x}{B^4tx^2} + \frac{\ln x^2}{B^4t^2x^2}\Big) + \Big(-\frac{1}{2x^2} \\
&+ \frac{C}{B^2x^2} + \frac{(1-\ln x)}{B^2tx^2}\Big)\Big] - Cx\Big(\frac{1}{2x} - \frac{C}{B^2x} + \frac{\ln x}{B^2tx}\Big) + D \qquad (6.3.19) \\
&= -\frac{\ln x^2}{2B^2t^2} - \frac{B^2}{8} + \frac{C}{4} - \frac{\ln x}{4t} + \frac{C}{4} - \frac{C^2}{2B^2} + \frac{C\ln x}{2B^2t} - \frac{\ln x}{4t} \\
&+ \frac{C\ln x}{2B^2t} + \frac{\ln x^2}{2B^2t^2} + \frac{B^2}{4} - \frac{C}{2} - \frac{(1-\ln x)}{2t} - \frac{C}{2} + \frac{C^2}{B^2} \\
&- \frac{C\ln x}{B^2t} + D \\
&= \frac{B^2}{8} + \frac{C^2}{2B^2} - \frac{1}{2t} - \frac{C}{2} + D.
\end{aligned}$$

We now have the following equation from (6.3.19) ,

$$\frac{d\phi}{\phi} = \Big(\frac{B^2}{8} + \frac{C^2}{2B^2} - \frac{1}{2t} - \frac{C}{2} + D\Big)dt.$$

We then obtain the following equation after integrating,

$$\begin{aligned}
\ln\phi &= \frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{1}{2}\ln t - \frac{Ct}{2} + Dt + K \\
\phi &= \frac{K}{\sqrt{t}}exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{Ct}{2} + Dt\Big).
\end{aligned}$$

The invariant solution of $X_4$ is

$$u(x,t) = \frac{K}{\sqrt{t}}exp\Big(\frac{\ln x}{2} - \frac{C\ln x}{B^2} + \frac{\ln x^2}{2B^2t} + \frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{Ct}{2} + Dt\Big). \qquad (6.3.20)$$

5. For symmetry generator

$$\begin{aligned}
X_5 &= u(t(B^4t + 4C^2t - 4B^2(1 + Ct - 2Dt)) + 4(B^2 - 2C)t\ln x \\
&+ 4(\ln x)^2)\frac{\partial}{\partial u} + 8B^2t^2\frac{\partial}{\partial t} + 8B^2tx\ln x\frac{\partial}{\partial x}
\end{aligned} \qquad (6.3.21)$$

the characteristics system of $X_5$ is,

$$dt/8B^2t^2 = dx/8B^2tx\ln x = du/u(B^4t^2 + 4C^2t^2 - 4B^2t - 4B^2Ct^2 + 8B^2Dt^2 + 4B^2t\ln x - 8Ct\ln x + 4(\ln x)^2)$$

which can be written in the same way as,

$$1. \quad \frac{dt}{8B^2t^2} = \frac{dx}{8B^2tx\ln x} \longrightarrow \frac{dt}{t} = \frac{dx}{x\ln x},$$

and

2. $$\frac{dt}{8B^2t^2} = \frac{du}{u(B^4t^2 + 4C^2t^2 - 4B^2t - 4B^2Ct^2 + 8B^2Dt^2 + 4B^2t\ln x - 8Ct\ln x + 4(\ln x)^2)}.$$

Taking into account the first equation and making a substitution $v = \ln x$, $dv = \frac{1}{x}dx$,

$$\frac{dt}{t} = \frac{dv}{v}.$$

We obtain the following,

$$M_1 + \ln t = \ln v \longrightarrow M_1 t = \ln x \longrightarrow M_1 = \frac{\ln x}{t}.$$

Now, lets take a look at the second equation,

$$\Big(\frac{B^2}{8} + \frac{C^2}{2B^2} - \frac{1}{2t} - \frac{C}{2} + D + \frac{\ln x}{2t} + \frac{C\ln x}{B^2 t} + \frac{(\ln x)^2}{2B^2t^2}\Big)dt = \frac{du}{u}. \tag{6.3.22}$$

We then integrate to get,

$$\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{\ln t}{2} - \frac{Ct}{2} + Dt + \frac{\ln x \ln t}{2} + \frac{C\ln x\ln t}{B^2} - \frac{(\ln x)^2}{2B^2t} + M_2 = \ln u \tag{6.3.23}$$

therefore,

$$u = M_2 \exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{\ln t}{2} - \frac{Ct}{2} + Dt + \frac{\ln x\ln t}{2} + \frac{C\ln x\ln t}{B^2} - \frac{(\ln x)^2}{2B^2t}\Big) \tag{6.3.24}$$

expressing $M_2$ as a function of $M_1$, we have that $M_2 = \phi(M_1) = \phi(\frac{\ln x}{t})$, as a result, equation (6.3.24) can be rewritten as,

$$u = \phi\Big(\frac{\ln x}{t}\Big)exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{\ln t}{2} - \frac{Ct}{2} + Dt + \frac{\ln x\ln t}{2} + \frac{C\ln x\ln t}{B^2} - \frac{(\ln x)^2}{2B^2t}\Big). \tag{6.3.25}$$

We now differentiate $u$ with respect to "$t$" and "$x$",

$$\begin{aligned} u_t = {} & \phi'(\frac{\ln x}{t})\Big(-\frac{\ln x}{t^2}\Big)\exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{\ln t}{2} - \frac{Ct}{2} + Dt + \frac{\ln x\ln t}{2} \\ & + \frac{C\ln x\ln t}{B^2} - \frac{(\ln x)^2}{2B^2t}\Big) + \phi(\frac{\ln x}{t})\exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{\ln t}{2} - \frac{Ct}{2} + Dt \\ & + \frac{\ln x\ln t}{2} + \frac{C\ln x\ln t}{B^2} - \frac{(\ln x)^2}{2B^2t}\Big)\Big(\frac{B^2}{8} + \frac{C^2}{2B^2} - \frac{1}{2t} - \frac{C}{2} + D + \frac{\ln x}{2t} \\ & + \frac{C\ln x}{B^2t} + \frac{(\ln x)^2}{2B^2t^2}\Big). \end{aligned} \tag{6.3.26}$$

$$\begin{aligned} u_x = {} & \phi'(\frac{\ln x}{t})\Big(\frac{1}{tx}\Big)\exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{\ln t}{2} - \frac{Ct}{2} + Dt + \frac{\ln x\ln t}{2} \\ & + \frac{C\ln x\ln t}{B^2} - \frac{(\ln x)^2}{2B^2t}\Big) + \phi(\frac{\ln x}{t})\exp\Big(\frac{B^2t}{8} + \frac{C^2t}{2B^2} - \frac{\ln t}{2} - \frac{Ct}{2} \\ & + Dt + \frac{\ln x\ln t}{2} + \frac{C\ln x\ln t}{B^2} - \frac{(\ln x)^2}{2B^2t}\Big)\Big(\frac{\ln t}{2x} + \frac{C\ln t}{B^2x} - \frac{\ln x}{B^2tx}\Big). \end{aligned} \tag{6.3.27}$$

$$
\begin{aligned}
u_{xx} = {} & \phi''(\frac{\ln x}{t})\Big(\frac{1}{t^2x^2}\Big)\exp\Big(\frac{B^2t}{8}+\frac{C^2t}{2B^2}-\frac{\ln t}{2}-\frac{Ct}{2}+Dt+\frac{\ln x\ln t}{2} \\
& +\frac{C\ln x\ln t}{B^2}-\frac{(\ln x)^2}{2B^2t}\Big)+\phi'(\frac{\ln x}{t})\Big(-\frac{1}{tx^2}\Big)\exp\Big(\frac{B^2t}{8}+\frac{C^2t}{2B^2}-\frac{\ln t}{2} \\
& -\frac{Ct}{2}+Dt+\frac{\ln x\ln t}{2}+\frac{C\ln x\ln t}{B^2}-\frac{(\ln x)^2}{2B^2t}\Big)+\phi'(\frac{\ln x}{t})\Big(\frac{1}{tx}\Big) \\
& \exp\Big(\frac{B^2t}{8}+\frac{C^2t}{2B^2}-\frac{\ln t}{2}-\frac{Ct}{2}+Dt+\frac{\ln x\ln t}{2}+\frac{C\ln x\ln t}{B^2} \\
& -\frac{(\ln x)^2}{2B^2t}\Big)\Big(\frac{\ln t}{2x}+\frac{C\ln t}{B^2x}-\frac{\ln x}{B^2tx}\Big)+\phi'(\frac{\ln x}{t})\Big(\frac{1}{tx}\Big)\exp\Big(\frac{B^2t}{8} \\
& +\frac{C^2t}{2B^2}-\frac{\ln t}{2}-\frac{Ct}{2}+Dt+\frac{\ln x\ln t}{2}+\frac{C\ln x\ln t}{B^2}-\frac{(\ln x)^2}{2B^2t}\Big) \\
& \Big(\frac{\ln t}{2x}+\frac{C\ln t}{B^2x}-\frac{\ln x}{B^2tx}\Big)+\phi(\frac{\ln x}{t})\exp\Big(\frac{B^2t}{8}+\frac{C^2t}{2B^2}-\frac{\ln t}{2} \\
& -\frac{Ct}{2}+Dt+\frac{\ln x\ln t}{2}+\frac{C\ln x\ln t}{B^2}-\frac{(\ln x)^2}{2B^2t}\Big)\Big(\frac{\ln t}{2x}+\frac{C\ln t}{B^2x} \\
& -\frac{\ln x}{B^2tx}\Big)^2+\phi(\frac{\ln x}{t})\exp\Big(\frac{B^2t}{8}+\frac{C^2t}{2B^2}-\frac{\ln t}{2}-\frac{Ct}{2}+Dt+\frac{\ln x\ln t}{2} \\
& +\frac{C\ln x\ln t}{B^2}-\frac{(\ln x)^2}{2B^2t}\Big)\Big(-\frac{\ln t}{2x^2}-\frac{C\ln t}{B^2x^2}+\frac{\ln x}{B^2tx^2}\Big).
\end{aligned}
\tag{6.3.28}
$$

Now we substitute into the Black Scholes-equation (4.3.1) and divide by $\exp\Big(\frac{B^2t}{8}+\frac{C^2t}{2B^2}-\frac{\ln t}{2}-\frac{Ct}{2}+Dt+\frac{\ln x\ln t}{2}+\frac{C\ln x\ln t}{B^2}-\frac{(\ln x)^2}{2B^2t}\Big)$ to obtain,

$$
\begin{aligned}
& \phi'(\frac{\ln x}{t})\Big(-\frac{\ln x}{t^2}\Big)+\phi(\frac{\ln x}{t})\Big(\frac{B^2}{8}+\frac{C^2}{2B^2}-\frac{1}{2t}-\frac{C}{2}+D+\frac{\ln x}{2t} \\
& +\frac{C\ln x}{B^2t}+\frac{(\ln x)^2}{2B^2t^2}\Big)+\phi''(\frac{\ln x}{t})\Big(\frac{B^2}{2t^2}\Big)+\phi'(\frac{\ln x}{t})\Big(-\frac{B^2}{2t}\Big)+\phi'(\frac{\ln x}{t})\Big(\frac{B^2x}{2t}\Big) \\
& \Big(\frac{\ln t}{2x}+\frac{C\ln t}{B^2x}-\frac{\ln x}{B^2tx}\Big)+\phi'(\frac{\ln x}{t})\Big(\frac{B^2x}{2t}\Big)\Big(\frac{\ln t}{2x}+\frac{C\ln t}{B^2x}-\frac{\ln x}{B^2tx}\Big) \\
& +\phi(\frac{\ln x}{t})\Big(\frac{\ln t}{2x}+\frac{C\ln t}{B^2x}-\frac{\ln x}{B^2tx}\Big)^2\Big(\frac{B^2x^2}{2}\Big)+\phi(\frac{\ln x}{t})\Big(-\frac{\ln t}{2x^2}-\frac{C\ln t}{B^2x^2} \\
& +\frac{\ln x}{B^2tx^2}\Big)\Big(\frac{B^2x^2}{2}\Big)+\phi'(\frac{\ln x}{t})\Big(\frac{C}{t}\Big)+\phi(\frac{\ln x}{t})\Big(\frac{\ln t}{2x}+\frac{C\ln t}{B^2x}-\frac{\ln x}{B^2tx}\Big) \\
& Cx-D\phi(\frac{\ln x}{t})=0
\end{aligned}
\tag{6.3.29}
$$

which is then written as,

$$
\begin{aligned}
& \phi''(4B^4)+\phi'(-4B^4t+8B^2Ct+4B^4t\ln t-16B^2\ln x+8B^2Ct\ln t) \\
& +\phi(-8B^2t+B^4t^2-4B^2ct^2+4c^2t^2+8tc\ln x-2B^4t^2\ln t+4B^2ct^2\ln t \\
& +B^4t^2(\ln t)^2+8B^2t\ln x-8ct\ln x-4B^2t\ln t\ln x+8(\ln x)^2+4t\ln t(2ct \\
& +B^2t\ln t-2\ln x)+8Ct^2x(\ln t)^2-4CB^2t^2\ln t=0.
\end{aligned}
\tag{6.3.30}
$$

**Note** that operators $X_6$ and $X_\phi$ do not provide invariant solutions because $X_6$ and $X_\phi$ are independent of the variables $t$ and $x$.

# 7 Conclusion

The research began with a study of the Black Scholes equation, which included a derivation from basic principles (a derivation that does not require pre-knowledge in calculus). Then the Adomian Decomposition Method was then introduced as a method for solving first-order differential equations. The research also shows an unpopular basis (computed using SYM) that spans the Lie algebra of symmetries of the Black-Scholes equation. Lastly, we determined the invariant solutions of the Black Scholes equation and the Heat equation.

For future work, we will construct the optimal system of invariant solutions of the Black scholes equations and also study the practical and theoretical implications of the optimal system.

# References

[1] G.Adomian. A review of the the decomposition method and some recent results for nonlinear equations. Center for applied Mathematics, University of Georgia. Athens. GA 30602.sec.1-4.

[2] D.J Ariggo. Nonclassical Contact Symmetries and Charpit's method of compatibility. Journal of Nonlinear Mathematical Physics. Volume 12, Number 3, pp.321-329, 2005.

[3] A.Z Azmi. Numerical Solutions of the Heat equation. Lecturer notes. Fakulti Sains. Universiti Teknologi Malaysia, 2009.

[4] G.Baumann. Symmetry Analysis of Differential Equations with Mathematica. Mathl Comput. Modelling Vol. 25, No. 8/9, pp. 25-37. Copyright©1997 Elsevier Science Ltd. Printed in Great Britain. All rights reserved.

[5] F.Black and M. Scholes. The Pricing of Options and Corporate Liabilities. The University of Chicago Press. The Journal of Political Economy, Vol. 81, No. 3 (May - Jun., 1973), pp. 637-654 London. J.Wiley and Sons Ltd, 1999.

[6] A James Boness A theory and measurement of stock-option value. University of Chicago. Ph. D. Thesis. 1962

[7] L.A. Bordag and I.P. Yamshchikov. *Lie Group Analysis of Nonlinear Black Scholes Models*. Springer International Publishing, 2017.

[8] H.J Buettler. Ito Fundamental PDE-02. http://www.hj-buettler.ch/papers/12/Ito.

[9] C. Carleos, M.C. Lopez-Diaz and M.A. Lopez-Diaz. Stochastic Order of Shape Variability with an Application to Cell Nuclei Involved in Mastitis. J Math Imaging Vis 38, 95–107, 2010.

[10] J.Chen. Delta-Hedging.2019. Available at: https://www.investopedia.com/terms/d/deltahedging.asp. [Accessed February 2020].

[11] Y.Cherrault. Convergence of the Adomian's Method. Kybernetes 18(2), 31-38, 1989.

[12] Y.Cherrault and G.Adomian. Decomposition Method: A new Proof of convergence. Mathl. Comput. Modelling Vol. 18, No.12, pp. 103-106, 1993.

[13] C.P Chitalkar-Dhaigude and V.N Bhadgaonkar. Adomian Decomposition Method Over Charpits Method for solving Nonlinear First Order Partial differential Equations. Bulletin of the Marathwada Mathematical Society, Vol.18, No.1, pp. 01-18, 2017

[14] S.Dimas and D.Tsoubelis. SYM: A new symmetry - finding package for Mathematica. In N.H.Ibragimov, C.Sophocleous, and P.A.Damianou, editors, The 10 the International Conference in Modern Group Analysis, pp. 64-70, Nicosia, 2005.

[15] L.C Evans. An Introduction to Stochastics Differential Equations. Version 1.2. Department of Mathematics, U.C Berkeley.8

[16] J.Fernando. Derivative. 2021. Available at:https://www.investopedia.com/terms/d/derivative.asp. [Accessed January 2022].

[17] R.K Gazizov and N.H Ibragimov. Lie Symmetry Analysis of Differential Equations in Finance. Nonlinear Dynamics 17:387 - 407, 1998.

[18] O.Gonzalez-Gaxiola and R.Bernal-Jaquez. Applying Adomian Decomposition Method to Solve Burgess Equation with a Non-linear Source. International Journal of Applied and Computational Mathematics pp. 213, 2017.

[19] A. Gorgius. Charpit and Adomian for solving integral equations. Applied mathematics and computations 193. pp. 446-454,2007.

[20] S.Grandville. Derivation of the Black Scholes equation from Basic Principles. The Mathematical Association of America. The college of mathematics Journal. VOL.49, No.3 May, 2018.

[21] F.Granstrom. Symmetry method and some nonlinear differential equations. Karlstad University, Department of Mathematics and Computer Science, 2017.

[22] M.M Hosseini and H. Nasabzadeh. On the Convergence of the Adomian Decomposition Method. Department of Mathematics,Yazd University , P.O Box 89195-741,Yazd, Iran.

[23] J.C Hull. *Options, Futures, and Other Derivatives.* 8th Edition. University of Toronto. ©2012, Pearson.

[24] Z.Hussain, M.Sulaiman and E.K.E.Sackey. Optimal System of Subalgebras of Invariant solutions of the Blacks-Scholes Equation.Blekinge Institute of Technology School of Engineering. Department of Mathematics and Science. MSc. thesis, November 2009.

[25] N.H Ibragimov. *Elementary Lie Group Analysis and Ordinary Differential Equations.* London. J.Wiley and Sons Ltd, 1999.

[26] N.H Ibragimove (ed). *CRC handbook of Lie group Analysis of Differential equations.* Vol.1. 1994,Vol.2, 1995, Vol.3, 1996 CRC Press. Boca Raton FL.

[27] K. Ito *On stochastic processes (infinitely divisible laws of probability)* (Doctoral thesis). Japan Journ. Math. XVIII, 261-301.

[28] M.Joshi. The concepts and practice of mathematical finance. Sec.5.6, pp 119-120, 2004.

[29] T.Kardi. Stochastic Process Tutorial.(2017).http://people.revoledu.com/kardi/tutorial/StochasticProcess/

[30] K.Marek. Black Scholes model under Arithmetic Brownian Motion. University of Economics. Prague, December 2013.

[31] M.C. Nucci. The Role of Symmetries in Solving Differential Equations. Dipartimento di Matematica, Universitk di Perugia.06123 Perugia, Italy. Mathl. Comput. Modelling Vol. 25, No. 819, pp. 181-193, 1997.

[32] A. Müller and D. Stoyan *Comparison Methods for Stochastic Models and Risks.* Wiley Series in Probability and Statistics. Wiley, Chichester, 2002.

[33] A.S Shinde and K.C Takale. Study of Black Scholes Model and its Applications. 1877-7058©2012, Elsevier, Procedia Engineering 38. pp. 270-279, 2012.

[34] N.G Van Kampen. *Stochastic Processes in Physics and Chemistry.* (Third edition), 2007.

[35] D.V Widder. *The Heat Equation.* Academic Press. 22 January 1976.