Mod p points on Shimura varieties of parahoric level

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Abstract. We study the $\overline{\mathbb{F}}_p$-points of the Kisin-Pappas integral models of abelian type Shimura varieties with parahoric level structure. We show that if the group is quasi-split and unramified, then the mod p isogeny classes are of the form predicted by the Langlands-Rapoport conjecture (c.f. Conjecture 9.2 of [**59**]). We prove the same results for quasi-split and tamely ramified groups when their Shimura varieties are proper. The main innovation in this work is a global argument that allows us to reduce the conjecture to the case of a very special parahoric, which is handled in earlier work of Rong Zhou. This way we avoid the complicated local problem of understanding connected components of affine Deligne-Lusztig varieties for general parahoric subgroups. Along the way, we give a simple irreducibility criterion for Ekedahl-Oort and Kottwitz-Rapoport strata.

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CHAPTER 1

Introduction

1. Hasse-Weil zeta functions of smooth projective varieties

Let E be a number field and let X be a smooth projective variety over E. The Hasse-Weil zeta function of X is (see [**63**]), at least conjecturally, a meromorphic function $\zeta_X(s)$ on the complex numbers which encodes deep global arithmetic information about X. For example the conjecture of Birch and Swinnerton-Dyer predicts for an elliptic curve X over $\mathbb{Q}$ that its zeta function knows the Mordell-Weil rank of X. The zeta function is defined as an Euler product over all primes $\mathfrak{p}$ of E

$$\zeta_X(s) = \prod_{\mathfrak{p}} Z_{X,\mathfrak{p}}(s).$$

of local Zeta functions $Z_{X,\mathfrak{p}}(s)$. For primes $\mathfrak{p}$ where X has good reduction, the local zeta function $Z_{X,\mathfrak{p}}(s)$ encodes information about the number of points of the special fiber $X_{\mathfrak{p}}$ over finite fields. The local zeta functions at places of bad reduction are harder to define, and it is not always clear what arithmetic information they encode, except in special cases.

Let S be the set of places of bad reduction. The partial product

$$\zeta_{X,S}(s) = \prod_{\mathfrak{p} \notin S} Z_{X,\mathfrak{p}}(s)$$

converges absolutely for $\mathrm{Re}(s) > 1 + d$, where d is the dimension of X (see Section 1.2 of [**63**]) and defines a holomorphic function. Proving this absolute convergence relies on the Hasse-Weil bounds for the number of points of X over finite fields. It now makes sense to conjecture that $\zeta_{X,S}(s)$ has analytic continuation to a meromorphic function on $\mathbb{C}$, which is one half of the Hasse-Weil conjecture. It will follow that $\zeta_X(s)$ has meromorphic continuation to $\mathbb{C}$, because the local Euler factors $Z_{X,\mathfrak{p}}(s)$ for places $\mathfrak{p}$ of bad reduction are meromorphic functions by construction.

The other half of the Hasse-Weil conjecture is a functional equation for $\zeta_X(s)$. Just as with the functional equation for the Riemann zeta function, this is best stated in terms of a completed zeta function. Define

$$\xi_X(s) = A^{s/2}\zeta_X(s) \cdot \prod_{v \in \Sigma_E^\infty} \Gamma_{X,v}(s).$$

Here $A \in \mathbb{Q}_{>0}$ is the conductor of X (see Section 4.1 of [**63**]), the symbol Σ_E^∞ denotes the infinite places of E and $\Gamma_{X,v}(s)$ denotes the Gamma-factor of X at an infinite place (see Section 3 of [**63**]). We can now state a formal conjecture, in which we will implicitly assume that X is equidimensional.

Conjecture 1.0.1. *The function $\xi_X(s)$ has meromorphic continutation to all of $\mathbb{C}$ and satisfies*

$$\xi_X(s) = \pm\xi_X(d+1-s),$$

where d is the dimension of X.

This conjecture is completely open in general, and most of the known results all follow the same strategy: Show that $\zeta(X,s)$ is equal to a product of "automorphic L-functions", and prove that these automorphic L-functions have meromorphic continuation and a functional equation. Unfortunately, it is far beyond the scope of this thesis to define automorphic representations and automorphic L-functions, so we'll settle for an example:

Example 1.0.2. Let X be the elliptic curve over $\mathbb{Q}$ defined by the equation $y^2 + y = x^3 - x^2$, then the Hasse-Weil zeta function of X is equal to

$$\zeta_X(s) = \frac{\zeta(s)\zeta(s-1)}{\mathcal{L}_f(s)},$$

where $\zeta(s) = \zeta_{\operatorname{Spec}\mathbb{Q}}(s)$ is the usual Riemann-zeta function, and $\mathcal{L}_{f,s}$ is the L-function of the modular form

$$f(z) = \prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11})^2 \in S_2[\Gamma_0(11)].$$

The Hasse-Weil conjecture for $\zeta_X(s)$ now follows from the functional equation and meromorphic continuation for $\zeta(s)$ and $\mathcal{L}_{f,s}$.

For a general variety X the approach sketched above seems hopeless, because it is not clear 'where' the automorphic representations should come from. This is different for Shimura varieties, because automorphic representations are closely related to Shimura varieties. For example it follows from work of Eichler [**14**] and Shimura [**66**] that the Hasse-Weil zeta functions of modular curves are products of L-functions of modular forms. However, their approach does not easily generalise to higher-dimensional Shimura varieties.

In [**46**], Langlands outlines a three-part approach to prove that the Hasse-Weil zeta functions of Shimura varieties are related to L-functions of automorphic representations. The first and third part are 'a matter of harmonic analysis', we refer the reader to [**76**] for an introduction. The second part is about describing the mod p points of suitable (smooth) integral models of Shimura varieties. His original strategy is only suitable for studying the local zeta functions at places of good reduction, but it was generalised by Rapoport and Kottwitz to include places of parahoric[1] bad reduction [**59**].

Disregarding the very interesting and very complicated harmonic analysis that will no doubt have to be used, computing the (semisimple) [2] local zeta functions of Shimura varieties at primes of parahoric bad reduction requires two ingredients: The first is constructing reasonable integral models and describing their singularities, or rather computing the (semisimple) trace of Frobrenius on the sheaf of nearby cycles. The integral models were constructed by Kisin-Pappas [**37**] and the recent work of Haines-Richarz [**23**] solves the problem of understanding the nearby cycles. The second ingredient is describing the mod p-points of these Kisin-Pappas integral models, which is the central topic of this thesis. A conjectural description of the mod p points of (conjectural) integral models of Shimura varieties was first given by Langlands in [**45**] and was later refined by Langlands-Rapoport [**47**] and by Rapoport [**59**] to include the case of parahoric bad reduction.

2. The Langlands-Rapoport conjecture

The Langlands-Rapoport conjecture gives a conjectural description of the $\overline{\mathbb{F}}_p$-points of conjectural integral models of a Shimura variety associated to a Shimura datum (G, X). Stating the conjecture is technically quite involved and we will postpone that to Chapter 3. The goal of this section is to show that beneath all the technicalities lies a beautiful motivic story. We will start by discussing mod p points on the modular curve.

[1]This means that the level at p is a parahoric subgroup.

[2]The semisimple local zeta function is a variant of the local zeta function defined in [**11**] from which the usual local zeta function can be recovered, if one assumes the weight-monodromy conjecture.

2.1. The modular curve. There are great and detailed explanations of Langlands-Rapoport for the modular curve elsewhere (e.g. [**60**]), here we will only give a basic overview. Consider the tower of schemes

$$\{Y_N/\mathbb{Z}_{(p)}\}_N$$

where N runs over positive integers coprime to p ordered by divisibility and Y_N is the moduli space of elliptic curves E together with an isomorphism of group schemes $E[N] \simeq (\mathbb{Z}/N\mathbb{Z})^2$. This is most easily done by considering all N at once, or working with the inverse limit. Define

$$Y(\overline{\mathbb{F}}_p) := \varprojlim_{(N,p)=1} Y_N(\overline{\mathbb{F}}_p),$$

which has a natural action of $\mathrm{GL}_2(\hat{\mathbb{Z}}^p)$ that extends to an action of $\mathrm{GL}_2(\mathbb{A}_f^p)$ via Hecke correspondences. We want a 'group theoretic' description of $Y(\overline{\mathbb{F}}_p)$, which takes this action into account. We will give a description in two steps:

(1) Divide elliptic curves into isogeny classes and classify them (Honda-Tate theory).

(2) Count elliptic curves inside a fixed isogeny class.

2.1.1. *The structure of isogeny classes.* If we fix an elliptic curve $E_0/\overline{\mathbb{F}}_p$, then its isogeny class $\mathscr{I}_\phi \subset Y(\overline{\mathbb{F}}_p)$ has a description in terms of linear- and semi-linear algebra. Let V^pE_0 be the rational prime-to-p adic Tate module of E_0, in other words it is

$$\left(\prod_{\ell \neq p} T_\ell E_0\right) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $T_\ell E_0 = \varprojlim_m E_0[\ell^m](\overline{\mathbb{F}}_p)$. Since each $T_\ell E_0$ is a free $\mathbb{Z}_\ell$ module of rank two, we see that V^pE_0 is a free module of rank two over the ring

$$\mathbb{A}_f^p := \left(\prod_{\ell \neq p} \mathbb{Z}_\ell\right) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Let V_pE_0 be $T_pE_0[1/p]$, where T_pE_0 is the covariant Dieudonné module of E_0. In short, T_pE_0 is a free $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$-module Λ of rank two equipped with a Frobenius semi-linear map $F : \Lambda \to \Lambda$ satisfying $p\Lambda \subset F\Lambda \subset \Lambda$. An isogeny $f : E_0 \to E$ induces bijections $V^pE_0 \simeq V_pE$ and $V_pE_0 \simeq V_pE$, but the lattices inside will be different; in fact f will be determined by the induced lattices in V^pE_0 and V_pE_0. We define

$$X^p(\phi) = \{\hat{\mathbb{Z}}^p\text{-lattices } \Lambda^p \subset V^pE_0 \text{ together with an isomorphism } \Lambda^p \simeq (\hat{\mathbb{Z}}^p)^{\oplus 2}\}$$

$$X_p(\phi) = \{\text{Dieudonné-lattices in } V_pE_0\}.$$

Then $X^p(\phi)$ is a $\mathrm{GL}_2(\mathbb{A}_f^p)$-torsor and there is a $\mathrm{GL}_2(\mathbb{A}_f^p)$-equivariant map

$$X^p(\phi) \times X_p(\phi) \to \mathscr{I}_\phi,$$

sending a pair of lattices (Λ^p, Λ_p) corresponding to an isogeny $f : E_0 \to E$ to the elliptic curve E together with its trivialisation. This induces an isomorphism

$$\mathscr{I}_\phi \simeq I_\phi(\mathbb{Q})\backslash \left(X^p(\phi) \times X_p(\phi)\right),$$

where $I_\phi(\mathbb{Q})$ is the set of self quasi-isogenies of E_0.

2.1.2. *Classification of isogeny classes.* Classical Honda-Tate theory describes isogeny classes of abelian varieties over $\mathbb{F}_q$ in terms of q-Weil numbers. Equivalently, we can describe isogeny classes of elliptic curves E by the characteristic polynomial of Frob_q acting on the ℓ-adic Tate module of E for some $\ell \neq p$ (this is an element of $\mathbb{Z}[X]$ independent of ℓ). This gives us a (semisimple) conjugacy class of matrices in $\mathrm{GL}_2(\mathbb{Q})$ and its stabiliser is an inner form of the group I_ϕ of self quasi-isogenies of E. Another perspective is that the isogeny class of E determines a two-dimensional pure motive (say with numerical equivalence) over $\mathbb{F}_q$, or a motive with GL_2-structure.

2.1.3. *Conclusion.* In conclusion, we see that the mod p points on the modular curve can be described as

$$\varprojlim_{(N,p)=1} Y_N(\overline{\mathbb{F}}_p) \simeq \coprod_\phi I_\phi(\mathbb{Q}\backslash X_p(\phi) \times X^p(\phi), \tag{2.1.1}$$

equivariant for the action of $\mathrm{GL}_2(\mathbb{A}_f^p)$, where ϕ ranges over the set of isogeny classes of elliptic curves over $\overline{\mathbb{F}}_p$. Moreover it turns out that the action of Frobenius on the left hand side corresponds to the action of a certain operator Φ on $X_p(\phi)$. The Langlands-Rapoport conjecture for a general Shimura variety has the same shape as (1.2.1.1).

2.2. The Langlands-Rapoport conjecture in general. Let (G, X) be a Shimura datum, let p be a prime number, let $U_p \subset G(\mathbb{Q}_p)$ be a parahoric subgroup. Consider the tower of Shimura varieties $\{\mathbf{Sh}_{G,U^pU_p}\}_{U^p}$ over the reflex field E with its action of $G(\mathbb{A}_f^p) \times Z_G(\mathbb{Q}_p)$, where U^p varies over compact open subgroups of $G(\mathbb{A}_f^p)$ and where Z_G is the center of the algebraic group G. Then we conjecture that this tower has a $G(\mathbb{A}_f^p) \times Z_G(\mathbb{Q}_p)$-equivariant extension to a tower of flat (normal) schemes $\{\mathscr{S}_{G,U^pU_p}\}_{U^p}$

over $\mathcal{O}_{E_{(v)}}$. When U_p is hyperspecial, the integral model should be smooth and satisfy a certain extension property, which determines it uniquely if it exists (c.f. **[51]**). Recent work **[53]** of Pappas defines a notion of canonical integral models when U_p is an arbitrary parahoric and (G, X) is of Hodge type, and proves that they are unique if they exist. Moreover, there should be a bijection

$$\varprojlim_{U^p} \mathscr{S}_{U^pU_p}(\overline{\mathbb{F}}_p) \simeq \varprojlim_{U^p} \coprod_{[\phi]} I_\phi(\mathbb{Q}) \backslash X^p(\phi) \times X_{p,U_p}(\phi)/U^p$$

compatible with the action of $G(\mathbb{A}_f^p) \times Z_G(\mathbb{Q}_p)$. Here $X^p(\phi)$ is a $G(\mathbb{A}_f^p)$-torsor as before, and we are left to explain the indexing set ϕ, the sets $X_p(\phi)$ and the groups $I_\phi(\mathbb{Q})$. The indexing set should be a generalisation of the notion of isogeny class; we'll explore this in the next section.

2.3. Mod p isogeny classes on general Shimura varieties. We would like to say that the mod p isogeny classes on the special fiber of a Shimura variety associated to a Shimura datum (G, X) are isogeny classes of "abelian varieties with G-structure" or "motives with (G, X)-structure". There are various ways of making this precise, the simplest one works only for Shimura varieties of Hodge type.

Suppose that (G, X) is of Hodge type and let $i : (G, X) \hookrightarrow (\mathrm{GSp}_{2g}, S^{\pm})$ be a Hodge embedding. Then an abelian variety with G structure over $\overline{\mathbb{F}}_p$ is a g-dimensional abelian variety A together with a finite collection of tensors

$$\{s_{\alpha,\ell}\}_{\alpha \in C} \in V_\ell(A)^{\otimes}$$

for all ℓ (including $\ell = p$) such that the stabiliser of the tensors in $\mathrm{GL}\, V_\ell(A)$ is given by $G_{\mathbb{Q}_\ell}$ (note that the indexing set C is independent of ℓ). Here $V_\ell(A)^{\otimes}$ is the direct sum of all modules obtained from the rational ℓ-adic Tate-module (or rational Dieudonné-module if $\ell = p$) $V_\ell(A)^{\otimes}$ using the operations of duals, tensor products, symmetric powers and exterior powers. This is the kind of "abelian variety with G-structure" that one actually gets from a $\overline{\mathbb{F}}_p$-point on the special fiber of the Kisin-Pappas integral models of Hodge type Shimura varieties. In fact there will a finite field $\mathbb{F}_q$ such that the abelian variety is defined over $\mathbb{F}_q$ and such that the tensors are Galois invariant.

This notion of "abelian variety with G-structure" is not well behaved because we are not asking for any compatibility between tensors for different ℓ. Indeed, the Tate conjecture for motives over finite fields predicts that our tensors come from algebraic cycles and we would obviously like to say that the $s_{\alpha,\ell}$ are the ℓ-adic realisations the same cycle s_α. Another issue is that we would like to get rid of the choice of Hodge embedding.

2.3.1. *Motives.* Let $\mathcal{C}_q$ be the category of motives with numerical equivalence over $\mathbb{F}_q$, see [**62**] for an introduction. A priory this is just a pseudo-abelian category with a tensor product, but it follows from [**34**] that this is actually a semisimple abelian category. Moreover it follows from [**35**] that it is a semisimple Tannakian category, see Section 1 of [**49**]. If we assume the Tate conjecture in the form of Conjecture 1.14 of [**49**], then for $\ell \neq p$ there is a fully faithful tensor functor

$$\mathcal{C}_q \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to \mathbb{V}_\ell(\mathbb{F}_q)$$

to the category of semisimple continuous representations of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$ on finite dimensional vector spaces over $\mathbb{Q}_\ell$, given by ℓ-adic étale cohomology. Similarly if we assume the crystalline version of the Tate conjecture, then there are fully faithful tensor functors

$$\mathcal{C}_q \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathbb{V}_p(\mathbb{F}_q)$$

to the category of F-isocrystals over $W(\mathbb{F}_q)[1/p]$. When we pass to the category $\mathcal{C}$ of motives over $\overline{\mathbb{F}}_p$, we get fully faithful tensor functors

$$\omega_\ell : \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to \mathbb{V}_\ell(\overline{\mathbb{F}}_p)$$
$$\omega p : \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathbb{V}_p(\overline{\mathbb{F}}_p),$$

where $\mathbb{V}_p(\overline{\mathbb{F}}_p)$ is the category of isocrystals over $W(\overline{\mathbb{F}}_p)[1/p]$ and the category $\mathbb{V}_\ell(\overline{\mathbb{F}}_p)$ is the category of "germs of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-representations". Its objects are equivalence classes of Galois representations of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n})$ for some n, with equivalence given by $\rho \sim \rho'$ if there is some open subgroup of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ on which they agree. The morphisms are given by

$$\hom(\rho, \rho') = \varinjlim_n \hom_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p^n)}(\rho, \rho').$$

2.3.2. *Motives with G-structure.* If $G/\mathbb{Q}$ is an algebraic group then a motive with G-structure is an exact tensor-functor

$$\alpha : \mathrm{Rep}_{\mathbb{Q}}(G) \to \mathcal{C}$$

such that for all ℓ the following diagram commutes

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_\ell}(G) & \xrightarrow{\alpha_\ell} & \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \\ \downarrow & & \downarrow \omega_\ell \\ \mathrm{Vect}_{\mathbb{Q}_\ell} & = & \mathrm{Vect}_{\mathbb{Q}_\ell}. \end{array}$$

In other words, for each representation V of G we get a motive $\alpha(V)$ such that the ℓ-adic étale cohomology of $\alpha(V)$ is of dimension equal to $\mathrm{Dim}\, V$, and the same for the crystalline cohomology.

Associated to a motive with G-structure is the composition

$$\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathbb{V}_p(\overline{\mathbb{F}}_p).$$

This is an isocrystal with G-structure in the sense of Kottwitz [**41**], and these are classified up to isomorphism by the set $B(G)$. If we are also given a Shimura datum (G, X), then it makes sense to ask that the element of $B(G)$ we obtain is admissible with respect to this Shimura datum, in other words, to ask that it lies in $B(G, X) \subset B(G)$ (see Section 2.1.0.2). Let us call a motive with G-structure *admissible* (with respect to (G, X)) if this is the case. When (G, X) is the Siegel Shimura datum, this comes down to asking that the isocrystal (with alternating form) comes from a p-divisible group (with a polarisation).

2.3.3. *Circumventing the Tate conjecture.* Assuming the Tate conjecture, Milne [**49**] gives an explicit description of the category $\mathcal{C}$ with its tensor structure. In fact this description is so explicit that it is possible to write down a Tannakian category $\tilde{\mathcal{C}}$ together with faithful tensor functors

$$\tilde{\omega}_\ell : \tilde{\mathcal{C}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to V_\ell(\overline{\mathbb{F}}_p)$$

for all ℓ *without* assuming the Tate conjecture. If the Tate conjecture does hold, then there is an equivalence of categories $\mathcal{C} \simeq \tilde{\mathcal{C}}$, compatible with all the tensor functors. The category $\tilde{\mathcal{C}}$ is exactly the category of representations of the pseudo-motivic groupoid $\mathfrak{P}$ introduced in Chapter 3. Let us call $\tilde{\mathcal{C}}$ the category of *fake motives* over $\overline{\mathbb{F}}_p$; the notion of fake motive with G-structure and admissible fake motive with G-structure is now obvious.

2.3.4. *Back to the Langlands-Rapoport conjecture.* Let (G, X) be a Shimura datum such that Z_G^0 satisfies the Serre condition, i.e., such that Z_G^0 is isogenous to a product $T_1 \times T_2$ where $T_1/\mathbb{Q}$ is a split torus and where T_2 is a torus with $T_2(\mathbb{R})$ compact. This condition automatically holds when (G, X) is of Hodge type because then

$Z_G^0/w(\mathbb{G}_m)(\mathbb{R})$ is compact, where $w : \mathbb{G}_m \to G$ is the weight homomorphism obtained from X.

Under these assumptions, the indexing set of the Langlands-Rapoport conjecture is closely related to the set of equivalence classes of admissible fake motives with G-structure. In fact there are precisely two extra conditions we have to put on an admissible fake motive with G-structure in order for it to give rise to an admissible morphism (which are the objects that index the isogeny classes). To explain these, let us furthermore assume that G^{der} is simply connected.

- The first condition has to do with the induced motive with G^{ab}-structure. To be precise, it should agree with the one coming via "reduction modulo p" of the CM motive associated to the CM torus $(G^{\mathrm{ab}}, X^{\mathrm{ab}})$, see Section 4 of [**49**]).
- One condition at infinity, having to do with fully faithful tensor functors (the first is constructed by Milne assuming the Tate conjecture)
$$\omega_\infty : \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{R} \to V_\infty(\overline{\mathbb{F}}_p)$$
$$\tilde{\omega}_\infty : \tilde{\mathcal{C}} \otimes_{\mathbb{Q}} \mathbb{R} \to V_\infty(\overline{\mathbb{F}}_p).$$
Here $V_\infty(\overline{\mathbb{F}}_p)$ is the $\mathbb{R}$-linear Tannakian category of $\mathbb{Z}$-graded $\mathbb{C}$-vector spaces equipped with a τ-linear map F respecting the grading such that F^2 acts as $(-1)^m$ on the m-th graded piece, and τ is the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

For a general Shimura datum (G, X), we have to replace the pseudo-motivic Galois gerb $\mathfrak{P}$ with the quasi-motivic Galois gerb $\mathfrak{Q}$, which comes equipped with a map $\mathfrak{Q} \to \mathfrak{P}$. In other words we are replacing the category $\tilde{\mathcal{C}}$ with a category $\mathcal{D}$, which comes with a natural functor $\tilde{\mathcal{C}} \to \mathcal{D}$.

Remark 2.3.5. The perspective in Chapter 3 is in terms of Galois gerbs and morphisms of Galois gerbs rather than Tannakian categories and functors between them. The reason for this shift is that Kisin's paper [**38**] is written entirely in the former perspective. Since our proofs are merely generalisations of his to ramified groups, we often refer to his work for certain details and arguments, and its therefore natural to adopt his notation and perspective.

2.4. Affine Deligne-Lusztig varieties. In order to generalise the sets $X_p(\phi)$, we recall that in the case of the modular curve the set $X_p(\phi)$ is a subset of the space of all "$\breve{\mathbb{Z}}_p$-lattices" in V_pE_0, which can be identified with

$$\mathrm{GL}_2(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_2(\breve{\mathbb{Z}}_p).$$

In other words, the set of Dieudonné lattices in $V_PE_0 \simeq \breve{\mathbb{Q}}_p^{\oplus 2}$ sits inside the set of all lattices in $\breve{\mathbb{Q}}_p^{\oplus 2}$. The condition that a lattice $\Lambda \subset \breve{\mathbb{Q}}_p^{\oplus 2}$ is a Dieudonné lattice is the condition that

$$p\Lambda \subset F\sigma^*\Lambda \subset \Lambda,$$

here $F : \sigma^*\breve{\mathbb{Q}}_p^{\oplus 2} \simeq \breve{\mathbb{Q}}_p^{\oplus 2}$ is the map coming from the fact that V_pE_0 is an isocrystal. Let us write $F = b \otimes \sigma$ as a σ-linear map, with $b \in \mathrm{GL}_2(\breve{\mathbb{Q}}_p)$. Then a lattice Λ is a Dieudonné-lattice if under the relative position map

$$\mathrm{Inv} : \mathrm{GL}_2(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_2(\breve{\mathbb{Z}}_p) \times \mathrm{GL}_2(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_2(\breve{\mathbb{Z}}_p) \to \Big(\mathrm{GL}_2(\breve{\mathbb{Z}}_p)\backslash\,\mathrm{GL}_2(\breve{\mathbb{Q}}_p)/\,\mathrm{GL}_2(\breve{\mathbb{Z}}_p)\Big) \simeq \mathbb{Z}^2/S_2$$

the image $\mathrm{Inv}(\Lambda, b\Lambda) = (1, 0)$. If we think of $\mathbb{Z}^2/S_2$ as the set of conjugacy classes of cocharacters of GL_2, then the element $(1, 0)$ corresponds precisely to the inverse of the Hodge cocharacter associated to the Shimura datum of the modular curve.

It is important to note that $X_p(\phi)$, unlike $X^p(\phi)$ depends on the isogeny class of E_0, or rather it depends on the isogeny class of $E_0[p^\infty]$ or equivalently on the σ-conjugacy class of b. For a connected reductive group $G/\mathbb{Q}_p$ and choice of parahoric $\mathcal{G}$, we consider

$$G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p),$$

which we think of as the space of '$\mathcal{G}$-lattices' inside the standard G-isocrystal given by $b \in B(G, X)$. To define the affine Deligne-Lusztig variety we again have a relative position map (see Section 2.5)

$$\mathrm{Rel} : G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) \times G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) \to \Big(\mathcal{G}(\breve{\mathbb{Z}}_p)\backslash G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)\Big) \simeq W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}},$$

and $X_p(\phi)$ will be a subset of $G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)$ defined by a condition on $\mathrm{Rel}(g, bg)$. It remains for us to define this condition, which will take the form of a finite set, called the admissible set

$$\mathrm{Adm}(\mu)_{\mathcal{G}} \subset \mathcal{G}(\breve{\mathbb{Z}}_p)\backslash G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) \simeq W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$$

When $\mathcal{G}$ is hyperspecial then $W_{\mathcal{G}}\backslash\tilde{W}/W_{\mathcal{G}}$ is just the set of conjugacy classes of cocharacters of G, and the admissible set will consists of a single element corresponding to the inverse of the Hodge cocharacter associated to the Shimura datum. When $\mathcal{G}$ is a general parahoric subgroup, the admissible set will have more than one element and we will define it in Section 2. This definition is motivated by considerations from local harmonic analysis, see [**22**] for an introduction.

3. Previous results

Kottwitz describes the mod p points of Shimura varieties of PEL type A and C, at primes $p > 2$ of hyperspecial good reduction in [**42**]. His description of the isogeny classes is essentially the same, but his classification of the isogeny classes takes a slightly different form than the one in the Langlands-Rapoport conjecture.

Kisin [**38**] proves a slightly weaker version of the Langlands-Rapoport conjecture for abelian type Shimura varieties under the assumption that $G_{\mathbb{Q}_p}$ is quasi-split and split over an unramified extension, and that U_p is hyperspecial. An important idea in his proof is to show that both admissible morphisms and isogeny classes 'come from special points'. He deduces the former from Satz 5.3 of [**47**] and the latter is deduced, after a lengthy dévissage from the abelian type to the Hodge type case, from uniformisation of isogeny classes (we'll discuss his strategy for proving uniformisation of isogeny classes when we discuss our own proof strategy).

In the parahoric case, uniformisation of isogeny classes was proven by Zhou in [**73**], under the assumption that $G_{\mathbb{Q}_p}$ is residually split. We remind the reader that split implies residually split implies quasi-split and that residually split + unramified implies split.

There is also important work of Reimann [**61**], which not only proves the Langlands-Rapoport conjecture for certain quaternionic Shimura varieties but also corrects the definition of the quasi-motivic Galois gerb given by Langlands-Rapoport.

4. Main results

Let (G, X) be a Shimura datum of abelian type and let $p > 2$ be a prime such that $G_{\mathbb{Q}_p}$ is quasi-split and splits over an unramified extension. Let $U_p \subset G(\mathbb{Q}_p)$ be a parahoric subgroup and consider the tower of Shimura varieties $\{\mathbf{Sh}_{G,U^pU_p}\}_{U^p}$ over the reflex field E with its action of $G(\mathbb{A}_f^p)$, where U^p varies over compact open subgroups of $G(\mathbb{A}_f^p)$. Then by Theorem 0.1 of [**37**], this tower of Shimura varieties has a $G(\mathbb{A}_f^p)$-equivariant extension to a tower of flat normal schemes $\{\mathscr{S}_{G,U^pU_p}\}_{U^p}$ over $\mathcal{O}_{E_{(v)}}$, where $v \mid p$ is a prime of the reflex field E.

Theorem 1. *Let (G, X) be as above and suppose that (G^{ad}, X^{ad}) has no factors of type $D^{\mathbb{H}}$ or that U_p is contained in a hyperspecial subgroup.*[3] *Then there is an $G(\mathbb{A}_f^p)$-equivariant bijection*

$$\varprojlim_{U^p} \mathscr{S}_{G,U^pU_p}(\overline{\mathbb{F}}_p) \simeq \coprod_{\phi} \varprojlim_{U^p} I_\phi(\mathbb{Q})\backslash X_p(\phi) \times X^p(\phi)/U^p$$

respecting the action of Frobenius, where the action of $I_\phi(\mathbb{Q})$ on $X_p(\phi) \times X^p(\phi)$ is the natural action conjugated by some $\tau(\phi) \in I_\phi^{ad}(\mathbb{A}_f)$. Here $X_p(\phi)$ is the affine Deligne-Lustzig variety of level U_p associated to ϕ, see Section 2.1.0.4. The indexing set runs over conjugacy classes of admissible morphisms $\mathfrak{Q} \to G$, see Section 3.2.

As a byproduct of our arguments, we obtain the following result:

Theorem 2. *Let (G, X) be as above and let U_p denote a hyperspecial parahoric. Assume that G^{ad} is $\mathbb{Q}$-simple and let $\mathscr{S}_{U,\overline{\mathbb{F}}_p}\{w\}$ be an Ekedahl-Oort stratum that is not contained in the basic locus (the smallest Newton stratum). Then*

$$\mathscr{S}_{U,\overline{\mathbb{F}}_p}\{w\} \to \mathscr{S}_{U,\overline{\mathbb{F}}_p}$$

induces a bijection on connected components.

Our methods will also prove versions of Theorems 1 and 2 without the assumption that $G_{\mathbb{Q}_p}$ splits over an unramified extension, but always under the assumption that $G_{\mathbb{Q}_p}$ is quasi-split. Moreover we prove irreducibility of Kottwitz-Rapoport strata at Iwahori level. The generalisations of Theorems 1 and 2 are Theorems 5.4.0.1 and 5.4.0.3, respectively, which assume that the Shimura varieties in question are proper and not of type A. Our proof of Theorem 5.4.0.1 proceeds by reduction to the case of an very special parahoric. This case is handled by Rong Zhou in Appendix A of [**32**], by studying connected components of affine Deligne-Lusztig varieties of very special level and applying the main results of his earlier paper [**73**].

Ekedahl-Oort strata contained in the basic locus are highly reducible, for example the number of points in the supersingular locus of the modular curve goes to infinity with p. Similarly the basic locus itself is highly reducible. This means that the theorem is false for products of Shimura varieties with b basic in one factor and non-basic in the other; this is where the assumption that G^{ad} is $\mathbb{Q}$-simple comes from. It can be

[3]See Appendix B of [**50**] for a classification of abelian type Shimura varieties into types $A, B, C, D^{\mathbb{R}}$ and $D^{\mathbb{H}}$

replaced with the assumption that b is $\mathbb{Q}$-non basic, which means that the image of b in $B(G_{i,\mathbb{Q}_p})$ is basic for any $\mathbb{Q}$-factor G_i of G^{ad} (this terminology comes from [**44**]).

It follows from Theorem D of [**65**] that each Newton stratum contains a minimal EO stratum, that is, an EO stratum that is a central leaf. Central leaves that are not contained in the basic locus are expected to be irreducible, this is often referred to as the 'discrete part' of the Hecke-orbit conjecture (c.f. [**9, 72**]). In a previous version of my paper [**32**] I claimed to prove this conjecture, however my proof contained an error.

Instead the conjecture follows from my joint work [**33**] with Luciena Xiao Xiao, where we prove irreducibility of Igusa varieties. Our proof of this irreducibility builds on Theorem 2, and combines recent work of D'Addezio on monodromy of compatible local systems with a generalisation of a method of Hida. Our results on the irreducibility of Igusa varieties were independently obtained by Kret and Shin [**44**], using completely different methods. Their proof uses point counting methods, automorphic forms and harmonic analysis.

Remark 4.0.1. Theorem 2 was proven for Siegel modular varieties varieties by Ekedahl and van der Geer [**15**]. There is also work of Achter [**1**] concerning certain $GU(1,n-1)$ Shimura varieties (his results are stated as irreducibility of Newton strata, but it his case the Newton strata in question are also Ekedahl-Oort strata).

Remark 4.0.2. The assumption that $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$ or that U_p is contained in a hyperspecial subgroup is also present in the statement of Theorem 0.4 of [**37**] and for the same reason: We can reduce Theorem 1 for (G,X) to the case of Shimura varieties (H,Y) of Hodge type with H^{der} simply-connected, except if $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has factors of type $D^{\mathbb{H}}$.

5. Overview of the proof

Both Kisin and Zhou employ roughly the same strategy, which we will now briefly sketch: The integral models $\mathscr{S}_G$ of Hodge type Shimura varieties come equipped, by construction, with finite maps $\mathscr{S}_G \to \mathscr{S}_{\mathrm{GSp}}$ to Siegel modular varieties. Given a point $x \in \mathscr{S}_G(\overline{\mathbb{F}}_p)$, classical Dieudonné theory produces a map

$$X_p(\phi) \to \mathscr{S}_{\mathrm{GSp}}(\overline{\mathbb{F}}_p)$$

and it suffices to show that it factors through $\mathscr{S}_G$. A deformation theoretic result shows that it suffices to prove this result for one point on each connected component of $X_p(\phi)$, and therefore we need to understand these connected components. In the hyperspecial case, this is done in [**10**], and in the parahoric case this is done in [**31**], under the assumption that $G_{\mathbb{Q}_p}$ is residually split. The main obstruction to extend Zhou's methods beyond the residually split case, is that we do not understand connected components of affine Deligne-Lusztig varieties for more general groups.

Our proof of Theorem 1 does not address connected components of affine Deligne-Lusztig varieties. Instead, we prove the theorem at parahoric level by reducing to the case of a hyperspecial parahoric, where we can use Kisin's result. Our argument makes crucial use of moduli spaces of mixed characteristic shtukas (see [**64**, **71**]) and the incarnation of special fibers of local models as subvarieties of *mixed characteristic* affine Grassmannians (see [**30**]).

We will now give a brief overview of the strategy of our proof: It turns out that it suffices to work with Hodge type Shimura varieties such that G^{ad} is $\mathbb{Q}$-simple. Let U_p denote a hyperspecial parahoric and let U_p' denote an Iwahori subgroup contained in U_p, then by Section 7 of [**73**] there is a proper morphism of integral models $\mathscr{S}_{U^pU_p'} \to \mathscr{S}_{U^pU_p}$ and we let $\mathrm{Sh}_{U_p'} \to \mathrm{Sh}_{U_p}$ be the induced morphism on the perfections of their special fibers. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}_{U_p'} & \longrightarrow & \mathrm{Sht}_{\mu,U_p'} \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{U_p} & \longrightarrow & \mathrm{Sht}_{\mu,U_p}, \end{array} \tag{5.0.1}$$

where Sht_{μ,U_p} is the stack of U_p-shtukas of type μ introduced by Xiao-Zhu [**71**] (c.f. Section 2.3, 2.5 and Section 4 of [**64**]), with μ the inverse [4] of the Hodge cocharacter induced by the Shimura datum. The horizontal morphisms in (1.5.0.1) are the Hodge type analogues of the morphism from the moduli space of abelian varieties to the moduli space of p-divisible groups (or the moduli spaces of Dieudonné-modules, since we are over a perfect base). If $G = \mathrm{GSp}$, then this diagram is Cartesian and in general it follows from 'local uniformisation' of $\mathrm{Sht}_{\mu,U_p'}$, that $\mathrm{Sh}_{U_p'}$ has the correct $\overline{\mathbb{F}}_p$ points if and only if (1.5.0.1) is Cartesian. So our main theorem, in the Hodge type case, is equivalent to showing that this diagram is Cartesian.

[4]We will make our precise conventions on the Hodge cocharacter clear in Chapter 3.

The morphism $\mathrm{Sht}_{\mu,U_p'} \to \mathrm{Sht}_{\mu,U_p}$ is representable by perfectly proper algebraic spaces, and we let $\tilde{Sh}_{U_p'}$ be the fiber product of (1.5.0.1). There is a natural morphism $\iota : \mathrm{Sh}_{U_p'} \to \tilde{Sh}_{U_p'}$ given by the universal property of the fiber product. To prove the main theorem, it suffices to show that ι is an isomorphism, which we do in three steps:

- We first show that $\iota : \mathrm{Sh}_{U_p'} \to \tilde{Sh}_{U_p'}$ is a closed immersion.
- We then show that $\tilde{Sh}_{U_p'}$ is equidimensional of the same dimension as $\mathrm{Sh}_{U_p'}$ and that it has a Kottwitz-Rapoport stratification with the expected properties.
- We conclude by showing that $\tilde{Sh}_{U_p'}$ has the same number of irreducible components as $\mathrm{Sh}_{U_p'}$.

It is this last step that requires by far the most work. The second bullet points tells us that $\tilde{Sh}_{U_p'}$ and $\mathrm{Sh}_{U_p'}$ are unions of closures of Kottwitz-Rapoport (KR) strata and therefore it suffices to count irreducible components in each KR stratum separately. A result of Zhou [**73**] tells us that ι is an isomorphism on basic KR strata, and so it suffices to analyse irreducible components of nonbasic KR strata. We will show that the nonbasic KR strata of $\tilde{Sh}_{U_p'}$ are 'irreducible', by which we mean that they have one irreducible component lying over each connected component of $\mathrm{Sh}_{U_p,\overline{\mathbb{F}}_p}$. It follows from Section 8 of [**73**] that the KR strata of $\mathrm{Sh}_{U_p'}$ have at least this many irreducible components, and we conclude that $\tilde{Sh}_{U_p'}$ is isomorphic to $\mathrm{Sh}_{U_p'}$ and that KR strata of $\mathrm{Sh}_{U_p'}$ are 'irreducible'. Theorem 2 now follows because every EO stratum is the image of a KR stratum under the forgetful map. In the introduction to Section 4, we will give a more detailed overview of our connectedness argument. For now, we just mention that it combines the connectedness argument of [**21**], the connectedness argument of [**31**] and strong approximation. To deal with noncompact Shimura varieties, we make use of results of Wedhorn-Ziegler [**70**] and a recent result of Andreatta [**2**].

6. Outline

We start with some preliminaries in the local representation theory of $G_{\mathbb{Q}_p}$ such as Iwahori-Weyl groups, affine Grassmannians and affine Deligne-Lusztig varieties. We recommend the reader skip Section 2.6 on the first reading, as it is very technical.

In Chapter 3 we define Galois gerbs and state the Langlands-Rapoport conjecture. Afterwards, we study how the Langlands-Rapoport conjecture behaves under central isogenies of Shimura data. This latter part of the chapter is only needed to deduce the conjecture for abelian type Shimura varieties from the conjecture for Hodge type

Shimura varieties. Its contents can be summed in one sentence as follows: Everything in Section 3 of [**38**] generalises to arbitrary quasi-split groups without too much trouble. We encourage the reader skip this part of the chapter upon first reading.

Chapter 4 is, from the author's point of view, the most interesting chapter and the chapter where all the original mathematics happens. The main goal of this Chapter is to show that we can deduce Theorem 1 for a general parahoric subgroup from the case of a very special parahoric subgroup, for a Shimura variety of Hodge type. This chapter contains the connectedness argument sketched above, and from it we deduce irreducibility for EKOR strata.

Chapter 5 is where we state and prove all the theorems for abelian type Shimura varieties, mostly following Section 4 of [**38**].

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CHAPTER 2

Local Preliminaries

In this chapter we collect definitions and results about representation theoretic objects associated to a connected reductive group $G/\mathbb{Q}_p$. Some of these are of a combinatorial nature, such as Iwahori-Weyl groups and Frobenius conjugacy classes, and some of these are of a geometric nature, such as affine flag varieties and affine Deligne-Lusztig varieties. The main of the chapter is to introduce moduli "spaces" of mixed characteristic shtukas and to prove some results about them for later use.

1. Iwahori-Weyl groups

The main reference for this section will be Section 2 of [**27**] and Section 2 of [**73**]. Let G be a connected reductive group over $\mathbb{Q}_p$ and let $\{\mu\}$ be a conjugacy class of homomorphisms $\mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$. Let L be the completion of the maximal unramified extension of $\mathbb{Q}_p$, with Frobenius σ. Let $S \subset G_L$ be a maximal L-split torus and let T be its centraliser, which is a maximal torus of G by a theorem of Steinberg. Choose a σ-invariant alcove $\mathfrak{a}$ in the apartment of the Bruhat-Tits building of G associated to S over L. We define the *relative Weyl group* by

$$W_0 = N(L)/T(L)$$

and the *Iwahori-Weyl group* (or extended affine Weyl group) by

$$\tilde{W} = N(L)/\mathcal{T}(\mathcal{O}_L),$$

where $\mathcal{T}/\mathcal{O}_L$ is the connected component of the identity of the Néron model of T. There is a short exact sequence

$$0 \to X_*(T)_I \to \tilde{W} \to W_0 \to 0,$$

where $I = \mathrm{Gal}(\overline{L}/L)$ is the inertia group and $X_*(T)_I$ denotes the inertia coinvariants of the cocharacter lattice of T. The map $X_*(T)_I \to \tilde{W}$ is denoted on elements by $\lambda \mapsto t^\lambda$. Let $\mathbb{S} \subset \tilde{W}$ denote the set of simple reflections in the walls of $\mathfrak{a}$ and let $\tilde{W}_a$ be the subgroup of $\tilde{W}$ generated by $\mathbb{S}$, which we will call the *affine Weyl group*. Parahoric subgroups $\mathcal{K}$ of G_L that contain the Iwahori subgroup corresponding to $\mathfrak{a}$, correspond to subsets $K \subset \mathbb{S}$ such that the subgroup $\tilde{W}_K$ generated by K is finite. This identification is Frobenius equivariant in the sense that $\sigma(\mathcal{K})$ corresponds to $\sigma(K)$. In particular, a subset $K \subset \mathbb{S}$ corresponds to a parahoric subgroup of G if and only if $\sigma(K) = K$, note that our fixed Iwahori subgroup corresponds to $\emptyset \subset \mathbb{S}$. There are parahoric group schemes $\mathcal{G}_K$ over $\mathcal{O}_L$ associated to subsets $K \subset \mathbb{S}$ as above, and we have identifications $\sigma^*\mathcal{G}_K \simeq \mathcal{G}_{\sigma(K)}$. In particular, if K is stable under σ then $\mathcal{G}_K$ is defined over $\mathbb{Z}_p$. The maximal reductive quotient of the special fiber of $\mathcal{G}_K$ is a reductive group over the residue field k of L with Dynkin diagram K.

Parahoric group schemes are the connected component of the identity of so-called Bruhat-Tits stabiliser group schemes. We will call a parahoric subgroup *connected* if it is equal to such a Bruhat-Tits stabiliser group scheme. When working with (Hodge type) Shimura varieties of parahoric level, we will always assume that the corresponding parahoric subgroup is connected. This is automatically true if G^{der} is simply connected and $X_*(G_{\mathrm{ab}})_I$ is torsion free or if $G_{\mathbb{Q}_p}$ is unramified and our parahoric is contained in a hyperspecial subgroup.

A Bruhat-Tits stabiliser group scheme is called *special* if it is the stabiliser of a special vertex of the Bruhat-Tits building of G^{ad}, and *very special* if it is the stabiliser of a very special vertex, that is, a special vertex that remains special in the Bruhat-Tits building of G^{ad}_L.

There is a split short exact sequence

$$0 \to \tilde{W}_a \to \tilde{W} \to \pi_1(G)_I \to 0, \tag{1.0.1}$$

where $\pi_1(G)$ is the algebraic fundamental group of G (c.f. the introduction of [**4**]). The affine Weyl group $\tilde{W}_a$ has the structure of a Coxeter group, and this can be used to define a Bruhat order and a notion of length on $\tilde{W}$, by splitting (2.1.0.1) and regarding $\pi_1(G)_I \subset \tilde{W}$ as the subset of length zero elements. We can now define the set of μ-admissible elements as

$$\mathrm{Adm}(\mu) := \{w \in \tilde{W} \ : \ w \le t^{x\overline{\mu}} \text{ for some } x \in W_0\},$$

where $\overline{\mu}$ is the image of a dominant representative (with respect to the choice of some Borel of G over L) of $\{\mu\}$ in $X_*(T)_I$. There is a unique element $\tau = \tau_\mu \in \mathrm{Adm}(\mu)$ of length zero and in fact $\mathrm{Adm}(\mu) \subset \tilde{W}_a\tau$. For K a σ-stable type we define $\mathrm{Adm}(\mu)_K$ as the image of $\mathrm{Adm}(\mu)$ under $\tilde{W} \to \tilde{W}_K\backslash\tilde{W}/\tilde{W}_K$. We write ${}^K\mathrm{Adm}(\mu)$ for $\mathrm{Adm}(\mu) \cap {}^K\tilde{W}$, where ${}^K\tilde{W}$ denotes the subset of elements that are of minimal length in their left $\tilde{W}_K$-coset.

1.0.1. *σ-conjugacy classes.* There is a Kottwitz homomorphism $\tilde{k}_G : G(L) \to \pi_1(G)_I$, and we write k_G for the composition with $\pi_1(G)_I \to \pi_1(G)_\Gamma$, where $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. If we let $B(G)$ denote the set of σ-conjugacy classes in $G(L)$, then k_G induces a functorial map

$$k_G : B(G) \to \pi_1(G)_\Gamma$$

and there is also a functorial map (the Newton map)

$$\nu_G : B(G) \to \mathcal{N}(G),$$

where $\mathcal{N}(G) = (X_*(T)_\mathbb{Q}/W)^\Gamma$. More canonically, we can describe $\mathcal{N}(G)$ as the set of $G(\overline{\mathbb{Q}}_p)$-conjugacy classes of morphisms $\nu : \mathbb{D}_{\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ defined over $\mathbb{Q}_p$, where $\mathbb{D}$ is the pro-torus over $\mathbb{Q}_p$ with character group $\mathbb{Q}$. Moreover, the following diagram commutes

$$\begin{array}{ccc} B(G) & \xrightarrow{k_G} & \pi_1(G)_\Gamma \\ \big\downarrow{\scriptstyle \nu_G} & & \big\downarrow \\ \mathcal{N}(G) & \longrightarrow & (\pi_1(G) \otimes_\mathbb{Z} \mathbb{Q})^\Gamma, \end{array}$$

where the right vertical map is the isomorphism

$$(\pi_1(G) \otimes_\mathbb{Z} \mathbb{Q})_\Gamma \to (\pi_1(G) \otimes_\mathbb{Z} \mathbb{Q})^\Gamma,$$

defined by averaging over Γ-orbits, see p. 162 of [**57**]. We also recall that the product

$$(k_G, \nu_G) : B(G) \to \pi_1(G)_\Gamma \times \mathcal{N}(G)$$

is injective. There is a natural partial order on $\mathcal{N}(G)$ and we can use this to define a partial order on $B(G)$ by setting $[b] \leq [b']$ if $k_G([b]) = k_G([b'])$ and $\nu_{[b]} \leq \nu_{[b']}$.

1.0.2. *Admissible σ-conjugacy classes.* Let $\psi : G \to G^*$ be an inner twisting, where G^* is the quasi-split inner form of G. If $\{\mu\}$ is a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of morphisms $\mathbb{D}_{\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ defined over $\mathbb{Q}_p$, then so is $\{\psi \circ \mu\}$. This gives us a map

$$N_\psi : \mathcal{N}(G) \to \mathcal{N}(G^*),$$

which only depends on the $G(\breve{\mathbb{Q}}_p)$-conjugacy class of ψ. Our conjugacy class $\{\psi \circ \mu\}$ of cocharacters $\mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G^*_{\overline{\mathbb{Q}}_p}$ has a dominant representative $\mu^* \in X_*(T^*)$, for some choice of maximal torus and Borel $T^* \subset B^*$ defined over $\mathbb{Q}_p$. We set

$$N\mu^* = \frac{1}{[\Gamma : \Gamma_{\mu^*}]} \sum_{\sigma \in \Gamma/\Gamma_{\mu^*}} \sigma\mu^* \in X_*(T^*)_{\mathbb{Q}}^{\Gamma},$$

where Γ_{μ^*} is the stabiliser of μ^* in Γ. We will write $\overline{\mu}^*$ for the image of $N\mu^*$ in $\mathcal{N}(G^*)$ and $\mu^\sharp$ for the image of μ in $\pi_1(G)_\Gamma$.

DEFINITION 1.0.3. *We define $B(G, \{\mu\})$ or just $B(G, \mu)$ as the set $[b] \in B(G, \mu)$ such that $k_G([b]) = \mu^\sharp$ and such that $N_\psi(\nu_G([b])) \leq \overline{\mu}^*$.*

1.0.4. *Affine Deligne-Lusztig sets.* Let $K \subset \mathbb{S}$ be a σ-stable *type*, that is, a subset such that $\tilde{W}_K$ is finite. Then for $b \in G(L)$ we define the affine Deligne-Lusztig set

$$X_\mu(b)_K = \{g \in G(L)/\mathcal{G}_K(\mathcal{O}_L) \mid g^{-1} b \sigma(g) \in \bigcup_{w \in \mathrm{Adm}(\mu)} \mathcal{G}_K(\mathcal{O}_L) w \mathcal{G}_K(\mathcal{O}_L)\}.$$

This set has an action of $J_b(\mathbb{Q}_p)$, where J_b is the algebraic group over $\mathbb{Q}_p$ whose R-points are given by

$$J_b(R) = \{g \in G(L \otimes_{\mathbb{Q}_p} R) \mid g^{-1} b \sigma(g) = b\}.$$

Moreover, it only depends on the class of b in $B(G)$ as a set with $J_b(\mathbb{Q}_p)$ action, up to isomorphism. The following nonemptiness result for the sets $X_\mu(b)_K$ was conjectured by Kottwitz and Rapoport and proven by He.

THEOREM 1.0.5 (Theorem 1.1 of [**29**]). *The set $X_\mu(b)_K$ is nonempty if and only if $b \in B(G, \mu)$. Moreover, for $K' \subset K$ another σ-stable type, the natural projection $G(L)/\mathcal{G}_{K'}(\mathcal{O}_L) \to G(L)/\mathcal{G}_K(\mathcal{O}_L)$ induces a $J_b(\mathbb{Q}_p)$-equivariant surjection*

$$X_\mu(b)_{K'} \to X_\mu(b)_K.$$

We will later see that $G(L)/\mathcal{G}_K(\mathcal{O}_L)$ can be identified with the set of $\overline{\mathbb{F}}_p$-points of a perfect ind-scheme Gr_K over $\overline{\mathbb{F}}_p$, and that there is a closed subscheme of Gr_K with an action of $J_b(\mathbb{Q}_p)$ such that its $\overline{\mathbb{F}}_p$-points can be identified with $X_\mu(b)_K$, equivariant for the action of $J_b(\mathbb{Q}_p)$.

2. Some perfect algebraic geometry

We will use the language of perfect algebraic geometry from Appendix A of [**75**]. In this section we will collect some important definitions and results that we will make use of regularly. Let k be the residue field of L as above, then we call a k-algebra R

perfect if the map $\sigma : R \to R$ defined by $r \mapsto r^p$ is an isomorphism. Let $\mathbf{Aff}_k^{\mathrm{perf}}$ denote the category of perfect k-algebras, this will be the 'test category' on which many of our geometric objects are defined. If X is a scheme over k, considered as presheaf on the category of k-algebras, then the restriction of X to the $\mathbf{Aff}_k^{\mathrm{perf}}$ precisely remembers the perfection

$$X^{\mathrm{perf}} := \varprojlim_{\sigma} X$$

of X. There is a well behaved notion of pfp (perfectly of finite presentation) algebraic spaces as functors on $\mathbf{Aff}_k^{\mathrm{perf}}$. One can prove that every pfp (perfectly of finite presentation) algebraic space is in fact the perfection of a (weakly normal) algebraic space of finite presentation over k, and similarly that every morphism $f : X \to Y$ between pfp algebraic spaces arises from a morphism between algebraic spaces of finite presentation (this is called a 'deperfection'). One way to define properness of such morphisms is by asking that every deperfection of it is proper. The most important notion we need is that of perfectly smooth morphism:

Definition 2.0.1 (Definition A.18 of [**75**]). *Let $f : X \to Y$ be a morphism between pfp algebraic spaces over k. We say that f is perfectly smooth at $x \in X$ if there is an étale morphism $U \to X$ whose image contains x and an étale morphism $V \to Y$ whose image contains $f(x)$ such that: The map $U \to Y$ factors as $U \to V \to Y$ and the morphism $h : U \to V$ factors as an étale morphism $h' : U \to V \times (\mathbb{A}^n)^{perf}$ followed by the projection to V. We say that f is perfectly smooth if it is perfectly smooth at all points $x \in X$.*

Appendix A of [**71**] defines the notion of a perfect algebraic stack: Basically we take fpqc stacks that have perfectly smooth covers by schemes and diagonals represented by a perfect algebraic spaces. There is then a well defined notion of pfp (perfectly of finitely presentation) algebraic stack.

Definition 2.0.2 (Definition A.1.13 of [**71**]). *A morphism $f : X \to Y$ of pfp algebraic stacks is called perfectly smooth if there is a perfectly smooth morphism $U \to X$ from a scheme U such that the composition $U \to X \to Y$ is perfectly smooth (this makes sense because $U \to Y$ is representable).*

Lemma 2.0.3. *Perfectly smooth morphisms are stable under composition and base change.*

PROOF. Standard diagram chase, using the fact that étale morphisms are stable under composition and base change. □

REMARK 2.0.4. A perfectly smooth morphism $f : X \to Y$ has a relative dimension that is locally constant on Y. This relative dimension is preserved by base change and 'adds up' under composition.

LEMMA 2.0.5. *Let $f : X \to Y$ be a perfectly proper morphism between pfp algebraic spaces over k that induces a bijection on k-points, then f is an isomorphism.*

PROOF. Let $f' : X' \to Y'$ be a proper morphism of locally of finite type algebraic spaces over k whose perfection gives f. Then the fact that f induces a bijection on k-points tells us that f' induces a bijection on k-points and so it is surjective (since k-points are dense) and universally injective (because our morphisms are locally of finite type, see [**69**]). We now deduce that f' is a universal homeomorphism because it is universally injective, universally injective and universally closed. This implies that f is a separated universal homeomorphism between pfp algebraic spaces, and so it is an isomorphism by Corollary A.16 of [**75**]. □

3. Affine flag varieties and moduli spaces of local shtukas

In this section we will quickly recall some definitions from [**64, 71, 75**] and state some results. Let G be a connected reductive group over $\mathbb{Q}_p$ as above and let $\mathcal{G}_K, \mathcal{G}_J/\mathcal{O}_L$ be parahoric group schemes corresponding to types $K, J \subset \mathbb{S}$. For an object R of $\mathbf{Aff}_k^{\mathrm{perf}}$ we define

$$D_R = \operatorname{Spec} W(R), \qquad D_R^* = \operatorname{Spec} W(R)[1/p],$$

which are the mixed characteristic analogues of the disk $\operatorname{Spec} R[[t]]$ and the punctured disk $\operatorname{Spec} R[[t]][1/t]$. We consider the following functors on $\mathbf{Aff}_k^{\mathrm{perf}}$

$$LG(R) := G(D_R^*)$$
$$L^+\mathcal{G}_K(R) := \mathcal{G}(D_R).$$

DEFINITION 3.0.1. *Let R be an object of $\mathbf{Aff}_k^{perf}$, let $\mathcal{E}$ be a $\mathcal{G}_K$-torsor on D_R and let $\mathcal{F}$ be a $\mathcal{G}_J$-torsor on D_R. A modification*

$$\beta : \mathcal{E} \dashrightarrow \mathcal{F}$$

is an isomorphism of G-torsors

$$\beta : \mathcal{E}\big|_{D_R^*} \simeq \mathcal{F}\big|_{D_R^*}.$$

Here we mean torsor in the usual sense, i.e., a scheme $\mathcal{E} \to \operatorname{Spec} D_R$ with an action of $\mathcal{G}_K$ such that the action map

$$\mathcal{G}_K \times_{D_R} \mathcal{E} \to \mathcal{E} \times_{D_R} \mathcal{E}$$
$$(g, x) \mapsto (gx, x)$$

is an isomorphism and such that $\mathcal{E} \to \operatorname{Spec} D_R$ has a section fpqc locally on $\operatorname{Spec} D_R$. Since $\mathcal{G}_K$ is a smooth group scheme, this implies that $\mathcal{E} \to \operatorname{Spec} D_R$ is smooth and hence has a section étale locally on $\operatorname{Spec} D_R$. In fact, it follows from the proof of Lemma 1.3 of **[75]** that $\mathcal{E}$ can be trivialised after an étale cover $\operatorname{Spec} D_{R'} \to \operatorname{Spec} D_R$ coming from an étale cover $\operatorname{Spec} R' \to \operatorname{Spec} R$.

DEFINITION 3.0.2. *We define the (partial) affine flag variety Gr_K to be the functor on $\mathbf{Aff}_k^{perf}$ which sends R to the set of isomorphism classes of modifications*

$$\beta : \mathcal{E} \dashrightarrow \mathcal{E}^0,$$

where $\mathcal{E}$ is an $\mathcal{G}_K$-torsor on D_R and $\mathcal{E}^0$ is the trivial $\mathcal{G}_K$-torsor on D_R.

THEOREM 3.0.3 (**[75]**, **[3]**). *The functor Gr_K can be represented by an inductive limit of perfectly proper perfect schemes, and the transition morphisms in this inductive limit are closed embeddings. Moreover Gr_K is the étale sheafification of the sheaf $R \mapsto LG(R)/L^+\mathcal{G}_K(R)$*

DEFINITION 3.0.4. *We define the prestack Sht_K of $\mathcal{G}_K$-shtukas to be the functor on $\mathbf{Aff}_k^{perf}$ which sends a perfect k-algebra R to the groupoid of modifications*

$$\beta : \sigma^*\mathcal{E} \dashrightarrow \mathcal{E},$$

where $\sigma : D_R \to D_R$ denotes the Frobenius morphism induced from the relative Frobenius on R and where $\mathcal{E}$ is a $\mathcal{G}_K$-torsor on D_R. Here we consider $\sigma^*\mathcal{E}\big|_{D_R^*}$ as a G-bundle via the isomorphism $\sigma : \sigma^*G \to G$, coming from the fact that G is defined over $\mathbb{Q}_p$.

LEMMA 3.0.5 (Lemma 4.1.4 of [**64**]). *We have an isomorphism*

$$\mathrm{Sht}_K \simeq \left[\frac{LG}{\mathrm{Ad}_\sigma L^+\mathcal{G}_K}\right],$$

where Ad_σ *denotes* $L^+\mathcal{G}$ *acting on* LG *via* σ*-conjugation. To be precise let* $\sigma : L^+\mathcal{G}_K \to L^+\mathcal{G}_{\sigma(K)}$ *be the relative Frobenius morphism, then we let* $L^+\mathcal{G}_K(R)$ *act on* $LG(R)$ *via*

$$h \cdot g = (h^{-1} g \sigma(h)).$$

Here the quotient notation means quotient stack (in the étale topology or equivalently the fpqc topology).

4. Forgetful maps

If $L^+\mathcal{G}_J \subset L^+\mathcal{G}_K$ is an inclusion of parahoric subgroups corresponding to an inclusion of types $J \subset K$, then there is a forgetful map

$$\mathrm{Sht}_J \to \mathrm{Sht}_K .$$

Our goal is to show that these forgetful maps are representable by perfectly proper algebraic spaces. The basic idea is to show that the fibers are étale locally isomorphic to partial flag varieties for the maximal reductive quotient of the special fiber of $\mathcal{G}_K$, c.f. Proposition 8.7 of [**54**]. Let H_J be the image of $\mathcal{G}_J$ in $(\overline{\mathcal{G}}_K)^{\mathrm{red}}$, it is a parabolic subgroup of type $J \subset K$.

LEMMA 4.0.1. *The forgetful map* $\mathbf{B}L^+\mathcal{G}_J \to \mathbf{B}L^+\mathcal{G}_K$ *is an* $(\overline{\mathcal{G}}_K)^{red}/H_J$*-bundle, in particular it is representable by perfectly proper algebraic spaces.*

PROOF. Let R be an object of $\mathbf{Aff}_k^{\mathrm{perf}}$ and let X be an $L^+\mathcal{G}_K$ torsor on R represented by a map $\mathrm{Spec}\, R \to \mathbf{B}L^+\mathcal{G}_K$. Then it follows from general nonsense that the top square in the following diagram of prestacks is Cartesian

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ [X/L^+\mathcal{G}_J] & \longrightarrow & [*/L^+\mathcal{G}_J] \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, R & \longrightarrow & [*/L^+\mathcal{G}_K]. \end{array}$$

Lemma 1.3 of [**75**] tells us that there is an étale cover $T \to \mathrm{Spec}\, R$ such that X_T is isomorphic to the trivial $L^+\mathcal{G}_K$ torsor, hence $[X/L^+\mathcal{G}_J]$ is étale locally isomorphic to $[(L^+\mathcal{G}_K)_X / L^+\mathcal{G}_J]$. Therefore it suffices to show that the latter is representable by perfectly proper schemes. Now consider the following commutative diagram of perfect group schemes

$$\begin{array}{ccc} L^+\mathcal{G}_J & \twoheadrightarrow & H_J \\ \downarrow & & \downarrow \\ L^+\mathcal{G}_K & \xrightarrow{\varphi} & (\overline{\mathcal{G}}_K)^{\mathrm{red}}. \end{array}$$

If we could show that this diagram was Cartesian, then it would follow that

$$[L^+\mathcal{G}_K/L^+\mathcal{G}_J] \simeq [(\overline{\mathcal{G}}_K)^{\mathrm{red}}/H_J]$$

and the latter is a perfectly proper scheme because it is the perfection of a partial flag variety. Because both the fiber product of the diagram and $L^+\mathcal{G}_J$ are closed subschemes of $L^+\mathcal{G}_K$, we just have to check that the underlying topological spaces are the same. So it suffices to prove that the diagram is Cartesian on K-points for all algebraically closed fields K of characteristic p, which is Theorem 4.6.33 of [**6**]. ☐

COROLLARY 4.0.2. *The map* $\mathrm{Sht}_J \to \mathrm{Sht}_K$ *is a* $(\overline{\mathcal{G}}_K)^{red}/H_J$*-bundle, in particular it is representable by perfectly proper algebraic spaces.*

PROOF. This would be immediate if we could show that the following diagram were Cartesian

(4.0.1)
$$\begin{array}{ccc} \mathrm{Sht}_J & \longrightarrow & \mathrm{Sht}_K \\ \downarrow & & \downarrow \\ \mathbf{B}L^+\mathcal{G}_J & \longrightarrow & \mathbf{B}L^+\mathcal{G}_K. \end{array}$$

Given an $\mathcal{G}_K$ shtuka $(\mathcal{E}, \beta) \in \mathrm{Sht}_K(R)$ together with an $\mathcal{G}_J$-torsor $\mathcal{E}'$ and an isomorphism $\alpha : \mathcal{E}' \times_{\mathcal{G}_J} \mathcal{G}_K \to \mathcal{E}$, i.e. an element of the fiber product, we want to produce an $\mathcal{G}_J$ shtuka. But we can just take $(\mathcal{E}', \beta)$, because the LG-torsor induced from $\mathcal{E}'$ is identified with the LG-torsor induced from $\mathcal{E}$ via α. This gives a map from the fiber product to Sht_J, and one can check that it is an inverse to the map coming from the universal property. ☐

5. Relative position

Let $\mathcal{E}$ and $\mathcal{E}'$ be two $\mathcal{G}_K$-torsors over $\mathcal{O}_L$ together with a modification

$$\beta : \mathcal{E} \dashrightarrow \mathcal{E}'.$$

There is a relative position $\mathrm{Inv}(\beta) \in \tilde{W}_K\backslash\tilde{W}/\tilde{W}_K$ attached to β as follows: Choosing isomorphisms $\mathcal{E} \cong \mathcal{E}^0$ and $\mathcal{E}' \cong \mathcal{E}^0$ we see that β becomes an isomorphism of the trivial G-torsor over L, i.e., an element of $G(L)$. However, this element is only well defined up to our choice of trivialisations of $\mathcal{E}$ and $\mathcal{E}'$, and so gives us a well defined double coset in

$$\mathrm{Inv}(\beta) \in \mathcal{G}_K(\mathcal{O}_L)\backslash G(L)/\mathcal{G}_K(\mathcal{O}_L).$$

The Bruhat-Tits decomposition then tells us that

$$\mathcal{G}_K(\mathcal{O}_L)\backslash G(L)/\mathcal{G}_K(\mathcal{O}_L) = \tilde{W}_K\backslash\tilde{W}/\tilde{W}_K.$$

This works verbatim for modifications of $L^+\mathcal{G}_K$-bundles over any algebraically closed field of characteristic p. Now let $\mathcal{E}$ and $\mathcal{E}'$ be two $\mathcal{G}$-torsors over D_R for some $R \in \mathbf{Aff}_k^{\mathrm{perf}}$ together with a modification

$$\beta : \mathcal{E} \dashrightarrow \mathcal{E}'.$$

Given such a modification, we get for each geometric point x of $\mathrm{Spec}\, R$ a relative position $\mathrm{Inv}(\beta)_x \in \tilde{W}_K\backslash\tilde{W}/\tilde{W}_K$. We write $\mathrm{Inv}(\beta) \preceq w$ if for all geometric points x we have $\mathrm{Inv}(\beta)_x \preceq w$, where $\preceq$ denotes the Bruhat order on $\tilde{W}_K\backslash\tilde{W}/\tilde{W}_K$ induced from the Bruhat order on $\tilde{W}$. We will write $\mathrm{Inv}(\beta) = w$ if $\mathrm{Inv}(\beta)_x = w$ for all geometric points x. It follows from the discussion after Remark 3.5 of [**31**] (c.f. Lemma 1.2.2 of [**75**] for the hyperspecial case) that the subspace

$$\mathrm{Spec}\, R(\preceq w) \subset \mathrm{Spec}\, R$$

consisting of points x such that $\mathrm{Inv}(\beta)_x \preceq w$ is a closed subscheme, and that the subscheme $\mathrm{Spec}(R)(w)$ where $\mathrm{Inv}(\beta)_x = w$ is locally closed. There is a stratification

$$\mathrm{Gr}_K = \bigcup_{w \in \tilde{W}_K\backslash\tilde{W}/\tilde{W}_K} \mathrm{Gr}_K(w),$$

where each $\mathrm{Gr}_K(w)$ is locally closed in Gr_K and it follows from loc. cit. that the closure of $\mathrm{Gr}_K(w)$ is equal to $\mathrm{Gr}_K(\leq w)$. Similarly there is a stratification

$$\mathrm{Sht}_K = \bigcup_{w \in \tilde{W}_K\backslash\tilde{W}/\tilde{W}_K} \mathrm{Sht}_K(w),$$

defined by a relative position condition on geometric points. We would like to say that $\mathrm{Sht}_K(w) \to \mathrm{Sht}_K$ is a "locally closed substack", except we don't have a good notion of topological spaces for Sht_K and $\mathrm{Sht}_K(w)$. An alternative definition is to ask that for every $R \in \mathbf{Aff}_k^{\mathrm{perf}}$ and every morphism $\operatorname{Spec} R = X \to \mathrm{Sht}_K$, the fiber product

$$X(w) = X \times_{\mathrm{Sht}_K} \mathrm{Sht}_K(w)$$

is a scheme and the morphism $X(w) \to X$ is a locally closed immersion. This follows from the discussion after Remark 3.5 of [**31**] (c.f. Lemma 1.2.2 of [**75**] for the hyperspecial case) as before. Finally, we define

$$\mathrm{Sht}_{K,\mu} = \bigcup_{w \in \mathrm{Adm}(\mu)_K} \mathrm{Sht}_K(w).$$

Remark 5.0.1 (Remark 5.2.2.(1) of [**71**]). If $\mathcal{G} = \mathrm{GL}_n$ and $\mu = \omega_i$ is the ith fundamental cocharacter, then $\mathrm{Sht}_{\mathcal{G},i}$ can be regarded as the moduli space of p-divisible groups of height n and dimension $n-i$. This uses the fact that a modification of GL_n bundles can be thought of as a morphism of vector bundles (or projective modules). The fact that the modification is of type μ then tells us that this is actually a Dieudonné module corresponding to a p-divisible group of the right height and dimension (using a result of Gabber about Dieudonné theory over perfect bases).

Since $\mathrm{Adm}(\mu)_K$ is closed in the partial order on $\tilde{W}_K \backslash \tilde{W} / \tilde{W}_K$, the morphism $\mathrm{Sht}_{K,\mu} \subset \mathrm{Sht}_K$ is representable by closed immersions. If $J \subset K$ is another σ-stable type then the following diagram commutes by definition of $\mathrm{Adm}(\mu)_J$ and $\mathrm{Adm}(\mu)_K$ (but it is not Cartesian!)

(5.0.1)
$$\begin{array}{ccc} \mathrm{Sht}_{J,\mu} & \hookrightarrow & \mathrm{Sht}_J \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{K,\mu} & \hookrightarrow & \mathrm{Sht}_K . \end{array}$$

It follows from Corollary 2.4.0.2 that the forgetful morphism $\mathrm{Sht}_{J,\mu} \to \mathrm{Sht}_{K,\mu}$ is representable by perfectly proper algebraic spaces.

6. Restricted local shtukas

In this section we will introduce restricted local shtukas which we will use to state and prove a technical lemma that will be very important in Chapter 4. The reader is advised to skip this section on the first read-through.

We will quickly recall some of the things we need from Section 4.2 of [**64**]. Let $K \subset \mathbb{S}$ a σ-stable type and let $M_K^{\mathrm{loc},\infty} \subset LG$ be the closed subfunctor of LG defined by the

Cartesian diagram

$$\begin{array}{ccc} M_K^{\mathrm{loc},\infty} & \longrightarrow & LG \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{K,\mu} & \longrightarrow & \mathrm{Sht}_K\,. \end{array}$$

It follows from (2.5.0.1) that there is an inclusion $M_J^{\mathrm{loc},\infty} \subset M_K^{\mathrm{loc},\infty}$ for $J \subset K$. Let

$$\beta : L^+\mathcal{G}_K \to (\overline{\mathcal{G}}_K)^{\mathrm{red}}$$

be the natural map, let

$$L^+\mathcal{G}_K^{1-\mathrm{rdt}} := \ker \beta$$

and let $M_K^{\mathrm{loc},1-\mathrm{rdt}}$ be the image of $M^{\mathrm{loc},\infty}$ under the projection

$$LG \to L^+\mathcal{G}_K^{1-\mathrm{rdt}} \backslash LG.$$

We then define

$$\mathrm{Sht}_{K,\mu}^{(\infty,1)} := \left[\frac{M_K^{\mathrm{loc},1-\mathrm{rdt}}}{\mathrm{Ad}_\sigma\, L^+\mathcal{G}_K}\right].$$

In the discussion in subsection 4.2.2 of [**64**] it is shown that the twisted conjugation action of $L^+\mathcal{G}_K$ on $M_K^{\mathrm{loc},1-\mathrm{rdt}}$ factors through $L^m\mathcal{G}_K$ for $m \gg 0$, and for such m we define

$$\mathrm{Sht}_{K,\mu}^{(m,1)} := \left[\frac{M_K^{\mathrm{loc},1-\mathrm{rdt}}}{\mathrm{Ad}_\sigma\, L^m\mathcal{G}_K}\right].$$

It is important to note that an inclusion $J \subset K$ leads to an inclusion $L^+\mathcal{G}_J \subset L^+\mathcal{G}_K$, which leads to an inclusion $L^+\mathcal{G}_K^{1-\mathrm{rdt}} \subset L^+\mathcal{G}_J^{1-\mathrm{rdt}}$ (in the 'wrong' direction!). This means that there is no natural forgetful map $\mathrm{Sht}_{K,\mu}^{(m,1)} \to \mathrm{Sht}_{J,\mu}^{(m,1)}$. There is however a correspondence between them, which we can use to prove the following lemma (here we only deal with the case that $J = \emptyset$).

Lemma 6.0.1. *There is a prestack Y such that the following diagram commutes, such that the left square is Cartesian and such that the map $Y \to \mathrm{Sht}_{\emptyset,\mu}^{(n,1)}$ is perfectly smooth.*

$$\begin{array}{ccccc} \mathrm{Sht}_{\emptyset,\mu} & \longrightarrow & Y & \longrightarrow & \mathrm{Sht}_{\emptyset,\mu}^{(n,1)} \\ \downarrow & & \downarrow & & \\ \mathrm{Sht}_{K,\mu} & \longrightarrow & \mathrm{Sht}_{K,\mu}^{(m,1)} & & \end{array}$$

Here we need to assume that $m \gg n \gg 0$.

Proof of Lemma 2.6.0.1. Consider the inclusion $L^+\mathcal{G}_\emptyset \subset L^+\mathcal{G}_K$, which induces an inclusion $B \subset (\overline{\mathcal{G}}_K)^{\mathrm{red}}$, where B is the image of $\mathcal{G}_\emptyset$. In fact the left square in the following diagram of perfect group schemes is Cartesian (c.f. the proof of Lemma 2.4.0.1)

$$\begin{array}{ccccc} & & \overset{\alpha}{\frown} & & \\ L^+\mathcal{G}_\emptyset & \twoheadrightarrow & B & \longrightarrow & (\overline{\mathcal{G}}_\emptyset)^{\mathrm{red}} \\ \downarrow & & \downarrow & & \\ L^+\mathcal{G}_K & \xrightarrow{\beta}\!\!\!\!\to & (\overline{\mathcal{G}}_K)^{\mathrm{red}}. & & \end{array}$$

This gives us an inclusion $\ker\beta \subset \ker\alpha$ and we consider the following Cartesian diagram

$$\begin{array}{ccccc} \mathrm{Sht}_{\emptyset,\mu} & = & \left[\frac{M_\emptyset^{\mathrm{loc},\infty}}{\mathrm{Ad}_\sigma L^+\mathcal{G}_\emptyset}\right] & \longrightarrow & \left[\frac{\ker\beta\backslash M_\emptyset^{\mathrm{loc},\infty}}{\mathrm{Ad}_\sigma L^+\mathcal{G}_\emptyset}\right] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sht}_{K,\mu} & = & \left[\frac{M_K^{\mathrm{loc},\infty}}{\mathrm{Ad}_\sigma L^+\mathcal{G}_K}\right] & \longrightarrow & \left[\frac{\ker\beta\backslash M_K^{\mathrm{loc},\infty}}{\mathrm{Ad}_\sigma L^+\mathcal{G}_K}\right] \end{array}$$

The action of $\mathrm{Ad}_\sigma L^+\mathcal{G}_\emptyset$ on $\ker\beta\backslash M_K^{\mathrm{loc},\infty}$ factors through $L^n\mathcal{G}_\emptyset$ for $n \gg 0$. The action of $L^+\mathcal{G}_K$ factors through $L^m\mathcal{G}_K$ for $m \gg 0$ and if we choose $m \gg n \gg 0$ we can arrange that the action of $\mathrm{Ad}_\sigma L^+\mathcal{G}_\emptyset$ factors through the image H of $L^+\mathcal{G}_\emptyset$ in $L^m\mathcal{G}_K$ and such that H surjects onto $L^n\mathcal{G}_\emptyset$. We then we get a Cartesian diagram

$$\begin{array}{ccc} \left[\frac{\ker\beta\backslash M_\emptyset^{\mathrm{loc},\infty}}{\mathrm{Ad}_\sigma L^+\mathcal{G}_\emptyset}\right] & \longrightarrow & \left[\frac{\ker\beta\backslash M_\emptyset^{\mathrm{loc},\infty}}{\mathrm{Ad}_\sigma H}\right] \\ \downarrow & & \downarrow \\ \left[\frac{\ker\beta\backslash M_K^{\mathrm{loc},\infty}}{\mathrm{Ad}_\sigma L^+\mathcal{G}_K}\right] & \longrightarrow & \left[\frac{\ker\beta\backslash M_K^{\mathrm{loc},\infty}}{\mathrm{Ad}_\sigma L^m\mathcal{G}_K}\right]. \end{array}$$

Now we consider the morphism

$$\left[\frac{\ker\beta\backslash M_\emptyset^{\mathrm{loc},\infty}}{\mathrm{Ad}_\sigma H}\right] \to \left[\frac{\ker\alpha\backslash M_\emptyset^{\mathrm{loc},\infty}}{\mathrm{Ad}_\sigma H}\right] = \left[\frac{M_\emptyset^{\mathrm{loc},1-\mathrm{rdt}}}{\mathrm{Ad}_\sigma H}\right].$$

This is an $\ker\alpha/\ker\beta = \ker(B \to (\overline{\mathcal{G}}_\emptyset)^{\mathrm{red}}) = U$-torsor, which is perfectly smooth because U is the perfection of a smooth group scheme. The natural map

$$\left[\frac{M_\emptyset^{\mathrm{loc},1-\mathrm{rdt}}}{\mathrm{Ad}_\sigma H}\right] \to \left[\frac{M_\emptyset^{\mathrm{loc},1-\mathrm{rdt}}}{\mathrm{Ad}_\sigma L^n\mathcal{G}_\emptyset}\right] = \mathrm{Sht}_{\emptyset,\mu}^{(n,1)}$$

is perfectly smooth because it is a gerbe for the smooth group scheme $\ker(H \to L^n\mathcal{G}_\emptyset)$. We can now conclude the proof by choosing

$$Y := \left[\frac{\ker\beta\backslash M_\emptyset^{\mathrm{loc},\infty}}{\mathrm{Ad}_\sigma H}\right].$$

□

7. Newton stratification

Let $\mathcal{E}$ be an LG-torsor over K, with K an algebraically closed field of characteristic p, and let $\beta : \sigma^*\mathcal{E} \to \mathcal{E}$ be an isomorphism where σ is the absolute Frobenius. After choosing a basis, we see that β can be represented by an element $b \in G(L)$ well defined up to σ-conjugacy; hence b gives rise to a σ-conjugacy class $[b_\beta] \in B(G)$. Recall that the set $B(G)$ of σ-conjugacy classes in $LG(K)$ does not depend on K and moreover that $B(G)$ is equipped with a partial order (c.f. [**57**]).

LEMMA 7.0.1. *Let $R \in \mathbf{Aff}_k^{perf}$, let $\mathcal{E}$ be an LG-torsor over R and let $\beta : \sigma^*\mathcal{E} \to \mathcal{E}$ be an isomorphism. Then for $b_0 \in B(G)$, the subset*

$$(\mathrm{Spec}\, R)_b := \{x \in \mathrm{Spec}\, R \ :\ [b_\beta(x)] \leq b_0\}$$

is closed in $\mathrm{Spec}\, R$.

PROOF. This is Theorem 3.6 (ii) of [**57**]. □

This gives us a stratification

$$\mathrm{Sht}_K := \bigcup_{b \in B(G)} \mathrm{Sht}_{K,b},$$

where $\mathrm{Sht}_{K,b}$ denotes the locally closed substack of Sht_K consisting of modifications $\beta : \sigma^*\mathcal{E} \dashrightarrow \mathcal{E}$ such that $b_\beta(x) = b$ for all geometric points x. We will write $\mathrm{Sht}_{K,\mu,b}$ for the intersection of $\mathrm{Sht}_{K,\mu}$ and $\mathrm{Sht}_{K,b}$, we will later see that this is nonempty if and only if $b \in B(G,\mu)$.

8. Affine Deligne-Lusztig varieties and uniformisation

In this section we introduce affine Deligne-Lusztig varieties and relate them to moduli spaces of shtukas, following Section 4.3 of [**64**]. Let K be a σ-stable type, let $b \in G(L)$ and consider the moduli functor $X_\mu(b)_K$ on $\mathbf{Aff}_k^{\mathrm{perf}}$ sending R to commutative diagrams of modifications of $\mathcal{G}_K$-bundles on D_R

$$\begin{array}{ccc} \sigma^*\mathcal{E}_1 & \overset{\beta_1}{\dashrightarrow} & \mathcal{E}_1 \\ \downarrow{\scriptstyle \sigma^*\beta_0} & & \downarrow{\scriptstyle \beta_0} \\ \sigma^*\mathcal{E}^0 & \overset{b}{\dashrightarrow} & \mathcal{E}^0. \end{array} \tag{8.0.1}$$

We will sometimes refer to such a diagram as a *quasi-isogeny* of shtukas from $(\mathcal{E}_1, \beta_1) \to (\mathcal{E}^0, b)$. Here b is the modification of the trivial G_K-bundle $\sigma^*\mathcal{E}^0 \simeq \mathcal{E}^0$ given by multiplication by b and β_1 is required to have relative position $\leq \mathrm{Adm}(\mu)_K$. We start with a basic result:

Lemma 8.0.1. *The morphism $X_\mu(b)_K \to \mathrm{Gr}_K$ which sends a diagram as in* (2.8.0.1) *to $\beta_0 : \mathcal{E}_1 \to \mathcal{E}^0$ is a closed immersion.*

Proof. Consider the functor $X(b)$ sending R commutative diagrams of modifications of $\mathcal{G}_K$-bundles on D_R

$$\begin{array}{ccc} \sigma^*\mathcal{E}_1 & \overset{\beta_1}{\dashrightarrow} & \mathcal{E}_1 \\ \downarrow{\scriptstyle \sigma^*\beta_0} & & \downarrow{\scriptstyle \beta_0} \\ \sigma^*\mathcal{E}^0 & \overset{b}{\dashrightarrow} & \mathcal{E}^0. \end{array} \tag{8.0.2}$$

as before, but now *without* the condition that β_1 has relative position bounded by $\mathrm{Adm}(\mu)_K$. It follows from the discussion after Remark 3.5 of [**31**] (c.f. Lemma 1.2.2 of [**75**] for the hyperspecial case) that $X_\mu(b)_K$ is a closed subfunctor of $X(b)$ and the lemma would follow if we could show that the map

$$f : X(b) \to \mathrm{Gr}_K$$

sending a diagram as in (2.8.0.2) to $\beta_0 : \mathcal{E}_1 \to \mathcal{E}^0$ is an isomorphism. The map f is an isomorphism because the map $g : \mathrm{Gr}_K \to X(b)$ sending $\beta_0 : \mathcal{E}_1 \to \mathcal{E}^0$ to the diagram

$$\begin{array}{ccc} \sigma^*\mathcal{E}_1 & \overset{\beta_1}{\dashrightarrow} & \mathcal{E}_1 \\ \downarrow{\scriptstyle \sigma^*\beta_0} & & \downarrow{\scriptstyle \beta_0} \\ \sigma^*\mathcal{E}^0 & \overset{b}{\dashrightarrow} & \mathcal{E}^0. \end{array}$$

with $\beta_1 = \beta_0^{-1} b\sigma^*\beta_0$ is an inverse to f. □

It follows that $X_\mu(b)_K$ is an an inductive limit of perfectly proper perfect schemes, because Gr_K is. In particular, topological notions like connected components and irreducible components make sense for $X_\mu(b)_K$. It should be true that $X_\mu(b)_K$ is actually a perfect scheme that is locally perfectly of finite type although a precise proof in this level of generality seems to be missing from the literature. It is shown in equal characteristic in the case of a hyperspecial parahoric in Section 6 of [**26**], and according to the proof of Lemma 1.1 of [**25**] this proof generalises to mixed characteristic. From there we can deal with Iwahori level ADLV's for unramified reductive groups using the forgetful maps, c.f. Corollary 2.5.3 of [**74**].

If b' is σ-conjugate to b, that is if $b' = gb\sigma(g)^{-1}$ with $g \in G(L)$, then $X_\mu(b)_K \simeq X_\mu(b')_K$ via the map

$$\begin{array}{ccc} \sigma^*\mathcal{E}_1 & \overset{\beta_1}{\dashrightarrow} & \mathcal{E}_1 \\ \downarrow{\scriptstyle \sigma^*\beta_0} & & \downarrow{\scriptstyle \beta_0} \\ \sigma^*\mathcal{E}^0 & \overset{b}{\dashrightarrow} & \mathcal{E}^0. \end{array} \quad \mapsto \quad \begin{array}{ccc} \sigma^*\mathcal{E}_1 & \overset{\beta_1}{\dashrightarrow} & \mathcal{E}_1 \\ \downarrow{\scriptstyle \sigma(g)\sigma^*\beta_0} & & \downarrow{\scriptstyle g^{-1}\beta_0} \\ \sigma^*\mathcal{E}^0 & \overset{b'}{\dashrightarrow} & \mathcal{E}^0. \end{array}$$

We note that this map is nothing more than the action of $g \in LG(\overline{\mathbb{F}}_p)$ on $X_\mu(b)_K \subset \mathrm{Gr}_K$ via the natural action of LG on Gr_K. For $b' = b$ this induces an action of the closed subgroup $F_b \subset LG$ on $X_\mu(b)_K$, where F_b is defined as the subfunctor of LG sending $R \in \mathbf{Aff}_k^{\mathrm{perf}}$ to

$$F_b(R) = \{g \in LG(R) \mid gb\sigma(g)^{-1} = b\}.$$

The $\overline{\mathbb{F}}_p$-points of F_b are in bijection with $J_b(\mathbb{Q}_p)$, where $J_b/\mathbb{Q}_p$ is the algebraic group over $\mathbb{Q}_p$ introduced in Section 2.1.0.4.

LEMMA 8.0.2. *Consider the morphism $\Theta_b : X_\mu(b)_K \to \mathrm{Sht}_{K,\mu}$ which sends a diagram as in* (2.8.0.1) *to* $(\mathcal{E}_1, \beta_1)$. *This morphism is F_b-invariant and induces an isomorphism of groupoids*

$$\mathrm{Sht}_{K,\mu,b}(\overline{\mathbb{F}}_p) \simeq \left[J_b(\mathbb{Q}_p)\backslash X_\mu(b)_K(\overline{\mathbb{F}}_p)\right].$$

PROOF. It is clear that the morphism is F_b-invariant since the action of F_b on $X_\mu(b)_K$ doesn't change $(\mathcal{E}_1, \beta_1)$ and in fact for every scheme $T \mapsto \mathrm{Sht}_{K,\mu}$ either $X_\mu(b)_K(T)$ is empty or the the action of F_b on $X_\mu(b)_K(T)$ is simply transitive. In other words, for the $\mathcal{G}_k$-shtuka $(\mathcal{E}_1, \beta_1)$ over T determined by $T \to \mathrm{Sht}_{K,\mu}$ the set of quasi-isogenies from $(\mathcal{E}_1, \beta_1)$ to $(\mathcal{E}^0, b)$ is either empty or has a simply transitive action by F_b. This uses the fact that F_b can be identified with the group scheme of self quasi-isogenies of $(\mathcal{E}^0, b)$. If we could show that every such $\mathcal{G}_K$-shtuka admits a quasi-isogeny fpqc-locally on T, then there would be an isomorphism (c.f. [**68**, Tag 0497]).

$$\mathrm{Sht}_{K,\mu,b} \simeq [F_b \backslash X_\mu(b)_K] \,.$$

To get the statement on $\overline{\mathbb{F}}_p$-points, we need to show that every $\mathcal{G}_K$-shtuka over $\overline{\mathbb{F}}_p$ in the Newton stratum determined by b is quasi-isogenous to $(\mathcal{E}^0, b)$, which is true by definition of the Newton stratification. □

REMARK 8.0.3. It should in fact be true that $F_b(R) \simeq J_b(\mathbb{Q}_p)$ for every perfect $\overline{\mathbb{F}}_p$-algebra R with $\operatorname{Spec} R$ connected, and that the locally profinite group $\pi_0(F_b)$ is isomorphic (as a topological group) to $J_b(\mathbb{Q}_p)$. Moreover every $\mathcal{G}_K$ should indeed admit, fpqc locally, a quasi-isogeny to a constant shtuka. Both of these statements follow from Theorem I.2.1 of [**16**], but we don't need them.

CHAPTER 3

The Langlands-Rapoport conjecture

In this chapter we follow Section 3 of [**38**] and Sections 8 and 9 of [**59**]. We will state the Langlands-Rapoport conjecture for an arbitrary Shimura datum (G, X), a prime p and a parahoric subgroup $U_p = \mathcal{G}_K(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$. Our version of the conjecture recovers Conjecture 3.3.7 of [**38**] when U_p is hyperspecial and Conjecture 9.2 of [**59**] when G^{der} is simply connected. Roughly speaking the conjecture predicts that there should be a 'nice' integral model of our Shimura variety, such that the set of $\overline{\mathbb{F}}_p$-points of its special fiber is a disjoint union of isogeny classes of the expected shape, with the isogeny classes parametrized by certain admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$ of Galois gerbs. Here $\mathfrak{Q}$ is the so-called quasi-motivic Galois gerb, and $\mathfrak{G}_G = G(\overline{\mathbb{Q}}) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

After stating the conjecture, we prove that special points $(T, X_T) \subset (G, X)$ give rise to admissible morphisms. When $G(\mathbb{Q}_p)$ is quasi-split, we show that every admissible morphism is conjugate to such a special morphism. Kisin proves this by reducing to the case where G^{der} is simply connected and Z_G^0 satisfies the Serre condition, where it is proven by Langlands-Rapoport (Satz 5.3 of [**47**]). Since the result of Langlands-Rapoport assumes that $G_{\mathbb{Q}_p}$ splits over an unramified extension, we have to do some work here. The nontrivial input that we need is Corollary 1.1.17 of [**36**], which replaces Lemma 5.11 of [**47**].

The rest of the chapter is devoted to studying how the conjecture behaves under central isogenies of Shimura data, which will be used to deduce the conjecture for Shimura data of abelian type from the conjecture for Shimura data of Hodge type. Following Sections 3.6 and 3.7 of [**38**], we formulate a refined version of the conjecture, in the style of [**55**]. This refined version implies the original conjecture but makes it easier to deduce the abelian type case from the Hodge type case. In the last section, we show that this refined conjecture behaves well with respect to central isogenies of Shimura data.

Essentially everything that happens in this section comes from Section 3 of [**38**] with minor modifications. Where possible, we refer to [**38**] for the proofs, or we indicate how to modify the proofs there to work in our setting.

1. Galois gerbs

In this section we define Galois gerbs, define the Dieudonné gerb and study its connections with isocrystals, and introduce the quasi-motivic Galois gerb.

DEFINITION 1.0.1. *Let k be a field of characteristic zero (usually a local or global field) and $\overline{k}$ an algebraic closure. Let $k \subset k' \subset \overline{k}$, then a k'/k Galois gerb is a linear algebraic group G/k' together with an extension of topological groups*

$$0 \longrightarrow G(k') \longrightarrow \mathfrak{G} \xrightarrow{q} \mathrm{Gal}(k'/k) \longrightarrow 0 \tag{1.0.1}$$

where $G(k')$ is equipped with the discrete topology and $\mathrm{Gal}(k'/k)$ with the Krull topology, such that:

(1) For every $\tau \in \mathrm{Gal}(k'/k)$ and every $g_\tau \in \mathfrak{G}$ lifting τ, conjugation by g_τ acts on $G(k')$ via an automorphism of algebraic groups

$$\tau^* G \to G.$$

(2) There is a finite extension $k \subset K \subset k'$ and a continuous group theoretic section

$$\mathrm{Gal}(k'/K) \to \mathfrak{G}.$$

EXAMPLE 1.0.2. Let G/k be an algebraic group, and let $G_{k'}$ be its base change to k'. Then the semi-direct product $G(k') \rtimes \mathrm{Gal}(k'/k)$ is a Galois gerb.

REMARK 1.0.3. Conditions (1) and (2) together imply that G/k' descends to an algebraic group G/K and that moreover $q^{-1}\,\mathrm{Gal}(k'/K)$ is isomorphic to the semi-direct product $G(k') \rtimes \mathrm{Gal}(k'/K)$. We can topologise $G(k') \rtimes \mathrm{Gal}(k'/K)$ with the product of the Zariski and Krull topology, and this induces a topology on $\mathfrak{G}$ because $q^{-1}\,\mathrm{Gal}(k'/K) \simeq G(k') \rtimes \mathrm{Gal}(k'/K)$ is finite index in $\mathfrak{G}$.

REMARK 1.0.4. If G is a commutative linear algebraic group over k, then extensions as in (3.1.0.1) are classified by the continuous Galois cohomology group $H^2(\mathrm{Gal}(k'/k), G(k'))$.

We will often refer to a Galois gerb $(G, \mathfrak{G})$ just by $\mathfrak{G}$, and write $\mathfrak{G}^\Delta$ for G, which we will call the *kernel* of $\mathfrak{G}$. If $k \subset k' \subset k'' \subset \overline{k}$ and $\mathfrak{G}$ is an k'/k-gerb, then we can construct a k''/k gerb by pulling back via $\mathrm{Gal}(k''/k) \to \mathrm{Gal}(k'/k)$ and pushing out via $\mathfrak{G}^\Delta(k') \to \mathfrak{G}^\Delta(k'')$. In particular, every k'/k Galois gerb gives rise to a $\overline{k}/k$-gerb, which we will just call a Galois gerb over k.

A morphism of k'/k Galois gerbs is a continuous homomorphism of groups $f : \mathfrak{G} \to \mathfrak{G}'$ inducing the identity on $\mathrm{Gal}(k'/k)$ and a morphism $f^\Delta : \mathfrak{G}^\Delta \to \mathfrak{G}'^\Delta$ such that f^Δ and f agree on $G(k')$. We say that two morphisms $f_1, f_2 : \mathfrak{G} \to \mathfrak{G}'$ are *conjugate* if there is $g \in G'(k')$ such that $f_1 = g^{-1} f_2 g$. The set of such $g \in G(k')$ is naturally the set of k points of a k-scheme

$$\underline{Isom}(f_1, f_2),$$

and if $f_1 = f = f_2$ then we will denote it by I_f. We record the following lemma for later use:

Lemma 1.0.5 (Lemma 3.1.2 of [**38**]). *Let G be a linear algebraic group over k, let $\mathfrak{G}'$ be a k'/k Galois gerb and let us consider a morphism of k'/k Galois gerbs $f : \mathfrak{G}' \to G(k') \rtimes \mathrm{Gal}(k'/k)$. Then*

(1) The base change $I_{f,k'}$ of I_f to k' is naturally isomorphic to the centraliser of $f^\Delta(\mathfrak{G}'^\Delta) \subset G_{k'}$.

(2) The set of maps $f' : \mathfrak{G}' \to G(k') \rtimes \mathrm{Gal}(k'/k)$ with $f'^\Delta = f^\Delta$ is in bijection with the set of continuous cocycles $Z^1(\mathrm{Gal}(k'/k), I_f(k'))$ and f' is conjugate to f precisely when the corresponding cocycle is trivial in $H^1(\mathrm{Gal}(k'/k), I_f(k'))$.

We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and consider $\overline{\mathbb{Q}}/\mathbb{Q}$-Galois gerbs (see 3.1.1 of [**38**]). If $G/\mathbb{Q}$ is an algebraic group, then we write $\mathfrak{G}_G$ for the Galois gerb $G(\overline{\mathbb{Q}}) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and if $f : \mathfrak{G} \to \mathfrak{G}'$ is a morphism of Galois gerbs then we denote by I_f the $\mathbb{Q}$ group scheme of automorphisms of f. We also fix algebraic closures $\overline{\mathbb{Q}}_v$ for all places v of $\mathbb{Q}$ together with embeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$ and we write $\mathbb{C}$ for $\overline{\mathbb{Q}}_\infty$.

Now fix a prime p. For every finite Galois extension $\mathbb{Q} \subset L \subset \overline{\mathbb{Q}}$ Kisin constructs (3.1.3 of [**38**]) a torus Q^L equipped with cocharacters $\nu(\infty)^L$ and $\nu(p)^L$, defined over $\mathbb{R}$ and $\mathbb{Q}_p$ respectively, and a morphism $Q^L \to R_{L/\mathbb{Q}}\mathbb{G}_m$. Lemma 3.14 of op. cit. tells us that $(Q^L, \nu(\infty)^L, \nu(p)^L)$ is an initial object in the category of triples (T, ν_∞, ν_p) consisting of a torus $T/\mathbb{Q}$ which splits over L and cocharacters ν_∞, ν_p defined over $\mathbb{R}$

and $\mathbb{Q}_p$ respectively, such that

$$\sum_{v\in\{\infty,p\}} \frac{1}{[L_v : Q_v]} \operatorname{tr}_{L/\mathbb{Q}}(\nu_v) = 0.$$

For $L \subset L'$, these fit into a projective system and we let Q be their inverse limit, which comes equipped with a morphism $Q \to \varprojlim_L R_{L/\mathbb{Q}}\mathbb{G}_m$, with a cocharacter $\nu(\infty)$ defined over a $\mathbb{R}$ and with a fractional cocharacter $\nu(p) : \mathbb{D} \to Q_{\mathbb{Q}_p}$. Here $R_{L/\mathbb{Q}}$ means restriction of scalars and $\mathbb{D} = \varprojlim D_n$, where $D_n = \mathbb{G}_{m,\mathbb{Q}_p}$ and the transition maps are given by $x \mapsto x^n$.

For $\ell \neq p$, we let $\mathfrak{G}_\ell$ be the trivial Galois gerb $\mathrm{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ and we let $\mathfrak{G}_\infty$ be the extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^\times$ coming from the fundamental class in $H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times)$. In 3.1.6 of [**38**], Kisin defines a pro-Galois gerb $\mathfrak{G}_p$ over $\mathbb{Q}_p$ with kernel $\mathbb{D}$, using local class field theory. It is induced from a $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$ pro-Galois gerb $\mathfrak{D}$, which is often called the *Dieudonné gerb*, this is the inverse limit of Galois gerbs $\mathfrak{D}_n$, see Section 3.1.6 of loc. cit.

The quasi-motivic Galois gerb $\mathfrak{Q}$, constructed in [**61**], is a pro-Galois gerb over $\mathbb{Q}$ with kernel Q. It comes equipped with morphisms

$$\zeta_v : \mathfrak{G}_v \to \mathfrak{Q}(v)$$

for all places of $\mathbb{Q}$, where $\mathfrak{Q}(v)$ is the basechange of $\mathfrak{Q}$ to $\mathbb{Q}_v$, and moreover there is a morphism $\psi : \mathfrak{Q} \to \mathfrak{G}_{R_{\overline{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m}$.

Given a torus $T/\mathbb{Q}$ and a cocharacter μ defined over a finite Galois extension L, Kisin constructs (3.1.10 of [**38**]) a morphism

$$\psi_\mu : \mathfrak{Q} \to \mathfrak{G}_{R_{\overline{\mathbb{Q}}/\mathbb{Q}}\mathbb{G}_m} \to \mathfrak{G}_{R_{L/\mathbb{Q}}\mathbb{G}_m} \to \mathfrak{G}_T.$$

Its composition with $\mathfrak{G}_v \to \mathfrak{Q}$ induces a morphism $\mathfrak{G}_v \to \mathfrak{G}_T$ and on kernels a morphism $\mathbb{D} \to T_{\overline{\mathbb{Q}}}$, which we can explicitly describe as a fractional cocharacter by the formula

$$\frac{1}{[L_p : \mathbb{Q}_p]} \sum_{\tau\in\mathrm{Gal}(L_p/\mathbb{Q}_p)} \tau(\mu).$$

1.0.6. *Isocrystals and the Dieudonné gerb.* Let $G/\mathbb{Q}_p$ be a connected reductive group and let $\phi : \mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}$ be a morphism of $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$ Galois gerbs. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ be the Frobenius, then there is a distinguished element $d_\sigma \in \mathfrak{D}$ that projects to σ in $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ (c.f. Kisin 3.3.3). This element is characterised by the fact that d_σ^n maps to p^{-1} in $\mathfrak{D}_n$. Let $\phi(d_\sigma) = b_\phi \rtimes \sigma$ in $\mathfrak{G}_G$, then conjugating ϕ by an element g of $G(\mathbb{Q}_p^{\mathrm{ur}})$ gives

$$b_{g^{-1}\phi g} = g^{-1} b_\phi \sigma(g)$$

and so we get a well defined element of $B(G)$, which is functorial in G. Given a morphism $\phi : \mathfrak{G}_p \to \mathfrak{G}_G$ of $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$ gerbs, we can always find a morphism $\phi^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}$ such that the induced map $\mathfrak{G}_p \to \mathfrak{G}_G$ is conjugate to ϕ (this is e.g. Lemma 2.1 of [**47**]). Moreover, if $g \in G(\overline{\mathbb{Q}}_p)$ is an element such that $g^{-1}\phi g$ is conjugate to the map induced by ϕ_0, then g must lie in $G(\mathbb{Q}_p^{\mathrm{ur}})$ (Lemma 3.3.4 of [**38**]). The upshot of this is that there is a well defined map

$$\hom(\mathfrak{G}_p \to \mathfrak{G}_G)/\sim \quad \to B(G),$$

which is functorial in G and where $\sim$ denotes conjugation. In [**40**], the set $B(G)$ is defined as the set of conjugacy classes of homomorphisms $\mathfrak{G}_p \to \mathfrak{G}_G$ and it is mentioned in its introduction that the above map is a bijection. Given $\phi : \mathfrak{D} \to \mathfrak{G}_G$ as above, there is an $n \gg 0$ such that ϕ factors through

$$\phi_n : \mathfrak{D}_n \to \mathfrak{G}_G.$$

Then

$$\begin{aligned}\phi_n^\Delta(1/p) &= \phi(d_\sigma^n)\\ &= (b_\phi \rtimes \sigma)^n\\ &= b_\phi\sigma(b_\phi)\cdots\sigma^{n-1}(b_\phi) \rtimes \sigma^n.\end{aligned}$$

It now follows from the definition of the Newton map $B(G) \to X_*(T)_{\mathbb{Q}}^\Gamma$ that $\nu_b = -\phi^\Delta$ (see 4.3 of [**41**]), the minus sign comes from the fact that $d^n = 1/p$ rather than p. Now if $G = T$ is a torus over $\mathbb{Q}$ equipped with a cocharacter μ, then there is a morphism

$$\psi_\mu \circ \nu(p) : \mathfrak{G}_p \to \mathfrak{G}_T(p)$$

which gives rise to an element in $B(T)$ as above. However, there is a tautological map

$$X_*(T) \to X_*(T)_\Gamma \simeq B(T),$$

and these two constructions are related as follows:

LEMMA 1.0.7. *The element of $B(T) = X_*(T)_\Gamma$ defined by $\psi_\mu \circ \nu(p)$ is equal to $-\tilde{\mu}$, where $\tilde{\mu}$ is the image of μ in $X_*(T)_\Gamma$.*

Proof. This follows from Lemma 2.2.(b) of [**41**] which states that the functors $T \mapsto X_*(T)_\Gamma$ and $T \mapsto B(T)$ are isomorphic, and moreover that a natural isomorphism between them is determined by what is does on $T = \mathbb{G}_m$. For $\mathbb{G}_m$, the lemma follows from the above result on Newton points because $B(\mathbb{G}_m) \to X_*(\mathbb{G}_m)^\Gamma_\mathbb{Q}$ is injective. $\square$

1.0.8. *Strictly monoidal categories.* Recall that a crossed module is a homomorphism of groups $\tilde{H} \to H$ together with an action of H on $\tilde{H}$ which lifts the action of $\tilde{H}$ on itself by conjugation, such that the action of H on itself via $H \to \mathrm{Aut}(\tilde{H}) \to \mathrm{Aut}(H)$ is also given by conjugation. The main example that we will need is $G^{\mathrm{sc}} \to G$, where $G^{\mathrm{sc}} \to G^{\mathrm{der}}$ is the simply connected cover of the derived group of a connected reductive algebraic group over $\mathbb{Q}$. Kisin writes $\tilde{G}$ for G^{sc} and we might sometimes do this as well.

Given a crossed module $\tilde{H} \to H$ we can form a category $H/\tilde{H}$, whose objects are given by the objects of H and where

$$\hom(h_1, h_2) = \{\tilde{h} \in \tilde{H} : h_2 = \tilde{h} h_1\}.$$

This category is strictly monoidal, with $H/\tilde{H} \times H/\tilde{H} \to H/\tilde{H}$ induced by multiplication on objects. We now define $\mathfrak{G}_{G/\tilde{G}}$ to be the strict monoidal category corresponding to the crossed module $\tilde{G}(k) \to \mathfrak{G}_G$. If G^{der} is simply connected, then this is isomorphic to $\mathfrak{G}_{G^{\mathrm{ab}}}$ and we encourage the reader to take this as as the main example.

Lemma 1.0.9 (Lemma 3.2.6 of [**38**]). *The natural morphisms*

$$G(\overline{\mathbb{Q}}) \to G^{ad}(\overline{\mathbb{Q}}) \times_{G^{ad}/\tilde{G}(\overline{\mathbb{Q}})} G/\tilde{G}(\overline{\mathbb{Q}})$$

$$\mathfrak{G}_G \to \mathfrak{G}_{G^{ad}} \times_{\mathfrak{G}_{G^{ad}/\tilde{G}}} \mathfrak{G}_{G/\tilde{G}}$$

are equivalences of strictly monoidal categories.

2. The Langlands-Rapoport conjecture

Let (G, X) be a Shimura datum with reflex field E, let $\mathcal{G}/\mathbb{Z}_{(p)}$ be a parahoric model of G, and let $\{\mu\}$ be the associated conjugacy class of cocharacters of G. To be precise, we consider the cocharacter $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to G_\mathbb{C}$ obtained from a choice of $h \in X$ given by

$$\mathbb{G}_m \xrightarrow{\mathrm{Id} \times 1} \mathbb{G}_m \times \mathbb{G}_m \simeq \mathbb{S}_\mathbb{R} \xrightarrow{h_\mathbb{C}} G_\mathbb{C}.$$

We call this the Hodge cocharacter $\mu = \mu_h$ associated to X (changing x by a conjugate amounts to conjugating μ_h). We define $B(G,X)$ as the subset of $B(G_{\mathbb{Q}_p})$ defined by $B(G, \{\mu^{-1}\})$.

Kisin defines a morphism (c.f. 3.3.1 of [**38**])

$$\phi_{\mu_{\tilde{ab}}} : \mathfrak{Q} \to \mathfrak{G}_{G/\tilde{G}}.$$

When G^{der} is simply connected this is the morphism $\mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ab}}}$ coming from the cocharacter μ^{ab}, where μ is the Hodge cocharacter associated to the Shimura datum. Moreover Kisin constructs morphisms $\xi_v : \mathfrak{G}_v \to \mathfrak{Q}(v)$ for all $v \neq p$ (the trivial morphism when $v \neq \infty$).

DEFINITION 2.0.1. *A morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is called admissible (w.r.t X) if*

A1 The composite (denoted by $\phi_{\tilde{ab}}$)

$$\mathfrak{Q} \xrightarrow{\phi} \mathfrak{G}_G \longrightarrow \mathfrak{G}_{G/\tilde{G}}$$

is conjugate isomorphic to $\psi_{\mu_{\tilde{ab}}}$ (see Section 3.2.1 of [**38**] *for the definition of conjugate isomorphic functors of strictly monoidal categories).*

A2 For $v \neq p$, the composite

$$\mathfrak{G}_v \xrightarrow{\zeta_v} \mathfrak{Q}(v) \xrightarrow{\phi(v)} \mathfrak{G}_G(v)$$

is conjugate to the morphism $\xi_v : \mathfrak{G}_v \to \mathfrak{G}_G(v)$.

A3 The image ϕ_b of ϕ in $B(G)$ defined by the composition

$$\theta : \mathfrak{G}_p \xrightarrow{\nu(p)} \mathfrak{Q}(p) \xrightarrow{\phi(p)} \mathfrak{G}_G(p)$$

lies in $B(G,X)$.

Our definition is equivalent to the definition in Section 3.3.6 of [**38**] when $G_{\mathbb{Q}_p}$ is quasi-split and splits over an unramified extension by Theorem 2.1.0.5, and equivalent to Definition 9.1 of [**59**] when G^{der} is simply connected. We now define

$$X^p(\phi) := \{(g_v)_{v \neq p,\infty} \in G(\overline{\mathbb{A}}_f^p) \ : \ \mathrm{Int}(g) \circ \xi_v = \phi(v) \circ \zeta_v\},$$

where $\overline{\mathbb{A}}_f^p$ is the restricted product of $\overline{\mathbb{Q}_\ell}$ for $\ell \neq p$ (recall that we've fixed these algebraic closures). The set $X^p(\phi)$ is nonempty by axiom A2 and in fact it is a $G(\mathbb{A}_f^p)$ torsor.

Write $\theta = \phi(p) \circ \zeta_p$ and define $X_p(\phi)$ to be the set of $g \in G(\overline{\mathbb{Q}}_p)/\mathcal{G}_K(\mathbb{Z}_p^{\mathrm{ur}})$ such that $g\theta g^{-1}$ is induced by a morphism $\theta_g^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}$ of $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$ gerbs satisfying $\theta_g^{\mathrm{ur}} = b_g \rtimes \sigma$ with $b_g \in B(G, X)$. This has an action of a p^r-Frobenius, where $r = [E_0 : \mathbb{Q}_p]$ with E_0 the maximal unramified extension of E_p, as follows: We define

$$\Phi(g) = g b_g \sigma(b_g) \cdots \sigma^{r-1}(b_g)$$

and note that

$$\theta_{\Phi(g)}^{\mathrm{ur}} = (1 \rtimes \sigma^r)\theta_g^{\mathrm{ur}}(1 \rtimes \sigma^r)^{-1},$$

so that $b_{\Phi(g)} = \sigma(b_g)$, which is still an element of $B(G, X)$. Lemma 2.1 of [**47**] tells us that there is a $g_0 \in G(\overline{\mathbb{Q}}_p)$ such that $g_0^{-1}\theta g_0$ is conjugate to a map of $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$ gerbs $\theta : \mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}$. It follows as in Lemma 3.3.4 of [**38**] that the map $g \mapsto g_0 g$ induces a bijection

$$X_{-\mu}(b)_K \simeq X_p(\phi),$$

where we consider both of them as subsets of $G(\overline{\mathbb{Q}}_p)/\mathcal{G}_K(\mathbb{Z}_p^{\mathrm{ur}})$, and this bijection is compatible with the action of Φ. Define

$$S(\phi) = \varprojlim_{U^p} I_\phi(\mathbb{Q})\backslash X(\phi)/U^p. \tag{2.0.1}$$

where $X(\phi) = X^p(\phi) \times X_p(\phi)$, where the action of $I_\phi(\mathbb{Q}) \subset G(\overline{\mathbb{Q}})$ is by left multiplication on $X^p(\phi) \times X_p(\phi) \subset G(\overline{\mathbb{A}}_f^p) \times G(\overline{\mathbb{Q}}_p)/\mathcal{G}_K(\mathbb{Z}_p^{\mathrm{ur}})$. Note that (3.2.0.1) is not necessarily in bijection with $I_\phi(\mathbb{Q})\backslash X(\phi)$ when Milne's axiom SV5 does not hold, i.e. when $Z_G(\mathbb{Q})$ is not discrete in $Z_G(\mathbb{A}_f)$. However this is not an issue in the Hodge type case, as this axiom will hold automatically. We are now ready to state the conjecture of Langlands and Rapoport, in a version that generalises both Conjecture 9.2 of [**59**] and Conjecture 3.3.7 of [**38**]:

Conjecture 2.0.2 (Langlands-Rapoport). *Let (G, X) and $\mathcal{G}$ be as above and let $U_p = \mathcal{G}(\mathbb{Z}_p)$. Consider the tower of Shimura varieties $\{\mathbf{Sh}_{G,U^pU_p}\}_{U^p}$ over the reflex field E with its action of $G(\mathbb{A}_f^p) \times Z_G(\mathbb{Q}_p)$, where U^p varies over compact open subgroups of $G(\mathbb{A}_f^p)$. Then this tower has a $G(\mathbb{A}_f^p) \times Z_G(\mathbb{Q}_p)$-equivariant extension to a tower of flat schemes $\{\mathscr{S}_{G,U^pU_p}\}_{U^p}$ over $\mathcal{O}_{E_{(v)}}$. Moreover, there is a bijection*

$$\varprojlim_{U^p} \mathscr{S}_{U^pU_p}(\overline{\mathbb{F}}_p) \simeq \coprod_{[\phi]} S(\phi),$$

compatible with the action of $G(\mathbb{A}_f^p) \times Z_G(\mathbb{Q}_p)$ *and the operator* Φ*, which acts on the left hand side as the geometric* p^r*-Frobenius. Here* ϕ *runs over conjugacy classes of admissible morphisms* $\mathfrak{Q} \to \mathfrak{G}_G$*.*

3. Special morphisms

In Section 3.5 of [**38**], Kisin constructs for every CM extension $\mathbb{Q} \subset L \subset \overline{\mathbb{Q}}$ a torus P^L with character group identified with the group of Weil numbers (modulo roots of unity) inside $L^\times$. For $L \subset L'$ there is a morphism $P^{L'} \to P^L$ and we let P be the inverse limit, which comes equipped with a map $Q \to P$. Pushing out $\mathfrak{Q}$ along this map gives rise to the *pseudo motivic groupoid* $\mathfrak{P}$, which is also a pro-Galois gerb. An admissible morphism $\mathfrak{Q} \to \mathfrak{G}_G$ factors through $\mathfrak{P}$ if Z_G^0 satisfies the Serre condition (Lemma 3.5.7 of [**38**]), this is automatic if (G, X) is of Hodge type.

If we assume the Tate conjecture for smooth projective varieties over finite fields, then the category of representations of $\mathfrak{P}$, i.e. the category of morphisms $\mathfrak{P} \to \mathfrak{G}_{\mathrm{GL}_n}$, is equivalent to the category of (numerical) pure motives over $\overline{\mathbb{F}}_p$, see [**49**]. This makes sense because motives over finite fields are conjecturally determined by q-Weil numbers, which are the characters of P.

Take a CM field L as in the previous paragraph, and let n be a sufficiently divisible natural number. Then there is an element $\delta_n \in P^L(\mathbb{Q})$ such for a $q = p^m$-Weil Number π, evaluating the character χ_π of P^L associated to π on δ_n gives

$$\chi_\pi(\delta_n) = \pi^{n/m}.$$

Moreover these elements satisfy $\delta_{n'} = \delta_n^{n'/n}$ if $n \mid n'$ and the are preserved by the maps $P^{L'} \to P^L$ for $L \subset L' \subset \overline{\mathbb{Q}}$. Given a morphism

$$\phi : \mathfrak{P} \to \mathfrak{G}_{\mathrm{GL}_n},$$

conjecturally corresponding to a motive over $\overline{\mathbb{F}}_p$ defined over $\mathbb{F}_q$, the image of δ_n in $\mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ should be thought of as $\mathrm{Frob}_{p^{m/n}}$ acting on the ℓ-adic realisation of our motive. Lemma 5.5 of [**47**] tells us that the collection of elements $\{\delta_n\}$ is Zariski-dense in P^L.

Let (G, X) be a Shimura datum and let $T \subset G$ be a torus of G over $\mathbb{Q}$ together with an $h : \mathbb{S} \to G_\mathbb{R}$ in X that factors through $T_\mathbb{R}$; we will call such a pair (T, h) a special point of (G, X). This gives us a cocharacter μ of T and hence a morphism $\psi_\mu : \mathfrak{Q} \to \mathfrak{G}_T$. It follows as in Lemma 3.5.8 of [**38**] that the composition of ψ_μ with $\mathfrak{G}_T \to \mathfrak{G}_G$ is an admissible morphism (to prove A3, use remark 5.2 of [**8**].) An admissible morphism $\phi : \mathfrak{Q} \to \mathfrak{G}$ is called *special* if it is conjugate to a morphism induced by $T \subset G$ as above.

THEOREM 3.0.1. *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism, and suppose that $G_{\mathbb{Q}_p}$ is quasi-split, then ϕ is special.*

PROOF. It follows as in the proof of Theorem 3.5.11 of [**38**] that we may reduce to the case that G^{der} is simply connected and that Z_G^0 satisfies the Serre condition, so that ϕ factors through $\mathfrak{P}^L$ for some CM field L. At this point Kisin invokes Satz 5.3 of [**47**], which proves the result under the assumption that $G_{\mathbb{Q}_p}$ is quasi-split and split over an unramified extension, we will indicate how the proof of loc. cit. generalises.

Let δ_n be the distinguished elements of $P^L(\mathbb{Q})$ discussed in the beginning of this section, and recall that they are Zariski dense in P^L by Lemma 5.5 of [**47**].

LEMMA 3.0.2. *After conjugating ϕ, there is a maximal torus $T \subset G$ such that $\phi(\delta_n) \in T(\mathbb{Q})$ and such that $T_{\mathbb{R}}^{ad}$ is anisotropic.*

PROOF. This is Lemma 5.4 of [**47**]. ☐

Let $b \in B(G)$ be the σ-conjugacy class defined by ϕ, which has Newton point

$$v_b = -\phi^{\Delta} \circ \nu(p),$$

where $\phi^{\Delta} \circ \nu(p)$ is the fractional cocharacter of G defined by $\phi \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G$. It follows as in Section 4.3.9 of [**38**] that for sufficiently divisible n we have $n\nu_b = \nu_{\delta_n}$. Using our assumption on $[b]$, we can invoke Corollary 1.1.17 of [**36**] which says that there is a cocharacter $\mu_T \in X_*(T) \cap \{\mu\}$ such that

$$\nu_b = [-\overline{\mu}_T] \in X_*(T)_{\mathbb{Q}}^{\Gamma},$$

where $\overline{\mu}_T$ is the $\mathrm{Gal}(K/\mathbb{Q}_p)$-average of μ with $K/\mathbb{Q}_p$ a Galois extension over which μ is defined. As usual there is a morphism of Galois gerbs

$$\psi_{\mu_T} : \mathfrak{Q} \to \mathfrak{G}_T,$$

which factors through $\mathfrak{P}$ because T satisfies the Serre condition and which, possibly after enlarging L, factors through $\mathfrak{P}^L$.

LEMMA 3.0.3. *We have an equality* $\phi^\Delta = \psi_{\mu_T}^\Delta$.

PROOF. This follows as in the proof of Lemma 5.11 of [**47**]. □

LEMMA 3.0.4. *After possibly conjugating* T *and* ϕ*, the cocharacter* μ_T *is equal to* μ_{h_T} *for some* $h_T : \mathbb{S} \to G_\mathbb{R}$ *factoring through* $T_\mathbb{R}$.

PROOF. This follows exactly as in the proof of Lemma 5.12 of [**47**] (c.f. the proof of Proposition 1.2.5 of [**36**]). □

Last we will show that we can conjugate T and ϕ such that $\phi = \psi_{\mu_T}$. Because we already know that $\phi^\Delta = \psi_{\mu_T}^\Delta$, Lemma 3.1.2 of [**38**] tells us that for a continuous section ρ of $\mathfrak{Q} \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with values $\rho(\tau) = \rho_\tau$ we have

$$\phi(\rho_\tau) = g_\tau \rtimes \tau$$
$$\psi_{\mu_{h_T}}(\rho_\tau) = k_\tau \rtimes \tau,$$

with $g_\tau = k_\tau a_\tau$ and where a_τ is a cocycle with values in T. We are going to show that the class α of a_τ is trivial in $H^1(\mathbb{Q}, G)$, represented by $v\tau(v)^{-1}$ and then consider $\phi' = v\phi g^{-1} v$ and $T' = v^{-1}\phi v$. Indeed, that would show that

$$\begin{aligned} v^{-1}\phi(\rho_\tau)v &= v^{-1}(g_\tau \rtimes \tau)v \\ &= v^{-1}(k_\tau a_\tau \rtimes \tau)v \\ &= (v^{-1}k_\tau v\tau(v)^{-1} \rtimes \tau)v \\ &= v^{-1}k_\tau v\tau(v)^{-1}\tau(v) \rtimes \tau \\ &= v^{-1}k_\tau v \rtimes \tau \\ &= \psi_{\mu'}(\rho_\tau), \end{aligned}$$

where $\mu' = v^{-1}h_T v$, so that $v^{-1}\phi v$ and $\psi_{\mu'}$ are conjugate.

First of all, admissibility tells us that the compositions of ϕ and $\psi_{\mu_{h_T}}$ with $\mathfrak{G}_G \to \mathfrak{G}_{G/\tilde{G}} = \mathfrak{G}_{G^{\mathrm{ab}}}$ are conjugate, so that the image of our cocycle in trivial in $H^1(\mathbb{Q}, G^{\mathrm{ab}})$. Using the long exact sequence in cohomology associated to $1 \to G^{\mathrm{der}} \to G \to G^{\mathrm{ab}} \to 1$, we find that $[a_\tau] = \alpha \in \mathrm{Im}(H^1(\mathbb{Q}, G^{\mathrm{der}}) \to H^1(\mathbb{Q}, G)$. Lemma 5.13 of [**47**] tells us that the Hasse principle holds for element in this image (using that G^{der} is simply connected). To be precise, we mean that the composition $H^1(\mathbb{Q}, G^{\mathrm{der}}) \to H^1(\mathbb{Q}, G) \to H^1(\mathbb{R}, G)$ is injective. So it suffices to show that the image of α is zero in $H^1(\mathbb{R}, G)$

and we will in fact show that the image of α is zero in $H^1(\mathbb{R}, T)$. Lemma 5.14 of [**47**] tells us that $H^1(\mathbb{R}, T) \subset H^1(\mathbb{R}, G')$ is injective, where G' is the anisotropic mod centre inner form of G, which can be realised as I_{ξ_∞}. But the class of α in $H^1(\mathbb{R}, I_{\xi_\infty})$ is zero, because the compositions $\zeta_\infty \circ \phi$ and $\zeta_\infty \circ \psi_{\mu_{h)_T}}$ are conjugate since they are both conjugate to ξ_∞ by A2. We conclude that the class of α in G is zero, and so by the above we have found a torus T and $h : \mathbb{S} \to \mathbb{T}_{\mathbb{R}}$ such that $\phi = \psi_{\mu_T}$. □

4. Connected components and a refined conjecture

Fix a morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_{G/\tilde{G}}$. In Section 3.6 Kisin defines sets $X^p(\phi)$ and $X^p(\phi)$ with product $X(\phi)$, analogous to our definitions for morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}$. Our definition of $X^p(\phi)$ is the same as his, but we have to slightly modify his definition of $X_p(\phi)$ for general groups by recalling that the Kottwitz map lands in $\pi_1(G)_I$ instead of $\pi_1(G)$. Then for an admissible $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ we define

$$\tilde{\pi}(G, \phi_0) := \coprod_{\phi^{\mathrm{ad}} = \phi_0} X(\phi_{\tilde{\mathrm{ab}}})$$

$$\pi(G, \phi_0) := \varprojlim_{U^p} \tilde{\pi}(G, \phi_0)/U^p G(\overline{\mathbb{Q}})^{\sharp}_{+},$$

where $\phi_{\tilde{\mathrm{ab}}}$ is the composition $\mathfrak{Q} \to \mathfrak{G}_G \to \mathfrak{G}_{\tilde{G}/G}$ and where $G(\overline{\mathbb{Q}})^{\sharp}_{+}$ is the inverse image of $G^{\mathrm{ad}}(\mathbb{Q})_+$ in $G(\overline{\mathbb{Q}})$ acting on $\tilde{\pi}(G, \phi_0)$ by right multiplication. Define

$$\pi(G) := G(\mathbb{Q})^{-}_{+} \backslash G(\mathbb{A}_f)/\mathcal{G}(\mathbb{Z}_p),$$

where the bar denotes closure, then arguing as in Lemma 3.6.2 of [**38**] we can show that $\pi(G, \phi_0)$ is a $\pi(G)$-torsor, keeping in mind that for any parahoric $\mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ we have

$$\pi_1(G)^{\sigma}_I \simeq G(\mathbb{Q}_p)/(\tilde{G}(\mathbb{Q}_p)\mathcal{G}(\mathbb{Z}_p))$$

by Lemma 5.18.(i) of [**73**].

4.0.1. Let $T/\mathbb{Q}$ be a torus and let μ_T be a cocharacter of T, then there is an induced morphism $\psi_{\mu_T} : \mathfrak{Q} \to \mathfrak{G}_T$ and in section 3.6.6 of [**38**] Kisin defines sets $X^p(\psi_{\mu_T}), X_p(\psi_{\mu_T})$ with product $X(\psi_{\mu_T})$. He then defines

$$S(\psi_{\mu_T}) = T(\mathbb{Q})^{-} \backslash X(\psi_{\mu_T}),$$

where $T(\mathbb{Q})^{-}$ denotes the closure of $T(\mathbb{Q})$ in $T(\mathbb{A}_f)$ and shows (Proposition 3.6.7 of [**38**]) that $X(\phi_{\mu_T})$ is a $T(\mathbb{A}_f^p)/\mathcal{T}(\mathbb{Z}_p)$-torsor and that there is a canonical isomorphism

$$S(\psi_{\mu_T}) \simeq T(\mathbb{Q})^{-} \backslash T(\mathbb{A}_f)/\mathcal{T}(\mathbb{Z}_p).$$

4.0.2. Consider the category $\mathscr{SH}_p$ whose objects consist of pairs $(\mathcal{G}, X)$, where $\mathcal{G}/\mathbb{Z}_{(p)}$ is a smooth affine group scheme with $G = \mathcal{G}_{\mathbb{Q}}$ connected reductive and $\mathcal{G}_{\mathbb{Z}_p}$ parahoric, and where (G, X) is a Shimura datum. Morphisms in this category are given by morphisms of group schemes $\mathcal{G} \to \mathcal{G}'$ over $\mathbb{Z}_{(p)}$, such that $(G, X) \to (G', X')$ is a morphism of Shimura data. Given an adjoint Shimura datum (H, Y), we will write $\mathscr{SH}_p(H, Y)$ for the subcategory of objects $(\mathcal{G}, X)$ such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is isomorphic to (H, Y). For $(\mathcal{G}, X)$ an object of our category, we set $U_p = \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ and write

$$\pi(\mathcal{G}, X) = \varprojlim_{U^p} \pi_0(\mathbf{Sh}_{U^p U_p}(\mathbb{C})),$$

which is a $\pi(G)$-torsor. Let $T/\mathbb{Q}$ be a torus together with a morphism $h_T : \mathbb{S} \to T_{\mathbb{R}}$ with associated cocharacter $\mu_{h_T} = \mu_T$. If we are given a morphism $i : (T, h_T) \to (G, X)$, then there is an induced morphism

$$T(\mathbb{Q})^- \backslash T(\mathbb{A}_f)/\mathcal{T}(\mathbb{Z}_p) \to \pi_0(G, X)$$

and similarly there is an induced morphism

$$S(\psi_{\mu_T}) \to \pi_0(G, \phi_0),$$

where $\phi_0 = (i \circ \psi_{\mu_T})_0$. The proof of Proposition 3.6.10 of [**38**] shows that if ϕ_0 is special, then there is a unique isomorphism of $\pi(G)$-torsors

$$\vartheta_G : \pi(G, \phi_0) \simeq \pi(G, X) \tag{4.0.1}$$

that is functorial with respect to morphisms in $\mathscr{SH}_p(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ and is compatible with the maps $S(\phi_{\mu_T}) \to \pi_0(G, \phi_0)$ induced by special points.

5. The refined conjecture

In this section we will compare admissible morphisms for (G, X) and admissible morphisms for the adjoint Shimura datum $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, following Section 3.4 of [**38**]. We will fix an admissible morphism $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ throughout this section. **From now on we will assume that $G_{\mathbb{Q}_p}$ is quasi-split, which implies that every admissible morphism is special.**

We will write $G^{\mathrm{ad}}(\mathbb{Q})^+$ for $G^{\mathrm{ad}}(\mathbb{Q}) \cap G^{\mathrm{ad}}(\mathbb{R})^+$, where the latter is the connected component of the identity in the real topology. Finally, we define $G(\overline{\mathbb{Q}})^{\sharp}_+$ as the inverse image of $G^{\mathrm{ad}}(\mathbb{Q})^+$ in $G(\overline{\mathbb{Q}})$ and $G(\mathbb{Q})_+$ as the inverse image of $G^{\mathrm{ad}}(\mathbb{Q})^+$ in $G(\mathbb{Q})$.

PROPOSITION 5.0.1 (Proposition 3.4.11 of [**38**]). *Let $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{ad}}$ be an admissible morphism. The set of admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ lifting ϕ_0 is naturally a $G(\overline{\mathbb{Q}})^\sharp_+/G(\mathbb{Q})_+$-torsor, and in particular nonempty. Moreover if we fix such a ϕ, then the set of admissible ϕ' that lift ϕ_0 and are conjugate to ϕ is naturally a torsor for $I_{\phi_0}(\overline{\mathbb{Q}})^\sharp_+/I_\phi(\mathbb{Q})$, where $I_{\phi_0}(\overline{\mathbb{Q}})^\sharp_+$ is the inverse image of $I_{\phi_0}(\mathbb{Q}) \subset G^{ad}(\overline{\mathbb{Q}})$ in $G(\overline{\mathbb{Q}})$.*

PROOF. The proof is essentially the same as the proof of Proposition 3.4.11 of [**38**]. □

5.0.2. Fix ϕ_0 as above. Then $I_{\phi_0}(\overline{\mathbb{Q}})^\sharp_+$ acts by left multiplication on the disjoint union

$$\coprod_{\phi^{\mathrm{ad}}=\phi_0} X(\phi) \tag{5.0.1}$$

and we set

$$S(G,\phi_0) = \varprojlim_{K^p} I_{\phi_0}(\overline{\mathbb{Q}})^\sharp_+ \backslash \coprod_{\phi^{\mathrm{ad}}=\phi_0} X(\phi)/K^p,$$

where K^p runs over compact open subgroups of $G(\mathbb{A}^p_f)$. Given $\tau \in I^{\mathrm{ad}}_\phi(\mathbb{A}_f)$ we define

$$S_\tau(\phi) = \varprojlim_{K^p} I_\phi(\mathbb{Q})\backslash X(\phi)/K^p$$

$$S_\tau(G,\phi_0) = \varprojlim_{K^p} I_{\phi_0}(\overline{\mathbb{Q}})^\sharp_+ \backslash \coprod_{\phi^{\mathrm{ad}}=\phi_0} X(\phi)/K^p,$$

where the action of $I_\phi(\mathbb{Q})$ on $X(\phi)$, respectively the action of $I_{\phi_0}(\overline{\mathbb{Q}})^\sharp_+$ on (3.5.0.1), is twisted by τ.

LEMMA 5.0.3 (Corollary 3.4.16 of [**38**]). *The natural map*

$$\coprod_{[\phi],[\phi^{ad}]=[\phi_0]} S_\tau(\phi) \to S_\tau(G,\phi_0)$$

is a bijection, where on the left hand side $[\phi]$ runs over conjugacy classes of admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ lifting ϕ_0.

PROOF. This is essentially immediate from Proposition 3.5, since

$$S_\tau(G,\phi_0) = I_{\phi_0}(\overline{\mathbb{Q}})^\sharp_+ \backslash \coprod_{\phi^{\mathrm{ad}}=\phi_0} S_\tau(\phi) \simeq \coprod_{[\phi],\phi^{\mathrm{ad}}} S_\tau(\phi)$$

because the set of ϕ in a single conjugacy class $[\phi]$ is in bijection with $I_{\phi_0}(\overline{\mathbb{Q}})^\sharp_+/I_\phi(\mathbb{Q})$. $\square$

Our parahoric model $\mathcal{G}$ defines parahoric models $\mathcal{G}^{\mathrm{der}}$ and $\mathcal{G}^{\mathrm{ad}}$ of G^{der} and G^{ad}, as in Section 4.6.1 of [**37**]. Let $\overline{\mathbb{Z}}_p$ be the ring of integers of $\overline{\mathbb{Q}}_p$ and write $\mathcal{G}(\overline{\mathbb{Z}}_{(p)})^\sharp_+$ for the preimage of $\mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ in $G(\overline{\mathbb{Q}})$. The following lemma is proven as in [**38**].

LEMMA 5.0.4 (Lemma 3.7.2 of [**38**]). *There is an action of* $\mathcal{G}(\overline{\mathbb{Z}}_{(p)})^\sharp_+$ *on*

$$\coprod_{\phi^{ad}=\phi_0} X(\phi),$$

which induces an of $\mathcal{G}^{ad}(\mathbb{Z}_{(p)})^+$ *on* $S_\tau(G,\phi_0)$.

5.0.5. We now have an action of $\mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ and of $G(\mathbb{A}_f^p)$ on the sets $S_\tau(G,\phi_0)$, which are compatible with the morphisms $\mathcal{G}(\mathbb{Z}_{(p)})_+ \subset G(\mathbb{Q})_+ \to G(\mathbb{A}_f^p)$ and $\mathcal{G}(\mathbb{Z}_{(p)})_+ \to \mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$, this will induce an action of

$$\mathcal{A}(\mathcal{G}) := G(\mathbb{A}_f^p) *_{\mathcal{G}(\mathbb{Z}_{(p)})_+} \mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$$

on $S_\tau(G,\phi_0)$. For the definition of $*$, see Section 3.7.3 of [**38**]. Intuitively, this is just a group that captures the action of $G(\mathbb{A}_f^p)$ and $\mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$, taking into account that $\mathcal{G}(\mathbb{Z}_{(p)})_+$ maps to both of them. We also introduce

$$\begin{aligned}\mathcal{A}(G) &= G(\mathbb{A}_f) *_{G(\mathbb{Q})_+} G^{\mathrm{ad}}(\mathbb{Q})^+\\ \mathcal{A}(G)^\circ &= G(\mathbb{Q})_+^- *_{G(\mathbb{Q})_+} G^{\mathrm{ad}}(\mathbb{Q})^+\\ \mathcal{A}(\mathcal{G})^\circ &:= \mathcal{G}(\mathbb{Z}_{(p)})_+^- *_{\mathcal{G}(\mathbb{Z}_{(p)})_+} \mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+,\end{aligned}$$

where the superscript $-$ denotes closure. Let us point out that there are natural maps

$$\begin{aligned}\mathcal{A}(G) &\twoheadrightarrow \pi(G,X)\\ \mathcal{A}(\mathcal{G}) &\to \pi(G,X)\end{aligned}$$

with $\mathcal{A}(G)^\circ$ contained in the kernel of the first map and $\mathcal{A}(\mathcal{G})^\circ$ equal to the kernel of the second map. There are natural projections $X(\phi) \to X(\phi_{\tilde{\mathrm{ab}}})$ which induce an $\mathcal{A}(\mathcal{G})$-equivariant map (c.f. Lemma 3.7.4 of [**38**])

$$c_G : S_\tau(G,\phi_0) \to \pi(G,\phi_0).$$

This map is surjective, but the proof of loc. cit. does not generalise. Instead, it will follow from the fact that $X_p(\phi) \to X_p(\phi_{\tilde{\mathrm{ab}}})$ is surjective, which is Lemma 6.1 of [**31**]. We now state the refined conjecture:

CONJECTURE 5.0.6. *Let (G, X) be a Shimura datum and let $\mathcal{G}/\mathbb{Z}_{(p)}$ be a model of G over $\mathbb{Z}_{(p)}$ such that its base change to $\mathbb{Z}_p$ is parahoric and let $U_p = \mathcal{G}(\mathbb{Z}_p)$. Consider the tower of Shimura varieties $\{\mathbf{Sh}_{G,U^pU_p}\}_{U^p}$ over the reflex field E with its action of $G(\mathbb{A}_f^p) \times Z_G(\mathbb{Q}_p)$, where U^p varies over compact open subgroups of $G(\mathbb{A}_f^p)$. Then this tower has a $G(\mathbb{A}_f^p) \times Z_G(\mathbb{Q}_p)$-equivariant extension to a tower of flat schemes $\{\mathscr{S}_{G,U^pU_p}\}_{U^p}$ over $\mathcal{O}_{E_{(v)}}$. Moreover, the action of $\mathcal{A}(\mathcal{G})$ on the generic fibre extends to the integral model. Furthermore there is an $\mathcal{A}(\mathcal{G}) \times Z_G(\mathbb{Q}_p)$-equivariant bijection fitting into a commutative diagram*

$$\begin{array}{ccc} \coprod_{[\phi_0]} S(G, \phi_0) & \xrightarrow{\sim} & \varprojlim_{U^p} \mathscr{S}_{U^pU_p}(\overline{\mathbb{F}}_p) \\ \downarrow & & \downarrow \\ \coprod_{[\phi_0]} \pi(G, \phi_0) & \xrightarrow{\vartheta_G} & \pi(G, X). \end{array}$$

compatible with the action of the operator Φ, which acts on the left hand side as the geometric p^r-Frobenius. Here $[\phi_0]$ runs over conjugacy classes of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_{G^{ad}}$. We remind the reader that the set $S(G, \phi_0)$ is the same as the set $S(G, \phi_0)_\tau$ for $\tau = 1$.

REMARK 5.0.7. Conjecture 3.2.0.2 follows immediately from Conjecture 3.5.0.6, using Lemma 3.4.16 of [**38**]. It follows as in Remark 3.7.10 of [**38**] that proving an $\mathcal{A}(\mathcal{G})$-equivariant bijection is enough if $Z_G(\mathbb{Q}) \cdot Z_{\mathcal{G}}(\mathbb{Z}_p) = Z_G(\mathbb{Q}_p)$. Indeed, in this case it follows as in loc. cit. that $Z_G(\mathbb{Q}_p)$ acts trivially on both sides of the conjectured isomorphism. In general, it is unclear to us how to construct an action of $Z_G(\mathbb{Q}_p)$ on the Kisin-Pappas integral models of Shimura varieties.

6. Connected components II

In this section we will build a theory of 'connected Shimura varieties' for the sets $S_\tau(G, \phi_0)$, following Section 3.8 of [**38**]. Let $h : (G, X) \to (G_2, X)$ be a surjective morphism of Shimura data that induces an isomorphism on derived groups. If $\mathcal{G}$ is a parahoric model of G, then h defines a parahoric model $\mathcal{G}_2$ of G_2 as in 1.1.3 of [**37**]. Let $X^+ \subset X^{\text{ad}}$ be a connected component, and consider the full subcategory $\mathscr{SH}_p(X^+)$ of $\mathscr{SH}_p$ consisting of objects (H, Y) such that $X^+ \subset Y$. Then for an object (G, X) of $\mathscr{SH}_p(X^+)$ we can consider the map

$$\pi_0(X) \to \pi(G, X) \xrightarrow{\vartheta_G^{-1}} \pi(G, \phi_0),$$

the image of X^+ under this map is a point $y \in \pi(G, \phi_0)$. Here ϑ is the map (3.4.0.1). Let $h : (\mathcal{G}, X) \to (\mathcal{G}_2, X)$ be a surjective morphism in $\mathscr{SH}_p$ that induces an isomorphism on derived groups. Define $S_\tau(G, \phi_0)^+$ and $S_\tau(G_2, \phi_0)^+$ to be the inverse image of y respectively y_2 in $\pi(G, \phi_0)$ respectively $\pi(G_2, \phi_0)$.

Lemma 6.0.1 (c.f. Lemma 3.8.2 of [**38**]). *The natural map $S_\tau(G, \phi_0)^+ \to S_\tau(G_2, \phi_0)^+$ is a bijection.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} S_\tau(G, \phi_0) & \xrightarrow{h} & S_\tau(G_2, \phi_0) \\ \downarrow & & \downarrow \\ \pi(G, \phi_0) & \longrightarrow & \pi(G_2, \phi_0), \end{array}$$

which is equivariant for the action of $\mathcal{A}(\mathcal{G})$ via $\mathcal{A}(\mathcal{G}) \to \mathcal{A}(\mathcal{G}_2)$. The bottom horizontal map can be identified (using our choice of y, y_2) with the map $\pi(G) \to \pi(G_2)$. It suffices to show that $h^{-1}(S_\tau(G_2, \phi_0)^+) \to S_\tau(G_2, \phi_0)^+$ is surjective, and that the fibers map bijectively to the fibers of $\pi(G) \to \pi(G_2)$. Both of these statements can be proven as in the proof of Lemma 3.8.2 of [**38**] (but the stronger statement that $\ker(\mathcal{A}(\mathcal{G}) \to \mathcal{A}(\mathcal{G}_2))$ acts transitively on the fibers does not follow, because 3.7.5 of [**38**] does not hold). □

Consider the action of $\langle \Phi \rangle \subset \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_{p^r})$, where $\mathbb{Q}_{p^r}$ is the maximal unramified extension contained in E_p. Then $\langle \Phi \rangle$ acts on $S_\tau(G, \phi_0)$ for any τ, and we let

$$\mathscr{E}_p^r(\mathcal{G}, \phi_0) \subset \mathcal{A}(\mathcal{G}) \times \langle \Phi \rangle$$

be the stabiliser of $S_\tau(G, \phi_0)^+$. In then follows as in the proof of Lemma 3.8.5 of [**38**] that the group $\mathscr{E}(\mathcal{G}, \phi_0)$ is an extension of $\langle \Phi \rangle$ by $\mathcal{A}(\mathcal{G})^\circ$ and depends only on $\mathcal{G}^{\mathrm{der}}, X^+$ and the integer r. Here we have to keep in mind that we only know injectivity (rather than bijectivity) of

$$\mathcal{A}(\mathcal{G})^\circ \backslash \mathcal{A}(\mathcal{G}) \simeq \mathcal{G}(\mathbb{Z}_{(p)})_+^- \backslash G(\mathbb{A}_f^p) \to G(\mathbb{Q})_+^- \backslash G(\mathbb{A}_f)/\mathcal{G}(\mathbb{Z}_p) \simeq \pi(G),$$

which is enough for the conclusion.

Lemma 6.0.2 (Lemma 3.8.8 of [**38**]). *There is a natural isomorphism*

$$\mathcal{A}(\mathcal{G}) *_{\mathcal{A}(\mathcal{G})^\circ} \mathscr{E}_p^r(\mathcal{G}, \phi_0) \simeq \mathcal{A}(\mathcal{G}) \times \langle \Phi \rangle.$$

Moreover, there is a natural isomorphism

$$S_\tau(G,\phi_0) \simeq \coprod_{s\in S}[\mathcal{A}(\mathcal{G}) \times S_\tau(G,\phi_0)^+ s]/\mathcal{A}(\mathcal{G})^\circ,$$

equivariant for the action of $\mathcal{A}(\mathcal{G}) \times \langle\Phi\rangle$*, where* $\mathcal{A}(\mathcal{G})$ *acts on the right hand via left multiplication on itself and* $S \subset \pi(G)$ *is a set of coset representatives for the inclusion*

$$\mathcal{A}(\mathcal{G})^\circ\backslash\mathcal{A}(\mathcal{G}) \to \pi(G).$$

Proof. The action of $\mathcal{A}(\mathcal{G})$ on $S_\tau(G,\phi_0)$ gives us a map

$$[\mathcal{A}(\mathcal{G}) \times S_\tau(G,\phi_0)^+]/\mathcal{A}(\mathcal{G})^\circ \hookrightarrow S_\tau(G,\phi_0),$$

which is injective because $\mathcal{A}(\mathcal{G})^\circ$ is the stabiliser of $S_\tau(G,\phi_0)^+$ in $\mathcal{A}(\mathcal{G})$. This map is not necessarily surjective, because the right hand side surjects onto $\pi(G,\phi_0)$ and the left hand side might not. Using the point y, we can identify the image of the left hand side in $\pi(G,\phi_0)$ with

$$\mathcal{A}(\mathcal{G})/\mathcal{A}(\mathcal{G})^\circ \subset \pi(G)$$

and the result follows. □

Lemma 6.0.3. *Let* $f:(G,X)\to(G_2,X)$ *be a surjective map with kernel* $Z\subset Z_G$ *and let* $\mathcal{G}\to\mathcal{G}_2$ *be the induced map on parahoric models. Suppose that there is an isomorphism* $Z_{\mathbb{Q}} \simeq R_{L/\mathbb{Q}}\mathbb{G}_m$*.*

(1) Then

$$S_\tau(G_2,\phi_0) \simeq S_\tau(G,\phi_0)/Z(\mathbb{A}_f^p) \simeq [\mathcal{A}(\mathcal{G}_2)\times S_\tau(G,\phi_0)]/\mathcal{A}(\mathcal{G}). \tag{6.0.1}$$

(2) There is a natural isomorphism

$$\mathscr{E}_p^r(\mathcal{G}_2) \simeq \mathcal{A}(\mathcal{G}_2)^\circ *_{\mathcal{A}(\mathcal{G})} \mathscr{E}_p^r(\mathcal{G}),$$

(3) The natural map of sets with $\mathscr{E}_p^r(\mathcal{G}_2)$ *action*

$$S_\tau(G_2,\phi_0)^+ \simeq [\mathcal{A}(\mathcal{G}_2)^\circ \times S_\tau(\mathcal{G},\phi_0)^+]/\mathcal{A}(\mathcal{G})^\circ,$$

is an isomorphism if $L/\mathbb{Q}$ *is Galois.*

(4) There is an $\mathcal{A}(\mathcal{G}_2)\times\langle\Phi\rangle$ *equivariant isomorphism*

$$S_\tau(G_2,\phi_0)^+ \simeq \coprod_{j\in J}[\mathcal{A}(\mathcal{G}_2)\times S_\tau(G,\phi_0)^+ j]/\mathcal{A}(\mathcal{G})^\circ,$$

where $J\subset\pi(G_2)$ *runs over a set of coset representatives for the inclusion*

$$\mathcal{A}(\mathcal{G})^\circ\backslash\mathcal{A}(\mathcal{G}_2)\hookrightarrow\pi(G_2).$$

Proof. The proof of the first part follows as in the proof of Lemma 3.8.10 of [**38**], except that we need to modify the argument showing surjectivity of the map on affine Deligne-Lusztig varieties. The map $\mathrm{Gr}_{\mathcal{G}}\to\mathrm{Gr}_{\mathcal{G}_2}$ induces an isomorphism $Y\simeq Y_2$ between a connected component Y in the source and a connected component Y_2 in the target (c.f. Section 4.2 of [**31**]). Moreover, it induces an isomorphism

$$X_\mu(b)_{G,K}\cap Y\to X_{\mu_2}(b_2)_{G_2,\mathcal{G}}\cap Y_2,$$

and so it suffices to check that connected components of $\mathrm{Gr}_{\mathcal{G}_2}$ that intersect with $X_{\mu_2}(b_2)_{G_2,\mathcal{G}}$ are in the image of connected components of Gr_K that intersect $X_\mu(b)_{G,K}$. In other words, we want to show that

$$\pi_1(G)_I^\sigma\to\pi_1(G_2)_I^\sigma$$

is surjective, which follows by considering the following diagram:

$$\begin{array}{ccc} G(\mathbb{Q}_p) & \twoheadrightarrow & G_2(\mathbb{Q}_p) \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ \pi_1(G)_I^\sigma & \longrightarrow & \pi_1(G_2)_I^\sigma \end{array}$$

Indeed, the top horizontal arrow is surjective because Z is an induced torus, and the vertical arrrows are surjective by the result of Section 7.7 of [**43**]. The isomorphism

$$S_\tau(G,\phi_0)/Z(\mathbb{A}_f^p)\simeq[\mathcal{A}(\mathcal{G}_2)\times S_\tau(G,\phi_0)]/\mathcal{A}(\mathcal{G})$$

similarly follows as in the proof of Lemma 3.8.10 of [**38**]. The proof (2) is the same as the proof of (2) in loc. cit. and moreover gives us an $\mathscr{E}_p^{cr}(\mathcal{G}_2^{\mathrm{der}})$-equivariant map

$$[\mathcal{A}(\mathcal{G}_2)^\circ\times S_\tau(G,\phi_0)^+]/\mathcal{A}(\mathcal{G})^\circ\to S_\tau(G_2,\phi_0)^+.$$

To prove (3), it suffices to prove that this map is a bijection. It is injective by the second isomorphism of (3.6.0.1) and so it suffices to prove surjectivity. From (i) we get that the natural map

$$f^{-1}(S_\tau(G_2,\phi_0)^+) \to S_\tau(G_2,\phi_0)^+$$

is surjective, and so it suffices to show that $f^{-1}(S_\tau(G_2,\phi_0)^+)$ surjects onto the preimage of y_2 in $\pi(G,\phi_0)$. If we identify $\pi(G,\phi_0) \to \pi(G_2,\phi_0)$ with $\pi(G) \to \pi(G_2)$ using y and y_2, it comes down to showing that

$$Z(\mathbb{A}_f^p) \to \ker(\pi(G) \to \pi(G_2)) \simeq Z(\mathbb{Q})^- \backslash Z(\mathbb{A}_f)/\mathcal{Z}(\mathbb{Z}_p) \tag{6.0.2}$$

is surjective, where $\mathcal{Z}$ is the kernel of $\mathcal{G} \to \mathcal{G}_2$. It follows from Proposition 2.4.12 of [**39**] that $\mathcal{Z}$ is in fact the connected Néron model of Z, and that $\mathcal{G} \to \mathcal{G}_2$ is surjective (here we use that Z is an induced torus). Moreover Remark 8.3 of [**12**] shows that $Z(\mathbb{Q})\mathcal{Z}(\mathbb{Z}_p) = \mathbb{Z}(\mathbb{Q}_p)$ (using the fact that $L/\mathbb{Q}$ is Galois). It follows that (3.6.0.2) is surjective. Part (4) of the lemma follows from Lemma 3.6.0.2 and the fact that $\mathcal{A}(\mathcal{G})^\circ \to \mathcal{A}(\mathcal{G}_2)^\circ$ is surjective because Z is an induced torus. □

The following analogue of Corollary 3.8.12 of [**38**] now follows:

Corollary 6.0.4. *Suppose that $\mathcal{G}^{der} \to \mathcal{G}_2^{der}$ is a central isogeny which induces an isomorphism of adjoint Shimura data. Then there is an isomorphism of sets with $\mathcal{A}(\mathcal{G}_2) \times \langle\Phi\rangle$ action*

$$S_\tau(G_2,\phi_0) \simeq \coprod_{j\in J} [\mathcal{A}(\mathcal{G}_2) \times S_\tau(G,\phi_0)^+ j]/\mathcal{A}(\mathcal{G})^\circ,$$

where J ranges over a set of coset representatives for

$$\mathcal{A}(\mathcal{G})^\circ \backslash \mathcal{A}(\mathcal{G}_2) \hookrightarrow \pi(G_2)$$

Proof. The proof is the same as the proof of Corollary 3.8.12 of [**38**]. □

Remark 6.0.5. There is a bijection

$$\pi(G_2) \simeq \mathcal{A}(G_2)^\circ \backslash \mathcal{A}(G_2)\mathcal{G}_2(\mathbb{Z}_p) \simeq \mathcal{A}(G)^\circ \backslash \mathcal{A}(G_2)\mathcal{G}_2(\mathbb{Z}_p),$$

which will be useful later when we compare with Lemma 4.6.13 of [**37**].

CHAPTER 4

Main result for Hodge type Shimura varieties

1. Main results

The main goal of this Chapter is to show that we can deduce Theorem 1 for a general parahoric subgroup from the case of a very special parahoric subgroup. Let (G, X) be a Shimura datum of Hodge type with reflex field E and conjugacy class of cocharacters μ (here we take the inverse of μ_h from the last section). Let $p > 2$ be a prime number such that $G = G_{\mathbb{Q}_p}$ is quasi-split and splits over a tamely ramified extension and such that p does not divide $\#\pi_1(G^{\mathrm{der}})$.

We will work with Shimura varieties of parahoric level at p and we will always assume that the parahoric subgroups are equal to Bruhat-Tits stabiliser group schemes; we will call such parahoric subgroups *connected*. We need this assumption because all the results in [**73**] use this assumption, and it is automatically satisfied either if G^{der} is simply connected and $X_*(G^{\mathrm{ab}})_I$ is torsion-free or if $G_{\mathbb{Q}_p}$ is unramified and the parahoric is contained in a hyperspecial parahoric.[1]

Let $U^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup and let $U_p = \mathcal{G}_K(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ be a connected parahoric subgroup corresponding to a σ-stable type $K \subset \mathbb{S}$. Then there is a smooth projective scheme $\mathbf{Sh}_U/\operatorname{Spec} E$, which is the Shimura variety associated to all the above data. Choose a place $v|p$ of E, then Kisin and Pappas (Theorem 0.1 of [**37**]) construct a flat integral model $\mathscr{S}_K/\mathcal{O}_{E_v}$ together with an action of $G(\mathbb{A}_f^p)$ by Hecke operators.

Write $G_{\mathbb{Q}_p}^{\mathrm{ad}} \simeq G_1 \times \cdots \times G_n$ with the G_i simple over $\mathbb{Q}_p$, this gives a corresponding decomposition $J_b^{\mathrm{ad}} = J_{b,1} \times \cdots \times J_{b,n}$, where $b \in B(G, X)$ is the unique basic element (because J_b is an inner form of $G_{\mathbb{Q}_p}$). Recall that we call $J_{b,i}$ of compact type if $J_{b,i}(\mathbb{Q}_p)$ is compact in the metric topology.

[1]When we deal with abelian type Shimura varieties later, we will always reduce to one of these two cases using Lemma 4.6.22 of [**37**], which is always possible unless $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has factors of of type $D^{\mathbb{H}}$.

THEOREM 1.0.1. *Suppose that there is a connected very special parahoric subgroup U_p corresponding to a σ-stable type K and a $\langle\Phi\rangle \times G(\mathbb{A}_f^p)$-equivariant bijection*

$$\mathscr{S}_K(\overline{\mathbb{F}}_p) \simeq \coprod_{[\phi]} I_\phi(\mathbb{Q}) \setminus G(\mathbb{A}_f^p) \times X_\mu(b)_K/U^p, \tag{1.0.1}$$

where $[\phi]$ runs over conjugacy classes of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$. Now let U_p' be any connected parahoric subgroup, corresponding to a σ-stable type J. Suppose that G^{ad} is $\mathbb{Q}$-simple, that for $1 \leq i \leq n$ the group $J_{b,i}$ is not of compact type, and that either $\mathbf{Sh}_U$ is proper or that Conjecture 4.7.0.5 holds. Then there is a $\langle\Phi\rangle \times G(\mathbb{A}_f^p)$-equivariant bijection

$$\mathscr{S}_J(\overline{\mathbb{F}}_p) \simeq \coprod_{[\phi]} I_\phi(\mathbb{Q}) \setminus G(\mathbb{A}_f^p) \times X_\mu(b)_J/U^p,$$

indexed by the same set of isogeny classes as (4.1.0.1).

REMARK 1.0.2. Kisin and Pappas do not construct an action of $Z_G(\mathbb{Q}_p)$ on their integral models, so we cannot say anything about $Z_G(\mathbb{Q}_p)$-equivariance of this bijection.

REMARK 1.0.3. The assumption on the groups $J_{b,i}$ is automatic when G is not of type A, because the only groups of compact type over $\mathbb{Q}_p$ are of type A and J_b is an inner form of G.

Along the way, we will prove the following version of Theorem 2:

THEOREM 1.0.4. *Let (G, X) as above and suppose that G^{ad} is $\mathbb{Q}$-simple, that for $1 \leq i \leq n$ the group $J_{b,i}$ is not of compact type, and that either $\mathbf{Sh}_U$ is proper or that Conjecture 4.7.0.5 holds. Let $w \in {}^K\mathrm{Adm}(\mu)$ and let $\mathscr{S}_{K,\overline{\mathbb{F}}_p}\{w\}$ be the corresponding Ekedahl-Kottwitz-Oort-Rapoport (EKOR) stratum. Suppose that it is not contained in the basic locus, then*

$$\mathscr{S}_{K,\overline{\mathbb{F}}_p}\{w\} \to \mathscr{S}_{K,\overline{\mathbb{F}}_p}$$

induces a bijection on connected components.

REMARK 1.0.5. Conjecture 4.7.0.5 predicts that irreducible components of closures of EKOR strata in $\mathscr{S}_{K,\overline{\mathbb{F}}_p}$, with K very special, intersect the unique 0-dimensional EKOR stratum. The conjecture follows from Proposition 6.20 of [**70**], combined with Theorem 1.2 of [**2**], when K is hyperspecial.

My paper [**32**] was written before Theorem 1.2 of [**2**] was available. In that paper, I prove conjecture 4.7.0.5 for many nonproper Hodge type cases with K hyperspecial, using a trick from [**17**] to make the arguments from [**70**] unconditional (my argument also involves proving condition 6.4.2 of [**17**], strengthening the main results of that paper).

Remark 1.0.6. Theorem 4.1.0.4 generalises Theorem 7.4 of [**21**], which discusses Siegel modular varieties with Iwahori level structure. Our proof partly generalises the proof of [**21**], but with one crucial difference. They use the results of [**15**] at hyperspecial level to deduce their results, by studying the fibers of the forgetful map. We will instead deduce the results at arbitrary parahoric level from the results at Iwahori level.

Let us now sketch the arguments that prove Theorem 4.1.0.1: By the arguments in Section 7 of [**73**], it suffices to handle the case that U_p' is an Iwahori subgroup. We will study the forgetful map $\mathscr{S}_{\emptyset,\overline{\mathbb{F}}_p} \to \mathscr{S}_{K,\overline{\mathbb{F}}_p}$, whose perfection fits in a commutative diagram of pre-stacks on the category of perfect k-algebras:

$$\begin{array}{ccc} \mathrm{Sh}_{G,\emptyset} & \longrightarrow & \mathrm{Sht}_{G,\emptyset,\mu} \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{G,K} & \longrightarrow & \mathrm{Sht}_{K,\mu}\,. \end{array}$$

It will follow from Lemma 2.8.0.2 that Theorem 4.1.0.1 holds if and only if this diagram is Cartesian. We let $\hat{\mathrm{Sh}}_{G,\emptyset}$ be the fiber product of this diagram, which is a pfp algebraic space by Corollary 2.4.0.2, and consider the induced map $\iota : \mathrm{Sh}_{G,\emptyset} \to \hat{\mathrm{Sh}}_{G,\emptyset}$. We will show that i is a closed immersion in Section 4.3, using results of [**73**]. In Section 4.4, we will construct a perfect local model diagram for $\hat{\mathrm{Sh}}_{G,\emptyset}$, compatible with the local model diagram of $\mathrm{Sh}_{G,\emptyset}$. The local model diagram tells us that both $\mathrm{Sh}_{G,\emptyset}$ and $\hat{\mathrm{Sh}}_{G,\emptyset}$ are the union of closures of maximal KR strata, which we will denote by $\mathrm{Sh}_{G,\emptyset}(\leq w)$ and $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$, and that KR strata are equidimensional of the correct dimension. This gives us equidimensionality of $\hat{\mathrm{Sh}}_{G,\emptyset}$ and so it suffices to prove that $\hat{\mathrm{Sh}}_{G,\emptyset}$ and $\mathrm{Sh}_{G,\emptyset}$ have the same number of irreducible components. The map $\iota : \mathrm{Sh}_{G,\emptyset} \hookrightarrow \hat{\mathrm{Sh}}_{G,\emptyset}$ is compatible with KR stratifications, therefore we can count irreducible components in the closure of each maximal KR stratum separately. We will distinguish between KR strata that are completely contained in the basic locus and KR strata that are not (nonbasic KR strata). If a KR stratum is completely contained in the basic locus, then so is its closure, because the basic locus is closed. It follows from [**73**] that

$$\iota : \mathrm{Sh}_{G,\emptyset} \to \hat{\mathrm{Sh}}_{G,\emptyset}$$

is an isomorphism over the basic locus, hence we can focus our efforts on nonbasic KR strata. We know that every KR stratum surjects onto $\pi_0(\mathrm{Sh}_{G,K})$ (by Section 8 of [**73**]), which gives a lower bound for the number of irreducible components of nonbasic KR strata. If $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ is not contained in the basic locus, we will show that the number of irreducible components of $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ is equal to the number of connected components of $\mathrm{Sh}_{G,K}$ [2]. This shows that $\hat{\mathrm{Sh}}_{G,\emptyset}$ and $\mathrm{Sh}_{G,\emptyset}$ have the same number of irreducible components.

To prove this 'irreducibility' of $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$, we will argue as follows: The local model diagram tells us that connected components of $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ are irreducible, so that it suffices to count connected components. Recall that the indexing set of the KR stratification is $\mathrm{Adm}(\mu) \subset \tilde{W}$, where $\tilde{W}$ is the Iwahori-Weyl group and $\mathrm{Adm}(\mu) \subset \tilde{W}_a\tau$. Write $w = v\tau$ and let $v = s_1 \cdots s_n$ be a reduced expression of v, then we define

$$Y_w = \hat{\mathrm{Sh}}_{G,\emptyset}(s_1\tau) \cup \cdots \cup \hat{\mathrm{Sh}}_{G,\emptyset}(s_n\tau) \cup \hat{\mathrm{Sh}}_{G,\emptyset}(\tau),$$

which is the union of all KR strata in $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ of dimension at most one. A technical argument, which requires either properness or Conjecture 4.7.0.5 and which generalises the proof of Theorem 6.4 of [**21**], shows that every connected component of $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ intersects Y_w, which means that it is enough to understand the connected components of Y_w. Our assumption that J_b^{ad} has no compact factors will then tell us that Y_w is contained in the basic locus (this will follow from Proposition 5.6 of [**20**]). We now proceed in two steps:

- We show that the basic locus of $\hat{\mathrm{Sh}}_{G,\emptyset}$ has the same number of connected components as $\mathrm{Sh}_{G,K}$. This uses strong approximation for I_ϕ^{sc}, where ϕ is the basic isogeny class, and the results of [**31**]. Since $(I_\phi^{\mathrm{sc}})_{\mathbb{R}}$ is compact and $(I_\phi^{\mathrm{sc}})_{\mathbb{Q}_p} = J_b^{\mathrm{sc}}$, we have to use the assumption that (most of) the $J_{b,i}$ are not of compact type in order to apply strong approximation.
- Now it remains to show that Y_w has the same number of connected components as the basic locus of $\hat{\mathrm{Sh}}_{G,\emptyset}$. This is now a local problem, and we reduce it to the connectedness of the Bruhat-Tits building of J_b^{ad} using Proposition

[2]Here we work with connected components of $\mathrm{Sh}_{G,K}$ rather than $\mathrm{Sh}_{G,\emptyset}$ for technical reasons. It will follow from our arguments that the natural map $\mathrm{Sh}_{G,\emptyset} \to \mathrm{Sh}_{G,K}$ induces a bijection connected components, but this is not clear a priori.

5.4 of [**20**] and results from [**28**] and [**19**]. Our arguments are essentially equivalent to the connectedness argument given in Section 6 of He-Zhou [**31**], although our perspective is different.

2. Local models and shtukas

2.1. Local models. Theorem 0.4 of [**37**] tells us that $\mathscr{S}_K/\mathcal{O}_{E_v}$ sits in a local model diagram

$$\begin{array}{ccc} & \widetilde{\mathscr{S}_K} & \\ \swarrow^{\pi} & & \searrow^{q} \\ \mathscr{S}_K & & M^{\mathrm{loc}}_{G,K,\mu}, \end{array}$$

where π is a $\mathcal{G}_K$-torsor and q is smooth of relative dimension equal to $\dim G$. We let $\mathrm{Sh}_{G,K}$ denote the perfection of the geometric special fiber of $\mathscr{S}_K$, which is a pfp scheme.

2.2. Shtukas. In order to construct a shtuka over $\mathrm{Sh}_{G,K}$, we will need to go into the details of the construction of $\mathscr{S}_K$. First we choose a Hodge-embedding $(G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$ and a parahoric $\mathcal{P}'$ of GSp such that $\mathcal{P}'(\breve{\mathbb{Z}}_p) \cap G = \mathcal{G}_K(\breve{\mathbb{Z}}_p)$. Then we get a finite morphism

$$\mathscr{S}_K \to \mathscr{S}_{\mathcal{P}'}(\mathrm{GSp}, S^{\pm}),$$

where the latter is a moduli theoretic integral model of a Siegel modular variety with parahoric level $\mathcal{P}'$ at p. This induces a finite morphism on the perfections of geometric special fibers

$$\mathrm{Sh}_{G,K} \to \mathrm{Sh}_{\mathrm{GSp},\mathcal{P}'}$$

and in particular a family of abelian varieties A over $\mathrm{Sh}_{G,K}$. Given $x \in \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$, Kisin and Pappas construct tensors $s_{\alpha,0,x}$ in Dieudonné-module $\mathcal{D}(A_x)$ of abelian variety A_x such that the stabiliser of the $s_{\alpha,0,x}$ in $\mathrm{GL}(\mathcal{D}(A)_x)$ is isomorphic to $\mathcal{G}_K$ (see Section 6.3 of [**73**]). This means that we can upgrade the Dieudonné module of A_x to a $\mathcal{G}_K$-shtuka over $\overline{\mathbb{F}}_p$, and this gives a map (see Section 8 of [**73**]).

$$\mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p) \to \mathrm{Sht}_{K,\mu}(\overline{\mathbb{F}}_p). \tag{2.2.1}$$

It is a result of Hamacher-Kim (Proposition 1 of [**24**], see Proposition 4.4.1 of [**64**]) that that there is actually a morphism $\mathrm{Sh}_{G,K} \to \mathrm{Sht}_{K,\mu}$ inducing (4.2.2.1) on $\overline{\mathbb{F}}_p$-points.

It follows from the discussion after Theorem 4.4.3. of [**64**] that the perfection of the special fiber of $M^{\mathrm{loc}}_{G,K,\mu}$ can be identified with a closed subscheme of the affine flag variety for $L^+\mathcal{G}_K$. To be precise it is isomorphic to

$$M^{\mathrm{loc}} := \bigcup_{w \in \mathrm{Adm}(\mu)_K} \mathrm{Gr}_K(w),$$

and under this isomorphism the right action of $L^+\mathcal{G}_K$ on M^{loc}, which factors through $\overline{\mathcal{G}}_K$, is identified with the $\overline{\mathcal{G}}_K$ action on the perfection of $M^{\mathrm{loc}}_{G,J,\mu}$. Furthermore, Theorem 4.4.3. of loc. cit. tells us that the perfectly smooth map $\mathrm{Sh}_{G,K} \to [M^{\mathrm{loc}}/\overline{\mathcal{G}}_K]$ induced from the local model diagram fits in a commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}_{G,K} & \longrightarrow & \mathrm{Sht}^{(m,1)}_{K,\mu} \\ & \searrow & \downarrow \\ & & [M^{\mathrm{loc}}/\overline{\mathcal{G}}_K], \end{array}$$

and that the map $\mathrm{Sh}_{G,K} \to \mathrm{Sht}^{(m,1)}_{K,\mu}$ is perfectly smooth.

3. Change of parahoric

Theorem 7.1 of [**73**] tells us that for $J \subset K$ there is a morphism $\mathscr{S}_J \to \mathscr{S}_K$ which induces the obvious forgetful morphism on generic fibers. Moreover it follows from Section 7.4 of op. cit. that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Sh}_{G,J} & \longrightarrow & \mathrm{Sht}_{J,\mu} \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{G,K} & \longrightarrow & \mathrm{Sht}_{K,\mu}\,. \end{array}$$

As explained in the introduction to this section, our goal is to show that this diagram is Cartesian. Now (and in the rest of this section) let K be a type corresponding to a very special parahoric such that the assumptions of Theorem 4.1.0.1 hold with $U_p = \mathcal{G}_K(\mathbb{Z}_p)$. Let $J = \emptyset$ be the fixed Iwahori subgroup and define $\hat{\mathrm{Sh}}_{G,\emptyset}$ via the following Cartesian diagram

$$\begin{array}{ccc} \hat{\mathrm{Sh}}_{G,\emptyset} & \longrightarrow & \mathrm{Sht}_{\emptyset,\mu} \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{G,K} & \longrightarrow & \mathrm{Sht}_{K,\mu}\,. \end{array}$$

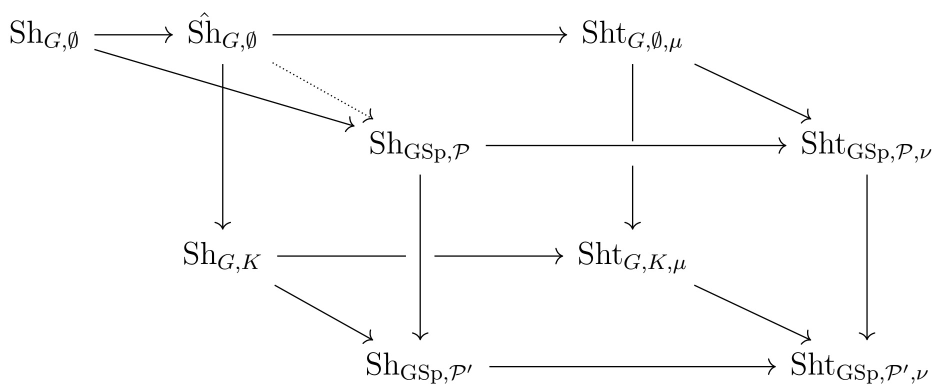
Corollary 2.4.0.2 tells us that $\hat{\mathrm{Sh}}_{G,\emptyset}$ is a perfect algebraic space which is perfectly proper over $\mathrm{Sh}_{G,K}$. The universal property of $\hat{\mathrm{Sh}}_{G,\emptyset}$ gives us a morphism $\mathrm{Sh}_{G,\emptyset} \to \hat{\mathrm{Sh}}_{G,\emptyset}$, which is proper because it is a morphism of perfect algebraic spaces that are perfectly proper over $\mathrm{Sh}_{G,K}$.

Proposition 3.0.1. *The morphism* $\iota : \mathrm{Sh}_{G,\emptyset} \to \hat{\mathrm{Sh}}_{G,\emptyset}$ *induced by the universal property of* $\hat{\mathrm{Sh}}_{G,\emptyset}$ *is a closed immersion.*

Proof. It suffices to prove that it is injective on k-points by Lemma 2.2.0.5 since $\mathrm{Sh}_{G,\emptyset} \to \hat{\mathrm{Sh}}_{G,\emptyset}$ is a morphism of perfectly proper $\mathrm{Sh}_{G,K}$-algebraic spaces and therefore perfectly proper. There is a commutative diagram (c.f. 8.1.1 of [**73**])

$$\begin{array}{ccc} \mathscr{S}_{\emptyset}(G,X) & \longrightarrow & \mathscr{S}_{\mathcal{P}}(\mathrm{GSp}, S^{\pm}) \\ \downarrow & & \downarrow \\ \mathscr{S}_{K}(G,X) & \longrightarrow & \mathscr{S}_{\mathcal{P}'}(\mathrm{GSp}, S^{\pm}), \end{array}$$

where $\mathscr{S}_{\mathcal{P}}(\mathrm{GSp}, S^{\pm}), \mathscr{S}_{\mathcal{P}'}(\mathrm{GSp}, S^{\pm})$ are moduli theoretic integral models of a Siegel modular variety with parahoric levels $\mathcal{P}$ and $\mathcal{P}'$ respectively at p, the right vertical map is the canonical forgetful map and the horizontal maps are finite. Taking geometric special fibers, perfecting and adding morphisms to moduli spaces of shtukas, we get a commutative cube



Here ν is the cocharacter of GSp corresponding to the Shimura datum $S^{\pm}$, and $\mathrm{Sh}_{\mathrm{GSp},\mathcal{P}}, \mathrm{Sh}_{\mathrm{GSp},\mathcal{P}'}$ are the perfections of the geometric special fibers of $\mathscr{S}_{\mathcal{P}}(\mathrm{GSp}, S^{\pm})$ and $\mathscr{S}_{\mathcal{P}'}(\mathrm{GSp}, S^{\pm})$ respectively. It suffices to show that the dotted arrow in the diagram exists, because Corollary 6.3 of [**73**] tells us that a point $x \in \mathrm{Sh}_{G,\emptyset}$ is determined by its image in $\mathrm{Sh}_{\mathrm{GSp},\mathcal{P}}$ and the tensors in the Dieudonné module of its p-divisible group, which are determined by the image of x in $\mathrm{Sht}_{G,\emptyset,\mu}$. The existence of the dotted arrow follows from the following claim: □

CLAIM 3.0.2. The front face of the cube, i.e., the square involving $\mathrm{Sh}_{\mathrm{GSp},\mathcal{P}}, \mathrm{Sht}_{\mathrm{GSp},\mathcal{P},\nu}$, $\mathrm{Sh}_{\mathrm{GSp},\mathcal{P}'}$ and $\mathrm{Sht}_{\mathrm{GSp},\mathcal{P}',\nu}$ is Cartesian.

PROOF. This follows from the moduli interpretation of the four objects in the front face of the cube. Indeed the Shimura varieties of level $\mathcal{P}'$ parametrises chains $A_0 \to A_1 \to \cdots \to A_n$ of abelian varieties, where the maps are p-power isogenies of fixed degree, and the Shimura variety of $\mathcal{P}$ similarly parametrises such chains $A_0 \to A_1 \to \cdots \to A_m$ with $m > n$. [3] The moduli spaces of shtukas parametrises the same kind of chains, but then of p-divisible groups. Since an isogeny $A \to B$ of abelian varieties is uniquely determined by the abelian variety A and the isogeny of p-divisible groups $A[p^\infty] \to B[p^\infty]$, the diagram is indeed Cartesian. □

4. A local model diagram

Recall that we have defined a perfect algebraic space $\hat{\mathrm{Sh}}_{G,\emptyset}$ via the Cartesian diagram

$$\begin{array}{ccc} \hat{\mathrm{Sh}}_{G,\emptyset} & \longrightarrow & \mathrm{Sht}_{\emptyset,\mu} \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{G,K} & \longrightarrow & \mathrm{Sht}_{K,\mu}\,. \end{array}$$

In this subsection we will show that the singularities of $\hat{\mathrm{Sh}}_{G,\emptyset}$ are controlled by the local model of $\mathrm{Sh}_{G,\emptyset}$. More precisely we will show that closures of KR strata are equidimensional and locally integral, by relating $\hat{\mathrm{Sh}}_{G,\emptyset}$ to the perfection of the local model. The local model diagram of $\mathrm{Sh}_{G,\emptyset}$ is encoded in a perfectly smooth morphism

$$\mathrm{Sh}_{G,\emptyset} \to \mathrm{Sht}_{\emptyset,\mu} \to \mathrm{Sht}_{\emptyset,\mu}^{(1,0)} := \left[M^{\mathrm{loc}}/\overline{\mathcal{G}}_{\emptyset}\right]$$

There is an obvious analogue of this morphism for $\hat{\mathrm{Sh}}_{G,\emptyset}$, and it suffices to show that this is also perfectly smooth.

PROPOSITION 4.0.1. *The morphism* $\hat{\mathrm{Sh}}_{G,\emptyset} \to \mathrm{Sht}_{\emptyset,\mu} \to \left[M^{loc}/\overline{\mathcal{G}}_{\emptyset}\right]$ *is perfectly smooth.*

[3]Of course the degrees of the isogenies are such that the forgetful map makes sense.

Proof. Lemma 2.6.0.1 proves that there is a pre-stack Y such that the following diagram commutes

$$\begin{array}{ccccccc}
\hat{\mathrm{Sh}}_{G,\emptyset} & \longrightarrow & \mathrm{Sht}_{\emptyset,\mu} & \longrightarrow & Y & \longrightarrow & \mathrm{Sht}_{\emptyset,\mu}^{(n,1)} \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Sh}_{G,K} & \longrightarrow & \mathrm{Sht}_{K,\mu} & \longrightarrow & \mathrm{Sht}_{K,\mu}^{(m,1)}. & &
\end{array}$$

Moreover it says that the middle square is Cartesian and that the map $Y \to \mathrm{Sht}_{\emptyset,\mu}^{(n,1)}$ is perfectly smooth. It follows that the rectangle containing $\hat{\mathrm{Sh}}_{G,\emptyset}, \mathrm{Sh}_{G,K}, Y, \mathrm{Sht}_{K,\mu}^{(m,1)}$ is Cartesian. Theorem 4.4.3 of [**64**] tells us that the map $\mathrm{Sh}_{G,K} \to \mathrm{Sht}_{K,\mu}^{(m,1)}$ is perfectly smooth, and because perfectly smooth morphisms are preserved under base change we deduce that the map $\hat{\mathrm{Sh}}_{G,\emptyset} \to Y$ is perfectly smooth and hence the map $\hat{\mathrm{Sh}}_{G,\emptyset} \to \mathrm{Sht}_{\emptyset,\mu}^{(n,1)}$ is perfectly smooth. Proposition 4.2.5 of [**64**] tells us that

$$\mathrm{Sht}_{\emptyset,\mu}^{(n,1)} \to \left[M^{\mathrm{loc}}/\overline{\mathcal{G}}_{\emptyset}\right]$$

is perfectly smooth, concluding the proof. □

Corollary 4.0.2. *The perfect scheme $\hat{\mathrm{Sh}}_{G,\emptyset}$ is equidimensional of the same dimension as $\mathrm{Sh}_{G,\emptyset}$, and closures $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ of KR strata are locally integral (complete local rings at closed points are integral) of dimension $l(w)$.*

Proof. The morphism $\hat{\mathrm{Sh}}_{G,\emptyset} \to \left[M^{\mathrm{loc}}/\overline{\mathcal{G}}_{\emptyset}\right]$ is (by definition) the same as a diagram

$$\begin{array}{ccccc}
 & & \tilde{\hat{\mathrm{Sh}}}_{G,\emptyset} & & \\
 & \swarrow^{s} & & \searrow^{t} & \\
\hat{\mathrm{Sh}}_{G,\emptyset} & & & & M^{\mathrm{loc}},
\end{array}$$

where $s : \tilde{\hat{\mathrm{Sh}}}_{G,\emptyset} \to \hat{\mathrm{Sh}}_{G,\emptyset}$ is a $\overline{\mathcal{G}_{\emptyset}}$-torsor. If we add the local model diagram for $\mathrm{Sh}_{G,\emptyset}$ to the diagram then we get

$$\begin{array}{ccccccc}
 & & \tilde{\mathrm{Sh}}_{G,\emptyset} & \hookrightarrow & \tilde{\hat{\mathrm{Sh}}}_{G,\emptyset} & \xrightarrow{t'} & \\
 & \swarrow^{s'} & & \swarrow^{s} & & \searrow^{t} & \\
\mathrm{Sh}_{G,\emptyset} & \hookrightarrow & \hat{\mathrm{Sh}}_{G,\emptyset} & & & & M^{\mathrm{loc}},
\end{array}$$

where s' is an $\overline{\mathcal{G}_\emptyset}$-torsor. Proposition 4.4.0.1 tells us that t is perfectly smooth and Theorem 4.4.3. of [**64**] tells us that t' is perfectly smooth. Since closed immersions have relative dimension zero, it follows that t and t' have the same relative dimension, which is constant because M^{loc} is connected. This also implies that the dimensions of KR strata and their closures are the same for $\hat{\text{Sh}}_{G,\emptyset}$ and $\text{Sh}_{G,\emptyset}$. The integrality of complete local rings follows from the local model diagram in a standard way, because it can be checked on a (perfectly) smooth cover. □

5. Connected components of the basic locus

In this section we will work with the basic locus of the Shimura variety at Iwahori level, we will always assume that our chosen Iwahori subgroup is a connected parahoric. In this section we will show, using Rapoport-Zink uniformisation, that the basic locus of $\text{Sh}_{G,\emptyset}$ has the same number of connected components as $\text{Sh}_{G,K}$ under the assumption that for $1 \leq i \leq n$ the group $J_{b,i}$ is not of compact type.

Recall that the connected components of our Shimura variety in characteristic zero have the following description (c.f. [**13**] 2.1.3):

$$\pi_0(\mathbf{Sh}_{U,\overline{\mathbb{Q}}}) = G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/U,$$

where $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$ with $G(\mathbb{R})_+$ the inverse image of the identity component (in the real topology) of $G^{\text{ad}}(\mathbb{R})$ under the natural map $G(\mathbb{R}) \to G^{\text{ad}}(\mathbb{R})$. Corollary 4.1.11 of [**48**] tells us that [4]

$$\pi_0(\text{Sh}_{G,\emptyset}) = \pi_0(\mathbf{Sh}_{K,\overline{\mathbb{Q}}})$$

and we will show that the natural map

$$\text{Sh}_{G,\emptyset,b} \to \text{Sh}_{G,K}$$

induces a bijection on connected components, where $\text{Sh}_{G,\emptyset,b}$ denotes the basic locus (the smallest Newton stratum). The main ingredient in the proof will be strong approximation and Rapoport Zink-uniformisation of the basic locus:

THEOREM 5.0.1 (Zhou). *Let $X_\mu(b)_\emptyset$ be the parahoric affine Deligne-Lusztig variety with $b \in B(G, X)$ the unique basic element. Then there is an isomorphism of perfect*

[4] The proof in loc. cit. seems to implicitly assume that $\mathscr{S}_{K,\overline{\mathbb{F}}_p}$ is geometrically normal, which is true in this case because K is very special (c.f. the proof of Proposition 4.6.28 of [**37**]). We therefore avoid using the result for the Shimura variety at Iwahori level.

schemes

$$I(\mathbb{Q})\backslash X_\mu(b)_\emptyset \times G(\mathbb{A}_f^p)/U^p \simeq \mathrm{Sh}_{G,\emptyset}(b),$$

where $I/\mathbb{Q}$ is an inner form of G which acts on $G(\mathbb{A}_f^p)$ via an isomorphism $G(\mathbb{A}_f^p) \simeq I(\mathbb{A}_f^p)$ and acts on $X_\mu(b)_\emptyset$ via an isomorphism $I_{\mathbb{Q}_p} \cong J_b(\mathbb{Q}_p)$. Moreover, the group $I(\mathbb{R})$ is compact mod centre.

Proof. Once we show that the basic locus contains a unique isogeny class, the result follows on the level of $\overline{\mathbb{F}}_p$-points follows from Proposition 7.7 of [**73**] as in the proof of Corollary 4.5.0.3. To get the statement on the level of perfect schemes, one can argue as in the proof of Lemma 7.2.12 of [**71**]. □

Proposition 5.0.2. *The basic locus $\mathrm{Sh}_{G,\emptyset}(b)$ contains a unique isogeny class.*

Our proof is similar to the proof of Corollary 7.2.16 of [**71**] and Proposition 6.11 of [**52**].

Proof. Suppose that $\mathrm{Sh}_{G,K,b}$ contains a unique isogeny class, then we would get an isomorphism

$$\mathrm{Sh}_{G,K,b} \simeq I(\mathbb{Q})\backslash X_\mu(b)_K \times G(\mathbb{A}_f^p)$$

by the above reasoning. On the level of $\overline{\mathbb{F}}_p$-points it would follow that

$$\hat{\mathrm{Sh}}_{G,\emptyset,b}(\overline{\mathbb{F}}_p) \simeq I(\mathbb{Q})\backslash X_\mu(b)_\emptyset(\overline{\mathbb{F}}_p) \times G(\mathbb{A}_f^p)$$

by the Cartesian diagram of basic loci

$$\begin{array}{ccccc} \hat{\mathrm{Sh}}_{G,\emptyset,b}(\overline{\mathbb{F}}_p) & \longrightarrow & \mathrm{Sht}_{\emptyset,\mu}(\overline{\mathbb{F}}_p) & = & \left[J_b(\mathbb{Q}_p)\backslash X_\mu(b)_\emptyset(\overline{\mathbb{F}}_p)\right] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sh}_{G,K,b}(\overline{\mathbb{F}}_p) & \longrightarrow & \mathrm{Sht}_{K,\mu}(\overline{\mathbb{F}}_p) & = & \left[J_b(\mathbb{Q}_p)\backslash X_\mu(b)_K(\overline{\mathbb{F}}_p)\right]. \end{array}$$

But we already know that the $\overline{\mathbb{F}}_p$-points of isogeny classes in $\mathrm{Sh}_{G,\emptyset,b}$ are also of this form, and therefore $\mathrm{Sh}_{G,\emptyset,b}$ cannot contain more than one isogeny class. As a corollary we find that $\hat{\mathrm{Sh}}_{G,\emptyset}(\overline{\mathbb{F}}_p) = \mathrm{Sh}_{G,\emptyset}(\overline{\mathbb{F}}_p)$ and therefore $\hat{\mathrm{Sh}}_{G,\emptyset} \simeq \mathrm{Sh}_{G,\emptyset}$ is an isomorphism of perfect schemes. So it remains to prove that $\mathrm{Sh}_{G,K,b}$ contains a unique isogeny class.

For this we are going to use results and notions from Chapter 5 about Shimura varieties at very special level, these do not depend on any previous results in this thesis until now. It follows from Theorem 5.2.0.1 that isogeny classes in $\mathrm{Sh}_{G,K}$ are parametrised by conjugacy classes of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$. Because (G, X) is of Hodge type such morphisms factor through $\mathfrak{P}$ and in fact through $\mathfrak{P}^L$ for some finite CM field $L \subset \overline{\mathbb{Q}}$.

Given $\phi : \mathfrak{P}^L \to \mathfrak{G}_G$ we obtain a Kottwitz triple $\mathfrak{t} = (\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta)$. Kottwitz triples are defined precisely in Section 5.1 and the Kottwitz triple associated to an admissible morphism ϕ is defined at the end of that section. For example $\gamma_0 \in G(\mathbb{Q})$ is, up to $G(\overline{\mathbb{Q}})$-conjugacy), given by $\phi^\Delta(\delta_n)$, where $\delta_n \in P^L(\mathbb{Q})$ is the distinguished elements discussed in Section 3.3. There are only finitely many conjugacy classes of admissible morphisms ϕ with the same equivalence class of Kottwitz triple $\mathfrak{t}$, the fibers of this map are in bijection with the set $Ш_G(\mathbb{Q}, I) \subset H^1(\mathbb{Q}, I)$ where $I = I_\phi = I_{\mathfrak{t}}$.

The Kottwitz triple $\mathfrak{t} = (\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta)$ associated to ϕ is basic precisely when the σ-conjugacy of δ is the basic element in $B(G, X)$. After replacing δ by a σ-conjugate, it follows from Section 4 of [**41**] that there is an $s \in \mathbb{Z}_{\geq 0}$ such that

$$\delta\sigma(\delta) \cdots \sigma^{s-1}(\delta) = (s\nu_b)(\pi),$$

where ν_b is the Newton cocharacter of b, which is central since b is basic. So it follows that

$$\delta\sigma(\delta) \cdots \sigma^{s-1}(\delta)$$

is central for some $s \in \mathbb{Z}_{\geq 0}$. Since γ_0 is conjugate to $\delta\sigma(\delta) \cdots \sigma^{s-1}(\delta)$ in $G(\overline{\mathbb{Q}}_p)$ by definition, it follows that γ_0 is central in $G(\mathbb{Q})$. Therefore the group I associated to the Kottwitz triple is an inner form of G. It follows from the definition that the group $Ш_G(\mathbb{Q}, I)$ is trivial and so that there is a unique conjugacy class of admissible ϕ with equivalence class of Kottwitz triples given by $(\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta)$.

It remains to explain why there is only one equivalence class of Kottwitz triples coming from such ϕ with δ basic. The image of γ_0 in $G^{\mathrm{ab}}(\mathbb{Q})$ is uniquely determined (up to torsion) by the admissibility of ϕ, using axiom A1 in Definition 3.2.0.1, and the kernel of

$$Z_G(\mathbb{Q}) \to G^{\mathrm{ab}}(\mathbb{Q})$$

is given by $Z_{G^{\mathrm{der}}}(\mathbb{Q})$, which is also torsion. It follows that if we have two Kottwitz triples $(\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta), (\gamma_0', (\gamma_\ell')_{\ell \neq p}, \delta')$ coming from two admissible morphisms ϕ, ϕ', that then $\gamma_0^{-1}\gamma_0'$ is torsion which means that the Kottwitz triples are equivalent by definition, see Definition 5.1.0.1. We conclude that there is a unique conjugacy class of admissible morphisms corresponding to basic isogeny classes, and hence by Theorem 5.2.0.1 that the basic locus contains a unique isogeny class. ☐

COROLLARY 5.0.3. *The morphism $\iota : \mathrm{Sh}_{G,\emptyset} \to \hat{\mathrm{Sh}}_{G,\emptyset}$ is an isomorphism over the basic locus.*

Assume from now on that $G^{\mathbf{ad}}$ is $\mathbb{Q}$-simple. Let $\rho : G^{\mathrm{sc}} \to G^{\mathrm{der}}$ be the simply connected cover. Then the classification of abelian type Shimura data in Appendix B of [**50**] tells us that G^{sc} is isomorphic to

$$G^{\mathrm{sc}} \simeq \mathrm{Res}_{F/\mathbb{Q}} H,$$

where $F/\mathbb{Q}$ is a totally real field and H/F is a connected reductive group that is absolutely simple. This implies that

$$I^{\mathrm{sc}} \simeq \mathrm{Res}_{F/\mathbb{Q}} H',$$

with H' an inner form of H. This gives us product decompositions

$$G^{\mathrm{sc}}_{\mathbb{Q}_p} = \prod_{\mathcal{O}_F \supset \mathfrak{p} | p} H_{\mathfrak{p}} = \prod_{i=1}^{n} G_i^{\mathrm{sc}} \qquad G^{\mathrm{ad}} = \prod_{i=1}^{n} G_i$$
$$I^{\mathrm{sc}}_{\mathbb{Q}_p} = \prod_{\mathcal{O}_F \supset \mathfrak{p} | p} H'_{\mathfrak{p}} = \prod_{i=1}^{n} I_i^{\mathrm{sc}} \qquad I^{\mathrm{ad}} = \prod_{i=1}^{n} I_i^{\mathrm{ad}} = \prod_{i=1}^{n} J_{b,i},$$

We are now ready to state the main result of this Section:

PROPOSITION 5.0.4. *Suppose that there exists an $1 \leq j \leq n$ such that μ is noncentral on G_j^{ad} and such that $I_j^{sc}(\mathbb{Q}_p)$ is not compact. Then the natural map*

$$\mathrm{Sh}_{G,\emptyset,b} \to \mathrm{Sh}_{G,K}$$

induces a bijection on connected components.

REMARK 5.0.5. The result is false for the modular curve because the supersingular locus is highly reducible. In this case I^{sc} is $\mathrm{SL}_1(D)$, where D is the unique quaternion algebra over $\mathbb{Q}$ that is ramified precisely at infinity and p. The group $I^{\mathrm{sc}}(\mathbb{Q}_p)$ is a unit ball in the unique nonsplit quaternion algebra over $\mathbb{Q}_p$, hence compact, so the assumptions are also not satisfied.

Recall that $G^{\mathrm{sc}}(\mathbb{R})$ is connected and so $\rho(G^{\mathrm{sc}}(\mathbb{Q})) \subset G(\mathbb{Q})_+$. Strong approximation for G^{sc}, using the fact that $G^{\mathrm{sc}}_{\mathbb{R}}$ has no compact factors, tells us that

$$G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/U = G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/UG^{\mathrm{sc}}(\mathbb{A}_f).$$

In order to compare this with the connected components of the basic locus, we need to understand what happens at p and what happens at infinity; the former is covered by the following lemma:

Lemma 5.0.6. *There is a natural isomorphism*

$$\frac{G(\mathbb{Q}_p)}{G^{sc}(\mathbb{Q}_p)U_p} \simeq \pi_1(G)_I^{\sigma}.$$

Proof. Recall that we have the Kottwitz homomorphism $\tilde{k}_G : G(L) \to \pi_1(G)_I$ with kernel given by (see Lemma 17 of the appendix of [**54**])

$$G^{\mathrm{sc}}(L) \cdot \mathcal{T}(\mathcal{O}_L) = G^{\mathrm{sc}}(L)\mathcal{G}_J(\mathcal{O}_L),$$

where $\mathcal{T}$ is the connected Néron model of a standard torus T of G. If we restrict $\tilde{k}_G$ to $G(\mathbb{Q}_p)$ we find that the kernel is given by (because $\mathcal{T}(\mathbb{Z}_p) \subset U_p$)

$$G^{\mathrm{sc}}(\mathbb{Q}_p)\mathcal{G}(\mathbb{Z}_p).$$

The result now follows from the fact that $G(\mathbb{Q}_p)$ surjects onto $\pi_1(G)_I^{\sigma}$, which is Lemma 5.18 (i) of [**73**]. □

We are going to use this lemma, in combination with the natural map

$$X_{\mu}(b)_J \to \pi_1(G)_I^{\sigma},$$

which induces a map $\pi_0(X_{\mu}(b)_J) \to \pi_1(G)_I^{\sigma}$. The main results of [**31**] describe the fibers of this map: First of all, from section 6.1 of op. cit. we get a Cartesian diagram

$$\begin{array}{ccc} X_{\mu}(b)_J & \longrightarrow & X_{\mu}(b)_J^{\mathrm{ad}} \\ \downarrow & & \downarrow \\ \pi_1(G)_I^{\sigma} & \longrightarrow & \pi_1(G^{\mathrm{ad}})_I^{\sigma}. \end{array}$$

The product decomposition

$$G^{\mathrm{ad}} = \prod_{i=1}^{n} G_i^{\mathrm{ad}}$$

induces a product decomposition

$$X_\mu(b)_J^{\mathrm{ad}} := \prod_{i=1}^n X_\mu(b)_{J,i}^{\mathrm{ad}}$$

and

$$\pi_1(G^{\mathrm{ad}})_I^\sigma = \prod_{i=1}^n \pi_1(G_i^{\mathrm{ad}})_I^\sigma$$

Moreover Theorem 6.3 of [**31**] tells us that

$$\pi_0\left(X_\mu(b)_{J,i}^{\mathrm{ad}}\right) \to \pi_1(G_i)_I^\sigma \tag{5.0.1}$$

is a bijection when μ is noncentral in G_i^{ad}. When μ is central in G_i^{ad}, then $X_\mu(b)_{J,i}^{\mathrm{ad}}$ is discrete and $J_{b,i}(\mathbb{Q}_p) = G_i(\mathbb{Q}_p)$-equivariantly isomorphic to

$$\frac{G_i(\mathbb{Q}_p)}{\mathcal{G}_{i,J}(\mathbb{Z}_p)}.$$

Moreover, in this case, the map (4.5.0.1) is given by the natural map

$$\frac{G_i(\mathbb{Q}_p)}{\mathcal{G}_{i,J}(\mathbb{Z}_p)} \to \frac{G_i(\mathbb{Q}_p)}{\mathcal{G}_{i,J}(\mathbb{Z}_p)G_i^{\mathrm{sc}}(\mathbb{Q}_p)} = \pi_1(G_i^{\mathrm{ad}})_I^\sigma.$$

In particular $G_i^{\mathrm{sc}}(\mathbb{Q}_p) = I_i^{\mathrm{sc}}(\mathbb{Q}_p)$ acts transitively on the fibers.

Proof of Proposition 4.5.0.4. Zhou's proof in Section 8 of [**73**] shows that $\mathrm{Sh}_{G,\emptyset,b}$ surjects onto $\pi_0(\mathrm{Sh}_{G,K})$, hence it suffices to show that the number of connected components are the same. By assumption we can choose $1 \le j \le n$ such that μ is noncentral on G_j^{ad}. Strong approximation for H' (Theorem 7.12 of [**56**]), away from the p-adic place of F corresponding to j, gives us

$$\begin{aligned} &I(\mathbb{Q})\backslash\left(\pi_0(X_\mu(b)_J) \times G(\mathbb{A}_f^p)\right)/U^p \simeq \\ &I(\mathbb{Q})\backslash\left(\pi_0(X_\mu(b)_J) \times G(\mathbb{A}_f^p)\right)/U^p G^{\mathrm{sc}}(\mathbb{A}_f^p)\prod_{i\neq j} I_i^{\mathrm{sc}}(\mathbb{Q}_p). \end{aligned} \tag{5.0.2}$$

By the discussion above, $\prod_{i\neq j} I_i^{\mathrm{sc}}(\mathbb{Q}_p)$ acts transitively on the fibers of

$$\pi_0(X_\mu(b)_J) \to \pi_1(G)_I^\sigma,$$

from which we conclude that (4.5.0.2) is in bijection with

$$\begin{aligned} &I(\mathbb{Q})\backslash\big(\pi_1(G)_I^\sigma \times G(\mathbb{A}_f^p)\big)/U^p G^{\mathrm{sc}}(\mathbb{A}_f^p) \\ &\simeq \frac{I(\mathbb{Q})}{I^{\mathrm{sc}}(\mathbb{Q})}\backslash\big(\pi_1(G)_I^\sigma \times G(\mathbb{A}_f^p)\big)/U^p G^{\mathrm{sc}}(\mathbb{A}_f^p), \end{aligned} \tag{5.0.3}$$

where the last equality follows from the fact that $I^{\mathrm{sc}}(\mathbb{Q})$ acts trivially on the left hand side. Applying Proposition 4.5.0.7 we see that (4.5.0.3) equals

$$\frac{G(\mathbb{Q})_+}{G^{\mathrm{sc}}(\mathbb{Q})}\backslash\left(\pi_1(G)_I^{\sigma}\times G(\mathbb{A}_f^p)\right)/U^pG^{\mathrm{sc}}(\mathbb{A}_f^p)\simeq\frac{G(\mathbb{Q})_+}{G^{\mathrm{sc}}(\mathbb{Q})}\backslash G(\mathbb{A}_f)/UG^{\mathrm{sc}}(\mathbb{A}_f), \tag{5.0.4}$$

where the second equality follows from Lemma 4.5.0.6. Now we unwind again and use strong approximation for G to deduce that (4.5.0.4) equals

$$G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/UG^{\mathrm{sc}}(\mathbb{A}_f)\simeq G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/U,$$

which is exactly equal to $\pi_0(\mathbf{Sh}_U)\simeq\pi_0(\mathrm{Sh}_{G,\emptyset})$. □

Proposition 5.0.7 (Borovoi). *There is a canonical isomorphism of abelian groups*

$$\frac{I(\mathbb{Q})}{I^{sc}(\mathbb{Q})}\simeq\frac{G(\mathbb{Q})_+}{G^{sc}(\mathbb{Q})}.$$

Proof. The following proof has been reproduced with permission from Mikhail Borovoi's Mathoverflow answer [**5**], we would like to thank him for his excellent answer.

We denote $K(G)=G(\mathbb{Q})_+/\rho G^{\mathrm{sc}}(\mathbb{Q})$. We compute $K(G)$; see the corollary below. It is clear from the corollary that $K(G)$ is canonically isomorphic to $K(I)$. Corollary 1 on page 121 of [**56**] tells us that $I^{\mathrm{ad}}(\mathbb{R})$ is connected and therefore $I(\mathbb{Q})=I(\mathbb{Q})_+$ which implies the lemma. We will use Section 3 of [**4**].

We consider the *crossed module* $(G^{\mathrm{sc}}\to G)$ and the *hypercohomology*

$$H^0_{\mathrm{ab}}(\mathbb{Q},G):=H^0(\mathbb{Q},G^{\mathrm{sc}}\to G),$$

where G is in degree 0; see [**4**]. By definition $H^0_{\mathrm{ab}}(\mathbb{Q},G)$ is a group. We consider the abelian crossed module $(Z^{\mathrm{sc}}\to Z)$, where $Z=Z(G)$ and $Z^{\mathrm{sc}}=Z(G^{\mathrm{sc}})$. The morphism of crossed modules

$$(Z^{\mathrm{sc}}\to Z)\longrightarrow(G^{\mathrm{sc}}\to G)$$

is a *quasi-isomorphism*, and hence it induces a bijection on hypercohomology, permitting us to identify $H^0_{\mathrm{ab}}(\mathbb{Q},G)$ with the abelian group $H^0(\mathbb{Q},Z^{\mathrm{sc}}\to Z)$. We conclude that $H^0_{\mathrm{ab}}(\mathbb{Q},G)$ is naturally an abelian group and that it does not change under inner twisting of G.

The short exact sequence

$$1\to(1\to G)\to(G^{\mathrm{sc}}\to G)\to(G^{\mathrm{sc}}\to 1)\to 1$$

(where $(G^{\mathrm{sc}} \to 1)$ is not a crossed module) induces a hypercohomology exact sequence

$$G^{\mathrm{sc}}(\mathbb{Q}) \to G(\mathbb{Q}) \to H^0_{\mathrm{ab}}(\mathbb{Q}, G) \to H^1(\mathbb{Q}, G^{\mathrm{sc}}),$$

where

$$\mathrm{ab}^0 \colon G(\mathbb{Q}) \to H^0_{\mathrm{ab}}(\mathbb{Q}, G)$$

is the *abelianization map*. This permits us to identify $G(\mathbb{Q})/\rho G^{\mathrm{sc}}(\mathbb{Q})$ with the kernel

$$\ker[H^0_{\mathrm{ab}}(\mathbb{Q}, G) \to H^1(\mathbb{Q}, G^{\mathrm{sc}})],$$

which is a subgroup of the abelian group $H^0_{\mathrm{ab}}(\mathbb{Q}, G)$. This kernel might change under inner twisting of G, because $H^1(\mathbb{Q}, G^{\mathrm{sc}})$ changes under inner twisting.

By definition, $G(\mathbb{R})_+ = Z(\mathbb{R}) \cdot \rho G^{\mathrm{sc}}(\mathbb{R})$, and hence

$$G(\mathbb{R})_+/\rho G^{\mathrm{sc}}(\mathbb{R}) = \mathrm{ab}^0(Z(\mathbb{R})) \subset \ker[H^0_{\mathrm{ab}}(\mathbb{R}, G) \to H^1(\mathbb{R}, G^{\mathrm{sc}})].$$

We see that $K(G) := G(\mathbb{Q})_+/\rho G^{\mathrm{sc}}(\mathbb{Q})$ can be identified with the preimage of $\mathrm{ab}^0(Z(\mathbb{R})) \subset H^0_{\mathrm{ab}}(\mathbb{R}, G)$ in $\ker[H^0_{\mathrm{ab}}(\mathbb{Q}, G) \to H^1(\mathbb{Q}, G^{\mathrm{sc}})]$.

LEMMA 5.0.8. *The preimage of* $\mathrm{ab}^0(Z(\mathbb{R})) \subset H^0_{\mathrm{ab}}(\mathbb{R}, G)$ *in* $\ker[H^0_{\mathrm{ab}}(\mathbb{Q}, G) \to H^1(\mathbb{Q}, G^{\mathrm{sc}})]$ *coincides with the preimage of* $\mathrm{ab}^0(Z(\mathbb{R}))$ *in* $H^0_{\mathrm{ab}}(\mathbb{Q}, G)$.

PROOF. Let $\xi \in H^0_{\mathrm{ab}}(\mathbb{Q}, G)$ lie in the preimage of

$$\mathrm{ab}^0(Z(\mathbb{R})) \subset \ker[H^0_{\mathrm{ab}}(\mathbb{R}, G) \to H^1(\mathbb{R}, G^{\mathrm{sc}})].$$

Then the image of ξ in $H^1(\mathbb{R}, G^{\mathrm{sc}})$ is trivial, and therefore, the image of ξ in $H^1(\mathbb{Q}, G^{\mathrm{sc}})$ lies in the kernel of the localization map

$$H^1(\mathbb{Q}, G^{\mathrm{sc}}) \to H^1(\mathbb{R}, G^{\mathrm{sc}}).$$

By the Hasse principle for simply connected groups, this kernel is trivial. Thus the image of ξ in $H^1(\mathbb{Q}, G^{\mathrm{sc}})$ is trivial, and hence ξ lies in the preimage of $\mathrm{ab}^0(Z(\mathbb{R}))$ in $\ker[H^0_{\mathrm{ab}}(\mathbb{Q}, G) \to H^1(\mathbb{Q}, G^{\mathrm{sc}})]$, as required. □

COROLLARY 5.0.9. *The abelianization map* $\mathrm{ab}^0 \colon G(\mathbb{Q}) \to H^0_{\mathrm{ab}}(\mathbb{Q}, G)$ *with kernel* $\rho G^{\mathrm{sc}}(\mathbb{Q})$ *induces a canonical isomorphism between the abelian groups* $K(G) := G(\mathbb{Q})_+/\rho G^{\mathrm{sc}}(\mathbb{Q})$ *and the preimage of* $\mathrm{ab}^0(Z(\mathbb{R})) \subset H^0_{\mathrm{ab}}(\mathbb{R}, G)$ *in* $H^0_{\mathrm{ab}}(\mathbb{Q}, G)$.

We see that $K(G)$ only depends on Z and $Z^{\mathrm{sc}} \to Z$ and therefore is the same for all inner forms. □

6. Connected components of unions of one-dimensional KR strata

In this section we will refine the results of the previous section, and show that certain unions of one-dimensional KR strata in the basic locus have the same number of connected components as the basic locus. It is good to point out that He and Zhou prove the results we used in the last section by studying these kinds of unions of one-dimensional KR strata, so in some sense these sections are in the wrong order. However our proof takes a slightly different perspective, which I prefer.

Let us put ourselves in the same situation as before, and let

$$\mathbb{S} = \coprod_{i=1}^{n} \mathbb{S}_i$$

denote the set of simple reflections of the Iwahori-Weyl group of $G_{\mathbb{Q}_p}$ coming from the product decomposition

$$G^{\mathrm{ad}}_{\mathbb{Q}_p} = \prod_{i=1}^{n} G_i.$$

In the rest of this section we fix a nonempty (!) subset $A \subset \mathbb{S}$ such that $A \cdot \tau \subset \mathrm{Adm}(\mu)$, which we will later specialise to be the set of simple reflections in the support of an element $w \in \tilde{W}_a$ such that $w\tau \in \mathrm{Adm}(\mu)$. From now we will specialise to the case $J = \emptyset$, and define

$$\mathrm{Sh}^{A}_{G,\emptyset} = \bigcup_{s \in A} \mathrm{Sh}_{G,\emptyset}(s\tau) \cup \mathrm{Sh}_{G,\emptyset}(\tau)$$

Lemma 6.0.1. *Suppose that for all $1 \leq i \leq n$ that $J_{b,i} := I^{ad}_{i,\mathbb{Q}_p}$ is not of compact type. Then $\mathrm{Sh}^{A}_{G,\emptyset}$ is contained in the basic locus of $\mathrm{Sh}_{G,\emptyset}$.*

Proof. Proposition 5.6 of **[20]** tells us that $\mathrm{Sh}_{G,\emptyset}(s\tau)$ is contained in the basic locus if and only if $\tilde{W}_{\mathrm{Supp}_{\sigma}(s)}$ is finite, where

$$\mathrm{Supp}_{\sigma}(s) = \bigcup_{n \in \mathbb{Z}} (\tau\sigma)^n s.$$

Choose $1 \leq i \leq n$ such that $s \in \mathbb{S}_i$, then by the above $s\tau$ is basic if and only if $\mathrm{Supp}_{\sigma}(s)$ does not contain a connected component of $\mathbb{S}_i$. But since σ acts transitively on the connected components of $\mathbb{S}_i$, that happens if and only if

$$\mathrm{Supp}_{\sigma}(s) \neq \mathbb{S}_i.$$

If $\mathrm{Supp}_\sigma(s) = \mathbb{S}_i$, then $(\tau\sigma)$ acts transitively on $\mathbb{S}_i$, which happens if and only if $J_{b,i}(\mathbb{Q}_p)$ is compact. Indeed, the action of $\sigma' = \tau\sigma$ on $\mathbb{S}$ corresponds to the action of Frobenius on the inner form I_i^{ad} of G_i^{ad} and such an action can only be transitive if G_i^{ad} is of type A and if $I_i^{\mathrm{ad}}(\mathbb{Q}_p)$ is compact. But $I_i^{\mathrm{ad}}(\mathbb{Q}_p) = J_{b,i}(\mathbb{Q}_p)$ is not compact by assumption. □

We now state the main result of this section:

Proposition 6.0.2. *Suppose that for all $1 \leq i \leq n$ the group $J_{b,i}$ is not of compact type. Moreover assume that there is an $1 \leq j \leq n$ such that $\mathrm{Supp}_\sigma(A \cap \mathbb{S}_j) = \mathbb{S}_j$. Then $\mathrm{Sh}^A_{G,\emptyset} \to \mathrm{Sh}_{G,K}$ induces a bijection on connected components.*

We start by collecting some notation before we will state our main local result, to simplify notation we will write $X_\emptyset$ for $X_\mu(b)_\emptyset$. For $s \in A$ we will write $X_\emptyset(s)$ for the locally closed subset of $X_\mu(b)_\emptyset$ corresponding to $s\tau \in \mathrm{Adm}(\mu)$, their union will be denoted by $X_\emptyset(A)$. There are obvious analogues when we replace G by G^{ad}, which will be denoted by adding the superscript ad. The decomposition

$$\mathbb{S} = \mathbb{S}_1 \coprod \cdots \coprod \mathbb{S}_n$$

induces $A = A_1 \coprod \cdots \coprod A_n$. This allows us to define

$$X_{\emptyset,i}(A_i)^{\mathrm{ad}}$$

for $1 \leq i \leq n$, using the product decomposition

$$X_\emptyset^{\mathrm{ad}} = \prod_{i=1}^n X_{\emptyset,i}^{\mathrm{ad}}.$$

The following Lemma is implicit in Section 6 of [**31**].

Lemma 6.0.3. *Choose $1 \leq j \leq n$ such that $\mathrm{Supp}_\sigma(A_j) = \mathbb{S}_j$, then*

$$X_{j,\emptyset}^{ad}(A_j) \to X_{\emptyset,j}^{ad}$$

induces a bijection on connected components.

Proof. The fact that $A \cap \mathbb{S}_j \neq \emptyset$ implies that μ is nontrivial on G_j^{ad} and so (by Theorem 6.3 of [**31**])

$$\pi_0(X_{\emptyset,j}^{\mathrm{ad}}) \simeq \pi_1(G_j^{\mathrm{ad}})_I^\sigma.$$

So it now suffices to prove that the natural map

$$\pi_0(X^{\mathrm{ad}}_{j,\emptyset}(A)) \simeq \pi_1(G^{\mathrm{ad}}_j)^{\sigma}_I$$

is a bijection, it is a surjection by because it is equivariance under the action of the twisted centraliser.

Let $s \in A_j$ and let $K_s \subset \mathbb{S}_j$ be the $\tau\sigma$-orbit of s, then the assumption that $\tau\sigma$ does not act transitively on $\mathbb{S}_j$ (because I^{ad}_j is not of compact type) tells us that $\tilde{W}_{K_s}$ is finite. It follows from Theorem 4.8 of [**28**] and its proof that the image of $X^{\mathrm{ad}}_{j,\emptyset}(\leq s\tau)$ under the forgetful map $X^{\mathrm{ad}}_{j,\emptyset} \to X^{\mathrm{ad}}_{j,K_s}$ is given by $X^{\mathrm{ad}}_{i,K_s}(\tau)$. Moreover, the fibers of the projection map

$$X^{\mathrm{ad}}_{j,\emptyset}(\leq s\tau) \to X^{\mathrm{ad}}_{j,K_s}(\tau)$$

are classical Deligne-Lusztig varieties

$$Y(\leq s) \subset (\overline{\mathcal{I}^{\mathrm{ad}}_{j,K_s}})^{\mathrm{red}}/B,$$

where $\mathcal{I}^{\mathrm{ad}}_{j,K_s}$ is the group scheme over $\mathbb{Z}_p$ associated to the $(\tau\sigma)$-stable type $K_s \subset \mathbb{S}_i$ for the group I^{ad}_i and B is a Borel subgroup (the image of $\mathcal{I}^{\mathrm{ad}}_{j,\emptyset}$). Since $\{s\} \subset K_s$ is not contained in a proper $(\tau\sigma)$-stable subset of K_s by construction, Theorem 1.1 of [**19**] tells us that $\overline{Y(s)}$ is connected. Theorem 3.5 of [**28**] tells us that $I^{\mathrm{ad}}_i(\mathbb{Q}_p)$ acts transitively on $X^{\mathrm{ad}}_{j,K_s}(\tau)$, which gives an identification

$$\pi_0\left(X^{\mathrm{ad}}_{j,\emptyset}(\leq s\tau)\right) \cong \frac{I^{\mathrm{ad}}_i(\mathbb{Q}_p)}{I^{\mathrm{ad}}_{j,K_s}(\mathbb{Z}_p)}, \tag{6.0.1}$$

where $I_{j,K_s}(\mathbb{Z}_p)$ is the parahoric subgroup of $I^{\mathrm{ad}}(\mathbb{Q}_p)$ corresponding to K_s. Similarly, we can identify

$$X^{\mathrm{ad}}_i(\tau) \simeq \frac{I^{\mathrm{ad}}_{j,\emptyset}(\mathbb{Q}_p)}{I^{\mathrm{ad}}_{j,\emptyset}(\mathbb{Z}_p)}. \tag{6.0.2}$$

Moreover, the map

$$X^{\mathrm{ad}}_{j,\emptyset}(\tau) \simeq \pi_0(X^{\mathrm{ad}}_{j,\emptyset}(\tau)) \to \pi_0\left(X^{\mathrm{ad}}_{j,\emptyset}(\leq s\tau)\right)$$

is given by the natural map

$$\beta : \frac{I^{\mathrm{ad}}_j(\mathbb{Q}_p)}{I^{\mathrm{ad}}_{j,\emptyset}(\mathbb{Z}_p)} \to \frac{I^{\mathrm{ad}}_j(\mathbb{Q}_p)}{I^{\mathrm{ad}}_{j,K_s}(\mathbb{Z}_p)}$$

coming from the inclusion $\mathcal{I}_{j,\emptyset} \subset \mathcal{I}_{j,K_s}$. Define a graph Γ with vertices given by (4.6.0.2), with edges given by (4.6.0.1) and with two vertices x, y connected by the edge $\beta(x)$ if $\beta(x) = \beta(y)$. We are going to show that the connected components of this graph are in bijection with $\pi_1(G_j^{\mathrm{ad}})_I^\sigma$.

If H is any connected reductive group over $\mathbb{Q}_p$ and $\mathcal{H}/\mathbb{Z}_p$ is a parahoric model, then there is a bijection

$$\frac{H(\mathbb{Q}_p)}{\mathcal{H}(\mathbb{Z}_p)} \simeq \frac{\tilde{\mathcal{H}}(\mathbb{Z}_p)}{\mathcal{H}(\mathbb{Z}_p)} \times \frac{H(\mathbb{Q}_p)}{\tilde{\mathcal{H}}(\mathbb{Q}_p)},$$

where $\tilde{\mathcal{H}}$ is the Bruhat-Tits stabiliser group scheme of which H is the identity component. This means that $\tilde{H}(\mathbb{Z}_p)$ is the stabiliser in $H(\mathbb{Q}_p)$ of a simplex of the Bruhat-Tits building of H^{ad}. Moreover there is a natural isomorphism

$$\frac{\tilde{\mathcal{H}}(\mathbb{Z}_p)}{\mathcal{H}(\mathbb{Z}_p)} \simeq \pi_1(G_j^{\mathrm{ad}})_I^\sigma.$$

It follows that (4.6.0.2) is a product of $\pi_1(G_j^{\mathrm{ad}})_I^\sigma$ and the set of alcoves in the aforementioned building, and (4.6.0.1) is a product of $\pi_1(G_j^{\mathrm{ad}})_I^\sigma$ and the set of codimension one facets of type K_s. Moreover, the vertices corresponding to two alcoves are connected by an edge if and only if the (closures of the) alcoves intersect in a codimension one facet.

The fact that $\mathrm{Supp}_\sigma(A_j) = \mathbb{S}_j$ tells us A_j contains an element in every $(\tau\sigma)$-orbit, so that the subsets K_s for $s \in A_j$ are precisely the orbits of simple reflections in $\mathbb{S}$ under $\tau\sigma$. It is clear that these correspond to all parahoric subgroups of I_j^{sc} that strictly contain the Iwahori subgroup and do not contain any other parahoric subgroups. In other words, these correspond to codimension one facets of the building. This means that Γ is isomorphic to the product of $\pi_1(G_j^{\mathrm{ad}})_I^\sigma$ and the adjacency graph for the set of alcoves in the building. Since the adjacency graph for alcoves in the building is connected, it follows that $\pi_0(\Gamma) = \pi_1(G_j^{\mathrm{ad}})_I^\sigma$. □

Proof of Proposition 4.6.0.2. As before it follows from Zhou's proof of axiom 5 in Section 8 of [**73**] that $\mathrm{Sh}_{G,\emptyset}^A$ surjects onto $\pi_0(\mathrm{Sh}_{G,K})$, hence it suffices to show that the number of connected components are the same. Lemma 4.6.0.1 tells us that $\mathrm{Sh}_{G,\emptyset}^A$ is contained in the basic locus and so it is isomorphic to

$$I(\mathbb{Q})\backslash X_\emptyset(A) \times I(\mathbb{A}_f^p)/U^p.$$

As in the previous section there is a Cartesian diagram

$$\begin{array}{ccc} \pi_0(X_\emptyset(A)) & \longrightarrow & \pi_0(X_\emptyset(A))^{\mathrm{ad}} \\ \downarrow & & \downarrow \\ \pi_1(G)_I^\sigma & \longrightarrow & \pi_1(G^{\mathrm{ad}})_I^\sigma. \end{array}$$

Let $1 \leq j \leq n$ be such that $\mathrm{Supp}_\sigma(A \cap \mathbb{S}_j) = \mathbb{S}_j$ (which exists by assumption), then Lemma 4.6.0.3 tells us that

$$\pi_0(X_{j,\emptyset}(A)^{\mathrm{ad}}) \to \pi_1(G_j^{\mathrm{ad}})_I^\sigma$$

is a bijection. For all $1 \leq i \leq n$ with $i \neq j$, we will show that the fibers of

$$\pi_0(X_{i,\emptyset}(A)^{\mathrm{ad}}) \to \pi_i(G_j^{\mathrm{ad}})_I^\sigma \tag{6.0.3}$$

are not too big, so that they can be dealt with using strong approximation. The inclusion

$$X_{i,\emptyset}(\tau)^{\mathrm{ad}} \to X_{i,\emptyset}(A)^{\mathrm{ad}}$$

induces a surjection on π_0, because every curve in $X_{i,\emptyset}(A)^{\mathrm{ad}}$ intersects $X_{i,\emptyset}(\tau)^{\mathrm{ad}}$ by Theorem 4.1 of [**31**]. Recall from the proof of Lemma 4.6.0.3 that

$$X_{i,\emptyset}(\tau)^{\mathrm{ad}} = \frac{I_i^{\mathrm{ad}}(\mathbb{Q}_p)}{I_{i,\emptyset}^{\mathrm{ad}}(\mathbb{Z}_p)},$$

which means that the fibers of (4.6.0.3) receive a surjection from the fibers of

$$\frac{I_i^{\mathrm{ad}}(\mathbb{Q}_p)}{I_{i,\emptyset}^{\mathrm{ad}}(\mathbb{Z}_p)} \to \pi_1(G_i)_I^\sigma \simeq \pi_1(I_i)_I^\sigma.$$

As in the previous section, these fibers are acted on transitively by $I_i^{\mathrm{sc}}(\mathbb{Q}_p)$ which implies that the fibers of

$$\pi_0(X_\emptyset(A)) \to \pi_1(G)_I^\sigma$$

are acted on transitively by $\prod_{i \neq j} I_i^{\mathrm{sc}}(\mathbb{Q}_p)$. The rest of the proof is the same as the proof of Proposition 4.5.0.4, using the fact that $I_j^{\mathrm{sc}}(\mathbb{Q}_p)$ is not compact. □

7. Connectedness of closures of KR strata

In this section we will show that for nonbasic $w \in \mathrm{Adm}(\mu)$, the KR stratum $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ is 'connected', by which we mean that it has the same number of connected components as $\mathrm{Sh}_{G,K}$. Recall that we have assumed that G^{ad} is $\mathbb{Q}$-simple.

THEOREM 7.0.1. *Suppose that G^{ad} is $\mathbb{Q}$-simple, that for $1 \leq i \leq n$ the group $J_{b,i}$ is not of compact type, and that either $\mathbf{Sh}_U$ is proper or that Conjecture 4.7.0.5 holds. Then for w nonbasic, the scheme $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ has the same number of connected components as $\mathrm{Sh}_{G,K}$.*

COROLLARY 7.0.2. *The closed immersion $\hat{\mathrm{Sh}}_{G,\emptyset} \hookrightarrow \mathrm{Sh}_{G,\emptyset}$ is an isomorphism.*

PROOF OF COROLLARY 4.7.0.2. We know that $\hat{\mathrm{Sh}}_{G,\emptyset}$ is a union of $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ for $w \in \mathrm{Adm}(\mu)$ of maximal length, therefore it is enough to prove that $\mathrm{Sh}_{G,\emptyset}(\leq w) \to \hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ is an isomorphism for nonbasic w (since $\hat{\mathrm{Sh}}_{G,\emptyset} \hookrightarrow \mathrm{Sh}_{G,\emptyset}$ is an isomorphism over the basic locus by Corollary 4.5.0.3 and the basic locus is closed). Because $\mathrm{Sh}_{G,\emptyset}(\leq w)$ and $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ are locally integral and equidimensional of the same dimension by the local model diagram, it suffices to prove that they have the same number of connected components. But we know that

$$\mathrm{Sh}_{G,\emptyset}(\leq w) \twoheadrightarrow \pi_0(\mathrm{Sh}_{G,K})$$

is surjective (this is true for $w = \tau$ by the arguments in Section 8 of **[73]**, and the general case follows from the proof of Theorem 4.1 of **[27]**). The closed immersion $\mathrm{Sh}_{G,\emptyset}(\leq w) \to \hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ implies that (the last equality follows from Theorem 4.7.0.1)

$$|\pi_0(\mathrm{Sh}_{G,\emptyset}(\leq w))| \leq |\pi_0(\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w))| = |\pi_0(\mathrm{Sh}_{G,K})|,$$

hence we are done. □

PROOF OF THEOREM 4.7.0.1. Our proof is a generalisation of the connectedness argument of Section 7 of **[21]**. Write $w = v\tau$ and let $v = s_1 \cdots s_n$ be a reduced expression of v, then we define

$$Y_w = \hat{\mathrm{Sh}}_{G,\emptyset}(s_1\tau) \cup \cdots \cup \hat{\mathrm{Sh}}_{G,\emptyset}(s_n\tau) \cup \hat{\mathrm{Sh}}_{G,\emptyset}(\tau),$$

which is the union of all KR strata in $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ of dimension at most one. It then suffices to prove the following two results, because of the inequalities

$$|\pi_0(\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w))| \geq |\pi_0(\mathrm{Sh}_{G,K})|$$

and (these follow from Proposition 4.7.0.3 and Lemma 4.7.0.4 respectively)

$$|\pi_0(\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w))| \leq |\pi_0(Y_w)| = |\pi_0(\mathrm{Sh}_{G,K})|.$$

□

PROPOSITION 7.0.3. *Suppose that Conjecture 4.7.0.5 holds or that* $\mathbf{Sh}_U$ *is proper. Let* $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ *be the closure of a KR stratum of* $\hat{\mathrm{Sh}}_{G,\emptyset}$ *and let* Z *be a connected component of* $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$*, then* Z *intersects* Y_w*.*

LEMMA 7.0.4. *Under the assumptions of Theorem 4.7.0.1, the closed subscheme* Y_w *has the same number of connected components as* $\mathrm{Sh}_{G,K}$*.*

CONJECTURE 7.0.5. *Let* V *be an irreducible component of the closure of an EKOR stratum in* $\mathrm{Sh}_{G,K}$*, then* V *intersects the unique* 0*-dimensional EKOR stratum* $\mathrm{Sh}_{G,K}\{\tau\}$*.*

REMARK 7.0.6. The conjecture follows from Proposition 6.20 of [**70**], combined with Theorem 1.2 of [**2**], when K is hyperspecial.

PROOF OF LEMMA 4.7.0.4. In order to show that $Y_w \to \mathrm{Sh}_{G,K}$ induces a bijection on connected components, it suffices to prove that

$$A = \mathrm{Supp}(w)$$

satisfies the assumptions of Proposition 4.6.0.2. Proposition 5.6 of [**20**] tells us that w is nonbasic if and only if

$$\tilde{W}_{\mathrm{Supp}_\sigma(w)}$$

is infinite, which only happens if there is an $1 \leq j \leq n$ such that $\mathbb{S}_j \subset \mathrm{Supp}_\sigma(w)$. The assumptions of Theorem 4.7.0.1 tell us that $J_{b,i}(\mathbb{Q}_p)$ is not compact and so we may apply 4.6.0.2. □

8. Proof of the main technical result

In this section we prove Proposition 4.7.0.3, we start by proving a lemma:

LEMMA 8.0.1. *Proposition 4.7.0.3 holds for* Z *if there exists a KR stratum* $\hat{\mathrm{Sh}}_{G,\emptyset}(x)$ *such that* $Z \cap \hat{\mathrm{Sh}}_{G,\emptyset}(x)$ *is nonempty, such that* $\overline{\hat{\mathrm{Sh}}_{G,\emptyset}(x)}$ *is proper and such that for every* $x' \leq x$ *the KR stratum* $\hat{\mathrm{Sh}}_{G,\emptyset}(x')$ *is quasi-affine.*

PROOF. Let $\hat{\mathrm{Sh}}_{G,\emptyset}(x)$ as in the statement of the lemma. Then there is an $x' \leq x$ of minimal length such that $\hat{\mathrm{Sh}}_{G,\emptyset}(x') \cap Z \neq \emptyset$, and it suffices to prove that this length is equal to zero. The minimality tells us that

$$\hat{\mathrm{Sh}}_{G,\emptyset}(x') \cap Z = \hat{\mathrm{Sh}}_{G,\emptyset}(\leq x') \cap Z, \tag{8.0.1}$$

since $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq x') \setminus \hat{\mathrm{Sh}}_{G,\emptyset}(x')$ is a union of KR strata associated to x'' of length strictly smaller than x'. Next, we note that $Z \cap \hat{\mathrm{Sh}}_{G,\emptyset}(x')$ is a union of connected components of $\hat{\mathrm{Sh}}_{G,\emptyset}(x')$, because $\hat{\mathrm{Sh}}_{G,\emptyset}(x') \subset \hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ and so connected components of $\hat{\mathrm{Sh}}_{G,\emptyset}(x')$ are either disjoint from Z or contained in Z. Since $\hat{\mathrm{Sh}}_{G,\emptyset}(x')$ is quasi-affine, we find that $\hat{\mathrm{Sh}}_{G,\emptyset}(x') \cap Z$ is quasi-affine. Moreover (4.8.0.1) implies that $\hat{\mathrm{Sh}}_{G,\emptyset}(x') \cap Z \subset \hat{\mathrm{Sh}}_{G,\emptyset}(x)$ is closed, hence proper. Therefore, $\hat{\mathrm{Sh}}_{G,\emptyset}(x') \cap Z$ is zero-dimensional, and since it is a union of connected components of $\hat{\mathrm{Sh}}_{G,\emptyset}(x')$, we find that x' has length zero. □

Proof of Proposition 4.7.0.3. The proof of Proposition 6.11 of [**27**] tells us that the image of $\hat{\mathrm{Sh}}_{G,\emptyset}(w)$ under the forgetful map $\pi : \hat{\mathrm{Sh}}_{G,\emptyset} \to \mathrm{Sh}_{G,K}$ is a union of EKOR strata. To elaborate, the paper [**27**] postulates a set of axioms for Shimura varieties of parahoric level (now known as the He-Rapoport axioms) and deduces various consequences from them. The scheme $\hat{\mathrm{Sh}}_{G,\emptyset}$ together with its forgetful map to $\mathrm{Sh}_{G,K}$ satisfies these axioms by construction, and therefore we can use the results proven from them. For this particular result, we remark that KR strata and EO strata on $\hat{\mathrm{Sh}}_{G,\emptyset}$ and $\mathrm{Sh}_{G,K}$ respectively are defined as the inverse images of KR strata and EO strata in $\mathrm{Sht}_{\emptyset,\mu}$ and $\mathrm{Sht}_{K,\mu}$ respectively. Therefore it suffices to prove that the forgetful map

$$\mathrm{Sht}_{\emptyset,\mu} \to \mathrm{Sht}_{K,\mu}$$

sends KR strata to unions of EO strata, and this is what is proven in Proposition 6.11 of [**27**]. To be precise, they prove the result on the level of $\overline{\mathbb{F}}_p$-points, but this is enough for our purposes since locally closed subsets of $\mathrm{Sh}_{G,K}$ are determined by their $\overline{\mathbb{F}}_p$-points.

It follows that the image of $\hat{\mathrm{Sh}}_{G,\emptyset}(\leq w)$ is a union of closures of EKOR strata (by properness of π), and $\pi(Z)$ is a union of irreducible components of closures of EKOR strata, Conjecture 4.7.0.5 tells us that $\pi(Z)$ intersects the zero-dimensional EKOR stratum $\mathrm{Sh}_{G,K}\{\tau\}$, and therefore Z intersects $\pi^{-1}(\mathrm{Sh}_{G,K}\{\tau\})$. The inverse image of $\pi^{-1}(\mathrm{Sh}_{G,K}\{\tau\})$ is proper because $\mathrm{Sh}_{G,K}\{\tau\}$ is finite and π is proper. It follows from Section 6.4 of [**27**] (as explained before) that this inverse image is a union of closures of KR strata. This means that the assumptions of Lemma 4.8.0.1 would be satisfied if we knew quasi-affineness of KR strata.

If $\mathbf{Sh}_U$ is proper, then Corollary 4.1.7 of [**48**] tells us that $\mathrm{Sh}_{G,K}$ is proper. It follows from this that $\hat{\mathrm{Sh}}_{G,\emptyset}$ is proper, and therefore by Lemma 4.8.0.1 it is enough to show that KR strata in $\hat{\mathrm{Sh}}_{G,\emptyset}$ are quasi-affine in this case.

Theorem 3.5.9 of [**64**] proves that KR strata in $\mathrm{Sh}_{G,\emptyset}$ are quasi-affine, which is not enough for our purposes. Our proof that KR strata in $\hat{\mathrm{Sh}}_{G,\emptyset}$ are quasi-affine in fact gives an alternative proof of their result.

LEMMA 8.0.2. *The morphism $f : \hat{\mathrm{Sh}}_{G,\emptyset} \to \mathrm{Sh}_{\mathrm{GSp},\mathcal{P}}$ constructed in the proof of Proposition 4.3.0.1 is finite.*

PROOF. By the proof of Proposition 4.3.0.1. there is commutative diagram

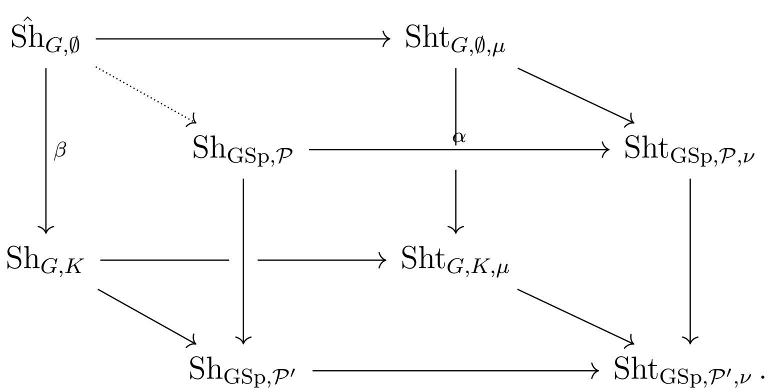
$$\begin{CD} \hat{\mathrm{Sh}}_{G,\emptyset} @>{f}>> \mathrm{Sh}_{\mathrm{GSp},\mathcal{P}} \\ @V{\beta}VV @VV{\alpha}V \\ \mathrm{Sh}_{G,K} @>{f'}>> \mathrm{Sh}_{\mathrm{GSp},\mathcal{P}'} \end{CD}$$

with f' finite. It suffices to show that f is quasi-finite, since its source and target are proper over $\mathrm{Sh}_{\mathrm{GSp},\mathcal{P}'}$. We will show that for $x \in \mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p)$ with image $y = f'(x)$ the map

$$f : \beta^{-1}(x) \to \alpha^{-1}(y)$$

is injective, which implies the quasi-finiteness. Indeed this implies that $z \in \alpha^{-1}(y)$ has at most one pre-image in $\beta^{-1}(x)$, and there are only finitely many possible x for which $\beta^{-1}(x)$ can map to $\alpha^{-1}(y)$ by quasi-finiteness of f'.

To prove this injectivity on fibers we return to the commutative cube from the proof Proposition 4.3.0.1, which we reproduce below for convenience.



Since the square involving the four objects with subscript G and the square involving the four objects with subscript GSp are Cartesian (by the moduli description, see

Claim 4.3.0.2), the statement on fibers can instead be proven for the square

$$\begin{array}{ccc} \mathrm{Sht}_{G,\emptyset,\mu} & \longrightarrow & \mathrm{Sht}_{\mathrm{GSp},\mathcal{P},\nu} \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{G,K,\mu} & \longrightarrow & \mathrm{Sht}_{\mathrm{GSp},\mathcal{P}',\nu} . \end{array}$$

Moreover, since the spaces of shtukas of type μ respectively ν sit inside the spaces of all shtukas, we can reduce to showing the statement (injectivity of the map on fibers) for

$$\begin{array}{ccc} \mathrm{Sht}_{G,\emptyset} & \longrightarrow & \mathrm{Sht}_{\mathrm{GSp},\mathcal{P}} \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{G,K} & \longrightarrow & \mathrm{Sht}_{\mathrm{GSp},\mathcal{P}'} . \end{array}$$

Recall from the proof of Corollary 2.4.0.2 the Cartesian diagrams (equation (2.4.0.1)

$$\begin{array}{ccc} \mathrm{Sht}_{G,\emptyset} & \longrightarrow & \mathrm{Sht}_{G,K} \\ \downarrow & & \downarrow \\ \mathbf{B}L^+\mathcal{G}_\emptyset & \longrightarrow & \mathbf{B}L^+\mathcal{G}_K . \end{array} \qquad \begin{array}{ccc} \mathrm{Sht}_{\mathrm{GSp},\mathcal{P}} & \longrightarrow & \mathrm{Sht}_{\mathrm{GSp},\mathcal{P}'} \\ \downarrow & & \downarrow \\ \mathbf{B}L^+\mathcal{P} & \longrightarrow & \mathbf{B}L^+\mathcal{P}' \end{array}$$

which fit into a commutative cube that we will not draw. This reduces the problem to showing the injectivity statement for the diagram

$$\begin{array}{ccc} \mathbf{B}L^+\mathcal{G}_\emptyset & \longrightarrow & \mathbf{B}L^+\mathcal{P} \\ \downarrow & & \downarrow \\ \mathbf{B}L^+\mathcal{G}_K & \longrightarrow & \mathbf{B}L^+\mathcal{P}', \end{array}$$

which comes down to showing injectivity of the map of partial flag varieties

$$\frac{L^+\mathcal{G}_K}{L^+\mathcal{G}_\emptyset} \to \frac{L^+\mathcal{P}'}{L^+\mathcal{P}}.$$

Finally, this is true because the intersection of $L^+\mathcal{P}$ with LG is equal to $L^+\mathcal{G}_\emptyset$ by construction (see Section 8 of **[73]** for the construction) and therefore the intersection of $L^+\mathcal{P}$ with $L^+\mathcal{G}_K$ is also equal to $L^+\mathcal{G}_\emptyset$. □

We can assume that $\mathcal{P}$ is an Iwahori subgroup, because all that is needed in Section 7 of **[73]** is that $\mathcal{P}(\breve{\mathbb{Z}}_p) \cap G(\mathbb{Q}_p) = \mathcal{G}_\emptyset(\breve{\mathbb{Z}}_p)$. Moreover, the image of $\mathcal{G}_\emptyset(\mathbb{Z}_p)$ in $\mathrm{GSp}(\mathbb{Q}_p)$ is automatically contained in an Iwahori subgroup. Indeed, consider the following diagram of perfect group schemes over $\overline{\mathbb{F}}_p$

$$\begin{array}{ccccc} \mathcal{G}_\emptyset & \dashrightarrow & \mathcal{P}_\emptyset & \longrightarrow & B \\ & \searrow & \downarrow & & \downarrow \\ & & \mathcal{P} & \longrightarrow & (\overline{\mathcal{P}})^{\mathrm{red}} \end{array}$$

where $\mathcal{P}_\emptyset$ is an Iwahori subgroup and B is a Borel subgroup of $(\overline{\mathcal{P}})^{\mathrm{red}}$. The square is Cartesian by (1.5.0.1) and the dotted arrow exists because the special fiber of $\mathcal{G}_\emptyset$ is solvable and therefore lands inside a Borel subgroup of $(\overline{\mathcal{P}})^{\mathrm{red}}$.

The morphism $f : \hat{\mathrm{Sh}}_{G,\emptyset} \to \mathrm{Sh}_{\mathrm{GSp},\mathcal{P}}$ is compatible with the maps $\mathrm{Sht}_{\emptyset,\mu} \to \mathrm{Sht}_{\mathrm{GSp},M_P,\nu}$, and we claim that this implies that f is compatible with KR stratifications. If we start with a modification $\beta : \mathcal{E} \to \mathcal{E}'$ of $\mathcal{G}_\emptyset$ torsors over $\overline{\mathbb{F}}_p$ of relative position λ, then there is an $f(\lambda)$ such that the induced modification of $\mathcal{P}$-torsors has relative position $f(\lambda)$. Indeed, the double coset

$$\mathcal{G}_\emptyset(\mathcal{O}_L)\lambda\mathcal{G}_\emptyset(\mathcal{O}_L) \subset G(L)$$

is mapped to a unique double coset

$$\mathcal{P}(\mathcal{O}_L)f(\lambda)\mathcal{P}(\mathcal{O}_L) \subset \mathrm{GSp}(L).$$

Theorem 5.4 of **[21]** tells us that $\mathrm{Sh}_{\mathrm{GSp},\mathcal{P}}(v)$ is quasi-affine, and because f is a finite morphism, we find that $f^{-1}(\mathrm{Sh}_{\mathrm{GSp},\mathcal{P}}(v))$ is quasi-affine. It follows that $\hat{\mathrm{Sh}}_{G,\emptyset}(w)$ is quasi-affine, because it is locally closed in something quasi-affine. □

9. Proofs of the main results for Hodge type Shimura varieties

In this section we will deduce Theorem 4.1.0.1 and Theorem 4.1.0.4 from 4.7.0.2.

PROOF OF THEOREM 4.1.0.1. We will first prove Theorem 4.1.0.1 when $J = \emptyset$, i.e., at Iwahori level. Theorem 4.7.0.1 gives us a Cartesian diagram

(9.0.1)
$$\begin{array}{ccc} \mathrm{Sh}_{G,\emptyset} & \longrightarrow & \mathrm{Sht}_{\emptyset,\mu} \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{G,K} & \longrightarrow & \mathrm{Sht}_{K,\mu} \end{array}$$

which in turn gives us a Cartesian diagram on the level of k-points. The assumption that the theorem holds for $\mathrm{Sh}_{G,K}$ tells us that

$$\mathrm{Sh}_{G,K}(\overline{\mathbb{F}}_p) = \coprod_{\phi} I_\phi(\mathbb{Q})\backslash X_p(\phi) \times X^p(\phi)/U^p,$$

where $X_p(\phi) = X_\mu(b_\phi)_K(\overline{\mathbb{F}}_p)$ is the set of $\overline{\mathbb{F}}_p$-points of an affine Deligne-Lusztig variety. Moreover, Lemma 2.8.0.2 tells us that

$$\mathrm{Sht}_{K,\mu}(\overline{\mathbb{F}}_p) = \coprod_{b \in B(G,X)} [J_{b,I}(\mathbb{Q}_p)\backslash X_\mu(b_\phi)_K(\overline{\mathbb{F}}_p)]$$
$$\mathrm{Sht}_{\emptyset,\mu}(\overline{\mathbb{F}}_p) = \coprod_{b \in B(G,X)} [J_{b,\emptyset}(\mathbb{Q}_p)\backslash X_\mu(b_\phi)_\emptyset(\overline{\mathbb{F}}_p)]$$

and the bottom morphism in (4.9.0.1) is given by projection to $X_\mu(b_\phi)_K(\overline{\mathbb{F}}_p)$. We conclude that

$$\mathrm{Sh}_{G,\emptyset}(\overline{\mathbb{F}}_p) = \coprod_{\phi} I_\phi(\mathbb{Q})\backslash X_\mu(b)_\emptyset \times \hat{\mathrm{Sh}}^p_{G,\emptyset}(\phi)/U^p.$$

If $J \subset \mathbb{S}$ is an arbitrary σ-stable type, then the result for $\mathrm{Sh}_{G,J}$ follows from Proposition 7.6 and Proposition 7.7 of **[73]** (assumption 6.18 of loc. cit. is Theorem 4.1.0.1 for $\mathrm{Sh}_{G,\emptyset}$). □

Proof of Theorem 4.1.0.4. Let $J \subset \mathbb{S}$ be a σ-stable type and consider the forgetful map $\pi : \mathrm{Sh}_{G,\emptyset} \to \mathrm{Sh}_{G,J}$. It is good to keep in mind throughout this proof that π induces a bijection on connected components because every connected component of $\hat{\mathrm{Sh}}_{G,\emptyset} = \mathrm{Sh}_{G,\emptyset}$ intersects $\mathrm{Sh}_{G,\emptyset,b}$ and $\pi_0(\mathrm{Sh}_{G,\emptyset,b}) \to \pi_0(\mathrm{Sh}_{G,K})$ is a bijection. If $w \in {}^J\mathrm{Adm}(\mu)$ is nonbasic then we know that

$$\mathrm{Sh}_{G,\emptyset}(w) \to \mathrm{Sh}_{G,\emptyset}$$

induces a bijection on connected components. It follows from the proof of Proposition 6.11 of **[27]** that $\mathrm{Sh}_{G,\emptyset}(w)$ surjects onto the EKOR stratum $\mathrm{Sh}_{G,J}\{w\}$ (c.f. Proposition 6.11 of **[27]**) and it follows that

$$\mathrm{Sh}_{G,J}\{w\} \to \mathrm{Sh}_{G,J}$$

induces a bijection on connected components, keeping in mind the above remark about forgetful maps. If $w \in \mathrm{Adm}(\mu)^J$ and $\mathrm{Sh}_{G,J}(w)$ is the corresponding KR stratum, then

$$\mathrm{Sh}_{G,J}(w) \to \mathrm{Sh}_{G,J}$$

induces a bijection on connected components, because there is an dense open EKOR stratum $\mathrm{Sh}_{G,J}\{v\} \subset \mathrm{Sh}_{G,J}(w)$ for which this holds. □

CHAPTER 5

Main results for abelian type Shimura varieties

In this Chapter we will discuss Kottwitz triples, prove our main theorem in the Hodge type case and end by deducing our main theorems in the abelian type case.

1. Kottwitz triples

Let (G, X) be a Shimura datum of Hodge type and let $\mathcal{G}/\mathbb{Z}_{(p)}$ be a parahoric model of G of type K.

Definition 1.0.1 (See 4.3.1 of **[38]**). *Let $r \geq 1$, set $K_0 = \mathrm{Fr}\, W(k)$ where $k = \mathbb{F}_{p^r}$. A Kottwitz triple $\mathfrak{t}$ of level r is a triple $(\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta)$ where*

- *$\gamma_0 \in G(\overline{\mathbb{Q}})$, defined up to conjugacy in $G(\overline{\mathbb{Q}})$.*
- *$(\gamma_\ell)_{\ell \neq p} \in G(\mathbb{A}_f^p)$*
- *$\delta \in G(K_0)$, defined up to σ-conjugacy by elements of $\mathcal{G}(W(k))$.*

These triples are required to satisfy the following conditions

(i) γ_0 is conjugate to $(\gamma_\ell)_{\ell \neq p}$ in $G(\overline{\mathbb{A}}_f^p)$.

(ii) γ_0 is conjugate to $\gamma_p := \delta\sigma(\delta) \cdots \sigma^{r-1}\delta$ in $G(\overline{\mathbb{Q}}_p)$.

(iii) The image of γ_0 in $G(\mathbb{R})$ is elliptic.

There is also a condition (iv), that will take some time to explain. First of all given such a triple $\mathfrak{t}$, we can define groups $I_{\ell/k}$ for finite primes ℓ. If $\ell \neq p$, then $I_{l/k}$ is the centraliser of $\gamma_\ell \in G(\mathbb{Q}_\ell)$ and if $l = p$ then we define

$$I_{p/k}(R) = \{g \in G(W(K) \otimes_{\mathbb{Z}_p} R) : g^{-1}\delta\sigma(g) = \delta\}.$$

We moreover let $I_{0/k}$ be the centraliser of γ_0 in G, then $I_{0/k} \otimes \mathbb{Q}_\ell$ is an inner form of $I_{p/k}$ for all p. Given a Kottwitz triple $(\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta)$ of level r and a positive integer m it is straightforward to see that $(\gamma_0^m, (\gamma_\ell^m)_{\ell \neq p}, \delta)$ is a Kottwitz triple of level rm. Moreover it is clear that $I_{0/k} \subset I_{0/k'}$ and $I_{l/k} \subset I_{l/k'}$, where $l' = \mathbb{F}_{p^{rm}}$. It turns out *that for m sufficiently large, the groups stabilise giving rise to groups I_0 and I_ℓ for all l. Condition (iv) is now the following:*

(iv) There is an inner twisting I of I_0 such that $I \otimes_{\mathbb{Q}} \mathbb{R}$ is anisotropic mod center, and such that $I \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq I_\ell$ and such that the following diagram commutes

$$\begin{array}{ccc} I_0 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_\ell} & \xrightarrow{\sim} & I_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} \\ \| & & \sim\uparrow \\ I_0 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_\ell} & \xrightarrow{\sim} & I \otimes_{\mathbb{Q}} \overline{\mathbb{Q}_\ell}. \end{array}$$

To be precise, we can choose inner twistings such that the above diagram commutes. Finally we consider the smallest equivalence relation on the set of Kottwitz triples of varying levels r such that $(\gamma_0, (\gamma_l)_{\ell \neq p}, \delta)$ is equivalent to $(\gamma_0^m, (\gamma_l^m)_{\ell \neq p}, \delta)$ and we define a Kottwitz triple to be an equivalence class under this relation.

Remark 1.1. The element γ_0 is determined up to $G(\overline{\mathbb{Q}})$-conjugacy by $(\gamma_l)_{\ell \neq p}$ and δ.

Given a Kottwitz triple $\mathfrak{t} = (\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta)$, we let $I_{\mathbb{A}_f^p} = I_{\mathbb{A}_f^p}(\mathfrak{t})$ denote the centraliser of $(\gamma_\ell^n)_{\ell \neq p}$ for $n \gg 0$ and we set $I_{\mathbb{A}_F} = I_{\mathbb{A}_f^p} \times I_p$. condition (iv) somehow tells us that there is an isomorphism $\iota : I \otimes_{\mathbb{Q}} \mathbb{A}_f \simeq I_{\mathbb{A}_f}$. The quadruple $\tilde{\mathfrak{t}} = (\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta, \iota)$ is called a refined Kottwitz triple.

1.2. Equivalences of Kottwitz triples. Let $\mathfrak{t}$ and $\mathfrak{t}'$ be Kottwitz triples, then we say that $\mathfrak{t}$ is equivalent to $\mathfrak{t}'$ and write $\mathfrak{t} \sim \mathfrak{t}'$ if there are representatives $\mathfrak{t} = (\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta)$ and $\mathfrak{t}' = (\gamma_0', (\gamma_\ell')_{\ell \neq p}, \delta')$ of the same level r such that: The elements $\gamma_\ell)_{\ell \neq p}$ and $\gamma_\ell')_{\ell \neq p}$ are conjugate in $G(\mathbb{A}_f^p)$ and δ is σ-conjugate to δ' in $G(W(\mathbb{F}_{q^r})[1/p])$.

If $\tilde{\mathfrak{t}} = (\gamma_0, (\gamma_\ell)_{\ell \neq p}, \delta, \iota)$ is a refined Kottwitz triple, then there is a set

$$S(\tilde{\mathfrak{t}}) = I(\mathbb{Q}) \backslash X_\mu(\delta)_K \times G(\mathbb{A}_f^p)$$

with an action of $Z_G(\mathbb{Q}_p) \times \langle \Phi \rangle \times G(\mathbb{A}_f^p)$. Recall from Section 4.3.2 of **[38]** that if $\mathfrak{t}' \sim \mathfrak{t}$ then we can transport the refinement ι to a refinement ι' of $\mathfrak{t}'$ and obtain an equivariant bijection

$$S(\tilde{\mathfrak{t}}) \simeq S(\tilde{\mathfrak{t}}').$$

Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism, then ϕ factors through $\phi : \mathfrak{P}^L \to \mathfrak{G}_G$ because Z_G^0 satisfies the Serre condition. Moreover, there is a morphism of $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$

gerbs $\theta : \mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}$ such that its inflation to a map $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$-gerbs is conjugate to ϕ. We then define a Kottwitz triple $\mathfrak{t}(\phi)$ by

$$\gamma_0 = \phi(\delta_n) = \gamma_\ell$$
$$\delta = \theta(d_\sigma^n).$$

It follows from Section 4.5 of [**38**] that this is indeed a Kottwitz triple with $I \simeq I_\phi$. Moreover, Lemma 4.5.2 of loc. cit. tells us that there is a $\langle \Phi \rangle \times G(\mathbb{A}_f^p)$-equivariant isomorphism

$$X(\phi) \simeq X_\mu(\delta)_K \times G(\mathbb{A}_f^p).$$

1.3. Admissible morphisms with the same Kottwitz triple. Because we are going to work with Kottwitz triples in the proof, we need to determine the fibers of

$$\{\text{ Admissible } \phi : \mathfrak{Q} \to \mathfrak{G}_G\}/\text{conjugacy} \to \{\text{triples}\}/\sim .$$

Recall first of all from Lemma 3.1.0.5 that the set of conjugacy classes of (not necessarily admissible) morphisms $\phi' : \mathfrak{Q} \to \mathfrak{G}_G$ with the same ϕ^Δ, and thus the same Kottwitz triple, is in bijection with $H^1(\mathbb{Q}, I_\phi)$. We would like to express the admissibility of such a (conjugacy class of) ϕ' in terms of conditions on the associated cohomology class. Kisin defines a certain Tate-Shafarevich group $\text{Ш}_G(\mathbb{Q}, I) \subset H^1(\mathbb{Q}, I)$ and shows that (Proposition 4.5.7 of [**38**]) that if the fiber of

$$\{\text{ Admissible } \phi : \mathfrak{Q} \to \mathfrak{G}_G\}/\text{conjugacy} \to \{\text{triples}\}/\sim \tag{1.3.1}$$

over a triple $\mathfrak{t}$ is nonempty, then it is a $\text{Ш}_G(\mathbb{Q}, I)$-torsor. A cohomology class $\alpha \in H^1(\mathbb{Q}, I)$ lies in $\text{Ш}_G(\mathbb{Q}, I)$ if it satisfies the following conditions:

- It is in the kernel of $H^1(\mathbb{Q}, I) \to \prod_v H^1(\mathbb{Q}_v, I)$, where the product runs over all finite places of $\mathbb{Q}$.
- It is in the kernel of the map $H^1(\mathbb{Q}, I) \to H^1(\mathbb{R}, I)$.
- It is in the kernel of the map

  $$\ker\left(H^1(\mathbb{Q}, I) \to H^1(\mathbb{R}, I)\right) \to \ker\left(H^1(\mathbb{Q}, G) \to H^1(\mathbb{R}, G)\right)$$

  defined in Section 4.4 of [**38**].

1.4. Kottwitz triples associated to special morphisms. If T is a torus over $\mathbb{Q}$ together with a cocharacter μ, then there is a morphism $\psi_\mu : \mathfrak{Q} \to \mathfrak{G}_T$. If T

satisfies the Serre condition, then this factors through a $\phi : \mathfrak{P} \to \mathfrak{G}_T$ and in fact through $\phi : \mathfrak{P}^L \to \mathfrak{G}_T$ for some sufficiently large L and we can define a Kottwitz triple $\mathfrak{t}(T, \mu)$ for T by

$$\gamma_0 = \phi(\delta_n) = \gamma_\ell$$
$$\delta = \theta(d_\sigma^n).$$

Since $\phi^\Delta : P^L \to T$ is defined over $\mathbb{Q}$, it follows that $\gamma_0 \in T(\mathbb{Q})$ for $n \gg 0$. Now suppose that $i : T \subset G$ such that $\mu = \mu_{h_T}$ for some $h_T : \mathbb{S} \to G_\mathbb{R}$ factoring through $T_\mathbb{R}$, then T satisfies the Serre condition by 4.3.9 of [**38**]. Furthermore there is an equivalence of Kottwitz triples

$$i_*(\mathfrak{t}(T, \mu) \sim \mathfrak{t}(i \circ \psi_\mu),$$

where $i_*(\gamma_0, (\gamma_\ell)_{l \neq p}, \delta) = (i(\gamma_0), (i(\gamma_\ell))_{\ell \neq p}, i(\delta))$. This is basically just saying that the construction of Kottwitz triples associated to an admissible morphism is functorial with respect to morphisms of Shimura data, in the special case that the source is a torus.

2. Mod p-points on Shimura varieties of Hodge type

Let (G, X) be a Shimura variety of Hodge type and let $p > 2$ be a prime such that $G_{\mathbb{Q}_p}$ is quasi-split and splits over a tamely ramified extension, such that p does not divide $\#\pi_1(G^{\mathrm{der}})$ and such that all parahorics of G are connected. Let $U^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup and let $U = U^p U_p$ with $U_p = \mathcal{G}(\mathbb{Z}_p)$, where $\mathcal{G}$ is a connected parahoric model of G. We let $\mathscr{S}_U$ be the Kisin-Pappas integral model of the Shimura variety $\mathbf{Sh}_{U^p U_P}$. Let $b \in B(G, X)$ be the unique basic element.

THEOREM 2.0.1.

(i) Suppose that $\mathcal{G}$ is a connected very special parahoric, then there is an $G(\mathbb{A}_f^p) \times \langle \Phi \rangle$-equivariant bijection

$$\mathscr{S}_{G, U^p U_p}(\overline{\mathbb{F}}_p) \simeq \coprod_\phi S_{\tau(\phi_0)}(\phi),$$

where ϕ runs over conjugacy classes of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$. Here the element $\tau(\phi_0) \in I_\phi^{ad}(\mathbb{A}_f)$ only depends on the conjugacy class of ϕ_0, the composition of $\mathfrak{Q} \to \mathfrak{G}_G$ with $\mathfrak{G}_G \to \mathfrak{G}_{G^{ad}}$.

(ii) Suppose that G^{ad} is $\mathbb{Q}$-simple, that J_b has no compact type factors and that $\mathbf{Sh}_U$ is proper or that Conjecture 4.7.0.5 holds. Then the same conclusion as in (i) holds.

Remark 2.0.2. We would like to point out that the first part of the theorem (for very special parahorics) is essentially due to Rong Zhou; it follows immediately from Appendix A of [**32**] and [**73**] and the argument in Section 4 of [**38**].

3. Proof of the Langlands-Rapoport conjecture for Hodge type Shimura varieties

In this section we will prove Theorem 5.2.0.1, following Section 4 of [**38**]. Because the arguments are so similar to the arguments in loc. cit., we will not give many details. The strategy of the proof can be summed up by the following diagram (we use the Kisin-Pappas integral models for Shimura varieties of abelian type)

$$\begin{array}{ccc} \{\text{Admissible morphisms}\}/\text{conjugacy} & & \\ \twoheaduparrow & \searrow & \\ \{\text{special points}\} & \longrightarrow & \{\text{Kottwitz triples}\}/\sim \\ \twoheaddownarrow & \nearrow & \\ \{\text{Isogeny classes}\}. & & \end{array}$$

The dotted arrow will be constructed by lifting an isogeny class to a special points, taking the associated Kottwitz triple, and then checking that the result does not depend on the choice of lift.

Proof of Theorem 5.2.0.1. It follows from Theorem 3.3.0.1 that all admissible morphisms are special and it follows from part (2) of Theorem A.4.5 of [**32**] that all isogeny classes come from special points. To go from isogeny classes to Kottwitz triples, we choose a special point landing in the isogeny class and then take the Kottwitz triple associated to that special point, which will be independent of the choice of special point up to equivalence as in 4.4.6 of [**38**]. Moreover, it follows as in loc. cit. that we can twist isogeny classes by elements of $Ш_G^{\infty}(\mathbb{Q}, I)$ [1] and the fibers of the map

$$\{\text{Isogeny classes}\} \to \{\text{Kottwitz triples}\}/\sim$$

[1] This is the same as $Ш_G(\mathbb{Q}, I)$ except that we don't impose any conditions at the finite places.

are either empty or $\text{Ш}_G(\mathbb{Q}, I)$-torsors (Proposition 4.4.13 of loc. cit.). We can now conclude that there is a bijection between isogeny classes and admissible morphisms, keeping in mind (5.1.3.1), so it suffices to deal with the uniformisation of isogeny classes. Part (1) of Theorem A.4.5 of **[32]** combined with Theorem 4.1.0.1 tells us that under our assumptions isogeny classes $\mathscr{I} \subset \mathscr{S}_U(\overline{\mathbb{F}}_p)$ have the following shape

$$\mathscr{I} \simeq I(\mathbb{Q})\backslash X_\mu(b)_K \times G(\mathbb{A}_f^p)/U^p,$$

and moreover this identification is $\langle \Phi \rangle \times G(\mathbb{A}_f^p)$-equivariant. Recall that there is an action of $\mathcal{A}(\mathcal{G})$ on $\mathscr{S}_U(\overline{\mathbb{F}}_p)$. It follows as in the proof of Proposition 4.4.14 of **[38]** that the stabiliser of $\mathscr{I}$ is given by $\mathcal{A}(\mathcal{G})^I \subset \mathcal{A}(\mathcal{G})$, where

$$(\mathcal{A}(\mathcal{G}))^I = G(\mathbb{A}_f^p) *_{\mathcal{G}(\mathbb{Z}_p)_+} \mathcal{G}^{\text{ad}}(\mathbb{Z}_p)_+^I$$

and where $\mathcal{G}^{\text{ad}}(\mathbb{Z}_p)_+^I$ is the kernel of

$$\mathcal{G}^{\text{ad}}(\mathbb{Z}_p)_+ \to H^1(\mathbb{Q}, Z_G) \to H^1(\mathbb{Q}, I).$$

Similarly, it follows as in Lemma 4.3.5 of loc. cit. that the stabiliser of $S_\tau(\phi) \subset S_\tau(G, \phi_0)$ under the action of $\mathcal{A}(\mathcal{G})$ is given by $(\mathcal{A}(\mathcal{G}))^I$. Let (T, h_T, i) be a special point mapping to $\mathscr{I}$ and let ϕ be an admissible morphism conjugate to $i \circ \psi_{\mu_{h_T}}$. We we now write

$$\mathscr{I}_0 = \bigcup_{h \in G^{\text{ad}}(\mathbb{Q})^+} \mathscr{I}^{[h]},$$

where $[h]$ is the class of h in $\text{Ш}_G^\infty(\mathbb{Q}, I)$. As in **[38]**, the theorem can be deduced from the following proposition: □

PROPOSITION 3.0.1 (c.f. Proposition 4.6.2 of **[38]**). *Let ϕ be as above, then there is an $\langle \Phi \rangle \times \mathcal{A}(\mathcal{G})$-equivariant bijection $\xi : \mathscr{I}_0 \simeq S_\tau(G, \phi_0)$ for some $\tau = \tau(\phi_0)$ fitting in a commutative diagram*

$$\begin{array}{ccc} \mathscr{I}_0 & \xrightarrow{\xi} & S_\tau(G, \phi_0) \\ \downarrow{\scriptstyle c_G} & & \downarrow \\ \pi(G, X) & \xrightarrow{\vartheta_G} & \pi(G, \phi_0). \end{array}$$

Moreover, each $\mathscr{I}^{[h]}$ is taken isomorphically to $S_\tau(\phi^{[h]})$.

PROOF. The proof is the same as the proof in loc. cit., except that we need the fact that $J_b(\mathbb{Q}_p) \to \pi_1(G)_I^\sigma$ is surjective. In the unramified case, this is Corollary 2.5.12

of [**10**] and we adapt their proof: Let T be the centraliser of a maximal split torus of $G_{\mathbb{Q}_p}$, this is a maximal torus since $G_{\mathbb{Q}_p}$ is quasi-split. The short exact sequence

$$0 \to X_*(T^{\mathrm{sc}}) \to X_*(T) \to \pi_1(G) \to 0$$

defining $\pi_1(G)$ induces a short exact sequence

$$0 \to X_*(T^{\mathrm{sc}})_I \to X_*(T)_I \to \pi_1(G)_I \to 0,$$

because $X_*(T^{\mathrm{sc}})_I$ is torsion-free since $X_*(T^{\mathrm{sc}})$ is an induced Galois module by 4.4.16 of [**7**]. Taking the long exact sequence in cohomology for the Frobenius action we see that the surjectivity of $X_*(T)_I^{\sigma} \to \pi_1(G)_I^{\sigma}$ is equivalent to the injectivity of

$$X_*(T^{\mathrm{sc}})_{\Gamma} \to X_*(T)_{\Gamma},$$

where Γ is now the full Galois group. This injectivity follows because $X_*(T^{\mathrm{sc}})_{\Gamma}$ is torsion free since $X_*(T^{\mathrm{sc}})$ is an induced Galois module. Finally the map $X_*(T)_I^{\sigma} \to \pi_1(G)_I^{\sigma}$ factors through $\pi_1(M)_I^{\sigma} \to \pi_1(G)_I^{\sigma}$ and since $\pi_1(M) \simeq \pi_1(J_b)$, we are done. $\square$

4. Main results for abelian type Shimura varieties

Let (G, X) be a Shimura datum of abelian type and let $p > 2$ be a prime such that $G_{\mathbb{Q}_p}$ is quasi-split and splits over a tamely ramified extension. Let $U_p \subset G(\mathbb{Q}_p)$ be a parahoric subgroup and consider the tower of Shimura varieties $\{\mathbf{Sh}_{G,U^pU_p}\}_{U^p}$ over E with its action of $G(\mathbb{A}_f^p)$, where U^p varies over compact open subgroups of $G(\mathbb{A}_f^p)$. Then by Theorem 0.1 of [**37**], this tower of Shimura varieties has a $G(\mathbb{A}_f^p)$-equivariant extension to a tower of flat normal schemes $\{\mathscr{S}_{G,U^pU_p}\}_{U^p}$ over $\mathcal{O}_{E_{(v)}}$, where $v \mid p$ is a prime of the reflex field E. Let μ' be the dominant representative of the conjugacy class $\{\mu_h^{-1}\}$ where μ_h is the Hodge cocharacter associated to X and let $\mu = \sigma(\mu')$. Let $b \in B(G, X)$ be the unique basic σ-conjugacy class, and let $J_b/\mathbb{Q}_p$ be its twisted centraliser. Consider the following sets of hypotheses on (G, X) and U_p.

(T1) The parahoric subgroup U_p is very special.

(T2) The group J_b^{ad} has no factors that are of compact type and either $\mathbf{Sh}_U$ is proper or Conjecture 4.7.0.5 holds for an auxiliary Hodge type Shimura datum of very special level.

(T3) The Shimura datum (G, X) admits an auxiliary Hodge type Shimura datum that is of PEL type A with $G_{\mathbb{Q}_p}$ unramified.

We need one more technical assumption which has to do with being able to reduce to the case of a Hodge type Shimura variety with a connected parahoric.

(P1) All factors of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ that are of type $D^{\mathbb{H}}$ split over an unramified extension (at p), and for those factors the parahoric subgroup $U_p = \mathcal{G}_J(\mathbb{Z}_p)$ is contained in an hyperspecial subgroup.

THEOREM 4.0.1. *Suppose that (G, X) and U_p satisfy (T3) or that they satisfy (P1) and either (T1) or (T2). Then there is an $G(\mathbb{A}_f^p) \times \langle \Phi \rangle$-equivariant bijection*

$$\mathscr{S}_{G,U^pU_p}(\overline{\mathbb{F}}_p) \simeq \coprod_{\phi} S_{\tau(\phi_0)}(\phi), \tag{4.0.1}$$

where ϕ runs over conjugacy classes of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$. The theorem in the (T1) case is essentially due to Rong Zhou and the theorem in the (T3) case is essentially due to Kottwitz [**38**].

REMARK 4.0.2. As in Theorem 4.1.0.1, we do not construct an action of $Z_G(\mathbb{Q}_p)$ on the left hand side. However if G splits over a metacyclic extension, then $Z_G(\mathbb{Q}_p)$ acts trivially on the right hand side of (5.4.0.1) (see Remark 3.5.0.7 and Remark 3.7.10.(2) of [**38**]), and so we get a $Z_G(\mathbb{Q}_p)$-equivariant statement for free.

THEOREM 4.0.3. *Let (G, X) be as above, let U_p denote an arbitrary parahoric and suppose that G^{ad} is $\mathbb{Q}$-simple and that (P1) and (T2) hold. Let $w \in {}^K\mathrm{Adm}(\mu)$ and let $\mathscr{S}_{U,\overline{\mathbb{F}}_p}\{w\}$ be the corresponding EKOR stratum, where K is the type of U_p (c.f. Section 2). Suppose that it is not contained in the basic locus, then*

$$\mathscr{S}_{U,\overline{\mathbb{F}}_p}\{w\} \to \mathscr{S}_{U,\overline{\mathbb{F}}_p}$$

induces a bijection on connected components.

Theorem 1 is a special case of Theorem 5.4.0.1, because Conjecture 4.7.0.5 holds for unramified groups by Proposition 6.20 of [**70**], combined with Theorem 1.2 of [**2**] and because all type A Shimura varieties admit auxiliary Hodge type data of PEL type (see Appendix B of [**50**] and Proposition 1.4 of [**67**]). By the same reasoning, Theorem 2 is a special case of Theorem 5.4.0.3, except that we have to prove irreducibility of nonbasic Ekedahl-Oort strata for unramified PEL type Shimura varieties of type A.

Proof of Theorem 2 for Shimura varieties of PEL type A. Let $\mathrm{Sh}_{G,K}\{w\}$ denote the nonbasic EO stratum that we are trying to show is 'connected'. By Proposition 4.4 of [**72**], it suffices to prove that the prime-to-Σ Hecke operators coming from G^{sc} act transitively on the fibers of

$$\pi_0(\mathrm{Sh}_{G,K}\{w\}) \to \pi_0(\mathrm{Sh}_{G,K}),$$

where Σ is a finite set of primes including p. There is a Hecke-equivariant and finite étale surjective map $\mathrm{Sh}_{G,\emptyset}(w) \to \mathrm{Sh}_{G,K}\{w\}$, so it suffices to show the same statement for $\pi_0(\mathrm{Sh}_{G,\emptyset}(w)) = \pi_0(\mathrm{Sh}_{G,\emptyset}(\leq w))$. Proposition 4.7.0.3 (see Remark 4.7.0.6) tells us that each connected component $\mathrm{Sh}_{G,\emptyset}(\leq w)$ intersects $\mathrm{Sh}_{G,\emptyset}(\tau)$. The closure relations then give us a surjective map $\mathrm{Sh}_{G,\emptyset}(\tau) \to \pi_0(\mathrm{Sh}_{G,\emptyset}(\leq w)$, hence it is enough to show that the prime-to-Σ Hecke operators act transitively on the fibers of

$$\mathrm{Sh}_{G,\emptyset}(\tau) \to \pi_0(\mathrm{Sh}_{G,\emptyset}) = \pi_0(\mathrm{Sh}_{G,K}).$$

Rapoport-Zink uniformisation (Theorem 4.5.0.1) and the discussion in Section 4.5 tells us that there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Sh}_{G,\emptyset}(\tau) & \xrightarrow{\sim} & I(\mathbb{Q})\backslash G(\mathbb{A}_f^p) \times \frac{J_b(\mathbb{Q}_p)}{J_{b,\emptyset}(\mathbb{Z}_p)}/U^p \\
\downarrow & & \downarrow \beta \\
\pi_0(\mathrm{Sh}_{G,\emptyset}) & \xrightarrow{\sim} & I(\mathbb{Q})\backslash \frac{G(\mathbb{A}_f^p)}{G^{\mathrm{sc}}(\mathbb{A}_f^p)} \times \frac{J_b(\mathbb{Q}_p)}{J_{b,\emptyset}(\mathbb{Z}_p) J_b^{\mathrm{sc}}(\mathbb{Q}_p)}/U^p.
\end{array}$$

Weak approximation (Theorem 7.8 of [**56**]) tells us that $I^{\mathrm{sc}}(\mathbb{Q})$ is dense

$$J_b^{\mathrm{sc}}(\mathbb{Q}_p) \times \prod_{p \neq \ell \in \Sigma} G^{\mathrm{sc}}(\mathbb{Q}_\ell),$$

which means that $G^{\mathrm{sc}}(\mathbb{A}_f^\Sigma)$ acts transitively on the fibers of β. ☐

5. Proofs

Theorem 5.4.0.3 follows from Theorem 4.1.0.4, because it can be checked on connected components of Shimura varieties. To be precise, EKOR strata on abelian type Shimura varieties are constructed from the EKOR strata on a single connected component of an auxiliary Hodge type Shimura variety, see Section 5.4 of [**64**].

Proof of Theorem 5.4.0.1. Theorem 5.4.0.1 in the (T1) and (T2) cases follows by the following chain of reasoning: As in [**38**], it suffices to show that the τ-version of Conjecture 3.5.0.6 holds for an auxiliary Hodge type Shimura datum, using Corollary

3.6.0.4, Remark 3.6.0.5 and Lemma 4.6.13 of [**37**]. We can take this auxiliary Hodge type datum to be a product of quasi-simple groups that all satisfy the assumptions of Theorem 5.2.0.1 by Lemma 4.6.22 of [**37**]. [To see that we can choose the parahoric to be connected in type $D^{\mathbb{H}}$-cases, use the argument in the proof of part (5) of Theorem 4.6.23 of op. cit. which relies on the assumption (P1)].

Theorem 5.2.0.1 tells us that the τ-version of Conjecture 3.5.0.6 holds for each of these groups, and it is not hard to see that this implies that it holds for their product.

In the (T3) case, we first reduce from the abelian type to the Hodge type case as above. In our situation, these Hodge type Shimura varieties can be chosen to be of PEL type by Appendix B of [**50**] in combination with Proposition 1.4 of [**67**], and we consider the Rapoport-Zink integral models. These are flat and normal by the main theorem of [**18**], and they come with tautological closed embeddings (for sufficiently small level away from p) into Siegel modular varieties of parahoric level. It follows that they are isomorphic to the (normalisation) of the Zariski closure of their generic fibre in the Siegel modular variety, and one can argue as in Section 7 of [**73**] that these models are isomorphic to the Kisin-Pappas integral models.

It follows from Proposition 4.4 of [**30**] that the $\overline{\mathbb{F}}_p$ points of Rapoport-Zink spaces of parahoric level agree with the $\overline{\mathbb{F}}_p$-points of the corresponding affine Deligne-Lusztig variety. It follows from the moduli description (c.f. Section 6 of [**58**]) that we can produce maps from the set of $\overline{\mathbb{F}}_p$ points of our Rapoport-Zink space into the set of $\overline{\mathbb{F}}_p$-points of our Shimura variety. To be precise, assumption 6.18 of [**73**] is satisfied and then Proposition 9.1.(i) gives us uniformisation of isogeny classes. □

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