Singular loci of polyhedral 3-manifolds

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SINGULAR LOCI OF POLYHEDRAL 3-MANIFOLDS

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To the glory of my Lord and Saviour Jesus Christ, and to Eleanor, the wonderful woman he has given me the honour of spending my life with.

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ABSTRACT

A *polyhedral manifold* is a manifold with a metric induced by a constant curvature triangulation. Polyhedral manifolds naturally inherit a Riemannian structure, which is well-defined outside of a subset of codimension at least 2 called the *singular locus.* The fundamental group of the complement to this singular locus has a natural representation called the *holonomy map,* whose image we term the holonomy *group.* The main aim of this thesis is to investigate how restrictions on the holonomy group of a polyhedral 3-manifold relate to properties of its singular locus.

In Chapter 2, we give most of the essential definitions and elementary results used throughout the thesis. These include the precise definitions of a *polyhedral manifold,* the *singular locus,* and *holonomy.*

In Chapter 3, we consider how restrictions on the holonomy group of a polyhedral 3-manifold affect the local and global properties of its singular locus. We study *Euclidean* polyhedral 3-manifolds that are *nonnegatively curved* and *integral,* two conditions motivated by Thurston's work in [Thu98]. In Theorem 1, we classify the 32 isometry types of codimension 3 singularities in such manifolds. We also show, in Theorem 2, that the number of these singularities is bounded.

Lastly, in Chapter 4, we consider the reverse problem: how restrictions on the topology of the singular locus result in constraints on the holonomy group. We study *spherical* polyhedral manifolds homeomorphic to the 3-sphere, and we require that the singular locus form a *Seifert link*—this is a slight generalisation of a torus link. Motivated by Panov's work in [Pan09], we investigate when such polyhedral 3-spheres can be shown to have unitary holonomy. In the case of the Hopf link, the investigation is comprehensive, allowing us in Theorem 3 to show that a polyhedral 3-sphere singular along the Hopf link has a very simple geometric structure in almost all cases. For a more general Seifert link, we impose only a very mild condition on the length of a singular component to show in Theorem 4 that the holonomy is unitary. This allows us to produce useful geometric formulae that apply to almost all polyhedral 3-spheres singular along Seifert links, generalising work of Kolpakov and Mednykh in [KM09], among others.

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Chapter 1

INTRODUCTION

The overarching theme of this thesis is the relationship between the geometry of a *polyhedral 3-manifold* and the topology of its *singular locus*. A *polyhedral n-manifold* is a topological n-manifold endowed with a complete metric induced by a triangulation by simplices of constant curvature—such a metric is called a *polyhedral metric*. The precise definition can be found in Definition 2.1.1.† The *singular locus* of a polyhedral manifold is the set of points having no neighbourhood that embeds isometrically into the relevant model space of constant curvature (see Definition 2.2.1).

The simplest nontrivial example of a polyhedral manifold is the surface of a convex polyhedron endowed with its intrinsic metric. The singular locus of such a polyhedral *surface* is the set of vertices of the polyhedron—these isolated singularities are known as *conical points*. Polyhedra have of course been studied since antiquity, but one of the earliest rigorous results concerning convex polyhedra is Cauchy's rigidity theorem (for which, see [AZ18, Ch. 14]), first formulated by Cauchy in 1813. It states that if two convex polyhedra are combinatorially equivalent and have congruent faces, then they are congruent in $\mathbb{R}^3$. An important fact about convex polyhedra is that the total angle around any vertex—the *conical angle*—is less than 2π. In 1942, Alexandrov vastly strengthened Cauchy's result by showing that any metric space glued from Euclidean triangles, homeomorphic to the 2-sphere, and with conical angles less than 2π can be realised as the surface of a unique convex polyhedron (see [BI08]). The two parts of this result became known as Alexandrov's existence and uniqueness theorems.

Thurston continued the trend of studying Euclidean polyhedral metrics on the 2-sphere with conical angles less than 2π, discovering a complex hyperbolic structure on the moduli space of such metrics in [Thu98]. *Spherical* polyhedral metrics on the 2-sphere have been studied, in increasing generality, by Troyanov in [Tro89], Eremenko in [Ere04], and Mondello and Panov in [MP16]. More recently, Eremenko, Mondello, and Panov have studied the moduli space of spherical polyhedral *tori* with one singular point in [EMP20].

In *three* dimensions, polyhedral manifolds have largely been studied because of their relationship to orbifolds. *Hyperbolic* polyhedral manifolds provide a crucial step in the

†Precise definitions of many of the terms used in this introduction can be found in Chapter 2 and will be referenced when relevant.

proof of Thurston's orbifold theorem, as demonstrated by Cooper, Hodgson, and Kerckhoff in [CHK00] and Boileau, Leeb, and Porti in [BLP05]—in these works, they are referred to as *cone-manifolds.* Indeed, much of the 3-dimensional work seems to be hyperbolic, and this is one of the reasons why we focus on Euclidean and spherical polyhedral 3-manifolds in this thesis.

Returning to the opening sentence of this introduction, we must explain what is meant by the 'geometry' of a polyhedral manifold. The way in which we quantify geometry in this context is by means of the *holonomy group.* This is the image of the *holonomy map,* a representation of the fundamental group of the complement to the singular locus that encapsulates how the metric changes as one moves around the manifold (see Definition 2.3.4). The question at the heart of this thesis is then,

> *How do restrictions on the holonomy group of a polyhedral 3-manifold relate to properties of its singular locus?*

This question is considered in two different ways and in two similar but distinct contexts in Chapters 3 and 4, the two research chapters of this thesis.

* * *

In Chapter 3, we consider one direction of the research question posed above: we impose restrictions on the holonomy group and then deduce resulting properties of the singular locus. We focus our attention on *Euclidean* polyhedral 3-manifolds—in this context, the holonomy group is a subgroup of the Euclidean group E(3). Motivated by Thurston's work in [Thu98], we impose two conditions on the Euclidean polyhedral 3-manifolds we consider: *nonnegative curvature* and *integrality.* We call a Euclidean polyhedral manifold *nonnegatively curved* if the conical angle of any edge of its singular locus is less than 2π (see Definition 2.2.5). A Euclidean polyhedral n-manifold is called *integral* if the orthogonal part of its holonomy group (known as the *monodromy group*) preserves a lattice in $\mathbb{R}^n$ (see Definition 2.3.9).

As will be shown in Proposition 2.2.16, the singular locus of a polyhedral 3-manifold is essentially a union of circles, lines, and graphs embedded in the manifold. The vertices of the graph components are called *singular vertices,* and the two main results of Chapter 3 concern singular vertices in *nonnegatively curved integral polyhedral 3-manifolds.* The first main result is a complete description of what the metric looks like close to any such singular vertex.

Theorem 1 (Classification of singular vertices)**.** *There are 32 possibilities for the local isometry type of a singular vertex in a nonnegatively curved integral polyhedral 3-manifold.*

This result is fleshed out and proven in Section 3.2, and the 32 local isometry types are given in the appendix. The central idea is that the local isometry type of a point in a polyhedral manifold can be encapsulated in the *link* of that point. The *link* of a point in a polyhedral manifold is essentially a small metric sphere about that point, scaled up so as to have curvature 1 (see Definition 2.1.7). In Section 3.2, we show that the link of any point in an integral polyhedral 3-manifold is a *ramified cover* (see Definition 3.2.1) of one of two spherical orbifolds. The nonnegative curvature requirement implies bounds on the order of ramification, which allow us to list all the relevant links algorithmically.

An important outworking of Theorem 1 is Corollary 3.2.13. This tells us that there is a maximal angle ε_0 in $(0, \pi)$ that can be subtended by two paths meeting at any singular vertex. This key property allows us to show that there is a universal bound on the number of singular vertices that can appear in any nonnegatively curved integral polyhedral 3-manifold. This global result about the topology of the singular locus is the apex of Chapter 3 and perhaps the most involved to prove of any result in the thesis.

Theorem 2 (Singular vertex bound)**.** *There is a constant B_{ver} in $\mathbb{N}$ such that* any *nonnegatively curved integral polyhedral 3-manifold has fewer than B_{ver} singular vertices.*

Any point with the maximal angle property mentioned above is called ε_0-*narrow* (see Definition 2.2.14). The majority of Section 3.3 is dedicated to proving Proposition 3.3.7, a result that gives a bound on the number of ε-narrow points in any *nonnegatively curved Alexandrov space* (see Subsection 2.2.2) depending only on ε and the dimension of the space. Theorem 2 then follows from Proposition 3.3.7, since, as we note in Proposition 2.2.9, any nonnegatively curved polyhedral manifold is a nonnegatively curved Alexandrov space. To prove Proposition 3.3.7—and therefore Theorem 2—we show that, in any nonnegatively curved Alexandrov space, any sufficiently large subset of points contains three distinct points forming a triangle with one angle close to π. The proof requires several results from Alexandrov geometry, including the Gromov–Bishop inequality ([BBI01, Thm. 10.6.6]) and Toponogov's theorem ([BBI01, Thm. 10.3.1]).

It should be mentioned at this point that Proposition 3.3.7 actually follows from a recent and very general result of Li and Naber, [LN20, Cor. 1.4]. However, the author did not learn of this until well after the content of Chapter 3 was written. Moreover, the proof of Proposition 3.3.7 presented in Section 3.3 is independent of their work and, due to the more specialised setting of our problem, is shorter than the proof of their result. The relationship to Li and Naber's work is briefly explored in Subsection 3.3.3.

The primary motivation for studying nonnegatively curved integral polyhedral 3-manifolds is Thurston's work in [Thu98]. There, he studies 2-spheres endowed with metrics induced by *equilateral* triangulations in which the degree of a vertex is at most 6. The shape of the triangles ensures that the holonomy group preserves the *Eisenstein lattice*—i.e., the lattice in $\mathbb{R}^2$ generated by $(1, 0)$ and $(1/2, \sqrt{3}/2)$—and the degree condition means that the conical angle of any vertex is at most 2π. Such a 2-sphere is therefore a *nonnegatively curved integral polyhedral surface.* Classifying and bounding the number of conical points on such a surface is almost trivial, as we demonstrate at the beginning of Chapter 3. Solving the analogous problems in *three* dimensions—i.e., proving Theorems 1 and 2—is far more complicated, and this is what we do in Chapter 3.

Another motivation comes from the notion of an *integral affine manifold with singularities,* although we do not explore the connection to these objects in this thesis. These are manifolds that, away from a subset of codimension at least 2, have an atlas with transition functions in affine transformations whose linear parts have integer entries (for more details, see [CBM09, Defs. 3.1 & 3.6]). Integral polyhedral manifolds are examples of these. The regular locus of such a manifold is naturally the base of a torus bundle, and compactifications of these bundles are often Calabi–Yau manifolds (see e.g., [KS01, GS03]). It is a well-known open question, whether or not there are finitely many families of Calabi–Yau threefolds (see [Wil21]). By analogy, this question motivates similar boundedness questions for the singular loci of nonnegatively curved integral polyhedral 3-manifolds, and Theorem 2 answers one such question.

Noncompact polyhedral 3-manifolds with conical angles at most π have been classified by Boileau, Leeb, and Porti in [BLP05, Thm. 4.1]. In the noncompact case, when the conical angles are bounded above by some constant c in $(0, 2\pi)$ and the singular locus is a submanifold, Cooper and Porti explain in [CP08] that they can give an explicit upper bound on the number of singular components, depending only on c. Brief reference is made to monodromy constraints by Porti and Weiss in [PW07, § 2]. But, to the author's knowledge, no classification (such as Theorem 1) or general singularity bound (such as Theorem 2) has been produced when integral monodromy is imposed. These results and their proofs bring together Alexandrov geometry, ramified covering theory, and the classification of crystallographic groups.

* * *

In Chapter 4, we investigate the other direction of the research question posed earlier: we impose restrictions on the topology of the singular locus and then deduce constraints on the holonomy group. Whereas in Chapter 3 we focussed on Euclidean polyhedral metrics, in this chapter, we restrict our attention almost entirely to *spherical* polyhedral metrics. A 3-sphere endowed with a spherical polyhedral metric is called a *polyhedral 3-sphere*—the holonomy group of such a space is a subgroup of SO(4). The restriction we impose on the singular locus is that it must form a *Seifert link* inside the 3-sphere, a slight generalisation of a torus link (see Definition 4.1.3). The central aim of this chapter is to demonstrate that, in the vast majority of cases, a *polyhedral 3-sphere singular along a Seifert link* is a *PK-link* and to consider the geometric implications of this fact. A 3-dimensional *PK-link* is, in a nutshell, a polyhedral 3-sphere whose holonomy group is conjugate to a subgroup of the unitary group U(2), assuming it has no conical angles divisible by 2π (see Definition 2.3.11 and Remark 4.1.15).

The simplest nontrivial example of a Seifert link is the *Hopf link*—any link in the 3-sphere that is equivalent to the union of two fibres of the Hopf map. The first main result of Chapter 4 is an almost complete description of polyhedral 3-spheres singular along the Hopf link.

Theorem 3 (Hopf link singularities)**.** *Let M be a polyhedral 3-sphere with no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$ whose singular locus is the Hopf link. Then M is the link of the product of two Euclidean 2-cones.*

This result may be viewed as a 3-dimensional analogue of a classical result of Troyanov, [Tro89, Thm. I], in which he classifies spherical polyhedral metrics on the 2-sphere with precisely two conical points. Theorem 3 follows from the fact, proven throughout Section 4.3, that any polyhedral 3-sphere satisfying the conditions of the theorem is a PK-link. Proving it from this point requires a careful but fairly short application of the classification of 3-dimensional PK-links (Theorem 4.1.16), which is but a reformulation of various results of Panov in [Pan09].

When no conical angles are divisible by 2π, showing that a polyhedral 3-sphere is a PK-link means showing that its holonomy group may be conjugated into $\mathrm{U}(2)$. The result that is most crucial to Chapter 4 is therefore Proposition 4.2.1, a purely linear-algebraic result stating that a subset of $\mathrm{SO}(4)$ may be conjugated into $\mathrm{U}(2)$ if all of its elements commute with something whose square is nontrivial. It is a classical result that the fundamental group of the Hopf link complement is $\mathbb{Z}^2$. Additionally, recall from earlier that the holonomy group of a polyhedral manifold is the image of the holonomy *map,* a representation of the fundamental group of the complement to its singular locus. All of this means that, to show that a polyhedral 3-sphere singular along the Hopf link is a PK-link, it is sufficient to show that its holonomy group contains an element with nontrivial square, which we do with very few exceptions in Lemma 4.3.2. The rest of Section 4.3 is devoted to dealing with the exceptional cases.

Proposition 4.2.1 can be applied, not only to polyhedral 3-spheres singular along the Hopf link, but to those singular along *any* Seifert link. This is because, as we show in Proposition 4.1.14, the fundamental group of any Seifert link complement has a distinguished central element. Our final main result therefore concerns a more general class of links called *generic* Seifert links. A Seifert link is called *generic* if it is not the unknot or the Hopf link. As was mentioned earlier, Seifert links are a slight generalisation of torus links. In fact, the components of a Seifert link are either unknots or (p, q)-torus knots, for some unique pair of coprime integers $q \geq p \geq 1$. Components of the latter kind are called *ordinary.* The last theorem of the thesis states that, subject to a very mild condition on the length of an ordinary component, almost all polyhedral 3-spheres singular along Seifert links are PK-links.

Theorem 4 (Seifert link singularities)**.** *Let M be a polyhedral 3-sphere with no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$ whose singular locus is a generic Seifert link. Let K be an ordinary component of the singular locus of M. If the length of K is not divisible by π, then M is a PK-link.*

To prove this result, we show that, with the hypotheses of the theorem, the square of the holonomy of the distinguished central element mentioned above is nontrivial. This is possible because a loop representing this central element is, in some sense, 'parallel' to any ordinary component of the singular locus. We show that the holonomy of the central element is an isometry of the unit 3-sphere that displaces a certain point along a great circle by a distance equal to the length of the ordinary component. Since this length is not divisible by π (and great circles have length 2π), this means that the square of the isometry is nontrivial. Thanks to Proposition 4.2.1, Theorem 4 follows.

The concept of a PK-link (short for *polyhedral Kähler link*) originates in Panov's work in [Pan09], where he introduces the notion of a *polyhedral Kähler manifold* and proves several results about them. As the name suggests, a *polyhedral Kähler manifold* is essentially a Euclidean polyhedral $2n$-manifold whose monodromy group (which, we recall, is the orthogonal part of its holonomy group) is conjugate to a subgroup of $\mathrm{U}(n)$. (There is an extra condition when there are conical angles divisible by 2π, but we largely exclude this case here—see Remark 4.1.15 for a brief discussion.)

One of Panov's main results is a classification of 4-dimensional polyhedral Kähler *cones*. These are polyhedral Kähler manifolds homeomorphic to $\mathbb{R}^4$ that provide local models for the singularities of arbitrary polyhedral Kähler 4-manifolds (see [Pan09, § 1.1]). All the geometry of such a polyhedral cone (see Definition 2.1.6) can be captured by the unit sphere about its tip—this is called the *link* of the cone (see Definition 2.1.7). The strict definition of a PK-link is therefore a polyhedral $(2n-1)$-sphere that is the link of a polyhedral Kähler $2n$-cone. Assuming that no conical angles are divisible by 2π, this is equivalent to the definition given earlier. One outworking of Panov's classification is that the singular locus of a 3-dimensional PK-link must be either empty or a Seifert link. In accordance with the overarching theme of this thesis, the primary motivation for Chapter 4 is the desire for a converse to this observation. Significant headway to such a converse is made in Theorems 3 and 4.

It has already been mentioned that Troyanov's result [Tro89, Thm. I] concerning spheres with two conical points is a motivation for Theorem 3, which can be seen as a 3-dimensional generalisation thereof. Significant motivation for Theorem 4 can be found in the work of Kolpakov and Mednykh in [KM09], of Kolpakov in [Kol13], and of Derevnin, Mednykh, and Mulazzani in [DMM14]. These works all contain geometric formulae for polyhedral 3-spheres singular along certain special torus links, which, for example, express the length of the singular components in terms of the conical angles. Theorem 4 allows us to generalise and unify these formulae (under the mild assumptions required in the theorem). The author knows of no geometric description of polyhedral 3-spheres singular along torus links more comprehensive than that which can be deduced from Theorems 3 and 4.

Chapter 2

POLYHEDRAL MANIFOLDS IN GENERAL

This chapter contains the precise definitions of all the key objects of study in this thesis, some examples to aid understanding, and a few important results that are used in later chapters. In Section 2.1, we give the definition of a polyhedral manifold, and in Subsection 2.1.1, we define spaces that model them locally. Section 2.2 contains material relevant to the singular loci of polyhedral manifolds. Subsection 2.2.1 contains the definitions of *codimension* and *conical angle,* and the *Gauss–Bonnet formula* for polyhedral surfaces. In Subsection 2.2.2, we make a brief detour into Alexandrov geometry, noting in particular that polyhedral manifolds with conical angles at most 2π are Alexandrov spaces with curvature bounded below, a fact that is central to Chapter 3. And in Subsection 2.2.3, we focus on the structure of singular loci in three dimensions, which is where most of the thesis is spent. Finally, in Section 2.3, we define some important geometric and algebraic objects, including the *developing map, holonomy,* and *monodromy,* and conclude in Subsection 2.3.1 by defining the precise holonomy restrictions that we consider in Chapters 3 and 4.

2.1. Polyhedral Manifolds and Local Models

The essential definition of a polyhedral manifold was given in Chapter 1—a manifold with a metric induced by a triangulation of constant curvature. However, the precise definition is not consistent throughout the literature. For example, in [AKP19, Def. 3.4.1], Alexander, Kapovitch, and Petrunin require a finite triangulation; whereas in [BBI01, Def. 3.2.4], Burago, Burago, and Ivanov do not even require the triangulation to be *locally* finite. Many authors use the term *cone-manifold,* such as Cooper, Hodgson, and Kerckhoff in [CHK00]. Our definition is most in line with de Borbon and Panov's in [dBP21, § 6.1.2]. We refer the reader to any of the above works for a good introduction to the theory.

The basic building blocks of a polyhedral manifold are simplices, living inside a model space of constant curvature. One can formulate a definition in full generality, allowing the curvature to be any real number, as is done in [BLP05, § 3], for example. However, in this thesis, we focus almost exclusively on the cases when the curvature is 0 or 1—i.e., when the polyhedral manifold is *Euclidean* or *spherical.* A Euclidean n-simplex is easy to define—it is the convex hull of $n+1$ points in $\mathbb{R}^n$ in general position, endowed with its intrinsic metric. Defining a *spherical* simplex takes a little more work.

Endow $\mathbb{S}^n$ with its intrinsic metric.[†] Any pair of points in $\mathbb{S}^n$ is joined by a shortest path whose length equals the distance between them. The only situation in which this path is not unique is when the points are antipodal, in which case there is an infinite family of shortest paths joining the two points, one for each point on the equator $\mathbb{S}^{n-1}$. The *convex hull* of a subset S of $\mathbb{S}^n$ is the smallest subset of $\mathbb{S}^n$ that contains S and is closed under taking shortest paths between points. This means that the convex hull of a pair of antipodal points is all of $\mathbb{S}^n$. In fact, the convex hull of S is *not* all of $\mathbb{S}^n$ if and only S is contained in an open hemisphere of $\mathbb{S}^n$. We say that a finite number of points in $\mathbb{S}^n$ are *in general position* if their convex hull has nonempty interior and they lie within an open hemisphere. It now makes sense to define a *spherical* n-simplex as the convex hull of $n+1$ points in $\mathbb{S}^n$ in general position, endowed with its intrinsic metric.

We now give the definition of a polyhedral manifold. The definition, while fairly technical, tries to abstract the notion of the surface metric of a polyhedron or polytope without requiring it to be embedded in any ambient space. The definition follows [dBP21, Def. 6.5], which is itself a variant of [AKP19, Def. 3.4.1] allowing locally finite triangulations.

Definition 2.1.1 (Polyhedral manifold)**.** Let M be a complete length space that is homeomorphic to a topological n-manifold (see [BBI01, Ch. 2] or [AKP19, § 1.4] for an introduction to length spaces). Suppose that M admits a locally finite triangulation in which each n-simplex is isometric to a Euclidean (or spherical) n-simplex. We then call M a *Euclidean* (or *spherical,* respectively) *polyhedral n-manifold*, and we refer to the metric on M as a *Euclidean* (or *spherical*, respectively) *polyhedral metric*. The aforementioned triangulation is called a *geometric triangulation* of M.

This definition implies that a polyhedral manifold is connected and *locally* compact, but not necessarily compact. The Hopf–Rinow theorem ([AKP19, Thm. 1.4.6]) states that a complete, locally compact length space is proper, and [AKP19, Prop. 1.4.5] states that a proper length space is *geodesic*—i.e., that any two points can be joined by a path whose length equals the distance between them (a *shortest path*). Therefore, a polyhedral manifold is a geodesic metric space. Let us now elucidate this definition by considering some examples.

Example 2.1.2. Let P be a Euclidean or spherical n-polytope—i.e., a subset of $\mathbb{R}^n$ or $\mathbb{S}^n$ that is homeomorphic to the closed n-ball and whose boundary is made up of a finite number of totally geodesic pieces. We can form the *double* DP of P by taking two identical copies of P and gluing them along their boundaries. The result is a Euclidean or spherical polyhedral metric on S^n (see Figure 2.1). When n is greater than 1, we can

[†]The unadorned symbol $\mathbb{S}^n$ will always be used to denote the *unit n-sphere endowed with its intrinsic metric,* while S^n will simply denote the underlying topological space.

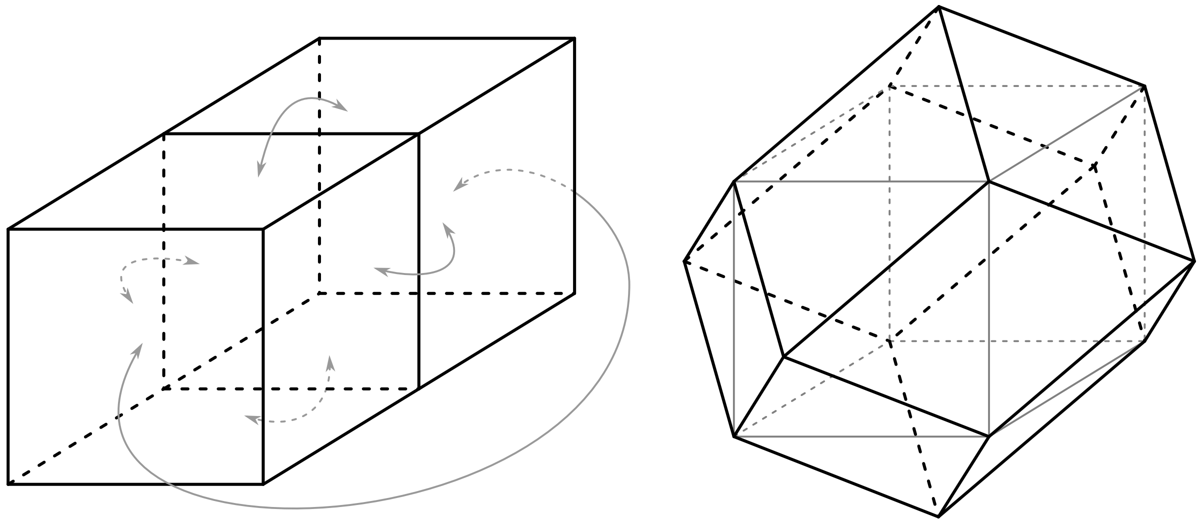


Figure 2.1. Two ways of presenting the double of a cube. Both figures represent solid polyhedra in $\mathbb{R}^3$. On the left, grey arrows denote face identifications. On the right is a *rhombic dodecahedron*, with face identifications performed by folding along the grey lines. Both give isometric polyhedral metrics on S^3.

also consider the intrinsic metric on the *boundary* ∂P of P—this is a polyhedral metric on S^{n-1}. In both of these examples, we are not concerned with how to subdivide the polytopes into simplices, as this will not affect the resulting polyhedral metric.

Example 2.1.3. More generally, any manifold obtained from a polytope by identifying its sides isometrically in pairs is a polyhedral manifold. Taking a parallelopiped in $\mathbb{R}^n$ and identifying opposite faces by translations yields a familiar polyhedral metric on T^n.

Example 2.1.4. Perhaps bizarrely, a polytope itself is in fact *not* a polyhedral manifold, as it has a boundary. However, many geometers have historically been interested in polyhedra because of their surface metrics, as evidenced by Cauchy's rigidity theorem (see [AZ18, Ch. 14]), Alexandrov's uniqueness and existence theorems in [Ale05], and Thurston's work in [Thu98]. If we view the geometry of a polytope as encapsulated in its boundary, then we can think of it as a polyhedral manifold, as explained in Example 2.1.2.

Example 2.1.5. The model spaces $\mathbb{R}^n$ and $\mathbb{S}^n$ are themselves polyhedral manifolds. This is because they can be decomposed into geodesic simplices. Circles of any length with their intrinsic metrics, as well as the real line $\mathbb{R}$, are both Euclidean and spherical polyhedral 1-manifolds. These are all the 1-dimensional examples. The Euclidean and spherical cases only overlap in dimension 1.

One reason for allowing Definition 2.1.1 to include *noncompact* spaces, such as $\mathbb{R}^n$, is the interesting work in that setting by Boileau, Leeb, and Porti in [BLP05, § 4] and by Cooper and Porti in [CP08]. Another key reason is that we wish to include an important class of spaces called *polyhedral cones*, which we define next.

2.1.1. Cones and Links

Polyhedral cones are defined in essentially the same way as polyhedral manifolds, except that the building blocks are simplicial cones rather than simplices. A *simplicial n-cone* in $\mathbb{R}^n$ (or $\mathbb{S}^n$) is constructed by taking a Euclidean (or spherical, respectively) n-simplex σ, choosing a vertex v of σ, and then taking the union of all shortest paths in $\mathbb{R}^n$ (or $\mathbb{S}^n$, respectively) that start at v and pass through another point of σ. The vertex v is called the *tip* of the simplicial cone. In $\mathbb{R}^n$, this results in an unbounded set, whereas in $\mathbb{S}^n$, the shortest paths all converge at the point antipodal to v, and we end up with a bounded set (see Figure 2.2).

Definition 2.1.6 (Polyhedral cone)**.** Let C be a complete length space that is homeomorphic to $\mathbb{R}^n$ (or S^n). Suppose that C admits a finite decomposition into simplicial n-cones in $\mathbb{R}^n$ (or $\mathbb{S}^n$, respectively) glued along their totally geodesic boundary pieces by isometries. We then call C a *Euclidean* (or *spherical*, respectively) *polyhedral n-cone*. The tips of all the simplicial cones are identified into one point, which is also called the tip of C.

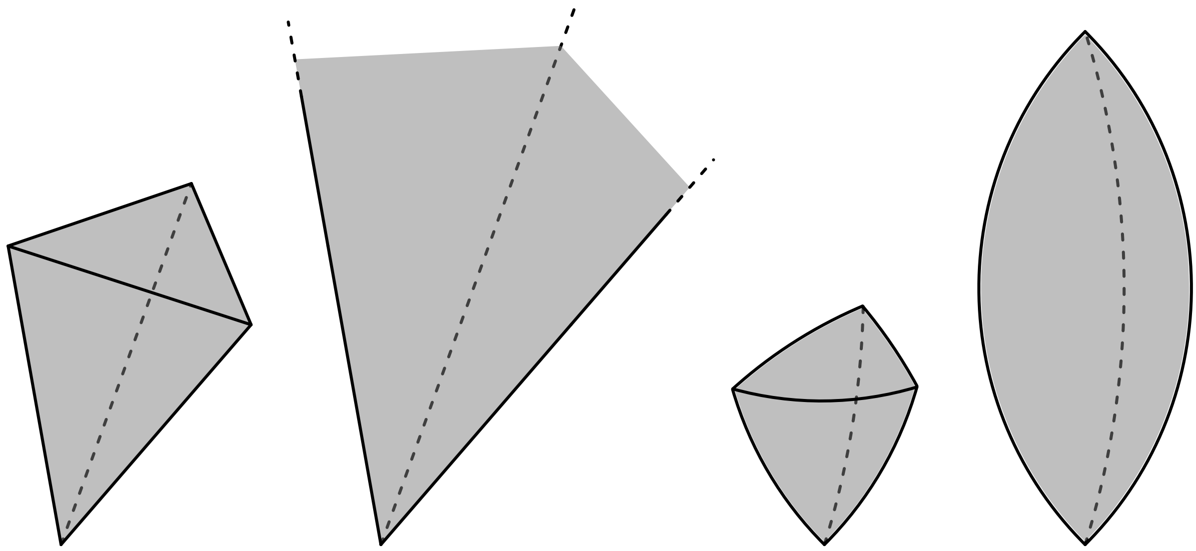


Figure 2.2. Simplices and simplicial cones in $\mathbb{R}^3$ (on the left) and $\mathbb{S}^3$ (on the right). The Euclidean cone is unbounded and thus only partially displayed, while the spherical cone is shown in full. The edges of the spherical cone are shortest paths between antipodal points of $\mathbb{S}^3$ and thus have length π.

Polyhedral cones are important because they locally model polyhedral manifolds, as we now show. Let M be a polyhedral n-manifold and x a point in M. Choose a geometric triangulation of M for which x is a vertex (this can always be done by taking *any* geometric triangulation and then subdividing the simplices to which x belongs, if x is not already a vertex). By extending the n-simplices to which x belongs to simplicial cones, we obtain a polyhedral n-cone of which x is the tip. This cone naturally has the following property: there is a small ball about its tip that is isometric to a ball of the same radius about x in M, the tip being sent to x. In other words, it *locally models M at x*. We summarise this in the following definition (see [dBP21, § 6.2]):

Definition 2.1.7 (Tangent cone and link)**.** Given a point x in a polyhedral manifold M, the polyhedral cone that locally models M at x, as described above, is called the *tangent cone* of M at x, denoted by T_xM. Since T_xM is a cone, all of its geometry is captured by a sphere about its tip. We choose the radius of this sphere to be 1 or $\pi/2$, depending on whether M is Euclidean or spherical respectively, and equip it with its intrinsic metric. This ensures it is a spherical polyhedral manifold. It is referred to as the *link* of M at x, denoted by Σ_xM. If M has dimension n, then Σ_xM is homeomorphic to S^{n-1}. One can recover T_xM from Σ_xM by taking a metric cone of curvature 0 or 1 over Σ_xM—see [BLP05, § 3.1] for an explanation of this.

Remark 2.1.8. It may seem that the definition of T_xM depends on an arbitrary choice of geometric triangulation for M. However, the fact that a small ball about its tip is pointed isometric to a small ball about x in M shows this not to be the case—the tangent cone is a local isometry invariant of M.

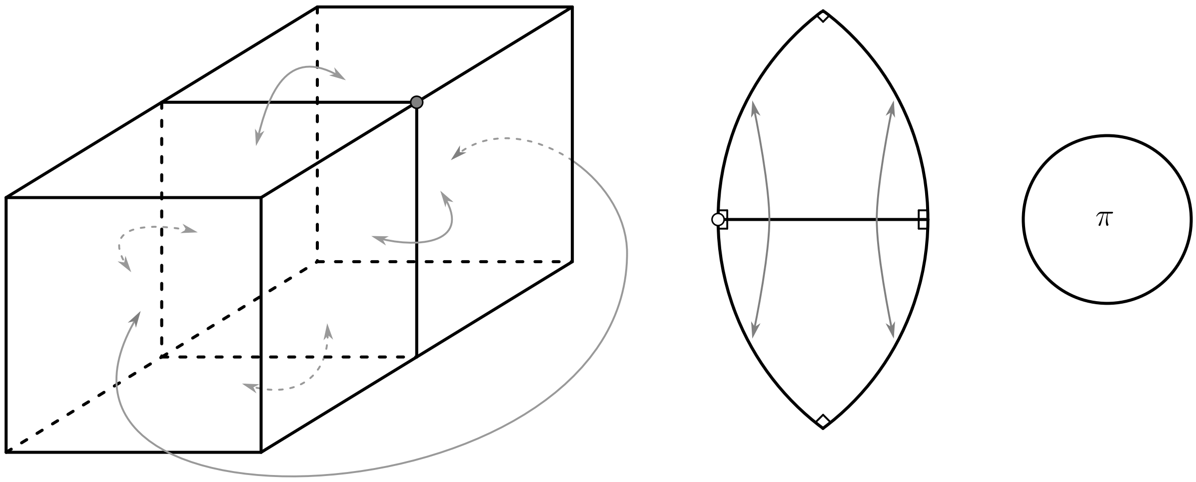


Figure 2.3. Taking iterated links. Grey arrows denote face or side identifications. On the left is the double of a cube. Taking the link at the grey vertex gives the middle figure, the double of a spherical triangle with three right angles. Taking the link at the white vertex gives the rightmost figure, a circle of length π.

Example 2.1.9. If M is the double of a polyhedron of which x is a vertex, then Σ_xM is the double of a spherical polygon. If M is the double of a polygon of which x is a vertex, then Σ_xM is a circle whose length is twice the internal angle of the polygon at x. A concrete example is shown in Figure 2.3.

Remark 2.1.10. In [BLP05, Def. 3.1], there is an alternative definition of polyhedral manifold (or *cone manifold*, to use their terminology). In summary, Boileau, Leeb, and Porti recursively define a cone manifold as a space that is locally modelled on metric cones over spherical cone manifolds one dimension down. The construction of the tangent cone in Definition 2.1.7 shows that our definition of polyhedral manifold satisfies their definition of cone manifold (provided we require that no *conical angle* exceed 2π—this will be explained in Definition 2.2.5). In fact, the reverse is also true: Boileau, Leeb, and

Porti's cone manifolds are polyhedral manifolds (provided we require that they have no boundary), as shown by Lebedeva and Petrunin in [LP15].†

2.2. Singular Loci and Local Properties

In this section, we define the singular locus of a polyhedral manifold and consider some basic properties of it. Fix a geometric triangulation of a polyhedral n-manifold M. For any point x lying in the interior of an n-simplex, the link $\Sigma_x M$ is isometric to $\mathbb{S}^{n-1}$. Because M is a manifold, an $(n-1)$-simplex belongs to precisely two n-simplices, and so $\Sigma_x M$ is still isometric to $\mathbb{S}^{n-1}$ when x lies in the interior of an $(n-1)$-simplex. This may cease to hold when x lies in a simplex of any codimension greater than 1. This gives rise to the notion of *singularity,* which we now define.

Definition 2.2.1 (Singular locus)**.** A point x in a polyhedral n-manifold M is called *singular* if $\Sigma_x M$ is *not* isometric to $\mathbb{S}^{n-1}$, and *regular* if it is. The *singular locus* of M is the set of all singular points of M, denoted by M_{s}, while $M \setminus M_{\mathrm{s}}$ is termed the *regular locus* of M. The above discussion shows that M_{s} must live in codimension 2 or higher. We will see in Lemma 2.2.8 that M_{s} is in fact a union of codimension 2 simplices.

Example 2.2.2. Continuing with the examples presented in Figure 2.3, the singular locus of the double of a cube is the union of the (closed) edges of the cube—its 'wireframe'. The singular locus of the double of a polygon P is the union of the vertices of the polygon. In general, if P is an n-polytope, then DP_{s} is the union of the (closed) codimension 2 facets of P (recall that DP denotes the double of P, as defined in Example 2.1.2).

Example 2.2.3. If P is an n-polytope, then ∂P_{s} is the union of the (closed) codimension 3 facets of P. For example, the singular locus of the boundary of a cube is the union of its vertices.

2.2.1. Local Invariants of Singularities

In Example 2.2.2, it becomes apparent that there are different levels of singularity. If x lies on the interior of an edge of the cube, then the tangent cone at x is the product of $\mathbb{R}$ with a 2-cone. On the other hand, if y is a vertex of the cube, then the tangent cone at y admits no such decomposition. We say that x is a *codimension 2* singularity, while y has *codimension 3.* This motivates the following definition (see [dBP21, Def. 6.11]).

Definition 2.2.4 (Codimension)**.** Let M be a polyhedral n-manifold and x a singular point of M. Let C_x be the Euclidean cone over the link $\Sigma_x M$ (if M is Euclidean then C_x is just $T_x M$). Let k be the smallest integer between 2 and n for which there is an

†Their main result only characterises *Euclidean* polyhedral manifolds with *finite* triangulations, but they comment in Section 5 that their proofs generalise with minimal change to the spherical and locally finite cases.

isometric decomposition $C_x \cong \mathbb{R}^{n-k} \times C^k$, where C^k is a Euclidean polyhedral k-cone. We then say that x is a *codimension* k singularity.

The set of all singularities of codimension at least k in M is a union of closed simplices of codimension at least k of any geometric triangulation of M (see [dBP21, Lem. 6.14 (1)] and its proof). However, it is not the case that a point lying on some codimension k simplex of M will be a codimension k singularity. This can be seen in Figure 2.1: none of the black edges of the rhombic dodecahedron end up contributing any singular points. This is because the sum of the dihedral angles—the *conical angle*—of any group of edges that get glued together is 2π. We now define this notion more precisely. Denote by $\mathbb{S}^1(\alpha)$ the circle of length α and by $C^2(\alpha)$ the Euclidean cone over $\mathbb{S}^1(\alpha)$—the *2-cone of angle* α.

Definition 2.2.5 (Conical angle)**.** Let σ be a codimension 2 simplex of a geometric triangulation of a polyhedral n-manifold M and let x be any interior point of σ. The Euclidean cone C_x defined in Definition 2.2.4 can be written as $\mathbb{R}^{n-2} \times C^2(\alpha)$, for some $\alpha \in (0, \infty)$ depending only on σ and not on the specific point x. This value α is called the *conical angle* of σ. Of course, if x is regular then α is 2π and $C^2(\alpha)$ is isometric to $\mathbb{R}^2$. If M is Euclidean and none of its conical angles exceed 2π, we say M is *nonnegatively curved*.

Example 2.2.6. The double or boundary of a *convex* Euclidean polytope is a nonnegatively curved polyhedral manifold.

A more concrete way of looking at the conical angle is as follows. Any codimension 2 simplex σ belongs to a finite number of top-dimensional simplices $\Delta_1, \ldots, \Delta_m$, and it has dihedral angles $\alpha_1, \ldots, \alpha_m$ associated to each of them. The conical angle of σ is then the sum $\alpha_1 + \ldots + \alpha_m$. In the example shown in Figure 2.3, the conical angle of any edge of the cube is π.

The conical angles of a polyhedral *surface* must satisfy the following well-known formula (see [CHK00, Thm. 3.15]). This formula is used in Chapter 3 when studying local properties of polyhedral metrics in three dimensions.

Theorem 2.2.7 (Gauss–Bonnet formula)**.** *Let $\alpha_1, \ldots, \alpha_k$ be the conical angles of a compact polyhedral surface S that differ from 2π and let A_S be its total area. Then*

$$KA_S + \sum_{i=1}^{k} (2\pi - \alpha_i) = 2\pi\chi(S), \tag{2.2.1}$$

where K is 0 or 1 if S is Euclidean or spherical respectively.

The notion of conical angle allows us to give the following alternative characterisation of the singular locus:

Lemma 2.2.8. *Let M be a polyhedral manifold of dimension at least 2. The singular locus of M is the union of all the (closed) codimension 2 simplices of M whose conical angles differ from 2π, in any geometric triangulation.*

Proof. Fix a geometric triangulation of M. We proceed by induction on the dimension n of M. First, suppose that $n = 2$ and let x be a point in M. If x lies in the interior of a 2-simplex, then $\Sigma_x M$ is $\mathbb{S}^1$ by definition. If x lies in the interior of a 1-simplex, then a small metric disc around x in M is glued from two half-discs of the same radius in $\mathbb{R}^2$ or $\mathbb{S}^2$, and therefore $\Sigma_x M$ is still isometric to $\mathbb{S}^1$. Finally, if x is a vertex of the triangulation, then $\Sigma_x M$ is $\mathbb{S}^1(\alpha)$, where α is the conical angle of the vertex. This is isometric to $\mathbb{S}^1$ if and only if α is 2π. Thus, the result holds for $n = 2$.

Suppose now that the result holds for some $n = m$ and let M have dimension $m + 1$. Firstly, let x be a singular point of M. Then $\Sigma_x M$ is a spherical polyhedral m-manifold with nontrivial singular locus. By the induction hypothesis, therefore, $(\Sigma_x M)_{\mathrm{s}}$ contains a spherical $(m-2)$-simplex of conical angle $\alpha \neq 2\pi$. This contributes a simplicial $(m-1)$-cone of conical angle α to $T_x M$, and thus x lies in an $(m-1)$-simplex of conical angle $\alpha \neq 2\pi$.

Conversely, suppose that x lies in some $(m-1)$-simplex of conical angle $\alpha \neq 2\pi$. By reversing the argument above, we see that $\Sigma_x M$ contains a corresponding spherical $(m-2)$-simplex of conical angle α. Thus, by the induction hypothesis, $(\Sigma_x M)_{\mathrm{s}}$ is nonempty, so $\Sigma_x M$ is not isometric to $\mathbb{S}^n$, and so x is a singular point of M. □

2.2.2. Relationship to Alexandrov Geometry

In Definition 2.2.5, we called a Euclidean polyhedral manifold *nonnegatively curved* if all its conical angles are at most 2π. One reason for this is that Euclidean polyhedral manifolds with conical angles at most 2π can be seen as discrete analogues of Riemannian manifolds of nonnegative sectional curvature, as is fleshed out in [LMPS15] and [Pet03]. A related reason is that they turn out to be *nonnegatively curved Alexandrov spaces.*

A thorough introduction to the theory of Alexandrov spaces with lower curvature bounds is given in [BBI01, Chs. 4 & 10]. Several definitions are given there—[BBI01, Defs. 4.1.2, 4.1.9 & 4.1.15]—all of which are shown to be equivalent in [BBI01, Thm. 4.3.5]. We briefly recall the definition most useful for our purposes—the *"angle condition"* ([BBI01, Def. 4.1.15]), modified to allow nonzero curvature bounds.

For any real number K, let $\mathbb{M}^2_K$ be the *model K-plane*—i.e., $\mathbb{R}^2$ when K is 0, the sphere of radius $K^{-1/2}$ endowed with its intrinsic metric when K is positive, or the hyperbolic plane scaled by $(-K)^{-1/2}$ when K is negative. Let X be a complete length space and a, b, and c points in X. A *triangle* $\triangle abc$ is the union of shortest paths $[ab]$, $[bc]$, and $[ac]$, and a *comparison triangle* in $\mathbb{M}^2_K$ for $\triangle abc$ is a triangle $\triangle \tilde{a}\tilde{b}\tilde{c}$ with the same side lengths as $\triangle abc$—i.e., with $|\tilde{a}\tilde{b}| = |ab|$, $|\tilde{b}\tilde{c}| = |bc|$, and $|\tilde{a}\tilde{c}| = |ac|$. Comparison triangles always exist (provided the original triangle is sufficiently small in the case when K is positive) and are unique up to congruence.

A triangle $\triangle abc$ in X is said to be *K-fat* if its angles are at least as big as the corresponding angles of the comparison triangle in $\mathbb{M}^2_K$—i.e., if $\measuredangle abc \geq \measuredangle \tilde{a}\tilde{b}\tilde{c}$, $\measuredangle bca \geq \measuredangle \tilde{b}\tilde{c}\tilde{a}$,

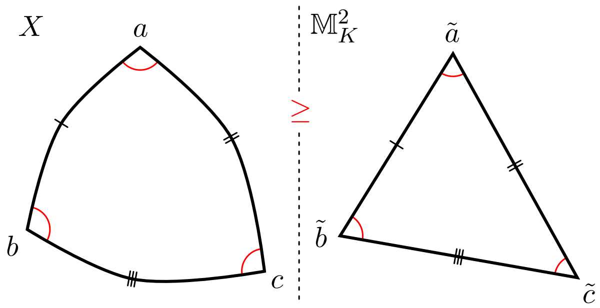


FIGURE 2.4. Angle comparison.

and $\measuredangle bac \geq \measuredangle \tilde{b}\tilde{a}\tilde{c}$, where $\measuredangle abc$ is the angle subtended at b by $[ab]$ and $[bc]$ etc. We say that X is an *Alexandrov space of curvature at least* K if every point of X has a neighbourhood in which every triangle is K-fat, and if additionally, for any points p, q, and s in X and r an interior point of $[pq]$, we have $\measuredangle prs + \measuredangle srq = \pi$. (The second condition is rather technical and will not be brought into question for any of the spaces we will consider.) We now state the result that relates Alexandrov spaces to our work (see [BLP05, Prop. 3.3 f.] for a summary of the proof).

Proposition 2.2.9. *A Euclidean or spherical polyhedral manifold of dimension at least 2 and with conical angles at most 2π is an Alexandrov space of curvature at least 0 or 1 respectively.*

This means that nonnegatively curved polyhedral manifolds of dimension at least 2 are nonnegatively curved Alexandrov spaces, a fact that is central to Chapter 3. There are two classical results from Alexandrov geometry that we use in Chapter 3: Toponogov's theorem ([BBI01, Thm. 10.3.1]) and the Gromov–Bishop inequality ([BBI01, Thm. 10.6.6]). The first we may now state easily using the terminology we have already established. The statement we give is more explicit than [BBI01, Thm. 10.3.1] but nonetheless equivalent to it.

Theorem 2.2.10 (Toponogov)**.** Every *triangle in an Alexandrov space of curvature at least K is K-fat.*

The second result concerns ratios of *volumes* of metric balls in nonnegatively curved Alexandrov spaces and so requires a little more set-up.

Definition 2.2.11 (Volume and area)**.** Let X be a metric space whose Hausdorff dimension is an integer $n \geq 3$ and let S be a (Hausdorff measurable) subset of X. The *volume* of S, denoted by $\operatorname{Vol} S$, is the n-dimensional Hausdorff measure of S, while the *area* of S is its $(n-1)$-dimensional Hausdorff measure. Regardless of the dimension of the ambient space, we will always refer to the 1 and 2-dimensional Hausdorff measures of a subset as its *length* and *area* respectively. If there may be any confusion, the dimension will be specified. Finally, as a shorthand, for $p \in X$ and $r > 0$, we use $V(p,r)$ to denote $\operatorname{Vol}(B(p,r))$.

We now give the Gromov–Bishop inequality in the form most useful for our purposes in Chapter 3. It essentially states that the volume of a metric ball in a nonnegatively curved Alexandrov space grows no faster than the volume of a Euclidean ball of the same radius. The form we give here is weaker than [BBI01, Thm. 10.6.6], but it is sufficient for our needs.

Theorem 2.2.12 (Gromov–Bishop inequality)**.** *Let p be a point in an n-dimensional nonnegatively curved Alexandrov space and let $R \geq r > 0$. Then*

$$\frac{V(p,R)}{V(p,r)} \leq \frac{R^n}{r^n}$$

Remark 2.2.13. In order to deduce the result just given from [BBI01, Thm. 10.6.6], the Alexandrov space in question needs to be locally compact, and in order for the volume to be well-defined, the Hausdorff dimension needs to be an integer. These problems are resolved by [BBI01, Thms. 10.8.1 & 10.8.2], which state (respectively) that all finite-dimensional Alexandrov spaces are locally compact and that the Hausdorff dimension of any Alexandrov space is either an integer or infinity.

We end this subsection by defining the notion of *ε-narrowness,* a concept central to the proof of Theorem 2. In preparation for this, we recall the following from [BBI01, § 9.1.8]. Given a point x in an Alexandrov space X of curvature bounded below, the *space of directions* of X at x is the completion of the space of equivalence classes of shortest paths emanating from x, where two paths are equivalent if their angle at x is 0. The metric is defined by taking angles at x, which means that the diameter of the space of directions is at most π. When X is a polyhedral manifold of dimension at least 2 and with conical angles at most 2π, the space of directions is the same as the link. This is because the link is simply the space of directions equipped with its *intrinsic* metric rather than its *angular* metric, and these metrics coincide for complete, intrinsic, finite-dimensional Alexandrov spaces with curvature bounded below (see [Hal00, Rem. 2.4 & Prop. 2.5 (a)]).

Definition 2.2.14 (ε-narrow)**.** Given $\varepsilon \in (0,\pi)$, we say that a point x in an Alexandrov space of curvature bounded below is *ε-narrow* if the space of directions at x has diameter at most ε. This is the same as saying that any angle subtended at x is at most ε.

2.2.3. Singular Loci in Three Dimensions

To finish this section, we focus on the singular loci of polyhedral 3-manifolds, as they are the objects of interest in the remaining chapters of the thesis. Our aim is understand the basic geometry and topology of the singular locus in three dimensions, in order to provide a framework for the later chapters. For the sake of simplicity, we will assume that no conical angles lie in $2\pi(\mathbb{N} \setminus \{1\})$.

Let M be a polyhedral 3-manifold, Euclidean or spherical. By Lemma 2.2.8, the singular locus of M is the union of all 1-simplices whose conical angles differ from 2π. The possible codimensions for a singular point x of M are 2 and 3, and the value may be determined by looking at the link $\Sigma_x M$. This is a spherical polyhedral manifold homeomorphic to the 2-sphere—such a space will henceforth be known as a *sphere with conical points,* the conical points being the singular points. Let us define a very simple type of sphere with conical points.

Definition 2.2.15. A closed region in $\mathbb{S}^2$ bounded by two great semicircles is called a *lune.* A lune is determined up to congruence by the angle subtended at its two antipodal vertices. If we take a finite collection of lunes of angles $\alpha_1, \ldots, \alpha_m$ and glue their sides together in pairs, we end up with a sphere with two conical points of the same angle $\alpha = \alpha_1 + \ldots + \alpha_m$. This space is called a *spherical football* of angle α, denoted by $\mathbb{S}^2(\alpha)$.

If x lies on a singular edge of conical angle α, then $\Sigma_x M$ is isometric to $\mathbb{S}^2(\alpha)$. If x lies at a vertex of the triangulation, then the conical points of $\Sigma_x M$ correspond to the singular 1-simplices incident to x, the conical angle of the 1-simplex being the same as the angle of the corresponding conical point of $\Sigma_x M$. Let us consider how the singular locus looks near x depending on the number of conical points of $\Sigma_x M$.

- If $\Sigma_x M$ has no conical points, then x is regular.
- It is impossible for $\Sigma_x M$ to have precisely one conical point—this is proven in Remark 2.3.8.
- If $\Sigma_x M$ has two conical points, then by [Tro89, Thm. I], it must be a spherical football of some angle α. This means that x has codimension 2 and is therefore metrically indistinguishable from an interior point of a 1-simplex of conical angle α.
- Finally, if $\Sigma_x M$ has three or more conical points, then x has codimension 3 and lies at the intersection of three or more singular 1-simplices.

These observations allow us to give the following structural result concerning the singular locus of a polyhedral 3-manifold. The result is classical but hard to find stated in full anywhere. It is a generalisation of Boileau, Leeb, and Porti's result [BLP05, Cor. 3.11].

Proposition 2.2.16. *The connected components of the singular locus of a polyhedral 3-manifold with no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$ are:*

1. *Locally finite graphs whose edges have constant conical angle and whose vertices have degree at least 3, possibly with infinitely long edges only incident to a vertex at one end;*
2. *Circles with constant conical angle; or*
3. *Lines of infinite length with constant conical angle.*

The vertices of the graph components are referred to as *singular vertices*—these are precisely the codimension 3 singularities. The finite edges of the graph components are called *singular segments,* and the infinite edges *singular rays.* The components in cases 2 and 3 above are called *singular circles* and *lines* respectively. The singular segments, rays, circles, and lines are collectively called the *singular edges.* Singular rays and lines cannot occur in compact manifolds, because if they did, they would have to accumulate somewhere, which would contradict the local finiteness condition in Definition 2.1.1.

Remark 2.2.17. If we allow conical angles divisible by 2π, then all that changes in the result above is that we must allow vertices of degree 2. To see this, let S be a sphere with precisely two conical points of angles $2n\pi$ and $2m\pi$, where n and m are integers greater than 1. It follows from [Tro89, Thm. 1] that $n = m$ and that S is a ramified cover of $\mathbb{S}^2$, ramified at two points. If these two points are not antipodal, then S is not a spherical football. This means that, if S is the link of a polyhedral 3-manifold M at a point x, then x can be metrically distinguished from interior points of the two singular edges incident to it and must therefore be viewed as a vertex of degree 2.

2.3. Holonomy

In this section, we define a crucial algebraic invariant of a polyhedral manifold called the *holonomy map.* This is a certain representation of the fundamental group of the regular locus that encodes information about how an observer's view would change as they moved around a polyhedral manifold without turning. Imposing conditions on the holonomy can have significant implications for the geometry and topology of the singular locus—this idea is explored in Chapter 3. Conversely, imposing conditions on the topology of the singular locus can also have implications for the holonomy—this idea is explored in Chapter 4. To define holonomy, we first need to define the *ramification* and *developing map* of a polyhedral manifold.

Definition 2.3.1 (Ramification)**.** Given a polyhedral manifold M, the *ramification* of M, denoted by $\operatorname{Ram} M$, is the metric completion of the universal cover of $M \setminus M_{\mathrm{s}}$, where the metric is defined by pulling back along the universal covering map (see [PP16]). Completing the metric of the universal cover $\widetilde{M \setminus M_{\mathrm{s}}}$ may be thought of as 'adding in' preimages to the singularities of M. The universal covering map extends to a map $p\colon \operatorname{Ram} M \to M$. This map has ramification (of countably infinite degree) along the preimages to the singular points.

Example 2.3.2. If M is the double of a triangle T, then $\operatorname{Ram} M$ is, topologically, an open disc plus countably many points on its boundary (see Figure 2.5). It is triangulated by triangles isometric to T, and the points on the boundary are the vertices of these triangles. There are infinitely many triangles incident to any vertex, and therefore $\operatorname{Ram} M$ is not a polyhedral manifold. However, to abuse the terminology of polyhedral manifolds, we might say that any vertex of $\operatorname{Ram} M$ has 'link $\mathbb{R}$' and 'conical angle ∞'.

Following the discussion just before Definition 2.2.1, any regular point of a Euclidean or spherical polyhedral n-manifold M has a neighbourhood that is isometric to $\mathbb{R}^n$ or $\mathbb{S}^n$ respectively. It follows that regular locus is a (G, X)*–manifold,* with G being the Euclidean group $\mathrm{E}(n)$ and X being $\mathbb{R}^n$ if M is Euclidean, or with G being the orthogonal group $\mathrm{O}(n+1)$ and X being $\mathbb{S}^n$ if M is spherical (see [CHK00, § 1.4] for a definition). In fact, M itself is a (G, X)*–cone-manifold* in the sense of Thurston (see [Thu98, pp. 523–524]—the notation (X, G)–cone-manifold is used there, and there does not seem to be

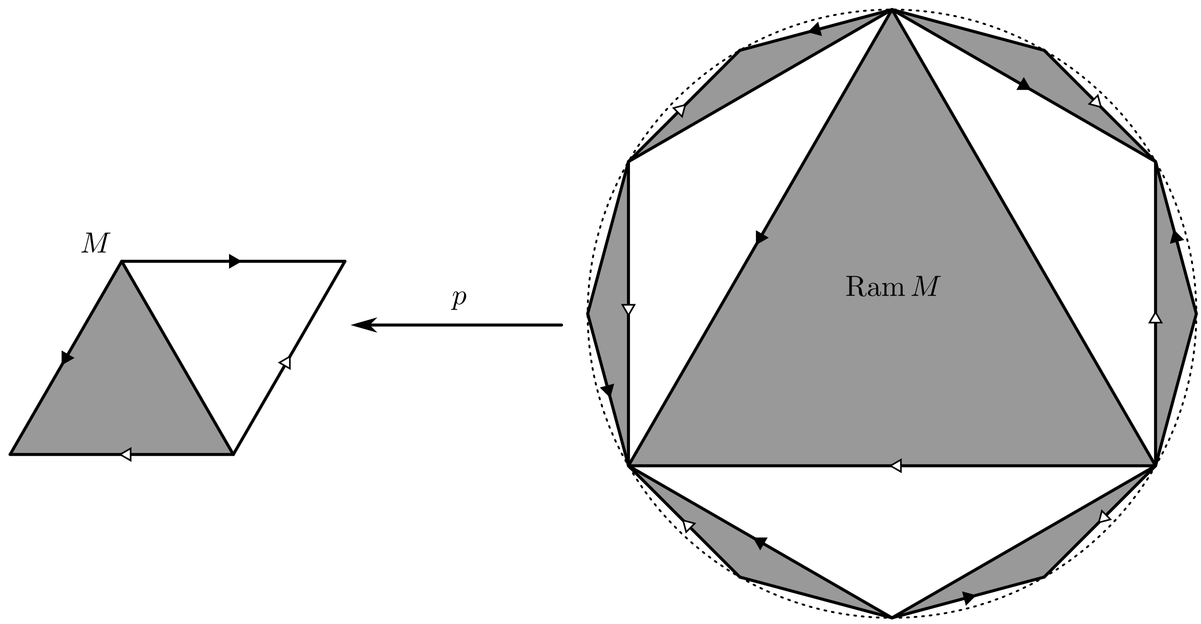


FIGURE 2.5. On the left is the double of a Euclidean equilateral triangle, with arrows marking side identifications. The figure on the right represents the ramification, although we can only show a finite number of triangles. The figure on the right is not to scale—every triangle drawn is isometric to the equilateral triangles shown on the left. The universal covering map p maps triangles to triangles according to their colour and edge labelling.

much consistency in the literature). This allows us to define a *developing map* for M (see [Thu97, § 3.4] or [CHK00, §§ 1.4 & 4.1] for a detailed explanation of what follows).

Definition 2.3.3 (Developing map)**.** Let M be a polyhedral n-manifold with some fixed geometric triangulation and let X be $\mathbb{R}^n$ or $\mathbb{S}^n$ if M is Euclidean or spherical respectively. Lift the triangulation to $\operatorname{Ram} M$ and choose an isometric embedding of one of the simplices into X. Analytically continue this map to all of $\operatorname{Ram} M$ so that it restricts to an isometric embedding on every simplex. This defines a map $\operatorname{Dev}\colon \operatorname{Ram} M \to X$—the *developing map* of M—that restricts to a local isometry on $\widetilde{M \setminus M_{\mathrm{s}}}$.

The developing map can also be thought of as a multivalued map $M \to X$. In this way, one can think of it as 'unfolding' or 'developing' M into the model space X, having chosen an initial embedding for one of the simplices Δ. If we develop along a closed path γ in $M \setminus M_{\mathrm{s}}$ based in Δ, we arrive at a possibly different embedding of Δ in X. Using this instead as our initial embedding, we get a different developing map, which is related to the first by transformation in G (where, recall, G is $\mathrm{E}(n)$ or $\mathrm{O}(n+1)$ if X is $\mathbb{R}^n$ or $\mathbb{S}^n$ respectively). This transformation is called the *holonomy* of γ. To make this precise, we use the language of deck transformations. Recall that $\pi_1(M \setminus M_{\mathrm{s}})$ acts on $\widetilde{M \setminus M_{\mathrm{s}}}$—and thus on $\operatorname{Ram} M$ by continuity—by deck transformations.

Definition 2.3.4 (Holonomy and monodromy)**.** The *holonomy map* of a polyhedral n-manifold M is the unique homomorphism $\operatorname{Hol}\colon \pi_1(M \setminus M_{\mathrm{s}}) \to G$ satisfying

$$\operatorname{Dev}(\gamma \cdot \widetilde{x}) = \operatorname{Hol}\gamma \cdot \operatorname{Dev}\widetilde{x}, \text{ for all } \gamma \in \pi_1(M \setminus M_{\mathrm{s}}) \text{ and } \widetilde{x} \in \operatorname{Ram} M.$$

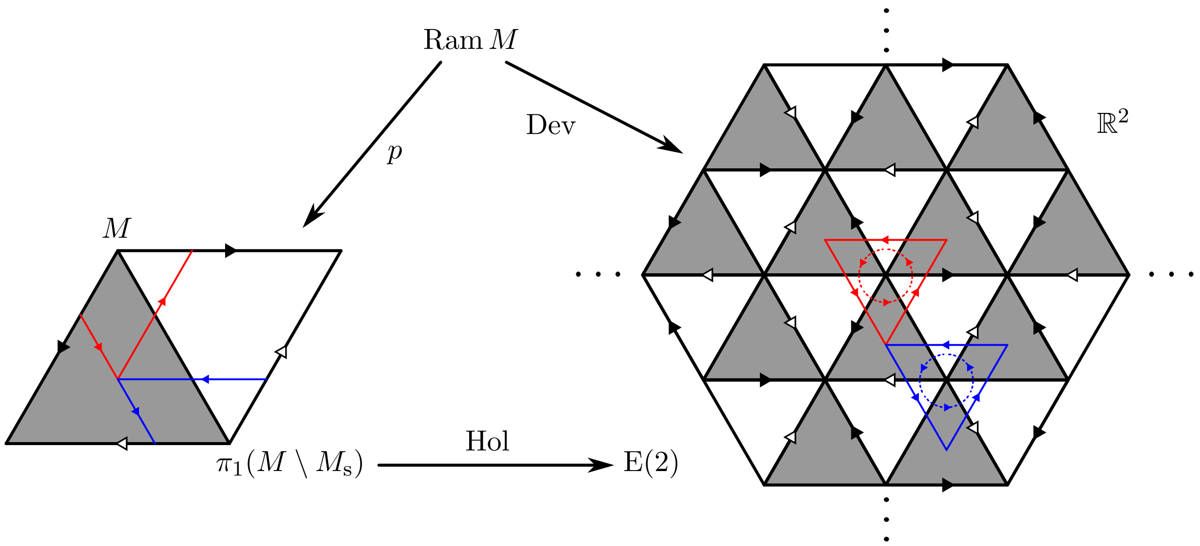


FIGURE 2.6. Developing map and holonomy of the double of a Euclidean equilateral triangle. The ramification is shown in Figure 2.5. Triangles map into $\mathbb{R}^2$ as shown on the right. The blue and red loops on the left are closed geodesics that generate $\pi_1(M \setminus M_\mathrm{s})$. These paths are developed, three times consecutively, to give the red and blue paths on the right. The holonomies of the blue and red loops are rotations of order 3 about the centres of the red and blue dotted circles respectively.

The *holonomy group* $\operatorname{Hol} M$ of M is the image the holonomy map. When M is Euclidean, we define the *monodromy map* $\operatorname{Mon}\colon \pi_1(M \setminus M_\mathrm{s}) \to \mathrm{O}(n)$ to be the composition of the holonomy map with the natural projection $\mathrm{E}(n) \to \mathrm{O}(n)$. In this case, the *monodromy group* $\operatorname{Mon} M$ of M is the image of the monodromy map.

Remark 2.3.5. We must, at this point, briefly discuss the difficulty of the terminology just introduced. In the majority of literature concerning (G, X)–structures, the term holonomy is used as above, following Thurston's example. But in Riemannian geometry, the term holonomy is used quite differently, to describe isometries of the tangent space at a point. The regular locus of a *Euclidean* polyhedral manifold is naturally a flat Riemannian manifold, and holonomy of the Levi–Civita connection of this Riemannian manifold is actually the same as the monodromy defined above. The key thing to remember is that *monodromy* is defined only for *Euclidean* polyhedral manifolds, and it fixes the origin, while *holonomy* does not (in general).

Example 2.3.6. The developing map of the double of an n-polytope P is defined by gluing copies of that polytope face to face to fill the whole of $X = \mathbb{R}^n$ or $\mathbb{S}^n$. The copies will overlap, unless the polytope tesselates. Suppose we have chosen an embedding of P into X, and that it has codimension 2 facets $\sigma_1, \ldots, \sigma_m$ with dihedral angles $\vartheta_1, \ldots, \vartheta_m$ respectively. The holonomy group of DP is the group generated by rotations about the σ_i by angle $2\vartheta_i$. In general, the image of a point under the action of this group is dense in X.

The above example demonstrates that holonomy groups of polyhedral manifolds can be very complicated. However, if we choose the polyhedral metric carefully, as in Figure 2.5, the developing map ends up exhibiting a high degree of regularity, as we explore in the following example. See Figure 2.6 for a visualisation of what follows.

Example 2.3.7. This example in inspired by Thurston's work in [Thu98, § 2]. Let T be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $u := (1,0)$, and $v := (1/2, \sqrt{3}/2)$. The developing map of DT is defined by tiling the plane with copies of T. The vertices of this tiling form the lattice $L := \mathbb{Z}\langle u, v\rangle$. The holonomy group is generated by rotations of order 3 about the lattice points, and the monodromy group is the cyclic group C_3. They both preserve the lattice L. In fact, DT may be recovered as $\mathbb{R}^2/\operatorname{Hol} DT$, since it is an orbifold (see [CHK00, Thm. 2.26]).

Remark 2.3.8. We can now prove something that was claimed back in Subsection 2.2.3: that a spherical polyhedral 2-sphere cannot have precisely one conical point. Indeed, suppose that we have such a sphere. Its regular locus is an open disc, which is simply connected, and so its holonomy group is trivial. This means that the developing map is a ramified cover, with ramification only at the conical point. But there is no map from the 2-sphere to itself ramified at only one point. Therefore, such a 2-sphere cannot exist.

2.3.1. Holonomy and Monodromy Restrictions

In this thesis, we impose two different restrictions on the holonomy of polyhedral manifolds. The first is modelled on Example 2.3.7 and is considered in Chapter 3. Recall that a *lattice* in $\mathbb{R}^n$ is $\mathbb{Z}\langle v_1, \dots, v_n\rangle$, where $\{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$.

Definition 2.3.9 (Integral)**.** A Euclidean polyhedral n-manifold is called *integral* if the action of its monodromy group (a subgroup of $\mathrm{O}(n)$) on $\mathbb{R}^n$ preserves a lattice in $\mathbb{R}^n$. This is equivalent to the monodromy group being conjugate to a subgroup of $\mathrm{GL}(n, \mathbb{Z})$.

Nonnegatively curved integral polyhedral 3-manifolds are studied in detail in Chapter 3. Specifically, their local properties are classified, and the existence of a bound on their number of singular vertices is demonstrated.

The other restriction we consider was defined by Panov in [Pan09] and relates to the unitary group. The standard definition of $\mathrm{U}(n)$ is as the subgroup of $\mathrm{GL}_n(\mathbb{C})$ of matrices whose inverses are their conjugate transposes. However, when considering holonomy groups—and therefore throughout this thesis—we identify $\mathrm{U}(n)$ with the subgroup of $\mathrm{O}(2n)$ of matrices commuting with the *standard complex structure* on $\mathbb{R}^{2n}$, which is defined as follows:

$$J_n := \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}.$$

This identification can be realised by the following embedding of $M_n(\mathbb{C})$ into $M_{2n}(\mathbb{R})$, under which J_n is the image of iI:

$$\begin{pmatrix} a_{11}+ib_{11} & \cdots & a_{1n}+ib_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}+ib_{n1} & \cdots & a_{nn}+ib_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} \begin{smallmatrix} a_{11} & -b_{11} \\ b_{11} & a_{11} \end{smallmatrix} & \cdots & \begin{smallmatrix} a_{1n} & -b_{1n} \\ b_{1n} & a_{1n} \end{smallmatrix} \\ \vdots & \ddots & \vdots \\ \begin{smallmatrix} a_{n1} & -b_{n1} \\ b_{n1} & a_{n1} \end{smallmatrix} & \cdots & \begin{smallmatrix} a_{nn} & -b_{nn} \\ b_{nn} & a_{nn} \end{smallmatrix} \end{pmatrix}.$$

Definition 2.3.10 (Polyhedral Kähler)**.** A Euclidean polyhedral $2n$-manifold (with no conical angles in $2\pi(\mathbb{N}\setminus\{1\})$) is called *polyhedral Kähler* if its monodromy group is conjugate to a subgroup of $\mathrm{U}(n)$. There is an extra condition for codimension 2 simplices having conical angles in $2\pi(\mathbb{N}\setminus\{1\})$, but we do not consider this situation much here (see [Pan09, Def. 1.1] for the full details).

In Chapter 4, we are particularly interested in spherical polyhedral metrics on S^3 that form local models for polyhedral Kähler 4-manifolds. With this in mind, we make the following definition in the spherical setting.

Definition 2.3.11 (PK-link)**.** Let M be a spherical polyhedral manifold, homeomorphic to S^{2n-1}. We say that M is a *PK-link* if the Euclidean cone over M is polyhedral Kähler. When none of the conical angles of M lie in $2\pi(\mathbb{N}\setminus\{1\})$, this is equivalent to the holonomy group of M (a subgroup of $\mathrm{SO}(2n)$) being conjugate to a subgroup of $\mathrm{U}(n)$.

Panov completely classified PK-link metrics on the 3-sphere in [Pan09, Thm. 1.7]. His result implies in particular that the singular locus of such a metric must be a *Seifert link,* a slight generalisation of a torus link. In Chapter 4, we work on a converse to this fact, by asking whether requiring the singular locus of a spherical polyhedral metric on the 3-sphere to be a Seifert link forces the metric to be a PK-link metric. We show that evaluating the holonomy of a single loop is sufficient to show that a spherical metric singular along a Seifert link is a PK-link metric. This result is then applied, first to the Hopf link, and then to Seifert links in general.

Chapter 3

NONNEGATIVELY CURVED INTEGRAL POLYHEDRAL 3-MANIFOLDS

In [Thu98], Thurston studied the moduli space of triangulations of the 2-sphere in which every vertex has degree at most 6. To do this, he endowed such a triangulated 2-sphere with a metric in which every triangle is isometric to a Euclidean equilateral triangle with unit sides. This is a Euclidean polyhedral metric. The degree condition ensures that it is nonnegatively curved, and the shape of the triangles ensures that its holonomy preserves the *Eisenstein* lattice—the lattice generated by $(1, 0)$ and $(1/2, \sqrt{3}/2)$ in $\mathbb{R}^2$. He later embeds the space of these triangulations inside the space of metrics on the 2-sphere that are locally Euclidean everywhere except at a finite number of conical points, whose conical angles must lie in $\{k\pi/3 \mid k \in \{1, \ldots, 5\}\}$. This is a space of nonnegatively curved integral polyhedral metrics on the 2-sphere. In this chapter, we consider nonnegatively curved integral polyhedral metrics on the *3-sphere* and other 3-manifolds. For the rest of this chapter, the term *polyhedral* will stand for *Euclidean polyhedral,* unless stated otherwise. The only spherical polyhedral manifolds we will consider here are links of points in polyhedral 3-manifolds. Such spaces will be referred to as spheres with conical points, as in Subsection 2.2.3.

Before summarising the layout of the chapter, let us briefly examine the 2-dimensional setting, showing that we can easily classify singular points and bound their number when the metric is nonnegatively curved and integral. Suppose we have a polyhedral surface S. If S is integral, then, in particular, the monodromy of a simple loop around any singular point must preserve a lattice in $\mathbb{R}^2$. The only lattice-preserving subgroups of $\mathrm{SO}(2)$ are the subgroups of C_6 (preserving the Eisenstein lattice) and of C_4 (preserving the coordinate lattice—see [Cox69, § 4.5]). The conical angles of S must therefore be integer multiples of either $\pi/3$ or $\pi/2$, if S is to be integral. This allows us to list the possible conical angles of a nonnegatively curved integral polyhedral surface: they are $\pi/3$, $\pi/2$, $2\pi/3$, π, $4\pi/3$, $3\pi/2$, and $5\pi/3$.

To classify singular points, simply note that the link—and thus the geometry of a small neighbourhood—is entirely determined by the conical angle. There are therefore seven possibilities for the link of a singular point in a nonnegatively curved integral polyhedral surface: circles of length α, denoted by $\mathbb{S}^1(\alpha)$, where α belongs to the list above.

To bound the number of singular points (at least in the compact case), recall from Theorem 2.2.7 that the conical angles $\alpha_1, \ldots, \alpha_k$ of a compact Euclidean polyhedral surface S must satisfy the Gauss–Bonnet formula:

$$\sum_{i=1}^{k}(2\pi - \alpha_i) = 2\pi\chi(S).$$

The minimum nonzero value of $2\pi - \alpha_i$ is $\pi/3$, and the maximum value of $\chi(S)$ is 2. The maximum number of singular points in a compact nonnegatively curved integral polyhedral surface is therefore 12, a bound that is achieved by the surface of a regular icosahedron, for example.

In this chapter, we generalise the observations made above to three dimensions, where far more involved techniques are required than in the 2-dimensional setting. It should be noted that most of the content of this chapter appears in the author's preprint, [Sha21]. The chapter layout is as follows.

- In Section 3.1, we classify the possible monodromy groups of *orientable* nonnegatively curved integral polyhedral 3-manifolds, allowing us in Subsection 3.1.1 to understand the monodromy in a neighbourhood of any point.

- In Section 3.2, we prove an expanded and more precise form of Theorem 1, thereby classifying all singular vertices in nonnegatively curved integral polyhedral 3-manifolds. We do this by showing that the link of any point is a ramified cover of one of two spherical orbifolds. We also deduce Corollary 3.2.13, a result needed for the proof of Theorem 2, which states that there is $\varepsilon_0 \in (0, \pi)$ such that any singular vertex is ε_0-narrow (recall Definition 2.2.14).

- Section 3.3 is dedicated to the proof of Theorem 2, which shows the existence of a universal bound on the number of singular vertices in *any* nonnegatively curved integral polyhedral 3-manifold. We actually prove a more general result, bounding the number of ε-narrow points in a nonnegatively curved Alexandrov space, and then deduce Theorem 2.

- Finally, in Section 3.4, we consider some possible extensions to the work done in this chapter.

3.1. Integral Monodromy Groups

In this section, we consider what it means for a polyhedral 3-manifold to be integral, at least in the orientable case. The reason we restrict to the orientable case is that much of the rest of the chapter is concerned with *local* invariants of singular points, and every polyhedral manifold is *locally* orientable. We start by giving a list of possible monodromy groups of orientable integral polyhedral 3-manifolds. These are, by definition, lattice-preserving subgroups of SO(3). In what follows, D_6 acts as the rotational symmetries of a regular hexagonal bipyramid, and S_4 as the rotational symmetries of a regular octahedron (shown on the left and right of Figure 3.1 respectively).

Lemma 3.1.1. *An orientable polyhedral 3-manifold is integral if and only if its monodromy group is isomorphic a subgroup of D_6 or S_4—i.e., to 1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, A_4, or S_4. All of these groups arise as monodromy groups of closed polyhedral 3-manifolds.*

Proof. The list given above agrees with the list of 3-dimensional lattice-preserving rotation groups in, e.g., [Cox69, § 15.6]. It follows from the classification of finite subgroups of SO(3) (for which, see [Arm88, Ch. 19]) that such subgroups are isomorphic if and only if they are conjugate. Therefore, the monodromy group of a polyhedral 3-manifold is conjugate to one of these standard groups, and therefore preserves a lattice, if and only if it is isomorphic to one of them. We exhibit every group in the list as the monodromy group of a closed polyhedral 3-manifold in Examples 3.1.3 to 3.1.9. □

Remark 3.1.2. By definition, the monodromy groups listed above should preserve lattices in $\mathbb{R}^3$. Which lattices do D_6 and S_4 (and their subgroups) actually preserve?

For D_6, consider the *hexagonal lattice* $\mathbb{Z}\langle(1,0,0),(1/2,\sqrt{3}/2,0),(0,0,1)\rangle$. The convex hull of the eight points closest to the origin is a regular hexagonal bipyramid centred on the origin. Any symmetry of this figure preserves the lattice.

For S_4, consider the *cubic lattice* $\mathbb{Z}\langle(1,0,0),(0,1,0),(0,0,1)\rangle$. The convex hull of the six points closest to the origin is a regular octahedron centred on the origin. Any symmetry of this figure preserves the lattice. (These lattice polytopes are shown in Figure 3.1.)

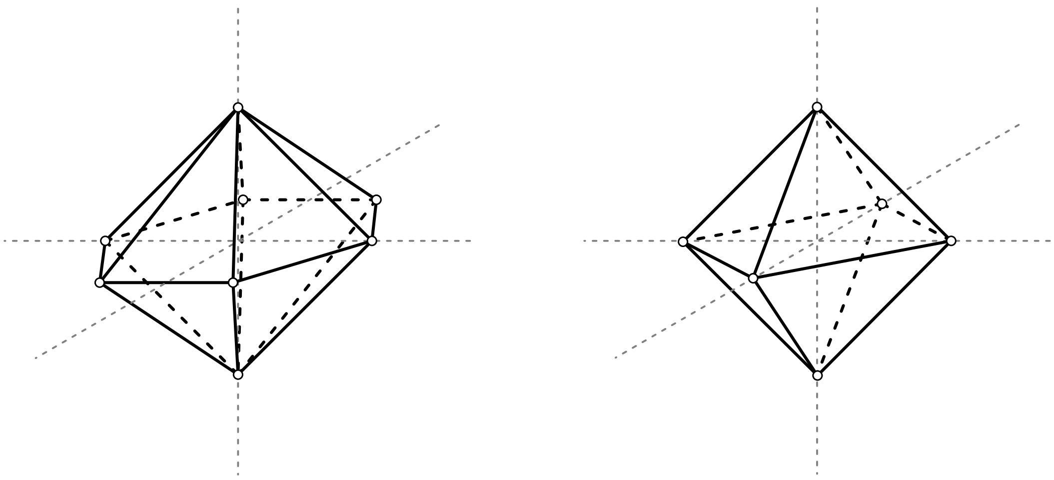


Figure 3.1. Lattice polytopes of hexagonal (right) and cubic (left) lattices. Coordinate axes are shown in dashed grey.

It was stated in Lemma 3.1.1 that every group listed actually arises as the monodromy group of a closed polyhedral 3-manifold. To demonstrate this, we give the following examples, showing that there is indeed no redundancy in Lemma 3.1.1.

Example 3.1.3 (Cyclic Monodromy)**.** The regular cubic 3-torus has trivial monodromy. For $n \in \{2, 3, 4, 6\}$, we may construct a polyhedral metric on the 2-sphere by doubling the regular $2n$-gon. This induces a polyhedral metric on $S^2 \times S^1$ with monodromy C_n.

The remaining, noncyclic examples are all homeomorphic to the 3-sphere.

Example 3.1.4 (Monodromy D_2)**.** Let C be a cube and DC its double. Then $\mathrm{Mon}(DC)$ is D_2 (which is isomorphic to $C_2 \times C_2$). To see this, observe that C may be arranged so that each edge is parallel to one of the coordinate axes. The dihedral angle of each edge of C is $\pi/2$, and so the conical angle of each edge of DC is π. Thus, the monodromy of a loop around each edge corresponds to a rotation of order 2 about one of the coordinate axes, and so $\mathrm{Mon}(DC)$ is generated by rotations by π about the coordinate axes.

Example 3.1.5 (Monodromy D_3)**.** Let P_1 be the (right) prism of an equilateral triangle. Then $\mathrm{Mon}(DP_1)$ is D_3. To see this, arrange P_1 so that the three lateral edges are parallel to the z-axis and one pair of the other edges is parallel to the x-axis. Then the remaining edges will be parallel to either the line $\{z = 0, \sqrt{3}x = y\}$ or the line $\{z = 0, \sqrt{3}x = -y\}$. The dihedral angle of the lateral edges is $\pi/3$, and the dihedral angle of the other edges is $\pi/2$. Thus, $\mathrm{Mon}(DP_1)$ is generated by a rotation of order 3 about the z-axis and rotations of order 2 about the other lines mentioned.

Example 3.1.6 (Monodromy D_4)**.** Let P_2 be the prism of a right-angled isosceles triangle. Then $\mathrm{Mon}(DP_2)$ is D_4. To see this, arrange P_2 so that the three lateral edges are parallel to the z-axis, and so that one pair of the other edges is parallel to the x-axis, one pair to the y-axis, and one pair to the line $\{z = 0, x = -y\}$. The dihedral angle of a lateral edge is either $\pi/2$ or $\pi/4$, and the dihedral angle of the other edges is $\pi/2$. Thus, $\mathrm{Mon}(DP_2)$ is generated by a rotation of order 4 about the z-axis and rotations of order 2 about the other lines mentioned.

Example 3.1.7 (Monodromy D_6)**.** Let P_3 be the prism of a triangle with angles $\pi/6$, $\pi/6$, and $2\pi/3$. Then $\mathrm{Mon}(DP_3)$ is D_6. To see this, arrange P_3 so that the three lateral edges are parallel to the z-axis, and so that one pair of the other edges is parallel to the x-axis, one pair to the line $\{z = 0, \sqrt{3}x = -y\}$, and one pair to the line $\{z = 0, x = -\sqrt{3}y\}$. The dihedral angle of a lateral edge is either $2\pi/3$ or $\pi/6$, and the dihedral angle of the other edges is $\pi/2$. Thus, $\mathrm{Mon}(DP_2)$ is generated by a rotation of order 6 about the z-axis and rotations of order 2 about the other lines mentioned.

Example 3.1.8 (Monodromy A_4)**.** This and the following example are slightly more complicated. Let Ω be the region in $\mathbb{R}^3$ defined by the inequalities $-1 \le x + z \le 1$, $-1 \le y + z \le 1$, $-1 \le x - z \le 1$, and $-1 \le y - z \le 1$. Then Ω is a convex, irregular octahedron. It has four edges lying in the (x, y)-plane, whose dihedral angles are $\pi/2$. One pair is parallel to the x-axis and the other to the y-axis. The other eight edges have dihedral angle $\pi/3$. One pair is parallel to the line $\{x = y = z\}$, another to the line $\{x = -y = -z\}$, another to the line $\{x = y = -z\}$, and the last pair to the line $\{x = -y = z\}$. Thus, $\mathrm{Mon}(D\Omega)$ is generated by rotations of order 2 about the x and y-axes and rotations of order 3 about the other lines mentioned. This gives $\mathrm{Mon}(D\Omega) \cong A_4$.

Example 3.1.9 (Monodromy S_4)**.** Let Σ be the region in $\mathbb{R}^3$ defined by the inequalities $0 \le y$, $0 \le x+z$, $0 \le x-z$, and $x+y \le 1$. Then Σ is a Euclidean simplex. It has three edges with dihedral angle $\pi/2$, all sharing a common vertex. One of these edges is parallel to the y-axis, another to the line $\{y=0, x=z\}$, and the last to the line $\{y=0, x=-z\}$. Of the remaining three edges, two have dihedral angle $\pi/3$—one of these is parallel to the line $\{-x=y=z\}$ and the other to the line $\{x=-y=z\}$. The final edge has dihedral angle $\pi/4$ and is parallel to the z-axis. In a similar manner to before, generators for $\operatorname{Mon}(D\Sigma)$ may be deduced, giving $\operatorname{Mon}(D\Sigma) \cong S_4$.

A natural question that arises from the above examples is, can we find a polyhedral metric on the 3-sphere with cyclic monodromy? The following result shows that, in almost all cases, we cannot.

Proposition 3.1.10. *Let M be a polyhedral 3-manifold with no conical angles in $2\pi(\mathbb{N}\setminus\{1\})$ and whose monodromy group is cyclic. Then M is not homeomorphic to the 3-sphere.*

Proof. If M_{s} is empty, then M is actually a flat Riemannian manifold and hence not homeomorphic to the 3-sphere. If M_{s} is nonempty, then because of the angle condition, $\operatorname{Mon} M$ is nontrivial. In this case, $\operatorname{Mon} M$ has a unique fixed axis in $\mathbb{R}^3$, which can be pulled back to a unit vector field X on $M \setminus M_{\mathrm{s}}$, unique up to sign. All the singular edges of M are parallel to X, and thus by Proposition 2.2.16, M_{s} is a union of circles and lines. If M is noncompact, then of course M cannot be the 3-sphere. If M is compact, then as noted just after Proposition 2.2.16, M contains no singular lines. Thus M_{s} is a union of circles.

The strategy from here is to smoothen the metric on M so that M becomes a Riemannian manifold. We will do this in such a way that X will be defined on all of M and still be parallel to the singular circles. This will allow us to define a closed 1-form ω on M for which $\omega(X)=1$. Suppose we have such a 1-form. Then, since X is parallel to any singular circle C, $\int_C \omega$ equals the length of C, which is nonzero. Therefore, ω is nonzero in $H^1(M)$. Since $H^1(S^3)$ is trivial, this implies that M is not homeomorphic to the 3-sphere. It is thus sufficient to smoothen M and produce the 1-form ω as described above.

Let x be a point in M. If x lies on a singular circle C, then let α be the conical angle of C; otherwise, let α be 2π. For $R>0$, let $C_R^2(\alpha)$ denote the open disc of radius R about the tip of the 2-cone $C^2(\alpha)$ (recall Definition 2.2.5). Then x has a neighbourhood U that is isometric to $C_R^2(\alpha) \times (-T,T)$, for some positive R and T, with x being sent to $(0,0)$. Parametrise U with cylindrical coordinates (r,ϑ,t) in $[0,R)\times[0,2\pi)\times(-T,T)$. Then X is just $\partial/\partial t$ locally, and we define the 1-form ω to be dt locally. These two objects are, for now, not defined for $r=0$ when α differs from 2π.

Away from $r=0$, the metric on U is induced by the following Riemannian metric (see [CHK00, § 3.3]):

$$g = dr^2 + \frac{\alpha}{2\pi} r\, d\vartheta^2 + dt^2.$$

When α is 2π, this metric is actually smooth everywhere. Otherwise, let $f\colon [0,R) \to \mathbb{R}$ be a smooth, monotonic function that equals 1 on $[0, R/3]$ and equals $\alpha/2\pi$ on $[2R/3, R)$. Then the following formula defines a smooth Riemannian metric on all of U:

$$g_{\text{smooth}} := dr^2 + f(r) r d\vartheta^2 + dt^2.$$

Given this smooth metric, X and ω can now be extended to $r = 0$, still being defined locally as $\partial/\partial t$ and dt respectively.

Using the recipe given above, we can smoothen a neighbourhood of every singular circle, choosing a single, sufficiently small value of R so as to avoid any overlap. This makes M into a Riemannian manifold with a globally defined 1-form ω for which $\omega(X) = 1$, as required. □

3.1.1. Local Monodromy and Conical Angles

It was mentioned at the start of this section that we restricted our attention to *orientable* polyhedral manifolds with the purpose of studying local properties. We explain this now by first defining the *local monodromy,* which is a useful tool in classifying singular vertices.

Definition 3.1.11 (Local monodromy)**.** Let M be a Euclidean polyhedral n-manifold and x a point in M. The *local monodromy map* of M at x is the monodromy map of the tangent cone at x, denoted by $\mathrm{Mon}_x\colon \pi_1((T_xM)\setminus(T_xM)_{\mathrm{s}}) \to \mathrm{SO}(n)$. The *local monodromy group* of M at x is the image of this map, denoted by $\mathrm{Mon}_x M$. The local monodromy map may also be viewed as the *holonomy* map of the link $\Sigma_x M$.

Remark 3.1.12. The developing map of the tangent cone T_xM restricted to a small ball about its tip is identical to the developing map of M restricted to a small ball about x. Therefore, the local monodromy map of M at x factors through the *global* monodromy map of M, and so $\mathrm{Mon}_x M$ is a subgroup of $\mathrm{Mon}\, M$. This implies, in particular, that T_xM is integral if M is. With this in mind, we can give the following immediate corollary of Lemma 3.1.1.

Corollary 3.1.13. *If x is a point in an integral polyhedral 3-manifold M, then $\mathrm{Mon}_x M$ is a subgroup of D_6 or S_4—i.e., it is one of the groups listed in Lemma 3.1.1.*

We finish this section by using the result above to list all the possible conical angles of a *nonnegatively curved* integral polyhedral 3-manifold.

Corollary 3.1.14. *Let x be a singular point in a nonnegatively curved integral polyhedral 3-manifold M. The possible conical angles of a singular edge containing x are*

1. *$\pi/3$, $2\pi/3$, π, $4\pi/3$, and $5\pi/3$, when $\mathrm{Mon}_x M$ is a subgroup of D_6; or*
2. *$\pi/2$, $2\pi/3$, π, $4\pi/3$, and $3\pi/2$, when $\mathrm{Mon}_x M$ is a subgroup of S_4.*

Proof. An edge of conical angle α containing x contributes a rotation by α to $\operatorname{Mon}_x M$, which by Corollary 3.1.13 is a subgroup of D_6 or S_4. The result thus follows by looking at possible orders of nontrivial elements of D_6 and S_4—i.e., 2, 3, and 6; and 2, 3, and 4 respectively—and then looking for rotations by less than 2π having such orders. All of these conical angles can be found in integral polyhedral metrics on the 3-sphere by taking doubles of prisms. □

3.2. Classification of Singular Vertices

A crucial step towards understanding the singular locus of a polyhedral manifold is examining how it looks locally—i.e., examining the *local isometry type* of an arbitrary singular point. Since the geometry of a small neighbourhood of any point in a polyhedral manifold is captured by the link at that point, we take the local isometry type of a point to mean the *isometry type of the link.* The main aim of this section is to classify the local isometry types of singular vertices in nonnegatively curved integral polyhedral 3-manifolds. Our approach requires the following definition (see [LZ04, Def. 1.2.18]).

Definition 3.2.1 (Ramified cover)**.** Let S_1 and S_2 be spheres with conical points, with a surjection $\varphi\colon S_1 \to S_2$. Suppose that φ is a locally isometric covering map, except at the preimages of the conical points of S_2, where it is locally the quotient by a finite group of isometries. We then say that S_1 is a *ramified cover* of S_2 and refer to φ as the *ramified covering map.* We will usually use a double-headed arrow, as in $\varphi\colon S_1 \twoheadrightarrow S_2$, to represent a ramified cover. The *degree* of φ, denoted by $\deg\varphi$, is its degree as an unramified covering map outside of the preimages of the conical points of S_2. In a sufficiently small neighbourhood of any point x_1 in S_1, φ is the quotient by a (possibly trivial) finite group of isometries, the order of which is called the *multiplicity* of x_1, denoted by $\operatorname{mult} x_1$. This means that the multiplicity of any preimage of a regular point of S_2 is 1. For any point x_2 in S_2, the following holds:

$$\sum_{x_1\in\varphi^{-1}(x_2)} \operatorname{mult} x_1 = \deg\varphi. \tag{3.2.1}$$

To classify the local isometry types in question, we show that each link can be expressed as a ramified cover of one of two spherical orbifolds, with nonnegative curvature implying restraints on the multiplicities. These ramified covers are determined by combinatorial data, which can be enumerated. The techniques for enumeration are explained in Subsections 3.2.1 and 3.2.2, and the results are summarised in Theorem 3.2.10 and given in full in the appendix. The two spherical orbifolds in question come from the possible local monodromy groups, as we will now see.

Proposition 3.2.2. *Let x be a point in an integral polyhedral 3-manifold M. Then $\Sigma_x M$ is a ramified cover of $\mathbb{S}^2/D_6$ or $\mathbb{S}^2/S_4$.*

Proof. By Corollary 3.1.13, $\operatorname{Mon}_x M$ is a finite subgroup of $\mathrm{SO}(3)$, and so the quotient $\mathbb{S}^2/\operatorname{Mon}_x M$ is an orbifold. Recall also from Definition 3.1.11 that $\operatorname{Mon}_x M$ is the *holonomy* group of $\Sigma_x M$. This allows us to construct a ramified covering map $\Sigma_x M \twoheadrightarrow \mathbb{S}^2/\operatorname{Mon}_x M$ as follows. Let $p\colon \operatorname{Ram}(\Sigma_x M) \to \Sigma_x M$ be the universal cover and $\operatorname{Dev}\colon \operatorname{Ram}(\Sigma_x M) \to \mathbb{S}^2$ the developing map. Given a point u in $\Sigma_x M$, consider the set $\operatorname{Dev}(p^{-1}(u))$. If v and v' both lie in $\operatorname{Dev}(p^{-1}(u))$, then the definition of holonomy implies that there is some isometry g in $\operatorname{Mon}_x M$ such that $v' = g \cdot v$. The following assignment therefore gives a well-defined ramified covering map:

$$
\begin{tikzcd}
\operatorname{Ram}(\Sigma_x M) \arrow[d, "p"'] \arrow[r, "\operatorname{Dev}"] & \mathbb{S}^2 \arrow[d, two heads] \\
\Sigma_x M \arrow[r, dashed, two heads, "\exists!"] & \mathbb{S}^2/\operatorname{Mon}_x M \arrow[d, two heads] \\
 & \mathbb{S}^2/G
\end{tikzcd}
$$

Figure 3.2

$$
\begin{aligned}
\Sigma_x M &\twoheadrightarrow \mathbb{S}^2/\operatorname{Mon}_x M, \\
u &\mapsto [v], \ \text{where } v \in \operatorname{Dev}(p^{-1}(u)).
\end{aligned}
$$

This map may be seen as the unique map in Figure 3.2 making the diagram commute. By Lemma 3.1.1, $\operatorname{Mon}_x M$ is a subgroup of $G = D_6$ or S_4, so we can compose this map with the quotient map $\mathbb{S}^2/\operatorname{Mon}_x M \twoheadrightarrow \mathbb{S}^2/G$ to get a ramified covering map $\Sigma_x M \twoheadrightarrow \mathbb{S}^2/G$. Note that $\mathbb{S}^2/D_6 \cong \mathbb{S}^2(\pi/3, \pi, \pi)$ and $\mathbb{S}^2/S_4 \cong \mathbb{S}^2(\pi/2, 2\pi/3, \pi)$—where $\mathbb{S}^2(\alpha, \beta, \gamma)$ is the double of the spherical triangle with angles $\alpha/2$, $\beta/2$, and $\gamma/2$—and that the areas of these orbifolds are $\pi/3$ and $\pi/6$ respectively. $\square$

This result implies that classifying links of singular vertices is equivalent to classifying spheres with at least three conical points and with conical angles at most 2π that are ramified covers of $\mathbb{S}^2/D_6$ or $\mathbb{S}^2/S_4$. The rest of this section is devoted to classifying these ramified covers. In order to aid in the classification, we make the following useful definition (see [LZ04, § 2.1]).

Definition 3.2.3 (*Dessin d'enfant*)**.** Let $\varphi\colon S_1 \twoheadrightarrow S_2$ be a ramified covering map, where S_2 is a sphere with precisely three conical points, p_1, p_2, and p_3. The preimage $\varphi^{-1}([p_1p_2])$ of the shortest path joining p_1 and p_2 is called the *dessin d'enfant* of φ. It is a finite graph embedded in S_1 that is 2-coloured by marking whether a vertex maps to p_1 or p_2. Because, by definition, φ only ramifies at p_1, p_2, and p_3, the restriction of φ to $\varphi^{-1}(S_2 \setminus [p_1p_2])$ is a covering of the open disc ramified at only one point, p_3. This means that $\varphi^{-1}(S_2 \setminus [p_1p_2])$ is a disjoint union of open discs, which implies that the *dessin* is connected.

3.2.1. Covers of $\mathbb{S}^2/D_6$

We begin by classifying ramified covers of $\mathbb{S}^2/D_6$. First, we give an upper bound on the number of conical points.

Proposition 3.2.4. *Let S be a sphere with conical points and with conical angles at most 2π that is a ramified cover of $\mathbb{S}^2/D_6$. Then S has at most three conical points.*

Proof. Let $\varphi\colon S \twoheadrightarrow \mathbb{S}^2/D_6$ be the ramified covering map and let z_1, z_2, and z_3 be the conical points of $\mathbb{S}^2/D_6$ of angle π, π, and $\pi/3$ respectively. Consider the *dessin d'enfant* $\varphi^{-1}([z_1 z_2])$; this is a graph embedded in S with the properties given in Definition 3.2.3. Since the conical angles of S are at most 2π, every preimage of z_1 or z_2 has multiplicity at most 2. This implies that every vertex of the graph has degree at most 2. The only connected 2-coloured graphs with maximum degree 2 are circles of even length and line segments. The preimages of z_3 correspond to the faces of the graph, the multiplicity being half the (graph-theoretic) degree of the face. When the graph is a circle, the only conical points are the two preimages of z_3, in which case S has two conical points—unless the circle has length 12, in which case S is just $\mathbb{S}^2$. The circle cannot have length greater than 12, or else the preimages of z_3 would have conical angle exceeding 2π. When the graph is a segment, it cannot have length greater than 6, or else the one preimage of z_3 would have conical angle exceeding 2π. When the segment has length 6, the preimage of z_3 is regular and so S has two conical points. When the segment has length less than 6, the conical points of S are the one preimage of z_3 and the two endpoints of the segment. □

The above proof gives us a way to construct the possible ramified covers: by finding graphs embedded in the sphere that could be the *dessin d'enfant.* This technique is used more thoroughly when considering $\mathbb{S}^2/S_4$. Since we are only interested in spheres with three or more conical points, we are restricted to the five cases where $\varphi^{-1}([z_1 z_2])$ is a segment of length at most 5. We summarise this in the following result.

Proposition 3.2.5. *Up to isometry, there are five spheres with at least three conical points and with conical angles at most 2π that are ramified covers of $\mathbb{S}^2/D_6$. They are $\mathbb{S}^2(n\pi/3, \pi, \pi)$, for $n \in \{1, \ldots, 5\}$.*

3.2.2. Covers of $\mathbb{S}^2/S_4$

We now move on to the classification of ramified covers of $\mathbb{S}^2/S_4$. The same *dessin d'enfant* technique as before is used, but this time it is combined with more involved considerations of the ramification data. To elaborate on this, suppose that $\varphi\colon S \twoheadrightarrow \mathbb{S}^2/S_4$ is a ramified cover and that y_1, y_2, and y_3 are the conical points of $\mathbb{S}^2/S_4$ with angles π, $2\pi/3$, and $\pi/2$ respectively. The conical angles of S allow us to calculate its area—and therefore the degree of φ—using the spherical Gauss–Bonnet formula (i.e., Formula (2.2.1) with $K = 1$). On the other hand, if the preimages $x_i^1, \ldots, x_i^{k_i}$ of y_i have multiplicities $m_i^1, \ldots, m_i^{k_i}$ respectively, then they have conical angles $m_i^1\alpha_i, \ldots, m_i^{k_i}\alpha_i$ respectively (where α_i is the conical angle of y_i), and, by Formula (3.2.1), the degree of φ is $m_i^1 + \ldots + m_i^{k_i}$. The list of multiplicities of the preimages of y_1, y_2, and y_3 is called a *multiplicity datum*, and calculations of the degree of φ these two different ways we refer to as *area-multiplicity calculations*. These two calculations allow us to exclude certain multiplicity data from appearing, thereby making it feasible to list all relevant ramified covers. This is explained in more detail later, but first, as before, we begin with a bound on the number of conical points.

Proposition 3.2.6. *Let S be a sphere with conical points and with conical angles at most 2π that is a ramified cover of $\mathbb{S}^2/S_4$. Then S has at most five conical points.*

Proof. Let $x_1, \dots, x_n$ be the conical points of S, having conical angles $\alpha_1, \dots, \alpha_n$ respectively. We know that S has some positive area A_S, and we know from Corollary 3.1.14 that $\alpha_i \leq 3\pi/2$ for all $i \in \{1, \dots, n\}$. Furthermore, S satisfies the spherical Gauss–Bonnet formula (from Theorem 2.2.7 with $K = 1$), which we recall here:

$$A_S + \sum_{i=1}^{n} (2\pi - \alpha_i) = 4\pi.$$

This gives the following inequality:

$$\begin{aligned} n\pi/2 \leq \sum_{i=1}^{n} (2\pi - \alpha_i) &= 4\pi - A_S, \\ &< 4\pi, \\ n &< 8. \end{aligned}$$

Thus, n is at most 7. Suppose first that n *is* 7. Then by Gauss–Bonnet, the possible values for the α_i are as follows. Firstly, $\alpha_i = 3\pi/2$ for all $i \in \{1, \dots, 7\}$—label such a sphere by S_1. Secondly, $\alpha_i = 3\pi/2$ for all $i \in \{1, \dots, 6\}$ and $\alpha_7 = 4\pi/3$—label this S_2. Lastly, $\alpha_i = 3\pi/2$ for all $i \in \{1, \dots, 5\}$ and $\alpha_6 = \alpha_7 = 4\pi/3$—label this S_3. We will show that none of these spheres can in fact occur as ramified covers of $\mathbb{S}^2/S_4$.

By Gauss–Bonnet, we have $A_{S_1} = \pi/2$, which is thrice the area of $\mathbb{S}^2/S_4$. Hence, the ramified covering map $\varphi\colon S \twoheadrightarrow \mathbb{S}^2/S_4$ has degree 3. But all the x_i are preimages of y_3, the conical point of angle $\pi/2$, and so y_3 has at least seven preimages. This is a contradiction. Similarly, we have $A_{S_2} = \pi/3$, so S_2 is a degree 2 cover of $\mathbb{S}^2/S_4$. But y_3 has at least six preimages: a contradication.
Finally, we have $A_{S_3} = \pi/6$, which implies that $S_3 \cong \mathbb{S}^2/S_4$. But this is clearly false. Thus, we cannot have $n = 7$.

Suppose now that n is 6. By Gauss–Bonnet, the possible values for the α_i are as follows. Firstly, $\alpha_i = 3\pi/2$ for all $i \in \{1, \dots, 6\}$—label such a sphere by S_4. Secondly, $\alpha_i = 3\pi/2$ for $i \in \{1, \dots, 5\}$ and $\alpha_6 = 4\pi/3$—label this S_5. Lastly, $\alpha_i = 3\pi/2$ for $i \in \{1, \dots, 5\}$ and $\alpha_6 = \pi$—label this S_6. In a similar way to before, we will show that none of these spheres correspond to genuine ramified covers.

By Gauss–Bonnet, we have $A_{S_4} = \pi$, so S_4 is a degree 6 cover of $\mathbb{S}^2/S_4$. But y_3 has six preimages of multiplicity 3, so the degree is at least 18: a contradiction.
Similarly, we have $A_{S_5} = 5\pi/6$, so S_5 is a degree 5 cover of $\mathbb{S}^2/S_4$. But y_3 has five preimages of multiplicity 3, so the degree is at least 15: a contradiction.
Finally, we have $A_{S_6} = \pi/2$, so S_6 is a degree 3 cover of $\mathbb{S}^2/S_4$. But y_3 has at least five preimages: a contradiction.

Thus, n is at most 5. But there is indeed a ramified cover of $\mathbb{S}^2/S_4$ with five conical points, as shown in Figure 3.3. It has two conical points of angle $3\pi/2$ and three of angle $4\pi/3$. □

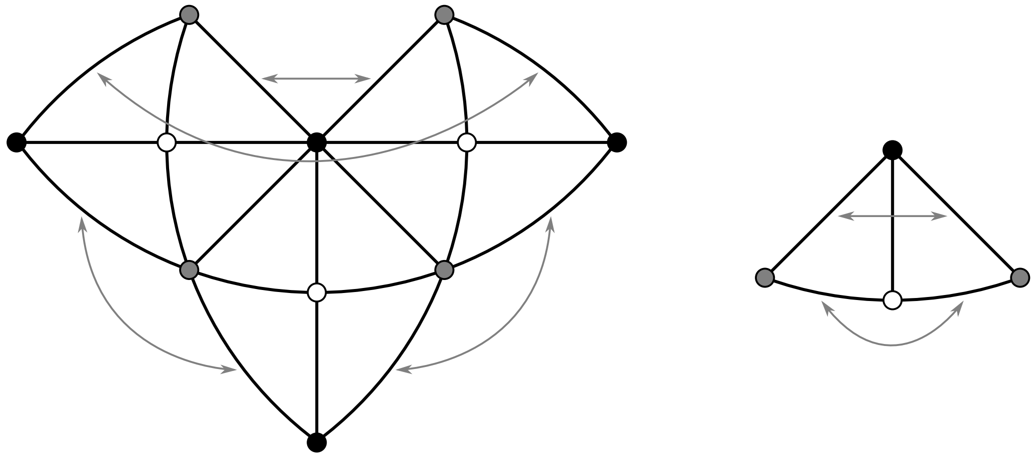


Figure 3.3. On the left is the only nonnegatively curved integral link with five conical points. Grey arrows denote edge identifications. It is *not* the double of a spherical polygon. It is constructed out of 12 copies of a spherical triangle, whose double is $\mathbb{S}^2/S_4$, shown on the right. This demonstrates the ramified covering map. The black, grey, and white vertices of the triangle have interior angles $\pi/4$, $\pi/3$, and $\pi/2$ respectively.

The above proof gives some useful examples of the area-multiplicity calculations that allow us to exclude certain tuples of conical angles from appearing in ramified covers of $\mathbb{S}^2/S_4$. We now explain the procedure for finding all the ramified covers of $\mathbb{S}^2/S_4$ that are spheres with at least three conical points and with conical angles at most 2π.

Procedure 3.2.7.

1. For each $n \in \{3, 4, 5\}$, we use the Gauss–Bonnet formula and our knowledge of the possible conical angles from Corollary 3.1.14 to give a finite list of possibilities for the (unordered) tuple $(\alpha_1, \ldots, \alpha_n)$ of conical angles. Explicitly, we solve

$$\sum_{i=1}^{n}(2\pi - \alpha_i) < 4\pi, \tag{3.2.2}$$

for $\alpha_1, \ldots, \alpha_n \in \{\pi/2, 2\pi/3, \pi, 4\pi/3, 3\pi/2\}$.

2. For any sphere S with conical points of angle less than 2π, we can consider the *Dirichlet domain* (for which, see [CHK00, § 3.6]) based at the point with smallest conical angle $\alpha_{\min}$. This has the same area as S and, by [CHK00, Prop. 3.14], is strictly contained in the spherical football of angle $\alpha_{\min}$, which has area $2\alpha_{\min}$. Therefore, we exclude tuples of angles that do not satisfy the following inequality (c.f. [Tro91, Thm. C] and [Ere04]):

$$\text{Area} = 4\pi - \sum_{i=1}^{n}(2\pi - \alpha_i) < 2\alpha_{\min}. \tag{3.2.3}$$

3. For each remaining tuple $(\alpha_1, \ldots, \alpha_n)$, we look for the possible multiplicity data that could give rise to these angles. Several tuples can be ruled out at this point by area-multiplicity calculations. Explicitly, for a multiplicity datum $m_j^1, \ldots, m_j^{k_j}$, $j \in \{1, 2, 3\}$, we check that

$$\text{Degree} = \sum_{l=1}^{k_j} m_j^l = \frac{1}{\pi/6}\left(4\pi - \sum_{i=1}^{n}(2\pi - \alpha_i)\right) = \text{Area}/\text{Area}(\mathbb{S}^2/S_4), \qquad (3.2.4)$$

for all $j \in \{1, 2, 3\}$.

4. For each remaining multiplicity datum, we look for the possible *dessins d'enfants* $\varphi^{-1}([y_1 y_2])$ in the 2-sphere, where $[y_1 y_2]$ is the shortest path joining the conical points of angles π and $2\pi/3$ in $\mathbb{S}^2/S_4$ and φ is the candidate ramified covering map. Either we construct all possible *dessins* for this datum, in which case we can construct all possible ramified covers with this datum; or we demonstrate that no such *dessin* can exist, in which case this multiplicity datum does not correspond to a genuine ramified cover.

5. Once all possible ramified covers are listed, we determine whether any of them are isometric. This is only possible when the tuples of conical angles are the same. Moreover, when n is 3, the isometry type is determined by the angles.

To better illustrate these steps, some examples of the above procedure are given in the Example in the appendix. We summarise the results of this procedure in the following result.

Proposition 3.2.8. *Up to isometry, there are 30 spheres with at least three conical points and with conical angles at most 2π that are ramified covers of $\mathbb{S}^2/S_4$. There are 17 with three conical points, 12 with four conical points, and precisely one with five conical points. With the exception of one pair of nonisometric spheres having conical angles $(\pi, \pi, 4\pi/3, 4\pi/3)$, these spheres may be distinguished purely by their conical angles.*

Remark 3.2.9. It is important at this point to note that there are three spheres in common between those listed in Propositions 3.2.5 and 3.2.8. They are the spheres $\mathbb{S}^2(2\pi/3, \pi, \pi)$, $\mathbb{S}^2(\pi, \pi, \pi)$ and $\mathbb{S}^2(4\pi/3, \pi, \pi)$.

3.2.3. Summary of the Classification

By combining the main results of this section, namely Propositions 3.2.2, 3.2.5, and 3.2.8, we can now fully classify the singular vertices of nonnegatively curved integral polyhedral 3-manifolds. The following result is a summary of the classification. It is an expanded form of Theorem 1. A complete geometric description of the links is given in Table A.1 in the appendix.

Theorem 3.2.10. *Let M be a nonnegatively curved integral polyhedral 3-manifold and x a singular vertex of M. There are 32 possibilities for the isometry type of $\Sigma_x M$. Of these, five have (local) monodromy in D_6, 30 in S_4, and three in both. Those with monodromy in D_6 all have three conical points. Of those with monodromy in S_4, 17 have three conical points, 12 have four conical points, and one has five conical points.*

Remark 3.2.11. Given that there are seven possible conical angles in a nonnegatively curved integral polyhedral 3-manifold and that the link of a regular point is $\mathbb{S}^2$, Theorem 3.2.10 implies that there are 40 possibilities in total for the local isometry type of a point in a nonnegatively curved integral polyhedral 3-manifold.

We finish with two immediate outworkings of this classification, which are useful when studying the global properties of the singular locus.

Corollary 3.2.12. *A singular vertex in any nonnegatively curved integral polyhedral 3-manifold has (graph-theoretic) degree at most 5.*

Corollary 3.2.13. *There is $\varepsilon_0 \in (0, \pi)$ such that, for any singular vertex x in a nonnegatively curved integral polyhedral 3-manifold M, the diameter of $\Sigma_x M$ is at most ε_0—i.e., x is ε_0-narrow.*

Proof. There are only finitely many possibilities for $\Sigma_x M$, as just shown, and by [CHK00, Lem. 3.10] any sphere with at least three conical points and with conical angles at most 2π has diameter strictly less than π. □

3.2.4. An Interesting Example and Alexandrov's Theorem

Consider the polyhedral manifold constructed as in Figure 3.4. It is homeomorphic to the 3-sphere, nonnegatively curved, and even integral. It has four singular vertices. Three of them have link $\mathbb{S}^2(4\pi/3, \pi/2, \pi/2)$—this is the double of a spherical triangle and therefore cannot appear as the link of a vertex of a 4-polytope. The other has the link with five conical points shown in Figure 3.3. This link is not the double of a spherical polygon, and therefore this vertex cannot appear in the double of a convex polyhedron. Therefore, this polyhedral manifold is neither the boundary of a convex 4-polytope nor the double of a convex polyhedron, and this can be determined purely by local considerations at the vertices.

This example demonstrates the stark constrast with the 2-dimensional setting, where we have the following result (see [BI08, Thm. 1.1]):

Theorem 3.2.14 (Alexandrov's existence/uniqueness theorem)**.** *A nonnegatively curved polyhedral surface, homeomorphic to the 2-sphere, is isometric to either the boundary of a convex polyhedron or the double of a convex polygon, and that polyhedron or polygon is unique up to rigid motions.*

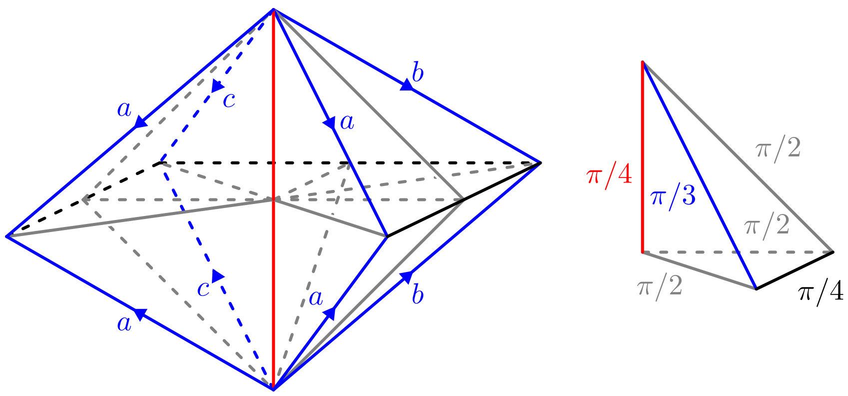


Figure 3.4. The polyhedral manifold in question is shown on the left, with face identifications shown by arrows and letters. It is built out of 12 copies (six with the same orientation and six mirror images) of the simplex on the right.

Alexandrov's theorem in fact generalises to polyhedral surfaces built out of triangles of *any* constant curvature. In the following statement, paraphrased from [Fil10, Thms. 1 & 2], a K*-polyhedral surface* is a polyhedral surface built out of triangles of curvature K, and $\mathbb{M}^n_K$ denotes the n-dimensional model space of constant curvature K, for $n \in \mathbb{N}$ and $K \in \mathbb{R}$.

Theorem 3.2.15 (Generalised Alexandrov theorem)**.** *A K-polyhedral surface, homeomorphic to the 2-sphere and with conical angles at most 2π, is isometric to either the boundary of a convex polyhedron in $\mathbb{M}^3_K$ or the double of a convex polygon in $\mathbb{M}^2_K$, and that polyhedron or polygon is unique up to rigid motions.*

This result makes studying polyhedral surfaces significantly simpler than studying polyhedral 3-manifolds.

3.3. Singular Vertex Bound

Section 3.2 dealt with the *local* geometry of singular vertices of nonnegatively curved integral polyhedral 3-manifolds. In this section, we answer a *global* question about singular vertices by proving Theorem 2—that there is a universal bound on the number of singular vertices in any nonnegatively curved integral polyhedral 3-manifold.

We begin in Subsection 3.3.1 with the more concrete case of doubles of convex polyhedra and boundaries of convex 4-polytopes. We will demonstrate that, in this setting, Theorem 2 may be proved using elementary techniques, which produce a fairly tight bound. We then move onto the general case in Subsection 3.3.2, giving the full proof of Theorem 2. The majority of the section is dedicated to the proof of Proposition 3.3.7, a more general result about nonnegatively curved Alexandrov spaces of finite dimension, from which we deduce Theorem 2. As was mentioned in Chapter 1, Proposition 3.3.7 can actually be deduced from a recent result of Li and Naber in [LN20], although the work presented here is completely independent of theirs and was written before the author became aware of the connection. We therefore finish in Subsection 3.3.3 by briefly exploring the connection with Li and Naber's work.

3.3.1. Convex Polytopes

We wish to bound the number of singular vertices in a nonnegatively curved integral polyhedral 3-manifold M. As was demonstrated in Subsection 3.2.4, we cannot assume that M is either the double of a convex polyhedron or the boundary of a convex 4-polytope. However, it is valuable to prove the existence of the bound in this simpler case for several reasons. Firstly, it motivates the more general bound. Secondly, it shows why it is significant that Alexandrov's theorem (Theorem 3.2.14) does not naïvely generalise to three dimensions. And thirdly, it provides techniques that may be adapted in future work. Therefore in this subsection, we show that Theorem 2 is true for doubles of polyhedra and boundaries of 4-polytopes.

A natural approach to proving Theorem 2 is to replace the somewhat enigmatic condition of integrality with a condition on a more tangible geometric quantity, satisfied by all nonnegatively curved integral polyhedral 3-manifolds. This is the approach taken throughout this chapter: in the introductory 2-dimensional setting, in this subsection, and in Subsection 3.3.2. The question is, which geometric quantity should be used? As highlighted in the opening paragraphs of this chapter, the quantity used in *two* dimensions is the *maximal conical angle*. The relevant condition satisfied by all nonnegatively curved integral polyhedral *surfaces* is that there is ε (equal to $\pi/3$) such that the maximal conical angle is at most $2\pi - \varepsilon$. We can use the Gauss–Bonnet formula to give a bound, depending only on ε, on the number of singular points in a polyhedral surface satisfying this simpler geometric condition.

As can be seen from Corollary 3.1.14, the aforementioned condition still holds in the 3-dimensional setting. However, it is no longer strong enough to produce a bound. More precisely, given $\varepsilon > 0$, there is no bound, depending only on ε, on the number of singular vertices in a polyhedral 3-manifold all of whose conical angles are at most $2\pi - \varepsilon$. The following example demonstrates this, indicating that a more subtle geometric quantity than the maximal conical angle must be used to prove Theorem 2, even in the special case of doubles or boundaries of convex polytopes.

Example 3.3.1. Given an integer n greater than 2, let P_n be the *rectified n-gonal prism* (see Figure 3.5). This is a polyhedron constructed, as follows, to have a number of vertices and edges that grows linearly with n, but whose maximal dihedral angle is always $3\pi/4$. It has two parallel n-gonal faces, n rhombic faces labelled B_k, and $2n$ isosceles-triangular faces—those n adjacent to the top n-gonal face are labelled A_k^+, while those n adjacent to the bottom are labelled A_k^-. When appropriately oriented, the outward facing normals of the top and bottom n-gonal faces are $(0,0,\pm 1)$, those of $A_k^\pm$ are

$$a_k^\pm = \frac{1}{\sqrt{2}}\left(\cos\frac{2k\pi}{n}, \sin\frac{2k\pi}{n}, \pm 1\right),$$

and that of B_k is

$$b_k = \left(\cos\frac{(2k+1)\pi}{n}, \sin\frac{(2k+1)\pi}{n}, 0\right).$$

The top and bottom n-gonal faces pass through $(0,0,\sin^2(\pi/n))$ and $(0,0,0)$ respectively, and the three faces $A_k^\pm$ and B_k all have the following point as a vertex:

$$\left(\cos\frac{2k\pi}{n}, \sin\frac{2k\pi}{n}, \sin^2\frac{\pi}{n}\right).$$

The angle between the normals of adjacent triangular and n-gonal faces is $\pi/4$. For $k \in \{1,\ldots,n\}$, B_k is adjacent to $A_k^\pm$ and $A_{k+1}^\pm$, with the convention that $A_{n+1}^\pm = A_1^\pm$. The angle $\vartheta_k^\pm$ between $a_k^\pm$ and b_k satisfies

$$\begin{aligned}
\cos\vartheta_k^\pm &= a_k^\pm \cdot b_k, \\
&= \frac{1}{\sqrt{2}}\left(\cos\frac{2k\pi}{n}\cos\frac{(2k+1)\pi}{n} + \sin\frac{2k\pi}{n}\sin\frac{(2k+1)\pi}{n}\right), \\
&= \frac{1}{\sqrt{2}}\cos\left(\frac{(2k+1)\pi}{n} - \frac{2k\pi}{n}\right), \\
&= \frac{1}{\sqrt{2}}\cos\frac{\pi}{n}.
\end{aligned}$$

By symmetry, the angle between $a_{k+1}^\pm$ and b_k is also $\vartheta_k^\pm$.

So let ϑ_n be the dihedral angle between adjacent rhombic and triangular faces in P_n. The discussion above shows that ϑ_n is strictly increasing and that $\vartheta_n \longrightarrow 3\pi/4$ as $n \to \infty$. Thus, the maximum conical angle in DP_n is $3\pi/2$ for any n, but DP_n has $3n$ singular vertices.

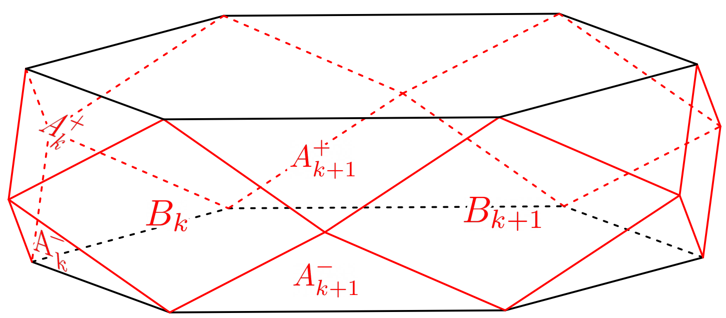


Figure 3.5. Rectified hexagonal prism. Black edges have dihedral angle $3\pi/4$; red edges, ϑ_6.

In the case of doubles and boundaries of convex polytopes, the geometric quantity we will use in place of the maximal conical angle is the *curvature* of a vertex. We define this after introducing some terminology relating to cones embedded in $\mathbb{R}^n$. What follows, up to Remark 3.3.5, is largely adapted from [Pak10, §§ 25.2–4].

Let C be a *polyhedral convex cone* in $\mathbb{R}^n$—i.e., the convex hull of a finite number of rays emanating from a common point v. We say that a hyperplane (of dimension $n-1$) H in $\mathbb{R}^n$ *supports* C at v if H contains v and $C \setminus H$ intersects at most one of the connected components of $\mathbb{R}^n \setminus H$. We say that a ray r in $\mathbb{R}^n$ is *normal* to C at v if it emanates from v, is orthogonal to a hyperplane H supporting C at v, and if $r \setminus H$ lies in a connected component of $\mathbb{R}^n \setminus H$ that $C \setminus H$ does not intersect. The *normal cone* $N(C)$ of C is then defined to be the union of all rays normal to C at v. Note that $N(C)$ is naturally also a polyhedral convex cone based at v. If P is a convex polytope in $\mathbb{R}^n$—i.e., the convex hull of a finite number of *points*—and v is a vertex of P, we define the *tangent cone* $T(P, v)$ of P at v to be the union of all rays emanating from v that intersect P at somewhere other than v. This allows us to define the normal cone of P at v simply as the normal cone of $T(P, v)$, denoted by $N(P, v)$. By convention, we define the normal cone of a single point in $\mathbb{R}^n$ to be the whole of $\mathbb{R}^n$.

Definition 3.3.2 (Vertex curvature)**.** Let P be a convex polytope in $\mathbb{R}^n$ and v a vertex of P. The *curvature* $\omega(P, v)$ of P at v is the solid angle of $N(P, v)$—i.e., the $(n-1)$-dimensional area of the intersection of $N(P, v)$ with the unit sphere centred on v (see Figure 3.6 for a visual example). We can similarly define the curvature of a cone: if C is a polyhedral convex cone, then the curvature $\omega(C)$ of C is the solid angle of $N(C)$.

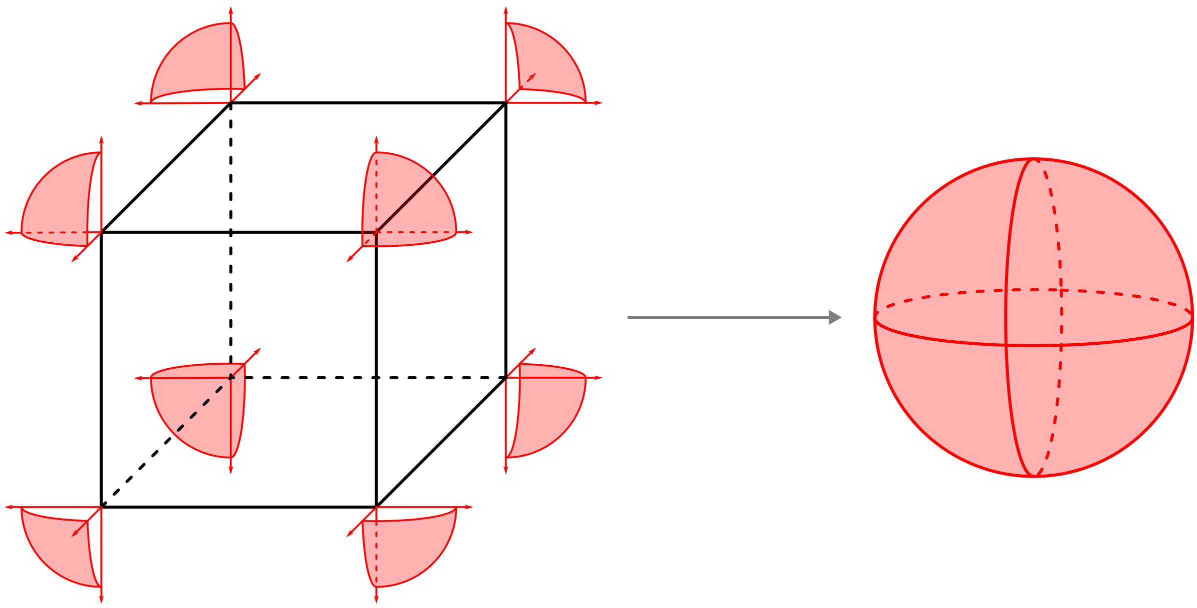


Figure 3.6. On the left is a cube, at each vertex of which is shown the intersection of the normal cone with $\mathbb{S}^2$. Each such intersection is a spherical triangle of area $\pi/2$. These triangles form a tiling of $\mathbb{S}^2$, shown on the right.

Remark 3.3.3. The normal cone of a polyhedral convex cone C in $\mathbb{R}^n$ has Hausdorff dimension less than n if and only if C contains a line. Therefore, $\omega(P, v)$ is strictly positive, for any convex n-polytope P in $\mathbb{R}^n$ and any vertex v of P.

The main result that allows us to use vertex curvature to bound singular vertices is the following generalised Gauss–Bonnet formula (see [Pak10, Thm. 25.4], or [Sch18, Eq. 1] for a published but very brief reference):

Theorem 3.3.4 (Generalised Gauss–Bonnet formula)**.** *Let P be a convex polytope in $\mathbb{R}^n$ with vertices $v_1, \ldots, v_k$. Then*

$$\omega(P, v_1) + \ldots + \omega(P, v_k) = A_{n-1},$$

where A_{n-1} is the $(n-1)$-dimensional area of $\mathbb{S}^{n-1}$.

Proof. For any $i \in \{1, \ldots, k\}$, the intersection of $N(P, v_i)$ with the unit sphere $\mathbb{S}^{n-1}$ centred on v_i is a convex spherical $(n-1)$-polytope R_i, whose area is, by definition, $\omega(P, v_i)$ and whose codimension 1 facets correspond bijectively to the edges of P incident to v_i. Sliding $\mathbb{S}^{n-1}$ along an edge to an adjacent vertex v_j, we get another convex spherical $(n-1)$-polytope R_j, which intersects R_i precisely along one common codimension 1 facet. Doing this for every vertex of P, we see that the spherical polytopes $R_1, \ldots, R_k$ form a tiling of $\mathbb{S}^{n-1}$ (see Figure 3.6). Since R_i has area $\omega(P, v_i)$, the result follows. □

Remark 3.3.5. It may be asked why the above result is referred to as a generalised Gauss–Bonnet formula. To see this, let P be a convex polyhedron (in $\mathbb{R}^3$) with vertices $v_1, \ldots, v_k$. The intersection of the normal cone $N(P, v_i)$ with the unit sphere centred on v_i is a convex spherical polygon R_i. Each corner of R_i corresponds to a face of P incident to v_i, and if the interior angle of the face at v_i is β, then the interior angle at the corresponding corner of R_i is $\pi - \beta$. Thus, if the sum of the interior face angles at v_i is α_i, it is a simple exercise in spherical geometry to check that $\omega(P, v_i)$, the area of R_i, is $2\pi - \alpha_i$. Note that α_i is the conical angle of v_i, viewed as a singular point in the polyhedral surface ∂P. So let $\alpha_1, \ldots, \alpha_k$ be the conical angles of $v_1, \ldots, v_k$ respectively. Then Theorem 3.3.4 reduces to

$$\sum_{i=1}^{n} (2\pi - \alpha_i) = 4\pi,$$

which is simply the statement of the usual Gauss–Bonnet formula (Theorem 2.2.7) for the polyhedral surface ∂P.

Now, if P is a convex polyhedron or 4-polytope and v is a singular vertex of DP or ∂P respectively, then v is also a vertex of P itself and so can be assigned the curvature $\omega(P, v)$. Moreover, thanks to the spherical variant of Alexandrov's theorem (i.e., Theorem 3.2.15 with $K = 1$), the link of any singular vertex in a polyhedral 3-manifold can be assigned a unique curvature. We now have all the technology we need to prove the singular vertex bound for doubles of polyhedra and boundaries of 4-polytopes.

Proposition 3.3.6. *Theorem 2 holds for doubles of polyhedra and boundaries of 4-polytopes.*

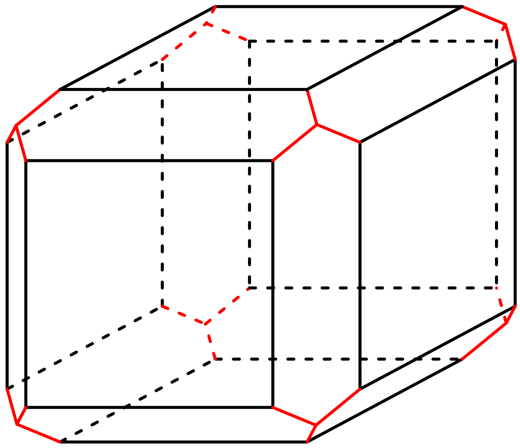


Figure 3.7. Chamfered cube. Black edges have dihedral angle $3\pi/4$; red edges, $2\pi/3$.

Proof. Every one of the 32 singular vertices mentioned in Theorem 3.2.10 can be assigned a positive curvature, depending on whether its link is the double of a spherical polygon or the boundary of a spherical polyhedron. Let ω_D be the minimal curvature among the doubles and ω_∂ the minimal curvature among the boundaries. Now let M be a nonnegatively curved integral polyhedral 3-manifold with k singular vertices. Suppose first that M is the double of some convex polyhedron P. Applying Theorem 3.3.4 to P, we see that $k\omega_D \leq A_2$. Suppose on the other hand that M is the boundary of some convex 4-polytope P. Similarly, applying Theorem 3.3.4 to P, we see that $k\omega_\partial \leq A_3$. Therefore, letting $B := \max(A_2/\omega_D, A_3/\omega_\partial)$, a positive constant, we have $k \leq B$. □

The proof above gives us an explicit way to calculate the vertex bound in the case of doubles and boundaries: find the minimal vertex curvature. The author is not aware of a general method to calculate the curvature of a vertex whose link is the boundary of a spherical polyhedron. However, if the link is the double of a spherical polygon Q, then the curvature of the vertex is $2\pi - p$, where p is the perimeter of Q. This is because if the vertex v belongs to the double of a polyhedron P, then the sides of Q correspond precisely to the faces of P incident to v, the length of the side being equal to the interior angle of the face at v. This allows us to calculate the minimal vertex curvature for doubles exactly using spherical trigonometry:

$$\omega_D = \frac{\pi}{2} - 2\arcsin\left(\frac{1}{\sqrt{3}}\right).$$

The minimal curvature is achieved by link #19 in Table A.1. This puts the value of $A_2/\omega_D = 4\pi/\omega_D$ at approximately 36.978. Therefore, the number of singular vertices in any integral polyhedral manifold that is the double of a convex polyhedron is at most 36. Combining this with the fact that the maximum degree of any such singular vertex is 4, this gives an upper bound of 72 on the number of singular edges in such polyhedral manifolds. The largest convex polyhedron known to the author whose double is integral is the *chamfered cube,* shown in Figure 3.7. It has 32 vertices and 48 egdes, which demonstrates that at least this special case of the vertex bound is not too far off.

3.3.2. Proof of the Singular Vertex Bound

In this section, we prove, in full generality, our second main result.

Theorem 2 (Singular vertex bound)**.** *There is a constant B_{ver} in $\mathbb{N}$ such that* any *nonnegatively curved integral polyhedral 3-manifold has fewer than B_{ver} singular vertices.*

As was mentioned at the start of this chapter, the property of nonnegatively curved integral polyhedral 3-manifolds that is used to prove this is the fact that there is $\varepsilon_0 \in (0, \pi)$ such that all singular vertices are ε_0-narrow. The proof relies on techniques from Alexandrov geometry—in fact, the majority of what follows applies to any finite-dimensional nonnegatively curved Alexandrov space. What we actually prove is the following.

Proposition 3.3.7. *For any integer n greater than 1 and $\varepsilon \in (0, \pi)$, there is a bound $B(n, \varepsilon)$ in $\mathbb{N}$ such that, in any (complete, connected) n-dimensional Alexandrov space of nonnegative curvature, the number of ε-narrow points is less than $B(n, \varepsilon)$.*

With this result established, we use Corollary 3.2.13 and Proposition 2.2.9, the latter of which states that any nonnegatively curved polyhedral manifold is an Alexandrov space of nonnegative curvature, to deduce Theorem 2. For the rest of this section, take n to be an integer greater than 1.

The skeletal logic of the proof of Proposition 3.3.7 is as follows. Recall from Definition 2.2.14 that, for $\varepsilon \in (0, \pi)$, a point x in an n-dimensional nonnegatively curved Alexandrov space M is called *ε-narrow* if any angle subtended at x is at most ε. This is equivalent to the space of directions at x having diameter at most ε. We will show that any sufficiently large subset S of M always contains three distinct points forming a triangle with one angle greater than ε, and so one of the points is not ε-narrow. 'Sufficiently large' here depends only on n and ε. Therefore, if we take S to be the set of all ε-narrow points in M, S cannot be 'sufficiently large'—i.e., it must be smaller than a certain bound depending only on n and ε.

In order to demonstrate that any sufficiently large set S contains such a triple, we first show that S contains a sufficiently long sequence of points $x_1, \ldots, x_m$ whose consecutive distances decay like a geometric progression. We then take m large enough to ensure that, for some $i < j < m$, the angle $\measuredangle x_i x_m x_j$ is less than $(\pi - \varepsilon)/2$. We then use Toponogov's theorem (Theorem 2.2.10) to compare the triangle $\triangle x_i x_j x_m$ with a Euclidean triangle. The geometric decay of the sequence ensures that the side ratio $|x_j x_m|/|x_i x_j|$ is small, which allows us to deduce that $\measuredangle x_i x_j x_m > \varepsilon$.

The aforementioned sequence of points $x_1, \ldots, x_m$ in S is constructed by an iterative process that relies on a certain series of nets for subsets of S. To determine how large S needs to be to continue this process, we begin by bounding the cardinality of these nets.

Lemma 3.3.8. *Let M be an n-dimensional nonnegatively curved Alexandrov space, $\alpha \in (0, 1)$, and X a finite subset of M with diameter at most d. There is a covering of X by closed balls of radius $\alpha d/4$ with at most $(1 + 8/\alpha)^n$ elements.*

Proof. This is adapted from a proof of Liu in [Liu92, § 3], although the argument is generally attributed to Gromov. Let $\{p_1, \ldots, p_k\}$ be a maximal set of points in X satisfying $d(p_i, p_j) > \alpha d/4$ for all $i \neq j$. Then we have a covering of X by k closed balls:

$$X \subseteq \bigcup_{i=1}^{k} \overline{B}(p_i, \alpha d/4).$$

By the definition of the p_i, we have

$$\overline{B}(p_i, \alpha d/8) \cap \overline{B}(p_j, \alpha d/8) = \varnothing, \text{ for all } i \neq j. \tag{3.3.1}$$

Also, by applying the triangle inequality, we see that, for any $j \in \{1, \ldots, k\}$,

$$\bigcup_{i=1}^{k} \overline{B}(p_i, \alpha d/8) \subseteq \overline{B}(p_j, (1 + \alpha/8)d). \tag{3.3.2}$$

Now let $\overline{B}(p_j, \alpha d/8)$ have minimal volume among the $\overline{B}(p_i, \alpha d/8)$. Then, recalling from Definition 2.2.11 that $V(p, r)$ denotes $\operatorname{Vol}(B(p, r))$, we have

$$\begin{aligned} k &= \sum_{i=1}^{k} \frac{V(p_i, \alpha d/8)}{V(p_i, \alpha d/8)}, \\ &\leq \frac{1}{V(p_j, \alpha d/8)} \sum_{i=1}^{k} V(p_i, \alpha d/8), \\ &\leq \frac{V(p_j, (1+\alpha/8)d)}{V(p_j, \alpha d/8)}, \text{ by Formulae (3.3.1) and (3.3.2) together,} \\ &\leq \frac{(1+\alpha/8)^n d^n}{(\alpha d/8)^n}, \text{ by the Gromov–Bishop inequality (Theorem 2.2.12),} \\ &= (1 + 8/\alpha)^n. \end{aligned}$$

□

We now describe one step of the iterative process.

Lemma 3.3.9. *Let M be an n-dimensional nonnegatively curved Alexandrov space, $\alpha \in (0, 1)$, and S a finite subset of M of cardinality at least 2 and diameter d. There is a point x in S and a subset S' of S such that $d(x, S') \geq d/2$, $\operatorname{diam} S' \leq \alpha d/2$, and $|S'| \geq |S| / \left(2\lfloor(1 + 8/\alpha)^n\rfloor\right)$.*

Proof. Choose points p and q in S such that $d(p, q) = d$ and define two subsets of S:

$$P := \{s \in S \mid d(s, p) \leq d(s, q)\} \text{ and } Q := S \setminus P.$$

If $|P| \geq |Q|$, let $x := q$ and $X := P$; otherwise, let $x := p$ and $X := Q$. We note three facts about X. Firstly, the definition of X and the triangle inequality imply that $d(x, X) \geq d/2$; secondly, $\operatorname{diam} X \leq d$; and thirdly, since X is chosen to be the larger of P and Q, we have

$$|X| \geq (|P| + |Q|)/2 = |S|/2. \tag{3.3.3}$$

Now let $k := \lfloor (1+8/\alpha)^n \rfloor$. By Lemma 3.3.8, we can cover X by k balls $B_1, \ldots, B_k$ of radius $\alpha d/4$ in M. Now let $X \cap B_j$ have maximal cardinality among the $X \cap B_i$, and let $S' := X \cap B_j$. Let us demonstrate that S' has the three required properties. Firstly, $d(x, S') \geq d(x, X) \geq d/2$; secondly, $\operatorname{diam} S' \leq \operatorname{diam} B_j \leq \alpha d/2$; and finally, since S' is chosen to maximise $|X \cap B_i|$, we have

$$\begin{aligned} |S'| \geq \frac{1}{k} \sum_{i=1}^{k} |X \cap B_i| &\geq \frac{|X|}{k}, \\ &\geq \frac{|S|}{2k}, \text{ by Formula (3.3.3)}, \\ &= \frac{|S|}{2\lfloor (1+8/\alpha)^n \rfloor}. \end{aligned}$$ ☐

This step is now applied recursively to produce a sequence of points whose consecutive distances decay like a geometric progression.

Lemma 3.3.10. *Let M be an n-dimensional nonnegatively curved Alexandrov space, m an integer greater than 2, and $\alpha \in (0,1)$. Any subset S of M of cardinality at least $2\left(2\lfloor (1+8/\alpha)^n \rfloor\right)^{m-2}$ contains a sequence of distinct points $x_1, \ldots, x_m$ satisfying*

$$d(x_{i+1}, x_{i+2}) \leq \alpha d(x_i, x_{i+1}), \quad \textit{for } i \in \{1, \ldots, m-2\}. \tag{3.3.4}$$

Proof. If S is infinite, we may replace S with any finite subset of cardinality at least $2\left(2\lfloor (1+8/\alpha)^n \rfloor\right)^{m-2}$. So assume that S is finite. We make $m-2$ applications of Lemma 3.3.9 to S. Specifically, starting with $S_1 := S$ and defining $S_{i+1} := S_i'$, we get a point x_i in S_i for each $i \in \{1, \ldots, m-2\}$. Then, letting $k := \lfloor (1+8/\alpha)^n \rfloor$, we have

$$|S_{m-1}| \geq \frac{|S_{m-2}|}{2k} \geq \ldots \geq \frac{|S_1|}{(2k)^{m-2}} \geq \frac{2(2k)^{m-2}}{(2k)^{m-2}} = 2.$$

Thus, we can arbitrarily choose two distinct points x_{m-1} and x_m in S_{m-1}. We thus have our sequence of distinct points $x_1, \ldots, x_m$ in S; we just need to check that it satisfies Formula (3.3.4). Indeed, let $i \in \{1, \ldots, m-2\}$. Then

$$\begin{aligned} d(x_{i+1}, x_{i+2}) &\leq \operatorname{diam} S_{i+1}, && \text{since } x_{i+1} \text{ and } x_{i+2} \in S_{i+1}, \\ &\leq \frac{\alpha}{2} \operatorname{diam} S_i, && \text{by Lemma 3.3.9}, \\ &\leq \alpha d(x_i, S_{i+1}), && \text{by Lemma 3.3.9}, \\ &\leq \alpha d(x_i, x_{i+1}), && \text{since } x_{i+1} \in S_{i+1}. \end{aligned}$$ ☐

We must now depart from the linear flow of the argument to give two technical results, Lemmas 3.3.11 and 3.3.12. These results do not follow from any of the earlier results in this section, but they are needed for the proof of the final lemma, Lemma 3.3.13, from which Proposition 3.3.7 follows. Thanks to Lemma 3.3.10, we now have our ‘geometrically decaying’ sequence $x_1, \ldots, x_m$. As was mentioned in the discussion at the start of this subsection, we need to use this geometric decay to ensure that an arbitrary distance ratio $d(x_j, x_m)/d(x_i, x_j)$ is sufficiently small. This is what we show in the first technical result, by careful applications of Formula (3.3.4) and the triangle inequality.

Lemma 3.3.11. *Let $x_1, \ldots, x_m$ be a sequence of distinct points in a metric space (X, d) satisfying Formula (3.3.4) for some $\alpha \in (0, 1/2)$. For any indices $1 \leq i < j < m$, we have*

$$\frac{d(x_j, x_m)}{d(x_i, x_j)} < \frac{\alpha}{1 - 2\alpha}.$$

Proof. For convenience, let $d := d(x_i, x_{i+1})$. On the one hand, we have

$$\begin{aligned}
d(x_j, x_m) &\leq d(x_j, x_{j+1}) + \ldots + d(x_{m-1}, x_m), && \text{by the triangle inequality,}\\
&\leq \alpha^{j-i}(d(x_i, x_{i+1}) + \ldots + d(x_{m+i-j-1}, x_{m+i-j})), && \text{by Formula (3.3.4),}\\
&\leq d\alpha^{j-i}(1 + \alpha + \ldots + \alpha^{m-j-1}), && \text{by Formula (3.3.4),}\\
&\leq d\alpha(1 + \alpha + \ldots + \alpha^{m-j-1}), && \text{since } i < j \text{ and } \alpha < 1,\\
&< \frac{d\alpha}{1 - \alpha}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
d(x_i, x_j) &\geq d(x_i, x_{i+1}) - d(x_{i+1}, x_j), && \text{by the triangle inequality,}\\
&\geq d - (d(x_{i+1}, x_{i+2}) + \ldots + d(x_{j-1}, x_j)), && \text{by the triangle inequality,}\\
&\geq d - d(\alpha + \alpha^2 + \ldots + \alpha^{j-i-1}), && \text{by Formula (3.3.4),}\\
&> d\left(1 - \frac{\alpha}{1 - \alpha}\right) = \frac{d(1 - 2\alpha)}{1 - \alpha}.
\end{aligned}$$

The final expression above is positive, since $\alpha \in (0, 1/2)$. Therefore, we can combine the two inequalities above to get the desired result. □

The second technical result states that a sufficiently large collection of points in an Alexandrov space of curvature at least 1 contains two points that are close to each other. This is shown by using a packing result of Grove and Wilhelm ([GW95, Prop. 1.3]) to reduce the problem to the sphere, where it can be solved using elementary spherical geometry. The result is used in the proof of Lemma 3.3.13 to ensure that, when our sequence $x_1, \ldots, x_m$ is long enough, we can find indices $1 \leq i < j < m$ such that the angle $\measuredangle x_i x_m x_j$ is small.

Lemma 3.3.12. *Let Σ be an $(n-1)$-dimensional Alexandrov space of curvature at least 1 and let $\delta \in (0, \pi)$. Any collection of $m - 1$ points in Σ contains two at a distance at most δ, provided that $m \geq m_n(\delta)$, where*

$$m_n(\delta) := \left\lfloor \frac{2}{I_{\sin^2(\delta/2)}\left(\frac{n-1}{2}, \frac{1}{2}\right)} \right\rfloor + 2, \tag{3.3.5}$$

and $I_t(a, b)$ is the regularised incomplete beta function.

Proof. We first show that it suffices to prove the result when $\Sigma = \mathbb{S}^{n-1}$. To do this, we define the q^{th} *packing radius* of a compact metric space X (see [GW95]). For a positive integer q, this is the quantity

$$\operatorname{pack}_q X := \frac{1}{2} \max_{x_1,\dots,x_q \in X} \left(\min_{i<j} (d(x_i, x_j)) \right).$$

Given this definition, observe that the following two statements are equivalent:

1. $\operatorname{pack}_{m-1} X \leq \delta/2$,
2. For all $v_1, \dots, v_{m-1} \in X$, there are indices $i < j$ such that $d(v_i, v_j) \leq \delta$.

The content of the lemma is that $m \geq m_n(\delta)$ implies the second statement for $X = \Sigma$. But [GW95, Prop. 1.3] states that $\operatorname{pack}_{m-1} \Sigma \leq \operatorname{pack}_{m-1} \mathbb{S}^{n-1}$, and so if the second statement holds for $X = \mathbb{S}^{n-1}$, then it holds for $X = \Sigma$. It therefore suffices to prove the lemma for $\Sigma = \mathbb{S}^{n-1}$, which we now do. Two points v_i and v_j in $\mathbb{S}^{n-1}$ being at a distance at most δ is the same as the two closed metric balls of radius $\delta/2$ centred on v_i and v_j intersecting. According to [Li11, Eq. 1], the ($(n-1)$-dimensional) area of a metric ball in $\mathbb{S}^{n-1}$ (there called a *hyperspherical cap*) of radius $\delta/2 \in (0, \pi/2)$ is

$$\frac{1}{2} I_{\sin^2(\delta/2)} \left(\frac{n-1}{2}, \frac{1}{2} \right) A_{n-1},$$

where A_{n-1} is the area of $\mathbb{S}^{n-1}$. By comparing these areas, it follows that any collection of at least $m_n(\delta) - 1$ hyperspherical caps of radius $\delta/2$ in $\mathbb{S}^{n-1}$ must contain an intersecting pair. The result follows. □

We now give the last lemma in the series, after which Proposition 3.3.7 is immediately deduced. It states that a sufficiently large subset of a nonnegatively curved Alexandrov space contains three points forming a triangle with an angle close to π. The proof combines Lemmas 3.3.10 to 3.3.12 and Toponogov's theorem.

Lemma 3.3.13. *Let M be an n-dimensional nonnegatively curved Alexandrov space and let $\varepsilon \in (0, \pi)$. Any subset S of M of cardinality at least $2 \cdot (2 \cdot 25^n)^{m_n((\pi-\varepsilon)/2)-2}$, where $m_n(-)$ is defined as in Formula (3.3.5), contains three points x, y, and z such that $\measuredangle xyz > \varepsilon$.*

Proof. We apply Lemma 3.3.10 with $m = m_n((\pi - \varepsilon)/2)$ and with $\alpha = 1/3$. We have $1 + 8 \cdot 3 = 25$, so to do this we need $|S| \geq 2 \cdot (2 \cdot 25^n)^{m_n((\pi-\varepsilon)/2)-2}$. Applying Lemma 3.3.10 to S gives a sequence of distinct points $x_1, \dots, x_m$ in S satisfying Formula (3.3.4) with $\alpha = 1/3$. Now, the space of directions Σ at x_m is, by [BBI01, Thm. 10.8.6], an $(n-1)$-dimensional Alexandrov space of curvature at least 1. Projecting $x_1, \dots, x_{m-1}$ onto Σ via shortest paths, we get a collection of points $v_1, \dots, v_{m-1}$ in Σ. By the choice of m, we can apply Lemma 3.3.12 to get indices $1 \leq i < j < m$ such that the distance between v_i and v_j in Σ is at most $(\pi - \varepsilon)/2$. By definition, this distance is equal to $\measuredangle x_i x_m x_j$. So take

$x = x_i$, $y = x_j$, and $z = x_m$—then $\measuredangle xzy \leq (\pi - \varepsilon)/2$. By Lemma 3.3.11 with $\alpha = 1/3$, we have $|yz| < |xy|$. Finally, we apply Toponogov's theorem (Theorem 2.2.10) to the triangle $T := \triangle xyz$ (see Figure 3.8 for a visual explanation of this). Let $\tilde{T} = \triangle \tilde{x}\tilde{y}\tilde{z} \subseteq \mathbb{R}^2$ be a comparison triangle for T. We have $\measuredangle xyz \geq \measuredangle \tilde{x}\tilde{y}\tilde{z}$ and $\measuredangle xzy \geq \measuredangle \tilde{x}\tilde{z}\tilde{y}$ by Toponogov's theorem, and by the Euclidean maxim *"The larger angle is opposite the larger side"*, we have $\measuredangle \tilde{x}\tilde{z}\tilde{y} > \measuredangle \tilde{y}\tilde{x}\tilde{z}$. Therefore,

$$\measuredangle xyz \geq \measuredangle \tilde{x}\tilde{y}\tilde{z} = \pi - \measuredangle \tilde{x}\tilde{z}\tilde{y} - \measuredangle \tilde{y}\tilde{x}\tilde{z} > \pi - 2\measuredangle \tilde{x}\tilde{z}\tilde{y} \geq \pi - 2\measuredangle xzy \geq \varepsilon. \qquad \square$$

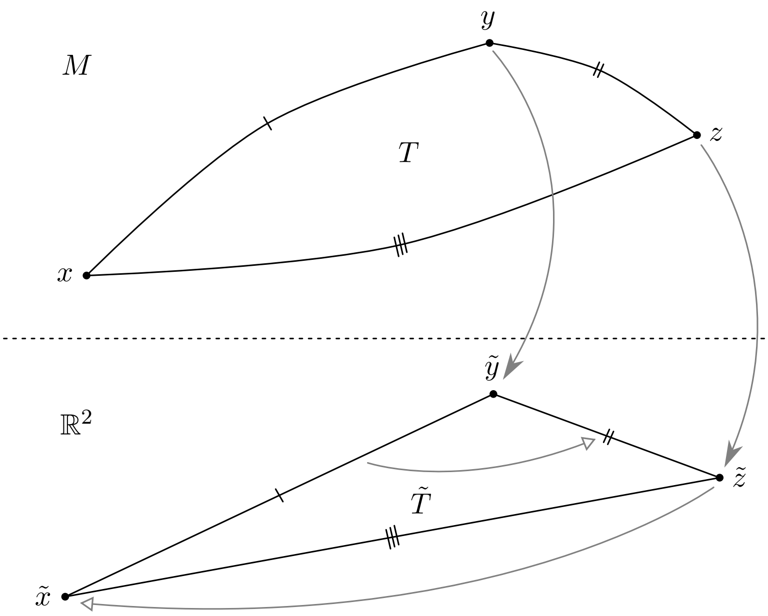


Figure 3.8. Grey arrows connect one point (or side) to another, signalling that the angle (or length, respectively) of the first is at least as big as that of the second. White headed arrows denote *strict* inequalities. Sides with the same number of notches (one, two, or three) have the same length.

Proof of Proposition 3.3.7. For n greater than 1 and $\varepsilon \in (0, \pi)$, let M be an n-dimensional nonnegatively curved Alexandrov space and let S be the set of ε-narrow points of M. We apply the contrapositive of Lemma 3.3.13 to S. The angles of any triangle with vertices in S are at most ε, so we must have $|S| < B(n, \varepsilon) := 2 \cdot (2 \cdot 25^n)^{m_n((\pi-\varepsilon)/2)-2}$. $\square$

We can now specialise back to the case of polyhedral 3-manifolds, by deducing the existence of a singular vertex bound—i.e., by proving Theorem 2.

Proof of Theorem 2. Let M be a nonnegatively curved integral polyhedral 3-manifold. Denote by M_{s}^0 the set of singular vertices of M. Corollary 3.2.13 tells us that every point of M_{s}^0 is ε_0-narrow, for some fixed ε_0. Therefore, since M is a 3-dimensional nonnegatively curved Alexandrov space (by Proposition 2.2.9), we may apply Proposition 3.3.7 to deduce that $|M_{\mathrm{s}}^0| < B_{\mathrm{ver}} := B(3, \varepsilon_0)$, which is constant. $\square$

Remark 3.3.14. The vertex bound B_{ver} is by no means effective from a numerical standpoint. Looking at the definition of $B(n,\varepsilon)$ from the proof of Proposition 3.3.7, we calculate that $B_{\text{ver}} = 2 \cdot 31250^{m_3((\pi-\varepsilon_0)/2)-2}$. When $n = 3$, Formula (3.3.5) reduces to

$$m_3(\delta) = \left\lfloor \frac{2}{1-\cos(\delta/2)} \right\rfloor + 2,$$

and for integral vertices ε_0 is at least $5\pi/6$ (this can be seen by noting that link #1 in Table A.1 is the double of a triangle having an edge of length $5\pi/6$). This bound is therefore at least in the region of 10^{1051}. As shown in Figure 3.7, the greatest number of singular vertices in a nonnegatively curved integral polyhedral 3-manifold known to the author is 32. Our aim has therefore been to demonstrate the *existence* of the bound. To produce any useful numerical bound, a different approach must be taken.

3.3.3. Connection to Li and Naber

It was mentioned at the start of Section 3.3 that Proposition 3.3.7 follows from a recent result of Li and Naber, [LN20, Cor. 1.4], although this was not known to the author until after the work in this section had been written. Let us briefly examine their result—in particular, why it implies ours. The central concept in [LN20] is *quantitative splitting*—a measure of how close a neighbourhood of a point is to being a metric product. We give a summary of the definitions relevant to our setting (see [LN20, Defs. 1.1–1.2]).

Definition 3.3.15 (Quantitative splitting)**.** Let M be a metric space, x a point in M, $r > 0$, k a nonnegative integer, and $\delta > 0$. We say that $B_r(x)$ is (k,δ)-*splitting* if there is a metric space Z and a point p in $\mathbb{R}^k \times Z$ such that $d_{\text{GH}}(B_r(x), B_r(p)) \le \delta r$, where d_{GH} is the Gromov–Hausdorff distance. We then define the (k,δ)-*singular set* to be

$$\mathcal{S}^k_\delta = \mathcal{S}^k_\delta(M) := \{x \in M \mid \forall r > 0, B_r(x) \text{ is not } (k+1,\delta)\text{-splitting}\}.$$

These definitions build on the established concept of singular sets. The *singular set* $\mathcal{S}(M)$ of an m-dimensional Alexandrov space M of curvature bounded below is defined as the set of points whose tangent cones are not isometric to $\mathbb{R}^m$. It is well known that this singular set has a natural stratification $\mathcal{S}(M) = \mathcal{S}^{m-1} \supseteq \ldots \supseteq \mathcal{S}^0$, where $\mathcal{S}^k$ is the set of points whose tangent cones are not isometric to $\mathbb{R}^{k+1} \times C$, for any metric cone C. (If M is a polyhedral manifold, then $\mathcal{S}^k$ is the set of singular points of codimension at least k.) One can think of $\mathcal{S}^k$ as the set of points where M does not locally look like $\mathbb{R}^{k+1} \times C$, for any metric cone C. The idea of quantitative splitting strengthens this notion, so that $\mathcal{S}^k_\delta$ is the set of points where M is, in a rigorous sense, δ far away from locally looking like $\mathbb{R}^{k+1} \times C$. As one would expect, $\mathcal{S}^k$ is equal to $\bigcup_{\delta>0} \mathcal{S}^k_\delta$.

The k-singular set $\mathcal{S}^k$ has Hausdorff dimension at most k and the same is true for $\mathcal{S}^k_\delta$. In particular, $\mathcal{S}^0_\delta$ is a discrete set of points. Li and Naber's result [LN20, Cor. 1.4] gives a bound on the k-dimensional Hausdorff measure of $\mathcal{S}^k_\delta \cap B_1(p)$ in any Alexandrov space M of curvature at least -1 and for any point p in M, depending only on δ and the dimension of M. When k is 0, this implies a bound on the *cardinality* of $\mathcal{S}^0_\delta \cap B_1(p)$. Proposition 3.3.7 concerns ε-narrow points, which always belong to $\mathcal{S}^0$, in *nonnegatively curved* Alexandrov spaces, so we now restrict to the case of $k = 0$ and curvature at least 0.

We first remove the dependence on $B_1(p)$ in the bounds given above, by rescaling the metric. For any positive scale λ, if M has nonnegative curvature, then so does λM. Note also that $\mathcal{S}^0_\delta(\lambda M)$ is the same as $\lambda\mathcal{S}^0_\delta(M)$. Any finite subset of $\mathcal{S}^0_\delta$ can thus be rescaled to lie within $B_1(p)$ for some point p in M. This means that, when M is nonnegatively curved, [LN20, Cor. 1.4] gives a bound on the cardinality of any finite subset of $\mathcal{S}^0_\delta$, depending only on δ and the dimension of M. This same bound then immediately applies to $|\mathcal{S}^0_\delta|$ itself. Therefore, showing that [LN20, Cor. 1.4] implies Proposition 3.3.7 reduces to showing that, for any $\varepsilon \in (0, \pi)$, there is $\delta = \delta(\varepsilon) > 0$ such that any ε-narrow point x of M belongs to $\mathcal{S}^0_\delta$. This is fairly simple: one can use Toponogov's theorem (Theorem 2.2.10) to show that, for any $r > 0$, the diameter of $B_r(x)$ is at most φr, where φ depends only on ε; then recall from [BBI01, Ex. 7.3.14] that $d_{\mathrm{GH}}(Y, Y') \geq \frac{1}{2}|\operatorname{diam} Y - \operatorname{diam} Y'|$, for bounded metric spaces Y and Y'.

Li and Naber's result therefore *does* imply Proposition 3.3.7, but nowhere near the full force of their result is needed in our specialised setting. Our proof is completely independent of their work and is (naturally) considerably shorter than the proof of [LN20, Thm. 1.3], from which [LN20, Cor. 1.4] follows.

3.4. Further Directions

Recall from Proposition 2.2.16 that the singular locus of a nonnegatively curved polyhedral 3-manifold is a union of graphs (of minimum degree 3 and possibly with some unbounded edges), circles, and lines embedded in the manifold. The vertices of the graphs are called *singular vertices* and the graph edges (bounded or unbounded), the circles, and the lines are collectively called *singular edges*. In this chapter, we have classified the local isometry types of singular vertices (and thus all points) when integrality is imposed and have shown that number of singular vertices is bounded. A natural extension is to attempt to control the size of the singular locus as a whole, rather than just its vertices.

Conjecture 3.4.1 (Singular edge bound)**.** *There is a constant B in $\mathbb{N}$ such that any nonnegatively curved integral polyhedral 3-manifold has fewer than B singular edges.*

This result, of course, implies Theorem 2, but Theorem 2 is probably needed to prove it. We know from [LN20, Cor. 1.4] that the sum of the lengths of the singular edges is less than some constant times the diameter of the space, but as yet we have no way to bound the total number. Conjecture 3.4.1 can also be weakened in another way, this time by neglecting the graph components of the singular locus.

Conjecture 3.4.2 (Singular circle/line bound)**.** *There is a constant B_{lin} in $\mathbb{N}$ such that the total number of singular circles and lines in any nonnegatively curved integral polyhedral 3-manifold is less than B_{lin}.*

Since, as noted in Corollary 3.2.12, the maximum degree of a singular vertex is 5, Theorem 2 implies a bound on the number of edges in the graph components of the singular locus. Conjecture 3.4.2 would therefore imply Conjecture 3.4.1.

Another way to extend Theorem 2 is to generalise it to higher dimensions. The proof did not actually require the full classification of singular vertices, but only that there are finitely many local isometry types. If we can show that, for any fixed dimension, there are only finitely many types of singular vertices (i.e., singularities of maximal codimension), then we should be able to apply Proposition 3.3.7 to nonnegatively curved integral polyhedral manifolds of any dimension.

Chapter 4

POLYHEDRAL 3-SPHERES SINGULAR ALONG SEIFERT LINKS

Much of the content of Chapter 3 may be viewed as a study of how algebraic constraints on the holonomy of a polyhedral 3-manifold have topological implications for the singular locus. We impose integrality (and nonnegative curvature), and we deduce, for example, Theorem 2 and Corollary 3.2.12—two results controlling the topology of the singular locus. In this chapter, we consider the reverse problem: we impose restrictions on the topology of the singular locus of a polyhedral 3-manifold and see what we can deduce about the holonomy. We also work in a different setting for this chapter; we restrict ourselves to *spherical* polyhedral metrics on Seifert fibre spaces wherein the singular locus is a union of fibres. As a concrete example, we study the holonomy of spherical polyhedral metrics on the 3-sphere wherein the singular locus is the Hopf link. We show that, in almost all cases, such a metric is a *PK-link* metric (recall Definition 2.3.11), and this allows us to give a simple description of almost all such metrics. For the rest of this chapter, we will refer to an n-sphere endowed with a spherical polyhedral metric as a *polyhedral n-sphere.*

The Hopf link is the simplest nontrivial 2-component link. An analogous setting in *two* dimensions is that of a spherical polyhedral metric on the 2-sphere with precisely two conical points. In [Tro89], Troyanov fully describes all such metrics, proving the following result:

Theorem ([Tro89, Thm. I])**.** *Let S be a polyhedral 2-sphere with with precisely two conical points of angles α and β. Then $\alpha = \beta$ and*

1. *If α is not divisible by 2π, then S is $\mathbb{S}^2(\alpha)$; whereas*
2. *If α is divisible by 2π, then S is a ramified cover of the unit sphere $\mathbb{S}^2$ with ramification at two distinct points.*

The first main result of this chapter can be viewed as a partial generalisation of this result to three dimensions.

Theorem 3 (Hopf link singularities)**.** *Let M be a polyhedral 3-sphere with no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$ whose singular locus is the Hopf link. Then M is the link of the product of two Euclidean 2-cones.*

The motivating idea behind this chapter comes from the fact that the Hopf link sits within a larger class of links that we refer to as as *Seifert links.* These will be defined precisely in Definition 4.1.3 but are essentially unions of fibres of a Seifert fibration of the 3-sphere. As was mentioned at the end of Subsection 2.3.1, Panov's work in [Pan09] implies that the singular locus of a PK-link metric on the 3-sphere forms a Seifert link, and in fact, *any* kind of Seifert link can appear in this way. This chapter arose as an attempt to demonstrate a converse to this observation: *"Given a polyhedral 3-sphere singular along a Seifert link, when can we show that it is a PK-link?"* We assume throughout that no conical angle belongs to $2\pi(\mathbb{N} \setminus \{1\})$. In Section 4.2, we reduce the question to a single sufficient criterion: the nontriviality of a certain holonomy element.

In the case of the Hopf link, we know exactly when this criterion holds. More than this, in Section 4.3, we show that a polyhedral 3-sphere singular along the Hopf link is *always* a PK-link, whether or not the criterion holds, provided that the conical angles do not belong to $2\pi(\mathbb{N} \setminus \{1\})$. From this, we deduce Theorem 3.

The other main result of this chapter is Theorem 4. This makes good headway in answering the question posed above in the case of *generic* Seifert links—i.e., those that are neither the unknot nor the Hopf link. Other than the usual requirement that the conical angles not be divisible by 2π, the only condition we impose is on the length of an *ordinary component,* a certain singular circle that will be defined just after Definition 4.4.1.

Theorem 4 (Seifert link singularities)**.** *Let M be a polyhedral 3-sphere with no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$ whose singular locus is a generic Seifert link. Let K be an ordinary component of the singular locus of M. If the length of K is not divisible by π, then M is a PK-link.*

This result allows us, under mild assumptions, to generalise and unify existing formulae for the volume, singular circle lengths, and conical angle restraints of polyhedral 3-spheres singular along torus links. Such formulae can be found in [DMM14] (for the trefoil knot), [Kol13] (for 3 and 4 component Hopf links), and [KM09] (for torus links of types $(2, 2n+1)$ and $(2, 2n)$).

The rest of the chapter is laid out as follows:

- In Section 4.1, we carefully define Seifert links and give a classification of them in Lemma 4.1.10. We note in Proposition 4.1.14 the existence of a special *central element* in the fundamental group of the complement to any Seifert link—this is the element whose holonomy will appear in the PK-link criterion mentioned above. We also recall the definition of PK-links and their classification in three dimensions and deduce that the singular locus of a PK-link is always a Seifert link.

- In Section 4.2, we derive the three closely related PK-link criteria, Corollaries 4.2.5 to 4.2.7, showing that the nontriviality of the holonomy of the square of the central element mentioned above is sufficient to ensure that the metric is a PK-link metric. The argument is entirely linear-algebraic.

- In Section 4.3, we prove Theorem 3. After fixing some terminology and notation, we make a quick application of Corollary 4.2.7 to metrics singular along the Hopf link. The rest of the section is largely devoted to proving that such a metric is a PK-link metric even when Corollary 4.2.7 does not apply—specifically, when the conical angles are odd multiples of π. We do this by direct consideration of the metric and developing map. With a little extra work, we then deduce Theorem 3.

- Finally, in Section 4.4, we make a brief investigation into when the criterion in Corollary 4.2.6 is satisfied for a generic Seifert link, which culminates in the proof of Theorem 4. Using results about polyhedral Kähler metrics from [Pan09] and about spheres with conical points from [MP16], we derive some explicit geometric formulae for polyhedral 3-spheres singular along generic Seifert links.

4.1. Preliminaries

In this section, we introduce the main objects required for this chapter—Seifert links, (p, q)-maps, and PK-links—and make some basic observations about them. We begin by recalling some key definitions from Chapter 2 in light of the spherical setting.

Let M be a spherical polyhedral n-manifold. Recall from Definition 2.3.3 that the *developing map* is a map $\mathrm{Dev}\colon \mathrm{Ram}\, M \to \mathbb{S}^n$ that is a local isometry outside of the singularities of $\mathrm{Ram}\, M$. It may be viewed as a multivalued map $M \to \mathbb{S}^n$ that sends geodesics to arcs of great circles. Recall from Definition 2.3.4 that the *holonomy map* is the map $\mathrm{Hol}\colon \pi_1(M \setminus M_{\mathrm{s}}) \to \mathrm{O}(n)$ defined by equivariance with the developing map. In most of this chapter, we will be dealing with polyhedral *3-spheres,* in which case the developing map maps into $\mathbb{S}^3$ and the holonomy map into $\mathrm{SO}(4)$, because the 3-sphere is orientable.

4.1.1. Seifert Links

Since this chapter concerns spherical polyhedral metrics on the 3-sphere whose singular locus forms a Seifert link, we must give a precise definition of a Seifert link. The exact definition is fairly simple but relies on the definition of a *Seifert fibre space.* We do not reproduce this classical definition here but instead refer the reader to [Hem76, Ch. 12] or [GL18, § 2] for a basic introduction. The one key piece of information we wish to recall is the notion of *ordinary* and *exceptional fibres.*

Definition 4.1.1 (Ordinary and exceptional fibres)**.** Let $f\colon M \to B$ be a Seifert fibration and p a point in B, so that $t := f^{-1}(p)$ is a fibre of f. If there is a neighbourhood U of t such that the restriction $f|_U$ looks like the projection $D^2 \times S^1 \to D^2$, where D^2 is the open 2-disc, we say that t is *ordinary.* Otherwise, we say that t is *exceptional.* All ordinary fibres are homotopic in the complement to the exceptional fibres, so by abuse of terminology we will often refer to the homotopy class of an ordinary fibre as *the* ordinary fibre. We will also freely switch between the homotopy class and a specific ordinary fibre.

Remark 4.1.2. Ordinary fibres are sometimes called *generic*, and exceptional fibres are sometimes called *singular.* We avoid using these terms in this way, as they are already used in other contexts—e.g., *singular* locus and edges and *generic* Seifert links (see Definition 4.4.1).

Definition 4.1.3 (Seifert link)**.** A link L in the 3-sphere is called a *Seifert link* if there is a Seifert fibration f of the 3-sphere such that L is equivalent to a union of fibres of f. By *equivalent,* we mean that there is a self-homeomorphism of the 3-sphere sending one link to the other. Links that are mirror images of each other are thus considered equivalent.

We now give some archetypal examples of Seifert links, which will form the basis of their classification.

Example 4.1.4. A Seifert fibration can be viewed as a circle bundle with isolated exceptional fibres. The only genuine circle bundle whose total space is the 3-sphere is the Hopf bundle (up to possibly orientation-reversing equivalence—see [GL18]). Therefore, any collection of Hopf circles—i.e., fibres of the Hopf bundle—inside the 3-sphere is a Seifert link (see Figure 4.1). A link that is equivalent to a collection of n Hopf circles is called a *Hopf n-link.* The Hopf 2-link is the usual Hopf link, and the Hopf 1-link is of course just the unknot.

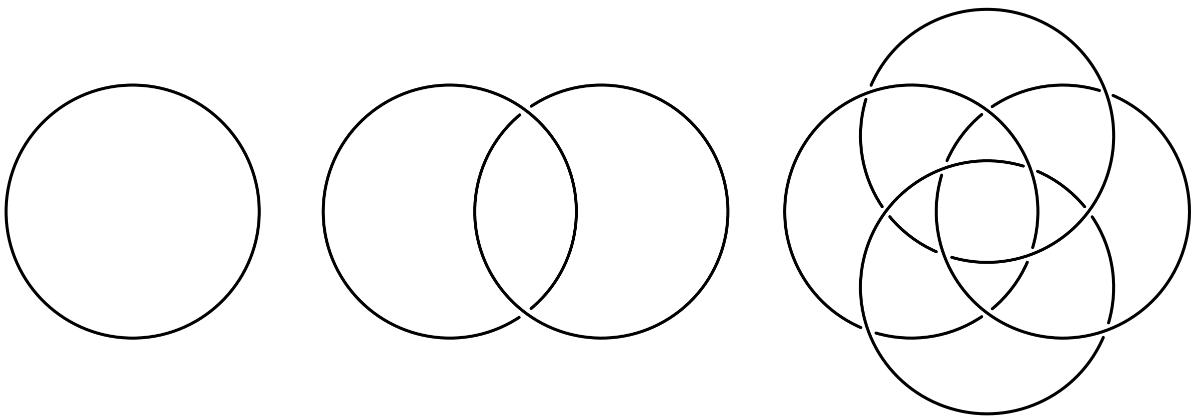


Figure 4.1. Hopf n-links, for $n = 1$, 2, and 4.

Example 4.1.5. Given coprime integers $1 \leq p \leq q$, a (p, q)-torus knot is a Seifert link. This is because it is equivalent to an orbit of the following circle action on the 3-sphere, which defines a Seifert fibration. Viewing S^3 as the unit sphere in $\mathbb{C}^2$ and S^1 as the unit circle in $\mathbb{C}$, the action is defined by

$$\lambda \cdot (z, w) := (\lambda^p z, \lambda^q w), \tag{4.1.1}$$

for $\lambda \in S^1$ and $(z, w) \in S^3$. The orbits of this action are the fibres of a Seifert fibration of the 3-sphere, and in fact, up to (possibly orientation-reversing) equivalence, every Seifert

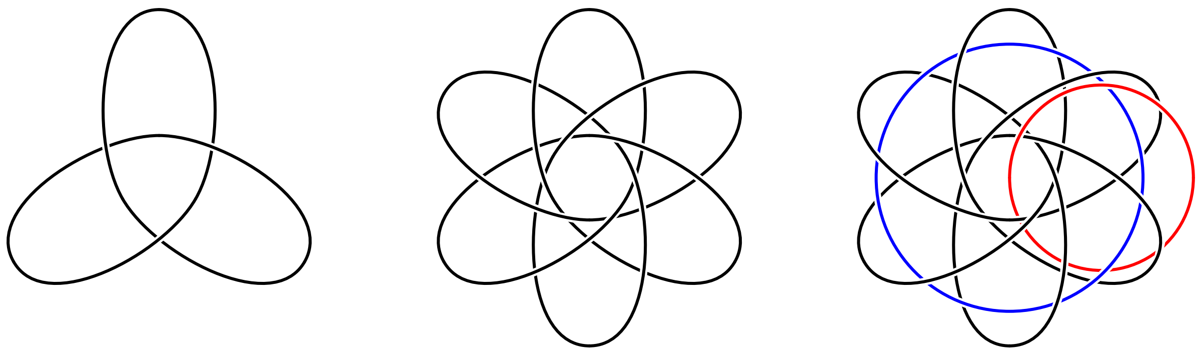


FIGURE 4.2. Left: $(2,3)$-torus knot (trefoil knot). Middle: $(4,6)$-torus link. Right: $(4,6)$-torus link with both axes, the 2-axis shown in blue and the 3-axis in red.

fibration of the 3-sphere arises this way (see [GL18, Prop. 5.2]). By taking n ordinary orbits of this action, we see that an (np, nq)-torus link is also Seifert link (see Figure 4.2).

The circle action defined above is worthy of special note, as it is central to the rest of the chapter.

Definition 4.1.6 ((p,q)-map)**.** For coprime integers $1 \leq p \leq q$, the action defined in Formula (4.1.1) (or any action conjugate to it) is called the (p,q)*-action.* As noted in Example 4.1.5, the orbits of this action are the fibres of a Seifert fibration of the 3-sphere. The orbit space is topologically a 2-sphere—this may be seen by identifying it with the weighted projective line $\mathbb{P}(p,q)$ (see [BFNR13, § 1]). The quotient map, denoted by $f_{(p,q)}\colon S^3 \to S^2$, is called the (p,q)*-map.*

Definition 4.1.7 (p and q-axes)**.** The following subsets of the 3-sphere are fibres of $f_{(p,q)}$—equivalently, orbits of the (p,q)-action:

$$\{(z,0) \in \mathbb{C}^2 \mid |z| = 1\} \quad \text{and} \quad \{(0,w) \in \mathbb{C}^2 \mid |w| = 1\}.$$

They are referred to as the p and q*-axes* and denoted by O_p and O_q respectively (see right of Figure 4.2). The images of O_p and O_q under $f_{(p,q)}$ are cyclic orbifold points of the 2-sphere of orders p and q respectively, denoted by P and Q.

Remark 4.1.8. The orders (or multiplicities) of O_p and O_q viewed as fibres of $f_{(p,q)}$ are p and q respectively. In other words, if all the other, ordinary fibres have period 1 under the (p,q)-action, then O_p and O_q have periods $1/p$ and $1/q$. This is reflected in the fact that the images of O_p and O_q in the base orbifold are cyclic of order p and q. Therefore, $f_{(1,1)}$ has no exceptional fibres; there is nothing special about the axes. (This is because $f_{(1,1)}$ is the Hopf fibration.) When $1 = p < q$, the only exceptional fibre is O_q, and when $1 < p < q$, the exceptional fibres are O_p and O_q.

Example 4.1.9. By adding in exceptional fibres, we arrive at our last family of examples of Seifert links. For integers n and q greater than 1, the union of O_q with n ordinary fibres of $f_{(1,q)}$ is a Seifert link, called an (n, nq)*-torus link with the* q*-axis.* For n a positive integer and $1 < p < q$, the union of n ordinary fibres of $f_{(p,q)}$ with one or both of O_p and O_q is a Seifert link, naturally referred to as an *(np, nq)-torus link with the p-axis, the q-axis,* or *with both axes.*

It turns out that Examples 4.1.4, 4.1.5, and 4.1.9 cover every possible Seifert link, as we show in the following result.

Lemma 4.1.10. *Let L be a Seifert link in the 3-sphere. Then L is equivalent to one of the following mutually exclusive possibilities:*

1. *The unknot;*
2. *The Hopf link;*
3. *A Hopf n-link, for $n > 1$;*
4. *An (n, nq)-torus link, with or without the q-axis, for n and $q > 1$; or*
5. *An (np, nq)-torus link, with none, one, or both of the p and q-axes, for $n \geq 1$ and $1 < p < q$ coprime.*

Proof. As noted in [GL18, Prop. 5.2], any Seifert fibration of S^3 is equivalent to $f_{(p,q)}$, for some coprime integers $1 \leq p \leq q$. Therefore, any Seifert link L is equivalent to a union of fibres of some $f_{(p,q)}$. We consider the different cases.

When $1 = p = q$, $f_{(1,1)}$ is just the Hopf fibration. If n is the number of components of L, then we fall into case 1, 2, or 3, according to whether n is 1, 2, or greater than 2.

When $1 = p < q$, let n be the number of components of L that are ordinary fibres of $f_{(1,q)}$. If $n = 0$, then L is O_q, and we are in case 1. Note that O_q forms a Hopf link with any other fibre of $f_{(1,q)}$, and that all the fibres of $f_{(1,q)}$ are unknots, as can be seen in Figure 4.3. Therefore, when $n = 1$, then if O_q is not a component of L, we are in case 1, whereas if it is, we are in case 2. When $n > 1$, we are in case 4.

When $1 < p < q$, again let n be the number of components of L that are ordinary fibres of $f_{(p,q)}$. If $n = 0$, then either L is O_p or O_q, in which case we are in case 1, or L is $O_p \cup O_q$, in which case we are in case 2. Finally, if $n \geq 1$, we are in case 5. □

Definition 4.1.11. If a Seifert link can be realised by a (p, q)-fibration, then we say it *has* or *is of type* (p, q). From the proof above, we see that in cases 3, 4, and 5, the type is unique. Case 3 always has type $(1, 1)$, case 4 has type $(1, q)$, and case 5 has type (p, q). The unknot and the Hopf link have type (p, q), for any coprime integers $1 \leq p \leq q$.

Let us now consider some nonexamples of Seifert links, which nonetheless in a sense come close to being Seifert links.

Example 4.1.12. An unlink of multiple components is not a Seifert link—its complement has free fundamental group of rank greater than 1, and therefore by [BM70, Thm. 1] is not Seifert fibred. However, it is a torus link; indeed, the components may be arranged as parallel meridians of a torus. Unlinks are the only torus links that are not Seifert links.

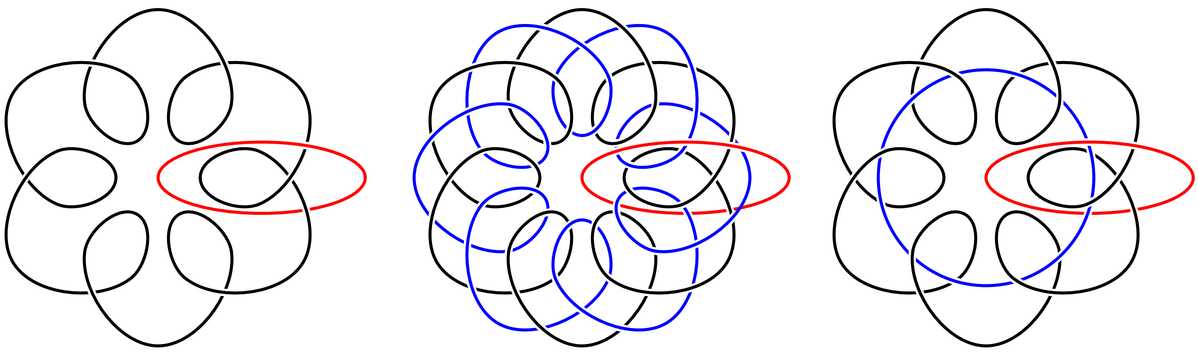


Figure 4.3. Left: $(1, 6)$-torus knot with the 6-axis (equivalent to the Hopf link). Middle: $(2, 12)$-torus link with the 6-axis. This is equivalent to the figure on the right: $(1, 6)$-torus knot with both axes, the blue ordinary fibre becoming the '1-axis'.

Example 4.1.13. A *keychain link* with multiple *keys* is not a Seifert link. It is constructed by taking at least two parallel meridians of a torus (the 'keys') and then adding in the core of the torus (the 'keyring'—see Figure 4.4). This is *almost* a Seifert link, in the sense that its complement is Seifert fibred—the complement to the keyring is an open solid torus, which admits a trivial fibration of which the keys are fibres. However, this fibration does not extend to the whole 3-sphere, as is noted in [BM70]. In fact, by [BM70, Thm. 1], keychain links are the only links with Seifert fibred complements that are not Seifert links.

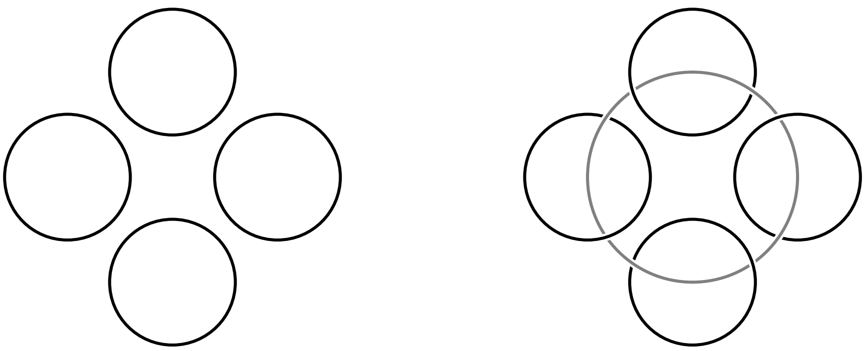


Figure 4.4. On the left is an unlink of four components. These components become the keys of a keychain link, shown on the right, with the keyring shown in grey.

We finish this subsection with a key result about Seifert links, which allows us to relate them to PK-links later in the chapter.

Proposition 4.1.14. *Let L be a Seifert link and let t be the ordinary fibre of a Seifert fibration of the 3-sphere realising L. Then $t \in Z(\pi_1(S^3 \setminus L))$.*

Proof. The Seifert fibration f of S^3 realising L restricts to a Seifert fibration of $S^3 \setminus L$ whose base Σ is a 2-sphere with n punctures, where n is the number of components of L. Then [Hem76, Thm. 12.1] gives an explicit presentation of $\pi_1(S^3 \setminus L)$. Since Σ is orientable and has genus 0 and n boundary components, we have

$$\pi_1(S^3 \setminus L) \cong \langle t, c_1, \ldots, c_m, d_1, \ldots, d_n \mid c_i t c_i^{-1} = t^{\eta_i}, d_j t d_j^{-1} = t^{\vartheta_j}, \text{ other relations}\rangle,$$

where $m \in \{0, 1, 2\}$ is the number of exceptional fibres of f *not* belonging to L, and each η_i and ϑ_j is either 1 or -1. The c_i represent meridians of the exceptional fibres not belonging to L, and the d_j represent meridians of the (removed) components of L. The end of the proof of [Hem76, Thm. 12.1] tells us that, since Σ is orientable, $S^3 \setminus L$ is orientable if and only if every η_i and ϑ_j is 1. Since $S^3 \setminus L$ *is* orientable, every η_i and ϑ_j *is* 1, and thus t commutes with everything. $\square$

4.1.2. PK-links

As noted in the opening of this chapter, the concept of a *PK-link* is central to it. Therefore, we now recall the definition of polyhedral Kähler manifolds and PK-links and state some key facts about 3-dimensional PK-links. The section culminates by circling back round to the question connecting Seifert links to PK-links that was stated at the beginning of this chapter. Most of the material in this subsection can be deduced from [Pan09].

Recall from Definition 2.3.10 that a Euclidean polyhedral $2n$-manifold M (with no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$) is called *polyhedral Kähler* if its monodromy group is conjugate to a subgroup of $\mathrm{U}(n)$—i.e., if there is a matrix P in $\mathrm{GL}_{2n}(\mathbb{R})$ such that $P^{-1}(\mathrm{Mon}\, M)P$ is a subgroup of $\mathrm{U}(n)$. Recall from Definition 2.3.11 that a (spherical) polyhedral $(2n-1)$-sphere M is called a *PK-link* if the Euclidean cone over M is polyhedral Kähler—i.e., if M is the *link* of a polyhedral Kähler *(PK)* cone. This is the same as saying that $\mathrm{Hol}\, M$ is conjugate to a subgroup of $\mathrm{U}(n)$.

Remark 4.1.15 (see [Pan09, Def. 1.1])**.** The above definitions assume that there are no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$. When the conical angle of a codimension 2 simplex does lie in $2\pi(\mathbb{N} \setminus \{1\})$, an additional condition is required. The requirement that the monodromy be conjugate to a subgroup of $\mathrm{U}(n)$ implies that there is a parallel complex structure J on $M \setminus M_{\mathrm{s}}$. We say that a codimension 2 simplex σ *has a holomorphic direction* if it is part of a holomorphic hyperplane with respect to J. This is the same as saying that the developing image of σ lies on a union of complex hyperplanes in $\mathbb{R}^{2n}$ (with respect to the basis that makes $\mathrm{Mon}\, M$ unitary). The full definition of a polyhedral Kähler manifold has the additional condition that every codimension 2 simplex with conical angle in $2\pi(\mathbb{N} \setminus \{1\})$ has a holomorphic direction (codimension 2 simplices with other conical angles automatically have holomorphic directions). This added condition is the reason why we must exclude the case of conical angles in $2\pi(\mathbb{N} \setminus \{1\})$ from many of the results of this chapter, particularly Theorems 3 and 4.

The relationship between PK-links and Seifert links becomes clear in the following classification of 3-dimensional PK-links, which, by means of (p, q)-maps, reduces the problem of understanding PK-links to understanding spheres with conical points. The classification is a synthesis of various results from [Pan09]—namely Theorems 1.7, 1.8, and Lemma 3.10, and material surrounding them—modified to pertain to PK-links, rather than the PK cones over them.

Theorem 4.1.16. *Let M be a 3-dimensional PK-link. Then either M is the link of a product $C^2(\alpha) \times C^2(\beta)$, which we will denote by $\Sigma(\alpha, \beta)$, or there is a 2-sphere S of curvature 4 with conical points and a (p, q)-map $f_{(p,q)} \colon M \to S$. In this latter case, we say that M* has *or* is of type *(p, q), and the following statements hold:*

1. *The fibres of $f_{(p,q)}$ are geodesics of M, and M_{s} is a union of fibres;*
2. *Any two fibres are parallel, and the distance between them is equal to the distance between their images in S;*
3. *All ordinary fibres have the same length—say l—such that S has area $l/2pq$ and M has volume $l^2/2pq$;*
4. *The exceptional fibres O_p and O_q have lengths l/p and l/q respectively;*
5. *If an ordinary fibre has conical angle α, then its image in S is a point of conical angle α; and*
6. *If O_p and O_q have conical angles β_p and β_q respectively, then their images P and Q have conical angles β_p/p and β_q/q respectively.*

Furthermore, the set of PK-links (up to isometry) of type (p, q) with ordinary fibres of length l is in bijective correspondence with the set of spheres (up to isometry) of curvature 4 with conical points and with area $l/2pq$, where one point is marked Q when $1 = p < q$ and two points are marked P and Q when $1 < p < q$.

Remark 4.1.17. This result might seem to suggest that the link of a product can never admit a (p, q)-fibration, but it in fact can, and this will become important in the proof of Theorem 3 in Section 4.3. As implied by [Pan09, Thm. 1.7], if α/β is irrational, then $\Sigma(\alpha, \beta)$ admits no such fibration. But suppose that $\alpha/\beta = p/q$, where $1 \leq p \leq q$ are coprime integers, and let $\omega := \alpha/p = \beta/q$. Then $\Sigma(\alpha, \beta)$ admits a (p, q)-map the spherical football of angle ω with curvature 4, rather than curvature 1, which we denote by $\mathbb{S}^2_4(\omega)$ (recall Definition 2.2.15—we may define $\mathbb{S}^2_4(\omega)$ as $\mathbb{S}^2(\omega)$ with the metric scaled by $1/2$). The fibres over the two points of conical angle ω are the two singular circles, the one of angle α having order p and length β, and the one of angle β having order q and length α. The ordinary fibres therefore have length $l := q\alpha = p\beta$, and $\mathbb{S}^2_4(\omega)$ has area $\omega/2 = l/2pq$.

Since the singular locus of $\Sigma(\alpha, \beta)$ is either empty (when both α and β are 2π), the unknot (when only one of them is), or the Hopf link (when neither of them is), the classification of PK-links in three dimensions has the following implication.

Corollary 4.1.18. *The singular locus of a 3-dimensional PK-link is either empty or a Seifert link, and if the PK-link has type (p, q), then so does its singular locus.*

We finish this section by asking when this corollary can be reversed—i.e., by restating the question we asked at the opening of the chapter.

Question. *Given a polyhedral 3-sphere singular along a Seifert link, when can we show that it is a PK-link?*

4.2. A Single Criterion for PK-links

In this section, we prove that, to show that a polyhedral 3-sphere singular along a Seifert link is a PK-link, it is sufficient to evaluate the holonomy of the ordinary fibre. Most of the argument is purely linear-algebraic, showing in Proposition 4.2.1 that a subset of $\mathrm{SO}(2n)$ is conjugate to a subset of $\mathrm{U}(n)$ if its centraliser contains a sufficiently nontrivial element. Only at the end of the section is this applied to the polyhedral setting: first to arbitrary even-dimensional Euclidean polyhedral manifolds, then to polyhedral 3-spheres singular along Seifert links, and finally to polyhedral 3-spheres singular along the Hopf link.

Before proceeding with the argument, we give some notational conventions to shorten and clarify proofs involving large matrices. We also recall how $\mathrm{U}(n)$ is defined as a subgroup of $\mathrm{SO}(2n)$.

Notation. Given square matrices M_i in $M_{k_i}(\mathbb{R})$, for $i \in \{1, \ldots, s\}$, we adopt the following block matrix shorthand:

$$M_1 \oplus \ldots \oplus M_s = \bigoplus_{i=1}^{s} M_i := \begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_s \end{pmatrix} \in M_n(\mathbb{R}),$$

where $n = k_1 + \ldots + k_s$. Let $R(\alpha)$ denote the 2×2 rotation matrix of angle α—i.e., $R(\alpha) := \left(\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\right)$. Generalising this, we define a $2k \times 2k$ rotation matrix:

$$R_k(\alpha) := \bigoplus_{i=1}^{k} R(\alpha). \tag{4.2.1}$$

We now recall the definition of $\mathrm{U}(n)$ given in Subsection 2.3.1. The *standard complex structure* on $\mathbb{R}^{2n}$ is defined as follows:

$$J_n := R_n(\pi/2) = \bigoplus_{i=1}^{n} \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

This allows us to define $\mathrm{U}(n)$ as the subgroup of those elements of $\mathrm{O}(2n)$ that commute with J_n. Any matrix that commutes with J_n has nonnegative determinant, and therefore $\mathrm{U}(n)$ is in fact a subgroup of $\mathrm{SO}(2n)$. When $n = 1$, $\mathrm{U}(1)$ and $\mathrm{SO}(2)$ are in fact identical.

We now precisely state the main result of this section, from which we will deduce the desired criterion for PK-links.

Proposition 4.2.1. *Let S be a subset of $\mathrm{SO}(2n)$ and suppose that there is an element Ω of $\mathrm{SO}(2n)$ with at most two real eigenvalues (counted with multiplicity) that commutes with every element of S. Then S is conjugate to a subset of $\mathrm{U}(n)$.*

To prove this, we make an orthonormal change of basis with respect to which Ω is block diagonal, each block having the form $R_k(\alpha)$ (as defined in Formula (4.2.1)). The fact that every element of S commutes with Ω implies that, with respect to this basis, S is a subset of $\mathrm{U}(n)$. We give the proof after two lemmas that are used in it.

Lemma 4.2.2. *If a matrix M in $M_{2k}(\mathbb{R})$ commutes with $R_k(\alpha)$, for some $\alpha \in (-\pi, \pi)\backslash\{0\}$, then M commutes with J_k.*

Proof. Note that $R_k(\alpha) = (\cos\alpha)I + (\sin\alpha)J_k$. Therefore, if M commutes with $R_k(\alpha)$, it also commutes with $(\sin\alpha)J_k$, and thus with J_k when $\sin\alpha$ is nonzero. □

Lemma 4.2.3. *Suppose that a matrix M in $\mathrm{SO}(2n)$ has the form $A \oplus B$, for some matrix A in $\mathrm{U}(n-1)$. Then $M \in \mathrm{U}(n)$.*

Proof. Since the columns of M are orthonormal, the columns of B are orthonormal, and so $B \in \mathrm{O}(2)$. As noted earlier, $\mathrm{U}(n-1)$ is a subgroup of $\mathrm{SO}(2(n-1))$, so $\det A = 1$. Then we have

$$1 = \det M = \det A \det B = \det B,$$

and so $B \in \mathrm{SO}(2) = \mathrm{U}(1)$. Now, A commutes with J_{n-1}, and B commutes with J_1, so M commutes with $J_{n-1} \oplus J_1 = J_n$. Therefore, $M \in \mathrm{U}(n)$. □

Proof of Proposition 4.2.1. We begin by putting Ω into *real Schur form* (see [HJ13, Thm. 2.3.4]). There is an orthogonal change-of-basis matrix P for which $\Omega' := P^T \Omega P$ has one of the following two forms:

1. $\bigoplus_{i=1}^{s} R_{k_i}(\alpha_i)$, where $k_i \in \mathbb{N}$, $k_1 + \ldots + k_s = n$, $a_i \in (-\pi, \pi) \setminus \{0\}$, and $\alpha_i \neq \pm\alpha_j$ for $i \neq j$; or
2. $(\bigoplus_{i=1}^{s} R_{k_i}(\alpha_i)) \oplus (\pm I)$, where $k_1 + \ldots + k_s = n-1$, and the α_i are as above.

Case 1 occurs when Ω has no real eigenvalues, and case 2 occurs when it does. We will show that, in both of these cases, every element of $P^T SP$ belongs to $\mathrm{U}(n)$. Indeed, let M be an arbitrary element of $P^T SP$.

Case 1. The eigenvalues of any block $R_{k_i}(\alpha_i)$ of Ω' are distinct from those of any other. Since M commutes with Ω', it preserves eigenspaces of Ω'. Therefore, M has the form $\bigoplus_{i=1}^{s} M_i$, where each M_i lies in $\mathrm{O}(2k_i)$ and commutes with $R_{k_i}(\alpha_i)$. By Lemma 4.2.2, each M_i commutes with J_{k_i}. Thus, M commutes with $\bigoplus_{i=1}^{s} J_{k_i} = J_n$ and thus lies in $\mathrm{U}(n)$.

Case 2. By the same reasoning as above, $M = A \oplus B$, where $A \in \mathrm{U}(n-1)$. Since $M \in \mathrm{SO}(2n)$, we can apply Lemma 4.2.3 to deduce that $M \in \mathrm{U}(n)$. □

Remark 4.2.4. As can be seen in the proof above, the eigenvalue condition on Ω in Proposition 4.2.1 essentially means that Ω does not act as $\pm I$ in any number of dimensions greater than two. It is vacuous when $n = 1$, and when $n = 2$, it reduces to the nontriviality of Ω^2—i.e., to $\Omega^2 \neq I$.

Now that we have Proposition 4.2.1, we can immediately deduce the following simple criterion for polyhedral Kähler manifolds, and thus for PK-links:

Corollary 4.2.5. *Let N be an even-dimensional orientable Euclidean polyhedral manifold with no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$. Suppose that there is an element g of $Z(\pi_1(N \setminus N_{\mathrm{s}}))$ such that* $\operatorname{Mon} g$ *has at most two real eigenvalues, counted with multiplicity. Then N is polyhedral Kähler. Similarly, if M is a polyhedral sphere of odd-dimension satisfying the same conditions as above (with* $\operatorname{Mon} g$ *replaced by* $\operatorname{Hol} g$*), then M is a PK-link.*

By combining this corollary with Proposition 4.1.14, we get the following, more explicit result in three dimensions:

Corollary 4.2.6. *Let M be a polyhedral 3-sphere singular along a Seifert link and with no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$ and let t be the ordinary fibre of a Seifert fibration of $M \setminus M_{\mathrm{s}}$. If* $\operatorname{Hol} t^2$ *is nontrivial, then M is a PK-link.*

Finally, using the classical fact that the fundamental group of the Hopf link complement is $\mathbb{Z}^2$, we get one more criterion:

Corollary 4.2.7. *Let M be a polyhedral 3-sphere singular along the Hopf link and with no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$. If there is any element g of $\pi_1(M \setminus M_{\mathrm{s}})$ for which* $\operatorname{Hol} g^2$ *is nontrivial, then M is a PK-link.*

4.3. Hopf Link Singularities

Now that we have Corollary 4.2.7, we focus our attention on polyhedral 3-spheres singular along the Hopf link, with the goal of proving the third main result of the thesis.

Theorem 3 (Hopf link singularities)**.** *Let M be a polyhedral 3-sphere with no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$ whose singular locus is the Hopf link. Then M is the link of the product of two Euclidean 2-cones.*

For the rest of this section, we will denote (the equivalence class of) the Hopf link inside the 3-sphere by H and any polyhedral 3-sphere singular along H with conical angles α and β by $H(\alpha, \beta)$. We begin by fixing some notation relating to $H(\alpha, \beta)$, which we will use throughout the rest of the section. We then make a simple application of Corollary 4.2.7, showing that $H(\alpha, \beta)$ is a PK-link for almost all values of α and β. The remaining cases are when α and β are both odd multiples of π, and when one of them lies in $2\pi(\mathbb{N} \setminus \{1\})$. By careful consideration of the metric and developing map, we show that, in the former case, $H(\alpha, \beta)$ is again a PK-link. This allows us, with a little additional work, to deduce Theorem 3. Finally, we give two examples to show that $H(\alpha, \beta)$ need not be a PK-link when both α and β lie in $2\pi(\mathbb{N} \setminus \{1\})$, and that, even if it is, it need not be $\Sigma(\alpha, \beta)$ (i.e., the link of the product $C^2(\alpha) \times C^2(\beta)$).

Notation. As mentioned above, any polyhedral 3-sphere singular along the Hopf link with conical angles α and β will be denoted by $H(\alpha,\beta)$. *A priori,* this means that $H(\alpha,\beta)$ could refer to multiple nonisometric objects. However, it follows from Theorem 3 that, as long as α and β do not lie in $2\pi(\mathbb{N}\setminus\{1\})$, $H(\alpha,\beta)$ is isometric to $\Sigma(\alpha,\beta)$, and therefore its isometry type is determined by α and β. Of course, the notation $H(\alpha,\beta)$ presupposes that neither α nor β is 2π, otherwise the singular locus would not be H. We will denote the singular circles of angles α and β by L_1 and L_2 respectively. Finally, fix a path γ of minimal length joining L_1 to L_2 and denote its midpoint by x. Note that γ is orthogonal to both L_1 and L_2. We now define standard generators for $\pi_1(S^3\setminus H)$.

Definition 4.3.1 (Meridians)**.** For $i=1$ or 2, we define the *meridian* μ_i of L_i as follows. Starting at x, follow γ towards L_i. When very close to L_i, circle once around L_i anticlockwise, remaining at a constant distance from L_i, and then travel back along γ to x. (See Figure 4.5 for a visual explanation.) Note that $\pi_1(S^3\setminus H)$ is the free Abelian group $\mathbb{Z}\langle\mu_1,\mu_2\rangle$. By considering a local model for L_1, we see that $\operatorname{Hol}\mu_1$ is a simple rotation of angle α—in other words, there is an orthonormal basis with respect to which it has the form $R(\alpha)\oplus I$. Similarly, $\operatorname{Hol}\mu_2$ is a simple rotation of angle β.

As promised, we now make a quick application of Corollary 4.2.7, covering $H(\alpha,\beta)$ for almost all values of α and β.

Lemma 4.3.2. *Let α and β be positive real numbers that are not divisible by 2π and suppose that at least one of them is not divisible by π. Then $H(\alpha,\beta)$ is a PK-link.*

Proof. Without loss of generality, assume that α is not divisible by π. Then there is an orthonormal basis with respect to which $\operatorname{Hol}\mu_1^2 = R(2\alpha)\oplus I$, which is nontrivial. We can thus apply Corollary 4.2.7 with $g=\mu_1$ to deduce that $H(\alpha,\beta)$ is a PK-link. □

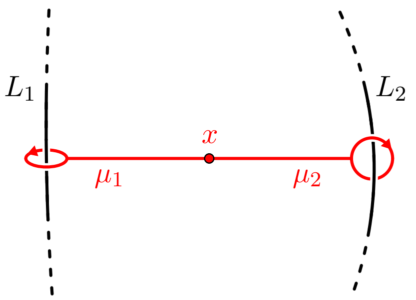


Figure 4.5. Meridians in a polyhedral 3-sphere singular along the Hopf link.

We now wish to show that $H(\alpha,\beta)$ is in fact *still* a PK-link even when both α and β are odd multiples of π. We do this in Proposition 4.3.6. The strategy is to prove something stronger: that, with respect to an appropriate basis, $\operatorname{Hol}\mu_1 = I\oplus(-I)$ and $\operatorname{Hol}\mu_2=(-I)\oplus I$. To do this, we begin in Lemma 4.3.3 by showing that the path of minimal length joining the two singular circles in $H(\alpha,\beta)$ has length at most $\pi/2$. This allows us in Lemma 4.3.4 to use the developing map to show that the fixed circles of $\operatorname{Hol}\mu_1$ and $\operatorname{Hol}\mu_2$ do not intersect. This, combined with the fact that $\operatorname{Hol}\mu_1$ and $\operatorname{Hol}\mu_2$ commute, implies that $\operatorname{Hol}\mu_1$ and $\operatorname{Hol}\mu_2$ have the desired form given above. This implication is demonstrated in Lemma 4.3.5, before being applied in Proposition 4.3.6 to deduce that $H(\alpha,\beta)$ is a PK-link.

Lemma 4.3.3. *Any path of minimal length joining the two singular circles in $H(\alpha,\beta)$ has length at most $\pi/2$.*

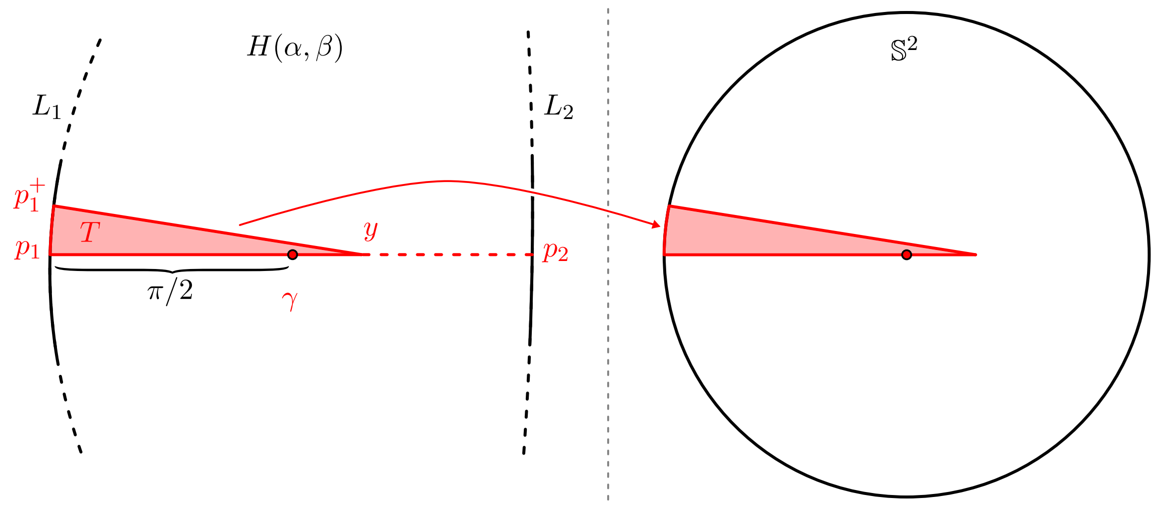


Figure 4.6. Isometrically embedding the geodesic surface T bounded by the triangle $\triangle p_1 y p_1^+$ into a unit hemisphere, in order to deduce that $[p_1^+ y]$ is shorter than $[p_1 y]$. The red arrow denotes the embedding.

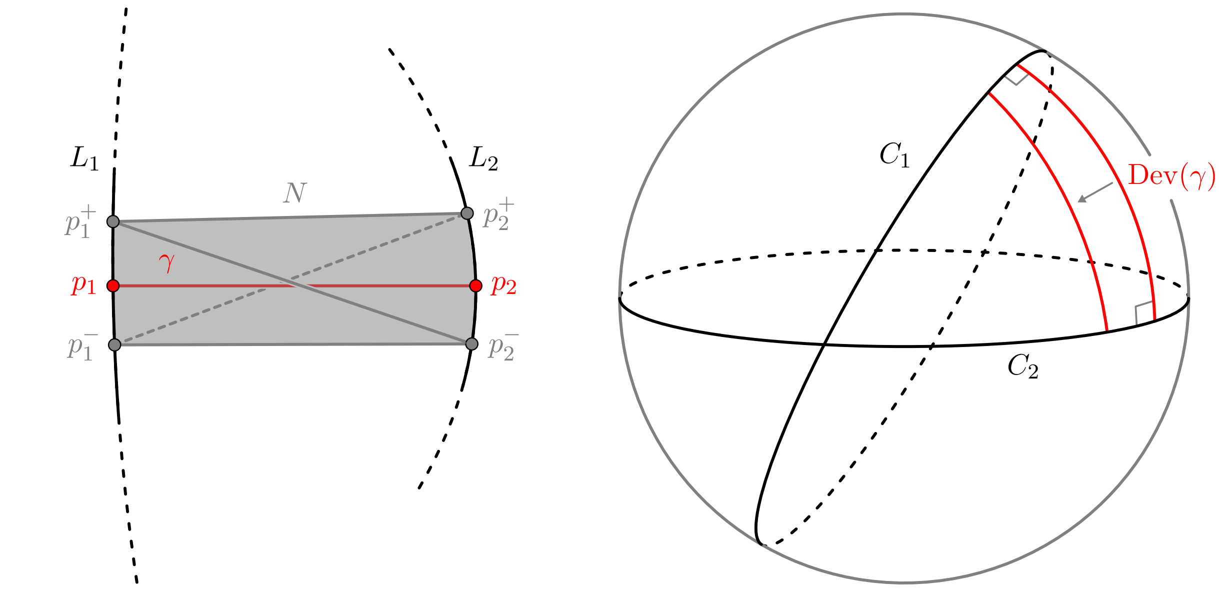
Proof. Suppose for a contradiction that such a path γ has length greater than $\pi/2$. Let p_1 and p_2 be the endpoints of γ lying on L_1 and L_2 respectively. Let y be a point on γ such that the subsegment $[p_1 y]$ has length in $(\pi/2, \pi)$. Let p_1^+ be a point on L_1 some small distance δ away from p_1. We show that there is a totally geodesic surface T whose boundary is the triangle $\triangle p_1 y p_1^+$ (see the left of Figure 4.6). Indeed, a portion of this surface exists inside a geometric simplex containing $[p_1 p_1^+]$ and small segments of $[p_1 y]$ and $[p_1^+ y]$. We can extend this portion all the way along $[p_1 y]$ by repeatedly taking adjacent geometric simplices that intersect both $[p_1 y]$ and $[p_1^+ y]$. Making δ sufficiently small ensures that $[p_1 y]$ and $[p_1^+ y]$ are close enough that we can cover them by such simplices.

Now, T embeds isometrically into the unit hemisphere, with $[p_1 p_1^+]$ being sent to the boundary of the hemisphere and $[p_1 y]$ being sent to an arc that passes through the midpoint of the hemisphere (see the right of Figure 4.6). At this point, we see that, because $[p_1 y]$ is longer than $\pi/2$, $[p_1^+ y]$ is shorter than $[p_1 y]$. Therefore, the path $[p_1^+ y] \cup [y p_2]$ is a path joining L_1 to L_2 that is shorter than γ, contradicting the minimality of the length of γ. $\square$

Recall from just before Definition 4.3.1 that γ is a path of minimal length joining the singular circles L_1 and L_2 of $H(\alpha,\beta)$. We now use the fact, just demonstrated, that γ has length at most $\pi/2$ to show that the fixed circles of $\operatorname{Hol}\mu_1$ and $\operatorname{Hol}\mu_2$ in $\mathbb{S}^3$ do not intersect. We do this by mapping γ isometrically to a geodesic in $\mathbb{S}^3$ that joins these two circles and then arguing by contradiction.

Lemma 4.3.4. *Let α and β be positive real numbers that are not divisible by 2π and let μ_1 and μ_2 be the meridians of the two singular circles in $H(\alpha,\beta)$. The fixed circles of $\operatorname{Hol}\mu_1$ and $\operatorname{Hol}\mu_2$ in $\mathbb{S}^3$ do not intersect.*

Proof. Let p_1 and p_2 be the endpoints of γ on L_1 and L_2 respectively. In what follows, take the midpoint x of γ as the basepoint of $\pi_1(S^3 \setminus H)$. Pick a branch $\operatorname{Dev}\colon H(\alpha,\beta) \to \mathbb{S}^3$



(A) Constructing a region containing γ and segments of L_1 and L_2 on which the developing map restricts to an isometric embedding.

(B) Shrinking $\mathrm{Dev}(\gamma)$ by moving it towards an intersection point of C_1 and C_2.

Figure 4.7

of the developing map, starting by embedding simplices containing x. We construct a region N in $H(\alpha, \beta)$ containing γ and two small segments of L_1 and L_2 on which Dev restricts to an isometric embedding (see Figure 4.7 (A) for a visualisation). For some small $\varepsilon > 0$, let p_1^+ and p_1^- be points on L_1 on either side of p_1 at a distance ε from p_1. Define $p_2^\pm$ similarly and denote the small segments $[p_1^- p_1^+]$ and $[p_2^- p_2^+]$ by l_1 and l_2 respectively. For any $q_1 \in l_1$ and $q_2 \in l_2$, there is a geodesic close to γ joining q_1 to q_2—let N be union of all such geodesics. This region is a geodesic simplex containing γ and the small segments l_1 and l_2 of L_1 and L_2 respectively. By taking ε sufficiently small, we can make the angles that N forms at L_1 and L_2 arbitrarily small, and we can ensure that the longest geodesic in N is arbitrarily close to γ. Since, by Lemma 4.3.3, γ has length at most $\pi/2$, this means that Dev embeds N into a convex subset of $\mathbb{S}^3$ (in fact, we only need that γ has length less than π). Therefore, $\mathrm{Dev}\,|_N$ is an isometric embedding.

Now, Dev sends geodesics to geodesics, and so $\mathrm{Dev}(l_1)$ and $\mathrm{Dev}(l_2)$ are segments of great circles C_1 and C_2 in $\mathbb{S}^3$ that, by the definition of holonomy, are the fixed circles of $\mathrm{Hol}\,\mu_1$ and $\mathrm{Hol}\,\mu_2$ respectively. Suppose that C_1 and C_2 intersect. Then C_1 and C_2 lie on a common $\mathbb{S}^2$, and since $\mathrm{Dev}(\gamma)$ is shorter than π, it must also lie on this same $\mathbb{S}^2$. Since $\mathrm{Dev}(\gamma)$ is orthogonal to both C_1 and C_2, we can decrease its length by slightly moving its endpoints along C_1 and C_2 towards their intersection point (see Figure 4.7 (B)). Pulling back along Dev, we can decrease the length of γ by moving its endpoints along L_1 and L_2, contradicting that γ is a path of minimal length joining L_1 and L_2. Thus, C_1 and C_2 cannot intersect. $\square$

What we have just shown is equivalent to saying that the fixed *planes* of the rotations $\mathrm{Hol}\,\mu_1$ and $\mathrm{Hol}\,\mu_2$ in $\mathbb{R}^4$ intersect only at the origin. The following abstract result concerns such rotations, with the added assumptions that they commute and have order 2. This is exactly the situation that arises in $H(\alpha, \beta)$ when both α and β are odd multiples of π.

Lemma 4.3.5. *Let R_1 and R_2 be simple rotations of order 2 in $\mathbb{R}^4$ whose fixed planes P_1 and P_2 intersect only at the origin and suppose that R_1 and R_2 commute. Then P_1 and P_2 are orthogonal, so there is an orthonormal basis with respect to which $R_1 = I \oplus (-I)$ and $R_2 = (-I) \oplus I$.*

Proof. Let v_1 and v_2 be vectors in P_1 and P_2, respectively, that minimise the angle $\measuredangle(v_1, v_2)$ and let V be the plane spanned by v_1 and v_2. Then V intersects P_1 orthogonally along a line and P_2 similarly. Thus, R_1 and R_2 both preserve V. In fact, $R_1|_V$ is a reflection in the line $\mathbb{R}v_1$, and $R_2|_V$ is a reflection in the line $\mathbb{R}v_2$. Since R_1 and R_2 commute, these two lines are either equal or orthogonal. They cannot be equal, as P_1 and P_2 intersect only at the origin. Therefore, $\mathbb{R}v_1$ and $\mathbb{R}v_2$ are orthogonal. The smallest angle between vectors in P_1 and P_2 is thus $\pi/2$, so they are orthogonal. The final part follows simply by rotating P_1 to be $\{(x, y, z, w) \in \mathbb{R}^4 \mid x = y = 0\}$. □

Finally, we combine Lemmas 4.3.4 and 4.3.5 to deduce that $\operatorname{Hol}\mu_1$ and $\operatorname{Hol}\mu_2$ have the desired form, and therefore that $H((2n-1)\pi, (2m-1)\pi)$ is a PK-link.

Proposition 4.3.6. *For natural numbers n and m, $H((2n-1)\pi, (2m-1)\pi)$ is a PK-link.*

Proof. By Lemma 4.3.4, the fixed circles of $\operatorname{Hol}\mu_1$ and $\operatorname{Hol}\mu_2$ in $\mathbb{S}^3$ do not intersect, and therefore their fixed planes in $\mathbb{R}^4$ intersect only at the origin. Since $\operatorname{Hol}\mu_1$ and $\operatorname{Hol}\mu_2$ have order 2 and commute, we can apply Lemma 4.3.5 with $R_1 := \operatorname{Hol}\mu_1$ and $R_2 := \operatorname{Hol}\mu_2$ to deduce that there is an orthonormal basis with respect to which $\operatorname{Hol}\mu_1 = I \oplus (-I)$ and $\operatorname{Hol}\mu_2 = (-I) \oplus I$. Both of these commute with J_2, and therefore $H((2n-1)\pi, (2m-1)\pi)$ is a PK-link. □

Now that we have shown $H(\alpha, \beta)$ is a PK-link whenever α and β are not divisible by 2π, we can deduce Theorem 3.

Proof of Theorem 3. Assume throughout this proof that the conical angles α and β are not divisible by 2π. Our aim is to show that $H(\alpha, \beta)$ is isometric to $\Sigma(\alpha, \beta)$. By Lemma 4.3.2 and Proposition 4.3.6, $H(\alpha, \beta)$ is a PK-link. Therefore, by the classification of PK-links (Theorem 4.1.16), the only situation in which $H(\alpha, \beta)$ might not be isometric to $\Sigma(\alpha, \beta)$ is when it admits a (p, q)-map $f_{(p,q)} \colon H(\alpha, \beta) \to S$, where S is a sphere of curvature 4 with conical points. We will show that, in this situation, the fibration, fibre lengths, and base sphere of $H(\alpha, \beta)$ are the same as those of $\Sigma(\alpha, \beta)$, and therefore $H(\alpha, \beta)$ is isometric to $\Sigma(\alpha, \beta)$, by the uniqueness of fibration data (i.e., the end of Theorem 4.1.16).

Indeed, suppose we have a (p, q)-fibration $f_{(p,q)} \colon H(\alpha, \beta) \to S$. Without loss of generality, assume that $\alpha \leq \beta$. Given that the singular locus of $H(\alpha, \beta)$ is the Hopf link, by the proof of Lemma 4.1.10, we are in one of three cases. Either $1 = p = q$, and S has two conical points of angles α and β (see Theorem 4.1.16 (5)); or $1 = p < q$, and

$L_2 = O_q$, so that S has two conical points, of angles α and β/q (see Theorem 4.1.16 (6)); or $1 < p < q$, $L_1 = O_p$, and $L_2 = O_q$, so that S has two conical points, of angles α/p and β/q. In every case, S has precisely two conical points, with angles α/p and β/q, so by [Tro89, Thm. I], we have $\alpha/p = \beta/q$—denote this quantity by ω. Then since α and β are not divisible by 2π, neither is ω, and so again by [Tro89, Thm. I], we have S is isometric to $\mathbb{S}^2_4(\omega)$. This has area $\omega/2$, and so by Theorem 4.1.16 (3), the ordinary fibres of $f_{(p,q)}$ have length $l = pq\omega = q\alpha = p\beta$. This fibration data is precisely the same as that of $\Sigma(\alpha, \beta)$ observed in Remark 4.1.17, and therefore $H(\alpha, \beta)$ is isometric to $\Sigma(\alpha, \beta)$. $\square$

Having now proven Theorem 3, we complete this section by demonstrating the necessity of the requirement that the conical angles do not lie in $2\pi\mathbb{N}$. We do this by giving two examples: one where $H(2a\pi, 2b\pi)$ is not a PK-link and therefore not the link of a product and one where $H(2a\pi, 2b\pi)$ *is* a PK-link and yet still not the link of a product. In both examples, a and b are integers greater than 1, and we assume that $a \leq b$.

Example 4.3.7. Let C_1 and C_2 be two nonintersecting great circles in $\mathbb{S}^3$ that *do not* lie at a constant distance from each other—this means that they *cannot* be fibres of a common Hopf fibration of $\mathbb{S}^3$. Still, C_1 and C_2 form a Hopf link, and so $N := \mathbb{S}^3 \setminus (C_1 \cup C_2)$ is homeomorphic to $T^2 \times (-1, 1)$. The fundamental group of N has two canonical generators, say λ and μ, corresponding to the meridians of C_1 and C_2 respectively. Let $f\colon M \to N$ be the cover corresponding to the subgroup $\langle a\lambda, b\mu \rangle$ of $\pi_1(N)$. Note that M is still homeomorphic to N. The covering map naturally extends to a ramified covering map $\bar{f}\colon S^3 \to \mathbb{S}^3$, which ramifies with orders a and b over C_1 and C_2 respectively. Let $H(2a\pi, 2b\pi)$ be S^3 endowed with the pullback metric of $\bar{f}$. Then $H(2a\pi, 2b\pi)$ is indeed singular along the Hopf link, but there is no complex structure with respect to which both singular circles have a holomorphic direction. This means that $H(2a\pi, 2b\pi)$ is not a PK-link and therefore not isometric to $\Sigma(2a\pi, 2b\pi)$.

Example 4.3.8. Write $a = np$ and $b = nq$, where $1 \leq p \leq q$ are coprime integers and n is a positive integer. Let S be a 2-sphere of curvature 4 that is a ramified cover of degree n over the 2-sphere of radius $1/2$, with ramification at two nonantipodal points. Then S is a sphere of curvature 4 with two conical points of angle $2n\pi$, but it is not $\mathbb{S}^2_4(2n\pi)$ (see part 2 of Troyanov's result at the start of this chapter). Let $H(2a\pi, 2b\pi)$ be the PK-link of type (p, q) with base sphere S, the two conical points being marked P and Q. This space is not isometric to $\Sigma(2a\pi, 2b\pi)$, because their fibrations have different base spheres.

It is unknown to the author whether there are cases when $H(\alpha, 2n\pi)$ is not a PK-link, where n is an integer greater than 1 and α is not divisible by 2π.

4.4. Generic Seifert Link Singularities

The results of the previous section provide one step towards answering the question posed at the beginning of this chapter, *"Given a polyhedral 3-sphere singular along a Seifert link, when can we show that it is a PK-link?"* Theorem 3 answers, *"For the Hopf link, when the conical angles are not divisible by 2π."* In this final section of the thesis, we attempt to answer it for more general Seifert links, by considering when we might be able to apply Corollary 4.2.6 in a general setting. The primary end of this consideration is Theorem 4, the fourth and final main result of this thesis. We also derive some geometric formulae for 3-dimensional PK-links, expressing volumes and singular edge lengths in terms of conical angles and giving domains of existence for PK-links with prescribed conical angles. (The derivations are very simple, thanks to the work of Panov in [Pan09] and Mondello and Panov in [MP16].) Because of Theorem 4, these formulae apply to the vast majority of polyhedral 3-spheres singular along Seifert links and therefore demonstrate the value of showing that such spaces are PK-links.

Recall from Corollary 4.2.6 that if t is the ordinary fibre of a polyhedral 3-sphere singular along a Seifert link, then the nontriviality of $\operatorname{Hol} t^2$ is a criterion ensuring that the space is a PK-link. After some basic remarks about the Seifert links that concern us in this final section, we begin by determining two purely metric properties that are sufficient to ensure that $\operatorname{Hol} t^2$ is nontrivial in a polyhedral 3-sphere that we *already assume* is a PK-link. One of them involves the *length* of a singular circle, and the other involves the *volume* of the space. We then ask whether these metric properties are still sufficient to ensure that $\operatorname{Hol} t^2$ is nontrivial in *any* polyhedral 3-sphere singular along a Seifert link. It turns out that the length condition *is* still sufficient, as we demonstrate in Theorem 4, while the sufficiency of the volume condition remains open.

Definition 4.4.1 (Generic Seifert link/PK-link)**.** We say that a Seifert link is *generic* if it is not the unknot or the Hopf link, and we call a PK-link *generic* if its singular locus is a generic Seifert link.

As noted in Definition 4.1.11, a generic Seifert link has a *unique* type (p, q). Furthermore, it must have a component that is a (p, q)-torus knot—i.e., an ordinary fibre of the (p, q)-fibration. We call such a component *ordinary.* This means that the type (p, q) of a generic PK-link is determined purely by the topology of its singular locus, which must have a (p, q)-torus knot as a component. Our first observation about the nontriviality criterion in Corollary 4.2.6 relates to the length of this ordinary singular component.

Lemma 4.4.2. *Let M be a generic 3-dimensional PK-link, let K be an ordinary component of M_{s}, and let $t \in \pi_1(M \setminus M_{\mathrm{s}})$ denote the ordinary fibre. If the length of K is not divisible by π, then $\operatorname{Hol} t^2$ is nontrivial.*

Proof. Let l denote the length of K. Since both K and t are ordinary fibres of the (p, q)-fibration of M, t also has length l. The developing map on M takes geodesics to geodesics, so developing along t^2 gives a curve of length $2l$ in $\mathbb{S}^3$ that travels along a great circle. This curve cannot be closed, since $2l$ is not divisible by 2π. But if $\operatorname{Hol} t^2$ were trivial, then this curve would be closed. Therefore, $\operatorname{Hol} t^2$ is nontrivial. □

The second observation relates to the volume of a PK-link and, given Theorem 4.1.16 (3), is essentially a restatement of the previous result.

Lemma 4.4.3. *Let M be a generic 3-dimensional PK-link of type (p, q) and let $t \in \pi_1(M \setminus M_{\mathrm{s}})$ denote the ordinary fibre. If the volume of M cannot be written as $\pi^2 k^2/2pq$ for some integer k, then* $\operatorname{Hol} t^2$ *is nontrivial.*

Proof. Recall from Theorem 4.1.16 (3) that $\operatorname{Vol} M = l^2/2pq$, where l is the length of an ordinary fibre. This means that if $\operatorname{Vol} M$ does not have the form $\pi^2 k^2/2pq$ for some integer k, then l is not divisible by π. It thus follows from Lemma 4.4.2 that $\operatorname{Hol} t^2$ is nontrivial. □

It turns out that the proof of Lemma 4.4.2 generalises without too much difficulty to any polyhedral 3-sphere singular along a generic Seifert link, provided that the conical angles are not divisible by 2π. This means that, in some sense, the vast majority of polyhedral 3-spheres singular along Seifert links are PK-links.

Theorem 4 (Seifert link singularities)**.** *Let M be a polyhedral 3-sphere with no conical angles in $2\pi(\mathbb{N} \setminus \{1\})$ whose singular locus is a generic Seifert link. Let K be an ordinary component of the singular locus of M. If the length of K is not divisible by π, then M is a PK-link.*

Proof. Let $N_\varepsilon(K)$ be the set of points in M at a distance less than ε from K. Choose ε small enough so that $N_\varepsilon(K)$ does not intersect itself or any other component of M_{s}. By definition, K is an ordinary fibre of the (topological) Seifert fibration of M. This means we can pick another ordinary fibre t lying in $N_\varepsilon(K) \setminus K$ and is homotopic to K inside $N_\varepsilon(K)$. We will show that $\operatorname{Hol} t^2$ is nontrivial by demonstrating that developing along t^2 gives a curve in $\mathbb{S}^3$ that is not closed.

Let l denote the length of K. Starting at a point on K and developing along it twice gives a curve of length $2l$ in $\mathbb{S}^3$ that travels along a great circle C in $\mathbb{S}^3$. Since l is not divisible by π, this curve is not closed—say it starts at some point x and ends at a different point y. Now, developing along t^2 gives another curve in the ε-tubular neighbourhood of C. Choose the basepoint of t so that this second curve starts on the geodesic disc normal to C at x. Then because t is homotopic to K inside $N_\varepsilon(K)$, the curve ends on the geodesic disc normal to C at y. It is therefore not closed, and so $\operatorname{Hol} t^2$ is nontrivial. By Corollary 4.2.6, this implies that M is a PK-link. □

We now give a statement that generalises Lemma 4.4.3 to polyhedral 3-spheres singular along generic Seifert links in the same way that Theorem 4 generalises Lemma 4.4.2. Unfortunately, we do not yet know how to prove it, so we leave it as a conjecture. It may be that it is actually equivalent to Theorem 4, just as Lemma 4.4.3 is equivalent to Lemma 4.4.2.

Conjecture 4.4.4. *Let M be a polyhedral 3-sphere with no conical angle in $2\pi(\mathbb{N} \setminus \{1\})$ whose singular locus is a generic Seifert link of type (p,q). If the volume of M cannot be written as $\pi^2k^2/2pq$ for some integer k, then M is a PK-link of type (p,q).*

Theorem 4 implies that almost all polyhedral 3-spheres singular along Seifert links are PK-links. Therefore, any formulae—concerning lengths, volumes, angle constraints, and domains of existence—that apply to PK-links also apply to such polyhedral 3-spheres in the majority of cases. Motivated by special cases of such formulae, we now give two results that apply to all generic PK-links and therefore to most polyhedral 3-spheres singular along Seifert links. In the first, we use results of Panov in [Pan09] to express singular circle lengths and volume in terms of conical angles.

Proposition 4.4.5. *Let M be a generic 3-dimensional PK-link of type (p,q). Let $\alpha_1,\ldots,\alpha_n$ be the conical angles of the ordinary components of M_{s} and let β_p and β_q be the conical angles of the p and q-axes respectively (we allow one or both of β_p and β_q to be 2π). Then the ordinary components of M_{s} all have the same length,*

$$l = \frac{pq}{2}\left(\sum_{i=1}^{n}(\alpha_i - 2\pi) + \frac{\beta_p - 2\pi}{p} + \frac{\beta_q - 2\pi}{q}\right) + (p+q)\pi,$$

while the p and q-axes have lengths l/p and l/q respectively. The volume of M is $l^2/2pq$.

Proof. See [Pan09, Thm. 1.9]. The result follows from Theorem 4.1.16 (3) and an application of the Gauss–Bonnet formula to the base sphere of the (p,q)-fibration of M. □

The second result combines results of Panov in [Pan09] with results of Mondello and Panov in [MP16] to give linear constraints on the conical angles of a PK-link. It also states that almost all tuples of angles satisfying these constraints exist as the conical angles of a PK-link. In order to state it, we require some notation.

Notation (see [MP16, § 1.3])**.** Let $\|\cdot\|_1$ denote the standard ℓ^1-norm on $\mathbb{R}^n$ and d_1 the associated metric. We denote by $\mathbb{Z}^n_{\mathrm{o}}$ the set of points $\boldsymbol{m} = (m_1,\ldots,m_n)$ in $\mathbb{Z}^n$ for which $\|\boldsymbol{m}\|_1$ is odd.

Proposition 4.4.6. *Let M be a generic 3-dimensional PK-link of type (p,q). Let $\alpha_1,\ldots,\alpha_n$ be the conical angles of the ordinary components of M_{s} and let β_p and β_q be the conical angles of the p and q-axes respectively (we allow one or both of β_p and β_q to be 2π). Finally, let $\boldsymbol{\alpha} := (\alpha_1,\ldots,\alpha_n,\beta_p/p,\beta_q/q)$ and $\boldsymbol{2\pi} := (2\pi,\ldots,2\pi) \in \mathbb{R}^{n+2}$. Then the following inequalities hold:*

$$\sum_{i=1}^{n}(\alpha_i - 2\pi) + (\beta_p/p - 2\pi) + (\beta_q/q - 2\pi) > -4\pi,$$

$$d_1\left(\boldsymbol{\alpha} - \boldsymbol{2\pi}, 2\pi\mathbb{Z}^{n+2}_{\mathrm{o}}\right) \geq 2\pi.$$

Furthermore, for any tuple $\alpha_1, \ldots, \alpha_n, \beta_p, \beta_q$ of positive real numbers satisfying the above inequalities strictly, *there is a generic 3-dimensional PK-link of type (p, q) with these conical angles.*

Proof. By Theorem 4.1.16, M fibres over a 2-sphere of curvature 4 with conical angles $\alpha_1, \ldots, \alpha_n, \beta_p/p, \beta_q/q$. By scaling the metric on this sphere by a factor of 2, we get a sphere of curvature 1 with the same conical angles. Applying [MP16, Thm. A] to this sphere gives the required inequalities. For the existence part, [MP16, Thm. C] guarantees the existence of a sphere of curvature 1 with conical angles $\alpha_1, \ldots, \alpha_n, \beta_p/p, \beta_q/q$ as long as the inequalities are satisfied strictly. We then scale this sphere to have curvature 4 and take the (p, q)-fibration over it, marking the conical points of angle β_p/p and β_q/q by P and Q respectively. This gives the required PK-link. □

To somewhat belabour the point, thanks to Theorem 4, the two propositions just given apply to a polyhedral 3-sphere singular along a generic Seifert link, as long as the length of one of its ordinary components is not divisible by π and none of its conical angles are divisible by 2π. The author believes that these extra conditions can in fact be removed, as we suggest in the following conjecture.

Conjecture 4.4.7. *The statements of Propositions 4.4.5 and 4.4.6 still hold when the phrase 'generic 3-dimensional PK-link' is replaced by 'polyhedral 3-sphere singular along a generic Seifert link'.*

We finish the chapter by giving two remarks explaining how our work in this section expands and clarifies existing literature about polyhedral 3-spheres singular along *torus* links. To avoid the repetitious exclusion of potentially pathological cases, the remarks assume the validity of Conjecture 4.4.7. They therefore, on the one hand, are certainly valid in almost all cases and, on the other, function as a motivation to prove Conjecture 4.4.7.

Remark 4.4.8. As was mentioned at the beginning of this chapter, the length and volume formulae given in Proposition 4.4.5 and the angle constraints and existence statement given in Proposition 4.4.6 both generalise and unify existing results in the literature. In what follows, we always assume that the exceptional fibres are nonsingular and that all the α_i lie in $(0, 2\pi)$.

- Taking $(p, q) = (2, 3)$ and $n = 1$, we recover Derevnin, Mednykh, and Mulazzani's results concerning a polyhedral 3-sphere singular along the trefoil knot, given in [DMM14, Prop. 8 & Thm. 10].

- Taking $(p, q) = (1, 1)$ and $n = 3$, we recover Kolpakov's results for the Hopf 3-link in [Kol13, Thm. 2].

- Taking $(p, q) = (1, 1)$, $n = 4$, and $\alpha_1 = \ldots = \alpha_4$, we recover Kolpakov's results for the Hopf 4-link in [Kol13, Thm. 3].

- Taking $(p, q) = (2, 2m+1)$, for any positive integer m, and $n = 1$, we recover Kolpakov and Mednykh's results for the $(2, 2m+1)$-torus knot in [KM09, Thm. 1].
- Lastly, taking $(p, q) = (1, m)$, for any positive integer m, and $n = 2$, we recover Kolpakov and Mednykh's results for the $(2, 2m)$-torus link in [KM09, Thm. 2].

The advantage of our PK-link approach is that it requires neither the explicit construction of a fundamental polyhedron nor the use of techniques valid for only certain torus links.

Remark 4.4.9. It should be noted at this point that the length and volume formulae given in Proposition 4.4.5 are equivalent to those for torus links given by Kolpakov in [Kol16, Thm. 4]. Furthermore, the angle constraints in Proposition 4.4.6 are almost identical to those he gives in [Kol16, Thm. 8], if one adds the assumption that the conical angles lie in $(0, 2\pi)$. However, it appears that Kolpakov makes three key assumptions that we do not. Firstly, as already mentioned, he assumes that the conical angles lie in $(0, 2\pi)$. Secondly, he assumes that the base sphere of the (p, q)-fibration is the double of a convex spherical polygon, which is not always the case. And thirdly, he seems to assume that the *topological* Seifert fibration realising the torus link has geodesic fibres, and that the holonomy preserves the Hopf fibration of $\mathbb{S}^3$. To make this last assumption is essentially to assume that a polyhedral 3-sphere singular along a torus link *is always* a PK-link. The validity of this jump is of course the central question of this chapter and cannot be taken for granted.

Appendix

INTEGRAL VERTICES

The aim of this appendix is to give examples of the steps in Procedure 3.2.7 and then to list the links of every possible singular vertex in nonnegatively curved integral polyhedral 3-manifolds in Table A.1.

Example. Here we give step-by-step examples of Procedure 3.2.7 for finding ramified covers. Theoretically speaking, nothing is added here, but the procedure is more clearly illustrated.

1. Let us consider the case of $n = 4$—i.e., of spheres with 4 conical points. Steps 1 and 2 are straightforward. There are 32 (unordered) quadruples satisfying Formula (3.2.2).
2. Of the 32 quadruples from step 1, only one of them does not satisfy Formula (3.2.3): $\left(\frac{\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{2}\right)$.
3. One of the remaining quadruples is $\left(\frac{4\pi}{3}, \frac{3\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{2}\right)$. Using the Gauss–Bonnet formula, we calculate that a sphere with these conical angles has area $11\pi/6$ and so would have to be a degree 11 ramified cover of $\mathbb{S}^2/S_4$. The three conical points of angle $3\pi/2$ must be multiplicity 3 preimages of the conical point of angle $\pi/2$. Since the multiplicities must sum to 11, there must also be one multiplicity 2 preimage. But this implies the existence a conical point of angle π: a contradiction. Thus, this tuple cannot satisfy Formula (3.2.4).
4. One of the 15 remaining quadruples at this stage is $\left(\pi, \frac{3\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{2}\right)$. We calculate that the degree of the cover must be 9, and the only possible multiplicity datum is: $m_1^1 = 1, m_1^2 = \ldots = m_1^5 = 2$; $m_2^1 = m_2^2 = m_2^3 = 3$; $m_3^1 = m_3^2 = m_3^3 = 3$. The *dessin d'enfant* $\varphi^{-1}([y_1y_2])$ would be a bipartite graph, with three faces of degree 6 (corresponding to preimages of y_3), with four vertices marked y_1 of degree 2 and one of degree 1, and with three vertices marked y_2 of degree 3. It can be shown that no such graph exists in the 2-sphere (see Figure A.1).
5. At this stage, we have 12 different ramified covers, only two of which have the same conical angles, $\left(\pi, \pi, \frac{4\pi}{3}, \frac{4\pi}{3}\right)$. They are distinguished by whether or not the points of angle π are preimages of y_1 or y_3. By expanding the two *dessins* to full

triangulations, we can see that they are doubles of two noncongruent spherical quadrilaterals and therefore cannot be isometric (see Figure A.2).

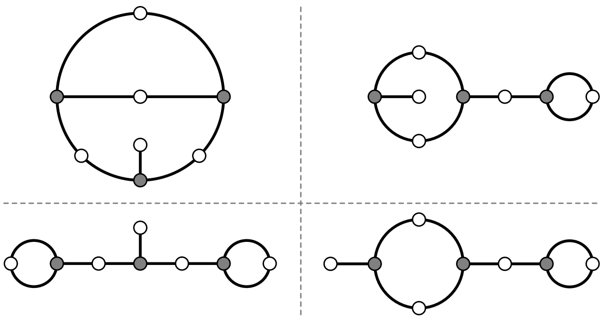


FIGURE A.1. White and grey vertices are marked y_1 and y_2 respectively. This figure demonstrates the search for the *dessin d'enfant* corresponding to the tuple $\left(\pi, \frac{3\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{2}\right)$. If the proposed graph existed, deleting the sole vertex of degree 1 and then temporarily ignoring those of degree 2 would leave a trivalent graph with two grey vertices. There are only two such graphs: the theta graph and the barbell graph. There are four ways to add back in the degree 1 and 2 vertices that give distinct graphs in the 2-sphere—these are shown above. None of them has all three faces having degree 6.

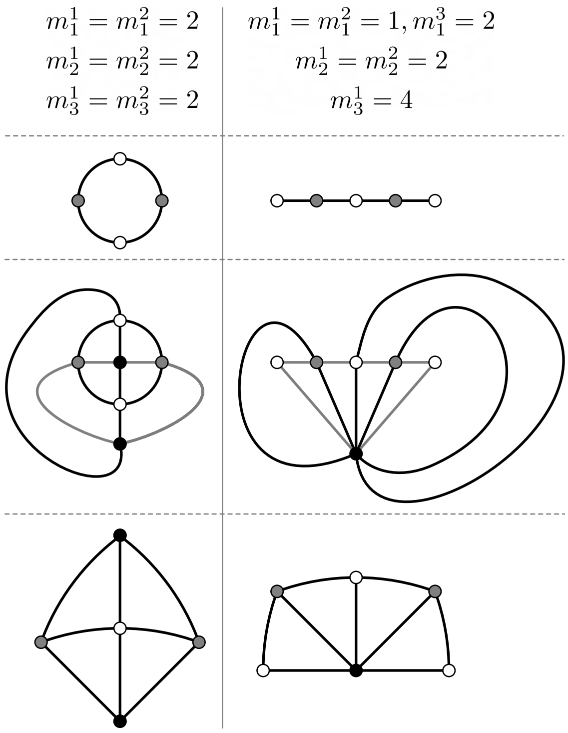


FIGURE A.2. White, grey, and black vertices are marked y_1, y_2, and y_3 respectively. This figure demonstrates the construction of the two nonisometric ramified covers of $\mathbb{S}^2/S_4$ corresponding to the tuple $\left(\pi, \pi, \frac{4\pi}{3}, \frac{4\pi}{3}\right)$. The two constructions are shown in parallel on the left and the right (corresponding to entries #23 and #24 in Table A.1 respectively). The first row gives the two possible multiplicity data. The second row shows the corresponding *dessins d'enfants*. The third row shows the full triangulations of the 2-sphere given by pulling back the triangulation of $\mathbb{S}^2/S_4$ along the ramified covering maps. At this point, we notice that both are doubles of spherical quadrilaterals, whose boundaries are marked in grey. These two noncongruent spherical quadrilaterals are shown more clearly in the final row.

We now give Table A.1, which lists the links of every possible singular vertex in a nonnegatively curved integral polyhedral 3-manifold. Each row in the table corresponds to the link of a unique vertex. The tuple $\boldsymbol{\vartheta}$ is defined so that $2\pi\boldsymbol{\vartheta}$ is the list of conical angles of the link in increasing order. The third column says whether or not the link is the double of a spherical polygon. The fourth column gives the local monodromy group. The purpose of the final column is to give a complete geometric description of the link where necessary. If $\boldsymbol{\vartheta} = (\alpha, \beta, \gamma)$, then this final column is left blank, as the link is completely determined as the double of the unique spherical triangle with angles $\pi\alpha$, $\pi\beta$, and $\pi\gamma$.

Otherwise, if the link is the double of a spherical quadrilateral, then that quadrilateral is shown in the final column, built out of copies of the spherical triangle with angles $\pi/4$, $\pi/3$, and $\pi/2$ (denoted by black, grey, and white vertices respectively). If the link is not a double, then a full triangulation is given, with grey arrows denoting edge identifications. The table is ordered increasingly, first by the number of conical points, then by the size of first conical angle (then second, third, etc.). The only two links with the same conical angles are links #23 and #24, which are ordered by the size of their monodromy groups.

TABLE A.1. Links of nonnegatively curved integral vertices

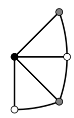
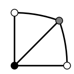
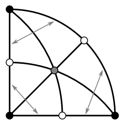
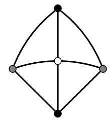
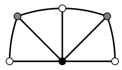
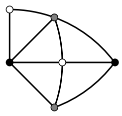
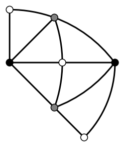
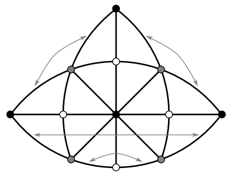
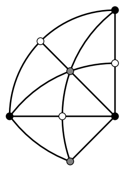
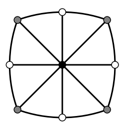
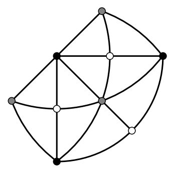
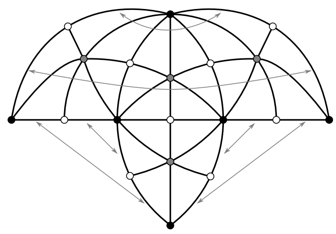
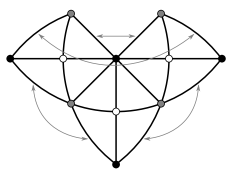
#: $\boldsymbol{\vartheta}$	**Double?**	**Monodromy**	**Description**
1: $\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}\right)$	Yes	D_6	
2: $\left(\frac{1}{4}, \frac{1}{4}, \frac{2}{3}\right)$	Yes	S_4	
3: $\left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right)$	Yes	S_4	
4: $\left(\frac{1}{4}, \frac{1}{3}, \frac{3}{4}\right)$	Yes	S_4	
5: $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right)$	Yes	D_4	
6: $\left(\frac{1}{4}, \frac{1}{2}, \frac{2}{3}\right)$	Yes	S_4	
7: $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}\right)$	Yes	A_4	
8: $\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$	Yes	A_4	
9: $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right)$	Yes	D_3	
10: $\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$	Yes	A_4	
11: $\left(\frac{1}{3}, \frac{1}{2}, \frac{3}{4}\right)$	Yes	S_4	
12: $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	Yes	D_2	
13: $\left(\frac{1}{2}, \frac{1}{2}, \frac{2}{3}\right)$	Yes	D_3	
14: $\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{4}\right)$	Yes	D_4	
15: $\left(\frac{1}{2}, \frac{1}{2}, \frac{5}{6}\right)$	Yes	D_6	
16: $\left(\frac{1}{2}, \frac{2}{3}, \frac{2}{3}\right)$	Yes	A_4	
17: $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right)$	Yes	S_4	
18: $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$	Yes	A_4	
19: $\left(\frac{2}{3}, \frac{3}{4}, \frac{3}{4}\right)$	Yes	S_4	
20: $\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right)$	Yes	S_4	
21: $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}\right)$	Yes	S_4	

Table A.1. (Continued)

#: $\boldsymbol{\vartheta}$	**Double?**	**Monodromy**	**Description**
22: $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}\right)$	No	S_4	
23: $\left(\frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}\right)$	Yes	A_4	
24: $\left(\frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}\right)$	Yes	S_4	
25: $\left(\frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right)$	Yes	S_4	
26: $\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right)$	Yes	S_4	
27: $\left(\frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}\right)$	No	S_4	
28: $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}\right)$	Yes	S_4	
29: $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$	Yes	A_4	
30: $\left(\frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}\right)$	Yes	S_4	
31: $\left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right)$	No	S_4	
32: $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}\right)$	No	S_4	

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