Towards exact holography in AdS3

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Abstract

In this thesis, we develop the program of supersymmetric localization for the computation of the functional integral of string theory on $\mathrm{AdS}_3\times\mathrm{S}^2$. We are placed in the framework of off-shell 5d $\mathcal{N}=2$ supergravity coupled to vector multiplets. We first present how to set up a consistent Euclidean version of this theory. We then show how the condition of supersymmetry in the Euclidean $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$ geometry naturally leads to a twist of the S^2 around the time direction of AdS_3. The twist gives us a five-dimensional Euclidean supergravity background which is dual to the elliptic genus of $(0,4)$ SCFT_2 at the semiclassical level. On this background we set up the off-shell BPS equations for one of the Killing spinors, such that the functional integral of five-dimensional Euclidean supergravity on $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$ localizes to its space of solutions. We obtain a class of solutions to these equations by lifting known off-shell BPS solutions of four-dimensional Euclidean supergravity on $\mathrm{AdS}_2\times\mathrm{S}^2$. In order to do this consistently, we construct and use a Euclidean version of the off-shell 4d/5d lift of arxiv:1112.5371, which could be of independent interest. We then assess the consistency of these localization solutions with the standard $\mathrm{AdS}_3\times\mathrm{S}^2$ boundary conditions on which the functional integral is defined. We find that the off-shell gauge fields respect their usual conditions, but that the off-shell metric in the AdS_3 directions is not compatible with the Brown-Henneaux conditions. We show instead that the metric fluctuations are consistent with a set of chiral boundary conditions recently constructed by Compere, Strominger and Song (CSS) in arxiv:1303.2662. We subsequently use this observation to propose a partial set of boundary terms for the 5d supergravity derived from these boundary conditions. We evaluate the bulk action and these boundary terms on the localization solutions, which yields a finite and tractable expression. Lastly, we perform a numerical search for additional localization solutions in the space of asymptotic metrics obeying CSS or Brown-Henneaux boundary conditions, using recursive methods analogous to those employed in holographic renormalization.

Publications

Much of the material presented in this thesis also features in the following publications by the author:

- **A. Ciceri, I. Jeon, S. Murthy**,

  Localization on $AdS_3 \times S^2$ I: the 4d/5d connection in off-shell Euclidean supergravity

  (2023), [arXiv:2301.08084].

- **A. Ciceri, I. Jeon, S. Murthy**,

  Localization on $AdS_3 \times S^2$ II: the action

  To Appear.

Additional collaborative research in Applied Mathematics conducted by the author but not presented in this thesis appears in the following publication:

- **A. Ciceri, T. Fischbacher**,

  On backpropagating Hessians through ODEs

  (2023), [arxiv:2301.08085].

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Contents

Chapter 1

Introduction

A curious notion that emerges in certain natural phenomena is that physical information contained in a $d + 1$- dimensional "volume" can be viewed as encoded into an associated d- dimensional "surface". For such systems, one might picture the dynamics playing out in the volume as a type of hologram that is being projected from said surface. The broad terminology used to describe this physical interplay is *Holography*.

Since the latter half of the 20^{th} century, holographic frameworks have played an increasingly important role in the description of gravitational physics beyond the classical models. In today's landscape, a powerful realization of this principle, the AdS/CFT correspondence [1], constitutes a concrete theoretical laboratory in which to describe and test the high-energy effects of many gravititational theories of interest.

Black Holes: the holographic stars

Black holes in Einstein's theory of General Relativity set the stage for an early holographic description of gravitational effects beyond the purely classical regime. In general relativity, gravity is an entirely geometric concept. More precisely, the spacetime-continuum is modeled as a manifold, whose geometric properties are encoded in a metric $g_{\mu\nu}$, and which is itself the dynamical field in the theory. This geometry is then able to acquire *curvature* through various classical physical processes that are either intrinsic or extrinsic to the spacetime. It is precisely the effect of this curvature, as felt by objects interacting with the spacetime, that is interpreted as the gravitational force.

Black holes represent a very special class of strongly curved Einstein geometries. They feature an "interior" subregion which is separated from the "exterior" by a causal surface known as the *event horizon*. The location of this horizon relative to the centre of the spacetime is entirely determined by the macroscopical properties of the black hole. In the simplest case, this is just the mass. In more complicated examples, there may also be electromagnetic charges and angular momentum. Now, at this classical (i.e. geometric) level, no physical process can cross the event horizon from the inside to the outside. In a thermodynamic sense, black holes in general relativity are therefore black bodies with zero temperature.

A paradigm shift towards a more modern understanding of black holes occurred in the 1970s. The underlying ideas revolved around dressing classical black holes with *semi-classical* processes involving matter particles that interact with the geometry. This program culminated in a set of elegant equations involving the macroscopic parameters of the black hole which turned out to be in striking analogy with the well-known laws of thermodynamics [2, 3, 4]. Most remarkably, Bekenstein and Hawking described an identification of the area of the event horizon with the usual notion of entropy in thermal systems [5, 6]. Their eponymous Bekenstein-Hawking entropy formula is given as [1]

$$\mathcal{S}_{\rm BH} \;=\; \frac{k_B}{G}\frac{c^3}{\hbar}\frac{A}{4}\,, \tag{1.1}$$

where A denotes the area of the event horizon, c is the speed of light, k_B is the Boltzmann constant, G is Newton's constant. We note the following remarks. Firstly, (1.1) has a distinctly holographic flavour. Indeed, it suggests that the some type of gravitational information $\mathcal{S}_{\rm BH}$ associated with the black hole is captured onto its "surface". Secondly, the presence of $\hbar$ and G together signals that this entropy stems from a interplay of quantum and gravitational effects. Perhaps then, one should expect a notion of further quantum corrections which could only be seen beyond the semi-classical analysis. Finally, if the quantity $\mathcal{S}_{\rm BH}$ on the left-hand-side of (1.1) is truly to be understood as a thermodynamic entropy of the black hole, one should seek an analogous statistical-mechanical description, *à la* Boltzmann, in terms of a counting of microstates of the system. The second and third remarks may be summarized in the following equation:

$$\mathcal{S}_{\rm BH} + \text{corrections} \;\stackrel{?}{=}\; k_B \log\Big(\#\text{microstrates}\Big)\,. \tag{1.2}$$

While (1.2) is only schematic, it is a good representative of the challenges faced in probing gravity beyond the semi-classical regime. Indeed, on the right-hand-side, an

[1] In (1.1), we include all fundamental constants for full transparency. In the rest of this thesis, we take $c = \hbar = 1$.

obvious point of contention is how to even characterize a "microstate" in a gravitational system. On the left-hand-side, computing corrections to the thermodynamic entropy might require control over various high-energy regimes of the macroscopic theory. One such regime could correspond to allowing for stronger curvature of the spacetime, which at the level of the Einstein theory would be incorportated by higher-derivative terms in the action. Another high-energy regime could correspond to the inclusion of quantum-gravitational effects in perturbation theory, i.e. graviton loops. However, since general relativity is not a renormalizable theory, a direct approach with the usual methods of quantum field theory will fall short.

One framework in which both sides of (1.2) have successfully been explored is string theory. In the architecture of this theory, black holes have a well-understood description in terms of certain D-brane [7] configurations. It is in such a set-up that Strominger and Vafa [8] showed an explicit agreement between the Bekenstein-Hawking entropy, computed from the area of the horizon associated to these objects, and a counting of the supersymmetric states ("microstates") on their world-volume theory. In fact, the microstate computation also included corrections to the Bekenstein-Hawking side. While we do not delve into a more detailed account of the calculation, we wish conclude with the following remark: The near-horizon geometry of the D-brane configuration that was considered in [8] is $\mathrm{AdS}_3\times\mathrm{S}^3$ (times an internal compact manifold). Meanwhile, the worldvolume theory of the branes is, in an appropriate limit, a supersymmetric gauge theory.

It was three years after Strominger and Vafa's result that, beyond the specific context of black hole entropy, a profound holographic duality between gravity on Anti-de-Sitter spaces and certain supersymmetric gauge theories was proposed.

The AdS/CFT correspondence

The AdS/CFT correspondence [1, 9, 10] is the duality between theories of quantum gravity in $d+1$ dimensions with asymptotically Anti-de-Sitter (AdS_{d+1}) boundary conditions and a d-dimensional conformal field theory (CFT_d) living on the conformal boundary of the bulk. Crucially, the conformal field theory is not a gravitational theory. In its strongest form, the duality can be taken as the very *definition* of a quantum theory of gravity (on asymptotically AdS spaces) as a quantum field theory with no gravity in one dimension less.

One important feature of the correspondence, among others, is the inverse relation between coupling strength on either side. In particular, the weak-coupling regime in the CFT is dual to the strong-coupling sector of the gravitational side, over which there is little control. Vice-versa, the strong-coupling regime of the CFT is in correspondence with the low-energy effective sector of the gravitational theory, which is typically tractable, and which in the case of string theory is described by two-derivative supergravity. The earliest concrete example of the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ duality was realized in the low-energy sector of string theory in $d = 4$. Here, we have the equivalence between type IIB string theory on an $\mathrm{AdS}_5\times\mathrm{S}^5$ background and the maximally supersymmetric $\mathcal{N} = 4$ Super-Yang-Mills theory in four dimensions. Another canonical example is with $d = 2$, where we have a duality between type IIB supergravity on an $\mathrm{AdS}_3\times\mathrm{S}^3$ background compactified on T^4 or $K3$ and a $\mathcal{N} = (4, 4)$ 2d conformal theory describing a system of D1-D5 branes [8, 11]. For an in-depth review of both the above examples, we refer to [12].

A hard problem in AdS/CFT: quantum corrections

A hard problem on both sides of the AdS/CFT correspondence is the computation of quantum corrections to dual quantities. In the language of the correspondence, these correspond to finite $1/N$ corrections on the CFT side and finite g_{string} corrections on the string theory side. The presence of supersymmetry, however, makes this problem more approachable. Firstly, we can focus on observables which are protected by the supersymmetry, and therefore do not change under a continuous deformation of the weak/strong-coupling constant. Secondly, there exist powerful computational tools to capture quantum corrections which are available precisely thanks to this additional fermionic symmetry.

The quantum computation of protected observables in this context has primarily been approached from the CFT side of the duality. In contrast, it has not been explored as much on the gravitational side. One successful example in AdS/CFT where a protected observable was successfully matched at the exact level (by which we mean including *all* quantum corrections) is the recent program of black hole entropy in $\mathrm{AdS}_2/\mathrm{CFT}_1$. In the gravitational interpretation, the protected observable computes the quantum entropy of supersymmetric black holes with $\mathrm{AdS}_2\times\mathrm{S}^2$ near-horizon geometry, and has a formal path-integral formulation given by the so-called quantum entropy function [13]. The exact calculation of this string- functional integral was performed in [14] using an adaptation of supersymmetric localization to the string fields (further analysis followed in [15, 16]). In fact, instead of a localizing onto a full action of string theory, the authors conducted the calculation on a *classical* off-shell action for supergravity [17, 18, 19] with a certain renormalization. The remarkable fact that, under localization, the classical action could capture quantum corrections without recourse to perturbation theory (see [14, 20] for details on this phenomenon) has motivated further research on the application of this technique to other supergravity theories. More recently, this has produced similarly accurate results for the quantum entropy of the higher-dimensional $\mathrm{AdS}_2\times\mathrm{S}^3$ black hole [21, 22].

As alluded to above, the overarching motivation of this thesis follows in the programme of computing exact protected observable in gravitational side of the duality using localization. In particular, we wish to find out whether this strategy can be applied to higher-dimensional examples in AdS_{d+1}. A humble but sensible starting point is simply one dimension higher. We are therefore placed in the setting of $\mathrm{AdS}_3/\mathrm{CFT}_2$, where the supersymmetric observable is known in the CFT side as the *elliptic genus* [23]. Before introducing this object in more detail, we make a small aside about general partition functions and thermality in $\mathrm{AdS}_3/\mathrm{CFT}_2$. Written schematically, the correspondence postulates that

$$Z_{\mathrm{CFT}_2}(\tau,\,\overline{\tau}) \;=\; Z_{\mathrm{AdS}_3}(\tau,\,\overline{\tau})\,, \tag{1.3}$$

where Z should be thought as a dual observable which, for now, is not supersymmetric. The parameter τ ($\overline{\tau}$) is part of the moduli of the theory that couple to the conserved charges (we will say more on these later). Focusing on the left-hand-side of the duality, the observable can usually be expressed in terms of a trace over the Hilbert space $\mathcal{H}$ of the quantum field theory. In the simplest case, we have a thermal partition function given as

$$Z_{\mathrm{CFT}_2}(\tau,\,\overline{\tau}) \;=\; \mathrm{Tr}_{\mathcal{H}}\Big[\mathrm{e}^{-\beta H}\mathrm{e}^{\ell P}\Big]\,, \tag{1.4}$$

where β is the inverse temperature $1/T$. The Hamiltonian H and angular momentum P of the theory are given in terms of the zero-modes L_0, $\overline{L}_0$ of the 2d conformal algebra as

$$H \;=\; L_0+\overline{L}_0\,, \qquad P \;=\; L_0-\overline{L}_0\,, \tag{1.5}$$

and β, ℓ are given in terms of the moduli τ, $\overline{\tau}$ as

$$\beta \;=\; \frac{1}{T} \;=\; -\mathrm{i}\pi(\tau-\overline{\tau})\,, \qquad \ell \;=\; \mathrm{i}\pi(\tau+\overline{\tau})\,. \tag{1.6}$$

The Hamiltonian in (1.5) naturally splits into Hamiltonians for the left- and right-moving part as $H=H_L+H_R$, with $H_L=L_0$, $H_R=\overline{L}_0$ (similarly for the angular momentum). The temperature T may also be split as $1/T=1/T_L+1/T_R$ with $T_L=-\mathrm{i}/(\pi\tau)$, $T_R=\mathrm{i}/(\pi\overline{\tau})$. Then, the trace (1.4) separates into an independent left- and right- moving sector, with temperatures given by T_L and T_R respectively. Now suppose taking $T_R \to 0$, which should be thought of as imposing supersymmetry on the right-moving sector. With this value of T_R, it is clear from the relation between T, T_L and T_R, that despite T_L still being non-trivial, T will nevertheless vanish. The message we wish to highlight is this: the computation of a right-moving supersymmetric version of the *finite-temperature* partition function (1.4) is a computation at *zero temperature*. Yet, on the left-movers, one still has a notion of thermality (and therefore states that are thermal-like excitations).

From here on, we turn to setting our problem up in a more technical manner. The approach to compute a supersymmetric protected observable in $\mathrm{AdS}_3/\mathrm{CFT}_2$ from the bulk using localization should naturally begin with the following steps. In Section 1.1 we introduce the elliptic genus as the protected observable of certain supersymmetric conformal field theories in two dimensions. We also discuss expectations on the form of the dual gravitational theory and its low-energy effective theory. In Section 1.2 we review the methodology of supersymmetric localization which, applied to the low-energy effective gravitational action, we hope can yield an exact result for the functional integral in the bulk. The chapter is concluded with Section 1.3, where we give an overview of the various upcoming steps taken in this thesis.

1.1 Elliptic genera in (0,4) SCFT_2 and the dual set-up

In this thesis we work in $\mathrm{AdS}_3/\mathrm{CFT}_2$. We take the boundary theory to be a Lorentzian (1+1)-d superconformal field theory (SCFT_2) with $\mathcal{N} = (0, 4)$ supersymmetry, living on $\mathrm{S}^1 \times \mathbb{R}$ (i.e. an infinite cylinder). The protected observable is the *elliptic genus*, which is a supersymmetric index of the theory. It is defined as the following trace over the Hilbert space of the theory (we suppress the subscript $\mathcal{H}$ for the Hilbert space):

$$\chi(\tau,\, \mu) \;:=\; \mathrm{Tr}_{\mathrm{R}}\Big[(-1)^F q^{L_0-\frac{c}{24}}\, \overline{q}^{\overline{L}_0-\frac{\bar{c}}{24}}\, \mathrm{e}^{q_I \mu^I}\Big]\,, \tag{1.7}$$

where $q \equiv \mathrm{e}^{2\pi\mathrm{i}\tau}$, $\overline{q} \equiv \mathrm{e}^{-2\pi\mathrm{i}\overline{\tau}}$, c and $\bar{c}$ are the central charges for the left- and right-moving Virasoro algebras respectively, and where we have a collection of left-moving U(1) charges, denoted as q_I, coupled to their corresponding chemical potentials μ^I. The operator $(-1)^F = (-1)^{2\overline{J_0^3}}$ is the fermion number operator, where $\overline{J_0^3}$ is the SU(2) R-current of the $\mathcal{N}=4$ superconformal algebra of the right-moving sector. The R subscript on the trace is related to fact that the supersymmetric theory has fermions, and so their periodicity around the S^1 need to be specified. The letter stands for the *Ramond* sector, which indicates that we choose *periodic* conditions. Alternatively, one could choose *anti-periodic* fermions, which corresponds to the *Neveu-Schwarz* (NS) sector. The two sectors are related by an automorphism of the $\mathcal{N}=4$ superconformal algebra known as the spectral flow [2]. The counting of states that the elliptic genus performs is as follows: In the right-moving sector, where we have supersymmetry, (1.7) is a Witten index with Hamiltonian $H_R = \overline{L}_0 - \bar{c}/24$. Therefore, the only right-moving states that contribute are the ground states $\overline{L}_0 = \bar{c}/24$. Note that this is why $\chi(\tau, \mu)$ is independent of $\overline{\tau}$. Meanwhile, all left-moving states can contribute.

Now, by the usual rules of statistical field theory, the trace (1.7) is equivalent to a Euclidean path integral. More precisely, we have a Wick-rotated (i.e. Euclidean) time direction which has been compactified as $t_E \sim t_E + \beta$, where $\beta = 2\pi\mathrm{Im}(\tau)$ as in (1.6). In this formalism, the SCFT_2 now lives on a $S^1 \times S^1$, i.e. a torus. We must then recall the periodicities of the fields when integrating along the new (time-) circle: generically, bosonic fields are periodic while fermions are anti-periodic. For the fermions, however, the $(-1)^F$ in the trace (1.7) translates to an additional subtlety. Indeed, this operator insertion flips the sign of the fermions as they are taken around the time circle. Spinors in the Euclidean path-integral for the elliptic genus (1.7) therefore have periodic boundary conditions around time.

What can we say about the gravitational dual?

Under the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, the functional integral for the elliptic genus (1.7) is dual to the functional integral of a quantum-gravitational theory with eight supercharges and AdS_3 boundary conditions. By AdS_3 boundary conditions, we mean that the conformal boundary should be fixed to a T^2 with complex structure τ. Furthermore, it is known that the $(0,4)$ SCFT_2 necessarily has an SU(2) R-symmetry, and this should therefore be reflected in the bulk. The most natural implementation of this comes in the form of an additional S^2 factor, which the SU(2) rotates. Our expectation for the asymptotic geometry in the bulk is therefore $\mathrm{AdS}_3 \times S^2$. Finally, the coupling in (1.7) to U(1) charges implies that the gravitational theory should also contain U(1) gauge fields W^I, where their values at the asymptotic boundary of the bulk will source the chemical potentials μ^I in the SFCT_2.

[2] A presentation of the $\mathcal{N}=4$ superconformal algebra, along with a discussion on spectral flow, is given in Appendix B.

It turns out that there are known examples in the full string theory, for which these considerations manifest themselves. A well-established family comes from a class of embeddings due to Maldacena, Strominger and Witten (MSW) [24]. One starts from M-theory compactified on $\mathbb{R}^{1,4}\times\mathrm{CY}$ and wraps a fivebrane (M5) around four-cycles of the Calabi-Yau. This gives a (1+1)-dimensional string in five dimensions. The string has a horizon (it is referred to as a *black string*). In the near-horizon limit, it exhibits an $\mathrm{AdS}_3\times\mathrm{S}^2$ geometry. Moreover, at low energies, the theory on the worldsheet of the string flows to a SCFT_2 with $\mathcal{N}=(0,4)$. This set up suggests that the suitable low-energy gravitational dual to the $\mathcal{N}=(0,4)$ theory is the 5d $\mathcal{N}=2$ supergravity (i.e. 8 supercharges) coupled to vector-multiplets that governs the dynamics of the black string.

Setting up the dual functional integral

These considerations suggest that the dual of the elliptic genus (1.7) can be derived exactly by solving the localization problem for the following functional integral:

$$Z^{PI}(\tau,\,\mu)\;:=\;\int_{T^2}[D\Phi]\,\exp\Big(S_{\rm ren}[\Phi]\Big)\,, \tag{1.8}$$

where $S_{\rm ren}$ denotes the action for the 5d supergravity theory described above, renormalized by potential boundary terms. Recall that τ and μ enter through the boundary conditions on the metric and the U(1) gauge fields respectively. Note that while we have suppressed the notation associated to sphere factor, readers should keep in mind that the topology of the conformal boundary is $T^2\times\mathrm{S}^2$ and not just T^2.

Two important immediate comments on the computation of (1.8) are in order. Firstly, as we will describe in upcoming review of localization in Section 1.2, the supergravity action entering $S_{\rm ren}$ should not be the Poincaré theory but rather its off-shell adaptation. Secondly, gravitational path integrals on AdS_3 such as (1.8) have been shown by Strominger and Maldacena [25] to admit a special structure, which we simply quote here (but also briefly review in Chapter 3). The statement is that the functional integral with T^2 boundary conditions splits into a discrete sum, where each term in the sum is itself a functional integral with T^2 boundary conditions *and fixed contractible cycle in the bulk*. There are infinitely many ways of fixing this cycle (and therefore infinitely many terms in the sum), and the choices are labelled by a distinct pair of *relatively prime* integers (c,d) with $c\geq 0$. More formally, this allow us to write

$$
\begin{aligned}
Z^{PI}(\tau,\,\mu) \;&=\; \sum_{(c,d)} \int_{\partial M_{c,d}} [D\Phi] \exp\Big(S_{\rm ren}[\Phi]\Big)\,, \\
&\equiv \sum_{(c,d)} Z^{PI}_{c,d}(\tau,\,\mu)\,.
\end{aligned}
\tag{1.9}
$$

Here $M_{c,d}$ denotes the solid torus with contractible cycle as $(c,d) = (c,d)$. The simplest representative of the $M_{c,d}$ geometries is $M_{0,1}$ geometry, which corresponds to the torus with contractible circle along the spatial direction of AdS_3. A further simplification of (1.9), which reduces the computation of the infinitely many functional integrals to the computation of one, comes from the fact that the $M_{c,d}$ geometries can obtained by action with the elements of $\mathrm{PSL}(2,\mathbb{Z})/\mathbb{Z}$ on the boundary T^2 of $M_{0,1}$ (or of any representative). In this way, (1.9) is in fact a sum over $\mathrm{PSL}(2,\mathbb{Z})/\mathbb{Z}$ images, and it becomes sufficient to compute the functional integral on the one representative $M_{0,1}$. From here on, our working expression for the functional is therefore

$$
Z^{PI}_{0,1} \;=\; \int_{\partial M_{0,1}} [D\Phi] \exp\Big(S_{\rm ren}[\Phi]\Big)\,. \tag{1.10}
$$

We close with a remark on the periodicities of the fermions in the computation of this contribution (1.10). Because the spatial cycle in $M_{0,1}$ is contractible in the bulk, the fermions are forced to be anti-periodic around that circle. Therefore, we expect that the calculation of $Z^{PI}_{0,1}$ is in the NS-sector of the elliptic genus. This is made explicit in Chapter 5.

1.2 Exact functional integrals with supersymmetric localization

In supersymmetric theories, the principle of (supersymmetric) localization leverages the presence of fermionic symmetries to reduce an a-priori infinite-dimensional functional integral to a finite number of ordinary integrals. This strategy was initially approached in the context of physics by Witten in [26]. Much later, concrete results in certain supersymmetric quantum field theories were derived by Pestun in [27]. In this section, we review the key features of the methodology, and discuss the particularities and challenges relevant to its application to supergravity theories.

We consider functional integrals of the form

$$W \;=\; \int [D\Phi] \exp\Big(S[\Phi]\Big)\,, \tag{1.11}$$

where S is the action functional for the system and Φ collectively denotes set of quantum fields of the theory. The basic assumption to begin the algorithm of localization is that the theory admits a fermionic charge $\mathcal{Q}$ under which the action is $\mathcal{Q}$-exact, i.e. $\mathcal{Q}S \,=\, 0$, and such that $\mathcal{Q}^2 \,=\, H$ where H is a compact bosonic generator on the isometry space of the background spacetime. Now, let V be a fermionic function satisfying $\mathcal{Q}^2 V \,=\, 0$. One makes the following deformation in the exponent of the path integral:

$$W(\lambda) \;=\; \int [D\Phi] \exp\Big(S[\Phi] + \lambda \mathcal{Q}V\Big)\,. \tag{1.12}$$

Assuming the integration measure is also $\mathcal{Q}$-invariant, it can be shown [26] that

$$\frac{dW(\lambda)}{d\lambda} \;=\; 0\,, \tag{1.13}$$

and so the functional invariant is invariant under the choice of λ. In particular, we have that

$$W(\lambda = 0) \;=\; W \;=\; W(\lambda \,\to\, \infty)\,. \tag{1.14}$$

In the expression on the right-hand-side, the λ-dependent term will dominate the exponent and so the functional integral localizes onto the saddle points of the functional

$$\mathcal{Q}V \;=\; 0\,. \tag{1.15}$$

Finally, one chooses

$$V \;=\; \sum_{\alpha} \Big(\mathcal{Q}\psi_\alpha,\, \psi_\alpha\Big)\,, \tag{1.16}$$

where ψ_α denotes the fermionic fields of the theory and $(\cdot,\cdot)$ is an appropriate inner product for the fermions. Restricting to a bosonic background (i.e. $\psi_\alpha \,=\, 0$), the localizing equation (1.15) reduces to

$$\mathcal{Q}V \;=\; 0 \quad\Longleftrightarrow\quad \mathcal{Q}\psi_\alpha \;=\; 0\,. \tag{1.17}$$

In the language of supersymmetric theories (including supergravity), note that the equation on the right is simply the set of BPS equations of the theory. Practically speaking, these are the equations given by setting the supersymmetry variation (with respect to the SUSY-parameter ϵ of $\mathcal{Q}$) of the fermions to zero, i.e. $\delta_\epsilon \psi_\alpha = 0$. The field configurations that solve the BPS equations are typically referred to as the BPS solutions. In the context of the localization problem, we also commonly use the term *localization solutions*. The set of all localization solutions for $\mathcal{Q}$ is called the localization manifold, and is denoted as $\mathcal{M}_\mathcal{Q}$. With (1.17), we therefore have that the infinite-dimensional path integral (1.12) simplifies to an integration over the submanifold $\mathcal{M}_Q$ of the full configuration space:

$$W \;=\; W(\lambda \to \infty) \;=\; \int_{\mathcal{M}_\mathcal{Q}} [d\phi_\mathcal{Q}]\, \mathrm{e}^{S[\phi_\mathcal{Q}]}\, W_{\text{1-loop}}\,, \tag{1.18}$$

where $\phi_\mathcal{Q}$ denotes coordinates on $\mathcal{M}_\mathcal{Q}$, $S[\phi_\mathcal{Q}]$ is the action evaluated on an arbitrary point on $\mathcal{M}_\mathcal{Q}$, and $W_{\text{1-loop}}$ is a one-loop functional determinant factor due to the fluctuations in the non-BPS directions around $\mathcal{M}_Q$.

A key point to highlight is that this technology requires an *off-shell* formulation of the supersymmetric theory at hand. Indeed, since the functional integral is computed over off-shell fluctuations, the superalgebra involving the localization supercharge $\mathcal{Q}$ must be realized off-shell (i.e. must close without imposing equations of motion). Now, compared to the case of supersymmetric QFTs, it turns out that for supergravities such off-shell formulations are notoriously involved. This constitutes a large part of the technical challenge when working with localization in these bulk theories. In our case, recall from Section 1.1 that the relevant low-energy holographic dual to a $(0,4)$ SCFT_2 with left-moving U(1) charges is minimal supergravity in five dimensions coupled to U(1) gauge fields (with certain $\mathrm{AdS}_3 \times \mathrm{S}^2$ boundary conditions). The off-shell realization of this theory exists and was constructed as part of the conformal supergravity program in [28, 29, 30] and [31, 32]. Its *Euclidean* version, whose relation the original *Lorentzian* theory we clarify in Section 5.1, is the supersymmetric bulk theory that we will work with throughout this thesis.

1.3 Strategy and outline of this thesis

In this thesis, we work towards the exact calculation of the functional integral (1.10) for the theory on $M_{0,1}$ (times S^2) using localization in the dual Euclidean

5d $\mathcal{N} = 2$ supergravity with U(1) vector multiplets. Here, we present our strategy in more detail, before giving an overview of each of the following chapters.

Breakdown of strategy

As discussed in Section 1.1, the $M_{0,1}$ geometry is the solid torus with contractible cycle in the spatial direction of AdS_3. We therefore work with the five-dimensional off-shell theory defined on global $\mathrm{AdS}_3 \times \mathrm{S}_2$ with a periodic Euclidean time coordinate, i.e. the manifold $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$. Following the methodology presented in Section 1.2, the problem then begins with finding all bosonic gravitational configurations that admit a Killing spinor whose asymptotic limit is one of the supercharges of the classical $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ vacuum. Secondly, one finds all matter configurations invariant under this supercharge. Thirdly, one constructs a suitable set of boundary terms for the bulk supergravity and evaluates the resulting renormalized action at a generic point in the resulting localization manifold. Finally one calculates the one-loop determinant of the non-BPS fluctuations around said manifold. In this thesis, we reach the third point.

To reach the first rung of the procedure, we require a consistent set of supersymmetry transformations in the off-shell five-dimensional Euclidean supergravity theory. Starting from the Lorentzian 5d $\mathcal{N} = 2$ conformal supergravity coupled to vector- and hyper-multiplets, which we review in Chapter 4, we explain how to obtain such transformations in Section 5.1. We soon run into another subtlety, also due to the Euclidean signature of the problem, which regards the set-up of the supersymmetric $\mathbb{H}^2/\mathbb{Z} \times \mathrm{S}^2$ vacuum configuration of the theory. Indeed, with the naive identifications of the coordinates on $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$, the Killing spinors are not well-defined with respect to the periodicities of fermions around the non-contractible circle of the torus. We resolve this problem in Section 5.2 by turning on a twist of the S^2 around this circle, which allows for spinors which are now constant in time and therefore well-defined.[3] In this same section, we also show that this twist reduces the asymptotic algebra to be a sub-algebra of the Brown-Henneaux-Coussaert [35, 36] $(0, 4)$ algebra on AdS_3. Finally, because the Killing spinors on $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ are anti-periodic along the spatial circle (which is expected due to its contractibility) we relate in Section 5.3 the bulk calculation of the functional integral to the NS-sector calculation of the elliptic genus (1.7). We also evaluate the 5d action on the $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ configuration and compare with the result of the effective 3d Einstein-Hilbert-Maxwell action evaluated on thermal AdS_3.

[3]A related supersymmetric set-up has been discussed in the literature in the context of supersymmetric black holes in AdS space [33] and, in particular, for BTZ black holes in [34].

With this non-trivial groundwork in place, we then approach the first and second steps of the localization procedure, namely the computation of localization solutions in the gravitational and matter-coupling sectors of the off-shell supergravity theory on $\mathbb{H}^2/\mathbb{Z}\times\mathrm{S}^2$. Our idea is to use the 4d/5d lift [37], which relates solutions of off-shell 4d supergravity to those of off-shell 5d supergravity compactified on a circle.[4] The localization manifold in 4d $\mathcal{N}=2$ supergravity on asymptotically $\mathrm{AdS}_2\times\mathrm{S}^2$ has been completely determined [39], and we can lift those solutions to obtain localization solutions to $\mathrm{AdS}_3\times\mathrm{S}^2$. Although this is not guaranteed to produce all BPS solutions, it should give all solutions that are independent of the circle of compactification. Similar ideas have been used successfully to make progress in the localization problems on $\mathrm{AdS}_2\times\mathrm{S}^3$ theories in [40], [21, 22].

It transpires that implementing this idea is not straightforward. Firstly, the 4d/5d map in [37] is given for Lorentzian backgrounds while we need it for Euclidean backgrounds. To this end, we modify the map to reflect the Euclidean supersymmetry transformations in both the four- and five-dimensional theory. In four dimensions we use the Euclidean supergravity discussed in [41, 42, 43, 44], [45], while in five dimensions we employ the Euclidean transformations discussed in Section 5.1. Here there is an additional problem compared the Lorentzian setting, namely that the 4d Euclidean theory carries a redundancy of allowed reality conditions which has no counterpart in the 5d Euclidean theory. We show that this redundancy can be absorbed in a parameter whose role is to implement the symmetry breaking $SO(1,1)_R\to\mathbb{I}$.[5] The second problem has to do with the global identifications of the background that we are interested in, i.e. $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$, which is not a Kaluza-Klein lift of Euclidean $\mathrm{AdS}_2\times\mathrm{S}^2$. The Kaluza-Klein condition was used in [37] for the off-shell 4d/5d lift and, indeed, general off-shell configurations do not consistently lift from Euclidean $\mathrm{AdS}_2\times\mathrm{S}^2$ to $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$. Nevertheless, the class of off-shell solutions relevant for the 4d black hole problem can be lifted to the supersymmetric $\mathbb{H}_3/\mathbb{Z}\times\mathrm{S}^2$, due to their enhanced rotational symmetry. Taking all these considerations into account, we obtain an adaptation of the 4d/5d lift relevant for the Euclidean $\mathrm{AdS}_3/\mathrm{CFT}_2$ problem, whose details we work out in Chapter 6. In Chapter 7, we apply this lift to find a highly non-trivial class of off-shell solutions in the theory on $\mathbb{H}_3/\mathbb{Z}\times\mathrm{S}^2$.

However, we promptly show in Chapter 8 that while the off-shell gauge fields in these solutions are consistent with the boundary conditions for the functional integral, the same is not true for the off-shell metric. Indeed, we find that the metric

[4] This is different from the 4d/5d lift of [38] which involves a lift on a Taub-NUT space.

[5] This parameter is the Euclidean analog of the parameter that enforces $SO(2)_R\to\mathbb{I}$ in [37].

fluctuations in the AdS_3 directions explicitly violate the Brown-Henneaux boundary conditions. We nevertheless persevere with an analysis of the solutions and find that the gravitational modes in fact obey a recent, more exotic construction of asymptotically AdS_3 boundary conditions developed by Compère, Strominger and Song (CSS) [46]. In Section 8.2, this finding encourages us to explore a renormalization scheme of the supergravity action with respect to these boundary conditions for the metric, following a set of boundary terms prescribed in the context of the pure three-dimensional theory in [46]. The gauge-field sector is also renormalized with respect to their standard boundary conditions. We then evaluate the localization solutions on this renormalized action and discuss its structure. In Section 8.3, we initialize a parallel study on the existence of additional BPS solutions, and in particular metric solutions that obey the Compere-Strominger-Song boundary conditions in the AdS_3 directions. Our idea is to explore the space of solutions to the off-shell Killing spinor equation in the asymptotic regime using Fefferman-Graham-like ansätze for the bosonic fields. We then recursively solve for the coefficients order-by-order in the Killing spinor equation. We find evidence for additional BPS modes obeying the CSS boundary conditions, as well as modes living in the left-moving Brown-Henneaux sector. We conclude with a short analysis of these Brown-Henneaux modes under the off-shell 4d/5d reduction to the theory on $\mathrm{AdS}_2\times\mathrm{S}^2$.

Outline of this thesis

Review chapters: The first three chapters that follow this introduction contain important background material in the set up of the localization problem. In Chapter 2, we review the classical theory of pure Einstein gravity in (2+1) dimensions with negative cosmological constant. We gain an understanding of the geometrical properties of AdS_3 solutions, their asymptotic structure, and boundary conditions. In Chapter 3, we move to a Euclidean setting of the same classical (2+1)-dimensional theory with suitable boundary conditions. In this setting, we briefly review the computation of the gravitational partition function on AdS_3, which is the dual of (1.4), as a sum over a Maldacena-Strominger family of geometries [25]. We then move towards the gravitational dual of the elliptic genus (1.7) by introducing into the Einstein action a coupling to U(1) matter gauge fields. In Chapter 4, the five-dimensional theory with eight supercharges ($\mathcal{N}=2$) and matter couplings is introduced. As required by the localization procedure, we focus on the off-shell superconformal formalism of this theory. In this framework, we review the classical Lorentzian $\mathrm{AdS}_3\times\mathrm{S}_2$ solution, focusing on the derivation of its Killing spinors and superalgebra. This has been studied in a series of insightful papers [47, 48].

Chapters containing new results: Upon conclusion of these review chapters, we move on to the novel elements that we have developed towards the localization computation, and which we described in the"Breakdown of strategy" discussion above. In Chapter 5 we first set up the Euclidean counterpart of the five-dimensional off-shell $\mathcal{N} = 2$ supergravity reviewed in Chapter 4. We then construct the twisted supersymmetric $\mathbb{H}^3/\mathbb{Z}$ background, and derive its Killing spinors and superalgebra. Finally, we present the relation of the path integral to the trace definition of the elliptic genus. In Chapter 6, we present the off-shell 4d/5d map modified to the Euclidean signature and present the lift from the Euclidean $\mathrm{AdS}_2{\times}\mathrm{S}^2$ background of the 4d $\mathcal{N} = 2$ off-shell supergravity to the $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ background constructed in the prior chapter. In Chapter 7 we apply our formalism to lift localization solutions around $\mathrm{AdS}_2{\times}\mathrm{S}^2$ to localization solutions on $\mathbb{H}^3/\mathbb{Z}{\times}\mathrm{S}^2$. In Chapter 8, we analyze the asymptotics of the new localization solutions and propose a boundary term structure for the bulk action. Lastly, we perform a numerical search in the asymptotic regime for additional localization solutions in the Weyl multiplet.

Chapter 2

AdS_3 gravity

In this chapter, we review the classical theory of general relativity with a cosmological constant, in Lorentzian signature, focusing on the (2+1)-dimensional setting. After emphasizing the unique subtleties of the theory in this dimensionality, we restrict to the sign of the cosmological constant compatible with locally Anti-de-Sitter (AdS) spacetimes, and highlight a selection of relevant solutions. Finally, we review aspects of the near-boundary regime of the theory, in particular the notions of asymptotically AdS_3 spaces and boundary conditions. This latter discussion includes a brief review of the Brown-Henneaux boundary conditions [35], which are the standard AdS_3 boundary conditions on which the gravitational functional integral is defined, as well as an alternative set of boundary conditions recently constructed by Compère, Strominger and Song [46].

2.1 3d Einstein-Hilbert theory

In this section we review the key features of classical Einstein-Hilbert gravity in (2+1) dimensions coupled to an arbitrary cosmological constant.

Action and conventions

The Einstein-Hilbert action for a three-dimensional Lorentzian spacetime $(\mathcal{M}, g)$, coupled to a cosmological constant $\Lambda \in \mathbb{R}$, is

$$S = \frac{1}{16\pi G_3} \int_{\mathcal{M}} d^3x \sqrt{-g}(R - 2\Lambda)\,, \tag{2.1}$$

where G_3 is the three-dimensional Newton's and R is the Ricci scalar, or scalar curvature. The metric determinant is denoted throughout this work as $g \equiv \mathrm{Det}(g_{\mu\nu})$.

The field equations derived from (2.1) strongly constrain the geometry of solutions, as we will shortly see. For instance, we will show that all solutions to the field equations have constant scalar curvature proportional to the cosmological constant Λ. The solution space is then further partitioned according to the sign of Λ. Before describing this partition, we note the following remark about our curvature conventions (which are summarized in Appendix A): our expression for the Riemann tensor (A.6) in terms of $g_{\mu\nu}$ has an overall opposite sign compared to the standard GR literature (e.g. see [49]), and so our Ricci scalar R is also related to that in standard conventions by an overall sign. For convenience [1], we then also flip the sign of the cosmological constant Λ. These differences result in our Lorentzian Einstein-Hilbert action (2.1) appearing with an overall opposite sign compared to the standard literature (which is minus).

Now, in our conventions, the choice $\Lambda > 0$ in (2.1) leads to solutions corresponding to locally *Anti-de-Sitter* (AdS) spacetimes, which have constant *positive* scalar curvature. The theory (2.1) with this sign of Λ is correspondingly referred to as *AdS gravity*. The $\Lambda < 0$ sector yields locally *de-Sitter* (dS) spacetimes, which have constant negative scalar curvature. Finally, setting $\Lambda = 0$ leads to solutions with zero curvature, i.e. locally Minkowski spacetimes. In this thesis, the focus is exclusively on the AdS sector.

Einstein's equations

To obtain the field equations for the metric tensor field $g_{\mu\nu}$, consider the first variation of (2.1) with respect to the inverse metric $g^{\mu\nu}$:

$$\begin{aligned}(16\pi G_3)\delta S \;=\; & \int_{\mathcal{M}} d^3x\sqrt{-g}\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\Big)\delta g^{\mu\nu} \\ & - \int_{\mathcal{M}} d^3x\sqrt{-g}\,\nabla_\sigma\Big(g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^{\nu}_{\nu\mu}\Big)\,. \end{aligned} \tag{2.2}$$

In this section only, we assume that the variational principle has been made well-defined, i.e. that fall-off conditions have been imposed on $g_{\mu\nu}$ such that the total derivative term in the second line vanishes (either identically or with the addition of suitable boundary terms). Then, setting the surviving bulk integrand in (2.2) to zero, one reads off the Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \;=\; -\Lambda g_{\mu\nu}\,. \tag{2.3}$$

[1]More precisely: it is convenient to have that solutions obtained from $\Lambda > 0$, $\Lambda < 0$, have scalar curvature $R > 0$, $R < 0$, respectively.

Note that equation (2.3) with $\mu,\,\nu = x^0,\,\cdots x^{d-1}$ is, in fact, the field equation for the generic d-dimensional[2] Einstein-Hilbert theory. This allows us to perform the following manipulations in this more general setting. Acting on (2.3) with $g^{\mu\nu}$ and using $g_{\mu\nu}g^{\mu\nu} = d$ gives

$$R \;=\; \frac{2d}{d-2}\Lambda\,. \tag{2.4}$$

This relation indicates that Einstein metrics have constant scalar curvature proportional to the cosmological constant Λ. In particular, solutions in the AdS-gravity sector ($\Lambda > 0$) have constant positive curvature. Substituting (2.4) into (2.3), the Einstein equation reduces to:

$$R_{\mu\nu} \;=\; \frac{2\Lambda}{d-2}g_{\mu\nu}\,. \tag{2.5}$$

Now, we return to the case of $d = 3$. A special feature of this dimensionality is that the Weyl tensor vanishes identically. This additional geometric constraint, which is not present in Einstein theory with $d > 3$, further restricts the solution space. A vanishing Weyl tensor indeed implies that the Riemann tensor is entirely determined by the Ricci tensor as

$$R_{\mu\nu\sigma\lambda} \;=\; 2\Big(g_{\mu[\sigma}R_{\lambda]\nu} - g_{\nu[\sigma}R_{\lambda]\mu}\Big) - g_{\mu[\sigma}g_{\lambda]\nu}R\,. \tag{2.6}$$

Substituting (2.4) and (2.5) into the above, we then have that

$$R_{\mu\nu\sigma\lambda} \;=\; \Lambda(g_{\mu\sigma}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\sigma})\,. \tag{2.7}$$

The implications of (2.7) are major: we have that all three-dimensional Einstein metrics of (2.1) are locally diffeomorphic to one another. In particular, they are locally diffeomorphic to the vacuum solution of the theory, which in the case of $\Lambda > 0$ is pure AdS_3.

The following equivalent statement is typically made: there are no *propagating local degrees of freedom* in three-dimensional Einstein gravity. This is can be illustrated by counting the physical degrees of freedom of a 3d Einstein-metric tensor: the metric in three dimensions has six independent components, but only three are dynamical (in the sense that only three appear with a timelike derivative in the Lagrangian). Taking the 3d diffeomorphisms into account, we are then left with zero physical de-

[2]Note that we now use d to denote the dimensionality of the bulk spacetime, and not the dimensionality of the boundary theory.

grees of freedom. In other words, the theory does not admit local excitations on the vacuum solution.

While these considerations seem to render 3d gravity trivial, a closer inspection reveals a more subtle story. Firstly, different *global* properties can be imposed on the boundary manifold which leads to physically distinct solutions, as we will see. It so turns out, in particular, that these "global degrees of freedom" can account for many features of gravity in higher dimensions, notably black hole solutions [50] in the case of $\Lambda > 0$. Secondly, studying the theory under certain choices of boundary conditions for $g_{\mu\nu}$ reveals the emergence of rich boundary dynamics, such as the so-called *boundary gravitons* discovered by Brown and Henneaux [35].

From here onwards, we fix the cosmological constant as

$$\Lambda = \frac{1}{4\ell^2}\,, \tag{2.8}$$

thus restricting to the AdS-gravity sector. By (2.4), all Einstein metrics have scalar curvature

$$R = \frac{3}{2\ell^2}\,. \tag{2.9}$$

Here, $\ell \in \mathbb{R}$ is a scale that will appear as an overall prefactor in the Einstein metrics.

We now turn to describing solutions to the theory with (2.8).

2.2 AdS$_3$ solutions

The spectrum of solutions to (2.1) with (2.8) consists of locally diffeormophic geometries with constant positive curvature, which are known as Anti-de-Sitter spaces. In this section, we briefly review the description of locally AdS$_3$ metrics through the so-called embedding formalism, before restricting our attention to two particular solutions of interest. These are, firstly, the global patch of the AdS$_3$ vacuum itself, or pure AdS$_3$, and secondly, the three-dimensional black hole solution. The distinction in the global properties of these solutions will be highlighted.

Embedding formalism

A powerful formalism for describing locally AdS$_3$ metrics (more generally AdS$_d$ metrics) is the *embedding formalism.*

Consider the space $\mathbb{R}^{2,2}$ covered by coordinates T_1, T_2 in the timelike directions and X_1, X_2 in the spacelike directions. With signature conventions $(-,-,+,+)$, the metric is

$$ds^2 \;=\; -(dT_1)^1 - (dT_2)^2 + (dX_1)^2 + (dX_2)^2\,. \tag{2.10}$$

AdS$_3$ is then defined as the hyperboloid in $\mathbb{R}^{2,2}$ given through the following embedding equation:

$$-(T_1)^1 - (T_2)^2 + (X_1)^2 + (X_2)^2 \;=\; -L^2\,, \tag{2.11}$$

where in our context we take $L = 2\ell$. The hyperboloid defined by (2.11) manifestly inherits boosts- and rotation-invariance from the $\mathbb{R}^{2,2}$ metric (2.10) (but breaks translations). The isometry group of locally AdS$_3$ spaces is therefore SO$(2,2)$.

A set x^μ of coordinates on the hyperboloid are obtained by choosing a parametrization for $T_{1,2}(x)$, $X_{1,2}(x)$ in terms of x^μ, such that (2.11) is satisfied. The corresponding AdS$_3$ metric is then the induced metric on the hyperboloid, obtained by substituting the chosen parametrization into (2.10). For instance, denoting $x^\mu = (\rho,\, \psi,\, t)$, one such parametrization is:

$$\begin{aligned} T_1 \;&=\; L\cosh\rho\,\cos t\,, \qquad & T_2 \;&=\; L\cosh\rho\,\sin t\,, \\ X_1 \;&=\; L\sinh\rho\,\sin\psi\,, \qquad & X_2 \;&=\; L\sinh\rho\,\cos\psi\,. \end{aligned} \tag{2.12}$$

with $\rho \in [0,\,\infty)$, $\psi \in [0, 2\pi)$, $t \in [0, 2\pi)$. The AdS$_3$ metric with the choice (2.12) is special, as we now discuss.

Global AdS$_3$

The vacuum solution to the Einstein's equations (2.5) with $\Lambda > 0$ is pure AdS$_3$. We analyze this space in the so-called global patch, or *global coordinates*.

A line element for pure AdS$_3$ in global coordinates is obtained as the induced metric on the hyperboloid (2.11) with embedding-space parametrization given in (2.12):

$$ds^2 \;=\; (2\ell)^2\Big(-\cosh^2\rho\,dt^2 + d\rho^2 + \sinh^2\rho\,d\psi^2\Big)\,. \tag{2.13}$$

where t is the time coordinate with range $(-\infty, \infty)$ [3], ρ is the radial coordinate with range $[0, \infty)$, and ψ is the compact angular direction with range $[0, 2\pi)$. The constant $\ell \in \mathbb{R}$, which appears through the choice of cosmological constant in (2.8), is referred to as the *radius*, or *scale* of AdS$_3$ [4]. These coordinates give a universal covering of the hyperboloid (hence *global* coordinates), and we note that the boundary

[3] Note that, compared to (2.12), we have unwrappped the range of t from $t \in \mathrm{S}^1$ to $t \in \mathbb{R}$. This is required to avoid closed timelike curves.

[4] The choice (2ℓ) for the radius in (2.13) is chosen for later convenience. The more conventional radius ℓ is obtained by choosing, instead of (2.8), the value $\Lambda = \ell^{-2}$.

topology of the space is $\mathbb{R}\times\mathrm{S}^1$, i.e. an infinite cylinder located at $\rho\to\infty$ and extending in the timelike direction.

From the embedding formalism, the symmetries AdS_3 are known. It is nevertheless useful to decouple from this construction and analyze them directly from (2.13). One finds that the metric (2.13) is maximally symmetric with the following six Killing vector fields:

$$\begin{aligned}
\ell_- &= \tfrac{1}{2}\left[\tanh\rho\,\mathrm{e}^{-\mathrm{i}(t-\psi)}\partial_t - \coth\rho\,\mathrm{e}^{-\mathrm{i}(t-\psi)}\partial_\psi + \mathrm{i}\mathrm{e}^{-i(t-\psi)}\partial_\rho\right]\\
\ell_0 &= -\tfrac{\mathrm{i}}{2}(\partial_t-\partial_\psi)\\
\ell_+ &= -\tfrac{1}{2}\left[\tanh\rho\,\mathrm{e}^{\mathrm{i}(t-\psi)}\partial_t - \coth\rho\,\mathrm{e}^{\mathrm{i}(t-\psi)}\partial_\psi - \mathrm{i}\mathrm{e}^{i(t-\psi)}\partial_\rho\right]\\
\bar\ell_- &= \tfrac{1}{2}\left[\tanh\rho\,\mathrm{e}^{-\mathrm{i}(t+\psi)}\partial_t + \coth\rho\,\mathrm{e}^{-\mathrm{i}(t+\psi)}\partial_\psi + \mathrm{i}\mathrm{e}^{-i(t+\psi)}\partial_\rho\right]\\
\bar\ell_0 &= -\tfrac{\mathrm{i}}{2}(\partial_t+\partial_\psi)\\
\bar\ell_+ &= -\tfrac{1}{2}\left[\tanh\rho\,\mathrm{e}^{\mathrm{i}(t+\psi)}\partial_t + \coth\rho\,\mathrm{e}^{\mathrm{i}(t+\psi)}\partial_\psi - \mathrm{i}\mathrm{e}^{i(t+\psi)}\partial_\rho\right]
\end{aligned} \tag{2.14}$$

It is important to highlight that these Killing vectors are all well-defined with respect to the identifications of (2.13). Under the Lie bracket, they form the following non-trivial commutation relations:

$$\begin{aligned}
&[\ell_0\,,\ell_\pm]_{\mathrm{Lie}} = \pm\ell_\pm\,, && [\ell_+\,,\ell_-]_{\mathrm{Lie}} = -2\ell_0\,,\\
&\left[\bar\ell_0\,,\bar\ell_\pm\right]_{\mathrm{Lie}} = \pm\bar\ell_\pm\,, && \left[\bar\ell_+\,,\bar\ell_-\right]_{\mathrm{Lie}} = -2\bar\ell_0\,,
\end{aligned} \tag{2.15}$$

corresponding to two commuting copies of $\mathrm{SL}(2,\mathbb{R})$. We therefore have that the isometry group of pure AdS_3 is, as expected, $\mathrm{SL}(2,\mathbb{R})_L\times\mathrm{SL}(2,\mathbb{R})_R\cong\mathrm{SO}(2,2)$. Here, the L- and R- subscripts is notation referring to the sets of left-moving (unbarred) and right-moving (barred) algebras respectively.

The BTZ black hole

Another solution to (2.3) of major interest is the 3d black hole, nowadays commonly referred to as the Banados-Teitelboim-Zanelli (BTZ) black hole [50].

The line element for the BTZ black hole is conveniently written as [5]

$$ds^2 \;=\; -N_r^2 d\tau^2 + N_r^{-2}dr^2 + r^2\Big(d\varphi + N_\varphi d\tau\Big)^2\,, \tag{2.16}$$

[5]Note that the τ symbol employed in this section is not to be confused with the modular parameter used in all other chapters.

where

$$N_r \equiv \sqrt{-8GM + \frac{r^2}{(2\ell)^2} + \frac{16G^2J^2}{r^2}}\,, \qquad N_\varphi \equiv -\frac{4GJ}{r^2}\,, \qquad M,\, J \in \mathbb{R}\,. \tag{2.17}$$

Here, the coordinates have ranges $\tau \in (-\infty,\, \infty)$, $r \in (0,\, \infty)$ and $\varphi \in [0,\, 2\pi)$. The parameters M and J are the mass and angular momentum of the black hole, respectively. The radial values for which $N_r = 0$ are two coordinate singularities

$$r_\pm = 2\ell\Big[4GM\Big(1 \pm \sqrt{1 - \Big(\frac{J}{2M\ell}\Big)^2}\Big)\Big]^{1/2}\,, \tag{2.18}$$

which correspond to an inner Cauchy surface (r_-) and an outer event horizon (r_+). Note that (2.18) is a constraint on the spectrum of allowed black holes, in that both $|J| \leq 2M\ell$ and $M > 0$ are required. The limiting case is $|J| = 2M\ell$ for which the surfaces r_+, r_- coincide, giving rise to the extremal BTZ black hole. We can also invert (2.18), so as to express the charges in terms of $r_\pm$:

$$M = \frac{r_+^2 + r_-^2}{8G_3(2\ell)^2}\,, \qquad J = \frac{r_+ r_-}{4G_3(2\ell)}\,. \tag{2.19}$$

Finally, we note that the black hole admits the semiclassical thermodynamic properties discussed in the introduction to this thesis. In particular, its Hawking entropy (1.1) (in units $c = \hbar = k_B = 1$) is

$$\mathcal{S}_{BTZ} = \frac{2\pi r_+}{4G_3}\,. \tag{2.20}$$

We now wish to highlight that the metric (2.16) of the black hole is locally pure AdS$_3$ at every point, but differs at the level of global identifications. To see this explicitly, we first substitute the expressions for the charges (2.19) back into the metric (2.16). Then, this BTZ metric is mapped into the form of the global AdS$_3$ metric (2.13), with coordinates that we denote $(\rho',\, \psi',\, t')$, by the following diffeomorphism: [6]

$$\sinh\rho' = \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}}\,, \qquad \psi' = \frac{\mathrm{i}}{2\ell}\Big(\frac{r_+}{2\ell}\tau - r_-\varphi\Big)\,, \qquad t' = -\frac{\mathrm{i}}{2\ell}\Big(\frac{r_-}{2\ell}\tau - r_+\varphi\Big)\,. \tag{2.21}$$

[6] Note that while the transformations (2.21), which directly relate (2.13) and (2.16), are complex, the two sets of transformations that respectively map from the embedding metric (2.10) to (2.13) and (2.16) are real (see (2.12) for the map to (2.13), while for the map to (2.16) we refer to [51]).

Given the periodicities of the black hole coordinates (φ, τ), it is clear that the periodicities of (ψ', t') in (2.21) are different to those of the true global AdS_3 coordinates (ψ, t). The BTZ black hole therefore differs from global AdS_3 by global identifications.

We conclude this discussion with a remark on the symmetries. Because the black hole is locally diffeomorphic to global AdS_3 via (2.21), the Killing vectors (2.14) of global AdS_3 are also *locally* Killing vectors of the black hole. However, only two of them ($\partial_{t'}$ and $\partial_{\psi'}$) are compatible with the periodicities of (ψ', t'). We therefore have that the $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ isometry group of pure AdS_3 is broken to the $\mathbb{R}\times\mathrm{SO}(2)$ subgroup. This exhibits the general fact that the set of global symmetries of an AdS_3 space with non-trivial identifications is typically smaller than its set of local symmetries.

2.3 Asymptotic structure of AdS_3 gravity

In addition to the interior solutions of the theory e.g. pure AdS_3 or the BTZ black hole, we are interested in an asymptotic (i.e. near-boundary) treatment of AdS_3 gravity. Our aim with this section is to explore this rich asymptotic structure. In particular, we discuss the notion of locally "asymptoticlly AdS_3 spaces" which, as we see in a later section, is crucial to the set-up of the gravitational path integral. More generally, we review how, even in the classical theory, AdS_3 gravity near the boundary already exhibits certain striking features suggestive of a quantum duality with a two-dimensional conformal field theory.

Two related principles enter this analysis. First, the asymptotic structure of the gravitational theory is constrained by Einstein's equations near the boundary. We explore this in detail in Section 2.3.1. Secondly, one requires a notion of consistent boundary conditions which must be imposed by hand. At the classical level, such conditions are required to fix degrees of freedom of the fields near the boundary and allow for a well-defined variational principle. We approach the topic of boundary conditions by describing two known examples, which both play an important role in this thesis. The first are the celebrated Brown-Henneaux boundary conditions [35], which we discuss in Section 2.3.2. The second, which we review in 2.3.3, are the (much more modern) Compère-Song-Strominger (CSS) boundary conditions [46].

2.3.1 Asymptotic Einstein's equations

We begin this analysis with a treatment of Einstein's equations near the boundary and how they constrain the asymptotic structure of their solutions. The starting point is the following theorem by Fefferman and Graham [52]. Let $g_{\mu\nu}$ denote a 3d metric that satisfies Einstein's equations with a positive cosmological constant $\Lambda = 1/(2\ell)^2$ and potential matter couplings. Then, there exists a distinguished coordinate system $y^\mu = (\rho, x^\alpha)$ such that the metric takes the form [7]

$$g_{\mu\nu} = (2\ell)^2 d\rho^2 + \gamma_{\alpha\beta}(\rho, x)\, dx^\alpha dx^\beta \,, \tag{2.22}$$

where ρ is an outgoing radial coordinate, and where the induced metric on a constant ρ slice $\gamma_{\alpha\beta}$ admits the following expansion in $\rho \gg 1$:

$$\gamma_{\alpha\beta}(\rho, x) \;=\; e^{2\rho}\gamma^{(0)}_{\alpha\beta}(x) + \gamma^{(2)}_{\alpha\beta}(x) + \mathcal{O}(e^{-2\rho})\,. \tag{2.23}$$

The above expansion is known as the *Fefferman-Graham* expansion, and will be used extensively throughout this thesis. Note that no assumptions are made on boundary manifold. Note also that, while the leading- and first subleading term in (2.23) are universal irrespective of matter couplings, the further subleading terms are not. (It is worth pointing out that for the pure 3d Einstein theory (2.1), the expansion terminates after the second subleading term $e^{-2\rho}\gamma^{(4)}_{\alpha\beta}(x)$ [53], though this is not an important feature of the analysis.)

A corresponding expansion for the inverse induced-metric $\gamma^{\alpha\beta}$ is determined by requiring that $\gamma^{\alpha\beta}\gamma_{\beta\lambda} = \delta^\alpha_\lambda$. One obtains

$$\gamma^{\alpha\beta}(\rho, x) \;= e^{-2\rho}\gamma^{(0)\alpha\beta}(x) - e^{-4\rho}\gamma^{(0)\alpha\lambda}\gamma^{(0)\beta\delta}\gamma^{(2)}_{\lambda\delta}(x) + \mathcal{O}(\mathrm{e}^{-6\rho})\,, \tag{2.24}$$

where $\gamma^{(0)\alpha\beta}$ is the inverse of $\gamma^{(0)}_{\alpha\beta}$, defined as $\gamma^{(0)\alpha\beta}\gamma^{(0)}_{\beta\lambda} = \delta^\alpha_\lambda$. It is useful to introduce the notation that indices on $\gamma^{(2)}$ are raised/lowered using $\gamma^{(0)}$, i.e. we write:

$$\gamma^{(2)\alpha\beta} \;\equiv\; \gamma^{(0)\alpha\lambda}\gamma^{(0)\beta\delta}\gamma^{(2)}_{\lambda\delta}\,, \tag{2.25}$$

[7] Note: Fefferman and Graham show that (2.22) is always available for generic AdS$_d$ spaces, not just AdS$_3$. This is highly non-trivial in $d > 3$. In $d = 3$ however, one can simply appreciate (2.22) from the fact that all solutions are locally diffeomorphic to pure AdS$_3$ (2.13), which is already in the form (2.22).

such that the Fefferman-Graham expansions (2.24) with (2.25) becomes

$$\gamma^{\alpha\beta}(\rho,x) \;=\; e^{-2\rho}\gamma^{(0)\alpha\beta}(x) - e^{-4\rho}\gamma^{(2)\alpha\beta} + \mathcal{O}(\mathrm{e}^{-6\rho})\,. \tag{2.26}$$

An immediate exercise is to substitute the metric (2.22) with (2.23) back into the Einstein's equations. This yields constraints between the various coefficients $\gamma^{(n)}$ of the expansion. We demonstrate this explicitly, focusing on the pure theory (2.1) with Einstein's equations given (2.3). The first step is the substitution of the Fefferman-Graham gauge (2.22) into (2.5), which gives Einstein's equations in terms of the induced metric $\gamma_{\alpha\beta}$ as

$$\frac{1}{2\ell}\partial_\rho K + K_{\alpha\beta}K^{\alpha\beta} - 2\Lambda \;=\; 0\,, \tag{2.27}$$

$$\nabla^\beta(K_{\alpha\beta} - \gamma_{\alpha\beta}K) \;=\; 0\,, \tag{2.28}$$

$$2K_{\alpha\lambda}K^{\lambda}{}_{\beta} - K_{\alpha\beta}K - \frac{1}{2\ell}\partial_\rho K_{\alpha\beta} - \tilde{R}_{\alpha\beta} + 2\Lambda\gamma_{\alpha\beta} \;=\; 0\,, \tag{2.29}$$

where, recalling from Section 2.1, we have $\Lambda = 1/(4\ell^2)$. The extrinsic curvature $K_{\alpha\beta}$ for Fefferman-Graham metrics (2.22) is

$$K_{\alpha\beta} \;=\; \frac{1}{2(2\ell)}\partial_\rho\gamma_{\alpha\beta}\,, \tag{2.30}$$

and $K \equiv \gamma^{\alpha\beta}K_{\alpha\beta}$. All indices in (2.27 - 2.29) are raised/lowered using $\gamma_{\alpha\beta}$. In (2.28), $\tilde{\nabla}_\alpha$ is the covariant derivative with respect to $\gamma_{\alpha\beta}$:

$$\tilde{\nabla}_\alpha V_\beta \;=\; \partial_\alpha V_\beta - \tilde{\Gamma}^{\lambda}_{\alpha\beta}V_\lambda\,, \quad \tilde{\Gamma}^{\lambda}_{\alpha\beta} \;\equiv\; \frac{1}{2}\gamma^{\lambda\delta}(\partial_\alpha\gamma_{\beta\delta} + \partial_\beta\gamma_{\alpha\delta} - \partial_\delta\gamma_{\alpha\beta})\,. \tag{2.31}$$

In (2.29), we have introduced $\tilde{R}_{\alpha\beta}$, the Ricci tensor of $\gamma_{\alpha\beta}$:

$$\tilde{R}_{\alpha\beta} \;=\; -2\Big(\partial_{[\lambda}\tilde{\Gamma}^{\lambda}_{\alpha]\beta} + \tilde{\Gamma}^{\delta}_{\lambda[\delta}\tilde{\Gamma}^{\lambda}_{\alpha]\beta}\Big)\,. \tag{2.32}$$

The second step is to substitute the Fefferman-Graham expansion (2.23) for the induced metric into the decomposed Einstein equations (2.27 - 2.29). A useful set of intermediate quantities is:

$$\begin{aligned} K_{\alpha\beta} &= \frac{1}{2\ell}(\mathrm{e}^{2\rho}\gamma^{(0)}_{\alpha\beta} - \mathrm{e}^{-2\rho}\gamma^{(4)}_{\alpha\beta}) + \mathcal{O}(\mathrm{e}^{-4\rho}) \,, \\ K^{\alpha\beta} &= \frac{1}{2\ell}(\mathrm{e}^{-2\rho}\gamma^{(0)\alpha\beta} - 2\mathrm{e}^{-4\rho}\gamma^{(2)\alpha\beta}) + \mathcal{O}(\mathrm{e}^{-6\rho}) \\ K &= \frac{1}{2\ell}(2 - \mathrm{e}^{-2\rho}\gamma^{(2)\alpha\beta}\gamma^{(0)}_{\alpha\beta}) + \mathcal{O}(\mathrm{e}^{-6\rho}) \,, \\ \tilde{R}_{\alpha\beta} &= \tilde{R}^{(0)}_{\alpha\beta} + \mathcal{O}(\mathrm{e}^{-2\rho}) \,, \end{aligned} \tag{2.33}$$

where $\tilde{R}^{(0)}_{\alpha\beta}$ is the Ricci tensor for $\gamma^{(0)}_{\alpha\beta}$, i.e.

$$\tilde{R}^{(0)}_{\alpha\beta} = -\Big(\partial_{[\lambda}\tilde{\Gamma}^{(0)\lambda}_{\alpha]\beta} + \tilde{\Gamma}^{(0)\delta}_{\lambda[\delta}\tilde{\Gamma}^{(0)\lambda}_{\alpha]\beta}\Big) \,, \quad \tilde{\Gamma}^{(0)\lambda}_{\alpha\beta} \equiv \frac{1}{2}\gamma^{(0)\lambda\delta}(\partial_\alpha\gamma^{(0)}_{\beta\delta} + \partial_\beta\gamma^{(0)}_{\alpha\delta} - \partial_\delta\gamma^{(0)}_{\alpha\beta}) \,. \tag{2.34}$$

With these substitutions, one finds the following three relations. The trace with respect to $\gamma^{(0)}$ of the leading order equation of (2.29) gives

$$\mathrm{Tr}_{(0)}[\gamma^{(2)}] = 2\ell^2\tilde{R}^{(0)} \,, \tag{2.35}$$

where $\tilde{R}^{(0)} = \gamma^{(0)\alpha\beta}\tilde{R}^{(0)}_{\alpha\beta}$ and $\mathrm{Tr}_{(0)}[\gamma^{(n)}] \equiv \gamma^{(0)\alpha\beta}\gamma^{(n)}_{\alpha\beta}$. Subleadings orders of (2.29) and (2.27) give relations that determine further subleading coefficients $\gamma^{(n)}_{\alpha\beta}$ in the Fefferman-Graham expansion. For instance, for the pure theory, we have:

$$\gamma^{(4)}_{\alpha\beta} = \frac{1}{4}(\gamma^{(2)}_{\alpha\lambda}\gamma^{(0)\lambda\delta}\gamma^{(2)}_{\delta\beta}) \,. \tag{2.36}$$

Finally, one finds that (2.28) is identically satisfied at leading order, but yields the following differential constraint at the first subleading order:

$$\tilde{\nabla}^{(0)\beta}\Big(\gamma^{(2)}_{\alpha\beta} - \mathrm{Tr}_{(0)}[\gamma^{(2)}]\gamma^{(0)}_{\alpha\beta}\Big) = 0 \,, \tag{2.37}$$

where $\tilde{\nabla}^{(0)}_\alpha$ is the covariant derivative constructed from $\gamma^{(0)}_{\alpha\beta}$ and its index is raised with $\gamma^{(0)\alpha\beta}$.

The relations (2.35) and (2.37) imply that for a given $\gamma^{(0)}$, the tensor $\gamma^{(2)}$ is only determined up to its trace and a constraint on its divergence. It is useful for a later discussion to introduce an equivalent representation of this fact, by making the following definition:

$$T_{\alpha\beta} := \gamma^{(2)}_{\alpha\beta} - \mathrm{Tr}_{(0)}[\gamma^{(2)}]\gamma^{(0)}_{\alpha\beta} = \gamma^{(2)}_{\alpha\beta} - 2\ell^2\tilde{R}^{(0)}\gamma^{(0)}_{\alpha\beta} \tag{2.38}$$

where we used the trace relation (2.35) to substitute for $\tilde{R}^{(0)}$. Re-arranging for $\gamma^{(2)}$, one has

$$\gamma^{(2)}_{\alpha\beta} \;=\; T_{\alpha\beta} + 2\ell^2 \tilde{R}^{(0)} \gamma^{(0)}_{\alpha\beta}\,. \tag{2.39}$$

and the constraints (2.35), (2.37) are, respectively,

$$\mathrm{Tr}_{(0)}[T] \;=\; -2\ell^2 \tilde{R}^{(0)}\,, \tag{2.40}$$

$$\tilde{\nabla}^{(0)\beta} T_{\alpha\beta} \;=\; 0\,. \tag{2.41}$$

In this picture, the ambiguity in $\gamma^{(2)}_{\alpha\beta}$ is recast in the form of a symmetric tensor $T_{\alpha\beta}$, which is determined by $\gamma^{(0)}$ only up to its trace and divergence as in (2.40), (2.41).

One concludes that the asymptotic form of Einstein metrics is entirely characterized by six functions of the boundary coordinates x^α, of which only five are independent. The six functions correspond to the components of the symmetric tensors $\gamma^{(0)}_{\alpha\beta}$ and $\gamma^{(2)}_{\alpha\beta}$. They are subjected to the algebraic trace condition (2.35), and are further constrained by the two differential equations (2.37). (Equivalently, in the language of (2.39), the inputs for $\gamma^{(2)}_{\alpha\beta}$ are traded for the two components of $T_{\alpha\beta}$ subjected to the trace condition (2.40) and the differential constraints (2.41).)

We now turn to the subject of boundary conditions. Boundary conditions correspond to fixing a certain amount of the input data summarized above to a specified reference value. In the classical theory, this reduces the overall ambiguity in the asymptotic metric (in fact, the metric becomes almost entirely determined by the equations of motion). In the quantum level, all degrees-of-freedom that are not fixed by the boundary conditions are allowed to fluctuate.

We study two particular sets of boundary conditions. The first is Brown-Henneaux, where the input data that is fixed are all the components of $\gamma^{(0)}$, and the only remaining degrees of freedom in the asymptotic solution appear as an arbitrary choice of one left-moving and one right-moving function for the two unfixed components of $\gamma^{(2)}$. The second set are the CSS conditions, where the input data that is fixed are two components of $\gamma^{(0)}$ and one component of $\gamma^{(2)}$. The third component of $\gamma^{(0)}$ is also partially fixed to be an arbitrary right-moving function. The remaining degrees of freedom in the asymptotic solution are then this very right-moving function and another independent arbitrary right-moving function for the unfixed $\gamma^{(2)}$ component.

For both choices of boundary conditions, the following two key criteria [54] are satisfied: the interior solutions of interest (e.g. global AdS_3 and the BTZ black hole) are allowed. Furthermore, a consistent variational principle is reached upon the addition of suitable boundary terms.

2.3.2 Boundary conditions I: Brown-Henneaux

The seminal works by Brown and Henneaux [35] on Einstein theory in asymptotically-AdS$_3$ spaces constitutes a pioneering step towards a holographic description of (semi-) quantum gravity.

Boundary conditions

Their treatment relies on the introduction of the now-called Brown-Henneaux boundary condition:

$$\gamma^{(0)} \;=\; \eta\,. \tag{2.42}$$

In other words, the conformal boundary metric is fixed to be locally Minkowski, while the subleading terms in (2.23) are left unconstrained. To proceed, it is useful to make an explicit choice of coordinates. The most convenient choice for our analysis is to take the global AdS$_3$ metric (2.13) as a reference, i.e. we identify its ρ coordinate with the ρ coordinate appearing in the Fefferman-Graham gauge (2.22). Then, by performing the asymptotic expansion in e^{ρ} of (2.13) and comparing to (2.23), we fix

$$ds^2_{(0)} \;\equiv\; \gamma^{(0)}_{\alpha\beta}\, dx^{\alpha}dx^{\beta} \;=\; \ell^2(-dt^2+d\psi)^2 \;=\; \ell^2 dx^+dx^-\,, \tag{2.43}$$

where we have introduced light-cone-type coordinates defined as $x^{\pm} \equiv \psi \mp t$. [8]

Asymptotic form of Brown-Henneaux metrics

With $\gamma^{(0)}$ fixed, the remaining degrees-of-freedom in the asymptotic metric expansion (2.23) are the components of $\gamma^{(2)}$ subjected to the Einstein-equation constraints (2.35), (2.37). The trace constraint is algebraic and entirely fixes one of the components:

$$\gamma^{(2)}_{+-} \;=\; 0\,, \tag{2.44}$$

Meanwhile, the two dynamical constraints imply that the remaining two modes, $\gamma^{(2)}_{++}$ and $\gamma^{(2)}_{--}$, are purely left-moving and right-moving functions respectively:

$$\partial_-\gamma^{(2)}_{++} \;=\; \partial_+\gamma^{(2)}_{--} \;=\; 0 \quad\Rightarrow\quad \gamma^{(2)}_{++}(x) \,\equiv\, \ell^2\mathcal{L}(x^+)\,, \qquad \gamma^{(2)}_{--}(x) \,\equiv\, \ell^2\bar{\mathcal{L}}(x^-)\,, \tag{2.45}$$

[8] Note that $x^{\pm}$ are not the usual light-cone coordinates $\mathrm{X}^{\pm} = t \pm \psi = \pm x^{\mp}$, for which the flat metric is $ds^2 = -dX^+dX^-$. Our choice is made such that under Wick rotation $t = -\mathrm{i}t_E$, the coordinates $x^{\pm}$ map directly to the standard complex coordinates as $(x^+, x^-) \mapsto (z, \bar{z})$, where $(z, \bar{z}) = (\psi + \mathrm{i}t_E,\, \psi - \mathrm{i}t_E)$.

where the factor of ℓ^2 is chosen for convenience. Note that, beyond their left/right-moving profile, these functions are entirely arbitrary. They are commonly referred to as the Brown-Henneaux modes, or the *boundary gravitons* of AdS$_3$. The direct substitution of (2.42), (2.44), (2.45), into (2.23) now gives the most general asymptotic form (up to trivial diffeomorphisms) of solutions to the vacuum AdS$_3$ theory (2.1) with $\Lambda = 1/(2\ell)^2$ and with Brown-Henneaux boundary condition (2.43):

$$ds^2 \;=\; (2\ell)^2 d\rho^2 + \ell^2\Big(\mathrm{e}^{2\rho}dx^+dx^- + \mathcal{L}(x^+)(dx^+)^2 + \bar{\mathcal{L}}(x^-)(dx^-)^2 + \mathcal{O}(\mathrm{e}^{-2\rho})\Big)\,. \quad (2.46)$$

One may check that the bulk solutions discussed in Section 2.2 take the asymptotic form (2.46) upon suitable choices for $\mathcal{L}$, $\bar{\mathcal{L}}$. For instance, global AdS$_3$ (2.13) is reached with the choice

$$\mathcal{L} \;=\; \bar{\mathcal{L}} \;=\; -1\,. \quad (2.47)$$

Variational principle

We turn to the variational principle of the theory with these boundary conditions. Since the Brown-Henneaux boundary conditions are Dirichlet conditions, one follows the usual treatment for Dirichlet problems in general relativity on non-compact spaces. Recall that this corresponds to the addition of the so-called Gibbons-Hawking-York boundary term and a cosmological counterterm. For metrics in the Fefferman-Graham gauge (2.22), these terms are respectively given as

$$S_{GH} \;=\; -\frac{1}{8\pi G_3}\int_{\partial\mathcal{M}} d^2x\,\sqrt{-\gamma}K\,, \quad (2.48)$$

$$S_{CC} \;=\; \frac{1}{8\pi G_3(2\ell)}\int_{\partial\mathcal{M}} d^2x\,\sqrt{-\gamma}\,, \quad (2.49)$$

where $\gamma \equiv \mathrm{Det}(\gamma_{\alpha\beta})$. For a stationary solution, the variation of (2.1) together with its boundary terms (2.48), (2.49), gives, upon substituting the Fefferman-Graham expansion (2.23):

$$\delta\Big(S + S_{GH} + S_{CC}\Big) \;=\; \frac{1}{16\pi\ell G_3}\int d^2x\sqrt{-\gamma^{(0)}}\Big(\gamma^{(0)\alpha\beta}\mathrm{Tr}[\gamma^{(2)}] - \gamma^{(2)\alpha\beta}\Big)\delta\gamma^{(0)}_{\alpha\beta}\,, \quad (2.50)$$

which indeed vanishes once the Brown-Henneaux conditions are imposed, i.e. that $\gamma^{(0)}$ is fixed ($\delta\gamma^{(0)}_{\alpha\beta} = 0$). The variational principle is therefore well-defined. Note the emergence of $T_{\alpha\beta}$, as introduced in (2.38), which now acquires the interpretation of a holographic stress tensor [55] (we have $\mathrm{Tr}[T] = 0$ by (2.40)).

Asymptotic symmetries

The symmetry group under which an asymptotically AdS_3 metric obeying Brown-Henneaux boundary conditions is mapped to another asymptotically AdS_3 metric obeying the same boundary conditions is much larger than the $\mathrm{SO}(2,2)$ isometry group of the bulk AdS_3 solutions. Most remarkably, the group in question turns out to be the 2d conformal group. Here, we review the first and main step in deriving this result, namely to compute the asymptotic Brown-Henneaux Killing vector fields that generate this asymptotic algebra.

Consider an arbitrary diffeomorphism $\xi_\mu(\rho,\, x^\alpha)$ on (2.22) where, recall, $(\rho,\, x^\alpha)$ are Fefferman-Graham coordinates. We say that ξ is an asymptotic Brown-Henneaux Killing vector if (a) it preserves the Fefferman-Graham gauge (2.22) [9] and (b) it leaves $\gamma^{(0)}$ in (2.43) invariant under the Lie derivative $\mathscr{L}_\xi$, but not necessarily the subleading metrics $\gamma^{(2)}$, $\cdots$ of $\gamma_{\alpha\beta}$. The condition (a) is equivalent to the constraints

$$\mathscr{L}_\xi g_{\rho\rho} \;=\; 0\,, \tag{2.51}$$

$$\mathscr{L}_\xi g_{\rho\alpha} \;=\; 0\,. \tag{2.52}$$

Meanwhile, (b) corresponds to

$$\mathscr{L}_\xi g_{\alpha\beta} \;=\; \mathcal{O}(1)\,. \tag{2.53}$$

We now solve these constraints for ξ. The first, (2.51), immediately gives

$$\partial_\rho\, \xi^\rho \;=\; 0 \quad\Rightarrow\quad \xi^\rho(\rho,x) \;\equiv\; C(x)\,. \tag{2.54}$$

Using this, the second constraint (2.52) gives

$$\gamma_{\alpha\beta}\, \partial_\rho \xi^\beta + (2\ell)^2 \partial_\alpha C(x) \;=\; 0 \quad\Rightarrow\quad \xi^\beta(\rho,x) \;=\; D^\beta(x) - \int d\rho\, \gamma^{\alpha\beta} \partial_\alpha C(x) \tag{2.55}$$

where we have introduced the arbitrary ρ-independent mode as $D^\alpha(x)$. Note that while this mode is $\mathcal{O}(1)$, the integral term is subleading because $\gamma^{\alpha\beta} = \mathrm{e}^{-2\rho}\gamma^{(0)\alpha\beta} + \cdots$ as in (2.26). For the third constraint (2.53), we have

$$2\gamma^{(0)}_{\alpha\beta}\, C(x) + D^\lambda(x) \partial_\lambda \gamma^{(0)}_{\alpha\beta} + 2\gamma^{(0)}{}_{\lambda(\alpha}\partial_{\beta)} D^\lambda(x) \;=\; 0\,, \tag{2.56}$$

[9]Note that this aspect can be relaxed. Indeed, Brown and Henneaux derive the asymptotic symmetry algebra while also allowing for subleading fluctuations in the $g_{\rho\rho}$ and $g_{\rho\alpha}$ components, which clearly violates the Fefferman-Graham gauge (2.22).

which is the result of the leading order $\mathcal{O}(e^{2\rho})$ of the equation with (2.54), (2.55), and with the fact that the Brown-Henneaux boundary conditions set $\partial_\lambda \gamma^{(0)}_{\alpha\beta} = 0$. Finally, taking the trace of (2.56) with $\gamma^{(0)\alpha\beta}$, we have

$$C(x) \;=\; -\frac{1}{2}\partial_\lambda D^\lambda(x)\,, \tag{2.57}$$

which we substitute back into (2.56) to give the equation

$$2\partial_{(\alpha}D_{\beta)}(x) \;=\; \gamma^{(0)}_{\alpha\beta}\partial_\lambda D^\lambda(x)\,. \tag{2.58}$$

Notice that this equation is nothing but the conformal Killing vector equation for 2d Minkowski space (recall $\gamma^{(0)} = \eta$). The components of the leading order $D^\alpha(x)$ of the diffeomorphism $\xi^\alpha(\rho, x)$ along the boundary directions are therefore the infinitesimal generators of the 2d conformal group. Explicitly, we have the usual infinite set of left/right-moving functions

$$D^+ \;=\; D^+(x^+)\,, \qquad D^- \;=\; D^-(x^-)\,. \tag{2.59}$$

The subleading orders of ξ^α are then determined by (2.55), and we have

$$\begin{aligned} \xi^+(\rho, x) \;&=\; D^+(x^+) + \frac{1}{2}\int d\rho\, e^{-2\rho}\gamma^{(0)-+}\partial_-\partial_- D^-(x^-) + \mathcal{O}(e^{-4\rho})\,,\\ \xi^-(\rho, x) \;&=\; D^-(x^-) + \frac{1}{2}\int d\rho\, e^{-2\rho}\gamma^{(0)+-}\partial_+\partial_+ D^+(x^+) + \mathcal{O}(e^{-4\rho})\,, \end{aligned} \tag{2.60}$$

where we used (2.57) to substitute for $C(x)$ in the integral terms. Finally, ξ^ρ is expressed from (2.54) with (2.57) as

$$\xi^\rho(\rho, x) \;=\; -\frac{1}{2}\Big(\partial_+ D^+(x^+) + \partial_- D^-(x^-)\Big)\,. \tag{2.61}$$

With (2.60) and (2.61), we have finally obtained the (infinite number of) asymptotic Brown-Henneaux Killing vector fields:

$$\begin{aligned} \xi \;=&\; D^+(x^+)\,\partial_+ + D^-(x^-)\,\partial_- - \frac{1}{2}\Big(\partial_+ D^+(x^+) + \partial_- D^-(x^-)\Big)\partial_\rho\\ &+ \frac{1}{2}\Big(\int d\rho\, e^{-2\rho}\gamma^{(0)-+}\partial_-\partial_- D^-(x^-)\Big)\partial_+\\ &+ \frac{1}{2}\Big(\int d\rho\, e^{-2\rho}\gamma^{(0)+-}\partial_+\partial_+ D^+(x^+)\Big)\partial_- + \mathcal{O}(e^{-4\rho})\,, \end{aligned} \tag{2.62}$$

which split into the left/right-moving parts

$$\begin{aligned}
\xi^{(+)} &= D^+(x^+)\,\partial_+ - \frac{1}{2}\partial_+ D^+(x^+)\,\partial_\rho + \frac{1}{2}\Big(\int d\rho\, e^{-2\rho}\gamma^{(0)+-}\partial_+\partial_+ D^+(x^+)\Big)\partial_- \,, \\
\xi^{(-)} &= D^-(x^-)\,\partial_- - \frac{1}{2}\partial_- D^-(x^-)\,\partial_\rho + \frac{1}{2}\Big(\int d\rho\, e^{-2\rho}\gamma^{(0)-+}\partial_-\partial_- D^-(x^-)\Big)\partial_+ \,,
\end{aligned} \tag{2.63}$$

where we suppressed the $\mathcal{O}(\mathrm{e}^{-4\rho})$.

The remaining step, which we simply state here, consists of computing the algebra associated to these asymptotic symmetries. A subtlety here is that directly computing the Lie bracket algebra of the vector fields (2.63) that generate these symmetries gives only the *classical* part of the 2d conformal algebra, namely two commuting copies of the *centerless* Virasoro algebra (a.k.a. the Witt algebra). This incompleteness stems from the fact that the canonical generator associated with a given vector field is not unique: it is only determined up to the addition of a constant (i.e. a "central extension"), which commutes with everything, and which the naive computation of the Lie bracket does not capture. To observe the *quantum* 2d conformal group, the algebra must be instead be computed from the Poisson brackets of the conserved charges $Q^{(\pm)}$ associated to these vector fields (recall that these charges are non-trivial because $\xi^{(\pm)}$ are not exact diffeomorphisms in the bulk). This approach correctly gives rise to the central extension of the Witt algebra, and comes from the $\mathcal{O}(\mathrm{e}^{-2\rho})$ piece of $\xi^{(\pm)}_n$. Upon introducing Fourier modes $Q^{(\pm)}_n$ for the generators, the resulting algebra is finally matched to $\mathrm{vir}_L \oplus \mathrm{vir}_R$ with central charge identified in terms of the gravitational constants as

$$c \;=\; \frac{3(2\ell)}{2G_3}\,. \tag{2.64}$$

This is the Brown-Henneaux central charge of AdS$_3$ gravity.

2.3.3 Boundary conditions II: Compere-Song-Strominger

We now present an alternative, more recent set of boundary conditions, the Compere-Song-Strominger (CSS) boundary conditions [46]. The construction of these conditions was at the time strongly motivated by the emergence of the so-called Kerr/CFT correspondence [56], but we underline that this is not the relevant context for our problem. Instead, we will later find these boundary conditions are compatible with a set of localization solutions around AdS$_3$.

The boundary conditions

The CSS boundary conditions are:

$$\gamma^{(0)}_{--} \;=\; \ell^2 \partial_- \bar{P}(x^-)\,, \qquad \gamma^{(0)}_{+-} \;=\; \frac{\ell^2}{2}\,, \qquad \gamma^{(0)}_{++} \;=\; 0\,, \qquad \gamma^{(2)}_{++} \;=\; 4G_3\Delta\ell\,, \tag{2.65}$$

where $\partial_-\bar{P}(x^-)$ is an arbitrary *fluctuating* function of the right-movers and Δ is a fixed constant that is related to the charges of specific BTZ-type solutions. Note that, as with Brown-Henneaux, the boundary conditions fix the conformal boundary metric to be Ricci-flat: $\tilde{R}^{(0)} = 0$. However, unlike with Brown-Henneaux, the CSS conditions are chiral, in the sense that they do not treat the left- and right- moving sectors on the same footing. They also allow for fluctuations in the conformal boundary metric, at the expense of fixing one component of the subleading order metric $\gamma^{(2)}$.

Asymptotic form of CSS metrics

We substitute the boundary conditions (2.65) into the asymptotic Einstein's equations (2.35), (2.37). The trace constraint (2.35) gives

$$\gamma^{(2)}_{+-} \;=\; 4\ell G_3 \Delta \partial_- \bar{P}(x^-)\,. \tag{2.66}$$

To solve the dynamical equations (2.37), note that the only non-trivial component of $\tilde{\Gamma}^{(0)}$ with the CSS conditions (2.65) and (2.66) is $\tilde{\Gamma}^{(0)+}{}_{--} = \partial_-\partial_-\bar{P}$. The choice $x^\alpha = x^+$ in (2.37) then gives

$$\partial_+\partial_-\bar{P} \;=\; 0\,, \tag{2.67}$$

which is trivially satisfied, and the choice $\alpha = x^-$ in (2.37) gives, using (2.66), that

$$\partial_+\gamma^{(2)}_{--} \;=\; 0 \quad \Rightarrow \quad \gamma^{(2)}_{--} \;\equiv\; 4G\ell(\bar{L}(x^-) + \Delta(\partial_-\bar{P})^2)\,. \tag{2.68}$$

Here, we have introduced an arbitrary right-moving function $\bar{L}$, and have split off a $(\partial_-\bar{P}(x^-))^2$ for convenience. The most general asymptotic metric obeying the Einstein's equations with the CSS conditions is therefore

$$\begin{aligned} ds^2 \;=\; & (2\ell)^2 d\rho^2 + \ell^2 e^{2\rho}\Big(dx^+ + \partial_-\bar{P}(x^-)dx^-\Big)dx^- \\ & + 4G_3\ell\Big(\bar{L}(x^-)dx^{-2} + \Delta(dx^+ + \partial_-\bar{P}(x^-)dx^-)^2\Big) + \mathcal{O}(\mathrm{e}^{-2\rho})\,. \end{aligned} \tag{2.69}$$

Global AdS$_3$ (2.13) is reached with the choice:

$$\partial_-\bar{P}(x^-) \;=\; 0\,, \qquad \bar{L} \;=\; \Delta \;=\; -\ell/(4G_3)\,. \tag{2.70}$$

More generally, note that setting $\partial_- \bar{P}$ to zero and keeping $\bar{L}$ arbitrary reduces the metric (2.69) to that of the right-moving sector of Brown-Henneaux, where $\bar{L}$ plays the role of the Brown-Henneaux mode $\bar{\mathcal{L}}$ in (2.46). In this picture, the left-moving Brown-Henneaux sector is to be seen as fixed to $\mathcal{L} \sim \Delta$.

Variational principle

The variational principle with these boundary conditions requires an additional boundary term on top of the Gibbons-Hawking term (2.48) and cosmological counterterm (2.49) of the Brown-Henneaux story. This is due to the fact that, unlike with Brown-Henneaux, we do *not* have $\delta\gamma^{(0)}_{\alpha\beta} = 0$ for any α, β, and so the surface term (2.50) does not vanish. Explicitly, (2.50) with the CSS boundary conditions (2.65) gives:

$$\begin{aligned} \delta\Big(S + S_{GH} + S_{CC}\Big)|_{CSS} &= -\frac{1}{16\pi\ell G_3}\int d^2x\sqrt{-\gamma^{(0)}}\gamma^{(2)\alpha\beta}\delta\gamma^{(0)}_{\alpha\beta}\,, \\ &= -\frac{\Delta}{\ell^4}\int d^2x\sqrt{-\gamma^{(0)}}\delta\gamma^{(0)}_{\bar{z}\bar{z}}\,, \end{aligned} \tag{2.71}$$

where used that $\gamma^{(0)\alpha\beta}\delta\gamma^{(0)}_{\alpha\beta} = 0$ and $\gamma^{(2)\bar{z}\bar{z}} = 16\Delta G_3/\ell^3$ by the boundary conditions (2.65). The extra boundary term that is added to cancel (2.71) is given in [46] as

$$S^{\text{bdry}}_{\text{CSS}} = -\frac{\Delta}{4\pi}\int_{\partial\mathcal{M}} d^2x\sqrt{-\gamma^{(0)}}\,\gamma^{(0)++}\,. \tag{2.72}$$

with which one can check that

$$\delta\Big(S + S_{GH} + S_{CC} + S^{\text{bdry}}_{\text{CSS}}\Big)|_{CSS} = 0\,. \tag{2.73}$$

Asymptotic symmetries

The derivation of the asymptotic symmetries preserving the CSS boundary conditions follows a strategy analogous to the derivation for Brown-Henneaux case. We refer to [46] for the details. Here, it suffices to state that the asymptotic symmetry algebra corresponds to a chiral (right-moving) Kac-Moody-Virasoro with central extension. The central charge of the Virasoro is $c_R = \frac{3(2\ell)}{2G_3}$.

Chapter 3

Aspects of AdS_3 partition functions

In Chapter 2, we reviewed the classical AdS_3 theory in Lorentzian signature, its solution space, and the topic of asymptotically AdS_3 boundary conditions. At this stage, we are in a good position to introduce basic aspects of partition functions in this classical and non-supersymmetric setting, in preparation for moving to the supergravity formalism of Chapter 4 and beyond.

The analysis in this chapter begins with a Wick rotation of the Einstein theory to Euclidean signature. This gives the semiclassical framework in which we then discuss the partition function dual to the generic CFT_2 trace given in (1.4). In this setting, we review the rewriting of the gravitational partition function as a sum over $\mathrm{PSL}(2,\mathbb{Z})/\mathbb{Z}$ geometries [25]. The simplest such geometry, $M_{0,1}$, or *thermal* AdS_3, is presented.

In the spirit of moving one step closer towards the gravitational dual of the elliptic genus (1.7), we introduce into the Einstein action a coupling to an arbitrary number of right-moving $\mathrm{U}(1)$ matter gauge fields. The long-distance dynamics of these fields are governed by a Chern-Simons action, and their boundary conditions require the addition of a boundary term. In this Einstein-Maxwell-Chern-Simons theory, we evaluate the action on the thermal AdS_3 configuration.

3.1 Semiclassical limit and sum over geometries

As the simplest example, we study the semi-classical limit of the gravitational path integral dual to the thermal CFT_2 partition function (1.4). In this limit, the path-integral is dominated by the (Euclidean) AdS$_3$ saddles of the Einstein theory:

$$Z(\tau) \;=\; \sum_{g_c} \exp\Big(S_E(g_c) + \cdots\Big)\,, \tag{3.1}$$

where S_E is the Euclidean action for the Einstein theory (with suitable renormalization), g_c denotes its saddles, and the $(+\cdots)$ represents suppressed quantum (loop) corrections to $S_E(g_c)$ in perturbation theory. Note that we suppress the $\overline{\tau}$ argument which should also enter in $Z()$. To proceed, two aspects need to be clarified. First, we need the Euclidean version of the Einstein action (2.1). Secondly, we need to take into consideration the boundary conditions that enter the definition of the path-integral, and that therefore restrict the types of saddles which appear in (3.1). For this discussion, our groundwork on Brown-Henneaux in Section 2.3.2 will help.

Euclidean Einstein-Hilbert action and conventions

First, we require the Euclidean version of the action for AdS$_3$ gravity given in (2.1). This is obtained through the Wick rotation:

$$t \;=\; -\mathrm{i}t_E\,, \tag{3.2}$$

where t_E is the Euclidean time coordinate. Note that light-cone coordinates (x^+, x^-) introduced in Section 2.3.2 are respectively mapped to the complex coordinates $(z, \overline{z})$, given as:

$$z \;=\; \psi + \mathrm{i}t_E\,, \qquad \overline{z} \;=\; \psi - \mathrm{i}t_E\,. \tag{3.3}$$

Here, let $\mathcal{L}$ and $S = \int dt\, d^2x\, \mathcal{L}$ generically denote the Lagrangian density and action functional of a Lorentzian theory, respectively. The Wick rotation (3.2) gives the Euclidean counterpart $\mathcal{L}_E$ of $\mathcal{L}$ as $\mathcal{L} = \mathcal{L}_E$. We then define the Euclidean action S_E in terms of $\mathcal{L}_E$ as

$$S_E \;=\; \int dt_E\, d^2x\, \mathcal{L}_E\,. \tag{3.4}$$

With this definition, S_E is related to the Lorentzian action as

$$\mathrm{i}S \;=\; S_E\,. \tag{3.5}$$

Note that our definition (3.4), through (3.5), implies that Euclidean path integrals are of the form $\int D[\Phi] \exp(S_E)$ (which is indeed the form we have been employing throughout this thesis). In these conventions, the path integral is perturbatively well-defined if S_E is negative-definite. Now, according to (3.4), the action for the

Euclidean Einstein-Hilbert action with cosmological constant (2.8) is

$$S_E \;=\; \frac{1}{16\pi G_3}\int dt_E\, d^2x\sqrt{g}\Big(R-\frac{1}{2\ell^2}\Big)\,. \tag{3.6}$$

The space of solutions to the field equations of (3.6) is also mapped by the Wick rotation (3.2) from the Lorentzian solutions. In particular, recalling from Chapter 2 that all Lorentztian solutions are locally diffeomorphic to AdS$_3$, we have that all Euclidean solutions metrics are locally diffeomorphic to the 3d hyperbolic space $\mathbb{H}^3$. The isometry group of $\mathbb{H}^3$ is SL$(2,\mathbb{C})$.

Maldacena and Strominger's sum of geometries

We turn to the boundary conditions for the path integral. As stated in Section 1.1, the metric configurations that contribute to the gravitational functional integral are those with asymptotic geometry corresponding to a T^2 with complex structure τ. More precisely, these are all configurations that obey the (Wick-rotated) Brown-Henneaux boundary conditions

$$\gamma_{\alpha\beta}\, dx^\alpha dx^\beta \;=\; \ell^2 \mathrm{e}^{2\rho} dz d\overline{z} + \mathcal{O}(1)\,, \tag{3.7}$$

where $(z,\overline{z})$ must be coordinates on a T^2, i.e. $z \sim 2\pi \sim 2\pi\tau$. In the semiclassical limit, where the path-integral takes the discrete form (3.1), the contributions are just saddles of the theory, which as described above are locally $\mathbb{H}^3$. By this virtue, they automatically have asymptotic form (3.7). The non-trivial question that remains is therefore how to classify all $\mathbb{H}^3$ spaces with a T^2 boundary. We review this in the following paragraphs.

Starting with $\mathbb{H}^3$, one takes the quotient $\mathbb{H}^3/\mathbb{Z}$ of the hyperbolic space with the discrete subgroup $\mathbb{Z}$ of its isometry group SL$(2,\mathbb{C})$. This defines the solid torus manifold, which is unique at the level of hyperbolic geometry. The conformal boundary of this torus is equipped with the modular parameter τ, which is defined only up to actions by the elements of the modular group PSL$(2,\mathbb{Z})$ =SL$(2,\mathbb{Z})/\mathbb{Z}_2$:

$$\tau \quad\mapsto\quad \frac{a\tau+b}{c\tau+d}\,,\qquad a,b,c,d\in\mathbb{R}\,,\qquad ad-bc \;=\; 1\,. \tag{3.8}$$

(The action of $\mathbb{Z}_2$ in the quotient is the simultaneous sign flip of (a,b,c,d).)

Although at the level of the geometry the modular group is a symmetry, its action does nevertheless change which cycle of the T^2 is contractible in the bulk. Explicitly, consider the two coordinates (ψ, t_E) on the T^2, which are related to $(z,\overline{z})$ of (3.7) as in (3.3). The geometry in which the ψ-circle is contractible is called the $M_{0,1}$ geometry, or the *thermal* geometry. The action on $M_{0,1}$ with an element

$$\gamma \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \in PSL(2,\mathbb{Z}) \tag{3.9}$$

produces a new configuration with modular parameter given in (3.8) and with a contractible cycle along the

$$c\,t_E + d\,\psi \tag{3.10}$$

direction (note that by Euclid's algorithm, the condition $ac-bd=1$ implies that (c,d) are relative primes). While it therefore seems that the g_c geometries that should be summed in (3.1) are all the PSL$(2,\mathbb{Z})$ images of the thermal geometry, this turns out to be too broad. Indeed, we note that after specifying (c,d) there is one further equivalence relation as $(a,b) \to (a,b)+t(c,d)$, $t\in\mathbb{Z}$. Therefore, the independent sum is only over distinct pairs (c,d). The associated geometries are the images of $M_{0,1}$ under PSL$(2,\mathbb{Z})/\mathbb{Z}$, which we denote as $M_{c,d}$. The gravitational partition function (3.1) becomes

$$Z(\tau) \;=\; \sum_{(c,d)} Z_{c,d}(\tau)\,, \tag{3.11}$$

where the summation is over all relatively prime c and d with $c\geq 0$, and where

$$Z_{c,d} \;\equiv\; \exp\Big(S_E(M_{c,d})+\cdots\Big)\,. \tag{3.12}$$

This family of $M_{c,d}$ geometries is the interpretation of what Maldacena and Strominger termed an "SL(2,$\mathbb{Z}$) family of black holes" [25]. Note that, because the $M_{c,d}$ are the PSL$(2,\mathbb{Z})/\mathbb{Z}$ images of $M_{0,1}$, we may also express (3.11) as

$$Z(\tau) \;=\; \sum_{\gamma^*\in \mathrm{PSL}(2,\mathbb{Z})/\mathbb{Z}} Z_{0,1}(\gamma^*\cdot\tau)\,. \tag{3.13}$$

Written in this way, it is clear that all contributions to the partition function are known once the contribution $Z_{0,1}$ of thermal AdS$_3$ is known. The exercise of computing the partition function therefore reduces to the calculation of $Z_{0,1}$. In particular, if the quantum $(+\cdots)$ corrections in (3.12) with $(c,d)=(0,1)$ can be computed, then the partition function is known exactly. Recall that theses are the considerations that were invoked in Section 1.1 when motivating the exact computation of $Z_{0,1}$ (1.10) in the supersymmetric theory.

The $M_{0,1}$ geometry: thermal AdS$_3$

As described above, the simplest classical configuration with boundary T^2 is thermal AdS$_3$, corresponding to the solid torus with contractible cycle along the spatial ψ direction. We will require an explicit form for this geometry. We realize it, as is usual, as the Wick rotation (3.2) of the global AdS$_3$ metric (2.13). This gives the line element

$$ds^2 \;=\; 4\ell^2\left(\cosh^2\rho\, dt_E{}^2 + d\rho^2 + \sinh^2\rho\, d\psi^2\right) . \tag{3.14}$$

We then impose the required thermal periodicities as

$$\left(t_{\scriptscriptstyle E}\,,\psi\right) \sim \left(t_{\scriptscriptstyle E} + 2\pi\tau_2\,, \psi + 2\pi\tau_1\right) \sim \left(t_{\scriptscriptstyle E}\,, \psi + 2\pi\right) . \tag{3.15}$$

Note that we introduced notation for the real and imaginary part of τ as

$$\tau \;\equiv\; \tau_1 + \mathrm{i}\tau_2\,. \tag{3.16}$$

3.2 Introducing U(1) gauge fields

In this section, we insert into the low-energy gravitational theory the relevant dual structure for constant chemical potentials μ^I coupled to a number of conserved U(1) charges $q_I = \int J_I$ in the boundary CFT$_2$. Here, J_I are the corresponding conserved currents in the CFT, which we will take to be right-moving. We then compute the action on thermal AdS$_3$ with these U(1) couplings. Once exponentiated, this action value corresponds to the leading order contribution to the function integral for $Z_{0,1}$.

In the presence of the U(1) charges q_I with chemical potentials μ^I, the thermal partition function (1.4) of the generic CFT$_2$ is modified as

$$Z_{\mathrm{CFT}_2}(\tau,\,\mu) \;=\; \mathrm{Tr}_{\mathcal{H}}\Big[\mathrm{e}^{-\beta H + \ell P + \mu^I q_I}\Big] , \tag{3.17}$$

where as before, H and P are the Hamiltonian and angular momentum operator coupled to their chemical potentials β, ℓ given as in (1.5), (1.6), respectively.[1] In the dual gravitational theory (3.6), this additional matter structure should be reflected by including the same number of U(1) gauge fields W^I. The most relevant term

[1] Again, we highlight that ℓ appearing in (1.4) and now (3.17) is *not* the AdS$_3$ radius appearing in expressions like (2.13).

governing their dynamics at low energies is given by the Chern-Simons (CS) action

$$-\frac{\mathrm{i}}{8\pi}k_{IJ}\int W^I\wedge dW^J \;=\; -\frac{\mathrm{i}}{8\pi}k_{IJ}\int d^3x\,\varepsilon^{\mu\nu\lambda}\,W^I_\mu\,\partial_\nu W^J_\lambda\,. \tag{3.18}$$

Gauge fields on asymptotically AdS$_3$ spaces admit a large ρ expansion analogous to the Fefferman-Graham expansion (2.23) as:

$$W^I_\alpha(\rho,x) \;=\; W^{I\,(0)}_\alpha(x)+\mathrm{e}^{-2\rho}\,W^{I\,(2)}_\alpha(x)+\cdots\,. \tag{3.19}$$

The asymptotic equations of motion further imply that $W^{I\,(0)}_\alpha$ is flat (i.e. independent of ρ). We choose the U(1) gauge $W^I_\rho=0$.

As is well-known, the CS term has a first order kinetic term so that the two legs $W^I_{z,\bar z}$ form canonical pairs in the Hamiltonian theory [57]. One should therefore impose Dirichlet boundary conditions on only one of the legs:

$$\delta W^{I\,(0)}_z \;=\; 0\,,\qquad W^{I\,(0)}_{\bar z}\text{ not fixed.} \tag{3.20}$$

Now, in accord with the bulk/boundary correspondence, the boundary source μ^I must be identified with the asymptotic value of the gauge field $W^{I\,(0)}_z$. Focusing on the thermal AdS$_3$ geometry, where the ψ-cycle is contractible, any smooth configuration must have $W^I_\psi=0$ at the origin. The saddle-point configurations have flat gauge fields due to the equations of motion, and therefore obey

$$W^I_z \;=\; -W^I_{\bar z} \;=\; -\mathrm{i}\,\mu^I\,. \tag{3.21}$$

The semiclassical thermal AdS$_3$ contribution

We turn to computing the semiclassical contribution of this U(1)-matter-coupled thermal AdS$_3$ configuration to the gravitational dual of the trace (3.17). As discussed in Section 3.1, this just involves computing the exponential of the renormalized action

$$S_{\text{ren}} \;\equiv\; S_{\text{bulk}}+S_{\text{bdry}} \tag{3.22}$$

on the field configuration. Here, S_{bulk} is the bulk Euclidean action of the Einstein-Hilbert-Chern-Simons theory given by the sum of (3.6) and (3.18). Note that a Maxwell term for W^I is of course also present in the theory, but it vanishes on the constant gauge fields (3.21). Meanwhile, the action S_{bdry} is the boundary action required to make the total action finite and well-defined under our choice of boundary conditions. This corresponds to the Gibbons-Hawking boundary term (2.48) and counter term (2.49), and a Chern-Simons boundary term given by

$$-\frac{\mathrm{i}}{8\pi}\,k_{IJ}\int dz\,d\bar{z}\,\Big[W_z^I\,W_{\bar{z}}^J\Big]_{\mathrm{bdry}}\,. \tag{3.23}$$

This last term is required to ensure the consistency of the variational principle of the gauge fields with the boundary conditions (3.20).

We may now evaluate S_{ren} on the field configuration (3.14), (3.21), with identifications (3.15). The result is

$$S_{\mathrm{ren}}(\tau,\,\mu) \;=\; -\pi\tau_2 k - \pi\tau_2\,k_{IJ}\,\mu^I\mu^J\,, \tag{3.24}$$

where $6k = \frac{3(2\ell)}{2G_3}$ is the Brown-Henneaux central charge of the gravitational theory for the AdS$_3$ space (3.14), and k_{IJ} is the level of the Chern-Simons term (3.18). We remind the reader that the boundary U(1) current obtained from (3.18), (3.23) is right-moving. The opposite chirality is described upon imposing opposite boundary conditions to (3.20), i.e. fixing $W_{\bar{z}}^{(0)}$ instead of $W_z^{(0)}$, which requires an opposite relative sign between (3.18) and (3.23) (for the variational principle to remain well-defined).

Chapter 4

5d conformal supergravity and $\mathrm{AdS}_3 \times \mathrm{S}^2$

In this section, we turn from the 3d Einstein-Hilbert setting studied in Chapters 2 and 3 to 5d Lorentzian supergravity on $\mathrm{AdS}_3 \times \mathrm{S}^2$, with $\mathcal{N} = 2$ (minimal) supersymmetry (i.e. 8 real supercharges) and coupled to U(1) vector multiplets. In the context of the localization formalism, we are particularly interested in the *off-shell* formulation of this theory, whose key features we review in Section 4.1. In Section 4.2, we move on to describe the classical global $\mathrm{AdS}_3 \times \mathrm{S}^2$ solution of this theory. We present the Killing spinors preserved by this configuration and the consequent superalgebra.

4.1 Off-shell 5d supergravity

Off-shell supergravity in the superconformal formalism in Lorentzian signature in various dimensions has been known for many decades (see the book [58]). The idea of this framework relies on the well-known fact that Poincaré *gravity* theories can consistently be described as conformal gravity theories coupled to compensating matter. It then turns out that this conformal description for *supergravity* theories allows to have off-shell representations for supersymmetry, albeit in the presence of additional auxiliary fields.

In this thesis, we are interested in the matter-coupled $\mathcal{N} = 2$ theory in five spacetime dimensions, which was constructed in [28, 29, 30], and in [32, 31, 59]. It is reviewed in the more recent [60, 37] whose conventions we follow.

Weyl	$E_M{}^A\,,\ \Psi^i_M\,,\ b_M\,,\ V_{M\,i}{}^j\,,\ T_{MN}\,,\ D\,,\ \chi^i$
Vector	$\sigma^I\,,\ W^I_M\,,\ \Omega^{I\,i}\,,\ Y^I_{ij}$
Hyper	$A_i{}^\alpha\,,\ \zeta^\alpha$
SUSY parameters	$\epsilon^i\,,\ \eta^i$

Table 4.1: Independent fields of the supersymmetric multiplets and Q, S-supersymmetry parameters in five-dimensional $\mathcal{N}=1$ conformal supergravity.

Supermultiplets

For the $\mathcal{N}=2$ 5d conformal supergravity theory, we follow the conventions of [60]. We consider the Weyl multiplet, which couples to $N_{\rm v}$ number of U(1) vector multiplets as well as a single hyper multiplet. One of the $N_{\rm v}$ vector multiplets and the single hypermultiplet constitute the two compensators to be added to the Weyl multiplet in order for the off-shell theory to correctly describe the $\mathcal{N}=2$ Poincaré supergravity. The reduction from the off-shell theory to the Poincaré theory is discussed at the end of this Section. We now review the field content each multiplet. For a summary, see Table 4.1.

The Weyl multiplet consists of the gauge fields corresponding to all the symmetry generators of the $\mathcal{N}=2$ superconformal algebra $\{P^A\,, M^{AB}\,, D\,, K^A\,, Q^i\,, S^i\,, V_j{}^i\}$, where D and K^A are the dilatation and special conformal transformation respectively. Among all the gauge fields, the gauge fields associated with $\{M^{AB}\,, K^A\,, S^i\}$ are composite, i.e. they are expressed in terms of other gauge fields. The independent gauge fields in Weyl multiplet are the vielbein $E_M{}^A$, dilatation gauge field b_M, gaugino ψ^i_M, and the SU(2)$_R$ gauge field $V_{Mj}{}^i$.[1] For the Weyl multiplet to be realized as an off-shell supermultiplet, it includes an auxiliary two-form tensor T_{AB}, an auxiliary fermion χ^i, and an auxiliary scalar D. Hence the independent fields of the Weyl multiplet are summarized as

$$\text{Weyl:}\quad \{E_M{}^A\,,\ \Psi^i_M\,,\ b_M\,,\ V_{M,i}{}^j\,;\,T_{MN}\,,\ \chi^i\,,\ D\}\,. \tag{4.1}$$

Here, the indices $\{A,B,\cdots\}$, $\{M,N,\cdots\}$, $\{i,j,\cdots\}$ are five-dimensional flat tangent space, curved spacetime, and SU(2) fundamental indices, respectively, which are summarized in Appendix A. We use the special conformal symmetry (that acts only

[1]Note that the SU(2)$_R$ of the supergravity is *not* the same as the SU(2) coming from the rotation of the S^2 geometry (which, recall, is dual of the R-symmetry of the (0,4) boundary theory.

on b_M) to gauge-fix $b_M = 0$, so that from here on this field will not appear. We consider $N_{\rm v}$ vector multiplets labeled by I, each of which consists of

$$\text{Vector:}\quad \{\sigma^I\,,\, W_M^I\,,\, \Omega^{I\,i}\,,\, Y_{ij}^I\}\,,\qquad I=1\,,2\,,\cdots,N_{\rm v}\,. \tag{4.2}$$

They corresponds to a scalar, a U(1) gauge field, gaugini, and an auxiliary symmetric SU(2) triplet. The i,j indices are raised and lowered using the $SU(2)$ symplectic metric ε, where, explicitely, $\varepsilon_{12}=\varepsilon^{12}=1$. In particular, we have $Y_{ij} = \varepsilon_{ik}\,\varepsilon_{j\ell}\,Y^{k\ell}$. We finally consider a single hypermultiplet, which consists of

$$\text{Hyper:}\quad \{A_i{}^\alpha\,,\,\zeta^\alpha\}, \tag{4.3}$$

corresponding to the hyper scalar, and the hyper fermion, where $\alpha = 1,2$. Note that this is an on-shell hypermultiplet. There is in fact no known off-shell Lorentz-covariant hypermultiplet with finite number of fields. While this limitation plays no role in the contents of this thesis, it is interesting to note that the construction of off-shell hypermultiplets for one supercharge in the context of localization has been studied [61, 20].

Supersymmetry algebra

The infinitesimal supersymmetry transformations of the various spinor fields under the Q and S supersymmetry are parametrized by the Q- and S- Killing spinors ϵ^i, η^i, respectively. Up to higher order in fermions, we have:

$$\begin{aligned}
\delta\Psi_M^i &= 2D_M\epsilon^i + \frac{i}{2}T_{AB}(3\gamma^{AB}\gamma_M - \gamma_M\gamma^{AB})\epsilon^i - \mathrm{i}\gamma_M\eta^i\,,\\
\delta\chi^i &= \tfrac12\epsilon^i D + \frac{1}{64}R_{MNj}{}^i(V)\gamma^{MN}\epsilon^j + \frac{3\mathrm{i}}{64}(3\gamma^{AB}\gamma^C + \gamma^C\gamma^{AB})\epsilon^i D_C T_{AB}\\
&\quad - \frac{3}{16}T_{AB}T_{CD}\gamma^{ABCD}\epsilon^i + \frac{3}{16}T_{AB}\gamma^{AB}\eta^i\,,\\
\delta\Omega^i &= -\frac12(F_{AB} - 4\sigma T_{AB})\gamma^{AB}\epsilon^i - \mathrm{i}\gamma^A\epsilon^i D_A\sigma - 2\varepsilon_{jk}Y^{ij}\epsilon^k + \sigma\eta^i\,,\\
\delta\zeta^\alpha &= -\mathrm{i}\gamma^A\epsilon^i D_A A_i{}^\alpha + \frac32 A_i{}^\alpha\eta^i\,.
\end{aligned} \tag{4.4}$$

where the curvature $R_{MNi}{}^j(V)$ is given by:

$$R_{MNi}{}^j(V) \;=\; 2\,\partial_{[M}V_{N]}{}_i{}^j - 2V_{[M}{}_i{}^k V_{N]}{}_k{}^j\,. \tag{4.5}$$

The relevant covariant derivatives acting on each field are covariant with respect to all bosonic gauge symmetries except conformal boosts:

$$\begin{aligned} D_M\,\epsilon^i &= \left(\partial_M - \frac{1}{4}\,\omega_M{}^{AB}\,\gamma_{AB} + \frac{1}{2}b_M\right)\epsilon^i + \frac{1}{2}V_{Mj}{}^i\,\epsilon^j\,, \\ D_M T_{AB} &= (\partial_M - b_M)T_{AB} - \omega_{MA}{}^C T_{CB} - \omega_{MB}{}^C T_{AC}\,, \\ D_M\,\sigma^I &= (\partial_M - b_M)\;\sigma^I\,, \\ D_M\,A_i{}^\alpha &= \left(\partial_M - \frac{3}{2}b_M\right)A_i{}^\alpha - \frac{1}{2}V_{Mi}{}^j A_j{}^\alpha\,. \end{aligned} \tag{4.6}$$

Two Q-supersymmetry transformations, parametrized by spinors ϵ_1 and ϵ_2 respectively, close into the bosonic symmetries of the theory as

$$[\delta_Q(\epsilon_1)\,,\delta_Q(\epsilon_2)] \;=\; \delta_{\rm cgct}(\xi^\mu) + \delta_M(\lambda) + \delta_S(\eta) + \delta_K(\Lambda_K) \tag{4.7}$$

where $\delta_{\rm cgct}$ are the covariant general coordinate transformations, δ_M is a local Lorentz transformation, δ_S is a conformal supersymmetry transformation, and δ_K is special conformal transformation. Explicitly, the relevant parameters to this thesis are

$$\begin{aligned} \xi^\mu &= 2\bar\epsilon_{2i}\gamma^\mu\epsilon_1^i\,, \\ \lambda^{AB} &= -\xi^\mu\omega_\mu{}^{AB} + \frac{\mathrm{i}}{2}T^{CD}\bar\epsilon_{2i}(6\gamma^{[A}\gamma_{CD}\gamma^{B]} - \gamma^{AB}\gamma_{CD} - \gamma_{CD}\gamma^{AB})\epsilon_1^i\,. \end{aligned} \tag{4.8}$$

The action

The bosonic Lagrangian at two-derivative level is

$$L_{\rm bulk} \;=\; E\;(\mathcal{L}_V + \mathcal{L}_{VW} + \mathcal{L}_H + \mathcal{L}_{HW} + \mathcal{L}_{CS})\;, \tag{4.9}$$

where $E\equiv\det(E_M{}^A)$, $\mathcal{L}_V$ contains purely vector multiplet terms, $\mathcal{L}_{VW}$ contains mixing between vector and Weyl, $\mathcal{L}_H$ is the kinetic hyper scalar piece, $\mathcal{L}_{HW}$ contains coupling of hyper to Weyl, and $\mathcal{L}_{CS}$ is the five-dimensional Chern-Simons action:

$$\begin{aligned}
\mathcal{L}_V &= \frac{1}{2}c_{IJK}\,\sigma^I\left(\frac{1}{2}D^M\sigma^J D_M\sigma^K+\frac{1}{4}F^J_{MN}F^{MN\,K}-3\sigma^J F^K_{MN}T^{MN}-Y^J_{ij}Y^{K\,ij}\right),\\
\mathcal{L}_{VW} &= -C(\sigma)\left(\frac{1}{8}R-4D-\frac{39}{2}T^2\right),\\
\mathcal{L}_H &= -\frac{1}{2}\Omega_{\alpha\beta}\,\varepsilon^{ij}D_M A_i{}^\alpha D^M A_j{}^\beta,\\
\mathcal{L}_{HW} &= \chi\left(\frac{3}{16}R+2D+\frac{3}{4}T^2\right),\\
\mathcal{L}_{CS} &= -\frac{i}{48\,E}\varepsilon^{MNOPQ}c_{IJK}W^I_M F^J_{NO}F^K_{PQ}\,.
\end{aligned} \tag{4.10}$$

In the Chern-Simons Lagrangian $\mathcal{L}_{CS}$, the object ε^{MNOPQ} is a fully antisymmetric tensor density taking values ±1. The scalar norms appearing in $\mathcal{L}_{VW}$ and $\mathcal{L}_{HW}$ are:

$$C(\sigma) := \frac{1}{6}c_{IJK}\,\sigma^I\sigma^J\sigma^K\,, \tag{4.11}$$

$$\chi := \frac{1}{2}\,\Omega_{\alpha\beta}\,\varepsilon^{ij}A_i{}^\alpha A_j{}^\beta\,. \tag{4.12}$$

The action of the theory is

$$S_{\mathrm{bulk}} = \frac{1}{8\pi^2}\int_{\mathcal{M}} d^5x\,L_{\mathrm{bulk}}\,, \tag{4.13}$$

for coordinates x^M on the 5d manifold $\mathcal{M}$.

Relation to Poincaré theory

In this thesis, while we work almost exclusively in the above off-shell formulation of the supergravity, we nevertheless require an understanding of its connection to the Poincaré frame. Here, we present this connection.

For our purposes, it is sufficient to describe the transition of only the *bosonic sector* of the off-shell theory, as given in (4.10), to the bosonic sector of the corresponding Poincaré frame.[2] We focus on the pure case, i.e. where the Poincaré theory has only a gravity multiplet. Recall that this multiplet should contain in the bosonic sector only the vielbein $E_M{}^A$ and the graviphoton W^g_M. To reach this frame from the off-

[2]The transition of the fermionic sector follows an analogous mechanism. See [30] for the complete treatment.

shell theory, one starts with the Weyl multiplet and the compensating multiplets. In the following steps, we therefore take $N_v = 1$ such that $I = 1$ denotes the vector-compensator.

Since we have already gauge-fixed the special conformal symmetry with $b_M = 0$, the only extra bosonic symmetry that is present in the off-shell theory (4.10) is the dilatational symmetry. We gauge-fix this symmetry by setting the scalar norm χ of the compensating hypermultiplet to a dimensionful constant (the "D-gauge")[3]. Note that in the Lagrangian (4.10), the auxiliary scalar D appears as the Lagrange multiplier of $C(\sigma)$ and χ. Its equation of motion gives the algebraic constraint:

$$C(\sigma) \;=\; -\frac{\chi}{2}\,, \tag{4.14}$$

and so applying the D-gauge *and* imposing the field equations will also fix $C(\sigma)$, which removes the scalar degree of freedom of the compensating vector multiplet. In this vector multiplet, the surviving bosonic degrees of freedom are now Y^1_{ij} and the gauge field W^1_M. The former is an auxiliary field and is removed through its algebraic field equation:

$$Y^1_{ij} \;=\; 0\,. \tag{4.15}$$

The gauge field $W^1_M \equiv W^g_M$ joins, as the graviphoton, the supergravity multiplet of the Poincaré frame. At this stage we have therefore eliminated the compensating vector multiplet and gained a graviphoton in exchange. The remaining extra fields of the off-shell multiplets are now the auxiliary T_{MN} and $V_{Mi}{}^j$ in the Weyl multiplet (note that D is absent from the action upon substituting (4.14)), and three hyper-scalar components of $A_i{}^\alpha$ in the compensating hypermultiplet. The two Weyl fields are eliminated by their equations motion, which are

$$T_{AB} \;=\; \frac{F^1_{AB}}{4\sigma^1}\,, \qquad \Omega_{\alpha\beta}\varepsilon^{ij}D_M A_i{}^\alpha \cdot A_k{}^\beta \;=\; 0\,. \tag{4.16}$$

Finally, the three $A_i{}^\alpha$ components are fixed to constants using the $SU(2)_R$ (the "SU(2)-gauge"). Putting all the above steps together, we are left with the bosonic

[3]The role of the compensating vector multiplet is to ensure consistent field equation for D in the presence of the compensating hypermultiplet: if the compensating vector multiplet were absent, then the D field would only appear in the Lagrangian as χD, and so the field equations for D would force $\chi = 0$.

action for pure 5d $\mathcal{N}=2$ Poincaré supergravity:

$$
\begin{aligned}
S^{\rm PC}_{\rm bulk} \;=\; \frac{1}{8\pi^2}\int d^5x\, E\,\Big[-\frac{C(\sigma)}{2}\,R-\frac{c_{111}\sigma^1}{16}F^1_{MN}\,F^{MN\,1} \\
-\frac{\mathrm{i}c_{111}}{48\,E}\varepsilon^{MNOPQ}W^1_M F^1_{NO}F^1_{PQ}\Big]\,.
\end{aligned}
\tag{4.17}
$$

Recall that, according to the D-gauge followed by the D-field equation (4.14), the quantity $C(\sigma)$ (and therefore σ) should be seen as a dimensionful constant. In particular, to reach the conventional $(16\pi G_5)^{-1}$ prefactor for the Ricci scalar in the action (4.17), one chooses

$$
\chi \;=\; \frac{2\pi}{G_5} \quad\Rightarrow\quad C(\sigma) \;=\; -\frac{\pi}{G_5}\,. \tag{4.18}
$$

4.2 Global Lorentzian AdS$_3\times$ S^2

We consider the fully supersymmetric AdS$_3\times$ S^2 solution of the Lorentzian off-shell supergravity described in Section 4.1, corresponding to the near-horizon geometry of the half-BPS magnetic black string [62]. To present the most general configuration, we reinstate an arbitrary number N_v of off-shell vector multiplets $I=1\,,\cdots,N_v$.

Field configuration

The metric in Lorentzian signature is

$$
ds^2 \;=\; 4\ell^2\,(-\cosh^2\rho\,dt^2+d\rho^2+\sinh^2\rho\,d\psi^2)+\ell^2\,(d\theta^2+\sin^2\theta d\phi^2)\,, \tag{4.19}
$$

where the coordinates of the AdS$_3$ have the ranges $\rho\in[0,\infty)$, $\psi\in[0,2\pi)$, $t\in(-\infty,\infty)$ and the angles on the S^2 have ranges $\theta\in[0,\pi)$, $\phi\in[0,2\pi)$. The radii of the AdS$_3$ and the S^2 are (2ℓ) and ℓ respectively, where this relative factor of 2 is determined by supersymmetry. Note that in the off-shell theory, ℓ is free and parametrizes the dilatations of the theory, while in the on-shell theory (where dilatations are broken) it is determined by the magnetic charges of the solution via the D-gauge condition. These magnetic charges p^I enter the solution through the vector multiplet. The non-trivial fields of the vector multiplet are:

$$
\sigma^I \;=\; -\frac{p^I}{\ell}\,,\qquad F^I_{\theta\phi} \;=\; p^I\,\sin\theta\,. \tag{4.20}
$$

Note that the solution does not have electric flux, which allows us to turn on flat gauge connections on the AdS$_3$. This aspect will become relevant in the following.

In the off-shell formalism of Section 4.1, one requires additional auxiliary fields. In the Weyl multiplet, the non-trivial fields are:

$$T_{\theta\phi} \;=\; -\frac{\ell}{4}\sin\theta\,. \tag{4.21}$$

In the compensating hypermultiplet, the BPS equation is solved by

$$A_i{}^{\alpha} \;=\; c_i{}^{\alpha}\,, \tag{4.22}$$

where the constants $c_i{}^{\alpha}$ are determined in terms of the charge p^I by the field equation for the auxiliary field D to be

$$\Omega_{\alpha\beta}\,\varepsilon^{ij}c_i{}^{\alpha}c_j{}^{\beta} \;=\; \frac{2}{3\ell^3}\,c_{IJK}\,p^I p^J p^K\,. \tag{4.23}$$

In this thesis, we fix an explicit choice for the $c_i{}^{\alpha}$ as

$$c_1{}^2 \;=\; c_2{}^1 \;=\; 0\,, \qquad c_1{}^1 \;=\; c_2{}^2 \;=\; \sqrt{\frac{p^3}{3\ell^3}}\,. \tag{4.24}$$

Relation to AdS$_3$

It useful to note the relation between the Brown-Henneaux central charge $(6k) = \frac{3(2\ell)}{2G_3}$ and the magnectic charges p^I of the black string. This involves the D-gauge procedure, described at the end of Section 4.1, whereby the vector-scalar norm $C(\sigma)$ is fixed in terms of the five-dimensional Newton's constant as in (4.18). Substituting into (4.18) the field configuration (4.20) for the background, we have

$$C(\sigma) \;=\; -\pi G_5^{-1} \quad\Rightarrow\quad 2p^3 \;=\; \frac{3(2\ell)\cdot 2\pi\ell^2}{G_5}\,, \tag{4.25}$$

where $p^3 \equiv c_{IJK}p^Ip^Jp^K$. A relation between G_5 and the three-dimensional Newton's constant G_3 can be identified by performing the on-shell reduction of the geometry onto the S^2 factor and comparing the resulting action with the 3d effective action (3.6). After substituting for the D-gauge (4.25), the Ricci-coupled part of the 5d action is

$$S_{\text{bulk}} \;=\; \frac{1}{16\pi G_5}\int d^3x\sqrt{g_{(3)}}\,d\theta d\phi\sin\theta(R^{(3)}+R^{(2)})+\cdots\,, \tag{4.26}$$

where we used that on AdS$_3$×S^2 (4.19), the 5d Ricci scalar is simply the sum of the Ricci scalar of each factor. We also split the 5d metric determinant into the

AdS$_3$ and S^2 part. Note that the $R^{(2)}$ factor contributes an additive constant that can be omitted for the sake of this argument. Performing the integration of the S^2 coordinates now gives for the right-hand-side of (4.26):

$$\frac{4\pi\ell^2}{16\pi G_5}\int d^3x\sqrt{g_{(3)}}\,R^{(3)}+\cdots \tag{4.27}$$

The comparison of (4.27) with the effective theory (3.6) in three dimensions then sets the relation between Newton's constants: $G_5 = \mathrm{Area}_{S^2}\times G_3 = 4\pi\ell^2 G_3$. Combining with (4.25), one reaches the relation for the central charge:

$$2p^3 \;=\; 6k\,. \tag{4.28}$$

This result has also been elegantly derived in [48] using the principle of c-extremization.

We will also require a relation between the level k_{IJ} of the U(1) current algebra and the p^I. This can be derived by on-shell reduction of the 5d Chern-Simons action on the S^2. We have

$$\begin{aligned}\frac{1}{8\pi^2}\int d^5x\,E\,\mathcal{L}_{CS} \;&=\; -\frac{4p^K}{8\pi^2}\int d^3x\,d\theta\,d\phi\,\sin\theta\,\frac{\mathrm{i}c_{IJK}}{48}\varepsilon^{\mu\nu\sigma\theta\phi}\,W_\mu^I F_{\nu\sigma}^J\\ &=\; -\frac{\mathrm{i}c_{IJK}p^K}{24\pi}\int d^3x\,\varepsilon^{\mu\nu\sigma}\,W_\mu^I F_{\nu\sigma}^J\,,\end{aligned} \tag{4.29}$$

where $\mathcal{L}_{CS}$ is given in (4.10). Comparing with the Chern-Simons action (3.18) of the three-dimensional effective theory, we identify:

$$\frac{2}{3}c_{IJK}\,p^K \;=\; k_{IJ}\,. \tag{4.30}$$

4.3 Supersymmetry algebra in Lorentzian AdS$_3\times$ S^2

Killing spinors

The Q- and S- supersymmetry parameters, ϵ^i and η^i respectively, that are preserved by the bosonic fields of the global AdS$_3\times S^2$ background are determined by setting the variation of the gravitino and the variation of the auxiliary fermion in (4.4) to zero. These two equations are, respectively,

$$0 \;=\; 2\,D_M\epsilon^i + \frac{\mathrm{i}}{2}T_{AB}(3\gamma^{AB}\gamma_M - \gamma_M\gamma^{AB})\epsilon^i - \mathrm{i}\gamma_M\eta^i\,. \tag{4.31}$$

$$\begin{aligned} 0 \;=\; & \frac{1}{2}\epsilon^i D + \frac{1}{64}R_{MN j}{}^{i}(V)\gamma^{MN}\epsilon^j + \frac{3}{64}\mathrm{i}(3\gamma^{AB}\not{D} + \not{D}\gamma^{AB})T_{AB}\epsilon^i \\ & - \frac{3}{16}T_{AB}T_{CD}\gamma^{ABCD}\epsilon^i + \frac{3}{16}T_{AB}\gamma^{AB}\eta^i\,. \end{aligned} \tag{4.32}$$

On our bosonic background, the second equation (4.32) immediately determines the S-supersymmetry spinor as $\eta^i = 0$. The first equation is referred to as the Killing spinor equation. We analyze its solutions in Appendix C and summarize the results below.

The complex basis of the Killing spinor on AdS$_3$× S^2 is given by the the following four Killing spinors,

$$\begin{aligned} \epsilon_+{}^+ &= \sqrt{\tfrac{\ell}{2}}\,\epsilon^+_{\mathrm{AdS}_3}\otimes\epsilon^+_{\mathrm{S}^2}\,, & \epsilon_+{}^- &= \sqrt{\tfrac{\ell}{2}}\,\epsilon^+_{\mathrm{AdS}_3}\otimes\epsilon^-_{\mathrm{S}^2}\,, \\ \epsilon_-{}^+ &= \sqrt{\tfrac{\ell}{2}}\,\epsilon^-_{\mathrm{AdS}_3}\otimes\epsilon^+_{\mathrm{S}^2}\,, & \epsilon_-{}^- &= \sqrt{\tfrac{\ell}{2}}\,\epsilon^-_{\mathrm{AdS}_3}\otimes\epsilon^-_{\mathrm{S}^2}\,, \end{aligned} \tag{4.33}$$

with

$$\begin{aligned} \epsilon^+_{\mathrm{AdS}_3} &= \mathrm{e}^{\frac{\mathrm{i}}{2}(t+\psi)}\begin{pmatrix}\cosh\frac{\varrho}{2}\\ -\sinh\frac{\varrho}{2}\end{pmatrix}, & \epsilon^-_{\mathrm{AdS}_3} &= \mathrm{e}^{-\frac{\mathrm{i}}{2}(t+\psi)}\begin{pmatrix}-\sinh\frac{\varrho}{2}\\ \cosh\frac{\varrho}{2}\end{pmatrix}, \\ \epsilon^+_{\mathrm{S}^2} &= \mathrm{e}^{\frac{\mathrm{i}}{2}\phi}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}, & \epsilon^-_{\mathrm{S}^2} &= \mathrm{e}^{-\frac{\mathrm{i}}{2}\phi}\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}. \end{aligned} \tag{4.34}$$

These four Killing spinors organize themselves into the eight pairs of symplectic Majorana spinors

$$\begin{aligned} &\epsilon^i_{(1)} = (-\mathrm{i}\epsilon_+{}^+, \epsilon_-{}^-)\,, \quad \epsilon^i_{(2)} = (\epsilon_+{}^+, -\mathrm{i}\epsilon_-{}^-)\,, \quad \epsilon^i_{(3)} = -(\epsilon_-{}^-, \mathrm{i}\epsilon_+{}^+)\,, \quad \epsilon^i_{(4)} = -(\mathrm{i}\epsilon_-{}^-, \epsilon_+{}^+)\,, \\ &\tilde\epsilon^i_{(1)} = (\epsilon_+{}^-, \mathrm{i}\epsilon_-{}^+)\,, \quad \tilde\epsilon^i_{(2)} = (\mathrm{i}\epsilon_+{}^-, \epsilon_-{}^+)\,, \quad \tilde\epsilon^i_{(3)} = (-\mathrm{i}\epsilon_-{}^+, \epsilon_+{}^-)\,, \quad \tilde\epsilon^i_{(4)} = (\epsilon_-{}^+, -\mathrm{i}\epsilon_+{}^-)\,, \end{aligned} \tag{4.35}$$

to form the 8 real basis of the Killing spinor on AdS$_3$×S^2. Each pair satisfies the symplectic-Majorana condition (A.11) appropriate to the 5d Lorentzian theory, i.e. $(\epsilon^i)^\dagger\gamma_{\hat t} = \varepsilon_{ij}(\epsilon^j)^T\mathcal{C}$ in the conventions of Appendix C.

Superconformal algebra: Let us denote

$$\overline{\mathcal{Q}}_a \;=\; \delta(\epsilon^i_{(a)})\,, \qquad\qquad \widetilde{\overline{\mathcal{Q}}}_a \;=\; \delta(\tilde{\epsilon}^i_{(a)})\,, \qquad a=1\,,2\,,3\,,4\,, \tag{4.36}$$

with the Grassmann even Killing spinors ϵ^i, $\tilde{\epsilon}^i$. Then,

$$\begin{aligned}
\{\overline{\mathcal{Q}}_a\,,\overline{\mathcal{Q}}_b\} \;&=\; -2\mathrm{i}\delta_{ab}(\overline{L}_0-\overline{J}^3)\,, \qquad \{\widetilde{\overline{\mathcal{Q}}}_a\,,\widetilde{\overline{\mathcal{Q}}}_b\} \;=\; -2\mathrm{i}\delta_{ab}(\overline{L}_0+\overline{J}^3)\,,\\
\{\overline{\mathcal{Q}}_a\,,\widetilde{\overline{\mathcal{Q}}}_b\} \;&=\; \begin{pmatrix} -2\mathrm{i}\overline{J}^2 & 2\mathrm{i}\overline{J}^1 & -(\bar{L}_+-\bar{L}_-) & \mathrm{i}(\bar{L}_++\bar{L}_-) \\ -2\mathrm{i}\overline{J}^1 & -2\mathrm{i}\overline{J}^2 & -\mathrm{i}(\bar{L}_++\bar{L}_-) & -(\bar{L}_+-\bar{L}_-) \\ \bar{L}_+-\bar{L}_- & \mathrm{i}(\bar{L}_++\bar{L}_-) & -2\mathrm{i}\overline{J}^2 & -2\mathrm{i}\overline{J}^1 \\ -\mathrm{i}(\bar{L}_++\bar{L}_-) & \bar{L}_+-\bar{L}_- & 2\mathrm{i}\overline{J}^1 & -2\mathrm{i}\overline{J}^2 \end{pmatrix}\,,
\end{aligned} \tag{4.37}$$

where the SL$(2,R)$ generators $\overline{L}_0\,,\overline{L}_\pm$ and SO(3) generators $\overline{J}^{\boldsymbol{a}}$, $\boldsymbol{a}=1,2,3$, satisfy

$$[\overline{L}_+,\,\overline{L}_-] \;=\; -2\overline{L}_0\,, \qquad [\overline{L}_0\,,\bar{L}_\pm] \;=\; \pm\overline{L}_\pm\,, \qquad [\overline{J}^{\boldsymbol{a}}\,,\overline{J}^{\boldsymbol{b}}] \;=\; \mathrm{i}\epsilon^{\boldsymbol{abc}}\overline{J}^{\boldsymbol{c}}\,. \tag{4.38}$$

Their representation as differential operators on the AdS$_3\times$S^2 is given in Appendix D. Note from (D.1) that the $\overline{L}_{-1,0,1}$ generators act in the $(t+\psi)$- sector, which in our conventions is the right-moving sector (hence the bar on the generators). The supercharges $\overline{\mathcal{Q}}_a$, $\widetilde{\overline{\mathcal{Q}}}_a$ also manifestly act in the right-moving sector.

Let us define the supercharges $\bar{G}^{i\alpha}_\gamma$

$$\begin{aligned}
\bar{G}^{++}_+ \;&\equiv\; \frac{\mathrm{i}\overline{\mathcal{Q}}_1+\overline{\mathcal{Q}}_2}{2}\,, \quad \bar{G}^{+-}_- \;\equiv\; \frac{-\overline{\mathcal{Q}}_3+\mathrm{i}\mathcal{Q}_4}{2}\,,\\
\bar{G}^{--}_- \;&\equiv\; \frac{\overline{\mathcal{Q}}_1+\mathrm{i}\overline{\mathcal{Q}}_2}{2}\,, \quad \bar{G}^{-+}_+ \;\equiv\; \frac{\mathrm{i}\overline{\mathcal{Q}}_3-\mathcal{Q}_4}{2}\,,\\
\bar{G}^{+-}_+ \;&\equiv\; \frac{\widetilde{\overline{\mathcal{Q}}}_1-\mathrm{i}\widetilde{\overline{\mathcal{Q}}}_2}{2}\,, \quad \bar{G}^{++}_- \;\equiv\; \frac{\mathrm{i}\widetilde{\overline{\mathcal{Q}}}_3+\widetilde{\mathcal{Q}}_4}{2}\,,\\
\bar{G}^{-+}_- \;&\equiv\; \frac{-\mathrm{i}\widetilde{\overline{\mathcal{Q}}}_1+\widetilde{\overline{\mathcal{Q}}}_2}{2}\,, \quad \bar{G}^{--}_+ \;\equiv\; \frac{\widetilde{\overline{\mathcal{Q}}}_3+\mathrm{i}\widetilde{\overline{\mathcal{Q}}}_4}{2}\,,
\end{aligned} \tag{4.39}$$

where γ is the sign of the $\overline{L}_0$ eigenvalue, i is the outer automorphism from the SU(2) R-symmetry of the supergravity, and α is the SU(2) R-symmetry index corresponding to isometries of the S^2. Then, we obtain the non-trivial commutation relations:

$$\left\{\bar{G}^{+\alpha}_{\pm}\,,\bar{G}^{-\beta}_{\mp}\right\} \;=\; \epsilon^{\alpha\beta}\overline{L}_0 \pm (\epsilon\boldsymbol{\tau_a})^{\beta\alpha}\overline{J}^{\boldsymbol{a}}\,, \qquad \left\{\bar{G}^{+\alpha}_{\pm}\,,\bar{G}^{-\beta}_{\pm}\right\} \;=\; \mp\mathrm{i}\epsilon^{\alpha\beta}\overline{L}_{\pm}\,, \tag{4.40}$$

where $\boldsymbol{\tau}_a$ are the Pauli sigma matrices, and $\epsilon^{\alpha\beta}$ has $\epsilon^{+-}=-\epsilon^{-+}=1$. We also have:

$$\begin{aligned} \left[\bar{L}_0\,,\bar{G}^{i\alpha}_{\pm}\right] \;&=\; \pm\tfrac{1}{2}\,\bar{G}^{i\alpha}_{\pm}\,, \qquad & \left[\bar{L}_{\pm}\,,\bar{G}^{i\alpha}_{\mp}\right] \;&=\; -\mathrm{i}\,\bar{G}^{i\alpha}_{\pm}\,, \\ \left[\bar{J}^3\,,\bar{G}^{i\pm}_{\gamma}\right] \;&=\; \pm\tfrac{1}{2}\,\bar{G}^{i\pm}_{\gamma}\,, \qquad & \left[\bar{J}^{\pm}\,,\bar{G}^{i\mp}_{\gamma}\right] \;&=\; \bar{G}^{i\pm}_{\gamma}\,, \end{aligned} \tag{4.41}$$

where $\bar{J}^{\pm}\equiv\overline{J}^1\pm\mathrm{i}\overline{J}^2$. The algebra (4.38), (4.40), (4.41) is $su(1,1|2)$ and corresponds to the global part of the NS-sector chiral $\mathcal{N}=4$ superconformal algebra. Denoting the super Virasoro charges as $\overline{\mathcal{L}}_n$, $n\in\mathbb{Z}$ and $\bar{\mathcal{G}}^{\alpha}_{\dot{A},r}$, $r\in\mathbb{Z}+\frac{1}{2}$, $\dot{A}=(+,-)$, the embedding into the $\mathcal{N}=4$ superconformal algebra as presented e.g. in [63] is given by $\overline{L}_{\pm}=\mp\mathrm{i}\overline{\mathcal{L}}_{\mp1}$, $\overline{L}_0=\overline{\mathcal{L}}_0$, $\bar{G}^{\pm\alpha}_{\pm}=\pm\bar{\mathcal{G}}^{\alpha}_{\mp,\mp1/2}$, $\bar{G}^{\mp\alpha}_{\pm}=\pm\bar{\mathcal{G}}^{\alpha}_{\pm,\mp1/2}$, and the $su(2)$ zero-modes are unchanged. The algebra in this form is also summarized in Appendix B.

Chapter 5

Supersymmetric $\mathbb{H}^3/\mathbb{Z}\times\mathbf{S}^2$ and twisting

In this section we finally develop the supergravity theory on $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$ relevant to the localization computation of (1.10). First, in Section 5.1, we discuss the Euclidean counterpart to the 5d off-shell matter-coupled conformal supergravity that we reviewed in Chapter 4. We explain that this Euclidean theory is obtained by redefinitions of fields of the Lorentzian theory that follow simply from the Wick rotation. In Section 5.2, we then construct the $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$ vacuum solution of this theory from the Lorentzian $\mathrm{AdS}_3\times\mathrm{S}^2$ configuration presented in Section 4.2. This is the supersymmetric version of thermal AdS_3 (the $M_{0,1}$ torus) discussed in Chapter 3. As described in Section 1.3, a non-trivial twist of the S^2 around the Euclidean time circle is required to define consistent Killing spinors on the torus. We compute the superalgebra generated by these Killing spinors. In Section 5.3, we identify a suitable localization supercharge $\overline{\mathcal{Q}}$. We then discuss the Hamiltonian trace interpretation of the functional integral on this twisted configuration, and discuss how this is related to the elliptic genus in the semi-classical limit. We conclude with an evaluation of the 5d supergravity action and boundary terms on the $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$ background.

5.1 A 5d Euclidean off-shell supergravity

Constructions of Euclidean supergravities are scarcely studied compared to their Lorentzian counterparts, and few references exist (e.g. [41, 42, 43, 44]). In these references the method of time-like reduction from a five-dimensional Lorentzian theory is used to systematically construct the Euclidean-signature theory in four dimensions.

One potential systematic approach to construct our five-dimensional Euclidean theory would be to perform a timelike reduction on a 6d theory. However, we use a less formal approach here: we start from a Wick rotation and make an appropriate set of transformations on all the fields of the Lorentzian theory so that we obtain a consistent 5d Euclidean theory. This approach was successfully employed for the $\mathcal{N}=2$ off-shell supergravity in four dimensions and the result agrees with the timelike reduction [45].

The starting point is the off-shell Lorentzian supergravity in Section 4.1. We consider a Wick rotation as $t = -\mathrm{i}t_E$, which relates the Lorentzian and Euclidean time coordinates. This is followed by the corresponding transformations of all tensors for this coordinate change (including the time-directional gamma matrices, which is related as $\gamma_t = \mathrm{i}\gamma_{t_E}$). Here, there can be subtleties involving the fermionic fields of the theory: indeed changing the signature of spacetime by this Wick rotation, in general, demands changing the nature of irreducible spinors. For instance, in 4d, while the Majorana representation of irreducible spinors is allowed in the Lorentzian theory, the same is not true in the Euclidean theory. In this dimensionality, an appropriate field redefinition of spinors is therefore needed. This can be achieved in by going to the symplectic-Majorana basis, which exists in both the Lorentzian and Euclidean theory, and in which the charge conjugation matrix is the same in both the theories (we refer to [45] for the full presentation of this 4d procedure). In our present five-dimensional case, the situation is simpler: 5d fermions are necessarily symplectic-Majorana in both Lorentzian and Euclidean signatures, and so the above spinorial subtleties are not present. We therefore carry on with the usual implementation of the Wick rotation.

The transformation of the Lagrangian and action functional of the Lorentzian theory under the Wick rotation follows as for the Einstein-Hilbert discussion around (3.4) and (3.5): the Lagrangian density is invariant under coordinate transformations, and so we obtain a Euclidean Lagrangian density $\mathcal{L}_E$ that is unchanged from its Lorentzian counterpart $\mathcal{L}$ given in (4.10). The Lorentzian action $S = \int dt\, d^4x\mathcal{L}$ then maps to $\mathrm{i}S = \int dt_E\, d^4x\mathcal{L}_E$, and we can identify the right-hand-side as the Euclidean action, i.e. $S_E \equiv \int dt_E\, d^4x\mathcal{L}_E$.[1] With this identification, note that the Euclidean action is formally identical to the Lorentzian action. Now consider the infinitesimal supersymmetry transformations (4.4) of the Lorentzian theory. Note that they are also manifestly invariant under coordinate transformations. Under the Wick-rotation, they therefore map to identical transformations in the Euclidean theory.

[1] Recall that our identification $S_E \equiv \int dt_E\, d^4x\mathcal{L}_E$ implies that $\exp(\mathrm{i}S) = \exp(S_E)$. Therefore, for path integrals to be perturbatively well-defined, S_E should be negative-definite.

In short, we have set up a Euclidean theory which, at the level of the action and supersymmetry transformations, is formally identical to the Lorentzian theory. Going forward, we may therefore refer with (4.10), (4.13) and (4.4) to the Lagrangian, the action, and the supersymmetry transformations of the off-shell 5d $\mathcal{N}=2$ supergravity in both Lorentzian *and* Euclidean signature.

We conclude with a brief remark on the reality conditions of the fields. Generically, the reality conditions for the fields in Lorentzian and Euclidean theories are different. For instance, in the five-dimensional case, an $\mathrm{SU}(2)_R$ spinor doublet ψ^i with $i=1,2$ in Lorentzian signature follows the symplectic-Majorana condition

$$(\psi^i)^\dagger\gamma_{\hat{t}} \;=\; \varepsilon_{ij}(\psi^j)^T\mathcal{C}\,, \tag{5.1}$$

where $\mathcal{C}$ is the unique choice of the charge conjugation matrix in five dimensions (this is more generally true in odd dimensions). Imposing (5.1) on the infinitesimal supersymmetry transformations of the fermions leads to a set of reality conditions for the bosons, e.g. the gauge fields and the metric are found to be real. In Euclidean signature, where the reality condition for 5d spinors is

$$(\psi^i)^\dagger \;=\; \varepsilon_{ij}(\psi^j)^T\mathcal{C}\,, \tag{5.2}$$

repeating this procedure of imposing supersymmetry leads to bosonic reality conditions which do not in fact guarantee a negative-definite sign for the kinetic terms in the Euclidean Lagrangian. These types of subtleties surrounding Euclidean reality conditions in localization are, in fact, already well known from the four-dimensional localization problem around $\mathrm{AdS}_2\times\mathrm{S}^2$ [14, 39, 64]. The resolution is understood to be as follows: one abandons the Euclidean fermion- reality condition, which in our 5d case would be (5.2), and treat ψ^1 and ψ^2 as two independent Dirac spinors instead. The bosonic reality conditions consistent with the desired sign of the Euclidean action can then consistently be imposed, at the cost of having formally doubled the fermionic degrees of freedom. At the level of the functional integral, this doubling has to be compensated by choosing a half-dimensional contour of integration for the fermions.[2] In this thesis, since we do not reach the computation of the quantum func-

[2]By *half-dimensional contour of integration*, we are referring to the following concept: consider a Lorentzian theory with one real scalar degree of freedom (d.o.f.) $X = X^*$. Now suppose there is a corresponding Euclidean theory where X and X^* are independent, i.e. with two real d.o.f. In this Euclidean setting, path integrals are in the complex plane spanned by (X, X^*). To describe the original number of d.o.f., we choose any straight-line contour in that plane, e.g. $X^* = X$ or $X^* = -X$. In short, we integrate the *two* d.o.f. along a *one*-dimensional (thus *half*-dimensional) contour.

tional integral, we postpone a study of these aspects in the five-dimensional theory to future work.

5.2 Twisted background and superalgebra

We now construct the supersymmetric vacuum solution of the 5d Euclidean theory. Under the Wick rotation $t = -\mathrm{i}t_E$, the global $\mathrm{AdS}_3\times \mathrm{S}^2$ metric (4.19) rotates to that of Euclidean $\mathbb{H}^3 \times S^2$. The non-trivial fields in the Weyl multiplet are:

$$\begin{aligned} ds^2 &= 4\ell^2\left(\cosh^2\rho\, dt_E^2 + d\rho^2 + \sinh^2\rho\, d\psi^2\right) + \ell^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), && (5.3)\\ T_{\theta\phi} &= -\frac{\ell}{4}\sin\theta\,. && (5.4)\end{aligned}$$

If the Euclidean time coordinate t_E runs from $(-\infty, \infty)$, the topology is that of a solid cylinder times a sphere, which we call the Euclidean cylinder frame. Although the Killing spinor equations (4.31) and (4.32) are formally solved by the same set of eight spinors (4.35) in this background, these spinors are no longer well-defined because they diverge at the ends of the Euclidean cylinder. The solution to this problem involves compactifying the Euclidean time on a circle and simultaneously rotating the S^2 as we go around the time circle. This twisted quotient makes for a well-defined background, as we now describe.

We start from the configuration (5.3) describing an infinite solid cylinder (times a sphere), and make the following identifications,

$$(t_E\,,\psi\,,\phi) \;\sim\; (t_E\,,\psi + 2\pi\,,\phi) \;\sim\; (t_E + 2\pi\tau_2\,,\psi + 2\pi\tau_1\,,\phi + \mathrm{i}2\pi\tau_2\Omega)\,. \qquad (5.5)$$

Equivalently, we can define a new set of "twisted" coordinates,

$$t'_E \;=\; t_E\,, \qquad \phi' \;\equiv\; \phi - \mathrm{i}\Omega t_E\,, \qquad (5.6)$$

which have the identification

$$(t'_E\,,\psi\,,\phi') \;\sim\; (t'_E\,,\psi + 2\pi\,,\phi') \;\sim\; (t'_E + 2\pi\tau_2\,,\psi + 2\pi\tau_1\,,\phi')\,. \qquad (5.7)$$

We denote the corresponding complex coordinates as $z' = \psi + \mathrm{i}t'_E\,,\ \bar{z}' = \psi - \mathrm{i}t'_E\,,$ which have the usual identifications on a T^2 as $(z'\,,\bar{z}') \;\sim\; (z' + 2\pi\tau\,,\bar{z}' + 2\pi\bar{\tau})$.

In these twisted coordinates, the on-shell background configuration is

$$\begin{aligned}
ds^2 &= 4\ell^2\left(\cosh^2\!\rho\, dt_E'^2 + d\rho^2 + \sinh^2\!\rho\, d\psi^2\right) + \ell^2\Big(d\theta^2 + \sin^2\theta\big(d\phi' + \mathrm{i}\Omega dt_E'\big)^2\Big)\,,\\
T_{\theta\phi'} &= -\frac{\ell}{4}\sin\theta\,, \qquad T_{\theta t_E'} = -\mathrm{i}\frac{\ell}{4}\Omega\sin\theta\,,\\
\sigma^I &= -\frac{p^I}{\ell}\,, \qquad W^I_{t_E'} = 2\mu^I - \mathrm{i}\Omega\, p^I\cos\theta\,, \qquad W^I_{\phi'} = -p^I\cos\theta\,,\\
A_1{}^1 &= A_2{}^2 = \sqrt{\frac{p^3}{3\ell^3}}\,.
\end{aligned} \tag{5.8}$$

The S^2 in (5.8) is fibered over the time circle of AdS_3, and we refer to this configuration as the *twisted torus* background. We also note that in the expression for $W^I_{t_E'}$ we have introduced an arbitrary constant μ^I which is allowed by the supersymmetry and equations of motion, and which we will interpret as the source of a $U(1)$ current in the boundary CFT. In fact, the BPS equations also allow W^I_ψ to take a constant value, but this constant is forced to be zero due to the contractibility of the ψ-cycle.

To see that the twisted torus background (5.8) has well-defined supersymmetry, we solve the Killing spinor equation from the variation of gravitino (4.4), which is rewritten now as

$$0 = 2D_M\varepsilon^i - \frac{\mathrm{i}}{4\ell}(3\gamma^{\hat\theta\hat\phi}\gamma_M - \gamma_M\gamma^{\hat\theta\hat\phi})\,\varepsilon^i\,, \tag{5.9}$$

where ε^i is the Killing spinors on this background. Here we use the following gamma matrices in the Euclidean theory, which follow from the Wick rotation,

$$\gamma_{\hat t_E} = \boldsymbol{\sigma}_3\otimes\boldsymbol{\tau}_3\,, \quad \gamma_{\hat\rho} = \boldsymbol{\sigma}_1\otimes\boldsymbol{\tau}_3\,, \quad \gamma_{\hat\psi} = \boldsymbol{\sigma}_2\otimes\boldsymbol{\tau}_3\,, \quad \gamma_{\hat\theta} = \mathbb{I}\otimes\boldsymbol{\tau}_1\,, \quad \gamma_{\hat\phi} = \mathbb{I}\otimes\boldsymbol{\tau}_2\,, \tag{5.10}$$

where $\boldsymbol{\sigma}_3$ is related to the Lorentzian gamma matrix $\boldsymbol{\sigma}_0$ in (C.4) by $\boldsymbol{\sigma}_3 \equiv -\mathrm{i}\boldsymbol{\sigma}_0$. We will take the representation $(\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = (-\boldsymbol{\tau}_3, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ with the Pauli sigma matrix $\boldsymbol{\tau}_a$. Note that unlike the case of global $AdS_3\times S^2$ in the Section 4.3, the Killing spinor equation (5.9) does not split into the equations of AdS_3 and S^2. This is because we have the following spin connections

$$\omega_{t_E'}^{\hat t_E\hat\rho} = -\sinh\rho\,, \quad \omega_{t_E'}^{\hat\theta\hat\phi} = \mathrm{i}\Omega\cos\theta\,, \quad \omega_{\psi}^{\hat\rho\hat\psi} = \cosh\rho\,, \quad \omega_{\phi'}^{\hat\theta\hat\phi} = \cos\theta\,, \tag{5.11}$$

where there is mixing between AdS_3 and S^2 directions through the non-zero twisting parameter Ω.

The solution of Killing spinors can be easily found by following the twisting construction. It is clear that the Euclidean continuation of the set of 8 Lorentzian Killing spinors (4.33), (4.35), followed by the coordinate transformation (5.6) obeys the new Killing spinor equation. Upon setting the parameter

$$\Omega \;=\; 1 + \mathrm{i}\frac{\tau_1}{\tau_2}\,, \tag{5.12}$$

the following[3] four of the original eight Killing spinors

$$\begin{aligned} \varepsilon^i_{(1)} &= (-\mathrm{i}\varepsilon_+{}^+\,, \varepsilon_-{}^-)\,, \qquad \varepsilon^i_{(2)} = (\varepsilon_+{}^+\,, -\mathrm{i}\varepsilon_-{}^-)\,, \\ \varepsilon^i_{(3)} &= -(\varepsilon_-{}^-\,, \mathrm{i}\varepsilon_+{}^+)\,, \qquad \varepsilon^i_{(4)} = -(\mathrm{i}\varepsilon_-{}^-\,, \varepsilon_+{}^+)\,, \end{aligned} \tag{5.13}$$

where

$$\begin{aligned} \varepsilon_+{}^+ \;&=\; \sqrt{\frac{\ell}{2}}\, \mathrm{e}^{\frac{1}{2}(1-\Omega)t'_E + \frac{\mathrm{i}}{2}(\psi+\phi')} \begin{pmatrix} \cosh\frac{\varrho}{2} \\ -\sinh\frac{\varrho}{2} \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}\,, \\ \varepsilon_-{}^- \;&=\; \sqrt{\frac{\ell}{2}}\, \mathrm{e}^{-\frac{1}{2}(1-\Omega)t'_E - \frac{\mathrm{i}}{2}(\psi+\phi')} \begin{pmatrix} -\sinh\frac{\varrho}{2} \\ \cosh\frac{\varrho}{2} \end{pmatrix} \otimes \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}\,, \end{aligned} \tag{5.14}$$

respect the periodicity (5.7) (they are periodic around the non-contractible circle and anti-periodic around the contractible circle).

One could also directly solve the Killing spinor equations (5.9) in the twisted coordinates $(\rho, \psi, \theta, \phi', t'_E)$. The only differences compared to solving them in the cylinder coordinates $(\rho, \psi, \theta, \phi, t_E)$ arise in the equation for the t'_E direction, which corresponds to:

$$\begin{aligned} 0 \;&=\; \big(2\partial_{t'_E} - \omega_{t'_E}^{\hat{t}_E\hat{\rho}}\gamma_{\hat{t}_E\hat{\rho}} - \omega_{t'_E}^{\hat{\theta}\hat{\phi}}\gamma_{\hat{\theta}\hat{\phi}}\big)\,\varepsilon_\pm{}^\pm \\ &\qquad -\frac{\mathrm{i}}{2\ell}E_{t'_E}{}^{\hat{t}_E}\gamma_{\hat{\theta}\hat{\phi}}\gamma_{\hat{t}_E}\,\varepsilon_\pm{}^\pm - \frac{\mathrm{i}}{\ell}E_{t'_E}{}^{\hat{\phi}}(\gamma_{\hat{\theta}\hat{\phi}}\gamma_{\hat{\phi}})\,\varepsilon_\pm{}^\pm\,. \end{aligned} \tag{5.15}$$

The difference with the equation for t_E in the cylinder frame is that in (5.15) above, $2\partial_{t'_E}$ acting on the Killing spinors (5.14) brings down $\pm(1-\Omega)$ instead of ± 1. Also, the third and the last terms are new. By the projection property along S^2 direction of the Killing spinor $(1\otimes \mathrm{e}^{-\mathrm{i}\tau_2\theta}\boldsymbol{\tau}_3)\varepsilon_\pm{}^\pm = \pm\varepsilon_\pm{}^\pm$, one can check that the effect of the

[3] The choice $\Omega = -1 - \mathrm{i}\frac{\tau_1}{\tau_2}$ also gives rise to a different set of four Killing spinors.

third and the last term indeed cancels the contribution of Ω from the time-derivative acting on the Killing spinor.

Supersymmetry algebra

The supercharges $\overline{\mathcal{Q}}_a = \delta(\varepsilon^i_{(a)})$, with the Killing spinors $\varepsilon^i_{(a)}$, $a = 1\,,2\,,3\,,4$ defined in (5.13), obey

$$\{\overline{\mathcal{Q}}_a\,,\overline{\mathcal{Q}}_b\} \;=\; -2\mathrm{i}\delta_{ab}(\overline{L}_0 - \overline{J}^3)\,, \qquad [\overline{L}_0 - \overline{J}^3\,,\overline{\mathcal{Q}}_a] \;=\; 0\,. \tag{5.16}$$

Consider the following four supercharges $\bar{G}^{i\alpha}_{\gamma}$,

$$\begin{aligned} \bar{G}^{++}_{+} &\equiv \frac{\mathrm{i}\overline{\mathcal{Q}}_1 + \overline{\mathcal{Q}}_2}{2}\,, \qquad \bar{G}^{--}_{-} \equiv \frac{\overline{\mathcal{Q}}_1 + \mathrm{i}\overline{\mathcal{Q}}_2}{2}\,, \\ \bar{G}^{+-}_{-} &\equiv \frac{-\overline{\mathcal{Q}}_3 + \mathrm{i}\overline{\mathcal{Q}}_4}{2}\,, \qquad \bar{G}^{-+}_{+} \equiv \frac{\mathrm{i}\overline{\mathcal{Q}}_3 - \overline{\mathcal{Q}}_4}{2}\,, \end{aligned} \tag{5.17}$$

where γ is the sign of the $\overline{L}_0$ eigenvalue, i is the doublet index under the outer automorphism coming from the SU(2) R-symmetry of the supergravity, and α is the doublet index under the SU(2) R-symmetry arising from the isometry of the S^2. They are charged under the bosonic generators of the right-moving generators $\overline{L}_0$ and $\overline{J}^3$ as

$$\left[\overline{L}_0\,,\bar{G}^{i\pm}_{\pm}\right] \;=\; \pm\tfrac{1}{2}\,\bar{G}^{i\pm}_{\pm}\,, \qquad \left[\overline{J}^3\,,\bar{G}^{i\pm}_{\pm}\right] \;=\; \pm\tfrac{1}{2}\,\bar{G}^{i\pm}_{\pm}\,, \tag{5.18}$$

so that

$$\left[\overline{L}_0 - \overline{J}^3\,,\bar{G}^{i\pm}_{\pm}\right] \;=\; 0\,, \tag{5.19}$$

and they obey the anticommutation relations

$$\left\{\bar{G}^{+\pm}_{\pm}\,,\bar{G}^{-\mp}_{\mp}\right\} \;=\; \pm\left(\overline{L}_0 - \overline{J}^3\right)\,, \qquad \left\{\bar{G}^{+\pm}_{\pm}\,,\bar{G}^{-\pm}_{\pm}\right\} \;=\; 0\,. \tag{5.20}$$

The above algebra (5.19), (5.20) forms a subalgebra of the global part of the $\mathcal{N} = 4$ superconformal algebra in the NS sector given in Section 4.3. Note that the subalgebra can also be thought of as the spectral flow[4], with parameter $\eta = 1$, to the following Ramond sector zero-modes as

$$\bar{L}_0 - \bar{J}^3 + c/24 \mapsto \bar{\mathcal{L}}^R_0\,, \qquad \bar{G}^{\pm\mp}_{\mp} \mapsto \mp\bar{\mathcal{G}}^{\mp}_{\mp,0}\,, \qquad \bar{G}^{\pm\pm}_{\pm} \mapsto \pm\bar{\mathcal{G}}^{\pm}_{\mp,0}\,. \tag{5.21}$$

[4]The spectral flow is taken on the charges with $\mathcal{N} = 4$ algebra as presented in Appendix B, which goes as as $\overline{\mathcal{L}}_n \mapsto \overline{\mathcal{L}}_n + \eta\overline{\mathcal{J}}^3_n + \eta^2\frac{c}{24}\delta_{n,0}$, $\overline{\mathcal{J}}^3_m \mapsto \overline{\mathcal{J}}^3_m + \eta\frac{c}{12}\delta_{n,0}$, $\bar{\mathcal{J}}^{\pm}_m \mapsto \bar{\mathcal{J}}^{\pm}_{m\pm\eta}$, $\bar{\mathcal{G}}^{\pm}_{\dot{A},r} \mapsto \bar{\mathcal{G}}^{\pm}_{\dot{A},r\pm\eta/2}$.

5.3 The trace interpretation and the semiclassical limit

In this section, we discuss the boundary dual of the gravitational functional integral Z^{PI} corresponding to the partition function on the twisted torus (5.7), (5.8). We then evaluate the supergravity action (with a set of boundary terms) on this torus and compare with the result for thermal AdS_3 in the untwisted theory in Section 3.2.

The trace interpretation of the functional integral

The bosonic generators corresponding to the translations (L_0, $\overline{L}_0$) around the torus and to the rotations of the sphere ($\overline{J}^3$) have the following representation in the twisted coordinates of (5.8):

$$L_0 \;=\; \mathrm{i}\frac{1}{2}(\mathrm{i}\partial_{t'_E} - \partial_\psi + \Omega\,\partial_{\phi'})\,, \qquad \overline{L}_0 \;=\; \mathrm{i}\frac{1}{2}(\mathrm{i}\partial_{t'_E} + \partial_\psi + \Omega\,\partial_{\phi'})\,, \qquad \overline{J}^3 \;=\; \mathrm{i}\partial_{\phi'}\,, \tag{5.22}$$

with $\Omega = 1 + \mathrm{i}\tau_1/\tau_2$. The Hamiltonian $H = -\partial_{t'_E}$ and angular momentum $P = -\mathrm{i}\partial_\psi$ on twisted torus are therefore

$$H \;=\; L_0 + \overline{L}_0 - \Omega\overline{J}^3\,, \qquad P \;=\; L_0 - \overline{L}_0\,. \tag{5.23}$$

Recall that the potentials β and ℓ that respectively couple to H and P are given in terms of the modular parameter $\tau = \tau_1 + \mathrm{i}\tau_2$ on the torus as $\beta = 2\pi\tau_2$ and $\ell = 2\mathrm{i}\pi\tau_1$. In addition, we have the chemical potentials μ^I coupling to U(1) current(s) q_I. Now consider the periodicities of the fermions. We have seen in Section 5.2 that they are anti-periodic around the contractible ψ circle and are periodic around the t'_E-circle. Respectively, these statements dictate that the partition function computes a Hamiltonian trace that is in the NS-sector and that has a $(-1)^F$ insertion (recall the concepts discussed in Section 1.1). Assembling these various statements, we therefore have:

$$\begin{aligned}
\mathrm{e}^{C(\tau,\mu)}\, Z^{PI}(\tau,\mu) \;&=\; \mathrm{Tr}_{\mathrm{NS}}\,(-1)^F\,\exp\bigl(2\pi\tau_2\,\partial_{t'_E} + 2\pi\tau_1\,\partial_\psi + \mu^I q_I\bigr)\\
&=\; \mathrm{Tr}_{\mathrm{NS}}\,(-1)^F\,\exp\bigl(-2\pi\tau_2\,(L_0 + \overline{L}_0 - \Omega\overline{J}^3) + 2\pi\mathrm{i}\tau_1\,(L_0 - \overline{L}_0) + \mu^I q_I\bigr)\,,\\
&=\; \mathrm{Tr}_{\mathrm{NS}}\,(-1)^F\,q^{L_0}\,\overline{q}^{\overline{L}_0 - \overline{J}^3}\,\mathrm{e}^{\mu^I q_I}\,,
\end{aligned} \tag{5.24}$$

with $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, $\overline{q} = \mathrm{e}^{-2\pi\mathrm{i}\overline{\tau}}$, and $\tau = \tau_1 + \mathrm{i}\tau_2$. We immediately recognize the right-hand side of (5.24) as the *elliptic genus in the NS sector*, given by the spectral flow of (1.7).[5] Notice the presence of the extra term $C(\tau,\mu)$ on the left-hand-side, which we write for the first time in this thesis. It corresponds to a Casimir-energy-type term which is needed to relate the functional integral form to the Hamiltonian trace form for generic partition functions in AdS$_3$/CFT$_2$ [65]. We will not discuss it further in this work.

From the anticommutator (5.20) we see that the above trace can additionally be written as

$$\mathrm{e}^{C(\tau,\mu)}\, Z^{PI}(\tau,\mu) \;=\; \mathrm{Tr}_{\mathrm{NS}}\,(-1)^F\, q^{L_0}\,\overline{q}^{\mathrm{i}\overline{\mathcal{Q}}^2}\,\mathrm{e}^{\mu^I q_I}\,, \tag{5.25}$$

where we have chosen a localization supercharge

$$\overline{\mathcal{Q}} \;\equiv\; \frac{1}{\sqrt{2}}\overline{\mathcal{Q}}_1 = \frac{1}{\sqrt{2}}(\bar{G}^{--}_{-} - \mathrm{i}\bar{G}^{++}_{+})\,. \tag{5.26}$$

The pairing of all non-BPS modes with respect to the supercharge $\overline{\mathcal{Q}}$ enforces that the elliptic genus is an anti-holomorphic function of τ.

On-shell action on the twisted torus background

Now that we have set up the twisted torus background, a natural step is to evaluate its semiclassical contribution to the functional integral, as we did for thermal AdS$_3$ in the 3d untwisted theory in Section 3.1. This follows the same steps: we evaluate the bulk action of the theory (in this case the 5d supergravity (4.13)) and a set of boundary terms corresponding to a Chern-Simons boundary term, a Gibbons-Hawking boundary term, and a gravitational counter term.

The bulk supergravity action (4.13) evaluated on the twisted torus (5.8) is

$$\begin{aligned} S_{\mathrm{bulk}}(\tau_2,p,\mu) \;&=\; \frac{1}{8\pi^2}\int_0^{\rho_0} d\rho \int_0^{2\pi} d\psi \int_0^{\pi} d\theta \int_0^{2\pi} d\phi' \int_0^{2\pi\tau_2} dt'_E\, L_{\mathrm{bulk}}\,, \\ &=\; -\frac{\pi\tau_2}{3}p^3 + \frac{\pi\tau_2}{6}p^3 e^{2\rho_0}\,, \end{aligned} \tag{5.27}$$

where we explicitly present the various integration ranges for clarity. The second term on the right-hand side denotes terms in the bulk action that diverge when the radial cutoff $\rho_0 \to \infty$, and is absorbed by standard boundary terms that we shortly present.

The boundary terms in the action of the gauge fields behave essentially in the same way as in the untwisted theory, but with slightly different details. In the coordinates

[5] The spectral flow acts on the right-moving generators. To match the left-moving generators of (1.7) to that of (5.24), one additionally requires a simple redefinition of L_0 by a constant shift as $L_0 \mapsto L_0 + c/24$.

of the cylinder frame (5.3), the gauge fields $W_{z,\bar{z}}$ on the AdS$_3$ factor have the boundary conditions (3.20), while the components $W_{\theta,\phi}$ on the S^2 are fixed at the boundary. Twisting these boundary conditions using (5.6) gives the boundary conditions for the gauge fields on the twisted torus:

$$\delta W^{I\,(0)}_{z'} \;=\; 0\,, \qquad W^{I\,(0)}_{\bar{z}'}\ \text{not fixed}, \qquad \delta W^{I\,(0)}_{\theta,\phi'} \;=\; 0\,, \tag{5.28}$$

where the (0) indicates the boundary values in the large-ρ expansion as in (3.19). The Chern-Simons boundary action consistent with these boundary conditions is:

$$S^{\text{bdry}}_{\text{CS}} = -c_{IJK}\frac{\mathrm{i}p^I}{48\pi^2}\int_{\partial\mathcal{M}} dz'd\bar{z}'d\theta d\phi'\,\sin\theta\Big[(W^J_{z'}-\frac{1}{2}\Omega W^J_{\phi'})W^K_{\bar{z}'}\Big]_{\text{bdry}}\,. \tag{5.29}$$

which on the twisted torus (5.8) evaluates to[6]

$$S^{\text{bdry}}_{CS} \;=\; -\frac{2\pi\tau_2}{3}c_{IJK}\,\mu^I\mu^J p^K\,. \tag{5.30}$$

The boundary terms in the gravitational sector also follow analogously from the three-dimensional theory. In particular, we recall the renormalization scheme with respect to the Brown-Henneaux conditions that was discussed in Section 2.3.2. This scheme dictates the addition of the Gibbons-Hawking boundary term (2.48) and a local counterterm on the boundary to cancel the divergences arising from the bulk action as in (5.27) as well as from the Gibbons-Hawking term. In the five-dimensional theory, these boundary terms are modified in the expected manner to include the volume form over the S^2, as well as a coupling to the dilaton fields of the off-shell 5d Lagrangian (4.10). We have [7]

$$S_{GH} \;=\; -\frac{1}{4\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\,\Phi\,K\,, \tag{5.31}$$

$$S_{CC} \;=\; \frac{1}{8\ell\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\,\Phi\,. \tag{5.32}$$

where

$$\Phi \equiv -\frac{C(\sigma)}{8}+\frac{3\chi}{16} \tag{5.33}$$

[6]Evaluating actions of this type is more conviently done by transforming back from the $(z',\,\bar{z}')$ to the $(\psi,\,t'_E)$ coordinates where the integration ranges are as in (5.27).

[7]In fact, to demonstrate that $S_{\text{bulk}}+S_{GH}+S_{CC}$ is well-defined under the variational principle of the 5d metric, the boundary conditions for the S^2 directions of the metric as well as for the dilaton Φ need to be specified. These considerations are discussed in Section 8.2.

is the dilaton that appears in the off-shell action action (4.13) as

$$S_{\text{bulk}} \;=\; \frac{1}{8\pi^2}\int d^5x\, E\Big(\Phi\, R + \cdots\Big)\,, \tag{5.34}$$

and $h = \det(h_{ij})$, $x^i = (\psi, \theta, \phi', t'_E)$, is the determinant of the induced metric h_{ij} of $\text{AdS}_3\times S^2$, i.e. which appears through $ds^2 = (2\ell)^2 d\rho^2 + h_{ij}\, dx^i dx^j$. The boundary terms (5.31) and (5.32) evaluate on the twisted torus to

$$S_{GH} \;=\; -\mathrm{e}^{2\rho_0}\frac{\pi\tau_2}{3}p^3 \qquad S_{CC} \;=\; \mathrm{e}^{2\rho_0}\frac{\pi\tau_2}{6}p^3\,. \tag{5.35}$$

In anticipation of Chapter 8, we note that the off-shell localization solutions computed in Chapter 7 will lead us to consider a different set of boundary conditions to Brown-Henneaux for the AdS_3 directions of the 5d metric. Correspondingly, we will propose a slightly different structure of gravitational boundary terms to (5.31), (5.32). However, these differences are only relevant when the metric goes off-shell, and do not change the on-shell background that we have discussed so far. Thus the value of the renormalized action on the twisted background (5.8) is

$$S_{\text{ren}} \,\equiv\, S_{\text{bulk}} + S^{\text{bdry}}_{CS} + S_{GH} + S_{CC} \;=\; -\pi k\tau_2 - \pi\tau_2\, k_{IJ}\,\mu^I\mu^J\,, \tag{5.36}$$

which matches the result (3.24) of the thermal AdS_3 computation. To express (5.36) in terms of k and k_{IJ}, note that we have used the relations (4.28), (4.30), which continue to be valid in the twisted theory. Indeed, the twisting procedure only affects global properties and does not change the Newton's constant. Therefore the central charge continues to be $c = 6k = 2p^3$ as in (4.28). Similarly, the level k_{IJ} of the boundary current algebra also does not change. To see this, note that the relation between the twisted and cylinder-frame fields is:

$$W_{z'} \;=\; W_z + \frac{1}{2}\Omega\, W^I_\phi(\theta)\,, \qquad W_{\bar z'} \;=\; W_{\bar z} - \frac{1}{2}\Omega\, W^I_\phi(\theta)\,, \qquad W_{\phi'} = W_\phi(\theta)\,, \tag{5.37}$$

where $W_{z,\bar z}$ are functions of the AdS_3 coordinates $(\rho, z, \bar z) = (\rho, z', \bar z')$ while $W^I_\phi = -p^I\cos\theta$. Substituting (5.37) into (5.29) gives:

$$S^{\text{bdry}}_{\text{CS}} \;=\; -c_{IJK}\frac{\mathrm{i}p^I}{12\pi}\int_{\partial\mathcal{M}} dz' d\bar z' \Big[W^J_z W^K_{\bar z}\Big]_{\text{bdry}}\,, \tag{5.38}$$

which is the same as the 3d Chern-Simons boundary term (3.23), since the integration ranges of $(z', \bar z')$ are the same as for $(z, \bar z)$. This shows that $k_{IJ} \;=\; \frac{2}{3}c_{IJK}\, p^K$ as in (4.30).

Chapter 6

The Euclidean 4d/5d lift

In this section we present a formalism to obtain off-shell localization solutions in 5d supergravity by lifting the localization manifold around Euclidean $\mathrm{AdS}_2 \times \mathrm{S}^2$. In particular, this allows us to obtain localization solutions around the supersymmetric twisted torus $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ background presented in (5.8).

We briefly recall the first step of the localization problem that the formalism addresses. We define the localization supercharge $\overline{\mathcal{Q}} = \frac{1}{\sqrt{2}}\overline{\mathcal{Q}}_1 = \frac{1}{\sqrt{2}}\delta(\varepsilon^i_{(1)})$ where the Killing spinor $\varepsilon^i_{(1)}$ is given in (5.13). (Equivalently, $\overline{\mathcal{Q}} = \frac{1}{\sqrt{2}}(\bar{G}^{--}_{-} - \mathrm{i}\bar{G}^{++}_{+})$ in terms of the super-Virasoro generators.) It acts only in the right-moving sector of the theory, where it obeys the algebra

$$\overline{\mathcal{Q}}^2 \;=\; -\mathrm{i}(\overline{L}_0 - \overline{J}^3)\,, \qquad [\overline{L}_0 - \overline{J}^3\,, \overline{\mathcal{Q}}] \;=\; 0\,. \tag{6.1}$$

We would like to study the space of solutions to the BPS equations given by setting the supersymmetry variations generated by $\overline{\mathcal{Q}}$ of all the fermions (4.4) to zero.

The BPS equations form a system of matrix-valued partial differential equations in terms of the bosonic fields of the theory. One systematic approach to solve them, assuming no fermionic backgrounds, begins by forming various Killing spinor bilinears [66, 67]. The BPS equations may then be expressed as a set of coupled first-order equations for these tensor fields, which describe the bosonic background of the solution. This approach was used in [39, 21] to solve the off-shell problem in the $\mathrm{AdS}_2 \times \mathrm{S}^2$ (and S^3) background. The general solutions to the resulting equations are, however, typically difficult to obtain, and we do not solve this problem of general classification in this paper. Instead, we leverage what is already known about the localization solutions in 4d supergravity around the Euclidean $\mathrm{AdS}_2 \times \mathrm{S}^2$ background [14, 39, 64], by lifting them to five dimensions. This involves the Kaluza-Klein (KK) lift of $\mathrm{AdS}_2 \times \mathrm{S}^2$

to AdS$_3\times$ S^2, which we describe in Section 6.1. Note, however, that while the 4d localization manifold has been determined completely, there may be additional solutions in 5d that *do* depend on the KK direction, and that will therefore not emerge from the lift. We postpone the discussion of such solutions to future work.

To lift the 4d localization solutions, we use the idea of the 4d/5d off-shell connection of [37]. However, as mentioned in the introduction, implementing this idea is not straightforward for the following reasons. Firstly, while the formalism in [37] was developed for Lorentzian supergravities, our 4d/5d connection needs to be adapted to accommodate the Euclidean supergravities in both four and five dimensions. A subtlety here, as we will shortly see, is that the 4d Euclidean theory has a redundancy in the choice of reality conditions and correspondingly a redundancy of AdS$_2\times$S^2 backgrounds, which has no counterpart in the 5d theory. Secondly, recall that the 4d/5d lift produces a five-dimensional background in the Kaluza-Klein ansatz and so, in order to reach the five-dimensional theory on the supersymmetric twisted torus $\mathbb{H}^3/\mathbb{Z}\times$S^2 from the four-dimensional theory on AdS$_2\times$S^2, we require a mapping of the twisted torus (5.8) into the Kaluza-Klein frame of AdS$_3\times$S^2. In Section 6.1 we present the mapping from the Kaluza-Klein frame to the cylinder frame. The twisted frame can then easily be mapped to the cylinder frame (5.3) by the local coordinate transformation (5.6). In Section 6.2 we review the 4d Euclidean supergravity and the AdS$_2\times$S^2 background. In Section 6.3, we present our construction of the Euclidean 4d/5d off-shell lift. Further, we show that the redundancy of the 4d theory mentioned above can be absorbed into the mapping parameter. We conclude the section by presenting the steps of lifting the 4d off-shell solutions to the 5d twisted torus.

6.1 The Kaluza-Klein coordinate frame

In this subsection we map the cylinder frame to the Kaluza-Klein frame. This mapping requires the local coordinate transformations as well as local Lorentz transformations. After presenting the general mechanism, we find the specific coordinate and Lorentz transformations, and the resulting background configuration and supercharges for AdS$_3\times$ S^2 in the Kaluza-Klein frame.

The general mechanism is as follows. Let $\{\dot{M},\dot{N},\cdots\}$ and $\{\dot{A},\dot{B},\cdots\}$ be the spacetime and tangent indices, respectively, in this Kaluza-Klein frame. The vielbein in the Euclidean cylinder frame $E_M{}^A$ maps to the vielbein in the KK frame $\dot{E}_{\dot{N}}{}^{\dot{A}}$ under a diffeomorphism together with some local rotation $L_A{}^{\dot{A}}$ which acts on the frame as [68]

$$E_M{}^A(x) \;=\; \frac{\partial \dot{x}^{\dot{N}}}{\partial x^M}\dot{E}_{\dot{N}}{}^{\dot{A}}(\dot{x})L^{-1}{}_{\dot{A}}{}^{A}\,. \tag{6.2}$$

Correspondingly, the spin connection transforms as

$$\omega_{MA}{}^{B} \;=\; \frac{\partial \dot{x}^{\dot{N}}}{\partial x^M}\left(L_A{}^{\dot{A}}\dot{\omega}_{\dot{N}\dot{A}}{}^{\dot{B}}L^{-1}{}_{\dot{B}}{}^{B} + (\partial_N L_A{}^{\dot{A}})\,L^{-1}{}_{\dot{A}}{}^{B}\right)\,. \tag{6.3}$$

Likewise, the remaining non-trivial background fields and the Killing spinors are mapped into the KK frame using the same diffeomorphism and local rotation $L_A{}^{\dot{A}}$, and a corresponding spinor rotation $\mathcal{L}$, as

$$T_{AB} \;=\; L_A{}^{\dot{A}}L_B{}^{\dot{B}}\dot{T}_{\dot{A}\dot{B}}\,, \qquad F_{AB} \;=\; L_A{}^{\dot{A}}L_B{}^{\dot{B}}\dot{F}_{\dot{A}\dot{B}}\,, \qquad \epsilon^i \;=\; \mathcal{L}\,\dot{\varepsilon}^j\,, \tag{6.4}$$

where $L_A{}^{\dot{A}}$ and $\mathcal{L}$ are related such that the gamma matrix is preserved:

$$L_A{}^{\dot{B}}\,\mathcal{L}\,\gamma_{\dot{B}}\,\mathcal{L}^{-1} \;=\; \gamma_A\,. \tag{6.5}$$

The diffeomorphism and local rotation in (6.2) should be chosen such that the vielbein in KK frame $\dot{E}_{\dot{N}}{}^{\dot{A}}$ has the following reduction ansatz. Decomposing the KK frame coordinate as $x^{\dot{M}} = \{x^\mu, x^5\}$ and $x^{\dot{A}} = \{x^a\,, 5\}$, the reduction ansatz of the vielbein is

$$\dot{E}_{\dot{M}}{}^{\dot{A}} \;=\; \begin{pmatrix} e_\mu{}^a & B_\mu\phi^{-1} \\ 0 & \phi^{-1} \end{pmatrix}, \qquad \dot{E}_{\dot{A}}{}^{\dot{M}} \;=\; \begin{pmatrix} e_a{}^\mu & -e_a{}^\mu B_\mu \\ 0 & \phi \end{pmatrix}, \tag{6.6}$$

where all the fields in the KK frame are independent of the compactified x^5 coordinate. Note that the KK ansatz (6.6) breaks the 5d diffeomorphisms to 4d diffeomorphisms and a $U(1)_{\text{gauge}}$, and breaks the 5d local rotation symmetry $O(5)$ to $O(4)\times\mathbb{Z}_2$. Using the $\mathbb{Z}_2$ we can fix the ϕ to have a fixed sign, say, positive. The vielbein (6.6) is equivalent to the following metric in the KK frame (with $x^5 \sim x^5 + 2\pi$), [1]

$$\dot{G}_{\dot{M}\dot{N}}\,dx^{\dot{M}}dx^{\dot{N}} \;=\; g_{\mu\nu}\,dx^\mu dx^\nu + \phi^{-2}(dx^5 + B_\mu\,dx^\mu)^2\,. \tag{6.7}$$

We see from the (6.6) and (6.7) that the five-dimensional vielbein $\dot{E}_{\dot{N}}{}^{\dot{A}}$ or metric $\dot{G}_{\dot{M}\dot{N}}$ are related to the four-dimensional veilbein $e_\mu{}^a$ or metric $g_{\mu\nu}$, a gauge field B_μ and a

[1] From this point onwards in the thesis, $g_{\mu\nu}$ denotes the *four-dimensional* metric tensor rather than the three-dimensional tensor in chapters 2 and 3.

scalar ϕ. The reduction ansatz leads to the following reduction of the spin connection as

$$\dot{\omega}_{\dot{A}}{}^{bc} = \begin{pmatrix} \omega_a{}^{bc} \\ \\ \frac{1}{2}\phi^{-1}F(B)^{bc} \end{pmatrix} \qquad \dot{\omega}_{\dot{A}}{}^{b5} = \begin{pmatrix} -\frac{1}{2}\phi^{-1}F(B)_a{}^b \\ \\ -\phi^{-1}D^b\phi \end{pmatrix}. \tag{6.8}$$

Here we see that the gauge field B_μ appears through its field strength $F(B)_{ab}$ in four dimensions.

Now we find the coordinate transformations and the local rotation in (6.2) that map the cylinder frame background in (5.3) to fit into the KK frame ansatz (6.6) and (6.7). The cylinder frame coordinates x^M and and the KK frame coordinates $\dot{x}^{\dot{M}}$

$$x^M = (\rho\,,\psi\,,\theta\,,\phi\,,t_E)\,, \quad \dot{x}^{\dot{M}} = (\eta\,,\chi\,,\theta\,,\phi\,,x^5)\,, \tag{6.9}$$

are related as

$$(\,\rho,\psi,t_E) = \left(\frac{\eta}{2},\,\chi+\mathrm{i}\frac{x^5}{2},\,\frac{x^5}{2}\right) \quad \Leftrightarrow \quad (\eta\,,\chi\,,x^5) = (2\rho\,,\psi-\mathrm{i}t_E\,,2t_E)\,, \tag{6.10}$$

with the coordinates (θ,ϕ) remaining the same. Note that the global conditions on the periodicities are not respected by this map (e.g. x^5 is compact whereas t_E is not). The corresponding local rotation matrix $L_A{}^{\dot{A}}$ is given as a rotation in the $2-5$ plane (along $\hat{\psi}$ and $\hat{t}_E$ direction) with angle $\omega = -\mathrm{i}\eta/2$:

$$L_1{}^{\dot{1}} = L_3{}^{\dot{3}} = L_4{}^{\dot{4}} = 1\,, \quad L_2{}^{\dot{2}} = L_5{}^{\dot{5}} = \cosh\frac{\eta}{2}\,, \quad L_2{}^{\dot{5}} = -L_5{}^{\dot{2}} = \mathrm{i}\sinh\frac{\eta}{2}\,. \tag{6.11}$$

In the exponential form, we have $L_A{}^{\dot{A}} = (\mathrm{e}^{\Omega})_A{}^{\dot{A}}$, where the $2-5$ component of the matrix in the exponent is $\Omega_{25} = -\Omega_{52} = -\omega = \mathrm{i}\eta/2$. [2] By the relation (6.5), the corresponding spinor rotation is

$$\mathcal{L} = \exp\left(\frac{1}{4}\Omega_{AB}\gamma^{AB}\right) = \exp\left(\frac{\mathrm{i}}{4}\eta\,\gamma_{25}\right) = \begin{pmatrix} \cosh\frac{\eta}{4} & -\sinh\frac{\eta}{4} \\ \\ -\sinh\frac{\eta}{4} & \cosh\frac{\eta}{4} \end{pmatrix} \otimes \mathbb{I}_2\,. \tag{6.12}$$

We note that although the spin connection in the cylinder frame has zero component for $\omega_{\hat{\rho}}{}^{\hat{\psi}\hat{t}_E}$, as can be seen in (C.2), the corresponding spin connection of KK frame

[2] The rotation with angle ω is $\exp\left(\omega\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{smallmatrix}\right)$. Here, we take $\omega = -\mathrm{i}\eta/2$ for the rotation in 2–5 plane. Note that the angle is imaginary, because the coordinate x^5 is Euclidean time.

mapped by (6.3), $\dot{\omega}_{\hat{\eta}}{}^{\hat{\psi}\hat{t}_E}(=\dot{\omega}_{\dot{1}}{}^{\dot{2}\dot{5}})$, is non-zero due to the contribution of the Lorentz transformation matrix in the second term of (6.3). According to (6.8), this non-zero component gives the non-zero value of the electric flux along AdS_2 . This explains why there is electric flux on AdS_2 even though the AdS_3 does not have any electric flux.

We now summarize the on-shell supersymmetric field configuration in the KK coordinates. By using the transformations (6.10) (6.11) (6.12) on the background Weyl multiplet as in (5.3) and matter multiplets as in (4.20) (4.22), we obtain the following configuration:

$$\begin{aligned}
ds_5{}^2 &= \ell^2\,(d\eta^2+\sinh^2\eta\,d\chi^2+d\theta^2+\sin^2\theta\,d\phi^2)+\ell^2\,(dx^5+\mathrm{i}(\cosh\eta-1)\,d\chi)^2\,,\\
\dot{T}_{34} &= -\frac{1}{4\ell}\,,\\
\dot{\sigma}^I &= -\frac{p^I}{\ell}\,, \qquad \dot{F}^I_{\theta\phi} = p^I\sin\theta\,, \qquad \dot{W}^I_{x^5} = \mu^I\,,\\
\dot{A}_1{}^1 &= \dot{A}_2{}^2 = \sqrt{\frac{p^3}{3\ell^3}}\,.
\end{aligned} \tag{6.13}$$

We note that the background geometry has an S^1 fibration over the four-dimensional base, which is Euclidean $AdS_2\times S^2$. The angular coordinate χ of Euclidean AdS_2 has periodicity 2π. By comparing the metric with the KK ansatz (6.7), we identify the following values for the KK one-form and scalar:

$$B = \mathrm{i}(\cosh\eta-1)\,d\chi \quad , \quad \phi = \ell^{-1}\,. \tag{6.14}$$

The background configuration given in (6.13) has well-defined supersymmetry. To see this, we look for the Killing spinors. By the Euclidean continuation of the Lorentzian Killing spinors (4.35) followed by the coordinate transformation (6.10) and the Lorentz transformation (6.12) we obtain

$$\begin{aligned}
\dot{\varepsilon}^i_{(1)} &= (-\mathrm{i}\dot{\varepsilon}_+{}^+\,,\dot{\varepsilon}_-{}^-)\,, & \dot{\varepsilon}^i_{(2)} &= (\dot{\varepsilon}_+{}^+\,,-\mathrm{i}\dot{\varepsilon}_-{}^-)\,,\\
\dot{\varepsilon}^i_{(3)} &= (-\dot{\varepsilon}_-{}^-\,,-\mathrm{i}\dot{\varepsilon}_+{}^+)\,, & \dot{\varepsilon}^i_{(4)} &= (-\mathrm{i}\dot{\varepsilon}_-{}^-\,,-\dot{\varepsilon}_+{}^+)\,,\\
\dot{\tilde{\varepsilon}}^i_{(1)} &= (\dot{\varepsilon}_+{}^-\,,\mathrm{i}\dot{\varepsilon}_-{}^+)\,, & \dot{\tilde{\varepsilon}}^i_{(2)} &= (\mathrm{i}\dot{\varepsilon}_+{}^-\,,\dot{\varepsilon}_-{}^+)\,,\\
\dot{\tilde{\varepsilon}}^i_{(3)} &= (-\mathrm{i}\dot{\varepsilon}_-{}^+\,,\dot{\varepsilon}_+{}^-)\,, & \dot{\tilde{\varepsilon}}^i_{(4)} &= (\dot{\varepsilon}_-{}^+\,,-\mathrm{i}\dot{\varepsilon}_+{}^-)\,.
\end{aligned} \tag{6.15}$$

where the spinors $\dot{\varepsilon}_{\pm}^{\,\pm}$ and $\dot{\varepsilon}_{\pm}^{\,\mp}$ are

$$\begin{aligned}
\dot{\varepsilon}_{+}^{\,+} &= \sqrt{\frac{\ell}{2}}\,\mathrm{e}^{\frac{\mathrm{i}}{2}(\chi+\phi)}\begin{pmatrix}\cosh\frac{\eta}{2}\\ -\sinh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix},\\
\dot{\varepsilon}_{+}^{\,-} &= \sqrt{\frac{\ell}{2}}\,\mathrm{e}^{\frac{\mathrm{i}}{2}(\chi-\phi)}\begin{pmatrix}\cosh\frac{\eta}{2}\\ -\sinh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix},\\
\dot{\varepsilon}_{-}^{\,+} &= \sqrt{\frac{\ell}{2}}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(\chi-\phi)}\begin{pmatrix}-\sinh\frac{\eta}{2}\\ \cosh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix},\\
\dot{\varepsilon}_{-}^{\,-} &= \sqrt{\frac{\ell}{2}}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(\chi+\phi)}\begin{pmatrix}-\sinh\frac{\eta}{2}\\ \cosh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}.
\end{aligned} \tag{6.16}$$

Note that they are well-defined with respect to the global structure of the geometry (6.7) because they do not depend on the x^5 direction (the spinors above are in fact precisely the four-dimensional Killing spinors on $\mathrm{AdS}_2\times\,\mathrm{S}^2$, as we spell out in Appendix E). Note also that, as in the twisted torus frame, we cannot impose any reality conditions on the Euclidean spinors. This is because although they formally satisfy $(\varepsilon^i)^\dagger\mathrm{i}\gamma_5 = \varepsilon_{ij}(\varepsilon^j)^T C$, which is formally the symplectic-Majorana condition of the Lorentzian theory (A.11), this condition is not compatible with the local Lorentz rotations of the Euclidean theory.

6.2 4d Euclidean supergravity and $\mathrm{AdS}_2\times\mathrm{S}^2$ background

The Kaluza-Klein formalism described in the previous subsection naturally connects the 5d supergravity on the $\mathrm{AdS}_3\times\mathrm{S}^2$ background in KK coordinates given in (6.13) to the 4d supergravity on an $\mathrm{AdS}_2\times\mathrm{S}^2$ background. In this subsection, we review the 4d Euclidean conformal supergravity and the $\mathrm{AdS}_2\times\mathrm{S}^2$ background in more detail. In the 4d Euclidean theory, there is a one-parameter redundancy for describing this background that comes from the possible choice of reality condition for the fermions.

4d $\mathcal{N} = 2$ supergravity

For the 4d $\mathcal{N} = 2$ Euclidean conformal supergravity, we consider the Weyl multiplet, coupled to $N_{\rm v} + 1$ vector multiplets and one hypermultiplet. One of the vector multiplets and the single hypermultiplet act as the compensators to consistently gauge-fix the dilatations of the off-shell theory (similarly to the five-dimensional theory). The fields of the Weyl multiplet are

$$\{e_\mu{}^a\,,\,\psi_a^i\,,\,A_\mu^D\,,\,A_\mu^R\,,\,\mathcal{V}_\mu{}^i{}_j\,,\,T_{ab}^\pm\,,\,\mathcal{D}\,,\,\chi_{4d}^i\}\,, \tag{6.17}$$

corresponding, respectively, to the vielbein, gravitino, dilatations gauge field, $\mathrm{SO}(1,1)_R$ gauge field, $\mathrm{SU}(2)_R$ gauge field [3], auxiliary self-dual/anti-self-dual two-form, auxiliary scalar, and the auxiliary fermion. As in the five-dimensional case, we fix $A_\mu^D = 0$ using the K-gauge. The fields of the $N_{\rm v} + 1$ vector multiplets are

$$\{X^{\mathcal{I}}\,,\,\overline{X}^{\mathcal{I}}\,,\,A_\mu^{\mathcal{I}}\,,\,\lambda^{\mathcal{I}\,i}\,,\,\mathcal{Y}^{\mathcal{I}\,ij}\}\,,\qquad \mathcal{I}\;=\;0\,,\,\cdots\,,\,N_{\rm v}\,, \tag{6.18}$$

corresponding to the complex scalar and its conjugate, the $U(1)$ gauge field, the gaugino, and the auxiliary $SU(2)$ triplet. Finally, the hypermultiplet consists of scalars and fermions,

$$\{\mathcal{A}_i{}^\alpha\,,\,\zeta_{4d}^\alpha\}\,. \tag{6.19}$$

The supersymmetry transformations on the spinor fields ψ_a^i, $\lambda^{\mathcal{I}\,i}$, ζ_{4d}^α are presented in (E.1), following the conventions of [45].

The 4d $\mathcal{N} = 2$ supergravity is governed by the prepotential $F(X)$ which is homogeneous of degree 2. Here, we choose the prepotential as [37]

$$F(X) \;=\; -\frac{1}{12}\,c_{IJK}\frac{X^I X^J X^K}{X^0}\,, \tag{6.20}$$

(the sum running over $I = 1, \ldots N_{\rm v}$), such that the vector multiplet sector of the 4d theory matches that of the 5d theory described in the section 4.1, according to the 4d/5d map that we will present shortly in Section 6.3.

Reality conditions

Note that in the Euclidean theory, the fields $X^{\mathcal{I}}$ and $\overline{X}^{\mathcal{I}}$—and, more generally, fields related by complex conjugation in the Lorentzian theory (e.g. T_{ab}^+ and T_{ab}^-)—are independent in the Euclidean theory. In order to preserve the number of degrees of

[3]The R-symmetry group of the Euclidean theory is $\mathrm{SU}(2) \times \mathrm{SO}(1,1)$ compared to $\mathrm{SU}(2) \times U(1)$ in the Lorentzian case.

freedom, we should impose reality conditions in the Euclidean theory. This may be done by imposing an appropriate reality condition on the spinors and using supersymmetry. Spinors in the four-dimensional Euclidean $\mathcal{N}=2$ theory can be chosen to obey the symplectic-Majorana condition. We note that there are actually an infinite number of such consistent conditions which, for any symplectic-Majorana spinor pair ψ^i, are parametrized by a real number α as

$$(\psi^i)^\dagger \mathrm{e}^{\mathrm{i}\alpha\gamma_5} = \epsilon_{ij}(\psi^i)^T C\,, \qquad \alpha \in \mathbb{R}\,. \tag{6.21}$$

This infinite choice stems from the fact that the chiral and anti-chiral spinors are independent in Euclidean 4d, and the symplectic-Majorana condition for the chiral and anti-chiral spinors can be imposed with relatively different phases. Two natural examples are:

$$\alpha \;=\; \pi/2: \quad (\psi^i)^\dagger \mathrm{i}\gamma_5 = \epsilon_{ij}(\psi^j)^T C\,, \qquad \alpha = 0: \quad (\psi^i)^\dagger \;=\; \epsilon_{ij}(\psi^j)^T C\,. \tag{6.22}$$

A spinor satisfying the general reality condition (6.21) (which we denote by $\psi^i(\alpha)$) is related to spinors satisfying (6.22)

$$\psi^i(\alpha) \;=\; \mathrm{e}^{\frac{\mathrm{i}}{2}\gamma_5(\alpha-\frac{\pi}{2})}\psi^i(\pi/2) \;=\; \mathrm{e}^{\frac{\mathrm{i}}{2}\alpha\gamma_5}\psi^i(0)\,. \tag{6.23}$$

Now, if we impose one such condition on all the spinors of the theory (including the Killing spinors), then the consistency of the supersymmetry transformations under this condition fixes specific reality conditions on the bosonic fields. For the two examples above we have, respectively, the following conditions for the relevant bosonic fields:

$$\begin{aligned}
\alpha \;=\; \pi/2: &\qquad \big(T^\pm_{ab}\big)^* \;=\; T^\pm_{ab}\,, \qquad \big(X^\mathcal{I}\big)^* \;=\; X^\mathcal{I}\,, \qquad \big(\overline{X}^\mathcal{I}\big)^* \;=\; \overline{X}^\mathcal{I}\,,\\
\alpha \;=\; 0: &\qquad \big(T^\pm_{ab}\big)^* \;=\; -T^\pm_{ab}\,, \quad \big(X^\mathcal{I}\big)^* \;=\; -X^\mathcal{I}\,, \quad \big(\overline{X}^\mathcal{I}\big)^* \;=\; -\overline{X}^\mathcal{I}\,.
\end{aligned} \tag{6.24}$$

However, note that imposing either reality condition in (6.24) does not necessarily make the kinetic terms of the action negative-definite [4], and therefore does not make the path integral perturbatively well-defined. In fact, this is the case for all bosonic

[4]Take the reality conditions $(X)^* = X$, $(\overline{X})^* = \overline{X}$, for instance. We can parameterize these real and independent X, $\overline{X}$ as $X = a+b$, $\overline{X} = a-b$, where a, b are real functions. The kinetic term for the scalars in the 4d Euclidean action is $\sim \partial_\mu X \partial^\mu \overline{X}$ which gives $(\partial a)^2 - (\partial b)^2$ on this parametrization. The sign of this term is clearly not definite.

reality conditions implied from supersymmetry by (6.21). As was discussed in Section 5.1, the resolution is to impose the standard reality condition on the bosonic fluctuations, e.g. $(\delta X^{\mathcal{I}})^* = \delta \overline{X}^{\mathcal{I}}$, so that path integral is well-defined, and to treat the fermion fluctuations ψ^1 and ψ^2 as being independent. For the background, however, the effect of the choice for α still remains: there is a one-parameter family of Killing spinors that satisfy the reality condition (6.21), and the supersymmetric bosonic background has a corresponding dependence on the choice of α as we will shortly see below.

4d AdS$_2\times$S^2 background

Here we present the Euclidean AdS$_2\times$ S^2 background, including the complete Weyl multiplet and matter multiplets. This solution can be obtained by Wick rotation of the Lorentzian AdS$_2\times$ S^2 solution, which carries both electric and magnetic charges $\big(q_{\mathcal{I}},\, p^{\mathcal{I}}\big)$. The non-trivial fields are:

$$
\begin{aligned}
ds_4{}^2 \;&=\; g_{\mu\nu}\, dx^\mu dx^\nu \;=\; \ell^2 \left(d\eta^2 + \sinh^2\eta\, d\chi^2 + d\theta^2 + \sin^2\theta d\phi^2\right) \,, \\
T^-_{12} \;&=\; -\mathrm{i}\,\omega\,, \qquad T^+_{12} \;=\; -\mathrm{i}\,\overline{\omega}\,, \qquad T^-_{34} \;=\; \mathrm{i}\,\omega\,, \qquad T^+_{34} \;=\; -\mathrm{i}\,\overline{\omega} \\
A^{\mathcal{I}} \;&=\; -\mathrm{i}\, e^{\mathcal{I}}(\cosh\eta - 1)\, d\chi - p^{\mathcal{I}}\cos\theta\, d\phi\,, \\
X^{\mathcal{I}} \;&=\; \frac{\omega}{8}(e^{\mathcal{I}} + \mathrm{i}p^{\mathcal{I}})\,, \qquad \overline{X}^{\mathcal{I}} \;=\; \frac{\overline{\omega}}{8}(e^{\mathcal{I}} - \mathrm{i}p^{\mathcal{I}})\,, \\
\mathcal{A}_i{}^\alpha \;&=\; a_i{}^\alpha \;=\; \text{constant}\,,
\end{aligned}
\tag{6.25}
$$

By the field equation for the auxiliary scalar $\mathcal{D}$, the $a_i{}^\alpha$ are constrained to obey:

$$
\Omega_{\alpha\beta}\, \varepsilon^{ij} a_i{}^\alpha a_j{}^\beta \;=\; -4\mathrm{i}(F_{\mathcal{I}}\overline{X}^{\mathcal{I}} - \overline{F}_{\mathcal{I}} X^{\mathcal{I}})\,. \tag{6.26}
$$

By the attractor equations [69], the electric field $e^{\mathcal{I}}$ is related to the electric charge $q_{\mathcal{I}}$ as

$$
4\mathrm{i}\left(\overline{\omega}^{-1}\frac{\partial \overline{F}(\overline{X})}{\partial \overline{X}^{\mathcal{I}}} - \omega^{-1}\frac{\partial F(X)}{\partial X^{\mathcal{I}}}\right) \;=\; q_{\mathcal{I}}\,, \tag{6.27}
$$

and the two independent complex parameters ω and $\bar{\omega}$ (unlike in the Lorentzian theory, they are not complex conjugate to each other) are related to the length scale of the metric ℓ as

$$
\ell^2 \;=\; \frac{16}{\omega\overline{\omega}}\,, \tag{6.28}
$$

which indeed scales consistently with Weyl weight (-2) and $SO(1,1)_R$ weight 0. Since the two complex parameters ω and $\bar{\omega}$ carry opposite charges under the $SO(1,1)_R$

gauge symmetry, we can set their magnitude to be same:

$$|\omega| \;=\; |\overline{\omega}| \;=\; 4/\ell\,. \tag{6.29}$$

Note that to match our 5d set-up, we uphold the dilatational symmetry, which is manifested here in the form of an arbitrary value for ℓ (one may break the symmetry by fixing ℓ to 1 for instance, as in [14]). The relation (6.28), (6.29) indicates that ω and $\bar{\omega}$ are now formally conjugate to each other so that we can rewrite them using the following parametrization:

$$\omega(\alpha) \;=\; \frac{4}{\ell}e^{\mathrm{i}\alpha}\,, \qquad \overline{\omega}(\alpha) \;=\; \frac{4}{\ell}e^{-\mathrm{i}\alpha}\,, \qquad \alpha\in\mathbb{R}\,. \tag{6.30}$$

Unlike in the 4d Lorentzian theory, where the phase α is fixed by the $U(1)_R$ gauge symmetry, in the Euclidean theory it remains as a free parameter. It is, in fact, precisely the parameter that determines the choice of reality condition for the spinors as in (6.21), i.e. the background described in (6.25) with generic α as in (6.30) preserves the supersymmetries generated by Killing spinors obeying the reality condition (6.21). For the case of $\alpha=\pi/2$ the 8 pairs of Killing spinors are presented in Appendix (E.22) and the Killing spinors for a generic α can be read off from (6.23). Note that the Killing spinors in (E.22) are exactly same Killing spinors as those of the 5d KK frame given in (6.15).

Now, by comparing the 4d background (6.25) to the 5d KK-frame background (6.13), it is clear that the AdS$_2\times$ S^2 metric in the former is the reduction of the AdS$_3\times$ S^2 metric in the latter, as mentioned in Section 6.1. However, it is not yet clear how the 4d/5d background values of the other fields are related (beyond just the metric), and how off-shell fluctuations are connected. In the next subsection, we will elucidate these points by describing the full off-shell map between the Euclidean 4d and 5d supergravity. Using this map, we will explicitly present how the 4d/5d backgrounds are mapped.

6.3 The off-shell Euclidean 4d/5d lift

In this subsection, we describe the off-shell connection between the 4d Euclidean and 5d Euclidean theory. We present how the AdS$_2\times$ S^2 on-shell background in (6.25) maps to the AdS$_3\times$ S^2 on-shell background in KK frame (6.13). This involves a choice of the relevant parameters of the 4d background, specifically (e^0, p^0) in (6.25), and depending on the choice of parameter ω and $\bar{\omega}$ (6.25), a proper mapping parameter is determined. We then show how to reach the 5d twisted torus background. We end the section with the steps to lift off-shell localization solutions to the 5d twisted torus.

To obtain the Euclidean 4d/5d connection, we use the Lorentzian 4d/5d relations of [37] and map the two theories to their consistent Euclidean counterparts. Getting the Euclidean 5d theory by the Wick rotation is straightforward, as explained in Section 4.1. We follow the conventions of the 4d Euclidean theory in [45]. Equivalently, one can start from the relations between the 5d Lorentzian and 4d Euclidean theories of [44], and Wick rotate the 5d theory. The map obtained in this approach differs from ours only in the way that the conventions of the 4d Euclidean theory of [44] differ from those of the 4d Euclidean theory of [45][5].

Under Kaluza-Klein reduction of the 5d conformal supergravity to 4d, the vector multiplets $I = 1, \ldots, N_{\mathrm{v}}$ reduce to the corresponding 4d matter vector multiplets $\mathcal{I} = 1, \ldots, N_{\mathrm{v}}$, and the Weyl multiplet reduced to the 4d Weyl multiplet and the additional Kaluza-Klein vector multiplet $\mathcal{I} = 0$.

One can expect that the Kaluza-Klein scalar ϕ associated with the 5d metric (6.6) falls into the scalar in the 4d Kaluza-Klein vector multiplet. However, directly performing this reduction only gives one real scalar degree-of-freedom, while there should be two real degree-of-freedom for the scalars of the vector multiplet. Additionally, the 4d $SO(1,1)_R$ symmetry factor is not realized in any of the multiplets. To recover the missing scalar d.o.f., an additional field φ is introduced [37, 44] to define the two 4d scalars in the KK vector multiplet as

$$X^0 = -\frac{\mathrm{i}}{2}\mathrm{e}^{-\varphi}\phi\,, \qquad \overline{X}^0 = \frac{\mathrm{i}}{2}\mathrm{e}^{\varphi}\phi\,. \tag{6.31}$$

The field φ transforms locally under SO(1,1)$_R$ as

$$\varphi \to \varphi + \Lambda^0\,, \tag{6.32}$$

where Λ^0 is real. One can then consistently couple φ to the remaining 4d fields, so that the $SO(1,1)_R$ of the 4d theory is realized.

We now present the explicit 4d/5d mappings, up to quadratic order in the fermions, keeping the general φ dependence. The 4d Weyl multiplet is related to the 5d Weyl

[5] The mapping between these conventions is also presented in [45].

multiplet as:

$$
\begin{aligned}
e_\mu{}^a &= \dot{E}_\mu{}^a\,, && (6.33)\\
\psi_a^i &= e^{-\frac{1}{2}\varphi\gamma_5}\dot{\Psi}_a{}^i\,, && (6.34)\\
A_a^R &= -6\mathrm{i}\dot{T}_{a5} + e_a{}^\mu\partial_\mu\varphi && (6.35)\\
\mathcal{V}_a{}^i{}_j &= \dot{V}_{a\,j}{}^i\,, && (6.36)\\
T_{ab}^{4d\pm} &= e^{\pm\varphi}(24\,\dot{T}_{ab}^{5d} + \mathrm{i}\phi^{-1}\varepsilon_{abcd}F(B)^{cd})^\pm\,, && (6.37)\\
\mathcal{D} &= 4\dot{D} + \frac{1}{4}\phi^{-1}e^{a\mu}D_\mu(e_a{}^\nu D_\nu\phi) + \frac{3}{32}\phi^{-2}F(B)^{ab}F(B)_{ab} && (6.38)\\
&\quad + \frac{3}{2}\dot{T}_{\dot{A}\dot{B}}\dot{T}^{\dot{A}\dot{B}} + \frac{1}{4}\phi^2\dot{V}_{x^5 i}{}^j\dot{V}_{x^5 j}{}^i\,, && (6.39)\\
\chi_{4d}^i &= 8\dot{\chi}^i + \frac{1}{48}\gamma^{ab}F(B)_{ab}\dot{\Psi}_{x^5}^i - \frac{3\mathrm{i}}{4}\phi\dot{T}_{ab}\gamma^5\gamma^{ab}\dot{\Psi}_{x^5}^i && (6.40)\\
&\quad + \frac{1}{4}\phi^{-1}\gamma_5\not{\mathcal{D}}(\phi^2\dot{\Psi}_{x^5}^i) - \frac{1}{2}\phi^2 V_{x^5}{}_j{}^i\dot{\Psi}_{x^5}^j - \frac{9}{4}\mathrm{i}\phi\dot{T}_{a5}\gamma^a\dot{\Psi}_{x^5}^i\,,
\end{aligned}
$$

where ε_{abcd} is the four-dimensional Levi-Civita symbol. The 4d supersymmetry parameters are given in terms of the 5d supersymmetry parameters and 5d Weyl multiplet fields as

$$
\begin{aligned}
\varepsilon_{4d}^i &= \mathrm{e}^{-\frac{1}{2}\varphi\gamma_5}\dot{\varepsilon}^i\,, && (6.41)\\
\eta_{4d}^i &= -\mathrm{i}\gamma_5\mathrm{e}^{\frac{1}{2}\varphi\gamma_5}\left(\dot{\eta}^i - 2\dot{T}_{a5}\gamma^a\gamma^5\dot{\varepsilon}^i + \frac{\mathrm{i}}{8}\phi^{-1}\gamma_5(F(B)_{ab} - 4\mathrm{i}\phi\dot{T}_{ab}\gamma_5)\gamma^{ab}\dot{\varepsilon}^i\right) && (6.42)
\end{aligned}
$$

Moving on to the vector multiplets, the 4d KK vector multiplet fields in terms of the 5d Weyl multiplet are:

$$
\begin{aligned}
X^0 &= -\frac{\mathrm{i}}{2}e^{-\varphi}\phi\,, \quad \overline{X}^0 = \frac{\mathrm{i}}{2}e^{\varphi}\phi\,, && (6.43)\\
A_a^0 &= e_a{}^\mu B_\mu\,, && (6.44)\\
\lambda^{0\,i} &= e^{-\frac{1}{2}\varphi\gamma_5}\dot{\Psi}_5{}^i\phi\,, && (6.45)\\
\mathcal{Y}^{0\,i}{}_j &= \phi\,\dot{V}_{5\,j}{}^i\,, && (6.46)
\end{aligned}
$$

and the 4d matter vector multiplet fields in terms of the 5d vector multiplet fields are:

$$X^I = \tfrac{1}{2}e^{-\varphi}(\sigma^I + \mathrm{i}\dot{W}_5^I)\,, \quad \overline{X}^I = \tfrac{1}{2}e^{\varphi}(\sigma^I - \mathrm{i}\dot{W}_5^I)\,, \tag{6.47}$$

$$A_a^I = \dot{W}_a^I\,, \tag{6.48}$$

$$\lambda^{I\,i} = e^{-\frac{1}{2}\varphi\gamma_5}\left(\dot{\Omega}^{I\,i} - \dot{W}_5^I\dot{\Psi}_5{}^i\right)\,, \tag{6.49}$$

$$\mathcal{Y}^{I\,i}{}_j = -2\left(Y^{I\,i}{}_j + \tfrac{1}{2}\dot{W}_5^I\dot{V}_{5\,j}{}^i\right)\,. \tag{6.50}$$

Finally, the 4d hypermultiplet in terms of the 5d hypermultiplet is

$$\mathcal{A}_i{}^\alpha = \phi^{-1/2}\dot{A}_i{}^\alpha\,. \tag{6.51}$$

Using the above maps, the 4d supersymmetry transformation is obtained from the 5d supersymmetry transformation together with a 5d local rotation,

$$\delta^{4d} = \delta^{5d} + \delta_M(\varepsilon)\,, \qquad \varepsilon_{5a} = -\varepsilon_{a5} = \bar{\dot{\varepsilon}}_i\gamma_a\Psi_5^i\,, \tag{6.52}$$

where the rotation parameter ε_{AB} is chosen to fix the gauge $\dot{E}_{x^5}{}^a = \dot{E}_5{}^\mu = 0$. We also need the supersymmetry transformation rule of φ,

$$\delta^{5d}\varphi = \bar{\dot{\varepsilon}}_i\dot{\Psi}_5^i\,. \tag{6.53}$$

For the purpose of lifting the 4d configuration to 5d, we use the inverse map, namely the 5d fields in terms of the 4d fields. The 5d Weyl multiplet fields are given in terms of the 4d Weyl multiplet and 4d KK multiplet as:

$$\dot{E}_\mu{}^a = e_\mu{}^a\,,\ \dot{E}_\mu{}^5 = \phi^{-1}B_\mu\,,\ \dot{E}_{x^5}{}^5 = \phi^{-1}\,, \tag{6.54}$$

$$\dot{\Psi}_a^i = e^{\frac{1}{2}\varphi\gamma_5}\psi_a^i\,,\ \dot{\Psi}_5{}^i = \phi^{-1}e^{\frac{1}{2}\varphi\gamma_5}\lambda^{0\,i}\,, \tag{6.55}$$

$$\dot{T}_{ab} = \frac{1}{24}\left(e^{-\varphi}T_{ab}^+ + e^{\varphi}T_{ab}^- - \mathrm{i}\phi^{-1}\varepsilon_{abcd}\,F(B)^{cd}\right)\,, \tag{6.56}$$

$$\dot{T}_{a5} = \frac{\mathrm{i}}{6}\left(A_a^R - e_a{}^\mu\partial_\mu\varphi\right)\,, \tag{6.57}$$

$$\dot{V}_{a\,j}{}^i = \mathcal{V}_a{}^i{}_j\,,\ \dot{V}_{5\,j}{}^i = \phi^{-1}\mathcal{Y}^{0\,i}{}_j\,, \tag{6.58}$$

$$\begin{aligned}\dot{D} = \frac{1}{4}\Big(&\mathcal{D} - \frac{1}{4}\phi^{-1}e^{a\mu}D_\mu(e_a{}^\nu D_\nu\phi) - \frac{3}{32}\phi^{-2}F(B)^{ab}F(B)_{ab}\\ &-\frac{3}{2}\dot{T}_{\dot{A}\dot{B}}\dot{T}^{\dot{A}\dot{B}} - \frac{1}{4}\phi^2\dot{V}_{x^5 i}{}^j\dot{V}_{x^5 j}{}^i\Big)\,,\end{aligned} \tag{6.59}$$

where

$$\phi = 2\mathrm{i}e^{\varphi}X^0 = -2\mathrm{i}e^{-\varphi}\overline{X}^0\,, \qquad B_\mu = A^0_\mu\,. \tag{6.60}$$

The 5d supersymmetry parameters are:

$$\begin{aligned} \dot{\varepsilon}^i &= \mathrm{e}^{\frac{1}{2}\varphi\gamma_5}\varepsilon^i_{4d}\,, & (6.61)\\ \dot{\eta}^i &= \gamma_5\left(\mathrm{i}e^{-\frac{1}{2}\varphi\gamma_5}\eta^i_{4d} + 2\dot{T}_{a5}\gamma^a\dot{\varepsilon}^i - \frac{\mathrm{i}}{8}\phi^{-1}(F(B)_{ab} - 4\mathrm{i}\phi\dot{T}_{ab}\gamma_5)\gamma^{ab}\dot{\varepsilon}^i\right)\,. & (6.62)\end{aligned}$$

The 5d vector multiplet is given in term of the 4d vector multiplet as:

$$\begin{aligned} \dot{\sigma}^I &= e^{\varphi}X^I + e^{-\varphi}\overline{X}^I\,, & (6.63)\\ \dot{W}^I_a &= A^I_a\,, \qquad \dot{W}^I_5 = -\mathrm{i}\left(e^{\varphi}X^I - e^{-\varphi}\overline{X}^I\right)\,, & (6.64)\\ \dot{\Omega}^{I\,i} &= e^{\frac{1}{2}\varphi\gamma_5}\lambda^{i\;I} + \dot{W}^I_5\dot{\Psi}_5{}^i\,, & (6.65)\\ \dot{Y}^{I\,i}{}_j &= -\tfrac{1}{2}\mathcal{Y}^{I\,i}{}_j - \tfrac{1}{2}\dot{W}^I_5\dot{V}_{5\,j}{}^i\,. & (6.66)\end{aligned}$$

The 5d hyper scalar given in terms of the 4d hypermultiplet is

$$\dot{A}_i{}^\alpha = \phi^{1/2}\mathcal{A}_i{}^\alpha\,. \tag{6.67}$$

Mapping 4d/5d classical backgrounds

By the above 4d/5d map, the relation between the 4d $\mathrm{AdS}_2\times\mathrm{S}^2$ backgrounds (6.25) and 5d $\mathrm{AdS}_3\times$ S^2 background in (6.13) in KK coordinates becomes more manifest. One important subtlety is about the choice of φ in (6.60). In the case of the Lorentzian 4d/5d connection, φ is just a $U(1)_R$ gauge parameter that fixes the gauge-redundant phase of X^0 and $\overline{X}^0$, making the ϕ automatically real. However, in the Euclidean case, the 4d theory has an $\mathrm{SO}(1,1)_R$ gauge symmetry instead of $\mathrm{U}(1)_R$, whereas the background values for X^0 and $\overline{X}^0$ have a relative phase coming from the choice of the parameter ω and $\bar{\omega}$ and value of the charge e^0 and p^0. Therefore, unlike in the Lorentzian case, the value of φ is not a 'gauge fixing' to kill the phase of X^0 and $\overline{X}^0$, but rather a 'choice' to cancel the phase of X^0 and $\overline{X}^0$. (By the $\mathrm{SO}(1,1)_R$ gauge redundancy and by the rule (6.32), we shift the φ to set the magnitude of X^0 and $\overline{X}^0$ to be same.)

Recalling the background value of X^0 and $\overline{X}^0$ as given in (6.25), where the ω and $\bar{\omega}$ are parametrized by α as in (6.30), the value of the mapping parameter φ is determined to be

$$\varphi^{\pm}(\alpha\,, e^0\,, p^0) = -\mathrm{i}\alpha \pm \mathrm{i}\frac{\pi}{2} - \mathrm{i}\arctan\Big(\frac{p^0}{e^0}\Big)\,, \tag{6.68}$$

by the condition that ϕ be real. There remains an ambiguity of $\pm\pi/2$ that is related to an overall sign choice for ϕ. We now consider specific examples for two distinct choices of $(e^0\,, p^0)$, keeping the choice of α to be generic. These are:

$$\begin{aligned} &(1) \quad (e^0,\, p^0) \;=\; (e^0,\, 0)\,, \qquad \varphi_1^{\pm}(\alpha) \;=\; -\mathrm{i}\alpha \pm \mathrm{i}\pi/2 \Rightarrow \phi \;=\; \mp\frac{e^0}{\ell}\,, \\ &(2) \quad (e^0,\, p^0) \;=\; (0,\, p^0)\,, \qquad \varphi_2^{\pm}(\alpha) \;=\; -\mathrm{i}(\alpha+\pi/2) \pm \mathrm{i}\pi/2 \Rightarrow \phi \;=\; \mp\frac{p^0}{\ell}\,. \end{aligned} \tag{6.69}$$

Here we see that, by the mapping parameter $\varphi^{\pm}$, the background value of the lifted 5d field ϕ is indeed real, but there is dependence on the choice $\pm$. We note that for both cases in (6.69) and, more generally, with any choice (6.68), all the lifted 5d fields are independent of the choice of phase $\omega \equiv \exp(\mathrm{i}\alpha)$ in the 4d background (6.25).

The resulting 5d background fields are listed in Table 6.1. The 4d configuration with $(e^0,\, p^0,\, \varphi) = (e^0,\, 0,\, \varphi_1^{\pm})$ as in (1) lifts to an $\mathrm{AdS}_3\times\mathrm{S}^2$ background, while the one with $(e^0,\, p^0,\, \varphi) = (0,\, p^0,\, \varphi_2^{\pm})$ as in (2) lifts to an $\mathrm{AdS}_2\times\mathrm{S}^3$ background. For the latter case, the localization solutions were studied in [21]. In both cases, the choice of the sign in $\varphi^{\pm}$ gives the opposite sign for the background values of ϕ, $\dot{T}_{\dot{A}\dot{B}}, \dot{\sigma}$ and hyper norm $\dot{\chi}$. At the level of the Killing spinor equation (that we review in Appendix C), choosing either sign gives a set of Killing spinors corresponding, respectively, to the right- or left-moving supercharges in terms of the 2d chiral $\mathcal{N}=4$ super algebra.

Now, for our problem, the full specification of parameters to lift the Euclidean $\mathrm{AdS}_2\times\mathrm{S}^2$ backgrounds (6.25) to the 5d KK frame (6.13) is

$$(e^0,\, p^0,\, \varphi) \;=\; (-1,\, 0,\, \varphi_1^{+})\,, \tag{6.70}$$

with identification $e^I = \mu^I$ and φ_1^{+} given in (6.69). To relate Euclidean $\mathrm{AdS}_2\times\mathrm{S}^2$ to the twisted torus (5.8), this lift is then followed by the following steps: taking the lifted 5d KK frame background (6.13) with Q- Killing spinors (6.15), one applies the local coordinate transformations (6.10), (5.6), the spinor Lorentz rotation in (6.4) with (6.12), and finally one imposes the periodicity conditions (5.7) with Ω given in (5.12). In this procedure, only four of the eight Q- Killing spinors mapped from (6.15) are well-defined on the twisted torus, as expected.

	$\varphi = \varphi_1^{\pm}$	$\varphi = \varphi_2^{\pm}$	
$(e^0,\, p^0) = (e^0,\, 0)$	$ds^2 = \mathrm{AdS}_3\times\mathrm{S}^2$ $\dot{T}_{34} = \mp 1/(4\ell)$ $\dot{\sigma}^I = \mp p^I/\ell\,,$ $\dot{F}^I_{34} = p^I/\ell^2$ $\dot{A}_{1,2}{}^{1,2} = \sqrt{\pm\frac{p^3}{3\ell^3}}$ $S_{\text{bulk}} = \frac{p^3}{12e^0}$	$ds^2 = \mathrm{AdS}_2\times\mathrm{S}^3$ $\dot{T}_{12} = \mp\mathrm{i}/(4\ell)$ $\dot{\sigma}^I = \pm e^I/\ell\,,$ $\dot{F}^I_{12} = -\mathrm{i}e^I/\ell^2$ $\dot{A}_{1,2}{}^{1,2} = \sqrt{\mp\frac{e^3}{3\ell^3}}$ $S_{\text{bulk}} = \frac{e^3}{12p^0}$	$(e^0,\, p^0) = (0,\, p^0)$

Table 6.1: The non-trivial 5d fields obtained by lifting the 4d backgrounds (6.25) with $(\omega,\,\overline{\omega})$ as given in (6.30) and with different choices for $(e^0,\, p^0)$ and φ. For the choice of $(e^0,\, p^0)$ on the left and right panel, the 4d hyper scalar that is lifted is determined by the $\mathcal{D}$-field equation constraint (6.26) as ${a_1}^1 = {a_2}^2 = 1/\ell\sqrt{-p^3/3e^0}$ and ${a_1}^1 = {a_2}^2 = 1/\ell\sqrt{e^3/3p^0}$ respectively. We also include the value for the finite piece of the bulk action (4.13). The field configurations on the right entry are solutions corresponding to the near-horizon of the supersymmetric Euclidean 5d black hole. The field configurations on the left entry are the Euclidean $\mathrm{AdS}_3\times\mathrm{S}^2$ solutions.

Mapping 4d localization solution to the 5d twisted torus frame

Having identified the relevant 4d background, together with the correct mapping parameter (6.70) that relates it to the 5d twisted torus background (5.8), we now want to map the *off-shell* localization solution of 4d supergravity on that background to the 5d localization solution around the twisted torus background. The strategy for this mapping follows the same steps as the mapping of the backgrounds presented above. Here, we assume that phase factors in the quantum fluctuation of the scalars X^0 and $\overline{X}^0$ are appropriately cancelled by a fluctuating value of φ around its value in (6.70), such that it makes the quantum fluctuation of the 5d field ϕ real.[6] It will turn out that for our off-shell localization solution, we can use the same value of φ as was chosen in (6.70).

Here, we summarize the steps as follows:

1. Start with the 4d localization manifold whose background is the Euclidean $\mathrm{AdS}_2\times\mathrm{S}^2$ solution (6.25) with $(e^0,\, p^0) = (-1, 0)$. Since the result does not depend on the choice of α in (6.30), without loss of generality we take $\alpha = \pi/2$ for convenience.

[6]Since we choose the reality condition for the fluctuation of X^0 and $\overline{X}^0$ to be complex conjugate to each other, as explained after (6.24), and since this condition is the same as the condition in the Lorentzian theory, it appears there may be some $\mathrm{U}(1)_R$ gauge symmetry hidden in the fluctuating field, and it may justify our assumption.

2. Apply the 4d/5d lift with the mapping parameter $\varphi = \varphi_1^+(\pi/2) = 0$ to obtain 5d localization solutions in the KK frame (6.13) of Euclidean $AdS_3 \times S^2$.

3. Transform these localization solutions to the twisted torus frame by applying the local coordinate maps (6.10), (5.6), the spinor Lorentz rotation in (6.4) with (6.12), and finally imposing the periodicity conditions (5.5) with Ω given in (5.12).

Note that a consistent lift to the twisted torus requires that the lifted solutions respect the periodicities (5.5). As an example of an inconsistent lift, consider a scalar field fluctuation on $AdS_2 \times S^2$ with non-zero momentum on χ, which therefore has 2π-periodicity in χ. Recalling that $\chi = \psi - it_E$, we see that such a mode, lifted to 5d, does not respect the second periodicity condition in (5.5). As we discuss in the next section, the fields in the four-dimensional localization manifold depend only the radial coordinate $\eta = 2\rho$ and therefore lift consistently to the 5d twisted torus.

Chapter 7

The lift of localization solutions on $\mathbf{AdS}_2\times\mathbf{S}^2$ to $\mathbb{H}^3/\mathbb{Z}\times\mathbf{S}^2$

In this chapter, we apply the lifting procedure constructed in Chapter 6 to obtain localization solutions around the supersymmetric $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$ background. We find a set of solutions to the BPS equations parametrized by $N_\mathrm{v}+1$ real coordinates $C^\mathcal{I}$, $\mathcal{I}=0,\ldots,N_\mathrm{v}$. These coordinates are inherited from the 4d localization manifold, where each $C^\mathcal{I}$ parametrizes the off-shell solution for the $\mathcal{I}^{th}$ vector multiplet. In the 4d $\mathrm{AdS}_2\times\mathrm{S}^2$ problem, the boundary conditions fix all the fields to their attractor values at infinity. The localization solution consists of the scalar fields $X^\mathcal{I}$ going off-shell in the interior, with a radially-decaying shape that is fixed by supersymmetry. The parameter $C^\mathcal{I}$ labels the size of deviation at the origin. In 5d, the C^I, $I=1,\cdots,N_\mathrm{v}$ parametrize the size of the off-shell solution in the vector multiplet, and C^0 parametrizes a certain excitation of the Weyl multiplet. Here, we have an $\mathrm{AdS}_3\times\mathrm{S}^2$ background, where one leg of the gauge field ($W_{z'}$) is fixed at infinity to its on-shell value while the other ($W_{\overline{z}'}$) is free to fluctuate, as we described in Section 5.3. The parameter C^I labels the deviation of both $W^I_{z'}$ and $W^I_{\overline{z}'}$ from their on-shell value at the origin as well as the boundary fluctuation of $W^I_{\overline{z}'}$. The precise solutions are presented in (7.6–7.9) for the Weyl multiplet, and in (7.12–7.15) for the vector multiplets. The hypermultiplet also fluctuates, and the solution is given in (7.16).

4d localization solutions

The most general solution in 4d around the $\mathrm{AdS}_2\times\mathrm{S}^2$ background is parametrized by one real parameter in each vector multiplet and one real parameter in the Weyl multiplet, before fixing the gauge for local scale transformations [39]. The gauge can be chosen so that there is no off-shell fluctuations in the Weyl multiplet [14]. The off-shell solution in the vector multiplets takes the following form:

$$X^{\mathcal{I}} = \frac{\mathrm{i}}{2\ell}\left(e^{\mathcal{I}} + \mathrm{i}\,p^{\mathcal{I}} + \frac{C^{\mathcal{I}}}{\cosh\eta}\right), \quad \overline{X}^{\mathcal{I}} = -\frac{\mathrm{i}}{2\ell}\left(e^{\mathcal{I}} - \mathrm{i}\,p^{\mathcal{I}} + \frac{C^{\mathcal{I}}}{\cosh\eta}\right) \tag{7.1}$$

$$A^{\mathcal{I}} = -\mathrm{i}\,e^{\mathcal{I}}(\cosh\eta - 1)\,d\chi - p^{\mathcal{I}}\cos\theta\,d\phi\,, \tag{7.2}$$

$$\mathcal{Y}^{\mathcal{I}\,1}{}_{1} = \mathcal{Y}^{\mathcal{I}}_{12} = \frac{-C^{\mathcal{I}}}{\ell^2\cosh^2\eta}\,, \tag{7.3}$$

where we use $(\omega(\pi/2), \overline{\omega}(\pi/2)) = (4\mathrm{i}/\ell,\, -4\mathrm{i}/\ell)$. The $C^{\mathcal{I}}$ are arbitrary constants and parametrize the off-shell fluctuations around the background (6.25).

Lift to the Weyl multiplet

For the lift to the Weyl multiplet, the relevant fields of the 4d localization solution (7.1) are those of the KK vector multiplet $\mathcal{I} = 0$. Using (6.60), we first obtain the off-shell values for the KK scalar and one-form:

$$\phi = \frac{1}{\ell}\left(1 - \frac{C^0}{\cosh\eta}\right), \qquad B_\chi = \mathrm{i}(\cosh\eta - 1)\,. \tag{7.4}$$

It is useful to define the function

$$\phi(x) := 1 - \frac{C^0}{\cosh x}\,. \tag{7.5}$$

Now, using the lifting equations (6.54 - 6.59) with $(e^0, p^0) = (-1, 0)$ and $\varphi = 0$, we obtain the full Weyl multiplet configuration in the KK frame. After applying the coordinate maps (6.10) and (5.6) to the twisted torus frame, the non-trival fields are:

$$E_{M'}{}^{A} = \ell\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sinh\rho\,(2-\frac{C^0}{\cosh 2\rho}-C^0)}{\phi(2\rho)} & 0 & 0 & \frac{2\mathrm{i}\frac{C^0}{\cosh 2\rho}\cosh\rho\sinh^2\rho}{\phi(2\rho)} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta & 0 \\ 0 & \frac{2\mathrm{i}\frac{C^0}{\cosh 2\rho}\cosh^2\rho\sinh\rho}{\phi(2\rho)} & 0 & \mathrm{i}\Omega\sin\theta & \frac{\cosh\rho\,(2-\frac{C^0}{\cosh 2\rho}+C^0)}{\phi(2\rho)} \end{pmatrix}, \tag{7.6}$$

$$T_{\theta\phi'} = \frac{\ell\sin\theta}{12}\left(\frac{1}{\phi(2\rho)}-4\right), \qquad T_{\theta t_E'} = \frac{\mathrm{i}\Omega\ell\sin\theta}{12}\left(\frac{1}{\phi(2\rho)}-4\right), \tag{7.7}$$

$$V_\psi = -2\mathrm{i}C^0\frac{\frac{\sinh^2\rho}{\cosh^2 2\rho}}{\phi(2\rho)^2}\boldsymbol{\tau}_3\,, \qquad V_{t_E'} = -\frac{2C^0\frac{\cosh^2\rho}{\cosh^2 2\rho}}{\phi(2\rho)^2}\boldsymbol{\tau}_3\,, \tag{7.8}$$

$$D = C^0\,\frac{\frac{\tanh^2 2\rho}{\cosh 2\rho}\left(3-\frac{2C^0}{\sinh 2\rho\tanh 2\rho}\right)}{24\ell^2\phi(2\rho)^2}\,. \tag{7.9}$$

Recall from Section 5.2 that $\Omega = 1+\mathrm{i}\tau_1/\tau_2$ in the twisted torus frame.

It remains to apply the lift to the Q- and S-Killing spinors. In principle, off-shell fluctuations in the bosonic fields of the Weyl multiplet may induce off-shell fluctuations in the 5d Killing spinors such that the BPS equations of the multiplet remain solved. Note however that the 4d Weyl multiplet in the 4d localization solution does not fluctuate, and so the 4d Q- and S- Killing spinors that we lift are just those of the 4d background, namely the eight spinors $\varepsilon^i_{4d}(\pi/2)$, given explicitly in (E.22), and $\eta^i_{4d}(\pi/2)=0$ (recall we have fixed $\alpha=\pi/2$). Further note that the lifting equation (6.61) for the 5d Q- spinors only involves the 4d Q- spinors (which are on-shell). We conclude that the lift of the Q- spinors is unchanged from the on-shell case, i.e. we obtain, in the twisted torus frame, the four well-defined on-shell Q -spinors $\varepsilon_{(a)}$, $a=1,2,3,4$, as given in (5.13). In contrast, the lifting equation (6.62) of the S- spinors η_{4d} involves bosonic 5d fields which do fluctuate. The 5d S- spinors, which are zero on-shell, therefore acquire a non-zero value off-shell. In the twisted frame, we obtain four well-defined S- spinors $\eta_{(a)}$, associated with the four Q-

spinors $\varepsilon_{(a)}$. The one associated to the localization supercharge $\varepsilon_{(1)}$ has value

$$\eta^1_{(1)} = -\frac{\frac{C^0}{\cosh(2\rho)}\,\mathrm{e}^{\frac{\mathrm{i}}{2}(\psi+\phi'+\mathrm{i}(\Omega-1)t'_E)}}{3\sqrt{2\ell}\phi(2\rho)}\begin{pmatrix}\cos\frac{\theta}{2}\cosh\frac{\varrho}{2}\\ -\sin\frac{\theta}{2}\cosh\frac{\varrho}{2}\\ -\cos\frac{\theta}{2}\sinh\frac{\varrho}{2}\\ \sin\frac{\theta}{2}\sinh\frac{\varrho}{2}\end{pmatrix}, \tag{7.10}$$

$$\eta^2_{(1)} = -\frac{\mathrm{i}\frac{C^0}{\cosh(2\rho)}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(\psi+\phi'+\mathrm{i}(\Omega-1)t'_E)}}{3\sqrt{2\ell}\,\phi(2\rho)}\begin{pmatrix}\sin\frac{\theta}{2}\sinh\frac{\varrho}{2}\\ \cos\frac{\theta}{2}\sinh\frac{\varrho}{2}\\ -\sin\frac{\theta}{2}\cosh\frac{\varrho}{2}\\ -\cos\frac{\theta}{2}\cosh\frac{\varrho}{2}\end{pmatrix}. \tag{7.11}$$

Lift of the vector multiplet

The relevant 4d fields are those of (7.1) with $\mathcal{I} = I$. Using the lifting equations (6.63 - 6.66) followed by the coordinate transformations (6.10) and (5.6), we obtain the following non-trivial fields of the vector multiplet configuration in the twisted torus frame:

$$\sigma^I = -\frac{p^I}{\ell}\,, \qquad W^I_{\phi'} = -p^I\cos\theta\,, \tag{7.12}$$

$$W^I_{\psi} = \frac{2\mathrm{i}\left(C^I/\mu^I + C^0\right)\frac{\sinh^2(\rho)}{\cosh 2\rho}}{\phi(2\rho)}\mu^I\,, \tag{7.13}$$

$$W^I_{t'_E} = -\mathrm{i}p^I\Omega\cos\theta + \frac{\frac{C^I/\mu^I-C^0}{\cosh 2\rho} + C^I/\mu^I + C^0 + 2}{\phi(2\rho)}\mu^I\,, \tag{7.14}$$

$$Y^I_{12} = \frac{1}{2\ell^2\phi(2\rho)}\frac{C^I/\mu^I + C^0}{\cosh^2 2\rho}\mu^I\,. \tag{7.15}$$

Lift of the hypermultiplet

Finally, the lift for the hypermultiplet (6.67) gives the following non-trivial components for the off-shell hyper scalar:

$$A_1{}^1 = A_2{}^2 = \left(\frac{\phi(2\rho)}{\ell}\right)^{1/2}\sqrt{\frac{p^3}{3\ell^3}}\,. \tag{7.16}$$

To summarize, the field configuration of the Weyl multiplet (7.6)–(7.9), the vector multiplet (7.12)–(7.15), and the hypermultiplet (7.16) are the 5d localization solutions. These configurations are off-shell fixed-points of the variations generated by the supercharge $\overline{\mathcal{Q}}$ given in (6.1), around the supersymmetric $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$ given in (5.8).

Chapter 8

Boundary conditions and action

As we have emphasized throughout this thesis, boundary conditions are a necessary and influential ingredient in the formulation of gravitational problems on non-compact spaces such as Anti-de-Sitter spacetimes. Already at the level of the classical theory, they are required to formulate meaningful notions of charge, asymptotic symmetries, and initial-value-problems. In the quantum theory, they become especially important since they determine which family of configurations will contribute to the functional integral. Boundary conditions also enter directly at the level of the action, where they are intrinsically linked with the construction of boundary terms. These terms are specifically chosen so as to achieve a well-defined variational principle of the theory with said boundary conditions (and this arises independently of other conditions one may wish to impose on the theory, e.g. supersymmetry or gauge invariance of the action). In this chapter, we begin to explore these boundary-condition and action principles in the context of the new off-shell BPS solutions around supersymmetric $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$, as presented in Chapter 7. In particular, we discuss how these solutions fit into the quantum functional integral formalism, and we initialize the construction of the renormalized action according to their obeyed boundary conditions.

In Section 8.1, what we soon find is that these new solutions in fact do *not* consistently fit into the quantum functional integral problem as we defined it in Chapters 1 and 3. More precisely, we find that while the off-shell gauge fields in these solutions remain consistent with the usual $\mathrm{AdS}_3\times\mathrm{S}^2$ boundary conditions (5.28), the metric fluctuations in the AdS_3 directions explicitly violate the standard Brown-Henneaux conditions. This is an uncomfortable fact, and a natural reaction would be to abandon these localization solutions entirely. However, in this chapter, we instead choose to persist with a further analysis on them. One reason for which this could constitute the right approach is the following: First, it is known at the semi-classical level that the quantum entropy function [13], which performs the macroscopic counting of the degeneracies of extremal black holes with near-horizon $AdS_2 \times S^2$ (or, more generally, times any compact manifold M), is intrinsically linked with the path integral on AdS_3 [70] [1]. Secondly, it is known that the 4d localization solutions around $AdS_2 \times S^2$, parameterized by $C^{\mathcal{I}}$, contribute to this quantum entropy function in the localization formalism [14, 15]. Therefore, since our $\mathbb{H}^3/\mathbb{Z} \times S^2$ localization solutions have been lifted from precisely these AdS_2 solutions, it is plausible that they should also contribute to the localization computation of the AdS_3 path integral.

Now, in deciding to continue forward with our new localization solution, the next question is whether there exists an alternative set consistent gravitational boundary conditions in which to embed these solutions. Supported by a rich literature on boundary conditions in AdS_3 (e.g. see [71, 72, 46]) we indeed find one such set: the Compère-Strominger-Song boundary conditions, which we reviewed in Section 2.3.3. In Section 8.2, this leads us to propose a boundary term structure for the 5d supergravity action, according to the renormalization scheme prescribed in [46]. While the resulting set of boundary terms is likely not complete, the value of this action on the localization solutions already displays certain interesting characteristics, which we discuss towards the end of the section.

In Section 8.3 we turn to the problem of exploring the existence of additional localization solutions, distinct from the class found in Chapter 7. Here, instead of using lifting principles, our approach is to perform a direct analysis of the supersymmetry equations in the large ρ regime. This strategy resembles the idea of holographic renormalization [53], where the field equations are solved recursively order-by-order for the coefficients of Fefferman-Graham expansions (see Section 2.3.1 for the example in pure 3d gravity). In our case, the same recursive approach is applied but to the *off-shell BPS equations* for our localization supercharge $\overline{\mathcal{Q}}$. Note that due to the inherent complexity of the BPS equations (we focus on the variation of the gravitino, which is especially complicated), we employ numerical strategies to extract and perform the recursive solving. The results of this procedure is evidence towards the existence of an *a priori* infinite class of new localization solutions, only a subclass of which is

[1] For instance, at the level of the $U(1)$ matter couplings, the AdS_2 and AdS_3 calculations are equivalent up to a difference in fixing the ensemble. In the AdS_2 case, the $U(1)$ *charges* are fixed at infinity, which places the calculation in the microcanonical ensemble. In the AdS_3 case, it is the *chemical potentials* for these charges (i.e. the μ^I) that are fixed instead, placing us in the canonical ensemble.

consistent with the CSS boundary conditions. The solutions which are not consistent with these boundary conditions are related to the left-moving Brown-Henneaux modes $\mathcal{L}_n$.

We close this chapter with a somewhat tangential but natural follow-up analysis on the topic of these BPS Brown-Henneaux modes in the left-sector, namely a study of their behaviour under the dimensional reduction to the theory on $\mathrm{AdS}_2\times\mathrm{S}^2$. Using the technology of the off-shell 4d/5d connection, we show that they reduce to non-normalizable modes in the four-dimensional theory.

Throughout this chapter, it should be noted that we work predominantly in the 5d cylinder-coordinate system $x^M = (\rho, z, \overline{z}, \theta, \phi)$, where we recall that the complex coordinates $(z, \overline{z})$ are related to (ψ, t_E) as:

$$z \;=\; \psi + \mathrm{i}t_E\,, \qquad \bar{z} \;=\; \psi - \mathrm{i}t_E\,. \tag{8.1}$$

We remind the reader that the $(\rho, \psi, \theta, \phi, t_E)$ coordinates are those related to the coordinates $(\rho, \psi, \theta, \phi', t'_E)$ of the $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$ background, as written in (5.8), by (5.6). We also take this chance to (re-)introduce additional index notation that will be relevant to the upcoming sections. We denote as $x^i = (z, \overline{z}, \theta, \phi)$ the transverse coordinates of the five-dimensional space, which are used on the induced metric h_{ij}. As we have seen in prior sections $x^\alpha = (z, \overline{z})$ are the boundary coordinates of the AdS_3 factor. Finally, $x^m = (\theta, \phi)$ are coordinates on the S^2 factor.

8.1 Boundary behaviour of the localization solutions

In this section, we analyze the asymptotic structure of the localization solutions lifted to $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$ in Chapter 7, focusing on the metric tensor and the gauge fields. In the interest of self-containement, we copy the relevant field expressions here, opting for the coordinates of the cylinder frame $x^M = (\rho, z, \overline{z}, \theta, \phi)$. The non-trivial components are as follows:

Metric tensor

$$
\begin{aligned}
G_{\rho\rho} &= (2\ell)^2\,, \qquad G_{\theta\theta} \;=\; \ell^2\,, \qquad G_{\phi\phi} \;=\; \ell^2\sin^2\theta\,,\\
G_{zz} &= -\frac{\ell^2}{\phi(2\rho)^2}\,, \qquad G_{z\overline{z}} \;=\; \frac{\ell^2\cosh(2\rho)}{\phi(2\rho)^2}\,,\\
G_{\overline{z}\overline{z}} &= -\ell^2\frac{1+(C^0)^2+\cosh(4\rho)-C^0\Big(\cosh(2\rho)+C^0\cosh(4\rho)-\cosh(6\rho)\Big)}{\phi(2\rho)^2\cosh^2(2\rho)}
\end{aligned}
\tag{8.2}
$$

U(1) gauge fields

$$
\begin{aligned}
W^I_\phi &= -p^I\cos\theta\,,\\
W^I_z &= -\frac{\mathrm{i}\mu^I\Big(1+\frac{C^I}{\mu^I\cosh(2\rho)}\Big)}{\phi(2\rho)}\,, \qquad W^I_{\overline{z}} \;=\; \frac{\mathrm{i}\mu^I\Big(1+\frac{C^I}{\mu^I}+C^0-\frac{C^0}{\cosh(2\rho)}\Big)}{\phi(2\rho)}\,.
\end{aligned}
\tag{8.3}
$$

Recall that we have defined

$$
\phi(x) \;=\; 1-\frac{C^0}{\cosh(x)}\,. \tag{8.4}
$$

Asymptotic form of off-shell gauge fields

Consider the off-shell U(1) gauge fields of the localization solution, as given in (8.3). At large ρ, the components obey the expansion

$$
W^I_i(\rho,x) \;=\; W^{(0)I}_i(x)+e^{-2\rho}W^{(2)I}_i(x)+\mathcal{O}(\mathrm{e}^{-4\rho})\,. \tag{8.5}
$$

In the AdS$_3$ directions $x^\alpha=(z,\overline{z})$, the first two expansion coefficients read

$$
\begin{aligned}
W^{(0)I}_z &= -\mathrm{i}\mu^I\,, \qquad W^{(0)I}_{\overline{z}} \;=\; \mathrm{i}\mu^I+\mathrm{i}\mu^IC^0+\mathrm{i}C^I\,,\\
W^{(2)I}_z &= -2\mathrm{i}\mu^I(C^0+C^I/\mu^I)\,, \qquad W^{(2)I}_{\overline{z}} \;=\; 2\mathrm{i}\mu^IC^0(C^0+C^I/\mu^I)\,.
\end{aligned}
\tag{8.6}
$$

We observe that W^I_z remains fixed to its classical value at the boundary while $W^I_{\overline{z}}$ has off-shell fluctuations. In the S^2 directions $x^m=(\theta,\phi)$, meanwhile, the gauge fields are also fixed to their on-shell values at the boundary (in fact, they do not fluctuate anywhere in the geometry):

$$
W^I_\theta \;=\; W^{(0)I}_\theta \;=\; 0\,, \qquad W^I_\phi \;=\; W^{(0)I}_\phi \;=\; -p^I\cos\theta\,. \tag{8.7}
$$

This behaviour for W^I_M is consistent with the standard $\mathrm{AdS}_3\times \mathrm{S}^2$ boundary condition for the gauge fields, which we recall from Section 5.3 are given as

$$\delta W^{(0)I}_{z} \;=\; 0\,, \qquad W^{(0)I}_{\bar z} \quad \text{not fixed}\,, \qquad \delta W^{(0)I}_{m} \;=\; 0\,. \tag{8.8}$$

Asymptotic form of off-shell metric

Consider now the BPS solution in the Weyl multiplet, labeled by C^0, and given in (8.2). The metric G_{MN} takes the form

$$\begin{aligned} G_{MN}\,dx^M dx^N \;&=\; (2\ell)^2 d\rho^2 + h_{ij}\,dx^i dx^j \\ &=\; (2\ell)^2 d\rho^2 + \gamma_{\alpha\beta}\,dx^\alpha dx^\beta + b_{mn}\,dx^m dx^n + 2c_{\alpha m}\,dx^\alpha dx^m \end{aligned} \tag{8.9}$$

where h_{ij} is the induced metric of the five-dimensional bulk metric, $\gamma_{\alpha\beta}$ is the metric over the AdS_3, b_{mn} is the metric over the S^2 and $c_{\alpha m}$ is the metric over the mixed $\mathrm{AdS}_3/\mathrm{S}^2$ directions. At large ρ, we have an expansion for the AdS_3 factor $\gamma_{\alpha\beta}$ as

$$\gamma_{\alpha\beta}(\rho,x) \;=\; e^{2\rho}\gamma^{(0)}_{\alpha\beta}(x) + \gamma^{(2)}_{\alpha\beta}(x) + \mathcal{O}(\mathrm{e}^{-2\rho})\,, \tag{8.10}$$

where the first few coefficients are

$$\begin{aligned} &\gamma^{(0)}_{zz} \;=\; 0\,, \qquad \gamma^{(0)}_{z\bar z} \;=\; \frac{\ell^2}{2}\,, \qquad \gamma^{(0)}_{\bar z\bar z} \;=\; -\ell^2 C^0\,, \\ &\gamma^{(2)}_{zz} \;=\; -\ell^2\,, \qquad \gamma^{(2)}_{z\bar z} \;=\; 2\ell^2 C^0\,, \qquad \gamma^{(2)}_{\bar z\bar z} \;=\; -\ell^2(1+3(C^0)^2)\,. \end{aligned} \tag{8.11}$$

The appearance of the off-shell mode C^0 in $\gamma^{(0)}_{\bar z\bar z}$ violates the Brown-Henneaux boundary conditions in AdS_3 which, in the complex coordinates (8.1), are given as

$$\gamma^{(0)}_{zz} \;=\; \gamma^{(0)}_{\bar z\bar z} \;=\; 0\,, \qquad \gamma^{(0)}_{z\bar z} \;=\; \frac{\ell^2}{2}\,. \tag{8.12}$$

Instead, the values for $\gamma^{(0)}_{\alpha\beta}$ and the subleading component $\gamma^{(2)}_{zz}$ in (8.11) together obey to the (Wick-rotated) Compere-Strominger-Song (CSS) boundary conditions of Einstein Gravity in three dimensions [46]. In Lorentzian signature, these boundary conditions were reviewed in Section 2.3.3. In the Wick-rotated setting, they are given as

$$\begin{aligned} &\gamma^{(0)}_{zz} \;=\; 0\,, \qquad \gamma^{(0)}_{\bar z\bar z} \;=\; \ell^2\partial_{\bar z}\bar P(\bar z)\,, \qquad \gamma^{(0)}_{z\bar z} \;=\; \frac{\ell^2}{2}\,, \\ &\gamma^{(2)}_{zz} \;=\; 4\ell G_3\Delta\,, \end{aligned} \tag{8.13}$$

where recall that $\bar{P}(\bar{z})$ is an arbitrary *fluctuating* function of $\bar{z}$, while the constant Δ is an input that, in the classical theory, is related to the charges of the BTZ black hole. In the quantum theory, all other Fefferman-Graham coefficients are allowed to fluctuate. Comparing (8.13) with (8.11), we identify

$$\partial_{\bar{z}}\bar{P}(\bar{z}) \;=\; -C^0\,, \qquad \Delta \;=\; -\frac{\ell}{4G_3}\,. \tag{8.14}$$

(Note that this means $\bar{P} \;=\; -C^0\bar{z}$ and so $\bar{P}$ is not periodic.)

Finally, we have metric components on the S^2 directions and mixed $\mathrm{AdS}_3/\mathrm{S}^2$ directions which do not fluctuate at all. It is nevertheless useful to introduce their asymptotic expansion as

$$\begin{aligned} b_{mn}(\rho,x) \;&=\; b^{(0)}_{mn}(x) + \mathrm{e}^{-2\rho} b^{(2)}_{mn}(x) + \cdots\,, \\ c_{\alpha m}(\rho,x) \;&=\; c^{(0)}_{\alpha m}(x) + \mathrm{e}^{-2\rho} c^{(2)}_{mn}(x) + \cdots\,. \end{aligned} \tag{8.15}$$

The boundary conditions for these metric components are that $b^{(0)}_{mn}\,, c^{(0)}_{\alpha m}$ are fixed to their on-shell values, and this is trivially true in our case. Recall that in the $(\rho, z, \overline{z}, \theta, \phi)$ coordinates, these on-shell values are:

$$\begin{aligned} b^{(0)}_{\theta\theta} \;&=\; \ell^2\,, \qquad b^{(0)}_{\phi\phi} \;=\; \ell^2\sin^2\theta\,, \\ c^{(0)}_{\alpha m} \;&=\; 0\,. \end{aligned} \tag{8.16}$$

For upcoming analyses concerning the metric, it will in fact also be required to specify boundary behaviour for the dilaton $\Phi = -C(\sigma)/8 + 3\chi/16$. The dilaton in our localization solutions obeys the expansion:

$$\Phi(x,\rho) \;=\; \Phi^{(0)}(x) + \mathrm{e}^{-2\rho}\Phi^{(2)}(x) + \cdots \tag{8.17}$$

with $\Phi^{(0)}$ fixed to the background value $\Phi^{(0)} = \frac{p^3}{12\ell^3}$. This corresponds to Dirichlet conditions.

8.2 Towards a renormalized action

In this section, we consider the boundary conditions that were identified for our localization solutions in Section 8.1 and use them to explore a corresponding boundary term structure for the 5d supergravity (4.13). We continue to focus on the metric sector and gauge-field sector. Since the metric in our localization solutions obeys the CSS boundary conditions, our approach to construct boundary terms for the gravitational sector is to follow the CSS renormalization scheme of the three-dimensional theory, as was reviewed in Section 2.3.3. Meanwhile, because the off-shell gauge fields obey the standard $\mathrm{AdS}_3\times\mathrm{S}^2$ boundary conditions, the renormalization in their sector is identical to the construction presented in Section 5.3. Before we begin, we emphasize that the proposal reached does not constitute a *fully* renormalized action for the boundary conditions. For instance, we have not computed boundary terms for the hypermultiplets, which we expect should contribute to the action on the localization solutions. We postpone a comprehensive analysis of these aspects to future work. We nevertheless note that our partial action already displays certain interesting characteristics. In particular, its value on the localization solution exhibits a tractable structure that is both finite and that can be compared to the analogous action for the 4d localization problem on $\mathrm{AdS}_2\times\mathrm{S}^2$ [14], as we briefly discuss towards the end of the section.

Boundary terms for the gauge fields

We begin with the gauge-field sector, where we have shown in (8.6) that the boundary conditions obeyed by the off-shell W^I_M are the usual $\mathrm{AdS}_3\times\mathrm{S}^2$ boundary conditions (8.8). Therefore, the boundary term to use here is unchanged from the five-dimensional Chern-Simons boundary term (5.29) presented in Section 5.3. For convenience, we copy it here:

$$S^{\text{bdry}}_{\text{CS}} = -c_{IJK}\frac{\mathrm{i}p^I}{48\pi^2}\int_{\partial\mathcal{M}} dz'd\bar{z}'d\theta d\phi'\,\sin\theta\,(W^J_{z'} - \frac{1}{2}\Omega W^J_{\phi'})W^K_{\bar{z}'}\,, \tag{8.18}$$

Note that for uniformity with Section 5.3, we continue to write this term in the coordinates of the twisted torus as in (5.8). Any other non-covariant boundary term presented in this section will be written in the cylinder coordinates $x^M = (\rho, z, \bar{z}, \theta, \phi, t_E)$, as mentioned earlier in the chapter.

Boundary terms for the metric

We now turn to the renormalization for the metric field. The regularization scheme in this sector should naturally follow that which was prescribed in the pure three-dimensional theory by Compère, Strominger and Song in [46], and which we also reviewed in Section 2.3.3. Recall that this scheme specifies the addition of a CSS-specific (chiral) boundary term (2.72) on top of the usual Gibbons-Hawking and counter term, given in (2.48) and (2.49) respectively. The five-dimensional equivalent of this CSS boundary term is

$$S^{\text{bdry}}_{CSS} = \frac{\ell}{8\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h^{(0)}}\left(\Phi^{(0)}\,\gamma^{(0)zz}\right), \tag{8.19}$$

where $h^{(0)} \equiv \det(\gamma^{(0)}b^{(0)}c^{(0)})$. The five-dimensional Gibbons-Hawking and counter term were already employed in the on-shell analysis of Section 5.3, and we copy them here for convenience:

$$S_{GH} = -\frac{1}{4\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\,\Phi\,K\,, \tag{8.20}$$

$$S_{CC} = \frac{1}{8\ell\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\,\Phi\,. \tag{8.21}$$

All together: a proposal for the renormalized action

Combining the bulk action (4.13) with the gauge field and metric boundary terms presented above, we propose a (partially) renormalized action as:

$$S_{\text{ren}} = S_{\text{bulk}} + S_{GH} + S_{CC} + S^{\text{bdry}}_{CSS} + S^{\text{bdry}}_{CS}\,. \tag{8.22}$$

We now turn to the variation of S_{ren} (8.22) with respect the gauge fields and the metric, constrasting to the on-shell construction of Section 5.3.

Variational principle of S_{ren} with respect to W^I

For the gauge fields, the boundary conditions and hence the boundary term S^{bdry}_{CS} are identical to those imposed in 5.3, and so δS_{ren} vanishes as it did there. As far as regularizing the bulk action with respect to their boundary conditions, we therefore expect no further boundary terms for W^I.

Variational principle of S_{ren} with respect to G_{MN}

For the metric, the situation is different to Section 5.3 for two reasons. Firstly, the boundary conditions in the AdS_3 directions are different: we are imposing the CSS conditions rather than Brown-Henneaux. We accordingly have the extra boundary term S^{bdry}_{CSS} (8.19) in analogy with the 3d term (2.72). Secondly, because the dilaton Φ is no longer constant in the off-shell configuration, there are additional surface terms in the variation of the bulk, which are of the form $\delta g(\nabla\Phi)$, that could potentially contribute non-trivially (whereas they vanished identically in the on-shell case where Φ=constant). These considerations justify an explicit analysis of the variational problem for the metric, to which we now turn.

The first-order variation of the bulk supergravity action (4.13) with respect to the metric tensor G_{MN} is

$$\begin{aligned}\delta S_{\rm bulk} \;=&\; \frac{1}{8\pi^2}\int_{\mathcal{M}} d^5x\sqrt{G}\Big[\Phi\,\mathcal{G}_{MN} + \Big(\nabla_M\nabla_N - G_{MN}\nabla^Q\nabla_Q\Big)\Phi - \mathcal{T}_{MN}\Big]\delta G^{MN}\\ &- \frac{1}{8\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\,n^Q\Big[\Phi\Big(\nabla^M\delta G_{QM} - G^{MN}\nabla_Q\delta G_{MN}\Big)\\ &\qquad\qquad + G^{MN}\delta G_{MN}\nabla_Q\Phi - \delta G_{QM}\nabla^M\Phi\Big]\,,\end{aligned} \tag{8.23}$$

where $\mathcal{G}_{MN}$ is the Einstein tensor

$$\mathcal{G}_{MN} \;=\; R_{MN} - \frac{G_{MN}}{2}R\,, \tag{8.24}$$

and where we packaged the remaining matter-couplings in a stress-tensor $\mathcal{T}_{MN}$:

$$\begin{aligned}\mathcal{T}_{MN} \equiv&\, \frac{G_{MN}}{2}\Big(\mathcal{L}_V + \mathcal{L}_H + C(\sigma)(4D + \frac{39}{2}T^2) + \chi(2D + \frac{3}{4}T^2)\Big)\\ &- \frac{c_{IJK}}{2}\sigma^I\Big(\frac{1}{2}D_M\sigma^J\,D_N\sigma^K + \frac{1}{2}F^{J\;Q}_{M}F^K_{NQ} - 6\sigma^J F_{(M}{}^Q T_{N)Q}\Big)\\ &- T_M{}^Q T_{NQ}\Big(39C(\sigma) + \frac{3}{2}\chi\Big) + \frac{1}{2}\Omega_{\alpha\beta}\varepsilon^{ij}D_M A_i{}^\alpha\,D_N A_j{}^\beta\,.\end{aligned} \tag{8.25}$$

(The expressions for $\mathcal{L}_V$ and $\mathcal{L}_H$ are given in (4.10).) The variation (8.23) has a bulk and a boundary piece. The bulk piece corresponds to the Einstein's equations in the off-shell theory when set to zero. The boundary piece is the relevant starting point to analyze the variational principle. We label it as:

$$\begin{aligned}\delta S_{\rm bulk}|_{\rm bdry} \equiv&\, -\frac{1}{8\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\,n^Q\Big[\Phi\Big(\nabla^M\delta G_{QM} - G^{MN}\nabla_Q\delta G_{MN}\Big)\\ &\qquad + G^{MN}\delta G_{MN}\nabla_Q\Phi - \delta G_{QM}\nabla^M\Phi\Big]\,,\end{aligned} \tag{8.26}$$

The first line of (8.26) can be written as

$$\begin{aligned}-\frac{1}{8\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\Big[\Phi\Big(Kn^Mn^N\delta G_{MN} - K^{MN}\delta G_{MN} - G^{MN}n^Q\nabla_Q\delta G_{MN}\Big)\\ - n^Q\delta G_{MQ}(\nabla^M\Phi)\Big]\,,\end{aligned} \tag{8.27}$$

(where a total derivative term $\nabla^M(\Phi\, n^N \delta_{MN})$ was discarded), so that

$$\begin{aligned}\delta S_{\text{bulk}}|_{\text{bdry}} &= -\frac{1}{8\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\Big[\Phi\Big(K n^M n^N \delta G_{MN} - K^{MN}\delta G_{MN} - G^{MN} n^Q \nabla_Q \delta G_{MN}\Big) \\ &\qquad + n^Q\Big(G^{MN}\delta G_{MN}(\nabla_Q \Phi) - 2\delta G_{MQ}(\nabla^M \Phi)\Big)\Big]\,. \end{aligned} \tag{8.28}$$

Now, the variation of the 5d Gibbons-Hawking (5.31) and 5d CC (5.32) terms are, respectively:

$$\begin{aligned}\delta S_{GH} &= -\frac{1}{8\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\Big[\Phi\Big(K G^{MN}\delta G_{MN} - K n^M n^N \delta G_{MN} + g^{MN} n^Q \nabla_Q \delta G_{MN}\Big) \\ &\qquad + 2n^Q \delta G_{MQ}(\nabla^M \Phi)\Big] \\ \delta S_{CC} &= \frac{1}{16\pi^2 \ell}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\,\Phi G^{MN}\delta G_{MN}\,, \end{aligned} \tag{8.29}$$

where a total derivative term $\nabla^M(\Phi\, n^N \delta_{MN})$ was again discarded in δS_{GH}. We therefore have that

$$\begin{aligned}\delta\Big(S_{\text{bulk}}|_{\text{bdry}} + S_{GH} + S_{CC}\Big) &= \frac{1}{8\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\Big[\Phi\Big(K^{MN} - K G^{MN} + \frac{1}{2\ell}G^{MN}\Big)\delta G_{MN} \\ &\qquad - n^Q G^{MN}\delta G_{MN}(\nabla_Q \Phi)\Big]\,. \end{aligned} \tag{8.30}$$

The first line is the dilaton-coupled Brown-York term with an additional contribution from the S_{CC}. The second line is an effect of the dilaton. We can now develop the (M, N) contractions over the indices (α, β), (m, n) and (α, m) and start imposing boundary conditions. As an intermediate step, we only substitute the boundary conditions for the metric components $c_{\alpha m}$ and b_{mn}, as given in (8.15), (8.16), as well as those for the dilaton Φ as in (8.17). This gives for (8.30):[2]

$$\begin{aligned}\delta\Big(S_{\rm bulk}|_{\rm bdry}+S_{GH}+S_{CC}\Big) \;=\;& \frac{1}{8\pi^2\ell}\int_{\partial\mathcal{M}} d^4x\sqrt{h^{(0)}}\,\Phi^{(0)}\Big(\gamma^{(0)\alpha\beta}{\rm Tr}[\gamma^{(2)}]-\gamma^{(2)\alpha\beta}\Big)\delta\gamma^{(0)}_{\alpha\beta}\\ &-\frac{1}{8\pi^2\ell}\int_{\partial\mathcal{M}} d^4x\sqrt{h^{(0)}}\,\Phi^{(0)}b^{(0)mn}\Big(b^{(2)}_{mn}\gamma^{(0)\alpha\beta}\delta\gamma^{(0)}_{\alpha\beta}-\delta b^{(2)}_{mn}\Big)\\ &-\frac{1}{8\pi^2}\int_{\partial\mathcal{M}} d^4x\sqrt{h^{(0)}}{\rm e}^{-2\rho}n^\rho\Big(\gamma^{(0)\alpha\beta}\delta\gamma^{(2)}_{\alpha\beta}+b^{(0)mn}\delta b^{(2)}_{mn}\Big)\partial_\rho\Phi^{(2)}\,.\end{aligned} \tag{8.31}$$

The last line above is the one corresponding to the last line in (8.30), i.e. the dynamical dilaton term. It is suppressed by $\mathrm{e}^{-2\rho}$ and therefore drops out. The second line in (8.30) has an interpretation coming from viewing the S^2 metric components b_{mn} as scalars in the effective AdS_3 theory. In this picture, one should add scalar-type boundary terms for these b_{mn}, which would be of the form $\int d^2x\sqrt{\gamma}\,n^\rho b^{mn}\partial_\rho b_{mn}$. In principle we should then correspondingly include these terms in the 5d theory, and one expects that their variation would cancel the second line of (8.30). It is however not necessary to do so for our practical purposes, which is ultimately to evaluate the renormalized action on the localization solution. Indeed, since b_{mn} in these solutions is independent of ρ, the aforementioned boundary terms would not contribute. In what follows we therefore suppress the second line in (8.30). It then only remains the first line, which is immediately recognized as the holographic stress-tensor of the three-dimensional theory, as given in (2.50), coupled to the dilaton. The treatment of this expression under our two relevant choices of AdS_3 metric boundary conditions (Brown-Henneaux and CSS) follows in an entirely analogous way to the three-dimensional considerations of Sections 2.3.2 and 2.3.3: under Brown-Henneaux boundary conditions ($\delta\gamma^{(0)}_{\alpha\beta}=0$), we trivially have

$$\delta\Big(S_{\rm bulk}|_{\rm bdry}+S_{GH}+S_{CC}\Big)|_{\rm BH} \;=\; 0\,, \tag{8.32}$$

while for CSS boundary conditions (8.13), (8.14),

$$\delta\Big(S_{\rm bulk}|_{\rm bdry}+S_{GH}+S_{CC}\Big)|_{\rm CSS} \;=\; \frac{1}{2\pi^2\ell^3}\int_{\partial\mathcal{M}} d^4x\sqrt{h^{(0)}}\Phi^{(0)}\delta\gamma^{(0)}_{\bar z\bar z}\,. \tag{8.33}$$

For this latter case, one then readily checks that the variation of $S^{\rm bdry}_{CSS}$ (8.19) can-

[2] Recall that we are also fixing the asymptotic metric to be in Fefferman-Graham gauge, for which the unit normal vector n^Q is only non-trivial in the radial direction n^ρ.

cels (8.33). Indeed:

$$\delta S^{\rm brdy}_{CSS} \;=\; -\frac{1}{2\pi^2\ell^3}\int_{\partial\mathcal{M}} d^4x\sqrt{h^{(0)}}\Phi^{(0)}\delta\gamma^{(0)}_{\bar z\bar z}\,. \tag{8.34}$$

We conclude that $S_{\rm ren}$, as given in (8.22), is well-defined with respect to the boundary conditions of the metric field consistent with the localization solutions.

Evaluation of $S_{\rm ren}$

We now turn to the more concrete exercise of evaluating $S_{\rm ren}$ (8.22) on the localization solution. The individual pieces of the renormalized action give the following values (where the radial integral is performed up to the cut-off $\rho_0 \gg 1$):

$$\begin{aligned} S_{\rm bulk} \;&=\; -\frac{\pi\tau_2\mu^0}{3\phi^0}(p^3 - 2c_{IJK}\phi^I\phi^J p^K) + \frac{\pi\tau_2 C^0}{3\mu^0}p^3 \\ &\qquad -\frac{2\pi\tau_2}{3}c_{IJK}\mu^I\phi^J p^K + \frac{\pi\tau_2}{6}p^3 e^{2\rho_0}\,, \end{aligned}\tag{8.35}$$

$$S_{GH} \;=\; \frac{\pi\tau_2 C^0}{2}p^3 - \mathrm{e}^{2\rho_0}\frac{\pi\tau_2}{3}p^3 \tag{8.36}$$

$$S_{CC} \;=\; \frac{\pi\tau_2 C^0}{12}p^3 + \mathrm{e}^{2\rho_0}\frac{\pi\tau_2}{6}p^3\,, \tag{8.37}$$

$$S^{\rm bdry}_{CSS} \;=\; \frac{2\pi\tau_2 C^0}{3}p^3\,, \tag{8.38}$$

$$S^{\rm bdry}_{CS} \;=\; -\frac{2\pi\tau_2}{3}c_{IJK}\,\mu^I\phi^J p^K - \frac{\pi\tau_2 C^0}{3}c_{IJK}\,\mu^I\mu^J p^K\,, \tag{8.39}$$

where we have redefined the localization modes as:

$$\phi^{\mathcal{I}} \;:=\; C^{\mathcal{I}} + \mu^{\mathcal{I}}\,, \tag{8.40}$$

and where, recall, $\mathcal{I} = (0, I)$ and $\mu^0 = -1$. All together, the value for $S_{\rm ren}$ is therefore:

$$\begin{aligned} S_{\rm ren} =&\frac{\pi\tau_2}{3\phi^0}(p^3 - 2\,c_{IJK}\phi^I\phi^J p^K) - \frac{4\pi\tau_2}{3}\,c_{IJK}\,\mu^I\phi^J p^K \\ &+ \frac{11\pi\tau_2\phi^0}{12}p^3 + \frac{11\pi\tau_2}{12}p^3 \\ &- \frac{2\pi\tau_2\phi^0}{3}c_{IJK}\,\mu^I\mu^J p^K - \frac{2\pi\tau_2}{3}c_{IJK}\,\mu^I\mu^J p^K\,. \end{aligned}\tag{8.41}$$

Note that the ϕ^I terms form a perfect square:

$$\begin{aligned} S_{\text{ren}} =& \pi\tau_2 p^3 \left(\frac{1}{3\phi^0} + \frac{11}{12}\phi^0 + \frac{11}{12}\right) \\ &- \frac{2\pi\tau_2}{3} c_{IJK}\, \mu^I \mu^J p^K - \frac{2\pi\tau_2}{3\phi^0} c_{IJK} (\phi^I + \phi^0\mu^I)(\phi^J + \phi^0\mu^J)\, p^K \,. \end{aligned} \tag{8.42}$$

When (8.42) is exponentiated and inserted into the path integral over the localization modes, the integration over the ϕ^I will therefore be Gaussian in these variables. This results in a power of ϕ^0 being brought down in front of the exponential. Let us assume that this integration has been done, such that the surviving action piece in the exponential is

$$\pi\tau_2 p^3 \left(\frac{1}{3\phi^0} + \frac{11}{12}\phi^0 + \frac{11}{12}\right) - \frac{2\pi\tau_2}{3} c_{IJK}\, \mu^I \mu^J p^K \,. \tag{8.43}$$

We close this section with some remarks and speculations on the value (8.41). An immediate comment is that it is finite and that it correctly reduces to the background contribution (5.36) under $C^0 = C^I = 0$. As a result, a reasonable expectation is that any further boundary terms to S_{bulk} are likely to make only finite contributions proportional to $C^{\mathcal{I}}$. A second point, which should receive further analysis in future work, concerns the relation of (8.41) with the renormalized action for the four-dimensional problem on $\text{AdS}_2 \times \text{S}^2$ evaluated on the $C^{\mathcal{I}}$-localization solutions [14]. Here we simply note some similarities and differences. The common structure among the two is the presence of a term linear in ϕ^0 and ϕ^I, a term as $1/\phi^0$ and a term with $\phi^I\phi^J/\phi^0$. However, (8.41) contains two additional contributions which are entirely absent from the 4d result. The first is the constant term proportional to $\tau_2\, p^3$, i.e. the second term on the second line. The second is the constant term proportional to $\tau_2\, \mu^I \mu^J$, i.e. the second term on the third line. (Here, by constant, we mean independent of $\phi^{\mathcal{I}}$.) Recall that the latter term already arises in the on-shell result (5.36), and is related to the Casimir-energy-type prefactor $C(\tau, \mu)$ that connects the path integral to the canonical trace form [65] as denoted in (5.24). The former constant term $\tau_2\, p^3$ does not, on the other hand, have an immediate interpretation. One possibility is that it couples to an additional gravitational localization mode that has no counterpart in the four-dimensional localization manifold, and which was therefore not detectable from our lift in Chapter 7. In the following section, this motivates an investigation in the existence of additional such BPS modes.

8.3 A numerical search for further localization solutions

We have found in Section 8.1 that the off-shell metric (8.2) in our lifted localization solution of Chapter 7 is not compatible with the standard Brown-Henneaux boundary conditions (8.12) of the path-integral, but is instead compatible with the CSS boundary conditions (8.13). This lead to a line of questioning on whether the functional integral for the localized supergravity action on $\mathrm{AdS}_3\times\mathrm{S}^2$ should, perhaps, receive contributions from BPS configurations obeying CSS rather than Brown-Henneaux. In this light, we initialized in Section 8.2 a study of a renormalization scheme for the bulk action (4.10) according to the CSS prescription. One obvious question that can be approached in parallel is to ask about the existence of additional localization solutions for $\overline{\mathcal{Q}}$ with non-trivial fluctuations in the metric, and how these fluctuations might fit into the candidate boundary conditions.

In this section we address this question by studying the Killing spinor equation (KSE) of the off-shell 5d supergravity in the large ρ regime, with Killing spinor fixed to our localization spinor $\varepsilon^i_{(1)}$ given in (5.13). This requires firstly introducing an ansatz for the form of the asymptotic expansions of the fluctuating bosonic fields, as well as for the fluctuating S- Killing spinor. Our ansatz for the 5d metric restricts fluctuations to be in the AdS_3 directions $\gamma_{\alpha\beta}$, and we allow these fluctuations to obey either CSS or Brown-Henneaux boundary conditions. The approach to solve the Killing spinor equation for the various expansion coefficients then follows a recursive order-by-order strategy. This is reminiscent of the procedure for solving the Einstein field equations for Fefferman-Graham coefficients in asymptotically AdS spacetimes, as reviewed in the pure AdS_3 case in Chapter 2. In this present analysis, because the recursive BPS equations quickly become very involved, we choose to employ the help of numerical tools instead of solving them by hand.

As we will see, the results of this analysis suggests the existence of an infinitely large family of localization solutions, with fluctuations parameterized by two arbitrary radial functions $a^+(\rho)$, $a^-(\rho)$. In the $\gamma_{\alpha\beta}$ factor of the 5d metric, these fluctuations appear at most at subleading order $\mathcal{O}(1)$. At the level of boundary conditions, we find that the a^+ fluctuations at this order are consistent with the CSS boundary conditions, but the a^- are not.

Killing spinor equation recap

Recall that the Killing spinor equation is the supersymmetry variation of the gravitino ψ^i_M set to zero. It is given in (4.31), but we copy it here for convenience

$$(\partial_M - \frac{1}{4}\omega_M{}^{AB}\gamma_{AB})\epsilon^i + \frac{1}{2}(V_M)_j{}^i\epsilon^j + \frac{\mathrm{i}}{4}T^{AB}(3\gamma_{AB}\gamma_M - \gamma_M\gamma_{AB})\epsilon^i \;=\; 0\,. \tag{8.44}$$

The bosonic fields in this equation belong to the 5d Weyl multiplet. They are the vielbein $E_M{}^A$, the auxiliary two-form T_{AB}, and the auxiliary SU(2) R-symmetry gauge-field $(V_M)_j{}^i$. The spinors ϵ^i, η^i parametrize the Q- and S- supersymmetry. Our gamma-matrix conventions in the Euclidean theory are given in the tangent frame in (5.10). As mentioned in the introduction to this section, we continue to fix the Q-Killing spinor ϵ^i to the localization ($\overline{\mathcal{Q}}$) Killing spinor $\varepsilon^i_{(1)}$ given in (5.13).

Asymptotic ansatz for the metric

We require an ansatz for the asymptotic form of the off-shell fields to substitute into the Killing spinor equation (8.44). We begin by constructing that of the vielbein $E_M{}^A$. Here, it useful to introduce the following "light-cone" tangent frame for the boundary AdS$_3$ directions, which is defined in terms of the frame $(\hat{\psi}, \hat{t}_E)$ frame as

$$E^{(\pm)} \;\equiv\; E^{\hat{\psi}} \pm \mathrm{i}E^{\hat{t}_E}\,. \tag{8.45}$$

Note that the this frame rotation is not a Lorentz rotation. The metric tensor is computed in this basis as $ds^2 = (2\ell)^2 d\rho^2 + E^{(+)}E^{(-)} + ds^2(\mathrm{S}^2)$. Note also that vielbeine that obey the Brown-Henneaux boundary conditions (8.12) are of the form [3]

$$\begin{aligned} \frac{E^{(+)}}{\ell} &= e^{\rho}dz + \mathcal{O}(\mathrm{e}^{-\rho})\,, \\ \frac{E^{(-)}}{\ell} &= e^{\rho}d\bar{z} + \mathcal{O}(\mathrm{e}^{-\rho})\,, \end{aligned} \tag{8.46}$$

Meanwhile, vielbeine that obey the CSS boundary conditions (8.13) are of the form

$$\begin{aligned} \frac{E^{(+)}}{\ell} &= e^{\rho}(dz + \partial_{\bar{z}}\bar{P}(\bar{z})d\bar{z}) + \mathcal{O}(\mathrm{e}^{-\rho})\,, \\ \frac{E^{(-)}}{\ell} &= e^{\rho}d\bar{z} + e^{-\rho}\frac{4G\Delta}{\ell}dz + \mathcal{O}(\mathrm{e}^{-\rho})d\bar{z} + \mathcal{O}(\mathrm{e}^{-3\rho})dz\,, \end{aligned} \tag{8.47}$$

[3] To compare the Lorentzian metrics of Chapter 2 with the Euclidean metrics presented here, recall the Wick rotation which maps the spacetime coordinates as $(x^+, x^-) \mapsto (z, \bar{z})$.

where the $\mathcal{O}()$ structure is specified for each leg of $E^{(-)}$ individually in order to indicate that $E_z^{(-)}$ cannot fluctuate at order $\mathcal{O}(\mathrm{e}^{-\rho})$ (but $E_{\bar{z}}^{(-)}$ can).

We now introduce our ansatz for the off-shell vielbein. We choose to turn on fluctuations only in the AdS$_3$ directions of the 5d space. To cover a broader space of solutions, we allow these fluctuations to obey either Brown-Henneaux boundary conditions as in (8.46) or CSS boundary conditions as in (8.47). The ansatz is:

$$E^{\hat{\rho}} = E_*^{\hat{\rho}}\,, \qquad E^{\hat{\theta}} = E_*^{\hat{\theta}}\,, \qquad E^{\hat{\phi}} = E_*^{\hat{\phi}}\,, \tag{8.48}$$

$$E^{(+)} = E_*^{(+)} + \ell\Big(\bar{a}^+(\rho)d\bar{z} + a^+(\rho)dz\Big)\,, \tag{8.49}$$

$$E^{(-)} = E_*^{(-)} + \ell\Big(\bar{a}^-(\rho)d\bar{z} + a^-(\rho)dz\Big)\,, \tag{8.50}$$

where E_* denote the background values on $\mathbb{H}^3/\mathbb{Z}\times \mathrm{S}^2$ (in the cylinder coordinates):

$$\begin{aligned} E_*^{\hat{\rho}} &= 2\ell d\rho\,, \qquad E_*^{\hat{\theta}} = \ell d\theta\,, \qquad E_*^{\hat{\phi}} = \ell\sin\theta d\phi\,, \\ E_*^{(+)} &= \ell(e^{\rho}dz - e^{-\rho}d\bar{z})\,, \qquad E_*^{(-)} = \ell(e^{\rho}d\bar{z} - e^{-\rho}dz)\,. \end{aligned} \tag{8.51}$$

The functions a^+, $\bar{a}^+$, a^-, $\bar{a}^-$ are off-shell fluctuations that encode the following expansions:

$$\begin{aligned} \bar{a}^+(\rho) &= e^{\rho}\bar{a}_0^+ + e^{-\rho}\bar{a}_2^+ + \cdots \\ a^+(\rho) &= e^{\rho}a_0^+ + e^{-\rho}a_2^+ + \cdots \\ \bar{a}^-(\rho) &= e^{\rho}\bar{a}_0^- + e^{-\rho}\bar{a}_2^- + \cdots \\ a^-(\rho) &= e^{\rho}a_0^- + e^{-\rho}a_2^- - \cdots \end{aligned} \tag{8.52}$$

We assume that the coefficients $\bar{a}_N^+$, a_N^+, $\bar{a}_N^-$, a_N^- are numerical constant. On these coefficients, we impose the following conditions:

$$a_0^+ = a_0^- = \bar{a}_0^- = 0\,. \tag{8.53}$$

The vielbein ansatz (8.48 - 8.52) with conditions (8.53) can accommodate CSS and Brown-Henneaux boundary conditions. Indeed, if $\bar{a}_0^+$ is abitrary and $a_2^- = 0$, the resulting metric obeys CSS boundary conditions (8.47). On the other hand, if $\bar{a}_0^+ = 0$, the metric obeys Brown-Henneaux boundary conditions (8.46). Note that the ansatz naturally encorporates our C^0 localization solutions as

$$\begin{aligned}
\bar{a}_0^+ &= -C^0\,, \qquad \bar{a}_2^+ = -2(C^0)^2\,, \qquad \bar{a}_4^+ = -4(C^0)^3\,,\cdots \\
a_2^+ &= 2C^0\,, \qquad a_4^+ = 4(C^0)^2\,,\cdots \\
\bar{a}_2^- &= C^0\,, \qquad \bar{a}_4^- = 2(C^0)^2\,,\cdots \\
a_2^- &= 0\,, \qquad a_4^- = -2C^0\,,\cdots
\end{aligned} \tag{8.54}$$

Asymptotic ansatz for the remaining fields

The fields other than the metric that enter the Killing spinor equation are T_{AB}, V_A,. We also have the S- supersymmetry spinor η^i. The ansatze for these quantities are chosen so as to reflect the structure of our C^0-localization solutions. We let:

$$T_{\hat{\theta}\hat{\phi}} = T^*_{\hat{\theta}\hat{\phi}} + t(\rho)\,, \tag{8.55}$$

$$V_{\hat{\psi},\hat{t}_E} = V^*_{\hat{\psi},\hat{t}_E} + v_{\hat{\psi},\hat{t}_E}(\rho)\,, \tag{8.56}$$

$$\eta^i = \eta^i_* + s(\rho)\lambda^i\,, \tag{8.57}$$

with all else zero, and where the functions t, v, s are given as

$$\begin{aligned}
t(\rho) &= t_0 + e^{-2\rho}t_2 + \cdots\,, \\
v_{\hat{\psi},\hat{t}_E}(\rho) &= e^{-\rho}(v_0)_{\hat{\psi},\hat{t}_E} + e^{-3\rho}(v_2)_{\hat{\psi},\hat{t}_E}\,, \\
s(\rho) &= s_0 + e^{-2\rho}s_2 + e^{-4\rho}s_4 + \cdots\,.
\end{aligned} \tag{8.58}$$

The coefficients t_N, v_N, s_N are taken to be numerical constants. As before, the starred quantities in (8.55 - 8.57) denote the background values on $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$:

$$T^*_{\hat{\theta}\hat{\phi}} = -\frac{1}{4\ell}\,, \qquad V^*_{\hat{\psi},\hat{t}_E} = \eta^i_* = 0\,. \tag{8.59}$$

The spinors λ^i in (8.57) are fixed to

$$\begin{aligned}
\lambda^1 &= -\frac{\mathrm{e}^{\frac{1}{2}(t_E+\mathrm{i}(\psi+\theta))}}{3\sqrt{2}\ell}\begin{pmatrix} \cosh(\frac{\rho}{2})\cos(\frac{\theta}{2}) \\ -\cosh(\frac{\rho}{2})\sin(\frac{\theta}{2}) \\ -\sinh(\frac{\rho}{2})\cos(\frac{\theta}{2}) \\ \sinh(\frac{\rho}{2})\sin(\frac{\theta}{2}) \end{pmatrix}, \\
\lambda^2 &= -\mathrm{i}\frac{\mathrm{e}^{-\frac{1}{2}(t_E+\mathrm{i}(\psi+\theta))}}{3\sqrt{2}\ell}\begin{pmatrix} \sinh(\frac{\rho}{2})\sin(\frac{\theta}{2} \\ \sinh(\frac{\rho}{2})\cos(\frac{\theta}{2}) \\ -\cosh(\frac{\rho}{2})\sin(\frac{\theta}{2}) \\ -\cosh(\frac{\rho}{2})\cos(\frac{\theta}{2}) \end{pmatrix}.
\end{aligned} \tag{8.60}$$

Substitution of ansatz into the KSE and results

We substitute the complete ansatz (8.49 - 8.60) into the Killing spinor equation (8.44) with $\epsilon^i = \varepsilon^i_{(1)}$. Turning to a computer, we extract the first twelve orders of the equation, which ranges from $\mathcal{O}(\mathrm{e}^{3\rho/2})$ to $\mathcal{O}(\mathrm{e}^{-15/2\rho})$ in steps of $\mathcal{O}(\mathrm{e}^{-\rho})$. We then task a program to solve these equations simultaneously for the expansion coefficients of the off-shell fluctuation functions a^+, a^-, $\bar{a}^+$, $\bar{a}^-$, t, v, s in (8.49 - 8.57). The output of this procedure is a set of algebraic relations between a^+_N, a^-_N, $\bar{a}^+_N$, $\bar{a}^-_N$, t_N, v_N, s_N, for N reaching down to several subleading orders (we reach at least the $N = 8$ coefficients for all functions). To disentangle these relations, it is useful to consider extra assumptions on these coefficients, such as setting some of them to zero by hand. Here, we consider two such examples of interest.

Case (1): $t = v = s = 0$

This is the assumption that only the vielbein fluctuations are turned on. The fields T_{AB}, V_A and η^i are fixed to their respective background values, i.e. we set the functions t, v, s to zero in (8.58). In this case, the asymptotic KSE gives only one non-trivial relation for the fluctuations of the vielbein:

$$\bar{a}^-_2 = \bar{a}^+_0(a^-_2 - 1)\,. \tag{8.61}$$

All other fluctuation coefficients not involved in (8.61) are determined to be zero by the equations. In principle, we therefore have a two-parameter family of off-shell asymptotic BPS solutions in the Weyl multiplet, parametrized by the vielbein fluctuations $(\bar{a}_0^+,\, a_2^-)$. Note that, because the expansions for the vielbein fluctuations terminate, these would be exact solutions.

However, these solutions are not consistent with the global properties of the space. Indeed, recall that we have contractibility of the ψ-circle and so we require 1-forms to vanish along this direction. In particular, we require:

$$E_\psi^{(\pm)}|_{\rho=0} \;=\; 0\,. \tag{8.62}$$

The fact that there are no non-trivial values for the fluctuations $(\bar{a}_0^+,\, a_2^-)$ which allow for (8.62) is most easily seen by noting that $E_\psi^{(+)}$ cannot be zero at the origin unless $\bar{a}_0^+$ vanishes. If $\bar{a}_0^+$ vanishes, so does $\bar{a}_2^-$ by (8.61). Then, $E_\psi^{(-)}$ cannot vanish at the origin unless a_2^- vanishes. We conclude that there are no allowed $\overline{\mathcal{Q}}$-BPS solutions around the supersymmetric torus for the case where off-shell fluctuations only occur in the vielbein.

Case (2): Assume $\bar{a}_0^+ = -C^0$, and set $C^0 = 0$

Here, we study the case of imposing the additional boundary condition

$$\bar{a}_0^+ \;=\; -C^0 \tag{8.63}$$

for $\bar{a}^+$, which corresponds to the structure of our C^0 solution. We then set $C^0 = 0$, so that

$$\bar{a}_0^+ \;=\; 0\,. \tag{8.64}$$

This set-up implies that any allowed BPS solutions emerging from solving the asymptotic KSE must exist independently of the C^0 mode.

The asymptotic KSE with this set-up gives, up to all orders considered, the following constraints on the vielbein fluctuations:

$$\begin{aligned} &\bar{a}_n^+ \;=\; \bar{a}_n^- \;=\; 0\,, \qquad n = 2,\, 4,\, \cdots\,, \\ &a_n^+,\, a_n^- \qquad \text{arbitrary}\,. \end{aligned} \tag{8.65}$$

The fluctuations $t(\rho)$, $v(\rho)$, $s(\rho)$ in the remaining fields of the KSE are given to be *entirely determined* in terms of a^+ and a^-. We note the following first few relations:

$$
\begin{aligned}
\underline{t(\rho):} \qquad & t_0 \;=\; 0\,, \\
& t_2 \;=\; \frac{a_2^+}{12}\,, \\
& t_4 \;=\; \frac{2a_4^+ - a_2^{+2}}{12}\,, \\
\underline{v(\rho):} \qquad & (v_0)_{\hat{\psi}} \;=\; (v_0)_{\hat{t}_E} \;=\; 0\,, \\
& (v_2)_{\hat{\psi}} \;=\; -\mathrm{i}a_2^+\,, \qquad (v_2)_{\hat{t}_E} \;=\; -a_2^+\,, \\
& (v_4)_{\hat{\psi}} \;=\; \mathrm{i}(a_2^{+2} - 2a_4^+ - a_4^-)\,, \qquad (v_4)_{\hat{t}_E} \;=\; a_2^{+2} - 2a_4^+ + a_4^-\,, \\
\underline{s(\rho):} \qquad & s_0 \;=\; 0\,, \\
& s_2 \;=\; a_2^+\,, \\
& s_4 \;=\; 2a_4^+ - a_2^{+2}
\end{aligned}
\tag{8.66}
$$

The asymptotic BPS solutions described by equations (8.65), (8.66) can, upon suitable choices for a^+ and a^-, be made to respect contractability along the ψ-cycle of the torus. For instance, consider the following choice for a^+ and a^-:

$$
\begin{aligned}
a_2^+ \;&=\; -a_4^+ \;\equiv\; \frac{D^0}{2}\,, \qquad a_{n+4}^+ \;=\; 0\,, \qquad n = 2,\, 4,\, \cdots \\
a_n^- \;&=\; 0\,.
\end{aligned}
\tag{8.67}
$$

The off-shell vielbein is exact, and is given as:

$$
E^{(+)} \;=\; E_*^{(+)} + e^{-2\rho} D^0 \sinh\rho\,(d\psi + \mathrm{i}dt_E)\,, \qquad E^{(-)} \;=\; E_*^{(-)}\,, \tag{8.68}
$$

for which the $d\psi$ directions indeed vanish at $\rho = 0$. This choice also makes the $\mathrm{SU}(2)_R$ gauge field $V_\psi = E_\psi{}^A V_A$ contractible, as can be computed from (8.66).

In conclusion, the analysis of *Case (2)* produces a two-function family of allowed BPS solutions on the torus, parametrized by $(a^+,\, a^-)$. These are solutions that, by construction of this case, persist independently of the C^0-solution. We now wish to comment on their embedding into a set of boundary conditions (i.e. Brown-Henneaux or CSS). While this can easily be done at the level of the vielbein by comparing with (8.46) and (8.47), it is visually most direct to convert back to metric formalism.

In the AdS$_3$ boundary directions, we have for generic (a^+, a^-):

$$\begin{aligned}\gamma_{\alpha\beta}\, dx^\alpha dx^\beta &= E^{(+)}E^{(-)} \\ &= \ell^2\Big(\mathrm{e}^{2\rho}dzd\bar{z} - (1-a_2^-)dz^2 - d\bar{z}^2 + a_2^+\, dzd\bar{z}\Big) + \mathcal{O}(\mathrm{e}^{-2\rho})\end{aligned} \tag{8.69}$$

and so we identify the first few Fefferman-Graham coefficients as :

$$\begin{aligned}\gamma^{(0)}_{zz} &= \gamma^{(0)}_{\bar{z}\bar{z}} = 0\,, \qquad \gamma^{(0)}_{z\bar{z}} = \frac{\ell^2}{2}\,, \\ \gamma^{(2)}_{zz} &= -\ell^2(1-a_2^-)\,, \qquad \gamma^{(2)}_{\bar{z}\bar{z}} = -\ell^2\,, \qquad \gamma^{(2)}_{z\bar{z}} = \frac{\ell^2}{2}a_2^+\,.\end{aligned} \tag{8.70}$$

It is clear by comparison with (8.12) that both the a^+ and a^- modes are consistent with the Brown-Henneaux boundary conditions. By comparison with (8.13), one finds that the a^+ mode is also consistent with the CSS boundary conditions, but the a^- mode is not (due to the fluctuating $\gamma^{(2)}_{zz}$). We close with a remark: the mode a_2^- can be identified as a (constant) left-moving Brown-Henneaux mode $\mathcal{L}(z)$, as can be seen from (2.46).[4] The fact that this Brown-Henneaux mode is $\overline{\mathcal{Q}}$-BPS is, in fact, expected on theoretical grounds. Indeed, $\overline{\mathcal{Q}}$ is a supercharge that acts in the right-moving sector of the asymptotic symmetry algebra and so it trivially commutes with the left-moving Virasoro generators $\mathcal{L}_n$. Now, faced with the existence of this infinite family of BPS solutions in the left-moving Brown-Henneaux sector, one interesting question is how to incorporate them in the localized path integral. To this, the functional integral being defined over CSS boundary conditions would provide a convenient answer, namely that these modes are simply excluded from the problem due to their violation of the boundary conditions.

8.4 Brown-Henneaux modes and the 4d/5d lift

We close this chapter by taking a step back from the CSS-centered analysis and returning instead to the more familiar ground of Brown and Henneaux. One natural question the reader may have at this late stage of the thesis is the following: what happens to the Brown-Henneaux modes of Euclidean AdS$_3\times$S^2 in the KK reduction

[4] We emphasize that we are still working in a generically off-shell setting, so while a_2^- behaves like a Brown-Henneaux mode at the level of the $\mathcal{O}(1)$ metric components, it is not the case that the further subleading terms a_4^-, a_6^-, $\cdots$ are determined in terms of it (as they would be in the on-shell setting).

to the theory on Euclidean AdS$_2\times$S^2? After all, this should be an approachable exercise with the Euclidean 4d/5d connection that we developed in Chapter 6. The question of the 4d/5d connection on the Brown-Henneaux modes is, in fact, also relevant to the findings of Section 8.3 where recall that we have found the existence of BPS solutions around AdS$_3\times$S^2 corresponding to fluctuations in the left-moving Brown-Henneaux sector (parametrized by the $a^-(\rho)$ function). Understanding the reduction of these fluctuations to the four-dimensional theory may shine a light on why they did not appear in the systematic analysis of the 4d localization manifold of [39]. The answer, which we show in this section, is that under the 4d/5d reduction to AdS$_2\times$S^2, fluctuations corresponding to left-moving Brown-Henneaux modes around the AdS$_3$($\times$S^2) background violate the AdS$_2$ boundary conditions of the 4d supergravity. Fluctuations corresponding to right-moving Brown-Henneaux modes, on the other hand, are allowed with respect to these boundary conditions.

We begin this treatment by first recalling the initial steps of the algorithm, developed in Chapter 6, for connecting off-shell configurations around Euclidean AdS$_2\times$S^2 to configurations around the supersymmetric torus $\mathbb{H}^3/\mathbb{Z}\times$S^2. Starting from the 5d perspective, one first writes the torus background (5.8) as a Kaluza-Klein ansatz over AdS$_2\times$S^2. The metric in the KK ansatz takes the form

$$ds^2 \;=\; \dot{G}_{\dot{M}\dot{N}}\, dx^{\dot{M}} dx^{\dot{N}} \;=\; g_{\mu\nu}\, dx^\mu dx^\nu + \phi^{-2}\Big(dx^5 + B_\mu dx^\mu\Big)^2 , \tag{8.71}$$

where recall that the KK coordinates are $x^{\dot{M}} = (\eta, \chi, \theta, \phi, x^5)$, $g_{\mu\nu}$ is the metric tensor on AdS$_2\times$S^2, and ϕ, B_μ are the KK scalar and KK gauge-field respectively. The mapping of the $\mathbb{H}^3/\mathbb{Z}\times$S^2 torus background metric to the form (8.71) is then achieved by the following local coordinate transformations:

$$(\,\rho, z, \bar{z}) \;=\; \Big(\frac{\eta}{2},\, \chi + \mathrm{i}x^5,\, \chi\Big) \quad\Leftrightarrow\quad (\eta\,, \chi\,, x^5) \;=\; \Big(2\rho\,, \bar{z}\,, -\mathrm{i}(z-\bar{z})\Big)\,. \tag{8.72}$$

With these maps, the elements of the KK metric (8.71) are the metric for Euclidean AdS$_2\times$S^2:

$$g_{\mu\nu}\, dx^\mu dx^\nu \;=\; \ell^2\Big(d\eta^2 + \sinh^2\eta\, d\chi^2 + d\theta^2 + \sin^2\theta\, d\phi^2\Big) \tag{8.73}$$

and the background KK scalar and gauge-field:

$$\phi \;=\; \ell^{-1}\,, \qquad B \;=\; \mathrm{i}(\cosh\eta - 1)d\chi\,. \tag{8.74}$$

Under the 4d/5d connection, ϕ and B descend to form a matter vector multiplet in the 4d theory. In particular, ϕ forms a complex 4d vector-scalar $(X^0, \overline{X}^0)$ and B corresponds to a U(1) vector field A^0:

$$
\begin{aligned}
X^0 &= -\frac{\mathrm{i}}{2} e^{-\varphi}\phi\,, \qquad \overline{X}^0 = \frac{\mathrm{i}}{2} e^{\varphi}\phi\,, \\
A^0_\mu &= B_\mu\,.
\end{aligned}
\tag{8.75}
$$

The value of the lifting parameter φ is not relevant to the analysis of this section, so we set it to zero. For the upcoming discussion, we recall the boundary conditions for the relevant fields of the quantum theory on AdS_2 [14]:

$$
\begin{aligned}
ds^2_{AdS_2}|_{\mathrm{bdry}} &= \Big((g_{\eta\eta})_* + \mathcal{O}(e^{-2\eta})\Big) d\eta^2 + \Big((g_{\chi\chi})_* + \mathcal{O}(1)\Big) d\chi^2\,, \\
X|_{\mathrm{bdry}} &= X_* + \mathcal{O}(e^{-\eta})\,, \qquad A^0_\chi|_{\mathrm{bdry}} = (A^0_\chi) * + \mathcal{O}(1)\,,
\end{aligned}
\tag{8.76}
$$

where the starred quantities are the *fixed* background values on AdS_2, with $(g_{\eta\eta})_*$ a constant, $(g_{\chi\chi})_*$ proportional to $e^{2\eta}$, X_* a constant, and $(A^0_\chi)_*$ proportional to e^{η}.

We now turn on Brown-Henneaux-type fluctuations around the AdS_3 part of the 5d background. We do not impose the fact these fluctuations are BPS. We however do impose they are constant, so that they are independent of x^5 and therefore allowed within the framework of the 4d/5d connection. In the language of the vielbein in (8.49), (8.50), such Brown-Henneaux modes are the a_n^-, $\bar{a}_n^+$, $n = 2, 4\cdots$, corresponding to the left-movers and right-movers, respectively (recall that the leading orders a_0^-, $\bar{a}_0^+$ are not Brown-Henneaux). For this analysis, we therefore write the vielbein components as [5]

$$
\frac{E^{(+)}}{\ell} = e^{\rho} dz + e^{-\rho}(-1 + \bar{a}_2^+) d\bar{z} + \cdots\,,
\tag{8.77}
$$

$$
\frac{E^{(-)}}{\ell} = e^{\rho} d\bar{z} + e^{-\rho}(-1 + a_2^-) dz + \cdots\,,
\tag{8.78}
$$

with the remaining components $E^{\hat{\rho}}, E^{\hat{\theta}}, E^{\hat{\phi}}$ fixed to their on-shell $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ values as in (8.51). We then compute the associated metric tensor to this vielbein with $ds^2 = (2\ell)^2 d\rho^2 + E^{(+)}E^{(-)} + ds^2(\mathrm{S}^2)$, and then map this metric into the form of the KK metric (8.71) using the coordinate transformations (8.72). We can then read

[5] In the language of the 3d gravity review in Section 2.3.2, this corresponds to Brown-Henneaux metrics (2.46) with $\mathcal{L} = -1 + a_2^-$, $\bar{\mathcal{L}} = -1 + \bar{a}_2^+$.

off the following fluctuating 4d metric and KK fields:

$$\begin{aligned} \phi^{-2} &= \ell^2(1-a_2^-) + \mathcal{O}(e^{-\eta})\,, \\ B_\chi &= \mathrm{i}\frac{e^{\eta}}{2-2a_2^-} + \mathcal{O}(1)\,, \\ g_{\chi\chi} &= \ell^2 \frac{e^{2\eta}}{4(1-a_2^-)} + \mathcal{O}(1)\,. \end{aligned} \tag{8.79}$$

(The mode $\bar{a}_2^+$ appears in the subleading terms.) Given the relations (8.75) to the 4d fields on AdS_2, we see that the *left-moving* Brown-Henneaux fluctuation a_2^- in the AdS_3 metric violates all the AdS_2 boundary conditions given in (8.76). Note that the *right-moving* Brown-Henneaux fluctuation $\bar{a}_2^+$, on the other hand, is a priori consistent with these boundary conditions. Returning to the context of Section 8.3, where recall that a_2^- was found to be BPS with respect to $\overline{\mathcal{Q}}$, we conclude that this solution generically violates the AdS_2 boundary conditions. In fact, this can be seen as the reason why this mode did not appear in the systematic 4d localization manifold analysis of [39].

Summary and Outlook

In this thesis, we developed the program of supersymmetric localization for the functional integral of 5d supergravity with eight supercharges on $\mathrm{AdS}_3 \times \mathrm{S}^2$. While we did not reach the actual computation of the localized path integral, our analyses resolved a number of intermediate problems towards this goal. We review these here.

First, we saw how the set-up of the supersymmetric $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ torus background in the Euclidean 5d supergravity is itself a subtle task, due to the apparent incompatibility of supersymmetry with the periodicities around the time cycle of the torus. The resolution came in the form of a twist of the S^2 around the time direction of Euclidean AdS_3. Once this background was established, we derived its superalgebra and identified a supercharge on which to set up the localization. Next, we directed our efforts at computing the bosonic BPS configurations for this supercharge. Our approach was to employ the off-shell 4d/5d connection to lift the known localization manifold of the four-dimensional $\mathcal{N} = 2$ theory on $\mathrm{AdS}_2 \times \mathrm{S}^2$. To this end, we first presented a mapping of the twisted $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ configuration into a Kaluza-Klein lift of $\mathrm{AdS}_2 \times \mathrm{S}^2$. Then, while the 4d/5d connection has been known in the context of Lorentzian supergravities for some time, we required an adaptation of it for the analogous Euclidean theories. As it turned out, performing this modification required careful considerations on the four-dimensional reality conditions, which lead to subtle implications for certain lifting parameters. The Euclidean 4d/5d connection was eventually presented, and using it we found a class of highly non-trivial localization solutions around $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$. To cement these off-shell configurations as players in the functional integral, it was important to follow up with a detailed analysis of their boundary behaviour. We focused in particular on the metric field and the $\mathrm{U}(1)$ gauge fields which under the usual definition of the functional integral should have asymptotics fixed by the Brown-Henneaux boundary conditions and the standard chiral boundary conditions in Chern-Simons theory, respectively. We soon found that while the gauge fields were consistent with these boundary conditions, the metric field was not. Indeed, an off-shell mode was found to reach the conformal boundary of the

AdS_3 factor, thus explicitly violating Brown-Henneaux. Remarkably, it turned out that the presence of this mode was entirely consistent with a more recent, chiral set of AdS_3 metric boundary conditions: the Compere-Strominger-Song boundary conditions. This observation lead us to push onward with these localization solutions and further investigate these boundary conditions in the context of the localized path-integral problem. We initiated a study of boundary terms for the bulk 5d supergravity action with respect to CSS in the metric sector. While a full renormalization scheme was not reached, the evaluation of this partial action on the localization solutions already yielded a simple and tractable structure. We then presented a numerical search for additional localization solutions near the boundary, focusing on the fields of the Weyl multiplet. The findings of this analysis were positive: we found evidence for BPS fluctuations in the AdS_3 factor of the metric, that arise independently of the lifted modes, and that also obey the CSS boundary conditions. We also found evidence for BPS modes in the left-moving sector of the theory. These, in fact, correspond to left-moving Brown-Henneaux-type modes, which we know should indeed be BPS by virtue of $\overline{\mathcal{Q}}$ acting in the right-moving sector. We closed the thesis with a tangential analysis on the subject of these Brown-Henneaux modes: we studied their behaviour under reduction to $AdS_2{\times}S^2$ using the off-shell 4d/5d connection, and showed that they explicitly violate the four-dimensional boundary conditions.

The final sections of this thesis contained speculative material, which set the stage for many open questions. We hope to address these in future work. First and foremost, if the CSS boundary conditions (rather than Brown-Henneaux) are indeed those that are relevant to the localized functional integral, one should understand why. One possibility is that the linear deformation of the action in the functional integral with $\overline{\mathcal{Q}}V$, as in (1.12), induces a change from the standard boundary conditions for the metric (and perhaps other fields). It is in fact plausible that these "deformed boundary conditions" would be chiral, as the CSS conditions are, since we recall that $\overline{\mathcal{Q}}$ is a supercharge that acts in the right-moving sector. A second related task would then be to construct a fully renormalized action for the theory, beyond our partial proposal made in the final chapter of this thesis. Such an action should be well-defined with respect to the boundary conditions of all fields of the theory. Additionally, if we are to localize on this renormalized action, it should be made $\overline{\mathcal{Q}}$-supersymmetric. This will likely require its own set of boundary terms (see [14] for the four-dimensional case). Finally, there is reason to suggest the existence of extra localization modes, which are consistent with the CSS boundary conditions, and for which we have only established asymptotic evidence. Natural candidates for these missing BPS fluctuations, which we only alluded to in this thesis, might be boundary modes corresponding to the action of specific bosonic charges of the right-moving $\mathcal{N} = 4$ superconformal algebra on the $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$ vacuum. We note that these charges would necessarily have to commute with the localization supercharge $\overline{\mathcal{Q}}$. Also, if the CSS boundary conditions are to be insisted upon, their action should be consistent with said boundary conditions. These types of algebraic constraints could constitute a theoretical avenue for investigating the existence of such additional localization solutions.

Appendix A

Notations and conventions

We summarize the various index notations in Table A.1.

Index	**Range**	**Description**
$M,\, N,\, \cdots$	$(\rho,\, \psi,\, \theta,\, \phi,\, t_E)$	5d cylinder coordinates
$M',\, N',\, \cdots$	$(\rho,\, \psi,\, \theta,\, \phi',\, t'_E)$	5d twisted torus coordinates
$A,\, B,\, \cdots$	$(\hat{\rho},\, \hat{\psi},\, \hat{\theta},\, \hat{\phi},\, \hat{t_E})$	tangent frame for cylinder and twisted torus coordinates
$\dot{M},\, \dot{N},\, \cdots$	$(\eta,\, \chi,\, \theta,\, \phi,\, x^5)$	5d Kaluza-Klein coordinates
$\dot{A},\, \dot{B},\, \cdots$	$(1,\, 2,\, 3,\, 4,\, 5)$	tangent frame for Kaluza-Klein coordinates
$\mu,\, \nu,\, \cdots$	$(\eta,\, \chi,\, \theta,\, \phi)$	Euclidean $AdS_2 \times S^2$ coordinates*
$a,\, b,\, \cdots$	$(1, 2, 3, 4)$	tangent frame for Euclidean $AdS_2 \times S^2$ coordinates
$\alpha,\, \beta,\, \cdots$	$(z, \overline{z})$	thermal AdS_3 complex boundary coordinates
$m,\, n,\, \cdots$	(θ, ϕ)	S^2 coordinates
$i,\, j,\, \cdots$	$(1,\, 2)$ or $(+,\, -)$	Fundamental SU(2)**

Table A.1: Summary of index notation. (* In chapters 2, 3, and 4, $g_{\mu\nu}$ is used to denote three-dimensional metrics on AdS_3.) (** In Sections 5.3 and 8.2, the i, j indices are also used to denote the four transverse coordinates x^i of asymptotically $AdS_3 \times S^2$ spaces.)

Curvature

We summarize our curvature conventions. Our Minkowski signature is $(-,\, +,\, +,\, \cdots)$. All expressions below are converted to Euclidean signature by simply exhanching η_{ab} for δ_{ab}. In particular, tangent frame indices a, b are raised/lowered using η in Lorentzian signature, but using δ in Euclidean signature.

We denote $e_\mu{}^a$ and $g_{\mu\nu}$ to be a generic vielbein and its metric tensor (in arbitrary dimensions), respectively. As usual, we have

$$g_{\mu\nu} = e_\mu{}^a \eta_{ab}\, e_\nu{}^b\,. \tag{A.1}$$

The torsion free spin connection $\omega_\mu{}^{ab}$ is built from the vielbein as

$$\omega_\mu{}^{ab} = -2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\,\partial_\sigma e_\nu{}^c\,. \tag{A.2}$$

Note that the right-hand-side has an opposite overall sign compared to the standard literature. This definition for the spin connection is then related to the Christoffel symbol as

$$\omega_\mu{}^{ab} = e_{b\nu}\,\partial_\mu e_\nu{}^a - e^{b\nu}e_\sigma{}^a\Gamma^\sigma_{\mu\nu}\,, \tag{A.3}$$

where the Christoffel symbols are the usual

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\Big(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\Big)\,. \tag{A.4}$$

In terms of the spin connection, we define the Riemann tensor as

$$R_{\mu\nu}{}^{ab} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu\, c}{}^b + \omega_\nu{}^{ac}\omega_{\mu\, c}{}^b\,, \tag{A.5}$$

Substitution of (A.3) into (A.5) then gives the expression for the Riemann tensor in terms of Christoffel symbols as

$$R_{\mu\nu}{}^\sigma{}_\rho = -2\Big(\partial_{[\mu}\Gamma^\sigma_{\nu]\rho} + \Gamma^\sigma_{\lambda[\mu}\Gamma^\lambda_{\nu]\rho}\Big)\,. \tag{A.6}$$

Here again, we note that the right-hand-side has an opposite overall sign compared to the standard literature. This is what very unconventionally leads us to have Anti-de-Sitter spaces with *positive* scalar curvature.

Spinors and gamma matrices

We denote a basis for the the d-dimensional Clifford algebra as

$$\big\{\Gamma = \mathbb{I},\, \gamma^{A_1},\, \gamma^{A_1A_2}, \cdots \gamma^{A_1A_2\cdots A_d}\big\}\,, \tag{A.7}$$

where:

$$\gamma^{A_1\cdots A_k} \;=\; \gamma^{[A_1}\cdots\gamma^{A_k]}\,. \tag{A.8}$$

In five dimension with Lorentzian signature, a consistent choice of gamma matrix satisfies the following relations:

$$\begin{aligned}
\gamma_A^\dagger &= -A\gamma_A A^{-1}\,, \quad & A &= \gamma_0\,, \quad & A^\dagger &= A^{-1} = -\gamma_0\,, & & \\
\gamma_A^T &= \mathcal{C}\gamma_A\mathcal{C}^{-1}\,, \quad & \mathcal{C}^T &= -\mathcal{C}\,, \quad & \mathcal{C}^\dagger &= \mathcal{C}^{-1}\,, & & \\
\gamma_A^* &= -B\gamma_A B^{-1}\,, \quad & B^T &= \mathcal{C}A^{-1}\,, \quad & B^\dagger &= B^{-1}\,, \quad & B^*B &= -1\,.
\end{aligned} \tag{A.9}$$

This is followed by the property, regarding the charge conjugation matrix $\mathcal{C}$,

$$(\mathcal{C}\Gamma^{(r)})^T = -(-)^{r(r-1)/2}\mathcal{C}\Gamma^{(r)}\,, \tag{A.10}$$

where $\Gamma^{(r)}$ is a matrix of the set (A.7) with rank r. Due to the property of the charge conjugation matrix, we can use the spinor representation satisfying the symplectic-Majorana condition

$$\bar{\psi}_i \;=\; (\psi^i)^\dagger\gamma_0\,, \tag{A.11}$$

where i is an $\mathrm{SU}(2)_R$ index, and where $\bar{\psi}_i$ is the symplectic-Majorana conjugate of ψ^i, defined as

$$\bar{\psi}_i := \varepsilon_{ij}(\psi^j)^T\mathcal{C}\,, \tag{A.12}$$

with ε_{ij} being the SU(2) symplectic metric $\varepsilon_{12} = -\varepsilon_{21} = 1$.

The five-dimensional Euclidean case is obtained by the Wick rotation of the time direction x^0, using the redefinition: $x^0 = -\mathrm{i}x^5$. This consistently redefines the 0^{th} gamma matrix as the 5-th directional one as $\gamma_0 = \mathrm{i}\gamma_5$. The relations on the Lorentizan gamma matrices (A.9) then become, for the Euclidean case:

$$\begin{aligned}
\gamma_A^\dagger &= \gamma_A \\
\gamma_A^* &= \gamma_A^T = \mathcal{C}\gamma_A\mathcal{C}^{-1}\,, \quad \mathcal{C}^\dagger = \mathcal{C}^{-1}\,, \quad \mathcal{C}^T = -\mathcal{C} \;\Leftrightarrow\; \mathcal{C}^*\mathcal{C} = -1\,,
\end{aligned} \tag{A.13}$$

with charge-conjugation matrix property:

$$(\mathcal{C}\Gamma^{(r)})^T \;=\; -(-)^{r(r-1)/2}\mathcal{C}\Gamma^{(r)}\,. \tag{A.14}$$

In the main text, we often consider Lorentz scalars of the type

$$\bar{\lambda}_i \, \Gamma^{(r)} \epsilon^j \, . \tag{A.15}$$

For two Grassman even spinors ϵ^i, λ^j, the property (A.14) leads to the following Majorana-flip relations:

$$\bar{\lambda}_i \, \Gamma^{(r)} \epsilon^j \; = \; (-)^{r(r-1)/2} \left(\delta_i^j \, \bar{\epsilon}_k \, \Gamma^{(r)} \lambda^k - \bar{\epsilon}_i \, \Gamma^{(r)} \lambda^j \right) \, . \tag{A.16}$$

Note some useful consequences of (A.16) for $\lambda = \epsilon$:

$$\bar{\epsilon}_i \, \epsilon^j \quad = \; \tfrac{1}{2} \left(\bar{\epsilon}_k \, \epsilon^k \right) \, \delta_i^j \, , \tag{A.17}$$

$$\bar{\epsilon}_i \, \gamma^A \epsilon^j \quad = \; \tfrac{1}{2} \left(\bar{\epsilon}_k \, \gamma^A \epsilon^k \right) \, \delta_i^j \, , \tag{A.18}$$

$$\bar{\epsilon}_k \, \gamma^{AB} \epsilon^k \quad = \; 0 \, , \qquad \bar{\epsilon}_k \, \gamma^{ABC} \epsilon^k \; = \; 0 \, , \tag{A.19}$$

where we used $r = 0, 1, 2, 3$ respectively. The spinors in the Euclidean theory can also be chosen to be symplectic-Majorana, but differently from (A.11), satisfying

$$\bar{\psi}_i \; = \; (\psi^i)^\dagger \, , \tag{A.20}$$

with the same definition of the symmplectic Majorana conjugate $\overline{\psi}_i$ as (A.12). However, we note that, as is commented in the begining of section 4.1, we does not impose (A.20) for quantum theory.

Appendix B

$\mathcal{N} = 4$ superconformal algebra and spectral flow

In this appendix we present the $\mathcal{N} = 4$ superconformal algebra in the conventions of [63]. The non-trivial commutation relations are the following:

$$
\begin{aligned}
[\mathcal{L}_m,\, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,,\\
[J^{\boldsymbol{a}}_m,\, J^{\boldsymbol{b}}_n] &= \mathrm{i}\epsilon^{\boldsymbol{abc}}J^{\boldsymbol{c}}_{m+n} + \frac{c}{12}m\delta^{\boldsymbol{ac}}\delta_{m+n,0}\,,\\
[\mathcal{L}_m,\, J^{\boldsymbol{a}}_n] &= -nJ^{\boldsymbol{a}}_{n+m}\,,\\
[\mathcal{L}_m,\, \mathcal{G}^{\alpha}_{\dot{A},r}] &= (\frac{m}{2}-r)\mathcal{G}^{\alpha}_{\dot{A},r+m}\,,\\
[J^{\boldsymbol{a}}_m,\, \mathcal{G}^{\alpha}_{\dot{A},r}] &= \frac{1}{2}(\boldsymbol{\tau_a})_{\beta}{}^{\alpha}\mathcal{G}^{\beta}_{\dot{A},m+r}\,,\\
\{\mathcal{G}^{\alpha}_{\dot{A},r},\, \mathcal{G}^{\beta}_{\dot{B},s}\} &= \epsilon_{\dot{A}\dot{B}}\left[\epsilon^{\alpha\beta}\mathcal{L}_{r+s} - (\epsilon\boldsymbol{\tau_a})^{\beta\alpha}(r-s)J^{\boldsymbol{a}}_{r+s} + \epsilon^{\alpha\beta}\frac{c}{6}(r^2-\frac{1}{4})\delta_{r+s,0}\right]\,,\\
&\qquad \alpha,\,\beta=+,\,-\,,\qquad \dot{A}\,,\dot{B}=+\,,-\,,\qquad \boldsymbol{a},\,\boldsymbol{b},\,\boldsymbol{c} \;=\; 1,\,2,\,3\,,
\end{aligned}
\tag{B.1}
$$

where here $\epsilon^{+-} = -\epsilon^{-+} = 1$ and $\epsilon_{+-} = -\epsilon^{-+} = -1$. The subscripts r, s take integer or half-integer values for the Ramond and NS sector respectively. In the main body, we often employ the SU(2) generators $J^{\pm}_m$ instead of $J^{1,2}_m$ above. They are related as

$$
J^{\pm}_m \;=\; J^1_m \pm \mathrm{i}J^2_m\,. \tag{B.2}
$$

The non-trivial commutation relations involving the SU(2) generators become:

$$\begin{aligned} [J^3_m, J^\pm_n] &= \pm J^\pm_{m+n}\,, \qquad & [J^+_m, J^-_n] &= 2J^3_{m+n} + \frac{c}{6} m\delta_{m+n,0}\,, \\ [J^3_m, \mathcal{G}^\pm_{\dot{A},r}] &= \pm\frac{1}{2}\mathcal{G}^\pm_{\dot{A},r+m}\,, \qquad & [J^\pm_m, \mathcal{G}^\mp_{\dot{A},r}] &= \mathcal{G}^\pm_{\dot{A},r+m}\,. \end{aligned} \tag{B.3}$$

In the NS sector, the modes

$$\mathcal{L}_0,\, \mathcal{L}_{\pm1},\, J^\pm_0,\, J^3_0,\, \mathcal{G}^\alpha_{\dot{A},\pm1/2}\,. \tag{B.4}$$

generate the global (i.e. centerless) part of the algebra, and the isomorphism

$$\begin{aligned} L_0 &= \mathcal{L}_0\,, \qquad & L_\pm &= \mp\mathrm{i}\mathcal{L}_{\mp1}\,, \\ J^\pm &= J^\pm_0\,, \qquad & J^3 &= J^3_0\,, \\ G^{\pm\beta}_\pm &= \pm2\mathcal{G}^\beta_{\mp,\mp1/2}\,, \qquad & G^{\mp\beta}_\pm &= \pm2\mathcal{G}^\beta_{\pm,\mp1/2}\,. \end{aligned} \tag{B.5}$$

maps this subalgebra to the form presented in the main body in (4.38), (4.40), (4.41).

The $\mathcal{N}=4$ algebra is isomorphic under the so-called spectral flow, parametrized by a real number η:

$$\begin{aligned} \mathcal{L}_m &\to \mathcal{L}_m + \eta J^3_m + \eta^2\frac{c}{24}\delta_{m,0}\,, \\ J^3_m &\to J^3_m + \eta\frac{c}{12}\delta_{m,0}\,, \\ \mathcal{G}^\pm_{\dot{A},r} &\to \mathcal{G}^\pm_{\dot{A},r\pm\frac{\eta}{2}}\,, \\ J^\pm_m &\to J^\pm_{m\pm\eta}\,. \end{aligned} \tag{B.6}$$

In particular, the Ramond sector (i.e. integer r, s) flows to the NS sector upon choosing $\eta=1$, and vice-versa.

Appendix C

Killing spinors on AdS$_3$ and S^2

In this appendix, we present solution of the Killing spinor equation (4.31) on the AdS$_3\times$S^2 background given in (4.19) and (4.21). Here, let us decompose the spacetime and local indices into those for $3+2$ dimensions as $M=\{\mu\,,\mathrm{m}\}$ and $A=\{a\,,\mathrm{a}\}$. Then the Killing spinor equation (4.31) splits as

$$\mathcal{D}_\mu\epsilon^i \;=\; s\frac{\mathrm{i}}{4\ell}\gamma^{\hat{\theta}\hat{\phi}}\gamma_\mu\epsilon^i\,, \qquad \mathcal{D}_{\mathrm{m}}\epsilon^i = s\frac{\mathrm{i}}{2\ell}\gamma^{\hat{\theta}\hat{\phi}}\gamma_{\mathrm{m}}\epsilon^i\,, \tag{C.1}$$

where we inserted the sign factor $s=\pm1$ to keep track of the choice of the background value of T_{MN}; $s=+1$ is for our background value of T_{MN} in (4.21), and $s=-1$ is for another background value by changing $T_{MN}\to -T_{MN}$ from the (4.21) (which involves changing $\sigma\to-\sigma$ from (4.20) by the BPS equation of vector multiplet). Note that, since the background metric (4.19) is direct product of 3 and 2 dimensions, the spin connection is also well separated as $-\frac{1}{4}\omega_\mu^{AB}\gamma_{AB}=-\frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$ and $-\frac{1}{4}\omega_{\mathrm{m}}^{AB}\gamma_{AB}=-\frac{1}{4}\omega_{\mathrm{m}}^{\mathrm{ab}}\gamma_{\mathrm{ab}}$. This can be seen explicitly by noting that the non-zero spin connection components are

$$\omega_t^{\hat{t}\hat{\rho}}=-\sinh\rho\,, \qquad \omega_\psi^{\hat{\rho}\hat{\psi}}=\cosh\rho\,, \qquad \omega_\phi^{\hat{\theta}\hat{\phi}}=\cos\theta\,. \tag{C.2}$$

We now decompose the spinor as

$$\epsilon^i \;=\; \epsilon^i_{AdS_3}\otimes\epsilon^i_{S^2}\,, \tag{C.3}$$

and take the following decomposition for the gamma matrices

$$\gamma_{\hat{t}} \;=\; \boldsymbol{\sigma}_0 \otimes \boldsymbol{\tau}_3\,, \qquad \gamma_{\hat{\rho}} \;=\; \boldsymbol{\sigma}_1 \otimes \boldsymbol{\tau}_3\,, \qquad \gamma_{\hat{\psi}} \;=\; \boldsymbol{\sigma}_2 \otimes \boldsymbol{\tau}_3\,, \qquad \gamma_{\hat{\theta}} \;=\; \mathbb{I} \otimes \boldsymbol{\tau}_1\,, \quad \gamma_{\hat{\phi}} = \mathbb{I} \otimes \boldsymbol{\tau}_2\,, \tag{C.4}$$

where $\boldsymbol{\tau}_a$, $a = 1, 2, 3$, denotes the Pauli sigma matrix and $\boldsymbol{\sigma}_a$ with $a = 0, 1, 2$ denotes the 3 dimensional gamma matrix. Here we choose $\boldsymbol{\sigma}_0 = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ such that $\gamma_{\hat{t}\hat{\rho}\hat{\psi}\hat{\theta}\hat{\phi}} \;=\; \mathrm{i}$ for our convention. The charge conjugation matrix can also be set to

$$\mathcal{C} \;=\; -\mathrm{i}\boldsymbol{\sigma}_2 \otimes \boldsymbol{\tau}_1\,, \tag{C.5}$$

such that the gamma matrix relation (A.9) is satisfied. With this splitting of spinors and gamma matrices, we arrive at the Killing spinor equations for AdS$_3$ and S^2 with radii 2ℓ and ℓ respectively :

$$0 \;=\; \left(\mathcal{D}_\mu \epsilon^i_{AdS_3} + s\frac{1}{4\ell}\boldsymbol{\sigma}_\mu \epsilon^i_{AdS_3}\right) \otimes \epsilon^i_{S^2} \quad , \quad 0 = \epsilon^i_{AdS_3} \otimes \left(\mathcal{D}_\mathrm{m} \epsilon^i_{S^2} + s\frac{1}{2\ell}\boldsymbol{\tau}_3\boldsymbol{\tau}_\mathrm{m}\epsilon^i_{S^2}\right)\,. \tag{C.6}$$

The general solutions of these equations are well known [73], and the solutions are given by

$$\epsilon_{AdS_3} \;=\; \mathrm{e}^{-s\frac{1}{2}\boldsymbol{\sigma}_1\rho}\mathrm{e}^{-s\frac{1}{2}\boldsymbol{\sigma}_0 t}\mathrm{e}^{\frac{1}{2}\boldsymbol{\sigma}_{12}\psi}A\,, \tag{C.7}$$

$$\epsilon_{S^2} \;=\; \mathrm{e}^{-s\mathrm{i}\frac{1}{2}\boldsymbol{\tau}_2\theta}\mathrm{e}^{\mathrm{i}\frac{1}{2}\boldsymbol{\tau}_3\phi}B\,, \tag{C.8}$$

where A and B are constant two-component complex spinors.

Let us write down the Killing spinor explicitly. We set the sign factor $s = 1$, denote the chiral and anti-chiral component of the constant spinors as $A_\pm$ and $B_\pm$, and choose the 3 dimensional gamma matrix representation as

$$\boldsymbol{\sigma}_a \;=\; (-\mathrm{i}\,\boldsymbol{\tau}_3\,, \boldsymbol{\tau}_1\,, \boldsymbol{\tau}_2)\,. \tag{C.9}$$

Then we can rewrite the solutions as

$$\epsilon_{\mathrm{AdS}_3} \;=\; A_+\epsilon^+_{\mathrm{AdS}} + A_-\epsilon^-_{\mathrm{AdS}}\,, \qquad \epsilon_{\mathrm{S}^2} \;=\; B_+\epsilon^+_{\mathrm{S}^2} + B_-\epsilon^-_{\mathrm{S}^2}\,, \tag{C.10}$$

where

$$\epsilon^{+}_{\mathrm{AdS}_3} = \mathrm{e}^{\frac{\mathrm{i}}{2}(t+\psi)}\begin{pmatrix} \cosh\frac{\rho}{2} \\ -\sinh\frac{\rho}{2} \end{pmatrix}, \qquad \epsilon^{-}_{\mathrm{AdS}_3} = \mathrm{e}^{-\frac{\mathrm{i}}{2}(t+\psi)}\begin{pmatrix} -\sinh\frac{\rho}{2} \\ \cosh\frac{\rho}{2} \end{pmatrix}, \tag{C.11}$$

$$\epsilon^{+}_{\mathrm{S}^2} = \mathrm{e}^{\frac{\mathrm{i}}{2}\phi}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad \epsilon^{-}_{\mathrm{S}^2} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\phi}\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{C.12}$$

By direct product of the Killing spinors (C.11) and those of (C.12), we obtain four complex basis of Killing spinors as (4.33), or 8 pairs of symplectic Majorana spinors as in (4.35).

Note that the effect of the different sign s is to flip the sign of both ρ and t in the Killing spinors. We also note that in odd dimensions there are two inequivalent representations of gamma matrix. For instance, we can also choose $\boldsymbol{\sigma}_a = (+\mathrm{i}\boldsymbol{\tau}_3, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ instead of (C.9). Then this is equivalent to the changing the sign of t in the Killing spinors.

Appendix D

Isometries of AdS_3 and S^2

Here we present the explicit coordinate representation for the Killing vectors of global AdS_3 and S^2, as used in (D.5). The global AdS_3 metric in the part of (4.19) has isometries given by the following generators

$$
\begin{aligned}
\ell_- &= \tfrac{1}{2}\left[\tanh\rho\, \mathrm{e}^{-\mathrm{i}(t-\psi)}\partial_t - \coth\rho\, \mathrm{e}^{-\mathrm{i}(t-\psi)}\partial_\psi + \mathrm{i}\mathrm{e}^{-i(t-\psi)}\partial_\rho\right] , \\
\ell_0 &= -\tfrac{\mathrm{i}}{2}(\partial_t - \partial_\psi) , \\
\ell_+ &= -\tfrac{1}{2}\left[\tanh\rho\, \mathrm{e}^{\mathrm{i}(t-\psi)}\partial_t - \coth\rho\, \mathrm{e}^{\mathrm{i}(t-\psi)}\partial_\psi - \mathrm{i}\mathrm{e}^{i(t-\psi)}\partial_\rho\right] , \\
\bar{\ell}_- &= \tfrac{1}{2}\left[\tanh\rho\, \mathrm{e}^{-\mathrm{i}(t+\psi)}\partial_t + \coth\rho\, \mathrm{e}^{-\mathrm{i}(t+\psi)}\partial_\psi + \mathrm{i}\mathrm{e}^{-i(t+\psi)}\partial_\rho\right] , \\
\bar{\ell}_0 &= -\tfrac{\mathrm{i}}{2}(\partial_t + \partial_\psi) , \\
\bar{\ell}_+ &= -\tfrac{1}{2}\left[\tanh\rho\, \mathrm{e}^{\mathrm{i}(t+\psi)}\partial_t + \coth\rho\, \mathrm{e}^{\mathrm{i}(t+\psi)}\partial_\psi - \mathrm{i}\mathrm{e}^{i(t+\psi)}\partial_\rho\right] .
\end{aligned}
\tag{D.1}
$$

They form the $SL(2,R)_L \times SL(2,R)_R$ algebra through the Lie bracket:

$$
\begin{aligned}
[\ell_0\,, \ell_\pm]_{\mathrm{Lie}} &= \pm\ell_\pm\,, \qquad & [\ell_+\,, \ell_-]_{\mathrm{Lie}} &= -2\ell_0\,, \\
\left[\bar{\ell}_0\,, \bar{\ell}_\pm\right]_{\mathrm{Lie}} &= \pm\bar{\ell}_\pm\,, \qquad & \left[\bar{\ell}_+\,, \bar{\ell}_-\right]_{\mathrm{Lie}} &= -2\bar{\ell}_0\,,
\end{aligned}
\tag{D.2}
$$

The S^2 metric in the part of (4.19) has SO(3) generated by

$$
\begin{aligned}
j_1 &= \mathrm{i}(\sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi) , \\
j_2 &= -\mathrm{i}(\cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi) , \\
j_3 &= -\mathrm{i}\partial_\phi ,
\end{aligned}
\tag{D.3}
$$

which satisfy the algebra

$$[j_i\,, j_j]_{\text{Lie}} = \mathrm{i}\epsilon_{ijk} j_k\,. \tag{D.4}$$

In the supersymmetry algebra of AdS$_3\times$S^2 presented in Section 4.3, only right-moving SL$(2,R)_R$ and the SO(3) symmetry generators appear in the bosonic sector. Their representations are given by the combination of the coordinate representation $-\bar{\ell}_0\,, -\bar{\ell}_\pm$ and $-j^{\boldsymbol{a}}$ presented in (D.1), (D.3) with the corresponding local Lorentz transformation given as follows:

$$\begin{aligned}
J^1 &= -j^1 + \tfrac{\mathrm{i}}{2}\delta_M(\lambda_{2\tilde{1}})\,, & J^2 &= -j^2 + \tfrac{\mathrm{i}}{2}\delta_M(\lambda_{1\tilde{1}})\,, & J^3 &= -j^3\,,\\
\bar{L}_+ &= -\bar{\ell}_+ + \tfrac{1}{2}\delta_M(\mathrm{i}\lambda_{4\tilde{1}} + \lambda_{3\tilde{1}})\,, & \bar{L}_- &= -\bar{\ell}_- + \tfrac{1}{2}\delta_M(\mathrm{i}\lambda_{4\tilde{1}} - \lambda_{3\tilde{1}})\,, & \bar{L}_0 &= -\bar{\ell}_0\,.
\end{aligned} \tag{D.5}$$

Here, $\delta_M(\hat{\lambda}_{a\tilde{b}})$ is the local Lorentz transformation in the $\{\overline{\mathcal{Q}}_a\,, \tilde{\overline{\mathcal{Q}}}_b\}$ algebra, as it appears in (4.7), with field dependent parameters $(\lambda_{a\tilde{b}})_{AB}$. [1] On the background (4.19 - 4.22), their values are

$$\begin{aligned}
(\lambda_{1\tilde{1}})_{\hat{\theta}\hat{\phi}} &= 2\tfrac{\sin\phi}{\sin\theta}\,, & (\lambda_{2\tilde{1}})_{\hat{\theta}\hat{\phi}} &= 2\tfrac{\cos\phi}{\sin\theta}\,, & &\\
(\lambda_{3\tilde{1}})_{\hat{t}\hat{\rho}} &= \tfrac{\cos(t+\psi)}{\cosh\rho}\,, & (\lambda_{3\tilde{1}})_{\hat{t}\hat{\psi}} &= -\sin(t+\psi)\,, & (\lambda_{3\tilde{1}})_{\hat{\rho}\hat{\psi}} &= -\tfrac{\cos(t+\psi)}{\sinh\rho}\\
(\lambda_{4\tilde{1}})_{\hat{t}\hat{\rho}} &= \tfrac{\sin(t+\psi)}{\cosh\rho}\,, & (\lambda_{4\tilde{1}})_{\hat{t}\hat{\psi}} &= \cos(t+\psi)\,, & (\lambda_{4\tilde{1}})_{\hat{\rho}\hat{\psi}} &= -\tfrac{\sin(t+\psi)}{\sinh\rho}\,.
\end{aligned} \tag{D.6}$$

[1]The $\delta_M((\lambda_{a\tilde{b}})_{AB})$ acts on a spinor ψ as $\frac{1}{4}(\lambda_{a\tilde{b}})_{AB}\gamma^{AB}\psi$, and on a vector V^A as $(\lambda_{a\tilde{b}})^A{}_B V^B$.

Appendix E

Euclidean 4d supersymmetry and $\mathbf{AdS_2\times\ S^2}$

In this appendix, we present the supersymmetry transformation of the fermions in Euclidean 4d conformal supergravity, following the convention of [45], and setting all fermions to zero. The field content in Euclidean 4d superconformal gravity is given in Table E.1. We also present the Euclidean $\mathrm{AdS}_2\times\ \mathrm{S}^2$ background and its Killing spinors. All fields appearing in this section refer to four-dimensional ones, so we omit the 4d subscripts.

Euclidean 4d supersymmetry transformations

4d Weyl	$e_\mu{}^a$, ψ^i_a, A^D_μ, A^R_μ, $\mathcal{V}_\mu{}^i{}_j$, $T^\pm_{ab}$, $\mathcal{D}$, χ^i_{4d}
4d Vector	$X^\mathcal{I}$, $\overline{X}^\mathcal{I}$, $A^\mathcal{I}_\mu$, $\mathcal{Y}^\mathcal{I}_{ij}$, $\lambda^{\mathcal{I}\,i}$
4d Hyper	$\mathcal{A}_i{}^\alpha$, ζ^α_{4d}
4d SUSY parameters	ϵ^i_{4d}, η^i_{4d}

Table E.1: Independent fields of the supersymmetric multiplets and Q, S-supersymmetry parameters in four-dimensional $\mathcal{N}=2$ conformal supergravity.

The Q and S-supersymmetry transformations of the fermionic fields are

$$\begin{aligned}
\delta\psi^i_\mu &= 2D_\mu\varepsilon^i + \mathrm{i}\frac{1}{16}\gamma_{ab}(T^{ab+}+T^{ab-})\gamma_\mu\varepsilon^i + \gamma_\mu\gamma_5\eta^i\,,\\
\delta\chi^i &= \frac{\mathrm{i}}{24}\gamma_{ab}\not{D}(T^{ab+}+T^{ab-})\varepsilon^i + \frac{1}{6}\widehat{R}(\mathcal{V})^i{}_{j\mu\nu}\gamma^{\mu\nu}\varepsilon^j - \frac{1}{3}\widehat{R}(A^R)_{\mu\nu}\gamma^{\mu\nu}\gamma_5\varepsilon^i\\
&\qquad + \mathcal{D}\,\varepsilon^i + \mathrm{i}\frac{1}{24}(T^+_{ab}+T^-_{ab})\gamma^{ab}\gamma_5\eta^i\,,\\
\delta\lambda^i_+ &= -2\mathrm{i}\gamma^a D_a X\varepsilon^i_- - \tfrac{1}{2}\mathcal{F}_{ab}\gamma^{ab}\varepsilon^i_+ + \mathcal{Y}^{ij}\varepsilon_{jk}\varepsilon^k_+ + 2\mathrm{i}X\eta^i_+\,,\\
\delta\lambda^i_- &= -2\mathrm{i}\gamma^a D_a \overline{X}\varepsilon^i_+ - \tfrac{1}{2}\mathcal{F}_{ab}\gamma^{ab}\varepsilon^i_- + \mathcal{Y}^{ij}\varepsilon_{jk}\varepsilon^k_- - 2\mathrm{i}\overline{X}\eta^i_-\,,\\
\delta\zeta^\alpha &= \not{D}\mathcal{A}_i{}^\alpha\varepsilon^i - \mathcal{A}_i{}^\alpha\gamma_5\eta^i\,,
\end{aligned} \tag{E.1}$$

where:

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} - \left(\tfrac{1}{4}\overline{X}\,T^-_{\mu\nu} + \tfrac{1}{4}X\,T^+_{\mu\nu}\right)\,. \tag{E.2}$$

The covariant derivatives are:

$$D_\mu\varepsilon^i = (\partial_\mu - \frac{1}{4}\omega_{\mu ab}\gamma^{ab} + \frac{1}{2}A^D_\mu + \frac{1}{2}A^R_\mu\gamma_5)\varepsilon^i + \frac{1}{2}\mathcal{V}_\mu{}^i{}_j\varepsilon^j\,, \tag{E.3}$$

$$D_\mu X = (\partial_\mu - A^D_\mu + A^R_\mu)X\,, \tag{E.4}$$

$$D_\mu\overline{X} = (\partial_\mu - A^D_\mu - A^R_\mu)\overline{X}\,, \tag{E.5}$$

$$D_\mu\mathcal{A}_i{}^\alpha = (\partial_\mu\mathcal{A}_i{}^\alpha - b_\mu)\,\mathcal{A}_i{}^\alpha + \tfrac{1}{2}\mathcal{V}_\mu{}^j{}_i\mathcal{A}_j{}^\alpha\,, \tag{E.6}$$

and the curvatures are:

$$\begin{aligned}
\widehat{R}_{\mu\nu}(A^R) &= 2\partial_{[\mu}A^R_{\nu]}\,,\\
\widehat{R}_{\mu\nu}(\mathcal{V})^i{}_j &= 2\partial_{[\mu}\mathcal{V}_{\nu]}{}^i{}_j + \mathcal{V}_{[\mu}{}^i{}_k\mathcal{V}_{\nu]}{}^k{}_j\,.
\end{aligned} \tag{E.7}$$

Supersymmetric AdS$_2\times$S^2 background and Killing spinors

Recall the fully supersymmetric, Euclidean AdS$_2\times$S^2 solution of the 4d theory considered in (6.25):

$$ds^2 = \ell^2\left[d\eta^2 + \sinh^2\eta\, d\chi^2 + d\theta^2 + \sin^2\theta\, d\phi^2\right]\,, \tag{E.8}$$

$$A^{\mathcal{I}} = -\mathrm{i}e^{\mathcal{I}}(\cosh\eta - 1)d\chi - p^{\mathcal{I}}\cos\theta d\phi \tag{E.9}$$

$$X^{\mathcal{I}} = \frac{\omega}{8}(e^{\mathcal{I}} + ip^{\mathcal{I}})\,,\qquad \bar{X}^{\mathcal{I}} = \frac{\bar{\omega}}{8}(e^{\mathcal{I}} - ip^{\mathcal{I}})\,,\quad \mathcal{I} = 0, 1, \cdots, N_{\mathrm{v}} \tag{E.10}$$

$$T^-_{12} = -\mathrm{i}\omega\,,\qquad T^-_{34} = \mathrm{i}\omega\,,\qquad T^+_{12} = -\mathrm{i}\bar{\omega}\,,\qquad T^+_{34} = -\mathrm{i}\bar{\omega}\,. \tag{E.11}$$

Here, ℓ is the radius of AdS$_2$ and S^2, and ω, $\overline{\omega}$ are two independent complex constants satisfying

$$\ell^2 = \frac{16}{\bar{\omega}\omega}\,. \tag{E.12}$$

As discussed in Section 6.2, we may pick the $SO(1,1)_R$ gauge (6.29) such that (E.12) implies the following parametrization:

$$\omega(\alpha) \;=\; \frac{4}{\ell}e^{\mathrm{i}\alpha}\,,\qquad \overline{\omega}(\alpha) \;=\; \frac{4}{\ell}e^{-\mathrm{i}\alpha}\,,\qquad \alpha\in\mathbb{R}\,. \tag{E.13}$$

Here, we choose $\alpha = \pi/2$ and derive the corresponding Killing spinors.

We express the AdS$_2\times$ S^2 metric above in vielbein form:

$$e^1 = \ell\, d\eta\,,\quad e^2 = \ell\sinh\eta\, d\chi\,,\quad e^3 = \ell d\theta\,,\quad e^4 = \ell\sin\theta d\phi\,. \tag{E.14}$$

We also choose the following gamma matrix representation, where $\boldsymbol{\tau}_a$ and σ_a, $a = 1, 2, 3$ are the Pauli matrices

$$\gamma_1 = \boldsymbol{\tau}_1\otimes\sigma_3\,,\quad \gamma_2 = \boldsymbol{\tau}_2\otimes\sigma_3\,,\quad \gamma_3 = \mathbb{I}_2\otimes\sigma_1\,,\quad \gamma_4 = \mathbb{I}_2\otimes\sigma_2\,,\quad \gamma_5 = \gamma_{1234} = -\boldsymbol{\tau}_3\otimes\sigma_3\,. \tag{E.15}$$

With this representation, the four-dimensional Killing spinor equation, given in (E.1) as

$$\mathcal{D}_\mu\varepsilon = -\frac{\mathrm{i}}{32}(T^+_{ab} + T^-_{ab})\gamma_{ab}\gamma_\mu\varepsilon \;=\; -\frac{1}{2\ell}(\mathbb{I}_2\times\sigma_3)\gamma_\mu\varepsilon\,, \tag{E.16}$$

splits into the Killing spinor equations of AdS$_2$ and S^2. Indeed, decomposing the spinor $\varepsilon = \varepsilon_{\mathrm{AdS}_2}\otimes\varepsilon_{\mathrm{S}^2}$, one obtains the AdS$_2$ part as

$$(\partial_\mu + \omega_\mu)\varepsilon_{\mathrm{AdS}_2} \;=\; -\frac{1}{2}\boldsymbol{\tau}_\mu\,\varepsilon_{\mathrm{AdS}_2}\,,\quad \omega_\chi \;=\; -\frac{\mathrm{i}}{2}\cosh\eta\,\boldsymbol{\tau}_3\,, \tag{E.17}$$

and the S^2 as

$$(\partial_\mu + \omega_\mu)\varepsilon_{\mathrm{S}^2} \;=\; -\frac{1}{2}\sigma_3\sigma_\mu\varepsilon_{\mathrm{S}^2}\,, \quad \omega_\phi \;=\; -\frac{\mathrm{i}}{2}\cos\theta\,\sigma_3 \tag{E.18}$$

The Killing spinors for AdS$_2$ and S^2 are given by

$$\varepsilon^+_{\mathrm{AdS}_2} = e^{\frac{\mathrm{i}}{2}\chi}\begin{pmatrix} -\cosh\frac{\eta}{2} \\ \sinh\frac{\eta}{2}\end{pmatrix}, \qquad \varepsilon^-_{\mathrm{AdS}_2} = e^{-\frac{\mathrm{i}}{2}\chi}\begin{pmatrix} \sinh\frac{\eta}{2} \\ -\cosh\frac{\eta}{2}\end{pmatrix}, \tag{E.19}$$

and

$$\varepsilon^+_{\mathrm{S}^2} = e^{\frac{\mathrm{i}}{2}\phi}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\end{pmatrix}, \qquad \varepsilon^-_{\mathrm{S}^2} = e^{-\frac{\mathrm{i}}{2}\phi}\begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\end{pmatrix}. \tag{E.20}$$

Taking the direct product of the spinors (E.17) on AdS$_2$ with the spinors (E.18) on S^2, we obtain the following complex basis of Killing spinors on AdS$_2\times$ S^2:

$$\begin{aligned} \dot{\varepsilon}_+{}^+ &= \sqrt{\tfrac{\ell}{2}}\,\varepsilon^+_{\mathrm{AdS}_2}\otimes\varepsilon^+_{\mathrm{S}^2}\,, \qquad & \dot{\varepsilon}_+{}^- &= \sqrt{\tfrac{\ell}{2}}\,\varepsilon^+_{\mathrm{AdS}_2}\otimes\varepsilon^-_{\mathrm{S}^2}\,, \\ \dot{\varepsilon}_-{}^+ &= \sqrt{\tfrac{\ell}{2}}\,\varepsilon^-_{\mathrm{AdS}_2}\otimes\varepsilon^+_{\mathrm{S}^2}\,, \qquad & \dot{\varepsilon}_-{}^- &= \sqrt{\tfrac{\ell}{2}}\,\varepsilon^-_{\mathrm{AdS}_2}\otimes\varepsilon^-_{\mathrm{S}^2}\,. \end{aligned} \tag{E.21}$$

Note that, these spinors are identical to the Killing spinors on the Kaluza-Klein frame of AdS$_3\times$S^2, given in (6.16). The spinors (E.21) organize themselves to form the following 8 real set of basis for Killing spinors on AdS$_2\times$S^2,

$$\begin{aligned} \dot{\varepsilon}^i_{(1)} &= (-\mathrm{i}\dot{\varepsilon}_+{}^+\,, \dot{\varepsilon}_-{}^-)\,, \qquad & \dot{\varepsilon}^i_{(2)} &= (\dot{\varepsilon}_+{}^+\,, -\mathrm{i}\dot{\varepsilon}_-{}^-)\,, \\ \dot{\varepsilon}^i_{(3)} &= (-\dot{\varepsilon}_-{}^-\,, -\mathrm{i}\dot{\varepsilon}_+{}^+)\,, \qquad & \dot{\varepsilon}^i_{(4)} &= (-\mathrm{i}\dot{\varepsilon}_-{}^-\,, -\dot{\varepsilon}_+{}^+)\,, \\ \dot{\tilde{\varepsilon}}^i_{(1)} &= (\dot{\varepsilon}_+{}^-\,, \mathrm{i}\dot{\varepsilon}_-{}^+)\,, \qquad & \dot{\tilde{\varepsilon}}^i_{(2)} &= (\mathrm{i}\dot{\varepsilon}_+{}^-\,, \dot{\varepsilon}_-{}^+)\,, \\ \dot{\tilde{\varepsilon}}^i_{(3)} &= (-\mathrm{i}\dot{\varepsilon}_-{}^+\,, \dot{\varepsilon}_+{}^-)\,, \qquad & \dot{\tilde{\varepsilon}}^i_{(4)} &= (\dot{\varepsilon}_-{}^+\,, -\mathrm{i}\dot{\varepsilon}_+{}^-)\,, \end{aligned} \tag{E.22}$$

which is the same basis as for the 5d KK-frame (6.15). The spinors in (E.22) satisfy

$$(\varepsilon^i)^\dagger\mathrm{i}\gamma_5 = \epsilon_{ij}(\varepsilon^j)^T C\,. \tag{E.23}$$

which is indeed the reality condition, given in (6.22), for $\alpha = \pi/2$.

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