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Symmetries, tensors, and the Horrocks bundle

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A study of tensors on the quaternionic projective plane $\mathbb{HP}^2$ arising from a stable 3-form on $\mathbb{C}^6$ and an associated action of SU(3) is related to the existence of a holomorphic rank 3 vector bundle over $\mathbb{CP}^5$ discovered by Horrocks. It also leads to the construction of SU(3) invariant Spin(7) structures on $\mathbb{HP}^2$, which are characterised in terms of associated 4-forms.

Contents

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1. Introduction

This article explores special features of the quaternionic projective plane $\mathbb{HP}^2$ that are derived from a well-known action by the subgroup U(3) of its isometry group Sp(3). The cohomogeneity-one action by SU(3) was already highlighted by Gambioli in the context of quaternionic geometry [18]. It commutes with that of the centre U(1) of U(3), whose quotient was studied by Battaglia from the viewpoint of quaternionic geometry and Morse theory [7], and Atiyah and Witten in relation to G_2 geometry and M-theory [3].

We show that these actions, and the resulting tensors they define, can shed new light on quaternionic Kähler moment maps and the holomorphic rank 3 vector bundle Y over $\mathbb{CP}^5$ discovered by Horrocks [26] in 1977. The complex projective space $\mathbb{CP}^5$ admits a holomorphic contact form from a choice of symplectic form ω on $\mathbb{C}^6$, and the contact distribution is a twist of the related rank 4 null-correlation bundle S. The structure of $\mathbb{CP}^5$ can be reduced further by imposing additional forms on $\mathbb{C}^6$. In the Horrocks set-up, this means a 3-form ξ with an open orbit in $\mathrm{GL}(6,\mathbb{C})$, which is therefore stable in the sense of Hitchin [25]. The stabiliser of the pair (ω, ξ) is $\mathrm{SL}(3,\mathbb{C})$ and this action is what is needed to set up the monad realising Y. There is an analogy with the study of special geometries in real dimensions 6, 7 and 8, but in our context the stable forms characterise a reduction of the isometry group rather than that of a gauge or holonomy group.

In order to obtain our 'real' description, we impose the anti-holomorphic involution j of $\mathbb{CP}^5$ arising from an identification $\mathbb{C}^3 = \mathbb{H}^3$. This then exhibits $\mathbb{CP}^5$ as the twistor space of $\mathbb{HP}^2$ in the spirit of Roger Penrose [33]. Its j-invariant lines are the twistor fibres, and the kernel of the holomorphic contact form defines the horizontal space relative to the Levi-Civita connection on $\mathbb{HP}^2$. The groups $\mathrm{Sp}(3,\mathbb{C})$ and $\mathrm{SL}(3,\mathbb{C})$ are now reduced to Sp(3) and SU(3) respectively. The Horrocks bundle can be defined as the pullback of a vector bundle V on $\mathbb{HP}^2$ equipped with an instanton connection, in a generalisation of the Atiyah-Ward construction [5]. This was outlined by Mamone Capria and the second author in [30], but in this paper we succeed in defining V more directly using knowledge gleaned in the intervening years. Our approach can be viewed as a mere reinterpretation of the Horrocks construction, but the definition of V is more natural from the viewpoint of differential geometry. In a nutshell, we exhibit a section η_E of the tautological bundle E over $\mathbb{HP}^2$ with fibre $\mathbb{H}^2$ whose derivative $\nabla\eta_E$ defines the Horrocks monad.

The cohomogeneity-one action of SU(3) on $\mathbb{HP}^2$ has singular orbits S^5 and $\mathbb{CP}^2$. The former is the zero set of the Galicki-Lawson moment map for the action of U(1) [17], and fibres over a dual projective plane $\mathbb{CP}^{2*}$, which is the quaternionic Kähler quotient. The twistor space $F_{1,2}$ of $\mathbb{CP}^{2*}$ (together with its isometry group SU(3)) can now be regarded as a Kähler quotient of $\mathbb{HP}^2 \setminus \mathbb{CP}^2$ using the approach of [19], but we do not pursue this aspect in the present article. Instead, we focus on global tensors that are invariant by U(1) and SU(3), and the sections of vector bundles that they give rise to. All these sections satisfy versions of the twistor equation, and distill holomorphic objects on $\mathbb{CP}^5$.

Gray and Green had long ago posed the problem of finding explicit Spin(7) structures on $\mathbb{HP}^2$, and our methods provide an effective solution. We exhibit nowhere-vanishing sections of the spinor bundle Δ_+ over $\mathbb{HP}^2$. This already splits into real subbundles of rank 5 and 3, which can be further reduced to obtain various Spin(7) structures of cohomogeneity one. We construct families of such structures by modifying the locally symmetric metric.

Although $\mathbb{HP}^2$ cannot admit a metric with holonomy Spin(7), the rank-three subgroups Spin(7) and Sp(2)Sp(1) of SO(8) impose some common features on an 8-manifold. They both stabilise 4-forms on $\mathbb{R}^8$ whose coefficients differ only by certain sign changes, and our analysis involves the study of such 4-forms. This complements the approaches of [12,13], which characterise linear deformations of such forms described briefly in the final section.

2. Instanton bundles over $\mathbb{CP}^5$

In his paper [26], Horrocks constructs a rank 3 holomorphic vector bundle Y over $\mathbb{CP}^5$. This is the parent bundle from which others are derived, but we shall only deal with Y. It fits naturally in the context of the fibration

$$\pi\colon \mathbb{CP}^5 \longrightarrow \mathbb{HP}^2 \tag{1}$$

that realises 5-dimensional complex projective space as the twistor space of the quaternionic projective plane [30]. To understand the latter, we first define the more general class of quaterion-Kähler manifolds in real dimension 8.

A quaternion-Kähler (QK) 8-manifold M is a Riemannian 8-manifold whose holonomy lies in $\mathrm{Sp}(2)\mathrm{Sp}(1) := (\mathrm{Sp}(2)\times\mathrm{Sp}(1))/\{\pm 1\}$. It is *not* in general Kähler in the usual complex sense. Its complexified tangent bundle

$$T_{\mathbb{C}}M = E \otimes_{\mathbb{C}} H \tag{2}$$

decomposes locally as the tensor product of the vector bundle E with fibre $\mathbb{C}^4 \cong \mathbb{H}^2$ associated to the standard representation of Sp(2) and the vector bundle H with fibre $\mathbb{C}^2 \cong \mathbb{H}$ associated to the standard representation of Sp(1). Globally speaking, these vector bundles are subject to a $\mathbb{Z}_2$ ambiguity. The holonomy condition implies that the quaternionic structures on these vector bundles are preserved by the Levi-Civita connection. The twistor space of a QK manifold can be identified with the total space of the bundle $\mathbb{P}_{\mathbb{C}}(H)$, which is well defined even if H is not.

The unifying features of quaternionic symmetric spaces and their twistor spaces was realised by Wolf [40], and the discovery by Alekseevsky [2] of homogeneous non-symmetric QK spaces was a first step in generalising Wolf's theory. The only complete QK 8-manifolds with positive scalar curvature are the Wolf spaces $\mathbb{HP}^2$, $\mathrm{G}_2/\mathrm{SO}(4)$, and the Grassmannian $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$ (whose Kähler structure is largely irrelevant to the quaternionic geometry) [35]. These three manifolds share (along with the Lie group SU(3)) a cohomogeneity-one action by SU(3), which gives them many features in common [12]. Moreover, their twistor spaces incorporate an open orbit of a complex Heisenberg group, and are all birationally equivalent [11].

When $M = \mathbb{HP}^2$ has its standard QK structure, E and H are globally well defined, since the structure lifts to $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$. In fact, there is a decomposition

$$\boxed{\mathbb{U} \ = \ E \oplus H} \tag{3}$$

of the trivial bundle $\mathbb{U} = \mathbb{C}^6 \times \mathbb{HP}^2 = \mathbb{H}^3 \times \mathbb{HP}^2$. Here H corresponds to the tautological quaternionic line bundle, and $E = H^\perp$ is its orthogonal complement in $\mathbb{U}$, when the latter is endowed with an Sp(3) structure. The twistor space $\mathbb{P}_{\mathbb{C}}(H)$ can now be identified with $\mathbb{P}_{\mathbb{C}}(\mathbb{C}^6) = \mathbb{CP}^5$, and (1) maps a complex line to its quaternionic span. We have highlighted (3) because we shall apply successive operations to it to derive new tensors on $\mathbb{HP}^2$ starting from constant sections.

Equation (3) merely expresses the decomposition of the standard Sp(3) module with respect to its subgroup $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$. The geometries that we shall be concerned with all arise by imposing different structures on $\mathbb{C}^6$, and so on the trivial bundle $\mathbb{U}$. We give an example in (5) below, though most interest will arise when we choose structures that are not invariant by Sp(3). The identification $\mathbb{C}^6 \cong \mathbb{H}^3$ endows $\mathbb{C}^6$ with an anti-linear transformation j, and this extends (as $\otimes^d j$) to any exterior product $\Lambda^d(\mathbb{C}^6)$. The latter is again quaternionic when the degree d is odd, but it is the complexification of a real vector space when d is even; this is a basic observation in the study of representations of Lie groups [1].

The isometry group Sp(3) of $\mathbb{HP}^2$ is the stabiliser of the pair (ω, j), in which $\omega \in \Lambda^2(\mathbb{C}^6)^*$ is a real (i.e., j-invariant) symplectic form. We can choose a basis of $(\mathbb{C}^6)^*$, equivalently a constant basis of $\mathbb{U}^*$, such that

$$\begin{aligned} &\omega = u^1 \wedge u^4 + u^2 \wedge u^5 + u^3 \wedge u^6, \\ &ju^1 = u^4, \quad ju^2 = u^5, \quad ju^3 = u^6. \end{aligned} \tag{4}$$

At this juncture, the distinction between $\mathbb{U}$ and $\mathbb{U}^*$ is immaterial, since they are identified via ω. In the sequel, we shall indicate ju by $\tilde{u}$, and further streamline the notation to write $\omega = 1\tilde{1} + 2\tilde{2} + 3\tilde{3}$ to make computational proofs easier to visualise. The stabiliser of ω is the complexification of $\mathrm{Sp}(3)$ that we shall denote by $\mathrm{Sp}(3, \mathbb{C})$, rather than the equally logical notation $\mathrm{Sp}(6, \mathbb{C})$.

We shall denote the total Chern class of H as $c(H) = 1 - u$ in accordance with [36], where u is a generator of $H^4(\mathbb{HP}^2, \mathbb{Z})$. Then the pullback of u to $\mathbb{CP}^5$ via π corresponds to x^2, where $c(\mathcal{O}(1)) = 1 + x$. Under pullback, we have an isomorphism

$$\pi^* H \;\cong\; \mathcal{O}(-1) \,\oplus\, \mathcal{O}(1)$$

of smooth vector bundles, though $\pi^* H$ has no natural holomorphic structure. By contrast, E has an 'instanton' connection, and this induces a holomorphic structure on $\pi^* E$. The latter can in fact be identified with the null-correlation bundle (see below), denoted S by Horrocks.

Definition 2.1 ([37,30]). A connection on a vector bundle over $\mathbb{HP}^2$ is called quaternionic or (better) a self-dual instanton if its curvature 2-forms belong to the subbundle $S^2 E$ of $\Lambda^2 T^* \mathbb{HP}^2$ whose fibres are isomorphic to $\mathfrak{sp}(2)$.

It is an elementary fact (and part of the twistor space theory) that the pullback to $\mathbb{CP}^5$ of such a connection has $(1, 1)$ curvature, and is therefore integrable by a test case of the Newlander-Nirenberg theorem [15].

As a first application of (2), we take its second exterior power so as to obtain the decomposition

$$\boxed{\Lambda^2 \mathbb{U} \;\cong\; \Lambda^2 E \,\oplus\, E\,H \,\oplus\, \Lambda^2 H} \tag{5}$$

In the sequel, we shall often denote tensor products (over $\mathbb{C}$) by juxtaposition, and here $E\,H$ is shorthand for $E \otimes H$, to avoid confusion between $\oplus$ and $\otimes$. We may now decompose ω as a constant section of $\Lambda^2 \mathbb{U}^* \cong \Lambda^2 \mathbb{U}$ relative to (5). In fact,

$$\omega = \omega_E + 0 + \omega_H,$$

where ω_E and ω_H are the sections of $\Lambda^2 E$ and $\Lambda^2 H$ invariant by the holonomy group $\mathrm{Sp}(2)\mathrm{Sp}(1)$. This is because we can fix an origin $x \in \mathbb{HP}^2$ for which $H_x = \langle 1, \tilde{1} \rangle$ and $E_x = \langle 2, \tilde{2}, 3, \tilde{3} \rangle$. At x, we have $\omega_H = 1\tilde{1}$ and $\omega_E = 2\tilde{2} + 3\tilde{3}$, so ω has zero component in $E \otimes H$. The general statement holds because $\mathrm{Sp}(3)$ acts transitively on $\mathbb{HP}^2$. Using the subscript '0' to indicate orthogonal complements to the various symplectic forms, we now have

$$\Lambda^2_0 \mathbb{U} \;\cong\; \Lambda^2_0 E \,\oplus\, E\,H \,\oplus\, \underline{\mathbb{C}}, \tag{6}$$

where $\underline{\mathbb{C}} = \mathbb{C} \times \mathbb{HP}^2$ denotes a trivial bundle. The observations in this paragraph are purely algebraic, and follow from the fact that $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ fixes two linearly independent symplectic form on $\mathbb{C}^6$.

In this paper, we shall be dealing frequently with the vector bundle $\Lambda^2_0 E$ with fibre $\mathbb{C}^5$, and we shall denote it by F (for 'fundamental'). It is the complexification of a real vector bundle associated to the Euclidean representation of $\mathrm{SO}(5) = \mathrm{Sp}(2)/\mathbb{Z}_2$ on $\mathbb{R}^5$. In Horrocks's notation,

$$\pi^* F = \pi^*(\Lambda^2_0 E) \cong S\langle 1^2 \rangle,$$

the symbol 1^2 representing the character defining F, whose highest weight vector is $(1,1)$. The null-correlation bundle $S \cong \pi^* E$ is defined by the exact sequence

$$0 \to S(1) \longrightarrow T\mathbb{CP}^5 \xrightarrow{\theta} \mathcal{O}(2) \to 0$$

where θ is the contact 1-form that can be defined as follows. If we use $\mathbb{U}$ to denote also (its pullback) the trivial bundle on $\mathbb{CP}^5$, we have a holomorphic sequence

$$0 \to T^*\mathbb{CP}^5(1) \longrightarrow \mathbb{U}^* \longrightarrow \mathcal{O}(1) \to 0, \tag{7}$$

in which $\mathbb{U}^*$ can be identified with the bundle of 1-jets of holomorphic sections of the hyperplane section bundle $\mathcal{O}(1)$. We can now define a sequence

$$\mathcal{O}(-1) \xrightarrow{\iota} \mathbb{U} \xrightarrow{\omega} \mathbb{U}^* \xrightarrow{\iota^*} \mathcal{O}(1).$$

Since ω is skew-symmetric it follows that the image of the tautological line bundle $\mathcal{O}(-1)$ lies in $\ker(\iota^*)$, which we can identify with $T^*\mathbb{CP}^5(1)$, and θ arises from the dual of ι. As smooth vector bundles, we have $\mathbb{U} = \mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus S$.

A more concrete description can be given as follows: by dualising (7), we may identify $T\mathbb{CP}^5(-1)$ with the quotient of $\mathbb{U}$ consisting of vectors $\sum_{i=1}^6 a_i u_i$ modulo the relation $\sum_{i=1}^6 x_i u_i = 0$ over the point $[x_i] \in \mathbb{CP}^5$. Then S can be identified with the subbundle

$$\Big\{ \sum_{i=1}^6 a_i u_i : a_1x_4 - a_4x_1 + a_2x_5 - a_5x_2 + a_3x_6 - a_6x_3 = 0 \Big\}$$

of $T\mathbb{CP}^5(-1)$. The symplectic form ω induces one on S, and we obtain the complex rank 5 vector bundle $S\langle 1^2 \rangle = \Lambda^2 S/\langle \omega \rangle$.

In order to introduce the Horrocks bundle Y, we fix a basis of $\mathbb{U}$ consistent with (4), and distinguish the element

$$\xi = u^1 \wedge u^2 \wedge u^3 + u^4 \wedge u^5 \wedge u^6 = u^1 \wedge u^2 \wedge u^3 + \tilde{u}^1 \wedge \tilde{u}^2 \wedge \tilde{u}^3 \tag{8}$$

of $\Lambda^3 \mathbb{U}^*$. In abbreviated form, we write

$$\xi = 123 + \tilde{1}\tilde{2}\tilde{3}, \qquad \tilde{\xi} = j\xi = -123 + \tilde{1}\tilde{2}\tilde{3}. \tag{9}$$

The choice of ξ amounts to imposing a splitting

$$\mathbb{C}^6 = L \oplus jL, \tag{10}$$

where $L = \langle u^1, u^2, u^3 \rangle$, but without distinguishing the Lagrangian subspaces L and jL. The next result can be found in [26], but is also well known in the context of special geometries, see [25]:

Proposition 2.2.

(i) *The stabiliser of* ξ *in* $\mathrm{GL}(6,\mathbb{C})$ *is generated by* $\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})$ *and* j.

(ii) *The stabiliser of the pair* (ω, ξ) *in* $\mathrm{GL}(6,\mathbb{C})$ *is* $\mathrm{SL}(3,\mathbb{C})$ *acting diagonally on* (10)*, which is a subgroup of* $\mathrm{Sp}(3,\mathbb{C})$.

Having fixed (ω, j), the conditions $\xi \wedge \omega = 0$ and $\xi \wedge j\xi \neq 0$ on a 3-form ξ allow us to find a basis of $\mathbb{C}^6$ for which the equations (4) and (8) are simultaneously valid, and so (ii) holds. It follows that $\mathrm{GL}(6, \mathbb{C})$ has a dense orbit in the subspace

$$\{(\omega, \xi) \in \Lambda^2\mathbb{U}^* \times \Lambda^3\mathbb{U}^* : \omega \wedge \xi = 0\},$$

which is 28 dimensional.

The vector bundle Y is defined as the cohomology of a monad given by

$$\mathcal{O}(-1) \longrightarrow S\langle 1^2\rangle \longrightarrow \mathcal{O}(1) \tag{11}$$

where the two maps are induced by ξ and $-\tilde{\xi}$ respectively, as we shall explain below.

For background on monads, see [32]. Returning back to the construction we first see that twisting the first map of (11) by $\mathcal{O}(1)$ we get a trivial bundle embedding $\mathcal{O} \to S\langle 1^2\rangle(1)$ and thus to construct the first map it suffices to find a nowhere vanishing section of $S\langle 1^2\rangle(1)$. Since for this rank 5 vector bundle we have $c_5(S\langle 1^2\rangle(1)) = 0$ and the base $\mathbb{CP}^5$ is also 5 dimensional it follows that a generic section is nowhere vanishing.

Horrocks then shows that as $\mathrm{Sp}(6, \mathbb{C})$ representations, using a generalisation of Borel-Weil Theorem, we have

$$H^0(\mathbb{CP}^5, \mathcal{O}(S\langle 1^2\rangle(1))) \cong \langle 1^3\rangle = \Lambda^3_0(\mathbb{C}^6) = \{\alpha \in \Lambda^3(\mathbb{C}^6) : \alpha \wedge \omega = 0\}.$$

Since the orbit of ξ in $\Lambda^3_0(\mathbb{C}^6)$ is dense (note that it includes $-\tilde{\xi}$) it follows that any element in this orbit defines a nowhere vanishing section of $S\langle 1^2\rangle(1)$. Using the fact that $S\langle 1^2\rangle$ is self-dual the above argument also applies to the second map and thus any element in the orbit of ξ also defines a map $S\langle 1^2\rangle \to \mathcal{O}(1)$.

The final step of Horrocks's construction is to ensure that the composition $\mathcal{O}(-1) \to \mathcal{O}(1)$ determined by the pair $(\xi, -\tilde{\xi})$ is zero. A simple computation now shows that the total Chern class of the Horrocks bundle is $1 + 3y^2$, see Corollary 5.1.

3. Twistor sections over $\mathbb{HP}^2$

We now investigate the geometry on $\mathbb{HP}^2$ that results from reducing its isometry group $\mathrm{Sp}(3)$ to a subgroup acting with cohomogeneity one.

Let us fix a non-zero vector in $\mathbb{C}^6$, or equivalently a constant section u of $\mathbb{U}$. Using (3), we can write

$$u = u_E + u_H, \tag{12}$$

where $u_E \in \Gamma(E)$, $u_H \in \Gamma(H)$, and Γ denotes a space of smooth sections over $\mathbb{HP}^2$.

Proposition 3.1. *Up to* $\mathrm{Sp}(2)\mathrm{Sp}(1)$ *equivariant mappings,* ∇u_E *is a non-zero multiple of* u_H*, whilst* ∇u_H *is a non-zero multiple of* u_E*.*

Proof. Let D denote the flat connection on $\mathbb{U}$, and let x be an arbitrary point of $\mathbb{HP}^2$. Referring to (3), let e be a section of E. We can decompose

$$(De)_x = (\nabla e)_x + A(e) \ \in \ T^* \otimes \mathbb{U},$$

with $(\nabla e)_x \in T^* \otimes E$ and $A(e) \in T^* \otimes H$. Here, ∇ is an induced connection on E and A represents the second fundamental form of E as a subbundle of $\mathbb{U}$. Because $\mathbb{HP}^2 = \mathrm{Sp}(3)/(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$ is a Riemannian symmetric space, ∇ must coincide with the connection on E induced from the Levi-Civita connection on (2). Moreover A is a non-zero Sp(2)Sp(1) equivariant map (a tensor), and is therefore a multiple of the natural inclusion

$$E \hookrightarrow T^* \otimes H = EH \otimes H \cong E \,\oplus\, E\,S^2H. \tag{13}$$

We can decompose Du_H in the same way, and tabulate the results as follows:

	Du_E	Du_H
$T^* \otimes E$	∇u_E	$A(u_H)$
$T^* \otimes H$	$A(u_E)$	∇u_H

Since u is a constant tensor,

$$0 = Du = Du_E + Du_H,$$

and it follows that the sum of the terms in each row of the table is zero. $\square$

In the context of twistor theory, the condition that ∇u_H have no component in $E \otimes S^2H$ is expressed by saying that u_H is a solution of the 'twistor equation' [34]. Such solutions give rise to holomorphic data over $\mathbb{CP}^5$ as follows. Let $x \in \mathbb{HP}^2$, so that $H_x \cong \mathbb{C}^2$ is the fibre of H, and $\pi^{-1}(x) \cong \mathbb{P}_{\mathbb{C}}(H_x)$ is the corresponding twistor fibre in $\mathbb{CP}^5$. Then

$$H_x \cong H_x^* \cong H^0(\pi^{-1}(x), \mathcal{O}(1)). \tag{14}$$

It follows that u_H defines a smooth section of $\mathcal{O}(1)$ over $\mathbb{CP}^5$. It is known that such a section will be *holomorphic* if and only if $\nabla u_H \in \Gamma(E)$; see [24] for the analogous 4-dimensional statement.

Proposition 3.1 is asserting the existence of a complex 6-dimensional space of solutions of the twistor equation for sections of H. The twistor equation is analogous to the equation for Killing vector fields and is overdetermined; in fact any local solution must extend to u_H for some constant u. Note that the twistor space $\mathbb{CP}^5$ *is* the associated projective space of such solutions. We can also speak of u_E as a solution of a twistor equation, though a holomorphic interpretation would relate not to $\mathbb{CP}^5$ but to the flag manifold Sp(3)/(U(3)Sp(1)). We shall consider further variants of the twistor equations in this section.

The stabiliser of u in Sp(3) is the isotropy subgroup $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ of the point x_0 of $\mathbb{HP}^2$ representing the quaternionic line $\langle u, ju \rangle$. This group acts on $\mathbb{HP}^2$ with generic orbit

$$\frac{\mathrm{Sp}(2) \times \mathrm{Sp}(1)}{\mathrm{Sp}(1) \times \mathrm{Sp}(1)'} \cong S^7, \tag{15}$$

where $\mathrm{Sp}(1)'$ embeds diagonally, and singular orbits

(i) $\{x_0\}$, the 'origin', and the zero set of u_E;
(ii) the 'line at infinity' $\mathbb{HP}^1 \cong S^4$, and the zero set of u_H.

The existence of sections of E and H vanishing on disjoint subsets enables one to construct global Spin(7) structures on $\mathbb{HP}^2$ and find their associated 4-forms. We shall explain this in detail in the sequel, but with reference to the subgroup U(3) of Sp(3) that has the advantage of acting irreducibly on $\mathbb{C}^6$.

First, we fix ω, j, ξ as defined in Section 2.

Lemma 3.2. *The stabiliser of the triple* (ω, j, ξ) *is isomorphic to* SU(3)*.*

Proof. Define a non-degenerate Hermitian inner product f by

$$f(X,Y) = \omega(X, jY).$$

Since ω is j-invariant, we have

$$f(Y,X) = \omega(Y, jX) = -\omega(jX, Y) = -\overline{\omega(j^2X, jY)} = \overline{f(X,Y)},$$

as required. The stabiliser of the pair (f, ξ) is therefore the intersection of U(6) with diagonal $\mathrm{SL}(3,\mathbb{C})$, which is SU(3). □

We shall now regard ω and ξ as constant sections of the trivial bundles $\Lambda^2\mathbb{U}$ and $\Lambda^3\mathbb{U}$. Together with $j\colon \mathbb{U} \to \mathbb{U}$, they define a reduction of the structure group from $\mathrm{GL}(6,\mathbb{C})$ to SU(3), and in particular a decomposition (10) in which each summand *is* distinguished. These summands are also the eigenspaces for an action $e^{i\theta} \mapsto (e^{i\theta}, e^{-i\theta})$ of U(1) on $\mathbb{C}^6$ that commutes with j. Then

$$\mathrm{U}(1)\mathrm{SU}(3) = \mathrm{U}(3) \subset \mathrm{Sp}(3).$$

We can choose constant sections u^1, u^2, u^3 of $\mathbb{U}$ for which (4) and (8) hold, and the symmetric inner product $g = \mathrm{Re}\, f$ satisfies

$$g = i(u^1 \odot \tilde{u}^1 + u^2 \odot \tilde{u}^2 + u^3 \odot \tilde{u}^3). \tag{16}$$

Note that, like ω, the tensor g is j-invariant.

Each point $x \in \mathbb{HP}^2$ is specified by the fibre H_x of the tautological subbundle H of $\mathbb{U}$. At this point, we can choose a unitary basis $\{h, \tilde{h} = jh\}$ of H_x. Note that $h \wedge \tilde{h}$ is a globally-defined section of the trivial bundle $\Lambda^2 H$. Having chosen our standard basis of constant sections of $\mathbb{U}$, we can write

$$h = a1 + b2 + c3 + a'\tilde{1} + b'\tilde{2} + c'\tilde{3}$$

for $a, b, c, a', b', c' \in \mathbb{C}$. Acting by SU(3) on a unit vector $u \in \langle 1, 2, 3\rangle$ we can map u to 1, which itself has stabiliser isomorphic to SU(2). We can now act on $\langle \tilde{1}, \tilde{2}, \tilde{3}\rangle$ by this SU(2) and thus, we can map any $v \in \langle \tilde{1}, \tilde{2}, \tilde{3}\rangle$ to an element in $\langle \tilde{1}, \tilde{2}\rangle$. It follows that at each point x, we can always choose a unitary basis of $\mathbb{U}_x = \mathbb{C}^6$ leaving ω, ξ and $\tilde{\xi}$ unchanged so that

$$h = a1 + b\tilde{1} + c\tilde{2} \tag{17}$$

and $|a|^2 + |b|^2 + |c|^2 = 1$. It follows that $\tilde{h} = \bar{a}\tilde{1} - \bar{b}1 - \bar{c}2$.

The action of SU(3) on $\mathbb{H}^3 = \mathbb{C}^6$ induces a well-known cohomogeneity one action on $\mathbb{HP}^2$ with singular orbits a complex projective plane $\mathbb{CP}^2$ and a sphere S^5 (see also Section 5) [3]. We can characterise these in terms of (17) as follows:

(i) A point $x \in \mathbb{HP}^2$ belongs to $\mathbb{CP}^2 = \mathbb{P}_{\mathbb{C}}(L)$ iff $\dim(H_x \cap L) = 1$. Equivalently (since this dimension cannot exceed 1) $h \wedge \tilde{h} \wedge 123 = 0$, which means that $a = 0$ or $c = 0$.

(ii) A point $x \in \mathbb{HP}^2$ belongs to S^5 iff H_x is g-isotropic. This means that $b = 0$ and $|a| = |c|$, so that we can set $H_x = \langle 1 + e^{it}\tilde{2}, \tilde{1} - e^{-it}2\rangle$, and its projection to L equals $\langle 1, 2\rangle$. Thus S^5 fibres over the dual projective plane $\mathbb{P}_{\mathbb{C}}(L^*)$.

The $\mathbb{CP}^2$ in (i) is the fixed point set for the U(1) action, whereas the S^5 in (ii) is fibred by the maximal U(1) orbits. We shall quantify these facts below.

Consider the decomposition

$$\boxed{S^2\mathbb{U} \;\cong\; S^2E \,\oplus\, EH \,\oplus\, S^2H}, \tag{18}$$

analogous to (5), by writing

$$g = \zeta_E + X + \zeta_H, \tag{19}$$

where $\zeta_E \in \Gamma(S^2E)$, $X \in \Gamma(E \otimes H)$ and $\zeta_H \in \Gamma(S^2H)$. These components are all real and X can be viewed as a vector field on $\mathbb{HP}^2$.

Proposition 3.3. *Up to* Sp(2)Sp(1) *equivariant mappings,* ∇X *is a nontrivial linear combination of* ζ_E *and* ζ_H*, whilst* $\nabla\zeta_E$ *and* $\nabla\zeta_H$ *are both non-zero multiples of* X*.*

Proof. This is completely analogous to that of Proposition 3.1. We apply the argument leading to (13) to deduce that

$$(D\zeta_E)_x = (\nabla\zeta_E)_x + A(\zeta_E) \in T^* \otimes S^2\mathbb{U},$$

where $(\nabla\zeta_E)_x \in T^* \otimes S^2E$ and A embeds ζ_E in $T^* \otimes EH$ (since $T^* \otimes S^2H$ has no S^2E component).

We can decompose DX and $D\zeta_H$ in the same way, and tabulate their components in each of the three summands of $T^* \otimes \mathbb{U}$. For example, there are non-zero tensorial components $A_1(X)$ and $A_2(X)$ of DX in $T^* \otimes S^2E$ and $T^* \otimes S^2H$, since both these tensor products contain EH as a submodule:

	$D\zeta_E$	DX	$D\zeta_H$
$T^* \otimes S^2E$	$\nabla\zeta_E$	$A_1(X)$	
$T^* \otimes EH$	$A(\zeta_E)$	∇X	$A(\zeta_H)$
$T^* \otimes S^2H$		$A_2(X)$	$\nabla\zeta_H$

Since g is a constant tensor,

$$0 = Dg = D\zeta_E + DX + D\zeta_H,$$

and it follows that the sum of the terms in each row of the table is zero. □

Armed with Proposition 3.3 and SU(3) invariance, we can easily recognise the terms in (19). Namely, X can be identified with the Killing vector field determined by the action of U(1). As a Killing vector field, its covariant derivative ∇X takes values at each point in the holonomy bundle with fibre $\mathfrak{sp}(1) + \mathfrak{sp}(2)$, which is consistent with the equation

$$\nabla X = A(\zeta_E) + A(\zeta_H) \in S^2E \oplus S^2H.$$

In particular, ζ_H is the section of S^2H defined by the (Galicki-Lawson) analogue of the hyperkähler moment map for the action of U(1). We already know that X vanishes on $\mathbb{CP}^2$, the latter being the fixed point set for U(1). On the other hand it is well known that the zero set of the moment 'map' ζ_H is S^5, and that $S^5/\mathrm{U}(1) = \mathbb{CP}^{2*}$ is the QK quotient.

We can verify these facts using

Lemma 3.4. *With h of the form (17) we have*

$$\zeta_H = i(2\bar{a}\bar{b})h \otimes h - i(2ab)\tilde{h} \otimes \tilde{h} + i(|a|^2 - |b|^2 - |c|^2)(h \otimes \tilde{h} + \tilde{h} \otimes h)$$

and $X = h \otimes \alpha_1 + \tilde{h} \otimes \alpha_2 \in \Gamma(H \otimes E)$ where

$$\begin{aligned}\alpha_1 &= i((2\bar{a}|c|^2)\tilde{1} + (2|a|^2\bar{c})2 + (-2\bar{a}\bar{b}c)\tilde{2}),\\ \alpha_2 &= i((2a|c|^2)1 + (-2ab\bar{c})2 + (-2|a|^2c)\tilde{2}).\end{aligned}$$

Proof. Computing we have

$$\begin{aligned}h \cdot g &= (a\tilde{1} - b1 - c2) \cdot g = i(a1 - b\tilde{1} - c\tilde{2}),\\ \tilde{h} \cdot g &= (-\bar{a}1 - \bar{b}\tilde{1} - \bar{c}\tilde{2}) \cdot g = i(-\bar{a}\tilde{1} - \bar{b}1 - \bar{c}2).\end{aligned}$$

Therefore,

$$\begin{aligned}h \cdot \tilde{h} \cdot g = \tilde{h} \cdot h \cdot g &= i(-|a|^2 + |b|^2 + |c|^2),\\ h \cdot h \cdot g &= i(-2ab),\\ \tilde{h} \cdot \tilde{h} \cdot g &= i(2\bar{a}\bar{b}).\end{aligned}$$

It is now just a matter of identifying the components of g. Observe that $\alpha_2 = j(\alpha_1)$, confirming indeed that X is real. We can also directly check that $h \cdot \alpha_2 = \tilde{h} \cdot \alpha_2 = 0$, confirming that $\alpha_i \in E$. □

It now follows that the zero set of X corresponds to $a = 0$ or $c = 0$, and that of ζ_H to $b = 0$ and $|a| = |c|$. These two zero sets correspond to the singular orbits $\mathbb{CP}^2$ and S^5 respectively. Thus,

Corollary 3.5. *X is nowhere zero on $\mathbb{HP}^2 \setminus \mathbb{CP}^2$, and ζ_H is nowhere zero on $\mathbb{HP}^2 \setminus S^5$.*

An analogous result appears at the end of the next section.

4. Horrocks bundle revisited

Recall the definition of the Horrocks bundle Y over $\mathbb{CP}^5$ via the monad (11). The main result of this section is an application of (3) to construct Y (up to the action of $\mathrm{Sp}(3, \mathbb{C})$) as the pullback of a vector subbundle V of F over $\mathbb{HP}^2$ with an instanton connection. We shall show that this subbundle is entirely determined by the covariant derivative of a suitable section of E.

We start from the third exterior power

$$\boxed{\Lambda^3\mathbb{U} \;\cong\; \Lambda^3 E \,\oplus\, (\Lambda^2 E\, H) \,\oplus\, (E\, \Lambda^2 H)}$$

in place of (18), and consider the SU(3) invariant tensor ξ in place of g. Note that $\Lambda^3 E \cong E$ as Sp(2) modules, and $\Lambda^2 H$ is trivial, so $\Lambda^3\mathbb{U}$ really contains two copies of E. The symplectic form ω embeds $\mathbb{U}$ in $\Lambda^3\mathbb{U}$ and, by analogy to (6), its orthogonal complement is

$$\Lambda^3_0\mathbb{U} \;\cong\; E \,\oplus\, F\,H. \tag{20}$$

The first summand E here sits diagonally across the subspaces $\Lambda^3 E$ and $E\,\Lambda^2 H$ of $\Lambda^3\mathbb{U}$. Since $\omega \wedge \xi = \omega \wedge \tilde{\xi} = 0$ it follows that $\xi, \tilde{\xi} \in \Lambda^3_0\mathbb{U}$.

It is convenient to work with the simple SU(3)-invariant 3-form

$$\eta = \tfrac{1}{2}(\xi - \tilde{\xi}) = u^1 \wedge u^2 \wedge u^3$$

(abbreviated to 123) in place of ξ. Set

$$\eta = \eta_E + \eta_H,$$

where $\eta_E \in E$ and $\eta_H \in FH$ in (20).

Proposition 4.1. *Up to* Sp(2)Sp(1) *equivariant mappings,* $\nabla\eta_E$ *is a non-zero multiple of* η_H*, whilst* $\nabla\eta_H$ *is non-zero multiple of* η_E*.*

Proof. This proceeds almost exactly as in Proposition 3.1, with the table

	$D\eta_E$	$D\eta_H$
$T^* \otimes E$	$\nabla\eta_E$	$A(\eta_H)$
$T^* \otimes FH$	$A(\eta_E)$	$\nabla\eta_H$

Again, each column must sum to zero. □

Recall that the complex vector bundles H and F (equivalently, the underlying modules) are quaternionic and real respectively. In particular, there are Sp(1) and Sp(2) equivariant isomorphisms $H^* \cong H$ and $F^* \cong F$, and

$$HF \;\cong\; \mathrm{Hom}(H, F) \;\cong\; \mathrm{Hom}(F, H).$$

This makes it easier to grasp the significance of the rank of the tensor η_H. The next result asserts that it is everywhere maximal:

Lemma 4.2. *The section* η_H *has rank* 2 *at every point of* $\mathbb{HP}^2$*.*

Proof. We can write

$$\xi = \gamma + (h \wedge \beta + \tilde{h} \wedge \beta') + (h \wedge \tilde{h} \wedge \alpha),$$

where $\gamma \in \Lambda^3_0 E$, $\beta, \beta' \in F$ and $\alpha \in E$. We need to show that $\dim\langle \beta, \beta' \rangle = 2$. We shall denote contraction (interior product) using ω by a centred dot. Then

$$\tilde{h} \cdot \xi = -\beta - \tilde{h} \wedge \alpha,$$

so $\tilde{h} \wedge (\tilde{h} \cdot \xi) = -\tilde{h} \wedge \beta$, and

$$\begin{aligned} \beta &= -h \cdot (\tilde{h} \wedge (\tilde{h} \cdot \xi)) \\ &= -h \cdot (\tilde{h} \wedge (\tilde{h} \cdot 123)) - h \cdot (\tilde{h} \wedge (\tilde{h} \cdot \tilde{1}\tilde{2}\tilde{3})), \end{aligned}$$

using (9). Moreover,

$$\begin{aligned} \beta' &= -\tilde{h} \cdot (h \wedge (h \cdot \xi)) \\ &= -j\big[h \cdot (\tilde{h} \wedge (\tilde{h} \cdot j\xi))\big] \\ &= j\big[h \cdot (\tilde{h} \wedge (\tilde{h} \cdot 123)) - h \cdot (\tilde{h} \wedge (\tilde{h} \cdot \tilde{1}\tilde{2}\tilde{3}))\big]. \end{aligned}$$

In the following computations, we separate the two terms 123 and $\tilde{1}\tilde{2}\tilde{3}$, and effectively work out β and β' simultaneously.

We have

$$\begin{aligned}\tilde{h}\cdot 123 &= -\bar{a}23\\ \tilde{h}\cdot\tilde{1}\tilde{2}\tilde{3} &= -\bar{b}\tilde{2}\tilde{3}+\bar{c}\tilde{1}\tilde{3},\end{aligned}$$

and

$$\begin{aligned}\tilde{h}\wedge(\tilde{h}\cdot 123) &= -\bar{a}^2\tilde{1}23+\bar{a}\bar{b}123\\ \tilde{h}\wedge(\tilde{h}\cdot\tilde{1}\tilde{2}\tilde{3}) &= -\bar{a}\bar{b}\tilde{1}\tilde{2}\tilde{3}+\bar{b}^2 1\tilde{2}\tilde{3}+\bar{b}\bar{c}2\tilde{2}\tilde{3}-\bar{b}\bar{c}1\tilde{1}\tilde{3}+\bar{c}^2\tilde{1}2\tilde{3}.\end{aligned}$$

It follows that

$$\begin{aligned}h\cdot(\tilde{h}\wedge(\tilde{h}\cdot 123)) &= -\bar{a}|a|^2 23-\bar{a}|b|^2 23-\bar{a}^2 c\tilde{1}3+\bar{a}\bar{b}c13\\ h\cdot(\tilde{h}\wedge(\tilde{h}\cdot\tilde{1}\tilde{2}\tilde{3})) &= -\bar{b}|a|^2\tilde{2}\tilde{3}+a\bar{b}\bar{c}1\tilde{3}+a\bar{c}^2 2\tilde{3}-\bar{b}|b|^2\tilde{2}\tilde{3}+\bar{c}|b|^2\tilde{1}\tilde{3}-\bar{b}|c|^2\tilde{2}\tilde{3}+\bar{c}|c|^2\tilde{1}\tilde{3}.\end{aligned}$$

Therefore

$$\begin{aligned}\beta &= \bar{a}(|a|^2+|b|^2)23+\bar{a}^2 c\tilde{1}3-\bar{a}\bar{b}c13\\ &\quad+\bar{b}(|a|^2+|b|^2+|c|^2)\tilde{2}\tilde{3}-a\bar{b}\bar{c}1\tilde{3}-a\bar{c}^2 2\tilde{3}-\bar{c}(|b|^2+|c|^2)\tilde{1}\tilde{3}\\ \beta' &= b(|a|^2+|b|^2+|c|^2)23+\bar{a}bc\tilde{1}3+\bar{a}c^2\tilde{2}3-c(|b|^2+|c|^2)13\\ &\quad-a(|a|^2+|b|^2)\tilde{2}\tilde{3}+a^2\bar{c}1\tilde{3}+ab\bar{c}\tilde{1}\tilde{3}.\end{aligned}$$

One can also check that

$$h\cdot\beta=0,\quad \tilde{h}\cdot\beta=0,\quad h\cdot\beta'=0,\quad \tilde{h}\cdot\beta'=0$$

(the second and last equations are obvious), confirming that the elements β,β' both belong to F.

Now suppose that β,β' do not span 2 dimensions. Note that only β has a term $2\tilde{3}$, and only β' has a term $\tilde{2}3$, so both the coefficients must vanish. This means that $a=0$ or $c=0$. In the former case, it follows easily that $b=c=0$, so $h=0$. In the latter case, one obtains

$$\det\begin{pmatrix}\bar{a} & \bar{b}\\ b & -a\end{pmatrix}=0,$$

so again $h=0$, which is a contradiction. □

The Horrocks bundle V on $\mathbb{HP}^2$ can now be defined as the

$$V=\ker(\eta_H\colon F\to H).$$

Since $\mathrm{SO}(5)\subset\mathrm{SU}(5)$ acts on F and H has an $\mathrm{SU}(2)$-structure it follows that V inherits an $\mathrm{SU}(3)$-structure. Because the holonomy of $\mathbb{HP}^2$ lies in $\mathrm{Sp}(2)\mathrm{Sp}(1)$, the Riemannian connection induced on F is self-dual in the sense of Definition 2.1: its 2-forms lie in the subspace $\mathfrak{sp}(2)\cong S^2E$ of Λ^2T^* at each point of $\mathbb{HP}^2$.

The next result also appears in [30, Theorem 6].

Theorem 4.3. *The connection induced on V is itself self-dual.*

Proof. This follows from Proposition 4.1, because the section η_H that defines V as a subbundle of F satisfies the twistor equation, namely $\varpi(\nabla\eta_H) = 0$, where ϖ is the projection

$$T^* \otimes F\,H = EH \otimes F\,H \longrightarrow (E \,\oplus\, K) \otimes S^2H$$

obtained by symmetrising the factor $H \otimes H$. This fact can be used to prove directly that the subbundle V of F admits a self-dual connection, using the methods of [4].

Alternatively, we can complete the circle by showing η_H defines the required holomorphic map over $\mathbb{CP}^5$ as follows. It follows from (14) that η_H defines a section of $\pi^*F \otimes \mathcal{O}(1)$. Such a section will be holomorphic if and only if η_H satisfies the twistor equation $\varpi(\nabla\eta_H) = 0$. □

Returning to the proof of Lemma 4.2, an easier calculation gives

$$\alpha = h \cdot (\tilde{h} \cdot (123 - \tilde{1}\tilde{2}\tilde{3}) = \bar{a}c3 - a\bar{c}\tilde{3},$$

which vanishes if and only if $a = 0$ or $c = 0$, i.e. on the singular orbit $\mathbb{CP}^2$.

Corollary 4.4. *The section η_E is nowhere zero on $\mathbb{HP}^2 \setminus \mathbb{CP}^2$.*

One can use η_E to manufacture further tensors invariant by SU(3), namely

$$\begin{aligned} \phi_E &= (\eta_E \wedge j\eta_E)_0 &&\in \Lambda^2_0 E \cong F, \\ \psi_E &= i\eta_E \wedge j\eta_E &&\in S^2E. \end{aligned} \tag{21}$$

Observe that both are invariant by both i and j, meaning that the tensors are invariant by U(3), and are real, i.e. elements of the underlying real vector spaces. Like η_E itself, they will both be nowhere zero away from $\mathbb{CP}^2$.

Remark 4.5. The vector field X can be used to construct an invariant of the same type as ϕ_E, namely

$$\psi_E = \pi_5(X \otimes X) \;\in\; F \subset S^2T^*,$$

where π_5 denote projection to the 5-dimensional submodule of symmetric tensors. Calculations reveal that these two U(3) invariants are proportional. We also expect ψ_E and ζ_E to be proportional.

Corollaries 3.5 and 4.4 will be used to define explicit Spin(7) structures on the projective plane $\mathbb{HP}^2$.

5. Spinors and characteristic classes

It is well known that the quaternionic projective plane $\mathbb{HP}^2$ has zero integral cohomology in degrees 1 and 2. In particular, its first and second Stiefel-Whitney classes vanish, so it has a unique Spin(8)-structure. Actually, the same is true for any 8-manifold whose structure reduces to the subgroup Sp(2)Sp(1) of SO(8). Its spinor bundle $\Delta = \Delta_+ \,\oplus\, \Delta_-$ is given by

$$\Delta_+ \cong F \,\oplus\, S^2H, \qquad \Delta_- \cong E\,H, \tag{22}$$

and there is a lifting $\mathrm{Sp}(2)\mathrm{Sp}(1) \subset \mathrm{Spin}(8)$. (Recall that F is shorthand for Λ^2_0E, and $E\,H$ for $E \otimes H$.) In particular, $\Delta_- \cong T\mathbb{HP}^2 \cong T^*\mathbb{HP}^2$.

The splitting (22) reflects the similarity between an almost quaternionic structure (defined by Sp(2)Sp(1)) on an 8-manifold, and a Grassmannian structure (defined by SO(3)×SO(5)). Indeed, there is an isomorphism

$$\frac{\mathrm{Spin}(8)}{\mathrm{Sp}(2)\mathrm{Sp}(1)} \longrightarrow \frac{\mathrm{SO}(8)}{\mathrm{SO}(3)\times\mathrm{SO}(5)} \tag{23}$$

of simply-connected symmetric spaces induced by triality. This theme was developed by Witt [39].

Let us recall the basic characteristic class computations for $\mathbb{HP}^2$, using Chern characters. In the absence of cohomology in degrees 1 and 3, the Chern character of a vector bundle W satisfies

$$\mathrm{ch}\, W = \mathrm{rk}(W) - c_2 + \tfrac{1}{12}(c_2^2 - 2c_4).$$

In particular, with the previous notation in which $u = -c_2(H)$ generates $H^4(\mathbb{HP}^2, \mathbb{Z})$, we have

$$\mathrm{ch}(H) = 2 + u + \tfrac{1}{12}u^2.$$

It follows from (3) that

$$\mathrm{ch}(E) = 6 - \mathrm{ch}(H) = 4 - u - \tfrac{1}{12}u^2, \tag{24}$$

or equivalently the total Chern class of E is given by

$$c(E) = (1-u)^{-1} = 1 + u + u^2.$$

We can now compute the Chern character of $T_{\mathbb{C}} = T_{\mathbb{C}}\mathbb{HP}^2$ using (2):

$$\mathrm{ch}(T_{\mathbb{C}}) = \mathrm{ch}(E)\,\mathrm{ch}(H) = 8 + 2u - \tfrac{5}{6}u^2. \tag{25}$$

Using standard techniques, it also follows that

$$\begin{aligned} \mathrm{ch}(S^2H) = \mathrm{ch}(H)^2 - 1 &= 3 + 4u + \tfrac{4}{3}u^2, \\ \mathrm{ch}(F) = \mathrm{ch}(\Lambda^2 E) - 1 &= 5 - 2u + \tfrac{5}{6}u^2. \end{aligned} \tag{26}$$

Since $\mathrm{ch}(V) = \mathrm{ch}(F) - \mathrm{ch}(H)$, we have

Corollary 5.1. *The Horrocks instanton bundle has Chern character*

$$\mathrm{ch}(V) = 3 - 3u + \tfrac{3}{4}u^2.$$

Remark 5.2. As Horrocks points out, any rank 3 vector bundle on $\mathbb{CP}^5$ with $c_1 = c_3 = 0$ must have $c_2 = ax$ with a one of 3, 8 or 11 modulo 12 [23, p. 166]. Working back down on $\mathbb{HP}^2$, had we not proved Lemma 4.2 nor the existence of an embedding $H \subset F$, we could have recognised the possibility that the K-theory element $F - H$ is a genuine vector bundle by the fact that $c_4(F - H) = 0$. It may be that there are other virtual Sp(2)Sp(1) modules or low rank that have vanishing higher Chern classes.

Adding the two lines in (26) together yields

$$\mathrm{ch}(\Delta_+) = 8 + 2u + \tfrac{13}{6}u^2.$$

From (25) and (26), we obtain

$$\mathrm{ch}(\Delta_+ - \Delta_-) = \mathrm{ch}(\Delta_+) - \mathrm{ch}(T_{\mathbb{C}}) = 3u^2,$$

which integrates to give the Euler number χ of $\mathbb{HP}^2$. The last equation is a version of the Gauss-Bonnet theorem, reflecting the fact that $(\Delta_+)^2 - (\Delta_-)^2$ equals (in the sense of K-theory) the de Rham complex, see forward to (28). We also record

Proposition 5.3. *The spinor bundles have Pontrjagin classes*

$$p_1(\Delta_+) = -2u, \qquad p_2(\Delta_+) = -11u^2,$$
$$p_1(\Delta_-) = 2u, \qquad p_2(\Delta_-) = 7u^2.$$

The last two classes in the proposition are those of the quaternionic projective plane $\mathbb{HP}^2$ itself.

As regards Δ_+ over $\mathbb{HP}^2$, its Euler class vanishes because of the odd-dimensional summands. Since the rank of Δ_+ coincides with the dimension of the base, it must possess a smooth non-vanishing section. The choice of such a (say, unit) δ gives an explicit reduction of the structure group of $\mathbb{HP}^2$ to Spin(7), indeed to $\mathrm{Sp}(1)^3/\mathbb{Z}_2$, see Section 6. Now, S^2H cannot admit a nowhere-zero section since it would then split as the sum of a complex line bundle and a trivial bundle and have zero Chern classes on $\mathbb{HP}^2$. The same is true of the other summand of Δ_+:

Lemma 5.4. *The vector bundle F has no nowhere-zero section over $\mathbb{HP}^2$.*

Proof. Recall that F is the complexification of a real vector bundle. Without loss of generality, we may assume that any nowhere-zero section is real, and therefore determines a complex rank 4 subbundle F' with an SO(4) structure. By expressing $F' = A \otimes B$ locally as the tensor product of two spinor bundles, we obtain

$$\begin{aligned} \mathrm{ch}(F') &= (2 - au + \tfrac{1}{12}a^2u^2)(2 - bu + \tfrac{1}{12}b^2u^2) \\ &= 4 - 2(a+b)u + \tfrac{1}{6}(a^2 + 6ab + b^2)u^2, \end{aligned}$$

where $4a, 4b \in \mathbb{Z}$. This incidentally shows that $c_4(F') = (a-b)^2u^2$ is the square of an Euler class. From (26), we deduce that $a + b = 1$ and

$$5 = a^2 + 6ab + b^2 = 4a - 4a^2 + 1,$$

so $a^2 - a + 1 = 0$, which is impossible. □

The above proof is a variant of one given to the authors by Diarmuid Crowley. See [14] for related calculations.

Remark 5.5. Each fibre of unit elements in S^2H is the 2-sphere $\{aI + bJ + cK\}$ of almost complex structures defining the quaternionic structure of $\mathbb{HP}^2$ at that point. Thus, a section of S^2H defines an almost complex structure wherever it is non-zero. It is well known that $\mathbb{HP}^2$ admits no almost complex structure, essentially because $\chi - \sigma = 2$ is not divisible by 4, see for example [31]. See [20] for non-existence of almost complex structures on other QK symmetric spaces.

Using Proposition 5.3 and the fact that $\langle \mathbb{HP}^2, u^2 \rangle = 1$, the characteristic numbers of $\mathbb{HP}^2$ satisfy

$$4p_2 - {p_1}^2 = 8\chi. \tag{27}$$

In fact, any 8-manifold whose structure group reduces to Spin(7), Sp(2)Sp(1) or SU(4) satisfies (27), the SU(4) case being particularly easy [36]. The equality for Spin(7) dates back to [21] (except that '8' is accidentally missing), and our remarks confirm that it also holds for the QK manifolds $\mathrm{G}_2/\mathrm{SO}(4)$ and $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4) \cong \mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$, cf. [21, Theorem 4.5]. In fact, it is known that (27) is both a necessary and sufficient condition for the existence of a Spin(7) structure [29, Theorem 10.7].

Theorem 5.6. *The Riemannian symmetric space* $\mathbb{HP}^2$ *admits families of* Spin(7) *structures (compatible with the QK metric) invariant by the cohomogeneity-one action of* U(3) *and depending on two arbitrary functions.*

Proof. Corollaries 3.5 and 4.4 are tailored for this purpose. The former asserts that ζ_H vanishes only on S^5, and latter furnishes a section ϕ_E of F that vanishes only on $\mathbb{CP}^2$. Then any non-trivial linear combination

$$\delta = a\,\phi_E + b\,\zeta_H$$

is nowhere-zero on $\mathbb{HP}^2$. It remains to replace a and b by suitable functions.

One can parametrise the orbits of U(3) by the QK moment mapping $f = \|\zeta_H\|^2$ as in [7]. The derivative df of this function is essentially $\zeta_H(X)$. Moreover, f vanishes on S^5 and achieves a maximum f_1 on $\mathbb{CP}^2$. We are therefore free to take $a = a(f)$ and $b = b(f)$ to be smooth functions of f such that $a(0) \neq 0$ and $b(f_1) \neq 0$. □

An aim of the next two sections will be to realise the resulting Spin(7) structures more explicitly, and construct others compatible with different metrics.

6. Invariant 4-forms

A generalisation of (22) to higher dimensions was first described in [6]. At this juncture, let us work with representations rather than vector bundles, with little modification of notation. Recall that if $\Delta \cong \mathbb{C}^{2^n}$ denotes the faithful representation of Spin$(2n)$, there is an equivariant isomorphism

$$\Delta \otimes \Delta \;\cong\; \Lambda^*(\mathbb{R}^{2n})^* = \bigoplus_{k=0}^{2n} \Lambda^k(\mathbb{R}^{2n})^*, \tag{28}$$

with corresponding decompositions of $\Delta_\pm \otimes \Delta_\pm$ that split the exterior algebra into even and odd subspaces. In the case of $n = 4$, we can supplement (28) with the isomorphisms

$$\Lambda^4_+ \cong S^2_0(\Delta_+), \qquad \Lambda^4_- \cong S^2_0(\Delta_-), \tag{29}$$

where

$$\Lambda^4(\mathbb{R}^8)^* = \Lambda^4_+ \,\oplus\, \Lambda^4_- \tag{30}$$

is the decomposition into the ±1-eigenspaces of Hodge $*$.

Having fixed a spin structure on an 8-manifold, a nowhere-zero section δ of the positive spin bundle reduces the structure group from Spin(8) to Spin(7) (the pointwise stabiliser of δ). An application of (30) to Theorem 5.6 is

Lemma 6.1 ([29]). *If* δ *is a unit spinor, the corresponding* Spin(7) *4-form equals the component of* $\delta \otimes \delta$ *in* Λ^4_+.

From now on, we indicate the SO(8) module $\mathbb{R}^8$ by T, so that its dual T^* represents the cotangent space at an arbitrary point of $\mathbb{HP}^2$. Whilst S^2T^* has a submodule isomorphic to F, the summand S^2H in (22) is of course isomorphic to the submodule of

$$\Lambda^2 T^* \;\cong\; S^2E \,\oplus\, S^2H \,\oplus\, F\,S^2H \tag{31}$$

generated pointwise by a triple

$$\begin{array}{rcl}\omega_1 & = & 12+34+56+78,\\ \omega_2 & = & 13+42+57+86,\\ \omega_3 & = & 14+23+58+67\end{array} \tag{32}$$

of 2-forms associated to the quaternionic structure. In this section and the next, the indices $1,\dots,8$ form an abbreviation for an orthonormal basis of real 1-forms (in contrast to previous sections, where they stood for elements of $\mathbb{C}^3$). It follows from (29) that

$$\begin{array}{rcl}\Lambda^4_+ & = & \mathbb{C} \,\oplus\, S^4H \,\oplus\, F\,S^2H \,\oplus\, S^2_0F,\\ \Lambda^4_- & = & F \,\oplus\, S^2E\,S^2H.\end{array} \tag{33}$$

The 1-dimensional summand of Λ^4T^* is spanned by the QK 4-form

$$\Omega = \omega_1^2 + \omega_2^2 + \omega_3^2 \tag{34}$$

first studied extensively by Kraines [28]. The stabiliser of Ω in $\mathrm{GL}(8,\mathbb{R})$ is isomorphic to $\mathrm{Sp}(2)\mathrm{Sp}(1)$.

The full decomposition of the exterior algebra under $\mathrm{Sp}(n)\mathrm{Sp}(1)$ for arbitrary n was used by Swann [38] to prove that the closure of the 4-form (34) is sufficient to imply that the holonomy reduces to $\mathrm{Sp}(n)\mathrm{Sp}(1)$ provided $n \geqslant 3$. That is, $d\Omega \equiv 0$ implies $\nabla\Omega \equiv 0$ in dimensions $4n \geqslant 12$, although this is not true in dimension 8. Indeed, $\mathrm{G}_2/\mathrm{SO}(4)$ admits an $\mathrm{Sp}(2)\mathrm{Sp}(1)$ structure that is not locally symmetric but for which $d\Omega \equiv 0$ [12]; the associated metric has a cohomogeneity-one action by $\mathrm{SU}(3)$. A corresponding statement for $\mathbb{HP}^2$ remains open.

We can present the decomposition (31) as

$$\Lambda^2T^* \;=\; \Lambda^2_{10} \,\oplus\, \Lambda^2_3 \,\oplus\, \Lambda^2_{15},$$

in which subscripts indicate the dimensions of irreducible summands for $\mathrm{Sp}(2)\mathrm{Sp}(1)$. They can all be defined with reference to wedging with Ω:

$$\begin{array}{rcl}\Lambda^2_{10} & = & \{\alpha : *(\alpha\wedge\Omega) = -6\alpha\}\\ \Lambda^2_3 & = & \{\alpha : *(\alpha\wedge\Omega) = 10\alpha\}\\ \Lambda^2_{15} & = & \{\alpha : *(\alpha\wedge\Omega) = 2\alpha\}.\end{array}$$

The subspace Λ^2_3 is generated by the forms (32). Note that our definition of Ω has no constant $\frac{1}{2}$; if this were adopted, the eigenvalues above would be -3, 5, 1.

We can likewise present the decompositions (33) as

$$\begin{array}{rcl}\Lambda^4_+ & = & \Lambda^4_1 \,\oplus\, \Lambda^4_{5+} \,\oplus\, \Lambda^4_{15} \,\oplus\, \Lambda^4_{14},\\ \Lambda^4_- & = & \Lambda^4_{5'} \,\oplus\, \Lambda^4_{30},\end{array}$$

in which the spaces with subscripts $5+$ and $5-$ are not isomorphic. Most of these spaces are distinguished by their parity (self or anti-self dual) and the action of $\mathrm{Sp}(1)$ via $\Lambda^2_3 \cong \mathfrak{sp}(1)$. In the lines below '$\omega$' stands for an arbitrary element $\sum_1^3 a_i\omega_i$ in Λ^2_3:

$$\begin{array}{rcl}\Lambda^4_1 & = & \langle\,\Omega\,\rangle\\ \Lambda^4_5 & = & \{\alpha\in\Lambda^4_+ \mid *(\alpha\wedge\omega)\wedge\Omega = 10\alpha\wedge\omega\}\\ \Lambda^4_{15} & = & \{\alpha\in\Lambda^4_+ \mid *(\alpha\wedge\omega)\wedge\Omega = 2\alpha\wedge\omega\}\\ \Lambda^4_{14} & = & \{\alpha\in\Lambda^4_+ \mid \alpha\wedge\omega = 0\}\\ \Lambda^4_{5'} & = & \{\alpha\in\Lambda^4_- \mid *(\alpha\wedge\omega)\wedge\Omega = 2\alpha\wedge\omega\}\\ \Lambda^4_{30} & = & \{\alpha\in\Lambda^4_- \mid *(\alpha\wedge\omega)\wedge\Omega = -6\alpha\wedge\omega\}.\end{array}$$

In this section, we shall identify invariants in most of these subspaces, arising from a choice of section of Δ_+.

Let us begin with the first summand of Δ_+ in (22). Given a real unit spinor $\phi \in F$ (for example, $\phi = \phi_E$ from (21)), its stabiliser in Sp(2) is $\mathrm{Sp}(1)_\sigma \times \mathrm{Sp}(1)_\tau$ (where the Greek subscripts distinguish the two subgroups) acting as SO(4). Thus, a non-vanishing spinor with values in the rank 5 subbundle of Δ_+ defines a reduction of structure group from Sp(2)Sp(1) to

$$\mathrm{Sp}(1)^3/\mathbb{Z}_2 = \mathrm{Sp}(1)^2\mathrm{Sp}(1) = \frac{(\mathrm{Sp}(1) \times \mathrm{Sp}(1))\mathrm{Sp}(1)}{\{(1,1,1),(-1,-1,-1)\}}. \tag{35}$$

This allows us to break the spaces of 4-forms and vector bundles (33) under the action of $\mathrm{Sp}(1)^3/\mathbb{Z}_2$.

Invariant sections will only occur in those summands that do not involve H, i.e. $\Lambda^4_1 \cong \mathbb{C}$, $L^4_{14} \cong S^2_0 F$ and $\Lambda^4_{5-} \cong F$. In order to identify these tensors, we work at a point and choose an Sp(2)Sp(1) orthonormal basis of 1-forms. Adopting shorthand, set

$$\begin{aligned} \sigma_1 &= 12 + 34, \qquad & \sigma_2 &= 13 + 42, \qquad & \sigma_3 &= 14 + 23, \\ \tau_1 &= 56 + 78, \qquad & \tau_2 &= 57 + 86, \qquad & \tau_3 &= 58 + 67. \end{aligned}$$

Then $\phi \otimes \omega_1$ defines a 2-form in the submodule $\Lambda^2_{15} \cong F\, S^2 H$ that we may identify with $\sigma_1 - \tau_1$. To obtain the reincarnation of ϕ as a 4-form, we merely have to take its wedge product with ω_1, since $\omega_1 \otimes \omega_1 \in S^4 H \oplus \mathbb{C}$ yet $\Lambda^4 T^*$ has no component $F\, S^4 H$. This gives

$$(\sigma_1 - \tau_1) \wedge \omega_1 = \sigma_1^2 - \tau_1^2,$$

which equals twice $1234 - 5678$, and will be denoted in the next lemma by Ω_5^-. This 4-form calibrates the two 4-planes in the decomposition

$$T = \mathbb{R}^8 = \mathbb{R}^4_{1234} \oplus \mathbb{R}^4_{5678}, \tag{36}$$

for which (σ_i) and (τ_j) are triples of self-dual 2-forms on the respective summands.

Lemma 6.2. *Any 4-form invariant by* $\mathrm{Sp}(1)^3/\mathbb{Z}_2$ *is pointwise a linear combination of*

$$\begin{aligned} \Omega_1 &= 3(\sigma_1^2 + \tau_1^2) + 2(\sigma_1\tau_1 + \sigma_2\tau_2 + \sigma_3\tau_3) &&= \Omega, \\ \Omega_{14} &= \sigma_1^2 + \tau_1^2 - \sigma_1\tau_1 - \sigma_2\tau_2 - \sigma_3\tau_3 &&\in \Lambda^4_{14}, \\ \Omega_{5-} &= \sigma_1^2 - \tau_1^2 &&\in \Lambda^4_{5-}. \end{aligned}$$

Proof. The expression given for Ω can be verified directly.

Under the assumption that there exists a reduction from Sp(2)Sp(1) to $\mathrm{Sp}(1)^3/\mathbb{Z}_2$, there are decompositions

$$\begin{aligned} E &= A_\sigma \oplus A_\tau, \\ F &\cong A_\sigma A_\tau \oplus \mathbb{C}, \\ S^2_0 F &\cong (S^2 A_\sigma)(S^2 A_\tau) \oplus (A_\sigma A_\tau) \oplus \mathbb{C}, \end{aligned} \tag{37}$$

where A_σ, A_τ are each isomorphic to $\mathbb{C}^2 = \mathbb{H}$. It follows that F and $S^2_0 F$ each contain a unique invariant up to scaling. We have already discussed Ω_{5-}. The expression given for Ω_{14} is invariant by $\mathrm{Sp}(1)^3/\mathbb{Z}_2$, and one can verify that it belongs to Λ^4_{14} as defined above. Alternatively, observe that the wedge product $(\sigma_1 - \tau_1)^2$ must belong to $\Lambda^4_1 \oplus \Lambda^4_{5+} \oplus \Lambda^4_{14}$, and so

$$(\sigma_1 - \tau_1)^2 - a\Omega - b\omega_1^2 \in \Lambda^4_{14}$$

for some a, b. These constants can be found by wedging with $\omega_i \wedge \omega_i$, and setting the result zero (for $i = 1, 2$). □

Given a non-zero spinor in S^2H, its stabiliser in Sp(1), acting as SO(3) on S^2H, is U(1). Thus, a non-vanishing spinor that values in the rank 3 subbundle of Δ_+ defines a reduction of the structure group from Sp(2)Sp(1) to Sp(2)U(1). The unit spinor can be identified with a 2-form ω_1, which we may express at a point as

$$\omega_1 = 12 + 34 + 56 + 78 = \sigma_1 + \tau_1. \tag{38}$$

Combining ϕ_E with ω_1 enables us to construct more invariant 4-forms.

Suppose now that we have two spinors, one (such as ω_1) in S^2H and one (such as ϕ_E) in $\Lambda^2_0 E$, both nowhere zero on an open subset of $\mathbb{HP}^2$. Then we have a reduction of the structure group from Sp(2)Sp(1) to $\mathrm{Sp}(1)^2\mathrm{U}(1)$.

Lemma 6.3. *Any 4-form invariant by* $\mathrm{Sp}(1)^2\mathrm{U}(1)$ *is pointwise a linear combination of* $\Omega, \Omega_{14}, \Omega_{5-}$*, together with*

$$\begin{array}{rcll} \Omega_{5+} & = & -2(\sigma_1^2 + \tau_1^2) + \sigma_2\tau_2 + \sigma_3\tau_3 & \in \Lambda^4_{5+}, \\ \Omega_{15} & = & \sigma_2\tau_3 - \sigma_3\tau_2 & \in \Lambda^4_{15}. \end{array}$$

Proof. The reduction from $\mathrm{Sp}(1)^3/\mathbb{Z}_2 = \mathrm{Sp}(1)^2\mathrm{Sp}(1)$ to $\mathrm{Sp}(1)^2\mathrm{U}(1)$ will give rise to a 1-dimensional trivial summand in any submodule $S^{2k}H$, and therefore an extra invariant in both Λ^4_{5+} and L^4_{15}. There is no new invariant in Λ^4_{30} because, further to (37),

$$S^2E \;\cong\; S^2(A_\sigma \,\oplus\, A_\sigma) \;\cong\; S^2A_\sigma \,\oplus\, S^2A_\tau \,\oplus\, A_\sigma A_\tau$$

has no trivial summand.

The expression for Ω_{5+} is the linear combination of $\frac{1}{2}(\Omega - 3\omega_1^2)$, which wedges to zero with W.

The expression for Ω_{15} is harder to pin down, but arises as follows. Fix again an invariant element $\phi \in F$, and let $\widetilde{\phi}_i$ denote the 2-form in $\Lambda^2_{15} \cong F\,S^2H$ defined by $\phi \otimes \omega_i$. Then

$$\widetilde{\phi}_2 = \sigma_2 - \tau_2, \qquad \widetilde{\phi}_3 = \sigma_3 - \tau_3,$$

and

$$\Omega_{15} = \widetilde{\phi}_2\omega_3 = -\widetilde{\phi}_3\omega_2,$$

since $\phi \otimes \omega_2 \otimes \omega_3$ defines an element in $F\,S^4H \oplus F\,S^2H$, yet the first summand does not appear in Λ^4T^*. □

7. Applications to Spin(7) geometry

In this section, we shall pursue similarities that arise in the definition of Sp(2)Sp(1) and Spin(7) structures.

At the algebraic level, the first (and better-known) link arises from fixing a 2-form ω_1 on an open set of a QK 8-manifold M. This determines a reduction of structure groups

$$\mathrm{Sp}(2)\mathrm{U}(1) \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8),$$

for the tangent bundle, or equivalently negative spinor bundle Δ_-. Meanwhile, the positive spinor bundle acquires a reduction

$$\mathrm{SO}(5) \times \mathrm{SO}(2) \subset \mathrm{SO}(7) \subset \mathrm{SO}(8),$$

related to the previous one by triality, cf. (23). The associated Spin(7) 4-form is

$$\begin{aligned} -\omega_1^2 + \omega_2^2 + \omega_3^2 &= \Omega - 2\omega_1^2 \\ &= 2\Omega_{5+} + \omega_1^2, \end{aligned} \tag{39}$$

and was exploited by [8] in the context of hyperkähler geometry. The distinguished 7-dimensional subspace Λ^2_7 of 2-forms defined by the Spin(7) structure splits as $5 + 2$.

The second link (and more the focus of this paper) arises from fixing a section ϕ of $F = \Lambda^2_0 E$. This gives a different away of fitting Spin(7) 'into' a quaternionic structure, namely by means of the inclusions

$$\mathrm{Sp}(1)^3/\mathbb{Z}_2 \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8)$$

of $T \cong \Delta_-$, and

$$\mathrm{SO}(4) \times \mathrm{SO}(3) \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8)$$

of Δ_+. In this case, the associated Spin(7) 4-form will be a linear combination of Ω and Ω_{14}, and Λ^2_7 splits as $4 + 3$.

The description of both of these Spin(7) structures arises from special cases of Lemmas 6.2 and 6.3, and the 4-forms are derived from Lemma 6.1. The SO(8) is fixed, in the sense that the underlying Riemannian metric remains the one defined by the Sp(2)Sp(1) reduction. Below, we shall consider particular linear combinations of the forms introduced in the previous section that deform the QK metric across 4-dimensional distributions defined by (36).

Of the invariants in Lemma 6.2, the one whose stabiliser is closest to $\mathrm{Sp}(1)^3/\mathbb{Z}_2$ is Ω_{14}. Unlike an element of F, it does not enable us to distinguish the two summands of (36):

Proposition 7.1. *The stabiliser of Ω_{14} in $\mathrm{GL}(8,\mathbb{R})$ is isomorphic to $(\mathrm{Sp}(1)^3/\mathbb{Z}_2) \rtimes \mathbb{Z}_2$, where the final $\mathbb{Z}_2$ flips the summands in (36).*

Proof. Denote by G the stabiliser of Ω_{14}, which we know contains $\mathrm{Sp}(1)^3/\mathbb{Z}_2$. Observe that $G \subset \mathrm{SL}(8,\mathbb{R})$ since it preserves the volume form $\frac{1}{20}\Omega_{14} \wedge \Omega_{14}$. Consider now the G-equivariant map

$$\begin{aligned} L\colon \Lambda^4 &\longrightarrow \Lambda^8 \cong \mathbb{R} \\ \alpha &\mapsto \alpha \wedge \Omega_{14}. \end{aligned}$$

The spaces $\Lambda^4_{5+}, \Lambda^4_{15}, \Lambda^4_{30}$ belong to the kernel of L since they are acted on non-trivially by Λ^2_3. From (37), we conclude that (as $\mathrm{Sp}(1)^3/\mathbb{Z}_2$ modules) the only elements of Λ^4 that do not belong to the kernel of L must lie in the 3-dimensional subspace spanned by Ω_{14} and the simple forms 1234 and 5678. It also easy to see that $\Omega_5^- \wedge \Omega_{14} = 0$ and thus, we have shown that the only 4-forms stabilised by G are Ω_{14} and $1234 + 5678$. It also follows that G acts as -1 on Ω_5^- corresponding to the action of the outer $\mathbb{Z}_2$ automorphism. □

The Spin(7) 4-form (39) can be expressed as

$$\sigma_1^2 + \tau_1^2 - 2\sigma_1\tau_1 + 2\sigma_2\tau_2 + 2\sigma_3\tau_3. \tag{40}$$

We can flip the last two signs by changing those of the coordinates u_3, u_4. Since $\mathrm{Sp}(1)^3/\mathbb{Z}_2$ is a subgroup of Spin(7), it must be possible to define a Spin(7) 4-form by combining those of Lemma 6.2. We investigate this now. Let a, b, c be real constants. Consider the linear combination

$$\begin{aligned} \Psi_{a,b,c} &= a\Omega_{14} + b\Omega + c\Omega_{5-} \\ &= (3b+a+c)\sigma_1^2 + (3b+a-c)\tau_1^2 + (2b-a)(\sigma_1\tau_1 + \sigma_2\tau_2 + \sigma_3\tau_3). \end{aligned} \tag{41}$$

Setting $(a, b, c) = (8, -1, 0)$ in (41), we obtain the Spin(7) 4-form

$$\tfrac{1}{5}\Psi_{8,-1,0} = \sigma_1^2 + \tau_1^2 - 2\sigma_1\tau_1 - 2\sigma_2\tau_2 - 2\sigma_3\tau_3.$$

Up to scaling, this defines the same metric and orientation as the QK 4-form Ω, in accordance with Lemma 6.1. However, as promised, we can derive distinct metrics by choosing $c \neq 0$:

Proposition 7.2. *The stabiliser of $\Psi_{a,b,c}$ is isomorphic to*

(i) Spin(7) *if $a - 2b > 0$, $3b + a - c > 0$, $3b + a + c > 0$ and*

$$c^2 = \tfrac{1}{4}(8b + a)(4b + 3a).$$

If instead $a - 2b < 0$ then the stabiliser is isomorphic to Spin(4, 3).

(ii) Sp(2)Sp(1) *if $a - 2b < 0$, $3b + a - c > 0$, $3b + a + c > 0$ and*

$$c^2 = 5a(3b - \tfrac{1}{4}a).$$

If instead $a - 2b > 0$ then the stabiliser is Sp(1, 1)Sp(1).

Proof. This follows by rescaling $\Psi_{a,b,c}$ so that it can be expressed as

$$\sigma_1^2 + \frac{3b + a - c}{3b + a + c}\tau_1^2 - 2\frac{a - 2b}{2(3b + a + c)}(\sigma_1\tau_1 + \sigma_2\tau_2 + \sigma_3\tau_3)$$

The condition for Spin(7) is that the numerators and denominators are positive and that

$$\frac{3b + a - c}{3b + a + c} = \Big(\frac{a - 2b}{2(3b + a + c)}\Big)^2,$$

which simplifies to the equation in (i). If $a - 2b < 0$, the result follows from the definition of Spin(4, 3) given in [9].

The condition for $\Psi_{a,b,c}$ to define an Sp(2)Sp(1)-structure is that $3b + a - c > 0$ and $3b + a - c > 0$ as before, but now we need $a - 2b < 0$ and

$$\frac{3b + a - c}{3b + a + c} = \Big(\frac{3(a - 2b)}{2(3b + a + c)}\Big)^2.$$

This simplifies to the equation in (ii). If $a - 2b > 0$, then $\Psi_{a,b,c}$ is pointwise equivalent to

$$(\sigma_1 - \tau_1)^2 + (\sigma_2 - \tau_2)^2 + (\sigma_3 - \tau_3)^2$$

after a change of coordinates. Since the stabiliser of the triple $\sigma_i - \tau_i$ is Sp(1, 1), it follows that $\Psi_{a,b,c}$ is stabilised by Sp(1, 1)Sp(1). □

Part (i) of the proposition supplies a 2-parameter family of Spin(7) structures at each point. One such instance is

$$\tfrac{1}{10}\Psi_{82,1,75} = (4\sigma_1)^2 + \tau_1^2 - 2(4\sigma_1)\tau_1 - 2(4\sigma_2)\tau_2 - 2(4\sigma_3)\tau_3.$$

In both cases, the associated Riemannian metric has the form

$$g = (3b+a+c)^{1/2}\sum_{i=1}^{4} dx_i^2 \;\pm\; (3b+a-c)^{1/2}\sum_{i=5}^{8} dx_i^2.$$

This follows from the coefficients of σ_1^2 and τ_1^2 in (41).

Remark 7.3. The homogeneous space $\mathrm{Spin}(7)/(\mathrm{Sp}(1)^3/\mathbb{Z}_2)$ parametrises oriented 4-dimensional subspaces of $\mathbb{R}^8$ on which Φ restricts to give the volume form [22, p. 123]; these are the so-called 'Cayley planes'. This space has dimension 12, compared to 16 for $\mathbb{G}\mathrm{r}_4(\mathbb{R}^8)$. In a manifold (M^8, Ψ) with holonomy Spin(7), the Φ-calibrated submanifolds are called Cayley submanifolds. The $(\mathrm{Sp}(1)^3/\mathbb{Z}_2) \rtimes \mathbb{Z}_2$ structure determined by Ω_{14} corresponds to choosing a pair of (undistinguished) orthogonal Cayley planes on each tangent space of M^8 with respect to any of the Spin(7)-structures. Note that the deformation theory of Cayley submanifolds does not require $d\Phi = 0$ [27, p. 274].

One can use the reduction to $\mathrm{Sp}(1)^3/\mathbb{Z}_2$ to give an explicit relation between the QK structure of $\mathbb{HP}^2$ and the complete metric with holonomy Spin(7) on the spin bundle $\mathbb{S}$ over $S^4 = \mathbb{HP}^1$ defined by Bryant and the second author. If we regard $\mathbb{HP}^2$ as the projectivisation of $\mathbb{H}^2$, then $\mathbb{H}^2 \setminus \{0\}$ fibres over the line $\mathbb{HP}^1$ at infinity. The origin 0 defines a point $x_0 \in \mathbb{HP}^2$, the structure group Sp(3) reduces to $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$, and we have an equivariant embedding $\mathbb{S} \hookrightarrow \mathbb{HP}^2$. This exhibits $\mathbb{HP}^2$ as a cohomogeneity-one manifold with principal orbits S^7 and singular orbits S^4 and $\{x_0\}$, just as in the discussion around (15).

We continue to use Ω to denote the QK 4-form on $\mathbb{HP}^2$. Let

$$\Psi = f^2\psi_1 + fg\psi_2 + g^2\psi_3$$

denote the 4-form defining the Spin(7) metric in [10]. Here, ψ_1 denotes the volume form of the fibre, ψ_2 is a term that mixes 2-forms on the fibre and base S^4, and ψ_3 is the volume form of the base. Solving for $d\Phi = 0$ gives

$$f(r) = 4(1+r)^{-2/5}, \qquad g(r) = 5(1+r)^{3/5},$$

where r is a radial parameter on the fibre. In view of the pointwise descriptions of Ω and Φ in the previous sections, the 4-form

$$\Omega = -3u^2\psi_1 + uv\psi_2 - 3v^2\psi_3$$

defines a QK structure on $\mathbb{S}$. Solving for $d\Omega = 0$ shows that

$$u(r) = 4(r+1)^{-2}, \qquad v(r) = (r+1)^{-1}.$$

The associated metrics are given by

$$\begin{aligned} g_{\mathrm{BS}} &= 4(r+1)^{-2/5}(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2) + 5(r+1)^{3/5}(\omega_0^2+\omega_1^2+\omega_2^2+\omega_3^2),\\ g_{\mathrm{QK}} &= 4(r+1)^{-2}(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2) + (r+1)^{-1}(\omega_0^2+\omega_1^2+\omega_2^2+\omega_3^2). \end{aligned}$$

A coordinate transformation verifies that g_{QK} extends smoothly to the one point compactification of $\mathbb{S}$, whilst of course g_{BS} does not.

From our earlier discussion of $\mathrm{Sp}(1)^3/\mathbb{Z}_2$-structures on (an open set of) $\mathbb{HP}^2$, we see that we can regard

$$\Omega_{5-} = -u^2\psi_1 + v^2\psi_3 \in \Lambda^4_{5-} \cong F$$

as a section of Δ_+, one which only vanishes at the point at infinity. This spinor then determines

$$\Omega_{14} = -u^2\psi_1 - v^2\psi_3 - \tfrac{1}{2}uv\psi_2 \in \Lambda^4_{14}.$$

After some computation (which are omitted here), we recover

Proposition 7.4. *The metric g_{BS} with holonomy* Spin(7) *on $\mathbb{S}$ is defined by the 4-form $\Psi = \Psi_{a,b,c}$, where*

$$-5a = \frac{f^2}{u^2} + \frac{g^2}{v^2} + 6\frac{fg}{uv}, \quad -10b = \frac{f^2}{u^2} + \frac{g^2}{v^2} - 4\frac{fg}{uv}, \quad -2c = \frac{f^2}{u^2} - \frac{g^2}{v^2}.$$

We conjecture that the 4-form in this proposition is closely related to the section $(u_E \wedge ju_E)_0$ of F defined by (12) where $x_0 = \langle u, ju\rangle$ is the point removed from $\mathbb{HP}^2$.

We shall leave a discussion of 4-forms arising from Lemma 6.3 and their stabilisers for future investigation. One aim would be to extend the approach of this paper to compute exterior derivatives of the 4-forms in Lemma 6.3, with a view to studying properties of a 4-form of type

$$\begin{aligned}\Psi_{a,b,c,d,e} &= a\Omega_{14} + b\Omega + c\Omega_{5-} + d\Omega_{5+} + e\Omega_{15}\\ &= (a+3b-2d+c)\sigma_1^2 + (a+3b-2d-c)\tau_1^2 + (2b-a)(\sigma_1\tau_1)\\ &\quad +(2b-a+d)(\sigma_2\tau_2+\sigma_3\tau_3) + e(\sigma_2\tau_3 - \tau_3\sigma_2),\end{aligned}$$

with different coefficients (for example, c) set to zero. We conclude by describing one natural way of modifying an Sp(2)Sp(1) structure.

Let $\{e_i\}$, $\{e^j\}$ be dual orthonormal bases of $\mathbb{R}^8$, $(\mathbb{R}^8)^*$, compatible with a 4-form

$$\Omega = e^1 \wedge \gamma + \Upsilon \tag{42}$$

with stabiliser Sp(2)Sp(1), where $\gamma = e_1 \cdot \Omega$ and the dot indicates interior product. Then $e_1 \cdot \Upsilon = 0$, and

$$\Upsilon \in \Lambda^3\left\langle e^2, \ldots, e^8\right\rangle.$$

We can now replace e^1 in (42) by the 1-form $\tilde{e}^1 = e^1 + \alpha$, for any α, without affecting the stabiliser of Ω up to isomorphism. The modified 4-form equals

$$\tilde{\Omega} = \Omega + \alpha \wedge (X \cdot \Omega), \tag{43}$$

where $X = e_1$, and is associated with the metric

$$\tilde{g} = \tilde{e}^1 \otimes \tilde{e}^1 + \sum_{i=2}^{8} e^i \otimes e^i.$$

This is a special case of a nilpotent perturbation, as defined in [12].

Now suppose that Ω denotes a parallel QK 4-form. If $\alpha = dh$ is exact and X is a Killing vector field (like the one that generates the U(1) action in Section 3) then the Lie derivative $\mathcal{L}_{hX}\Omega = dh \wedge (X \cdot \Omega)$ coincides with the deformation in (43), and the latter arises via diffeomorphism. A more subtle choice of X was used to establish the existence of continuous families of closed (non-parallel) 4-forms with stabiliser Sp(2)Sp(1) on $G_2/SO(4)$ [12, section 5]. It remains an open question as to whether such a 4-form exists on $\mathbb{HP}^2$.

One might hope that the methods leading to Proposition 7.4 could be related to work in [16], or lead to new explicit incomplete metrics with holonomy Spin(7). However, there are no non-trivial Sp(2)-invariant linear deformations of the Spin(7) 4-form associated to g_{BS} [13]. That leaves the question of whether one can define useful canonical Spin(7) metrics for which the 4-form Ψ is not closed, perhaps by specifying Spin(7) orbits in which the intrinsic torsion $d\Psi$ should lie.

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