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Adaptive Step Size Stochastic Runge-Kutta Method of Order 1.5(1.0) for Stochastic Differential Equations (SDEs)

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Abstract The stiff stochastic differential equations (SDEs) involve the solution with sharp turning points that permit us to use a very small step size to comprehend its behavior. Since the step size must be set up to be as small as possible, the implementation of the fixed step size method will result in high computational cost. Therefore, the application of variable step size method is needed where in the implementation of variable step size methods, the step size used can be considered more flexible. This paper devotes to the development of an embedded stochastic Runge-Kutta (SRK) pair method for SDEs. The proposed method is an adaptive step size SRK method. The method is constructed by embedding a SRK method of 1.0 order into a SRK method of 1.5 order of convergence. The technique of embedding is applicable for adaptive step size implementation, henceforth an estimate error at each step can be obtained. Numerical experiments are performed to demonstrate the efficiency of the method. The results show that the solution for adaptive step size SRK method of order 1.5(1.0) gives the smallest global error compared to the global error for fix step size SRK4, Euler and Milstein methods. Hence, this method is reliable in approximating the solution of SDEs.

Keywords Embedded Stochastic Runge-Kutta, Adaptive Step Size, Stochastic Differential Equations

1. Introduction

Most of the physical systems around us are subjected by the uncontrolled factors, hence stochastic differential equations (SDEs) are needed to model these systems. The random function in SDEs can be modelled by perturbing a system with a Wiener process [1]. SDEs can be found in many fields including geology, demography, economics, physics, signal processing, modern control theory and more. However, the exact solution of SDEs is complicated to be solved. In this situation, a numerical method provides the solution to the systems. The numerical analysis of SDEs differs significantly from that of ODEs due to peculiarities of stochastic calculus [2].

The development of numerical methods for SDEs is far from complete. First step to this direction was done by Maruyama in 1950 by introducing Euler method with 0.5 order of convergence. Euler-Maruyama is the simplest method that based on the truncation of stochastic Taylor expansion followed by Milstein method developed by Milstein in 1974 [3]. It is well-known that Runge-Kutta (RK) method has been used widely to approximate the solution of ODEs [4]. The effectiveness of this method in solving SDEs has been demonstrated in [5]. Burrage introduced general formulation of stochastic Runge-Kutta (SRK) method [5]. The SRK method has been developed for 2-stage and 4-stage. The rooted tree theory of ODEs has been extended to SDEs to study the order of conditions.

Most of the numerical methods that have been developed thus far are fixed step size. Fix step size implementation failed in providing the most efficient solution in many physical systems. There are cases of stiff problems which the solution has sharp turning points and need for the use of a very small step size to obtain the approximation. Fix step size implementation is ineffective particularly in analyzing the behavior of the challenging area of the stiff problems. In order to solve the problems, variable step sizes have been employed in ODEs. Fehlberg proposed Runge-Kutta Fehlberg method, which is an embedded Runge-Kutta method for solving ODEs [6]. This adaptive fifth-order method has been constructed that uses only six function evaluations for each time-step. Variable step sizes implementation in the development of numerical methods for ODEs can be considered as successful in providing good approximation to the exact solution [7]. In SDEs, the embedded of SRK method has been developed by [8] and [9]. The proposed method converges to the analytical solution of stochastic problems [8] and [9].

This research focuses on the development of an adaptive step size stochastic Runge-Kutta method where the 2-stage stochastic Runge Kutta of order 1.0 is embedded into 4-stage stochastic Runge Kutta of order 1.5 for solving SDEs. The outline of this paper is; in Section 2, the SRK2 method with order of convergence of 1.0 and SRK4 method with order of convergence 1.5 will be introduced. Furthermore, the newly developed scheme for an adaptive step size and the numerical algorithm is then carried out in this section. In Section 3, the numerical result is presented to illustrate the efficiency of adaptive step size SRK method in solving stochastic model, then followed by the discussion and concluding remarks in Section 4.

2. Materials and Methods

2.1. Stochastic Runge-Kutta for SDEs

Consider the autonomous Stratonovich SDEs

$$dy(t) = f\left(y(t)\right)dt + g\left(y(t)\right)dW(t) \tag{1}$$

where f is drift function, g is diffusion function and $W(t)$ is a d-dimensional process having independent scalar Wiener process components $(t \geq 0)$ [5].

Equation (1) can be written in integral form of

$$y(t) = y(t_0) + \int_{t_0}^{t} f\left(y(s)\right)ds + \int_{t_0}^{t} g\left(y(s)\right)dW(s) \tag{2}$$

To obtain the stochastic Taylor expansion of the exact solution, the differential operators

$$\begin{aligned} L^0 &= f\frac{\partial}{\partial y} \\ L^1 &= g\frac{\partial}{\partial y} \end{aligned} \tag{3}$$

are substituted into equation (2), where L^0 and L^1 are differential operators in the form of Stratonovich calculus. The expansion of equation (2) is derived through the iterated application of the stochastic chain rule. By considering up 2.0 order of convergence, the following stochastic Taylor expansion is yielded

$$\begin{aligned} y(t) = {} & y_0 + f(y_0)J_0 + g(y_0)J_1 \\ & + f'(y_0)\left(f(y_0)\right)J_{00} \\ & + f'(y_0)\left(g(y_0)\right)J_{01} \\ & + g'(y_0)\left(f(y_0)\right)J_{10} \\ & + g'(y_0)\left(g(y_0)\right)J_{00} \\ & + g'(y_0)\left(g'(y_o)\left(f(y_0)\right)\right)J_{011} \\ & + g'(y_0)\left(f'(y_0)\left(g(y_0)\right)\right)J_{101} \\ & + f'(y_0)\left(g'(y_0)\left(g(y_0)\right)\right)J_{110} \\ & + g'(y_0)\left(g'(y_0)\left(g(y_0)\right)\right)J_{111} \\ & + g''(y_0)\left(g(y_0)g(y_0)\right)J_{111} \\ & + g''(y_0)\left(g(y_0)f(y_0)\right)J_{110} \\ & + f''(y_0)\left(g(y_0)g(y_0)\right)J_{110} \\ & + g''(y_0)\left(f(y_0)g(y_0)\right)J_{101} \\ & + g'(y_0)\left(g'(y_0)\left(g'(y_0)\left(g(y_0)\right)\right)\right)J_{1111} \\ & + g'(y_0)\left(g''(y_0)\left(g(y_0)g(y_0)\right)\right)J_{1111} \\ & + 3g''(y_0)\left(g(y_0)g'(y_0)\left(g(y_0)\right)\right)J_{1111} \\ & + g'''(y_0)\left(g(y_0)g(y_0)g(y_0)\right)J_{1111} + R \end{aligned} \tag{4}$$

where R is a remainder term and $J_{j_1, j_2, \cdots, j_k}$ represent the Stratonovich multiple integral. The integration is respected to ds if $j_i = 0$ or $dW(s)$ if $j_i = 1$. For example $J_1 = \int_{t_n}^{t_{n+1}} dW_s$ and $J_{10} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} dW_{s_1} ds_2$.

According to [1], the general form of SRK method can be written as

$$
\begin{aligned}
Y_i(t) &= Y_n(t_0) + \Delta \sum_{j=1}^{i-1} a_{ij} f\left(y_j(t)\right) \\
&\quad + \sum_{j=1} \left(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{h} \right) g\left(y_j(t)\right) \\
y_{n+1}(t) &= y_n + \Delta \sum_{j=1}^{s} \alpha_j f\left(y_j(t)\right) \\
&\quad + \sum_{j=1} \left(\gamma_j^{(1)} J_1 + \gamma_j^{(2)} \frac{J_{10}}{h} \right) g\left(y_j(t)\right)
\end{aligned} \tag{5}
$$

where $i = 1, \ldots, s$ represent the stage of SRK method. The scheme for the *s*-stage can be written in Butcher tableau [10] as follows

$$
\begin{array}{c|ccc}
A & a_{11} & \cdots & a_{1s} \\
 & \vdots & \ddots & \vdots \\
 & a_{s1} & \cdots & a_{ss} \\
\hline
\alpha^T & \alpha_1 & \cdots & \alpha_s
\end{array}
\qquad
\begin{array}{c|ccc}
 & b_{11}^{(1)} & \cdots & b_{1s}^{(1)} \\
B^{(1)} & \vdots & \ddots & \vdots \\
 & b_{s1}^{(1)} & \cdots & b_{ss}^{(1)} \\
\hline
\gamma^{(1)T} & \gamma_1^{(1)} & \cdots & \gamma_s^{(1)}
\end{array}
$$

$$
\begin{array}{c|ccc}
 & b_{11}^{(2)} & \cdots & b_{1s}^{(2)} \\
B^{(2)} & \vdots & \ddots & \vdots \\
 & b_{s1}^{(2)} & \cdots & b_{ss}^{(2)} \\
\hline
\gamma^{(2)T} & \gamma_1^{(2)} & \cdots & \gamma_s^{(2)}
\end{array} \tag{6}
$$

2.2. 2-Stage Stochastic Runge Kutta with High Strong Order 1.0

General form of SRK2 method with 1.0 order of convergence was developed based on the formulation (5) by [1] and can be presented as

$$
\begin{aligned}
y_{n+1}(t) &= y_n(t_0) + \Delta \alpha_1 f(Y_1) + \Delta \alpha_2 f(Y_2) \\
&\quad + \left(\gamma_1^{(1)} J_1 + \gamma_1^{(2)} \frac{J_{10}}{\Delta} \right) g(Y_1) \\
&\quad + \left(\gamma_2^{(1)} J_1 + \gamma_2^{(2)} \frac{J_{10}}{\Delta} \right) g(Y_2)
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
Y_1 &= Y_0^{(n)} \\
Y_2 &= Y_0^{(n)} + \Delta a_{21} f\left(Y_0^{(n)}\right) \\
&\quad + \left(b_{21}^{(1)} J_1 + b_{21}^{(2)} \frac{J_{10}}{\Delta} \right) g\left(Y_0^{(n)}\right)
\end{aligned}
$$

The order conditions of Taylor expansion can be described by using rooted tree [11]. The local truncation error of SRK method is computed by comparing the stochastic Taylor expansion of the actual solution in (4) with the stochastic Taylor expansion of SRK in (7). The comparison gives

$$
\begin{aligned}
&\alpha^T e = 1 \\
&\gamma^{(1)T} (e, d, b) = \left(1, -y^{(2)T} b, \frac{1}{2}\right), \\
&y^{(2)T} (e, d) = (0, 0)
\end{aligned} \tag{8}
$$

where $e^T = (1, \ldots, 1)$, $b = B^{(1)} e$ and $d = B^{(2)} e$.

By solving equation (8) simultaneously, the scheme for 2-stage SRK with strong order of convergence 1.0 can be presented in Butcher Tableau as

$$
\begin{array}{c|cc}
A & 0 & \\
 & \frac{2}{3} & 0 \\
\hline
\alpha^T & \frac{1}{4} & \frac{3}{4}
\end{array}
\qquad
\begin{array}{c|cc}
B^{(1)} & 0 & \\
 & \frac{2}{3} & 0 \\
\hline
\gamma^{(1)T} & \frac{1}{4} & \frac{3}{4}
\end{array}
\qquad
\begin{array}{c|cc}
B^{(2)} & 0 & \\
 & 0 & 0 \\
\hline
\gamma^{(2)T} & 0 & 0
\end{array} \tag{9}
$$

2.3. 4-Stage Stochastic Runge-Kutta with High Strong Order 1.5

SRK4 method with 1.5 order of convergence was developed based on the formulation (5) and can be presented as

$$
\begin{aligned}
y_{n+1}(t) &= y_n(t_0) + \Delta \alpha_1 f(Y_1) \\
&\quad + \Delta \alpha_2 f(Y_2) + \Delta \alpha_3 f(Y_3) + \Delta \alpha_4 f(Y_4) \\
&\quad + \left(\gamma_1^{(1)} J_1 + \gamma_1^{(2)} \frac{J_{10}}{\Delta} \right) g(Y_1) \\
&\quad + \left(\gamma_2^{(1)} J_1 + \gamma_2^{(2)} \frac{J_{10}}{\Delta} \right) g(Y_2) \\
&\quad + \left(\gamma_3^{(1)} J_1 + \gamma_3^{(2)} \frac{J_{10}}{\Delta} \right) g(Y_3) \\
&\quad + \left(\gamma_4^{(1)} J_1 + \gamma_4^{(2)} \frac{J_{10}}{\Delta} \right) g(Y_4)
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
Y_1 &= Y_0^{(n)} \\
Y_2 &= Y_0^{(n)} + \Delta a_{21} f\left(Y_0^{(n)}\right) \\
&\quad + \left(b_{21}^{(1)} J_1 + b_{21}^{(2)} \frac{J_{10}}{\Delta}\right) g\left(Y_0^{(n)}\right) \\
Y_3 &= Y_0^{(n)} + \Delta a_{31} f\left(Y_1^{(n)}\right) + \Delta a_{32} f\left(Y_2^{(n)}\right) \\
&\quad + \left(b_{31}^{(1)} J_1 + b_{31}^{(2)} \frac{J_{10}}{\Delta}\right) g\left(Y_1^{(n)}\right) \\
&\quad + \left(b_{32}^{(1)} J_1 + b_{32}^{(2)} \frac{J_{10}}{\Delta}\right) g\left(Y_2^{(n)}\right) \\
Y_4 &= Y_0^{(n)} + \Delta a_{41} f\left(Y_1^{(n)}\right) \\
&\quad + \Delta a_{42} f\left(Y_2^{(n)}\right) + \Delta a_{43} f\left(Y_3^{(n)}\right) \\
&\quad + \left(b_{41}^{(1)} J_1 + b_{41}^{(2)} \frac{J_{10}}{\Delta}\right) g\left(Y_1^{(n)}\right) \\
&\quad + \left(b_{42}^{(1)} J_1 + b_{42}^{(2)} \frac{J_{10}}{\Delta}\right) g\left(Y_2^{(n)}\right) \\
&\quad + \left(b_{43}^{(1)} J_1 + b_{43}^{(2)} \frac{J_{10}}{\Delta}\right) g\left(Y_3^{(n)}\right)
\end{aligned}
$$

The Stratonovich Taylor expansion of the actual solution in (4) and the Stratonovich Taylor expansion of numerical solution (10) are compared, and this leads to

$$
\begin{aligned}
&\alpha^T (d, b) = (1, 0), \\
&\gamma^{(1)T}\left(c, b^2, B^{(1)}b, d^2, B^{(2)}d\right) = \begin{pmatrix} 1, \frac{1}{3}, \frac{1}{6}, -2\gamma^{(2)T} bd, \\ -\gamma^{(2)T} B^{(2)}b + B^{(1)}d \end{pmatrix}, \\
&\gamma^{(2)T}\left(c, b^2, B^{(1)}b, d^2, B^{(2)}d\right) = \begin{pmatrix} -1, -2\gamma^{(1)T} bd, \\ -\gamma^{(2)T}\left(B^{(2)}b + B^{(1)}d\right), \\ 0, 0 \end{pmatrix}
\end{aligned} \tag{11}
$$

where $e^T = (1, \ldots, 1)$, $c = Ae$, $b = B^{(1)}e$ and $d = B^{(2)}e$.

By solving equation (11) simultaneously, the scheme for 4-stage SRK with strong order of convergence 1.5 can be obtained and it is presented as follows

$$
A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

$$
B^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.7242916356 & 0 & 0 & 0 \\ 0.4237353406 & -0.1994437050 & 0 & 0 \\ -1.578475506 & 0.840100343 & 1.738375163 & 0 \end{bmatrix}
$$

$$
B^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2.702000410 & 0 & 0 & 0 \\ 1.757261649 & 0 & 0 & 0 \\ -2.918524118 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\alpha^T = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}
$$

$$
\gamma^{(1)} = \begin{bmatrix} -0.7800788474 \\ 0.07363768240 \\ 1.486520013 \\ 0.2199211524 \end{bmatrix}, \quad \gamma^{(2)} = \begin{bmatrix} 1.693950844 \\ 1.636107882 \\ -3.024009558 \\ -0.3060491602 \end{bmatrix}
$$

2.4. Adaptive Step Size SRK Method of Order 1.5(1.0)

The adaptive SRK method of order 1.5(1.0) consists of 2-stage SRK and 4-stage SRK methods. The scheme of coefficient is written as

$$
\begin{array}{c|ccc}
A & a_{11} & \cdots & a_{1s} \\
 & \vdots & \ddots & \vdots \\
 & a_{s1} & \cdots & a_{ss} \\
\hline
\alpha^T & \alpha_1 & \cdots & \alpha_s \\
\alpha^T & \alpha_1 & \cdots & \alpha_s
\end{array}
\qquad
\begin{array}{c|ccc}
 & b_{11}^{(1)} & \cdots & b_{1s}^{(1)} \\
B^{(1)} & \vdots & \ddots & \vdots \\
 & b_{s1}^{(1)} & \cdots & b_{ss}^{(1)} \\
\hline
\gamma^{(1)T} & \gamma^{(1)} & \cdots & \gamma_s^{(1)} \\
\gamma^{(1)T} & \gamma^{(1)} & \cdots & \gamma_s^{(1)}
\end{array}
$$

$$
\begin{array}{c|ccc}
 & b_{11}^{(2)} & \cdots & b_{1s}^{(2)} \\
B^{(2)} & \vdots & \ddots & \vdots \\
 & b_{s1}^{(2)} & \cdots & b_{ss}^{(2)} \\
\hline
\gamma^{(2)T} & \gamma^{(2)} & \cdots & \gamma_s^{(2)} \\
\gamma^{(2)T} & \gamma^{(2)} & \cdots & \gamma_s^{(2)}
\end{array} \tag{12}
$$

such that equation (5) is of strong order p and

$$
\begin{aligned}
\hat{y}_{n+1}(t) &= y_n + \Delta \sum_{j=1}^{s} \hat{\alpha}_j f\left(y_j(t)\right) \\
&\quad + \sum_{j=1}^{s} \left(\hat{\gamma}_j^{(1)} J_1 + \hat{\gamma}_j^{(2)} \frac{J_{10}}{h}\right) g\left(y_j(t)\right)
\end{aligned} \tag{13}
$$

is of strong order $\hat{p} < p$. In order to embed SRK2 and SRK4, we choose $\hat{p} = 1$ and $p = 1.5$. Then, we substitute SRK2 of order 1.0 in (9) into (12), which gives

$$
\begin{array}{c|cccc}
 & 0 & 0 & 0 & 0 \\
A & \frac{2}{3} & 0 & 0 & 0 \\
 & a_{31} & a_{32} & 0 & 0 \\
 & a_{41} & a_{42} & a_{43} & 0 \\
\hline
\alpha^T & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\hat{\alpha}^T & \frac{1}{4} & \frac{3}{4} & 0 & 0
\end{array}
\quad
\begin{array}{c|cccc}
 & 0 & 0 & 0 & 0 \\
B^{(1)} & \frac{2}{3} & 0 & 0 & 0 \\
 & b_{31}^{(1)} & b_{32}^{(1)} & 0 & 0 \\
 & b_{41}^{(1)} & b_{42}^{(1)} & b_{43}^{(1)} & 0 \\
\hline
\gamma^{(1)T} & \gamma_1^{(1)} & \gamma_2^{(1)} & \gamma_3^{(1)} & \gamma_4^{(1)} \\
\hat{\gamma}^{(1)T} & \frac{1}{4} & \frac{3}{4} & 0 & 0
\end{array}
\tag{14}
$$

$$
\begin{array}{c|cccc}
 & 0 & 0 & 0 & 0 \\
B^{(2)} & 0 & 0 & 0 & 0 \\
 & b_{31}^{(2)} & b_{32}^{(2)} & 0 & 0 \\
 & b_{41}^{(2)} & b_{42}^{(2)} & b_{43}^{(2)} & 0 \\
\hline
\gamma^{(2)T} & \gamma_1^{(2)} & \gamma_2^{(2)} & \gamma_3^{(2)} & \gamma_4^{(2)} \\
\hat{\gamma}^{(2)T} & 0 & 0 & 0 & 0
\end{array}
$$

The unknown coefficients of the deterministic part of (14) are substituted with the coefficients in RK4 scheme. Using $a_{21} = \frac{2}{3}$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$, $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$ gives

$$
\begin{array}{c|cccc}
 & 0 & 0 & 0 & 0 \\
A & \frac{2}{3} & 0 & 0 & 0 \\
 & \frac{1}{12} & \frac{1}{4} & 0 & 0 \\
 & -\frac{5}{4} & \frac{1}{4} & 2 & 0 \\
\hline
\alpha^T & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8}
\end{array}
\tag{15}
$$

Another 17 equations with 18 parameters were computed using MAPLE and yield new embedded method of SRK1.5(1.0)

$$
\begin{array}{|cccc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{12} & \frac{1}{4} & 0 & 0 & -\frac{1}{2} & -\frac{1}{6} & 0 & 0 & 1 & 1 & 0 & 0 \\
-\frac{5}{4} & \frac{1}{4} & 2 & 0 & -\frac{3}{2} & 1 & \frac{1}{2} & 0 & \frac{5}{2} & -\frac{1}{2} & 0 & 0 \\
\hline
\frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & -\frac{1}{4} & \frac{3}{4} & 0 & \frac{1}{2} & \frac{3}{4} & -\frac{3}{4} & \frac{3}{4} & -\frac{3}{4} \\
\frac{1}{4} & \frac{3}{4} & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\tag{16}
$$

This scheme may increase the efficiency of the simulations problem in SDEs as this approach monitors the change of the solution in two subsequent discrete time of y and $\hat{y}$.

The approximation of Stratonovich integral of J_1 and J_{10} can be computed using

$$
\begin{aligned}
J_1 &= N_1\sqrt{h} \\
J_{10} &= \frac{1}{2}\sqrt{h^3}\left(N_1 + \frac{1}{\sqrt{3}}N_2\right)
\end{aligned}
\tag{17}
$$

where N_1 and N_2 are two independent standard normally distributed random variables [12].

2.5. Numerical Algorithm

Numerical algorithm performing the SRK of order 1.5(1.0) numerical schemes to the model is presented.

1) Define $N = 2^{m_{\max}}$, for $m_{\max} \in \mathbb{N}$ and the step size $h = \frac{T}{N}$, $t_i = (i-1)h$ for $i = 1, 2, \ldots, N$.
2) Define $J_1 = N_1\sqrt{h}$, $J_{10} = \frac{1}{2}\sqrt{h^3}\left(N_1 + \frac{1}{\sqrt{3}}N_2\right)$.
3) Define the initial condition, t_0 and y_0.
4) Perform numerical schemes of SRK of order 1.5(1.0) to SDEs.
5) Evaluate $R = |y_{i+1} - \hat{y}_{i+1}|$.
6) Define the relative error, $rtol$ and the absolute error, $atol$.
7) Evaluate $tol = \max\{|y_0|, |y_1|\} rtol + atol$.
8) Evaluate the adaptive step size for y_i using: If $R \le tol$, keep y_i as the current step solution and move to the next solution with step size, $0.8h\left(\frac{tol}{R}\right)^{\frac{1}{1.5}}$ else recalculate y_i with step size, $0.8h\left(\frac{tol}{R}\right)^{\frac{1}{1.5}}$. If the tolerance, tol, is not met more than once in the step, compute y_i with step size, $\frac{h}{2}$.
9) Compute the error between the adaptive step size SRK of order 1.5(1.0) with exact solution of SDEs model.
10) Compare the error between fix step size Euler method, Milstein method, SRK4 method and the adaptive step size SRK of order 1.5(1.0) with exact solution of SDEs model.

3. Results and Discussion

The fix step size SRK4, Euler and Milstein methods and the adaptive step size SRK of order 1.5(1.0) method are performed to the SDEs model,

$$dy(t) = \lambda y(t)dt + \sigma y(t)dW(t)$$

where λ and σ are real constant. The exact solution of SDE is

$$y(t) = y(0)\exp\left(\left(\lambda - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right) \tag{18}$$

The numerical results of Euler method, Milstein scheme, SRK4 and SRK of order 1.5(1.0) are compared with the exact solution (18).

Figure 1 shows the numerical solution of SRK of order 1.5(1.0) method for $\lambda = 2$ and $\sigma = 1$. Based on Fig. 1, we can see that the numerical solutions obtained show the solution of adaptive step size SRK of order 1.5(1.0) is converged to the exact solution.

Figure 2 shows the numerical solution of adaptive step size SRK of order 1.5(1.0) and fix step size SRK4 method and the exact solution. From Fig. 2, we can see that the numerical solution for SRK of order 1.5(1.0) is consistent with SRK4 method and both methods converge to the exact solution.

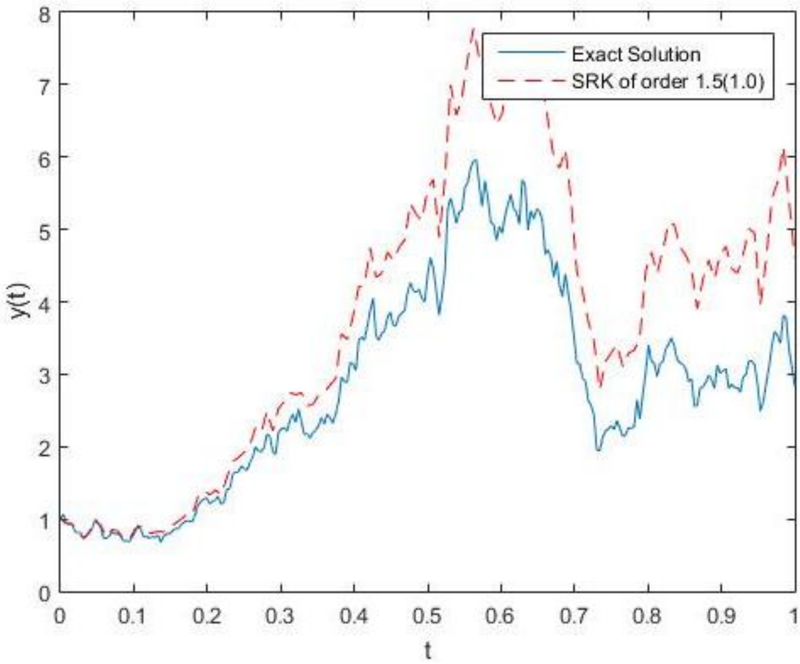

Figure 1. Numerical solution of SRK of order 1.5(1.0) method and the exact solution

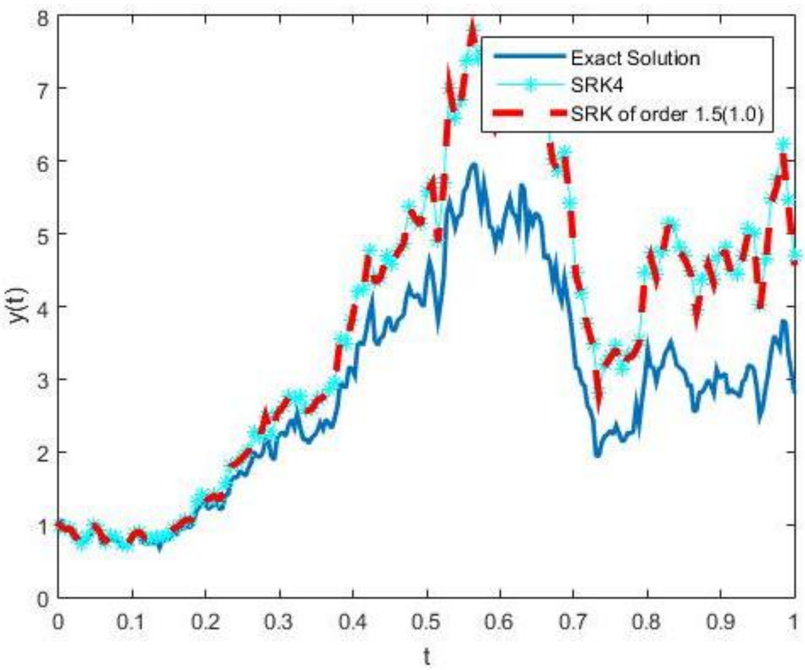

Figure 2. Numerical solution of SRK of order 1.5(1.0) method, SRK4 method and the exact solution

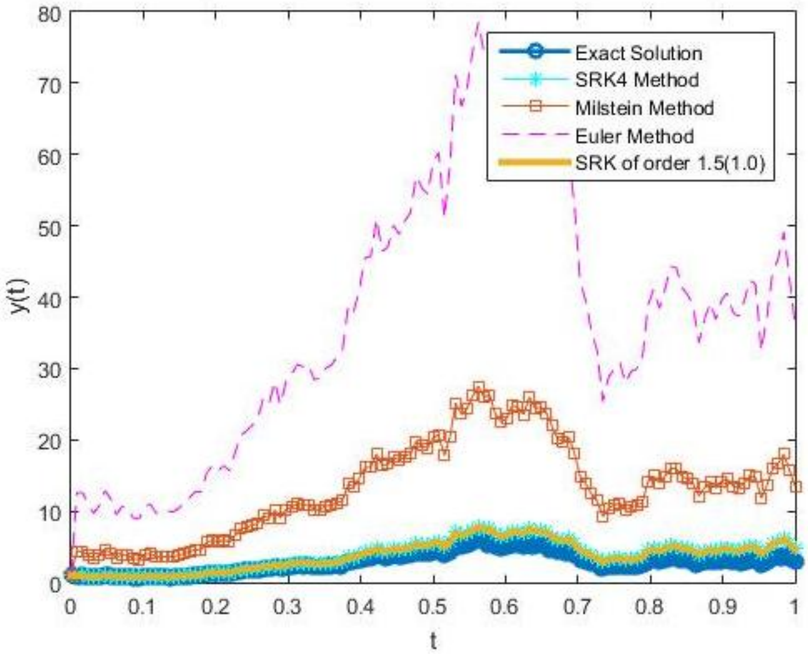

Figure 3. Numerical solution of SRK of order 1.5(1.0), SRK4, Euler, Milstein and the exact solution

The numerical solution illustrated in Figure 3 shows that the adaptive step size SRK of order 1.5(1.0) method gives the most nearly approximation to the exact solution in comparing to the fix step size SRK4, Milstein and Euler methods.

Table 1 shows the global error of SRK4, Euler, Milstein and SRK of order 1.5(1.0) methods for $t = 1$. The global error for adaptive step size SRK of order 1.5(1.0) is less than the global error for the solution obtained using fix step size SRK4, Milstein and Euler methods.

Table 1. Global error of SRK1.5(1.0), SRK4, Milstein and Euler methods

Numerical Method	Global Error
SRK1.5(1.0)	1.7739
SRK4	1.8555
Milstein	10.6079
Euler	33.1522

4. Conclusions

In summary, the solution of adaptive step size SRK of order 1.5(1.0) is converged to exact solution and the global error is less than the global error for fix step size implementation methods. We can conclude that SRK of order 1.5(1.0) method developed in this research gives better approximation for solving SDEs than 4-stage SRK (SRK4), Milstein and Euler methods. For future research, we can increase the order of convergence by embedding the order of 1.5 and 2.0. Stability analysis of the fix and variable step size methods can further investigate in the case of SDEs which require long time-integration.

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