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More Relaxed Stability Analysis and Positivity Analysis for Positive Polynomial Fuzzy Systems via Membership Functions Dependent Method

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Abstract

In this paper, the conservatism source of the positivity and stability analysis results of positive polynomial fuzzy-model-based (PPFMB) control system are studied. Also, in order to improve the flexibility of controller design, a fuzzy controller that does not depend on the membership functions of the fuzzy model is designed. In the existing literatures, it is proved that the LCLF can reduce the conservatism of stability results. However, the LCLF generally results in non-convex conditions which is still a conservatism source. To handle the non-convex conditions, the sector nonlinear concept is applied to handle non-convex terms in stability conditions, and the obstacles caused by mismatched membership functions can be eliminated by PLMF dependent method. In addition, to relax the conservatism caused by the lack of membership functions information, the PLMF dependent positivity analysis are performed for the first time. Meanwhile, PLMF dependent method is extended to stability conditions to obtained more relaxed conditions. Finally, a simulation example is presented to verify the feasibility of this method.

Keywords: positive polynomial fuzzy-model-based (PPFMB) control systems,

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linear copositive Lyapunov function (LCLF), sector nonlinear concept, piecewise linear membership functions (PLMF) dependent method.

1. Introduction

Positive systems are a class of systems whose states are always confined in the positive orthant whenever the initial conditions are non-negative. Such kind of systems are common in the practical industry and life, such as energy market [1], DC-DC power converters [2], and pharmacokinetics [3]. Considering the practical significance of investigating positive systems, some papers [4, 5, 6, 7, 8, 9] started the research process of positive systems from positive linear systems. Paper [5] established the necessary and sufficient conditions with quadratic form for the existence of positive observer. Papers [6] and [7] gave the linear necessary and sufficient conditions based on the consideration of the properties of the positive system to guarantee the existence of the positive observer and asymptotic stability of the system, respectively. Whereafter, in order to facilitate systematic research on the positive systems, the literature [4] reviewed some basic properties and applications of positive linear systems. In recent years, some papers [8, 9] investigated positive linear systems by using linear copositive Lyapunov function (LCLF) because this kind of Lyapunov function makes the analysis process more concise.

Although the research on positive linear systems has a good foundation, there are still some problems in the study of positive nonlinear systems. One problem is that the nonlinear terms in positive nonlinear system model increase the difficulty of system analysis. In order to deal with the nonlinearity of the positive nonlinear systems, Takagi-Sugeno (T-S) fuzzy model was used in [10] for the first time. Whereafter, T-S fuzzy model was widely used in various types of positive nonlinear systems with different control requirements, including the stability analysis of positive T-S fuzzy-model-based (FMB) continuous-time systems with time delay [11], stability analysis of positive T-S FMB discrete-time systems with time delay and bounded control [12], stochastic stability analysis of positive

T-S FMB Markov Jump systems [13, 14], positive L1 observer design for positive fuzzy Semi-Markovian switching systems [15], stability analysis and synthesis
30 for switched T-S fuzzy positive systems [16], observer-based control for positive T-S FMB systems [17], output-feedback control for positive T-S FMB systems [18], tracking control [19] and filter design [20] for positive T-S FMB systems, etc.

T-S fuzzy model demonstrates a strong expressing capability for model-
35 ing the nonlinear dynamics through fuzzy combination of local linear systems. When polynomial systems are used as local systems as proposed in polynomial fuzzy model [21, 22], its expressing capability is further enhanced with fewer number of rules in general. For polynomial fuzzy model based systems, sum of squares (SOS) based analysis approach is used instead of the linear-matrix-inequality (LMI) and linear programming (LP) based analysis approaches. Fol-
40 lowing the SOS based analysis approach, the conditions in terms of SOS are obtained in work [21], and the solutions of these SOS-based conditions can be found numerically by using the third-party MATLAB toolbox SOSTOOLS [23]. To the best of our knowledge, polynomial fuzzy model is rarely applied on posi-
45 tive systems in the existing literatures. The polynomial fuzzy model will be used in this paper to model positive nonlinear systems and perform polynomial-based fuzzy control for those significant advantages.

Another problem for the study of positive nonlinear systems is that the positivity analysis methods of positive linear systems are not perfect for positive
50 nonlinear systems. For example, although the positivity conditions of positive linear systems can guarantee the positivity of fuzzy positive systems [11, 12, 13, 14, 24, 25, 26], positivity analysis results of fuzzy positive systems under these positivity conditions are very conservative because these positivity conditions are membership function independent (MFI). To handle this prob-
55 lem, the membership-function-dependent (MFD) positivity conditions are given in this paper for the first time by adapting the positivity conditions shown in paper [4] and improving the piecewise linear membership functions (PLMF) dependent method [27].

In addition to the problems in positive analysis for fuzzy positive systems, there are also some problems in the stability analysis. For example, in order to relax the stability analysis results of positive nonlinear systems, by considering the positive characteristics of system states, linear-copositive-Lyapunov-function-based analysis method [11, 13, 14, 18, 28, 29], fuzzy-linear-copositive-Lyapunov-function-based analysis method [15] and quadratic-copositive-Lyapunov-function-based analysis method [19, 30, 31] were proposed to replace the quadratic-Lyapunov-function-based analysis method [32]. However, these Lyapunov functions generally lead to non-convex stability conditions which cannot be directly solved by convex programming technique such as LMI, LP, and SOS. Therefore, some iterative algorithms were applied in the exiting literatures [11, 18, 19, 30], which may bring computational complexity. In order to avoid this shortcoming of iterative algorithms, the work [14] adopted the convexification method in work [8] to handle the non-convex conditions derived by LCLF. However, this convexification method is only applicable to a special nonlinear positive systems where the input matrix of the sub-systems in the T-S fuzzy model is required to be a common matrix. In order to make the designed control strategy applicable to a wider range of systems, the work [29] designed a novel fuzzy controller which allows the input matrices of the sub-systems in the polynomial fuzzy model to be different, and it allows the imperfect premise matching (IPC) concept [33, 34, 24, 25, 35, 26, 36, 37] being used to increase the flexibility of controller design. IPC concept suggests that the membership functions between the fuzzy model and fuzzy controller can be different, which provides more freedom for the controller design and thus makes it possible to reduce the cost of controller implementation. However, the membership functions of the controller designed in work [29] are not completely different from the membership functions of the fuzzy model, they must same with the membership functions of the input matrices of the fuzzy model. Thus, the controller design in work [29] is less flexible, which motivates us to design a fuzzy controller that does not depend on the membership functions of fuzzy model at all, and give an effective convexification method to handle the generated non-convex conditions.

90 In this paper, the stability and positivity of positive polynomial fuzzy model
based (PPFMB) system are investigated. The polynomial fuzzy model is used to
represent a wider **range** of positive nonlinear systems, and IPC concept is applied
to provide the possible to reduce the controller implementation costs. In order to
relax the analysis results, the MFD positivity conditions are given by adapting
95 the positivity conditions in literature [4] and using the PLMF dependent method
[27]. Also, its ability to relax analysis results is compared with the ability of
parallel distributed compensation (PDC) method in this paper. Furthermore,
LCLF is applied in this paper to perform the stability analysis. Considering
that the existing literatures have not given effective convexification method for
100 the non-convex conditions caused by LCLF, an effective convexification method
combining sector nonlinear concept and PLMF dependent method is proposed
in this paper.

The contributions of this paper are listed as below:

1) Flexible controller design:

105 Under the stability analysis framework based on LCLF, a more flexible
controller design strategy is adopted. Different from the existing litera-
tures, the membership functions of the fuzzy controller in this paper are
allowed to be completely different from the membership functions of the
fuzzy model. This controller design strategy will effectively reduce the
110 implementation cost of the controller.

2) Convexification of positivity and stability conditions:

For the non-convex conditions caused by LCLF which is applied on the
PPFMB system under IPC concept, an effective convexification method
is first proposed by integrating sector nonlinear concept and PLMF de-
115 pendent method.

3) The relaxed MFD positivity conditions:

The MFD positivity conditions with a complete proof process are given for
the first time by adapting the existing positivity conditions and adopting
PLMF dependent method, so that more relaxed results are obtained.

120 The organization of this paper is as follows. In Section 2, the notations and **the formulation of** polynomial fuzzy model, polynomial fuzzy controller are described. In Section 3, **the LCLF is adopted to perform the stability analysis of the PPFMB system, and an effective convexification method is proposed to handle the non-convex stability conditions.** In Section 4, the PLMF dependent method is applied on positivity conditions and stability conditions to
125 obtain more relaxed analysis **results** of PPFMB system. Also, the PDC analysis method is applied on positive analysis to compare the characteristics of the PLMF dependent method and PDC analysis method. In Section 5, **a simulation example is provide to illustrate the advantages of the proposed control scheme.**
130 In Section 6, a conclusion is drawn. In Appendix, the proof of Lemma 1 is provided.

2. Preliminaries

2.1. Notation

The following notations are used throughout the paper. A monomial in
135 $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ is a function **in the form of** $x_1^{d_1}(t)x_2^{d_2}(t)\dots x_i^{d_i}(t)$, where $d_i \geq 0, i \in \{1, 2, \dots, n\}$ are nonnegative integers. The degree of a monomial is $d = \sum_{i=1}^n d_i$. A polynomial $f(\mathbf{x}(t))$ is an SOS if there exist polynomials $f_1(\mathbf{x}(t)), f_2(\mathbf{x}(t)), \dots, f_m(\mathbf{x}(t))$ such that $f(\mathbf{x}(t)) = \sum_{i=1}^m f_i^2(\mathbf{x}(t))$, where $f_i(\mathbf{x}(t))$ is a polynomial and m is a nonnegative integer. It is clear that $f(\mathbf{x}(t))$
140 being an SOS naturally implies $f(\mathbf{x}(t)) \geq 0$ for all $\mathbf{x}(t) \in \mathfrak{R}^n$. $\mathbf{A} \prec 0$ and $\mathbf{A} \succ 0$ mean that all elements of \mathbf{A} are negative and positive, respectively; $\mathbf{A} < 0$ and $\mathbf{A} > 0$ mean that \mathbf{A} is negative definite and positive definite, respectively. $\mathbf{A}^{(\alpha, \beta)}$ denotes the α^{th} row, β^{th} column element of \mathbf{A} . $\mathbf{A}^{(:, \beta)}$ is a vector denoting the β^{th} column of \mathbf{A} . $\mathbf{A}^{(\alpha, :)}$ is a vector denoting the α^{th} row of \mathbf{A} . \mathbf{A}^T denotes
145 the transpose of the matrix \mathbf{A} . Matrix \mathbf{Q} is called Metzler matrix [4], if its off diagonal elements are all nonnegative. \underline{p} represents $\{1, 2, \dots, p\}$, where p is a non-zero integer.

2.2. Polynomial Fuzzy Plant Model

The nonlinear system is described by a polynomial fuzzy model with p rules. The i^{th} rule is of the following format:

$$\begin{aligned} \text{Rule } i : & \text{ IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND } \cdots \text{ AND } f_\psi(\mathbf{x}(t)) \text{ is } M_\psi^i, \\ & \text{ THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \end{aligned}$$

where $\mathbf{x}(t) \in \mathfrak{R}^n$ and $\mathbf{u}(t) \in \mathfrak{R}^m$ are the state vector and control input vector of the system, respectively; n, m are their dimensions, $f_\vartheta(\mathbf{x}(t))$ is the premise variable and M_ϑ^i is the fuzzy set corresponding to its premise variable in rule i , $i \in \underline{p}$, $\vartheta \in \underline{\psi}$, and ψ is a positive integer; $\mathbf{A}_i(\mathbf{x}(t)) \in \mathfrak{R}^{n \times n}$, $\mathbf{B}_i(\mathbf{x}(t)) \in \mathfrak{R}^{n \times m}$ are the known polynomial system matrices and input matrices, respectively.

The dynamics of the nonlinear system is defined as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^p w_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)), \quad (1)$$

where $w_i(\mathbf{x}(t)) = \prod_{\vartheta=1}^{\psi} \mu_{M_\vartheta^i}(f_\vartheta(\mathbf{x}(t))) / \sum_{k=1}^p \prod_{\vartheta=1}^{\psi} \mu_{M_\vartheta^k}(f_\vartheta(\mathbf{x}(t)))$ is the normalized grade of membership, $w_i(\mathbf{x}(t)) \geq 0$, and $\sum_{i=1}^p w_i(\mathbf{x}(t)) = 1$; $\mu_{M_\vartheta^i}(f_\vartheta(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy term M_ϑ^i .

Definition 1. *The polynomial fuzzy system (1) is said to be positive only if for every nonnegative initial state, its state variables and outputs are all nonnegative.*

By adapting the proof of the positivity conditions in reference [4], the positivity conditions of the PPFMB system are given in the following lemma.

Lemma 1. *A polynomial fuzzy system (1) is guaranteed to be positive if $\sum_{i=1}^p w_i(\mathbf{x}(t))\mathbf{A}_i(\mathbf{x}(t))$ is a Metzler matrix; input matrices satisfy the conditions that $\sum_{i=1}^p w_i(\mathbf{x}(t))\mathbf{B}_i(\mathbf{x}(t)) \succ 0$ when $\mathbf{u}(t)$ is nonnegative.*

Proof 1. *The proof of Lemma 1 is given in the Appendix.*

2.3. Polynomial Fuzzy Controller

The IPC concept is adopted to design a polynomial fuzzy controller with p rules for the polynomial fuzzy system (1), the j^{th} rule of the polynomial fuzzy controller is as follows:

$$\begin{aligned} \text{Rule } j : & \text{ IF } g_1(\mathbf{x}(t)) \text{ is } N_1^j \text{ AND } \cdots \text{ AND } g_\phi(\mathbf{x}(t)) \text{ is } N_\phi^j, \\ & \text{ THEN } \mathbf{u}(t) = \mathbf{G}_j(\mathbf{x}(t))\mathbf{x}(t) \end{aligned}$$

where $g_\vartheta(\mathbf{x}(t))$ is the premise variable and N_ϑ^j is the fuzzy set corresponding to its premise variable in rule j , $j \in \underline{p}$, $\vartheta \in \underline{\phi}$, and ϕ is a positive integer. The polynomial fuzzy controller is defined as follows:

$$\mathbf{u}(t) = \sum_{j=1}^p m_j(\mathbf{x}(t)) \mathbf{G}_j(\mathbf{x}(t)) \mathbf{x}(t), \quad (2)$$

where $m_j(\mathbf{x}(t)) = \prod_{\vartheta=1}^{\phi} \mu_{N_\vartheta^j}(g_\vartheta(\mathbf{x}(t))) / \sum_{k=1}^p \prod_{\vartheta=1}^{\phi} \mu_{N_\vartheta^k}(g_\vartheta(\mathbf{x}(t)))$, $m_j(\mathbf{x}(t)) \geq 0$, and $\sum_{j=1}^p m_j(\mathbf{x}(t)) = 1$; $\mu_{N_\vartheta^j}(g_\vartheta(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy term N_ϑ^j . $\mathbf{G}_j(\mathbf{x}(t))$ is the polynomial fuzzy controller gain, which is defined as $\mathbf{G}_j(\mathbf{x}(t)) = \frac{\sum_{\iota=1}^m \mathbf{e}_m^\iota \mathbf{D}_{j\iota}(\mathbf{x}(t))}{\sum_{s=1}^p m_s(\mathbf{x}(t)) \boldsymbol{\lambda}^T \mathbf{B}_s(\mathbf{x}(t)) \mathbf{e}_m} \in \mathfrak{R}^{m \times n}$, where $\boldsymbol{\lambda}$ is the Lyapunov function variable which **will be** introduced in the following section; $\mathbf{e}_m = [1, \dots, 1]^T \in \mathfrak{R}^{m \times 1}$, \mathbf{e}_m^k denotes only the k^{th} element of \mathbf{e}_m is 1, other elements are 0; $\mathbf{D}_{j\iota}(\mathbf{x}(t)) \in \mathfrak{R}^{1 \times n}$ is to be determined, $\iota \in \underline{m}$. For example, $m = 2$, $n = 3$, $\mathbf{e}_2^1 = [1, 0]^T$, $\mathbf{e}_2^2 = [0, 1]^T$, $\mathbf{D}_{j1}(\mathbf{x}(t)) = [D_{j1}^{(1)}(\mathbf{x}(t)), D_{j1}^{(2)}(\mathbf{x}(t)), D_{j1}^{(3)}(\mathbf{x}(t))]$, $\mathbf{D}_{j2}(\mathbf{x}(t)) = [D_{j2}^{(1)}(\mathbf{x}(t)), D_{j2}^{(2)}(\mathbf{x}(t)), D_{j2}^{(3)}(\mathbf{x}(t))]$, $\sum_{\iota=1}^m \mathbf{e}_m^\iota \mathbf{D}_{j\iota}(\mathbf{x}(t)) =$

$$\begin{bmatrix} D_{j1}^{(1)}(\mathbf{x}(t)) & D_{j1}^{(2)}(\mathbf{x}(t)) & D_{j1}^{(3)}(\mathbf{x}(t)) \\ D_{j2}^{(1)}(\mathbf{x}(t)) & D_{j2}^{(2)}(\mathbf{x}(t)) & D_{j2}^{(3)}(\mathbf{x}(t)) \end{bmatrix}.$$

Remark 1. In reference [8], the linear positive system was investigated, and the controller was design as $\frac{\sum_{\iota=1}^m \mathbf{e}_m^\iota \mathbf{D}_\iota}{\boldsymbol{\lambda}^T \mathbf{B}_m}$ to avoid the non-convex terms in stability conditions. When T-S fuzzy positive system is investigated **in** [14], this kind of controller also can be applied to obtained convex conditions. However, it need to limit the input part of the system to be linear, which means that $\mathbf{B}_i = \mathbf{B}$

for any $i \in \underline{p}$. In this paper, in order to eliminate this restriction, the membership functions dependent controller gain $\frac{\sum_{\iota=1}^m \mathbf{e}_m^t \mathbf{D}_{j\iota}(\mathbf{x})}{\sum_{s=1}^p m_s(\mathbf{x}) \boldsymbol{\lambda}^T \mathbf{B}_s(\mathbf{x}) \mathbf{e}_m}$ is designed. It can be seen that the membership functions in the denominator of this designed controller gain are consistent with the controller membership functions, so this novel controller allows that the membership functions between the fuzzy model and the fuzzy controller are different.

3. Novel Stability Analysis Results for PPFMB System

In the following analysis, for simplicity, the time t is dropped for the situation without ambiguity. From (1) and (2), the closed-loop control system is rewritten as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^p \sum_{j=1}^p w_i(\mathbf{x}) m_j(\mathbf{x}) [(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \mathbf{G}_j(\mathbf{x})) \mathbf{x}]. \quad (3)$$

In order to make the PPFMB control system (3) positive and asymptotically stable, the polynomial fuzzy controller is designed through the following Theorem.

Theorem 1. For the PPFMB control system (3), if there exist $\boldsymbol{\lambda} \in \mathbb{R}^{n \times 1}$, polynomial vectors $\mathbf{D}_{j\iota}(\mathbf{x}) \in \mathbb{R}^{1 \times n}$ and $\tilde{\mathbf{D}}_j(\mathbf{x}) \in \mathbb{R}^{1 \times n}$, $\forall j \in \underline{p}$, $\iota \in \underline{m}$, polynomial scalars Y_{kv} and $R_{\hat{k}\varsigma}$, $\forall k \in \{1, 2, 3, 4\}$, $\hat{k} \in \{1, 2, 3\}$, $v \in \underline{p}$, $\varsigma \in \underline{\sigma}$, such that the following SOS-based conditions are satisfied:

$$\nu^T (\boldsymbol{\lambda}^{(\alpha,1)} - \varepsilon_1) \nu \text{ is SOS; } \forall \alpha \in \underline{n} \quad (4)$$

$$\nu^T (\tilde{\mathbf{D}}_j^{(1,\beta)}(\mathbf{x}) - \mathbf{D}_{j\iota}^{(1,\beta)}(\mathbf{x})) \nu \text{ is SOS, } \forall j \in \underline{p}, \iota \in \underline{m}, \beta \in \underline{n}; \quad (5)$$

$$- \nu^T (\boldsymbol{\Xi}_{1ij}^{(\alpha,1)}(\mathbf{x}) - \varepsilon_2(\mathbf{x})) \nu \text{ is SOS, } \forall i, j \in \underline{p}, \alpha \in \underline{n}; \quad (6)$$

$$- \nu^T (\boldsymbol{\Xi}_{2ij}^{(\alpha,1)}(\mathbf{x}) - \varepsilon_3(\mathbf{x})) \nu \text{ is SOS, } \forall i, j \in \underline{p}, \alpha \in \underline{n}; \quad (7)$$

$$\nu^T (Y_{kv}(\mathbf{x}) - \varepsilon_4(\mathbf{x})) \nu \text{ is SOS, } \forall k \in \{1, 2, 3, 4\}, v \in \underline{p}; \quad (8)$$

$$\nu^T (Y_{kv}(\mathbf{x}) - \theta_v(\mathbf{x}) - \varepsilon_5(\mathbf{x})) \nu \text{ is SOS, } \forall k \in \{1, 2, 3, 4\}, v \in \underline{p}; \quad (9)$$

$$\nu^T (R_{\hat{k}\varsigma}(\mathbf{x}) - \varepsilon_6(\mathbf{x})) \nu \text{ is SOS, } \forall \hat{k} \in \{1, 2, 3\}, \varsigma \in \underline{\sigma}; \quad (10)$$

$$\nu^T (\Lambda_{1i_1 i_2 \dots i_{\hat{n}} \varsigma}(\mathbf{x}) - \varepsilon_7(\mathbf{x})) \nu \text{ is SOS; } \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma} \quad (11)$$

$$\nu^T \Lambda_{2i_1 i_2 \dots i_{\tilde{n}} \varsigma}(\mathbf{x}) \nu \text{ is SOS; } \forall i_1, i_2, \dots, i_{\tilde{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma} \quad (12)$$

$$\nu^T \Lambda_{3i_1 i_2 \dots i_{\tilde{n}} \varsigma}(\mathbf{x}) \nu \text{ is SOS; } \quad \forall i_1, i_2, \dots, i_{\tilde{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma} \quad (13)$$

$$\nu^T (\Theta_{ijs}^{(\alpha, \beta)}(\mathbf{x}) - \varepsilon_8(\mathbf{x})) \nu \text{ is SOS, } \forall i, j, s \in \underline{p}, \alpha \neq \beta \in \underline{n}. \quad (14)$$

where $\Xi_{1ij}(\mathbf{x})$ and $\Xi_{2ij}(\mathbf{x})$ are defined in (19) and (20), respectively; $\Lambda_{1i_1 i_2 \dots i_{\tilde{n}} \varsigma}(\mathbf{x})$, $\Lambda_{2i_1 i_2 \dots i_{\tilde{n}} \varsigma}(\mathbf{x})$ and $\Lambda_{3i_1 i_2 \dots i_{\tilde{n}} \varsigma}(\mathbf{x})$ are defined in (44), (45) and (46), respectively; $\Theta_{ijs}^{(\alpha, \beta)}(\mathbf{x})$ is defined in (49); f_{min} and f_{max} are the predefined positive scalars; ν is an arbitrary vector independent of \mathbf{x} with appropriate dimensions; $\varepsilon_1 > 0$, $\varepsilon_2(\mathbf{x}) > 0, \dots, \varepsilon_8(\mathbf{x}) > 0$ are predefined scalar polynomials, then the system (3) is asymptotically stable and positive. The polynomial fuzzy controller gain can be obtained by $\mathbf{G}_j(\mathbf{x}) = \frac{\sum_{l=1}^m \mathbf{e}_m^l \mathbf{D}_{j_l}(\mathbf{x})}{\sum_{s=1}^p m_s(\mathbf{x}) \boldsymbol{\lambda}^T \mathbf{B}_s(\mathbf{x}) \mathbf{e}_m}$.

Proof 2. This proof contains two parts. The first part provides the derivation process of stability conditions, so the title of this part is Stability Analysis. Correspondingly, the part titled Positivity Analysis provides derivation process of the positivity conditions.

Part I: Stability Analysis

In order to perform stability analysis, a LCLF candidate $V(\mathbf{x}) = \mathbf{x}^T \boldsymbol{\lambda}$ is chosen, where every element of $\boldsymbol{\lambda} \in \mathbb{R}^{n \times 1}$ is positive. The time derivation of $V(\mathbf{x})$ is as follows:

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T \boldsymbol{\lambda} \\ &= \sum_{i=1}^p \sum_{j=1}^p w_i(\mathbf{x}) m_j(\mathbf{x}) \mathbf{x}^T (\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \mathbf{G}_j(\mathbf{x}))^T \boldsymbol{\lambda} \end{aligned} \quad (15)$$

The polynomial fuzzy controller gain $\mathbf{G}_j(\mathbf{x})$ has been *designed* as $\mathbf{G}_j(\mathbf{x}) = \frac{\sum_{l=1}^m \mathbf{e}_m^l \mathbf{D}_{j_l}(\mathbf{x})}{\sum_{s=1}^p m_s(\mathbf{x}) \boldsymbol{\lambda}^T \mathbf{B}_s(\mathbf{x}) \mathbf{e}_m}$. Suppose that there exist polynomial vector variables $\tilde{\mathbf{D}}_j(\mathbf{x})$ such that $\tilde{\mathbf{D}}_j(\mathbf{x}) \succ \mathbf{D}_{j_l}(\mathbf{x}), \forall l \in \underline{m}$, which means that the k^{th} element of every \mathbf{D}_{j_l} is less than the k^{th} element of $\tilde{\mathbf{D}}_j(\mathbf{x})$, where $k \in \underline{n}$, then (15) can be derived as follows:

$$\dot{V}(\mathbf{x})$$

$$\begin{aligned}
&= \sum_{i=1}^p \sum_{j=1}^p w_i(\mathbf{x}) m_j(\mathbf{x}) \mathbf{x}^T (\mathbf{A}_i^T(\mathbf{x}) \boldsymbol{\lambda} + \mathbf{G}_j^T(\mathbf{x}) \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}) \\
&= \sum_{i=1}^p \sum_{j=1}^p w_i(\mathbf{x}) m_j(\mathbf{x}) \mathbf{x}^T \left[\mathbf{A}_i^T(\mathbf{x}) \boldsymbol{\lambda} + \frac{\sum_{\ell=1}^m \mathbf{D}_{j\ell}^T(\mathbf{x}) (\mathbf{e}_m^\ell)^T}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}} \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda} \right] \\
&< \sum_{i=1}^p \sum_{j=1}^p w_i(\mathbf{x}) m_j(\mathbf{x}) \mathbf{x}^T \left(\mathbf{A}_i^T(\mathbf{x}) \boldsymbol{\lambda} + \frac{\sum_{\ell=1}^m \tilde{\mathbf{D}}_j^T(\mathbf{x}) (\mathbf{e}_m^\ell)^T}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}} \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda} \right) \\
&= \sum_{i=1}^p w_i(\mathbf{x}) \mathbf{x}^T \mathbf{A}_i^T(\mathbf{x}) \boldsymbol{\lambda} + \sum_{j=1}^p m_j(\mathbf{x}) \mathbf{x}^T \tilde{\mathbf{D}}_j^T(\mathbf{x}) \frac{\sum_{i=1}^p w_i(\mathbf{x}) \sum_{\ell=1}^m (\mathbf{e}_m^\ell)^T \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}}
\end{aligned} \tag{16}$$

where $\sum_{\ell=1}^m (\mathbf{e}_m^\ell)^T = \mathbf{e}_m^T$.

205 **Remark 2.** In reference [29], the non-convex term $\frac{\sum_{i=1}^p w_i(\mathbf{x}) \sum_{\ell=1}^m (\mathbf{e}_m^\ell)^T \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}}$ is avoided by making the membership functions of the fuzzy controller consistent with the membership functions of the input part of the fuzzy model. Although this method makes stability analysis easier, it reduces the flexibility of controller design. In this paper, in order to increase the flexibility of controller design, a
210 fuzzy controller that does not depend on the membership functions of the fuzzy model at all is designed, and an effective method to deal with the non-convex term $\frac{\sum_{i=1}^p w_i(\mathbf{x}) \sum_{\ell=1}^m (\mathbf{e}_m^\ell)^T \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}}$ is proposed in the following.

For the non-convex term $\frac{\sum_{i=1}^p w_i(\mathbf{x}) \sum_{\ell=1}^m (\mathbf{e}_m^\ell)^T \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}}$ in (16), the sector non-linear technique [38] is applied on the nonlinear term $f(\mathbf{x}) = \frac{\sum_{i=1}^p w_i(\mathbf{x}) \sum_{\ell=1}^m (\mathbf{e}_m^\ell)^T \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}}$. Assume that positive scalar f_{\min} and f_{\max} are the minimum and maximum value of $f(\mathbf{x})$ in the operating domain of \mathbf{x} defined in prior, respectively. Then, according to the sector nonlinear technique, the nonlinear term $f(\mathbf{x})$ is represented as follows:

$$f(\mathbf{x}) = \mu_{\mathbf{M}^1}(\mathbf{x}) f_{\min} + \mu_{\mathbf{M}^2}(\mathbf{x}) f_{\max}, \tag{17}$$

where $\mu_{\mathbf{M}^1}(\mathbf{x}) = \frac{f(\mathbf{x}) - f_{\max}}{f_{\min} - f_{\max}}$, $\mu_{\mathbf{M}^2}(\mathbf{x}) = 1 - \mu_{\mathbf{M}^1}(\mathbf{x})$. f_{\max} and f_{\min} are two constants that are slightly greater than and less than 1 respectively in a case
215 that $w_i(\mathbf{x})$ and $m_i(\mathbf{x})$ are closed to each other.

According to (17), it follows from (16) that:

$$\begin{aligned}
& \sum_{i=1}^p w_i(\mathbf{x}) \mathbf{x}^T \mathbf{A}_i^T(\mathbf{x}) \boldsymbol{\lambda} + \sum_{j=1}^p m_j(\mathbf{x}) \mathbf{x}^T \tilde{\mathbf{D}}_j^T(\mathbf{x}) \frac{\sum_{i=1}^p w_i(\mathbf{x}) \sum_{l=1}^m (\mathbf{e}_m^l)^T \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}} \\
&= \sum_{i=1}^p w_i(\mathbf{x}) \mathbf{x}^T \mathbf{A}_i^T(\mathbf{x}) \boldsymbol{\lambda} + \sum_{j=1}^p m_j(\mathbf{x}) \mathbf{x}^T \tilde{\mathbf{D}}_j^T(\mathbf{x}) \sum_{l=1}^2 \mu_{\mathbf{M}^l}(\mathbf{x}) f_l \\
&= \sum_{l=1}^2 \sum_{i=1}^p \sum_{j=1}^p \mu_{\mathbf{M}^l}(\mathbf{x}) w_i(\mathbf{x}) m_j(\mathbf{x}) \mathbf{x}^T (\mathbf{A}_i^T(\mathbf{x}) \boldsymbol{\lambda} + \tilde{\mathbf{D}}_j^T(\mathbf{x}) f_l), \tag{18}
\end{aligned}$$

where $f_1 = f_{\min}$ and $f_2 = f_{\max}$.

Defining

$$\boldsymbol{\Xi}_{1ij}(\mathbf{x}) = \mathbf{A}_i^T(\mathbf{x}) \boldsymbol{\lambda} + \tilde{\mathbf{D}}_j^T(\mathbf{x}) f_{\min}, \tag{19}$$

$$\boldsymbol{\Xi}_{2ij}(\mathbf{x}) = \mathbf{A}_i^T(\mathbf{x}) \boldsymbol{\lambda} + \tilde{\mathbf{D}}_j^T(\mathbf{x}) f_{\max}. \tag{20}$$

Since $0 \leq \mu_{\mathbf{M}^1}(\mathbf{x}), \mu_{\mathbf{M}^2}(\mathbf{x}) \leq 1$, the condition $\dot{V}(\mathbf{x}) < 0$ can be guaranteed by

$$\sum_{i=1}^p \sum_{j=1}^p w_i(\mathbf{x}) m_j(\mathbf{x}) \boldsymbol{\Xi}_{1ij}(\mathbf{x}) \prec 0, \tag{21}$$

$$\sum_{i=1}^p \sum_{j=1}^p w_i(\mathbf{x}) m_j(\mathbf{x}) \boldsymbol{\Xi}_{2ij}(\mathbf{x}) \prec 0, \tag{22}$$

$$f_{\min} \leq \frac{\sum_{i=1}^p w_i(\mathbf{x}) \sum_{l=1}^m (\mathbf{e}_m^l)^T \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}} \leq f_{\max}. \tag{23}$$

The conditions (21) and (22) can be guaranteed by $\boldsymbol{\Xi}_{1ij}(\mathbf{x}) \prec 0$ and $\boldsymbol{\Xi}_{2ij}(\mathbf{x}) \prec 0$, $\forall i \in \underline{p}, j \in \underline{p}$, which are expressed in terms of SOS in (6) and (7). However, the difficulty of analysis is that the restricted condition (23) cannot be guaranteed when membership functions are ignored. Thus, the MFD analysis method needs to be used. In this paper, the PLMF dependent method [27] is adapted to apply on the condition (23). The membership functions $w_i(\mathbf{x})$ and $m_s(\mathbf{x})$ are approximated by PLMFs $\hat{w}_i(\mathbf{x})$ and $\hat{m}_s(\mathbf{x})$. Suppose that there are $d_r + 1$ interpolation points for state variable x_r , the number of substate spaces of x_r is d_r , and the overall state space Ψ is divided into σ connected substate spaces which are denoted as Ψ_ς , $\varsigma \in \underline{\sigma}$, $\sigma = \prod_{r=1}^n d_r$. In substate space Ψ_ς , the original

membership functions are denoted as $w_{i\varsigma}(\mathbf{x})$ and $m_{s\varsigma}(\mathbf{x})$, and they are approximated by PLMFs $\hat{w}_{i\varsigma}(\mathbf{x})$ and $\hat{m}_{s\varsigma}(\mathbf{x})$. Then, in whole state space, the PLMFs are defined as follows:

$$\begin{aligned}\hat{w}_i(\mathbf{x}) &= \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \hat{w}_{i\varsigma}(\mathbf{x}) \\ &= \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_{\hat{n}}=1}^2 \prod_{r=1}^{\hat{n}} v_{ri_r\varsigma}(x_r) \zeta_{ii_1i_2\dots i_{\hat{n}}}\end{aligned}\quad (24)$$

$$\begin{aligned}\hat{m}_s(\mathbf{x}) &= \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \hat{m}_{s\varsigma}(\mathbf{x}) \\ &= \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_{\hat{n}}=1}^2 \prod_{r=1}^{\hat{n}} v_{ri_r\varsigma}(x_r) \eta_{si_1i_2\dots i_{\hat{n}}}\end{aligned}\quad (25)$$

where \hat{n} is the number of the system state variables on which the membership functions $w_i(\mathbf{x})$ and $m_i(\mathbf{x})$ depend; $\varphi_{\varsigma}(\mathbf{x}) = 1$ if $\mathbf{x} \in \Psi_{\varsigma}$; $\varphi_{\varsigma}(\mathbf{x}) = 0$ if $\mathbf{x} \notin \Psi_{\varsigma}$; the predefined interpolation functions $v_{ri_r\varsigma}(x_r)$ have the properties that $0 \leq v_{ri_r\varsigma}(x_r) \leq 1$ and $v_{r1\varsigma}(x_r) + v_{r2\varsigma}(x_r) = 1$ for $r \in \hat{n}$, $i_r \in \{1, 2\}$; constant scalars $\zeta_{ii_1i_2\dots i_{\hat{n}}}$ and $\eta_{si_1i_2\dots i_{\hat{n}}}$ denote the values of the membership functions $w_i(\mathbf{x})$ and $m_s(\mathbf{x})$ at the interpolation point $\mathbf{x} = [x_{1i_1}, x_{2i_2}, \dots, x_{\hat{n}i_{\hat{n}}}]$, respectively.

It is difficult to obtain an approximate function without error, so approximation errors need to be considered when approximation functions are introduced in conditions. Defining $\Delta w_{i\varsigma}(\mathbf{x}) = w_{i\varsigma}(\mathbf{x}) - \hat{w}_{i\varsigma}(\mathbf{x})$ and $\Delta m_{s\varsigma}(\mathbf{x}) = m_{s\varsigma}(\mathbf{x}) - \hat{m}_{s\varsigma}(\mathbf{x})$ as the approximated errors of $w_i(\mathbf{x})$ and $m_s(\mathbf{x})$ in the substate space Ψ_{ς} , respectively. The lower and upper bounds of $\Delta w_{i\varsigma}(\mathbf{x})$ are denoted as $\underline{\delta}_{i\varsigma}$ and $\bar{\delta}_{i\varsigma}$, the lower and upper bounds of $\Delta m_{s\varsigma}(\mathbf{x})$ are denoted as $\underline{\rho}_{s\varsigma}$ and $\bar{\rho}_{s\varsigma}$.

Referring to (23), denote $\theta_i(\mathbf{x}) = \sum_{i=1}^m (\mathbf{e}_m^i)^T \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}$ and $\theta_s(\mathbf{x}) = \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}$, then $\frac{\sum_{i=1}^p w_i(\mathbf{x}) \sum_{i=1}^m (\mathbf{e}_m^i)^T \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}} = \frac{\sum_{i=1}^p w_i(\mathbf{x}) \theta_i(\mathbf{x})}{\sum_{s=1}^p m_s(\mathbf{x}) \theta_s(\mathbf{x})}$.

Suppose that there exist positive decision variables $Y_{1i}(\mathbf{x})$ such that $Y_{1i}(\mathbf{x}) \geq \theta_i(\mathbf{x})$, based on the PLMF dependent analysis method, the following conditions can be obtained:

$$\sum_{i=1}^p w_i(\mathbf{x}) \theta_i(\mathbf{x})$$

$$\begin{aligned}
&= \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \sum_{i=1}^p [(\hat{w}_{i\varsigma}(\mathbf{x}) + \underline{\delta}_{i\varsigma})\theta_i(\mathbf{x}) + (w_i(\mathbf{x}) - \hat{w}_{i\varsigma}(\mathbf{x}) - \underline{\delta}_{i\varsigma})\theta_i(\mathbf{x})] \\
&\leq \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \sum_{i=1}^p [(\hat{w}_{i\varsigma}(\mathbf{x}) + \underline{\delta}_{i\varsigma})\theta_i(\mathbf{x}) + (\bar{\delta}_{i\varsigma} - \underline{\delta}_{i\varsigma})Y_{1i}(\mathbf{x})] \\
&= \overline{\mathbf{W}}_{\theta}(\mathbf{x}). \tag{26}
\end{aligned}$$

Similarly, suppose that there exist positive decision variables $Y_{2i}(\mathbf{x})$ such that $Y_{2i}(\mathbf{x}) \geq \theta_i(\mathbf{x})$, the following conditions can be obtained:

$$\begin{aligned}
&\sum_{i=1}^p w_i(\mathbf{x})\theta_i(\mathbf{x}) \\
&= \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \sum_{i=1}^p [(\hat{w}_{i\varsigma}(\mathbf{x}) + \bar{\delta}_{i\varsigma})\theta_i(\mathbf{x}) + (w_i(\mathbf{x}) - \hat{w}_{i\varsigma}(\mathbf{x}) - \bar{\delta}_{i\varsigma})\theta_i(\mathbf{x})] \\
&\geq \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \sum_{i=1}^p [(\hat{w}_{i\varsigma}(\mathbf{x}) + \bar{\delta}_{i\varsigma})\theta_i(\mathbf{x}) + (\underline{\delta}_{i\varsigma} - \bar{\delta}_{i\varsigma})Y_{2i}(\mathbf{x})] \\
&= \underline{\mathbf{W}}_{\theta}(\mathbf{x}). \tag{27}
\end{aligned}$$

Following the same line of the above, suppose that there exist positive decision variables $Y_{3s}(\mathbf{x})$ such that $Y_{3s}(\mathbf{x}) \geq \theta_s(\mathbf{x})$, the following conditions can be obtained:

$$\begin{aligned}
&\sum_{s=1}^p m_s(\mathbf{x})\theta_s(\mathbf{x}) \\
&\leq \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \sum_{s=1}^p [(\hat{m}_{s\varsigma}(\mathbf{x}) + \underline{\rho}_{s\varsigma})\theta_s(\mathbf{x}) + (\bar{\rho}_{s\varsigma} - \underline{\rho}_{s\varsigma})Y_{3s}(\mathbf{x})] \\
&= \overline{\mathbf{M}}_{\theta}(\mathbf{x}). \tag{28}
\end{aligned}$$

Suppose that there exist positive decision variables $Y_{4s}(\mathbf{x})$ such that $Y_{4s}(\mathbf{x}) \geq \theta_s(\mathbf{x})$, the following conditions can be obtained:

$$\begin{aligned}
&\sum_{s=1}^p m_s(\mathbf{x})\theta_s(\mathbf{x}) \\
&\geq \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \sum_{s=1}^p [(\hat{m}_{s\varsigma}(\mathbf{x}) + \bar{\rho}_{s\varsigma})\theta_s(\mathbf{x}) + (\underline{\rho}_{s\varsigma} - \bar{\rho}_{s\varsigma})Y_{4s}(\mathbf{x})] \\
&= \underline{\mathbf{M}}_{\theta}(\mathbf{x}). \tag{29}
\end{aligned}$$

Then, the following inequation can be obtained:

$$\frac{\mathbf{W}_\theta(\mathbf{x})}{\overline{\mathbf{M}}_\theta(\mathbf{x})} \leq \frac{\sum_{i=1}^p w_i(\mathbf{x}) \sum_{l=1}^m (\mathbf{e}_m^l)^T \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}} \leq \frac{\overline{\mathbf{W}}_\theta(\mathbf{x})}{\underline{\mathbf{M}}_\theta(\mathbf{x})}. \quad (30)$$

If $f_{min} \leq \frac{\mathbf{W}_\theta(\mathbf{x})}{\overline{\mathbf{M}}_\theta(\mathbf{x})}$ and $\frac{\overline{\mathbf{W}}_\theta(\mathbf{x})}{\underline{\mathbf{M}}_\theta(\mathbf{x})} \leq f_{max}$ are guaranteed, $f_{min} \leq \frac{\sum_{i=1}^p w_i(\mathbf{x}) \sum_{l=1}^m (\mathbf{e}_m^l)^T \mathbf{B}_i^T(\mathbf{x}) \boldsymbol{\lambda}}{\sum_{s=1}^p m_s(\mathbf{x}) \mathbf{e}_m^T \mathbf{B}_s^T(\mathbf{x}) \boldsymbol{\lambda}} \leq f_{max}$ can be satisfied. In addition, since $\sum_{s=1}^p m_s(\mathbf{x}) \theta_s(\mathbf{x}) > 0$, (28) can lead to $\overline{\mathbf{M}}_\theta(\mathbf{x}) > 0$. Thus, (23) can be guaranteed by the following inequality:

$$\underline{\mathbf{M}}_\theta(\mathbf{x}) > 0. \quad (31)$$

$$\underline{\mathbf{W}}_\theta(\mathbf{x}) - f_{min} \overline{\mathbf{M}}_\theta(\mathbf{x}) \geq 0, \quad (32)$$

$$f_{max} \underline{\mathbf{M}}_\theta(\mathbf{x}) - \overline{\mathbf{W}}_\theta(\mathbf{x}) \geq 0, \quad (33)$$

In addition, in order to avoid false approximation errors caused by the global searchability of the SOSTOOLS for state variables, *the threshold functions are defined as follows:*

$$\xi_{\hat{r}_\varsigma}(\mathbf{x}) = \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_{\hat{n}}=1}^2 \prod_{r=1}^{\hat{n}} v_{ri_r\varsigma}(x_r) (x_{\hat{r}} - x_{\hat{r}_\varsigma min})(x_{\hat{r}_\varsigma max} - x_{\hat{r}}), \quad \forall \varsigma \in \sigma \quad (34)$$

where $x_{\hat{r}_\varsigma min}$ and $x_{\hat{r}_\varsigma max}$ are the minimal and maximal value of the system state $x_{\hat{r}}$ in substate space Ψ_ς , respectively, so $\xi_{\hat{r}_\varsigma}(\mathbf{x})$ has the properties that $\xi_{\hat{r}_\varsigma}(\mathbf{x}) \geq 0$ if $\mathbf{x} \in \Psi_\varsigma$ and $\xi_{\hat{r}_\varsigma}(\mathbf{x}) < 0$ if $\mathbf{x} \notin \Psi_\varsigma$. *According to the S-procedure concepts, if there exist positive slack scalars $R_{1\varsigma}(\mathbf{x})$, $R_{2\varsigma}(\mathbf{x})$ and $R_{3\varsigma}(\mathbf{x})$ such that:*

$$\underline{\mathbf{M}}_\theta(\mathbf{x}) - \sum_{\varsigma=1}^{\sigma} \varphi_\varsigma(\mathbf{x}) \sum_{\hat{r}}^{\hat{n}} \xi_{\hat{r}_\varsigma}(\mathbf{x}) R_{1\varsigma}(\mathbf{x}) > 0 \quad (35)$$

$$\underline{\mathbf{W}}_\theta(\mathbf{x}) - f_{min} \overline{\mathbf{M}}_\theta(\mathbf{x}) - \sum_{\varsigma=1}^{\sigma} \varphi_\varsigma(\mathbf{x}) \sum_{\hat{r}}^{\hat{n}} \xi_{\hat{r}_\varsigma}(\mathbf{x}) R_{2\varsigma}(\mathbf{x}) \geq 0, \quad (36)$$

$$f_{max} \underline{\mathbf{M}}_\theta(\mathbf{x}) - \overline{\mathbf{W}}_\theta(\mathbf{x}) - \sum_{\varsigma=1}^{\sigma} \varphi_\varsigma(\mathbf{x}) \sum_{\hat{r}}^{\hat{n}} \xi_{\hat{r}_\varsigma}(\mathbf{x}) R_{3\varsigma}(\mathbf{x}) \geq 0, \quad (37)$$

then (31), (32) and (33) hold.

From (26), (27), (28) and (29), the inequalities (35), (36) and (37) are equivalent to the following:

$$\sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \left\{ \sum_{s=1}^p [(\hat{m}_{s\varsigma}(\mathbf{x}) + \bar{\rho}_{s\varsigma})\theta_s(\mathbf{x}) + (\underline{\rho}_{s\varsigma} - \bar{\rho}_{s\varsigma})Y_{4s}(\mathbf{x})] - \sum_{\hat{r}=1}^{\hat{n}} \xi_{\hat{r}\varsigma}(\mathbf{x})R_{1\varsigma}(\mathbf{x}) \right\} > 0 \quad (38)$$

$$\begin{aligned} & \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \left\{ \sum_{i=1}^p [(\hat{w}_{i\varsigma}(\mathbf{x}) + \bar{\delta}_{i\varsigma})\theta_i(\mathbf{x}) + (\underline{\delta}_{i\varsigma} - \bar{\delta}_{i\varsigma})Y_{2i}(\mathbf{x})] \right. \\ & \left. - f_{\min} \sum_{s=1}^p [(\hat{m}_{s\varsigma}(\mathbf{x}) + \underline{\rho}_{s\varsigma})\theta_s(\mathbf{x}) + (\bar{\rho}_{s\varsigma} - \underline{\rho}_{s\varsigma})Y_{3s}(\mathbf{x})] - \sum_{\hat{r}=1}^{\hat{n}} \xi_{\hat{r}\varsigma}(\mathbf{x})R_{2\varsigma}(\mathbf{x}) \right\} \geq 0, \end{aligned} \quad (39)$$

$$\begin{aligned} & \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \left\{ f_{\max} \sum_{s=1}^p [(\hat{m}_{s\varsigma}(\mathbf{x}) + \bar{\rho}_{s\varsigma})\theta_s(\mathbf{x}) + (\underline{\rho}_{s\varsigma} - \bar{\rho}_{s\varsigma})Y_{4s}(\mathbf{x})] \right. \\ & \left. - \sum_{\varsigma=1}^{\sigma} \varphi_{\varsigma}(\mathbf{x}) \sum_{i=1}^p [(\hat{w}_{i\varsigma}(\mathbf{x}) + \underline{\delta}_{i\varsigma})\theta_i(\mathbf{x}) + (\bar{\delta}_{i\varsigma} - \underline{\delta}_{i\varsigma})Y_{1i}(\mathbf{x})] - \sum_{\hat{r}=1}^{\hat{n}} \xi_{\hat{r}\varsigma}(\mathbf{x})R_{3\varsigma}(\mathbf{x}) \right\} \geq 0, \end{aligned} \quad (40)$$

In the definition (24) and (25), the positive scalars $\varphi_{\varsigma}(\mathbf{x})$ and $v_{ri_r\varsigma}(x_r)$ are independent of rule i and s , and $\sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_{\hat{n}}=1}^2 \prod_{r=1}^{\hat{n}} v_{ri_r\varsigma}(x_r) = 1$ in substate space Ψ_{ς} , so $\varphi_{\varsigma}(\mathbf{x})$ and $v_{ri_r\varsigma}(x_r)$ can be removed from the conditions (38), (39) and (40). Thus, these conditions can be guaranteed by the following conditions:

$$\Lambda_{1i_1i_2\dots i_{\hat{n}}\varsigma}(\mathbf{x}) > 0, \quad \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma}, \quad (41)$$

$$\Lambda_{2i_1i_2\dots i_{\hat{n}}\varsigma}(\mathbf{x}) \geq 0, \quad \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma}, \quad (42)$$

$$\Lambda_{3i_1i_2\dots i_{\hat{n}}\varsigma}(\mathbf{x}) \geq 0, \quad \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma}. \quad (43)$$

where

$$\begin{aligned} & \Lambda_{1i_1i_2\dots i_{\hat{n}}\varsigma}(\mathbf{x}) \\ & = \sum_{s=1}^p [(\eta_{si_1i_2\dots i_{\hat{n}}} + \bar{\rho}_{s\varsigma})\theta_s(\mathbf{x}) + (\underline{\rho}_{s\varsigma} - \bar{\rho}_{s\varsigma})Y_{4s}(\mathbf{x})] - \sum_{\hat{r}=1}^{\hat{n}} (x_{\hat{r}} - x_{\hat{r}\varsigma\min})(x_{\hat{r}\varsigma\max} - x_{\hat{r}})R_{1\varsigma}(\mathbf{x}), \end{aligned} \quad (44)$$

$$\begin{aligned}
& \Lambda_{2i_1 i_2 \dots i_{\hat{n}_\varsigma}}(\mathbf{x}) \\
&= \sum_{i=1}^p [(\zeta_{ii_1 i_2 \dots i_{\hat{n}_\varsigma}} + \bar{\delta}_{i_\varsigma})\theta_i(\mathbf{x}) + (\underline{\delta}_{i_\varsigma} - \bar{\delta}_{i_\varsigma})Y_{2i}(\mathbf{x})] - f_{min} \sum_{s=1}^p [(\eta_{si_1 i_2 \dots i_{\hat{n}_\varsigma}} + \underline{\rho}_{s_\varsigma})\theta_s(\mathbf{x}) \\
&+ (\bar{\rho}_{s_\varsigma} - \underline{\rho}_{s_\varsigma})Y_{3s}(\mathbf{x})] - \sum_{\hat{r}=1}^{\hat{n}} (x_{\hat{r}} - x_{\hat{r}_\varsigma min})(x_{\hat{r}_\varsigma max} - x_{\hat{r}})R_{2\varsigma}(\mathbf{x}), \tag{45}
\end{aligned}$$

$$\begin{aligned}
& \Lambda_{3i_1 i_2 \dots i_{\hat{n}_\varsigma}}(\mathbf{x}) \\
&= f_{max} \sum_{s=1}^p [(\eta_{si_1 i_2 \dots i_{\hat{n}_\varsigma}} + \bar{\rho}_{s_\varsigma})\theta_s(\mathbf{x}) + (\underline{\rho}_{s_\varsigma} - \bar{\rho}_{s_\varsigma})Y_{4s}(\mathbf{x})] - \sum_{i=1}^p [(\zeta_{ii_1 i_2 \dots i_{\hat{n}_\varsigma}} + \underline{\delta}_{i_\varsigma})\theta_i(\mathbf{x}) \\
&+ (\bar{\delta}_{i_\varsigma} - \underline{\delta}_{i_\varsigma})Y_{1i}(\mathbf{x})] - \sum_{\hat{r}=1}^{\hat{n}} (x_{\hat{r}} - x_{\hat{r}_\varsigma min})(x_{\hat{r}_\varsigma max} - x_{\hat{r}})R_{3\varsigma}(\mathbf{x}). \tag{46}
\end{aligned}$$

As a result, the condition (23) can be guaranteed by $Y_{kv}(\mathbf{x}) > 0$, $Y_{kv}(\mathbf{x}) -$
235 $\theta_v(\mathbf{x}) > 0$, $R_{\hat{k}\varsigma}(\mathbf{x}) > 0$, $\forall k \in \{1, 2, 3, 4\}$, $\hat{k} \in \{1, 2, 3\}$, $v \in \underline{p}$, $\varsigma \in \underline{\sigma}$ and
conditions (41)-(43), these conditions are expressed in terms of SOS in (8)-
(13).

Part II: Positivity Analysis

The PPFMB control system (3) can be regarded as a PPFMB system without
input matrices, with $\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x})$ being the system matrix. Similar to
the previous literature [14], the positivity of the PPFMB control system (3) is
achieved by conditions

$$\mathbf{A}_i^{(\alpha, \beta)}(\mathbf{x}) + \mathbf{B}_i^{(\alpha, \cdot)}(\mathbf{x})\mathbf{G}_j^{(\cdot, \beta)}(\mathbf{x}) > 0; \forall i, j \in \underline{p}, \alpha \neq \beta \in \underline{n}. \tag{47}$$

According to the definition of $\mathbf{G}_j(\mathbf{x})$, $\mathbf{G}_j^{(\cdot, \beta)}(\mathbf{x})$ in the above condition is
240 replaced by $\frac{\sum_{\iota=1}^m \mathbf{e}_m^\iota \mathbf{D}_{j\iota}^{(\cdot, \beta)}(\mathbf{x})}{\sum_{s=1}^p m_s(\mathbf{x})\boldsymbol{\lambda}^T \mathbf{B}_s(\mathbf{x})\mathbf{e}_m}$. Due to $\mathbf{e}_m \succ 0$, $\mathbf{B}_i(\mathbf{x}) \succ 0$, $\boldsymbol{\lambda} \succ 0$, as a result,
 $\sum_{s=1}^p m_s(\mathbf{x})\boldsymbol{\lambda}^T \mathbf{B}_s(\mathbf{x})\mathbf{e}_m > 0$.

Thus,

$$\begin{aligned}
& \mathbf{A}_i^{(\alpha, \beta)}(\mathbf{x}) + \mathbf{B}_i^{(\alpha, \cdot)}(\mathbf{x}) \frac{\sum_{\iota=1}^m \mathbf{e}_m^\iota \mathbf{D}_{j\iota}^{(\cdot, \beta)}(\mathbf{x})}{\sum_{s=1}^p m_s(\mathbf{x})\boldsymbol{\lambda}^T \mathbf{B}_s(\mathbf{x})\mathbf{e}_m} > 0 \\
& \Leftrightarrow \sum_{s=1}^p m_s(\mathbf{x})\boldsymbol{\Theta}_{ij s}^{(\alpha, \beta)}(\mathbf{x}) > 0. \tag{48}
\end{aligned}$$

where

$$\Theta_{ijs}^{(\alpha,\beta)}(\mathbf{x}) = \lambda^T \mathbf{B}_s(\mathbf{x}) \mathbf{e}_m \mathbf{A}_i^{(\alpha,\beta)}(\mathbf{x}) + \mathbf{B}_i^{(\alpha,:)}(\mathbf{x}) \sum_{\ell=1}^m \mathbf{e}_m^\ell \mathbf{D}_{j\ell}^{(:,beta)}(\mathbf{x}). \quad (49)$$

Then, the positivity of the PPFMB control system (3) can be guaranteed by the following conditions:

$$\Theta_{ijs}^{(\alpha,\beta)}(\mathbf{x}) > 0; \forall i, j, s \in \underline{p}, \alpha \neq \beta \in \underline{n}, \quad (50)$$

these positivity conditions are expressed in terms of SOS in (14).

4. Membership Functions Dependent Positivity and Stability Analysis

245 In the last Section, the positivity conditions and stability conditions all are MFI conditions, which means that every sub-condition of these basic conditions for any i, j and s needs to be positive or negative, so the MFI conditions lead to conservative results. It should be pointed that PLMF dependent method is only applied to the restricted conditions (23) for the purpose of handling non-convex
 250 problem in the last Section, the conservativeness of the results caused by the lack of membership functions information has not been eliminated. Thus, the PLMF dependent method is applied on all resultant conditions in this section to relax the results. In addition, considering the characteristics of positivity conditions, PDC-PLMF dependent method (combination of PDC analysis method
 255 and PLMF dependent method) is also applied to the positivity conditions to compare the ability of PDC analysis method and PLMF dependent method to relax the results.

4.1. PLMF Dependent Positivity Analysis

According to Lemma 1, the MFD positivity conditions are obtained as follows:

$$\sum_{i=1}^p \sum_{j=1}^p \sum_{s=1}^p w_i(\mathbf{x}) m_j(\mathbf{x}) m_s(\mathbf{x}) \Theta_{ijs}^{(\alpha,\beta)}(\mathbf{x}) > 0, \forall i, j, s \in \underline{p}, \alpha \neq \beta \in \underline{n}, \quad (51)$$

In order to handle the membership functions in (51), PLMF dependent method is performed in this section. Denote $q_{ijs}(\mathbf{x}) \equiv w_i(\mathbf{x})m_j(\mathbf{x})m_s(\mathbf{x})$, and the corresponding PLMFs is denoted as $\hat{q}_{ijs}(\mathbf{x})$. In substate space Ψ_ς , $q_{ijs}(\mathbf{x})$ and $\hat{q}_{ijs}(\mathbf{x})$ are denoted by $q_{ijs\varsigma}(\mathbf{x})$ and $\hat{q}_{ijs\varsigma}(\mathbf{x})$, respectively. Then, PLMFs $\hat{q}_{ijs}(\mathbf{x})$ can be defined as

$$\begin{aligned}\hat{q}_{ijs}(\mathbf{x}) &= \sum_{\varsigma=1}^{\sigma} \varphi_\varsigma(\mathbf{x}) \hat{q}_{ijs\varsigma}(\mathbf{x}) \\ &= \sum_{\varsigma=1}^{\sigma} \varphi_\varsigma(\mathbf{x}) \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_{\hat{n}}=1}^2 \prod_{r=1}^{\hat{n}} v_{ri,\varsigma}(x_r)^{\kappa_{ijsi_1i_2\dots i_{\hat{n}}}}.\end{aligned}\quad (52)$$

In addition, we denote $\Delta q_{ijs\varsigma}(\mathbf{x}) = q_{ijs\varsigma}(\mathbf{x}) - \hat{q}_{ijs\varsigma}(\mathbf{x})$ as the approximation error, and the minimum and maximum values of $\Delta q_{ijs\varsigma}(\mathbf{x})$ are denoted as $\underline{\rho}_{ijs\varsigma}$ and $\bar{\rho}_{ijs\varsigma}$, respectively. If there exist positive scalars $N_\varsigma(\mathbf{x})$ and positive scalars $\Gamma_{ijs}^{(\alpha,\beta)}(\mathbf{x})$ that satisfy $\Gamma_{ijs}^{(\alpha,\beta)}(\mathbf{x}) > \Theta_{ijs}^{(\alpha,\beta)}(\mathbf{x})$, $\forall i, j, s \in \underline{p}, \varsigma \in \underline{\sigma}, \alpha \neq \beta$, following the same line of Section 3, (51) can be guaranteed by the following conditions:

$$\begin{aligned}&\sum_{i=1}^p \sum_{j=1}^p \sum_{s=1}^p [(\kappa_{ijsi_1i_2\dots i_{\hat{n}}}(\mathbf{x}) + \bar{\rho}_{ijs\varsigma}) \Theta_{ijs}^{(\alpha,\beta)}(\mathbf{x}) + (\underline{\rho}_{ijs\varsigma} - \bar{\rho}_{ijs\varsigma}) \Gamma_{ijs}^{(\alpha,\beta)}(\mathbf{x})] \\ &- \sum_{\hat{r}=1}^{\hat{n}} (x_{\hat{r}} - x_{\hat{r}\varsigma\min})(x_{\hat{r}\varsigma\max} - x_{\hat{r}}) N_\varsigma(\mathbf{x}) > 0; \\ &\forall i, j, s \in \underline{p}, i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma}, \alpha \neq \beta \in \underline{n}.\end{aligned}\quad (53)$$

Theorem 2. For the PPFMB control system (3), if there exist $\lambda \in \mathbb{R}^{n \times 1}$, polynomial vectors $\mathbf{D}_{j\iota}(\mathbf{x}) \in \mathbb{R}^{1 \times n}$ and $\tilde{\mathbf{D}}_j(\mathbf{x}) \in \mathbb{R}^{1 \times n}$, $\forall j \in \underline{p}, \iota \in \underline{m}$, polynomial scalars $Y_{kv}(\mathbf{x})$ and $R_{\hat{k}\varsigma}(\mathbf{x})$, $\forall k \in \{1, 2, 3, 4\}$, $\hat{k} \in \{1, 2, 3\}$, $v \in \underline{p}, \varsigma \in \underline{\sigma}$, polynomial scalars $\Gamma_{ijs}^{(\alpha,\beta)}(\mathbf{x})$ and $N_\varsigma(\mathbf{x})$, $\forall i, j, s \in \underline{p}, \varsigma \in \underline{\sigma}$, such that the following SOS-based conditions are satisfied:

$$\nu^T (\lambda^{(\alpha,1)} - \varepsilon_1) \nu \text{ is SOS}, \forall \alpha \in \underline{n} \quad (54)$$

$$\nu^T (\tilde{\mathbf{D}}_j^{(1,\beta)}(\mathbf{x}) - \mathbf{D}_{j\iota}^{(1,\beta)}(\mathbf{x})) \nu \text{ is SOS}, \forall j \in \underline{p}, \iota \in \underline{m}, \beta \in \underline{n}; \quad (55)$$

$$- \nu^T (\Xi_{1ij}^{(\alpha,1)}(\mathbf{x}) - \varepsilon_2(\mathbf{x})) \nu \text{ is SOS}, \forall i, j \in \underline{p}, \alpha \in \underline{n}; \quad (56)$$

$$- \nu^T (\Xi_{2ij}^{(\alpha,1)}(\mathbf{x}) - \varepsilon_3(\mathbf{x})) \nu \text{ is SOS}, \forall i, j \in \underline{p}, \alpha \in \underline{n}; \quad (57)$$

$$\nu^T (Y_{kv}(\mathbf{x}) - \varepsilon_4(\mathbf{x})) \nu \text{ is SOS}, \forall k \in \{1, 2, 3, 4\}, v \in \underline{p}; \quad (58)$$

$$\nu^T (Y_{kv}(\mathbf{x}) - \theta_v(\mathbf{x}) - \varepsilon_5(\mathbf{x}))\nu \text{ is SOS}, \forall k \in \{1, 2, 3, 4\}, v \in \underline{p}; \quad (59)$$

$$\nu^T (R_{\hat{k}\varsigma}(\mathbf{x}) - \varepsilon_6(\mathbf{x}))\nu \text{ is SOS}, \forall \hat{k} \in \{1, 2, 3\}, \varsigma \in \underline{\sigma}; \quad (60)$$

$$\nu^T (\Lambda_{1i_1i_2\dots i_{\hat{n}}\varsigma}(\mathbf{x}) - \varepsilon_7(\mathbf{x}))\nu \text{ is SOS}; \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma} \quad (61)$$

$$\nu^T \Lambda_{2i_1i_2\dots i_{\hat{n}}\varsigma}(\mathbf{x})\nu \text{ is SOS}; \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma} \quad (62)$$

$$\nu^T \Lambda_{3i_1i_2\dots i_{\hat{n}}\varsigma}(\mathbf{x})\nu \text{ is SOS}; \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma} \quad (63)$$

$$\nu^T (\Gamma_{ijs}^{(\alpha, \beta)}(\mathbf{x}) - \varepsilon_8(\mathbf{x}))\nu \text{ is SOS}; \forall i, j, s \in \underline{p}, \alpha \neq \beta \in \underline{n}; \quad (64)$$

$$\nu^T (\Gamma_{ijs}^{(\alpha, \beta)}(\mathbf{x}) - \Theta_{ijs}^{(\alpha, \beta)}(\mathbf{x}) - \varepsilon_9(\mathbf{x}))\nu \text{ is SOS}; \forall i, j, s \in \underline{p}, \alpha \neq \beta \in \underline{n}; \quad (65)$$

$$\nu^T (N_\varsigma(\mathbf{x}) - \varepsilon_{10}(\mathbf{x}))\nu \text{ is SOS}; \forall \varsigma \in \underline{\sigma}; \quad (66)$$

$$\begin{aligned} & \nu^T \left\{ \sum_{i=1}^p \sum_{j=1}^p \sum_{s=1}^p [(\kappa_{ijsi_1i_2\dots i_{\hat{n}}\varsigma}(\mathbf{x}) + \bar{\varrho}_{ijs\varsigma})\Theta_{ijs}^{(\alpha, \beta)}(\mathbf{x}) + (\underline{\varrho}_{ijs\varsigma} - \bar{\varrho}_{ijs\varsigma})\Gamma_{ijs}^{(\alpha, \beta)}(\mathbf{x})] \right. \\ & \left. - \sum_{\hat{r}=1}^{\hat{n}} (x_{\hat{r}} - x_{\hat{r}\varsigma\min})(x_{\hat{r}\varsigma\max} - x_{\hat{r}})N_\varsigma(\mathbf{x}) - \varepsilon_{11}(\mathbf{x}) \right\} \nu \text{ is SOS}, \end{aligned}$$

$$\forall i, j, s \in \underline{p}, \alpha \neq \beta \in \underline{n}, \varsigma \in \underline{\sigma}, i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}. \quad (67)$$

where $\Xi_{1ij}(\mathbf{x})$ and $\Xi_{2ij}(\mathbf{x})$ are defined in (19) and (20), respectively; $\Lambda_{1i_1i_2\dots i_{\hat{n}}\varsigma}(\mathbf{x})$,
 260 $\Lambda_{2i_1i_2\dots i_{\hat{n}}\varsigma}(\mathbf{x})$ and $\Lambda_{3i_1i_2\dots i_{\hat{n}}\varsigma}(\mathbf{x})$ are defined in (44), (45) and (46), respectively;
 $\Theta_{ijs}^{(\alpha, \beta)}(\mathbf{x})$ is defined in (49); f_{\min} and f_{\max} are the predefined positive scalars;
 ν is an arbitrary vector independent of \mathbf{x} with appropriate dimensions; $\varepsilon_1 > 0$,
 $\varepsilon_2(\mathbf{x}) > 0, \dots, \varepsilon_{11}(\mathbf{x}) > 0$ are predefined scalar polynomials, then the system
 (3) is asymptotically stable and positive. The polynomial fuzzy controller gains
 265 can be obtained by $\mathbf{G}_j(\mathbf{x}) = \frac{\sum_{i=1}^m \mathbf{e}_m^t \mathbf{D}_{ji}(\mathbf{x})}{\sum_{s=1}^p m_s(\mathbf{x}) \boldsymbol{\lambda}^T \mathbf{B}_s(\mathbf{x}) \mathbf{e}_m}$.

In Theorem 2, all membership functions $w_i m_j m_s, \forall i, j, s \in \underline{p}$ are included in the positivity conditions. Considering that $w_i m_j m_s$ and $w_i m_s m_j$ are the same membership functions, PDC analysis method can be applied to relax the conditions. Then, the relaxed condition of (51) can be obtained as follows by grouping the terms with same membership functions:

$$\begin{aligned} & \sum_{i=1}^p \sum_{j=1}^p \sum_{s \geq j}^p w_i(\mathbf{x}) m_j(\mathbf{x}) m_s(\mathbf{x}) (\Theta_{ijs}^{(\alpha, \beta)}(\mathbf{x}) + \Theta_{isj}^{(\alpha, \beta)}(\mathbf{x})) > 0, \\ & \forall i, j \in \underline{p}, s \geq j \in \underline{p}, \alpha \neq \beta \in \underline{n}, \end{aligned} \quad (68)$$

Different from the full PLMF dependent positivity conditions (51), only the membership functions $w_i m_j m_s$, $\forall i, j, s \geq j \in \underline{p}$ need to be approximated by PLMFs and introduced into positivity conditions in (68), because the terms which are weighted by membership functions $w_i m_j m_s$, $\forall i, j, s \leq j \in \underline{p}$ have been grouped with the terms which are weighted by membership functions $w_i m_j m_s$, $\forall i, j, s \geq j \in \underline{p}$. Following the same line of derivation of Theorem 2, if there exist positive scalars $\hat{N}_\zeta(\mathbf{x})$ and positive scalars $\hat{\mathbf{\Gamma}}_{ijs}^{(\alpha, \beta)}(\mathbf{x})$ that satisfy the conditions $\hat{\mathbf{\Gamma}}_{ijs}^{(\alpha, \beta)}(\mathbf{x}) > \Theta_{ijs}^{(\alpha, \beta)}(\mathbf{x}) + \Theta_{isj}^{(\alpha, \beta)}(\mathbf{x})$, $\forall i, j, s \geq j \in \underline{p}$, the condition (68) can be guaranteed by the following condition:

$$\begin{aligned} & \sum_{i=1}^p \sum_{j=1}^p \sum_{s \geq j}^p [(k_{ijsi_1 i_2 \dots i_{\hat{n}}}(\mathbf{x}) + \bar{\varrho}_{ijs\zeta})(\Theta_{ijs}^{(\alpha, \beta)}(\mathbf{x}) + \Theta_{isj}^{(\alpha, \beta)}(\mathbf{x})) + (\underline{\varrho}_{ijs\zeta} - \bar{\varrho}_{ijs\zeta})\hat{\mathbf{\Gamma}}_{ijs}^{(\alpha, \beta)}(\mathbf{x})] \\ & - \sum_{\hat{r}=1}^{\hat{n}} (x_{\hat{r}} - x_{\hat{r}\zeta\min})(x_{\hat{r}\zeta\max} - x_{\hat{r}})\hat{N}_\zeta(\mathbf{x}) > 0; \end{aligned}$$

$$\forall i, j, s \geq j \in \underline{p}, i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \zeta \in \underline{\sigma}, \alpha \neq \beta \in \underline{n}. \quad (69)$$

Corollary 1. For the PPFMB control system (3), if there exist $\boldsymbol{\lambda} \in \mathbb{R}^{n \times 1}$, polynomial vectors $\mathbf{D}_{j\iota}(\mathbf{x}) \in \mathbb{R}^{1 \times n}$ and $\tilde{\mathbf{D}}_j(\mathbf{x}) \in \mathbb{R}^{1 \times n} \forall j \in \underline{p} \iota \in \underline{m}$, polynomial scalars $Y_{kv}(\mathbf{x})$ and $R_{\hat{k}\zeta}(\mathbf{x}) \forall k \in \{1, 2, 3, 4\}, \hat{k} \in \{1, 2, 3\}, v \in \underline{p}, \zeta \in \underline{\sigma}$, polynomial scalars $\hat{\mathbf{\Gamma}}_{ijs}^{(\alpha, \beta)}(\mathbf{x})$ and $\hat{N}_\zeta(\mathbf{x}) \forall i, j, s \geq j \in \underline{p}, \zeta \in \underline{\sigma}$, such that the following SOS-based conditions are satisfied:

$$\nu^T (\boldsymbol{\lambda}^{(\alpha, 1)} - \varepsilon_1) \nu \text{ is SOS}; \forall \alpha \in \underline{n} \quad (70)$$

$$\nu^T (\tilde{\mathbf{D}}_j^{(1, \beta)}(\mathbf{x}) - \mathbf{D}_{j\iota}^{(1, \beta)}(\mathbf{x})) \nu \text{ is SOS}, \forall j \in \underline{p}, \iota \in \underline{m}, \beta \in \underline{n}; \quad (71)$$

$$- \nu^T (\Xi_{1ij}^{(\alpha, 1)}(\mathbf{x}) - \varepsilon_2(\mathbf{x})) \nu \text{ is SOS}, \forall i, j \in \underline{p}, \alpha \in \underline{n}; \quad (72)$$

$$- \nu^T (\Xi_{2ij}^{(\alpha, 1)}(\mathbf{x}) - \varepsilon_3(\mathbf{x})) \nu \text{ is SOS}, \forall i, j \in \underline{p}, \alpha \in \underline{n}; \quad (73)$$

$$\nu^T (Y_{kv}(\mathbf{x}) - \varepsilon_4(\mathbf{x})) \nu \text{ is SOS}, \forall k \in \{1, 2, 3, 4\}, v \in \underline{p}; \quad (74)$$

$$\nu^T (Y_{kv}(\mathbf{x}) - \theta_v - \varepsilon_5(\mathbf{x})) \nu \text{ is SOS}, \forall k \in \{1, 2, 3, 4\}, v \in \underline{p}; \quad (75)$$

$$\nu^T (R_{\hat{k}\zeta}(\mathbf{x}) - \varepsilon_6(\mathbf{x})) \nu \text{ is SOS}, \forall \hat{k} \in \{1, 2, 3\}, \zeta \in \underline{\sigma}; \quad (76)$$

$$\nu^T (\Lambda_{1i_1 i_2 \dots i_{\hat{n}} \zeta}(\mathbf{x}) - \varepsilon_7(\mathbf{x})) \nu \text{ is SOS}; \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \zeta \in \underline{\sigma} \quad (77)$$

$$\nu^T \Lambda_{2i_1 i_2 \dots i_{\hat{n}} \zeta}(\mathbf{x}) \nu \text{ is SOS}; \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \zeta \in \underline{\sigma} \quad (78)$$

$$\nu^T \Lambda_{3i_1 i_2 \dots i_{\hat{n}} \varsigma}(\mathbf{x}) \nu \text{ is SOS}; \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma} \quad (79)$$

$$\nu^T (\hat{\Gamma}_{ijs}^{(\alpha, \beta)}(\mathbf{x}) - \varepsilon_8(\mathbf{x})) \nu \text{ is SOS}; \forall i, j, s \geq j \in \underline{p}, \alpha \neq \beta \in \underline{n}; \quad (80)$$

$$\nu^T (\hat{\Gamma}_{ijs}^{(\alpha, \beta)}(\mathbf{x}) - (\Theta_{ijs}^{(\alpha, \beta)}(\mathbf{x}) + \Theta_{isj}^{(\alpha, \beta)}(\mathbf{x})) - \varepsilon_9(\mathbf{x})) \nu \text{ is SOS};$$

$$\forall i, j, s \geq j \in \underline{p}, \alpha \neq \beta \in \underline{n}; \quad (81)$$

$$\nu^T (\hat{N}_\varsigma(\mathbf{x}) - \varepsilon_{10}(\mathbf{x})) \nu \text{ is SOS}; \forall \varsigma \in \underline{\sigma}; \quad (82)$$

$$\nu^T \left\{ \sum_{i=1}^p \sum_{j=1}^p \sum_{s \geq j}^p [(\kappa_{ijsi_1 i_2 \dots i_{\hat{n}}}(\mathbf{x}) + \bar{\varrho}_{ijs\varsigma}) (\Theta_{ijs}^{(\alpha, \beta)}(\mathbf{x}) + \Theta_{isj}^{(\alpha, \beta)}(\mathbf{x})) \right.$$

$$\left. + (\underline{\varrho}_{ijs\varsigma} - \bar{\varrho}_{ijs\varsigma}) \hat{\Gamma}_{ijs}^{(\alpha, \beta)} \right] - \sum_{\hat{r}=1}^{\hat{n}} (x_{\hat{r}} - x_{\hat{r}\varsigma\min})(x_{\hat{r}\varsigma\max} - x_{\hat{r}}) \hat{N}_\varsigma(\mathbf{x}) - \varepsilon_{11}(\mathbf{x}) \Big\} \nu \text{ is SOS},$$

$$\forall i, j, s \geq j \in \underline{p}, \alpha \neq \beta \in \underline{n}, \varsigma \in \underline{\sigma}, i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}. \quad (83)$$

where $\Xi_{1ij}(\mathbf{x})$ and $\Xi_{2ij}(\mathbf{x})$ are defined in (19) and (20), respectively; $\Lambda_{1i_1 i_2 \dots i_{\hat{n}} \varsigma}(\mathbf{x})$, $\Lambda_{2i_1 i_2 \dots i_{\hat{n}} \varsigma}(\mathbf{x})$ and $\Lambda_{3i_1 i_2 \dots i_{\hat{n}} \varsigma}(\mathbf{x})$ are defined in (44), (45) and (46), respectively; $\Theta_{ijs}^{(\alpha, \beta)}(\mathbf{x})$ is defined in (49); f_{\min} and f_{\max} are the predefined positive scalars; ν is an arbitrary vector independent of \mathbf{x} with appropriate dimensions; $\varepsilon_1 > 0$, $\varepsilon_2(\mathbf{x}) > 0, \dots, \varepsilon_{11}(\mathbf{x}) > 0$ are predefined scalar polynomials, then the system (3) is asymptotically stable and positive. The polynomial fuzzy controller gains can be obtained by $\mathbf{G}_j(\mathbf{x}) = \frac{\sum_{l=1}^m \mathbf{e}_l^T \mathbf{D}_{jl}(\mathbf{x})}{\sum_{s=1}^p m_s(\mathbf{x}) \lambda^T \mathbf{B}_s(\mathbf{x}) \mathbf{e}_m}$.

Remark 3. Compared with Theorem 2, Corollary 1 requires fewer decision variables. For example, $\hat{\Gamma}_{ijs}^{(\alpha, \beta)}(\mathbf{x})$ represents $(n^2 - n)pp!$ positive scalars, and $\Gamma_{ijs}^{(\alpha, \beta)}(\mathbf{x})$ represents $(n^2 - n)p^3$ positive scalars. Thus, Corollary 1 has a smaller computational burden than Theorem 2. However, since Theorem 2 introduces more membership functions information for the positive conditions, it has a stronger ability to relax the results than Corollary 1.

4.2. PLMF Dependent Stability Analysis

In addition to the positivity conditions, conservatism also exists in basic stability conditions due to the absent of membership functions information. Thus, the similar method which is used in Subsection 4.1 is applied to the stability conditions in this subsection. The original membership functions

$h_{ij}(\mathbf{x}) \equiv w_i(\mathbf{x})m_j(\mathbf{x})$ are approximated by PLMFs $\hat{h}_{ij}(\mathbf{x})$. In substate space Ψ_ς , $h_{ij}(\mathbf{x})$ and $\hat{h}_{ij}(\mathbf{x})$ are denoted by $h_{ij\varsigma}(\mathbf{x})$ and $\hat{h}_{ij\varsigma}(\mathbf{x})$, respectively. Then, PLMFs $\hat{h}_{ij}(\mathbf{x})$ can be defined as

$$\begin{aligned}\hat{h}_{ij}(\mathbf{x}) &= \sum_{\varsigma=1}^{\sigma} \varphi_\varsigma(\mathbf{x}) \hat{h}_{ij\varsigma}(\mathbf{x}) \\ &= \sum_{\varsigma=1}^{\sigma} \varphi_\varsigma(\mathbf{x}) \sum_{i_1=1}^2 \sum_{i_2=1}^2 \cdots \sum_{i_{\hat{n}}=1}^2 \prod_{r=1}^{\hat{n}} v_{ri_r\varsigma}(x_r) \chi_{ij i_1 i_2 \dots i_{\hat{n}}}.\end{aligned}\quad (84)$$

Denoting $\underline{\mu}_{ij\varsigma}$ and $\bar{\mu}_{ij\varsigma}$ as the minimum and maximum approximation errors, which satisfy the condition $\underline{\mu}_{ij\varsigma} < h_{ij\varsigma}(\mathbf{x}) - \hat{h}_{ij\varsigma}(\mathbf{x}) < \bar{\mu}_{ij\varsigma}$. **If there exist positive scalars $L_{\check{k}\varsigma}(\mathbf{x})$ and $\Omega_{\check{k}ij}^{(\alpha,1)}(\mathbf{x})$ that satisfy the conditions $\Omega_{\check{k}ij}^{(\alpha,1)}(\mathbf{x}) > \Xi_{\check{k}ij}^{(\alpha,1)}(\mathbf{x})$, $\forall \alpha \in \underline{n}$, $\check{k} \in \{1, 2\}$, $i, j \in \underline{p}$, based on the PLMF dependent method, the basic stability conditions $\Xi_{\check{k}ij}^{(\alpha,1)}(\mathbf{x}) < 0$, $\forall \check{k} \in \{1, 2\}$ can be relaxed by the following conditions:**

$$\begin{aligned}& \sum_{i=1}^p \sum_{j=1}^p [(\chi_{ij i_1 i_2 \dots i_{\hat{n}}}(\mathbf{x}) + \underline{\mu}_{ij\varsigma}) \Xi_{\check{k}ij}^{(\alpha,1)}(\mathbf{x}) + (\bar{\mu}_{ij\varsigma} - \underline{\mu}_{ij\varsigma}) \Omega_{\check{k}ij}^{(\alpha,1)}(\mathbf{x})] \\ & - \sum_{\hat{r}=1}^{\hat{n}} (x_{\hat{r}} - x_{\hat{r}\varsigma\min})(x_{\hat{r}\varsigma\max} - x_{\hat{r}}) L_{\check{k}\varsigma}(\mathbf{x}) < 0; \\ & \forall \check{k} \in \{1, 2\}, i, j \in \underline{p}, i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma}, \alpha \in \underline{n}.\end{aligned}\quad (85)$$

280 Combining Theorem 2 and the results of this subsection, the positivity and stability of the PPFMB control system (3) can be guaranteed by the following Theorem.

Theorem 3. *For the PPFMB control system (3), if there exist $\lambda \in \mathbb{R}^{n \times 1}$, polynomial vectors $\mathbf{D}_{j\iota}(\mathbf{x}) \in \mathbb{R}^{1 \times n}$ and $\tilde{\mathbf{D}}_j(\mathbf{x}) \in \mathbb{R}^{1 \times n}$, $\forall j \in \underline{p}$, $\iota \in \underline{m}$, polynomial scalars $Y_{kv}(\mathbf{x})$ and $R_{\check{k}\varsigma}(\mathbf{x})$, $\forall k \in \{1, 2, 3, 4\}$, $\check{k} \in \{1, 2, 3\}$, $v \in \underline{p}$, $\varsigma \in \underline{\sigma}$, polynomial scalars $\Gamma_{ijs}^{(\alpha,\beta)}(\mathbf{x})$ and $N_\varsigma(\mathbf{x})$, $\forall i, j, s \in \underline{p}$, $\varsigma \in \underline{\sigma}$, polynomial scalars $\Omega_{\check{k}ij}^{(\alpha,1)}(\mathbf{x})$, $L_{\check{k}\varsigma}(\mathbf{x})$, $\forall \check{k} \in \{1, 2\}$, $i, j \in \underline{p}$, $\varsigma \in \underline{\sigma}$ such that the following SOS-based conditions are satisfied:*

$$\nu^T (\lambda^{(\alpha,1)} - \varepsilon_1) \nu \text{ is SOS; } \forall \alpha \in \underline{n} \quad (86)$$

$$\nu^T (\tilde{\mathbf{D}}_j^{(1,\beta)}(\mathbf{x}) - \mathbf{D}_{j\iota}^{(1,\beta)}(\mathbf{x})) \nu \text{ is SOS, } \forall j \in \underline{p}, \iota \in \underline{m}, \beta \in \underline{n}; \quad (87)$$

$$\nu^T(\Omega_{\check{k}ij}^{(\alpha,1)}(\mathbf{x}) - \varepsilon_2(\mathbf{x}))\nu \text{ is SOS, } \forall i, j \in \underline{p}, \check{k} \in \{1, 2\}, \alpha \in \underline{n}; \quad (88)$$

$$\nu^T(\Omega_{\check{k}ij}^{(\alpha,1)}(\mathbf{x}) - \Xi_{\check{k}ij}^{(\alpha,1)}(\mathbf{x}) - \varepsilon_3(\mathbf{x}))\nu \text{ is SOS, } \forall i, j \in \underline{p}, \check{k} \in \{1, 2\}, \alpha \in \underline{n}; \quad (89)$$

$$\nu^T(L_{\check{k}\varsigma}(\mathbf{x}) - \varepsilon_4(\mathbf{x}))\nu \text{ is SOS, } \forall \check{k} \in \{1, 2\}, \varsigma \in \underline{\sigma}; \quad (90)$$

$$\begin{aligned} & - \nu^T \left\{ \sum_{i=1}^p \sum_{j=1}^p [(\chi_{ij i_1 i_2 \dots i_{\hat{n}}}(\mathbf{x}) + \underline{\mu}_{ij\varsigma}) \Xi_{\check{k}ij}^{(\alpha,1)}(\mathbf{x}) + (\bar{\mu}_{ij\varsigma} - \underline{\mu}_{ij\varsigma}) \Omega_{\check{k}ij}^{(\alpha,1)}(\mathbf{x})] \right. \\ & \quad \left. - \sum_{\hat{r}=1}^{\hat{n}} (x_{\hat{r}} - x_{\hat{r}\varsigma\min})(x_{\hat{r}\varsigma\max} - x_{\hat{r}}) L_{\check{k}\varsigma} - \varepsilon_5(\mathbf{x}) \right\} \nu \text{ is SOS,} \end{aligned}$$

$$\forall i, j \in \underline{p}, i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma}, \alpha \in \underline{n}, \check{k} \in \{1, 2\}; \quad (91)$$

$$\nu^T(Y_{kv}(\mathbf{x}) - \varepsilon_6(\mathbf{x}))\nu \text{ is SOS, } \forall k \in \{1, 2, 3, 4\}, v \in \underline{p}; \quad (92)$$

$$\nu^T(Y_{kv}(\mathbf{x}) - \theta_v - \varepsilon_7(\mathbf{x}))\nu \text{ is SOS, } \forall k \in \{1, 2, 3, 4\}, v \in \underline{p}; \quad (93)$$

$$\nu^T(R_{\hat{k}\varsigma}(\mathbf{x}) - \varepsilon_8(\mathbf{x}))\nu \text{ is SOS, } \forall \hat{k} \in \{1, 2, 3\}, \varsigma \in \underline{\sigma}; \quad (94)$$

$$\nu^T(\Lambda_{1i_1 i_2 \dots i_{\hat{n}}\varsigma}(\mathbf{x}) - \varepsilon_9(\mathbf{x}))\nu \text{ is SOS; } \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma} \quad (95)$$

$$\nu^T \Lambda_{2i_1 i_2 \dots i_{\hat{n}}\varsigma}(\mathbf{x}) \nu \text{ is SOS; } \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma} \quad (96)$$

$$\nu^T \Lambda_{3i_1 i_2 \dots i_{\hat{n}}\varsigma}(\mathbf{x}) \nu \text{ is SOS; } \forall i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}, \varsigma \in \underline{\sigma} \quad (97)$$

$$\nu^T(\Gamma_{ijs}^{(\alpha,\beta)}(\mathbf{x}) - \varepsilon_{10}(\mathbf{x}))\nu \text{ is SOS; } \forall i, j, s \in \underline{p}, \alpha \neq \beta \in \underline{n}; \quad (98)$$

$$\nu^T(\Gamma_{ijs}^{(\alpha,\beta)}(\mathbf{x}) - \Theta_{ijs}^{(\alpha,\beta)}(\mathbf{x}) - \varepsilon_{11}(\mathbf{x}))\nu \text{ is SOS; } \forall i, j, s \in \underline{p}, \alpha \neq \beta \in \underline{n}; \quad (99)$$

$$\nu^T(N_{\varsigma}(\mathbf{x}) - \varepsilon_{12}(\mathbf{x}))\nu \text{ is SOS; } \forall \varsigma \in \underline{\sigma}; \quad (100)$$

$$\begin{aligned} & \nu^T \left\{ \sum_{i=1}^p \sum_{j=1}^p \sum_{s=1}^p [(\kappa_{ijs i_1 i_2 \dots i_{\hat{n}}}(\mathbf{x}) + \bar{\varrho}_{ijs\varsigma}) \Theta_{ijs}^{(\alpha,\beta)}(\mathbf{x}) + (\underline{\varrho}_{ijs\varsigma} - \bar{\varrho}_{ijs\varsigma}) \Gamma_{ijs}^{(\alpha,\beta)}(\mathbf{x})] \right. \\ & \quad \left. - \sum_{\hat{r}=1}^{\hat{n}} (x_{\hat{r}} - x_{\hat{r}\varsigma\min})(x_{\hat{r}\varsigma\max} - x_{\hat{r}}) N_{\varsigma}(\mathbf{x}) - \varepsilon_{13}(\mathbf{x}) \right\} \nu \text{ is SOS,} \end{aligned}$$

$$\forall i, j, s \in \underline{p}, \alpha \neq \beta \in \underline{n}, \varsigma \in \underline{\sigma}, i_1, i_2, \dots, i_{\hat{n}} \in \{1, 2\}. \quad (101)$$

where $\Xi_{1ij}(\mathbf{x})$ and $\Xi_{2ij}(\mathbf{x})$ are defined in (19) and (20), respectively; $\Lambda_{1i_1 i_2 \dots i_{\hat{n}}\varsigma}(\mathbf{x})$, $\Lambda_{2i_1 i_2 \dots i_{\hat{n}}\varsigma}(\mathbf{x})$ and $\Lambda_{3i_1 i_2 \dots i_{\hat{n}}\varsigma}(\mathbf{x})$ are defined in (44), (45) and (46), respectively; $\Theta_{ijs}^{(\alpha,\beta)}(\mathbf{x})$ is defined in (49); f_{\min} and f_{\max} are the predefined positive scalars; ν is an arbitrary vector independent of \mathbf{x} with appropriate dimensions; $\varepsilon_1 > 0$, $\varepsilon_2(\mathbf{x}) > 0$, \dots , $\varepsilon_{13}(\mathbf{x}) > 0$ are predefined scalar polynomials, then the system (3) is asymptotically stable and positive. The polynomial fuzzy controller gains

can be obtained by $\mathbf{G}_j(\mathbf{x}) = \frac{\sum_{l=1}^m \mathbf{e}_m^l \mathbf{D}_{j^l}(\mathbf{x})}{\sum_{s=1}^p m_s(\mathbf{x}) \boldsymbol{\lambda}^T \mathbf{B}_s(\mathbf{x}) \mathbf{e}_m}$.

290 **Remark 4.** Theorem 1 provides basic positivity condition (14), basic stability conditions (4)-(7) and restricted conditions (8)-(13), where restricted conditions are the prerequisites for the stability conditions (4)-(7) to guarantee the *stability of system (3)*. In Theorem 1, basic positivity and stability conditions all do not depend on membership functions, so this theorem lead to conservative *results*. In
 295 order to relax the *results*, *PLMF* dependent method is tried to apply to positivity conditions in Theorem 2. In Corollary 1, the PDC method and *PLMF* dependent method are combined and applied to positivity conditions. To further relax the result, *PLMF* dependent method is applied to both positivity conditions and stability conditions in Theorem 3, the conditions (88)-(91) are stability condi-
 300 tions which are used to guarantee that the PPFMB system (3) is asymptotically stable, conditions (92)-(97) are restricted conditions. The positivity conditions (98)-(101) are used to guarantee that the PPFMB system (3) is positive.

5. Simulation Example

In this section, one example is provided to demonstrate the effectiveness and
 305 applicability of the analysis results. *The simulation results verify that the LCLF with the help of the proposed convexification method leads to less conservatism than quadratic Lyapunov function.* In addition, the simulation results show that the PLMF dependent analysis method has a stronger ability to relax results than PDC analysis method.

A three-rules PPFMB system is considered. The system and input matrices are as follows:

$$\mathbf{A}_1(x_1) = \begin{bmatrix} -0.039 & 28.82 \\ 1 & -2 - x_1^2 - x_1 \end{bmatrix},$$

$$\mathbf{A}_2(x_1) = \begin{bmatrix} -0.037 & 26.71 \\ 0.80 & -4 - 1.20x_1^2 \end{bmatrix},$$

$$\mathbf{A}_3(x_1) = \begin{bmatrix} -0.033 & 22.07 \\ a & -2 - x_1^2 - x_1 - b \end{bmatrix},$$

$$\begin{aligned}
\mathbf{B}_1(x_1) &= \begin{bmatrix} 3.27 + 0.05x_1^2 \\ 0.05 \end{bmatrix}, \mathbf{B}_2(x_1) = \begin{bmatrix} 2.90 + 0.02x_1^2 \\ 0.05 \end{bmatrix}, \\
\mathbf{B}_3(x_1) &= \begin{bmatrix} 2.09 + 0.10x_1^2 \\ b \end{bmatrix},
\end{aligned} \tag{102}$$

310 where a and b are constant scalars to be specified, the working range of both x_1 and x_2 are $[0, 4]$.

The membership functions of the PPFMB system are chosen as $w_1(x_1) = 1 - \frac{1}{1+e^{-5(x_1-1.6)}}$, $w_3(x_1) = \frac{1}{1+e^{-5(x_1-2.4)}}$, $w_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$. In this paper, we adopt IPC concept [33, 34, 24, 25, 35, 26, 36] to design the polynomial fuzzy controller, which means that the membership functions between the polynomial fuzzy model and controller are allowed to be different. The membership functions of the fuzzy controller are chosen as

$$\begin{aligned}
m_1(x_1) &= \begin{cases} 0 & \text{if } x_1 > 2.13 \\ -\frac{1}{1.06}x_1 + \frac{2.13}{1.06} & \text{if } 1.07 \leq x_1 \leq 2.13, \\ 1 & \text{if } x_1 < 1.07 \end{cases} \\
m_3(x_1) &= \begin{cases} 1 & \text{if } x_1 > 2.93 \\ \frac{1}{1.06}x_1 - \frac{1.87}{1.06} & \text{if } 1.87 \leq x_1 \leq 2.93, \\ 0 & \text{if } x_1 < 1.87 \end{cases} \\
m_2(x_1) &= 1 - m_1(x_1) - m_3(x_1).
\end{aligned} \tag{103}$$

In order to verify that the LCLF with the proposed convexification method can lead to more relaxed stability region than quadratic Lyapunov function, 315 three cases are considered. In the first case, Corollary 1 of [37] is applied, which adopted quadratic Lyapunov function and two controller membership functions rules; In the second case, Corollary 1 of [37] with the controller membership functions (103) is applied; In the third case, Theorem 1 with the controller membership functions (103) is applied. The constant parameters a and b are 320 chosen in the range of $8 \leq a \leq 22$ at the interval of 2 and $0 \leq b \leq 1$ at the

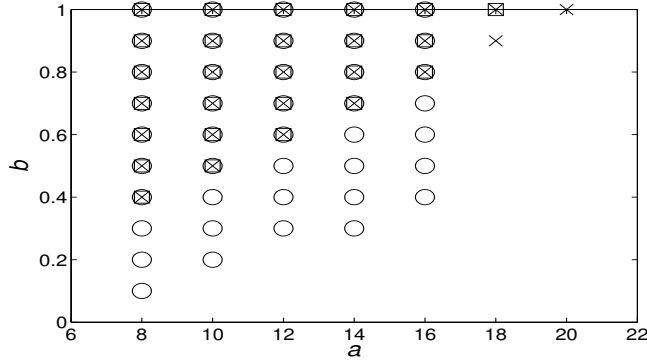


Figure 1: Comparison of stability regions which are obtained under case 1 (“x”), case2 (“□”) and case 3 (“o”).

interval of 0.1. For Theorem 1, the settings of $\varepsilon_1 = \dots = \varepsilon_8 = 1 \times 10^{-3}$ are the same as the settings of them in Corollary 1 of [37]; f_{min} and f_{max} are chosen as 0.92 and 1.3, respectively; $\mathbf{D}_{j\iota}(x_1)$ and $\tilde{\mathbf{D}}_j(x_1)$ are all of degrees from 0 to 2 in x_1 . The stability regions obtained under these three cases are shown in
 325 Fig. 1. The stability regions given by Corollary 1 of [37] with 2 and 3 controller membership functions rules are indicated by “x” and “□”, respectively, and the stability region given by Theorem 1 is indicated by “o”. It is obvious that the LCLF with the proposed convexification method gives larger stability region than quadratic Lyapunov function.

330 In order to verify that the MFD positivity conditions can be used to ensure that the system state is positive and effectively relax the stability region, Theorem 2 is applied, and the expansion points are chosen as $x_1 = \{0, 0.5, 1.07, 1.4, 1.87, 2.13, 2.6, 2.93, 3.5, 4\}$. The constant parameters a and b are chosen in the range of $10 \leq a \leq 74$ at the interval of 4 and $0 \leq b \leq 1$
 335 at the interval of 0.1. We choose $\varepsilon_1 = \dots = \varepsilon_{11} = 1 \times 10^{-3}$; $f_{min} = 0.92$, $f_{max} = 1.3$; $\tilde{\mathbf{D}}_j(x_1)$, $\mathbf{D}_{j\iota}(x_1)$, $Y_{kv}(x_1)$, $R_{\hat{\kappa}_\varsigma}(x_1)$, $\mathbf{\Gamma}_{ijs}^{(\alpha,\beta)}(x_1)$, and $N_\varsigma(x_1)$ are all of degree from 0 to 2 in x_1 . In order to compare the ability of PLMF dependent analysis method and the PDC analysis method to relax positivity conditions, the Corollary 1 is applied, the degree of $\hat{\mathbf{\Gamma}}_{ijs}^{(\alpha,\beta)}(x_1)$ and $\hat{N}_\varsigma(x_1)$ are the same

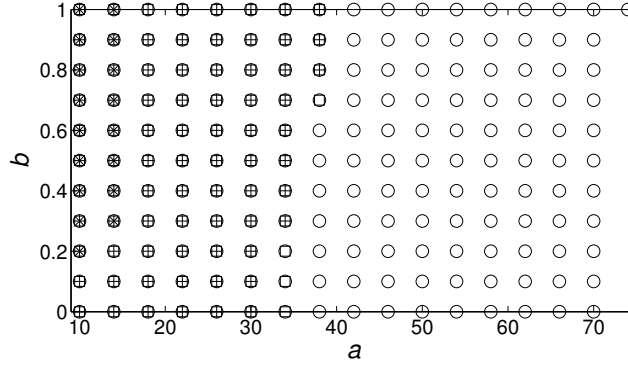


Figure 2: Stability regions for PPFMB system in Theorem 1 (“ × ”), Theorem 2 (“ □ ”), Corollary 1 (“ + ”) and Theorem 3 (“ o ”).

340 as the degree of $\mathbf{I}_{ijs}^{(\alpha,\beta)}(x_1)$ and $N_\zeta(x_1)$, and the other parameters and settings are kept the same as Theorem 2. In Theorem 3, both positivity conditions and stability conditions all are MFD conditions. $\Omega_{1ij}^{(\alpha,1)}(x_1)$, $\Omega_{2ij}^{(\alpha,1)}(x_1)$, $L_{1\zeta}(x_1)$ and $L_{2\zeta}(x_1)$ are all of degree from 0 to 2, and ε_{12} and ε_{13} are chosen as 1×10^{-3} . The other parameters and settings are the same as those in Theorem 2. The
 345 stability regions obtained by Theorems 1-3 and Corollary 1 are shown in Fig. 2. “ × ”, “ □ ” and “ o ” represent the stability regions which are given by Theorem 1, 2 and 3. “ + ” represents the stability region which is given by Corollary 1.

It can be seen in Fig. 2 that the stability regions given by Theorems 2 and 3, and Corollary 1 are larger than the stability region given by Theorem 1,
 350 which means that the PLMF dependent analysis method and the PDC analysis method all can effectively relax the analysis results. The stability region given by Theorem 2 is larger than the stability region given by Corollary 1, which means that PLMF dependent analysis method has ability to provide more relaxed positive analysis result than PDC analysis method. Thus, with the help of
 355 PLMF dependent analysis method, we can freely choose the membership functions of the controller for the flexibility of the controller design without worrying about the conservativeness brought by the mismatched premise variables.

In order to demonstrate that the system states with any initial states in

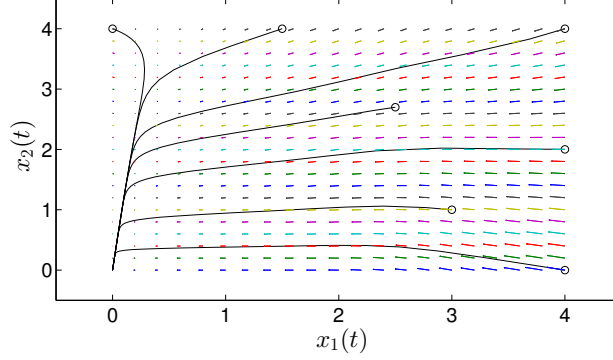


Figure 3: Phase plot of the states x_1 and x_2 for $a = 16$, $b = 0.4$ given by Theorem 1.

the stability regions can be **steered** to the equilibrium point by the designed
 360 controller and always remains in the positive quadrant, we draw phase plots
 for the boundary point of these stability regions. For example, we choose $a =$
 $16, b = 0.4$ in the stability region given by Theorem 1; $a = 34, b = 0$ in the
 stability region given by Theorem 2; $a = 38, b = 0.8$ in the stability region
 given by Corollary 1; $a = 74, b = 1$ in the stability region given by Theorem 3.
 365 For different sets of a and b , the conditions in the corresponding theorems or
 corollary are calculated by SOSTOOLS. The obtained results including $\mathbf{D}_{j_i}(x_1)$
 and λ are shown in the Table I. The phase plots for different sets of a and b
 are shown in Figs 3-6. It can be seen that the PPFMB system is positive and
 asymptotically stable under the proposed control strategy.

370 6. Conclusion

The stability and positivity of PPFMB fuzzy system have been investigated.
 The IPC concept has been adopted to increase the flexibility of the controller
 design. In order to obtain more relaxed analysis results, LCLF has been applied
 on the stability analysis. For the non-convex conditions derived by LCLF, the
 375 novel controller has been designed and the sector nonlinear concept has been
 used to handle the non-convex terms. Also, the PLMFs were adopted to remove
 obstacles to convexity caused by mismatched premise membership functions.

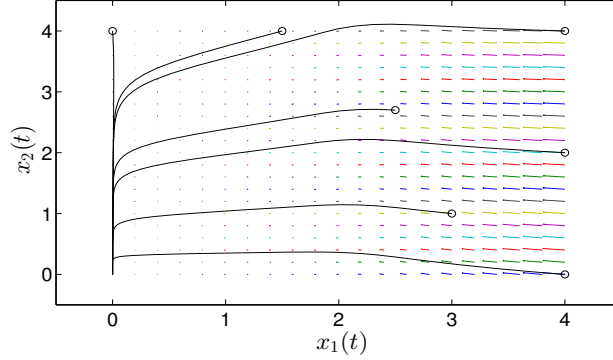


Figure 4: Phase plot of the states x_1 and x_2 for $a = 34$, $b = 0$ given by Theorem 2

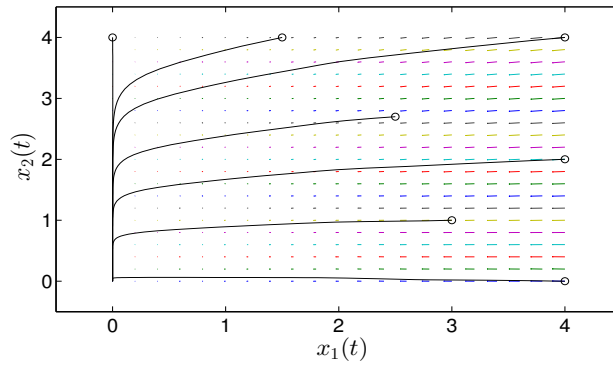


Figure 5: Phase plot of the states x_1 and x_2 for $a = 38$, $b = 0.8$ given by Corollary 1.

Relaxed analysis results have been obtained by LCLF with the proposed effective convexification strategy, meanwhile different premise membership functions
 380 between the fuzzy controller and model are allowed. In addition, the PLMF dependent positivity and stability conditions have been obtained by developing a systematic analysis method with the consideration of controller design, membership functions information, system positivity and stability, which leads to more relaxed analysis results. A simulation example has been presented to verify
 385 that the proposed method can effectively relax the results. This paper proposes an effective convexification method to handle the non-convex conditions when LCLF is adopted to investigate the stability of state feedback control of

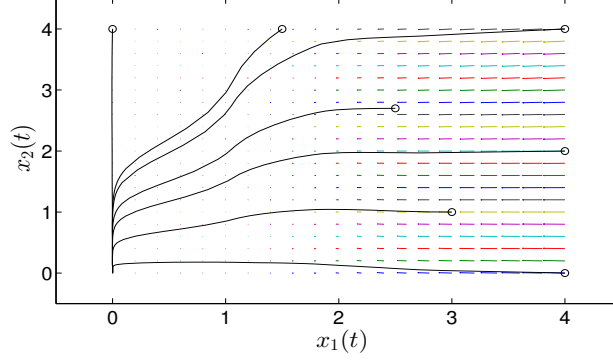


Figure 6: Phase plot of the states x_1 and x_2 for $a = 74$, $b = 1$ given by Theorem 3.

PPFMB system. In the future, LCLF with this convexification method can be used to investigate the stability of more extensive systems, such as switched
 390 positive fuzzy systems, positive Markov Jump fuzzy systems, positive tracking fuzzy systems, and so on.

Appendix

Proof of Lemma1

Necessity: For the case that system input $\mathbf{u} \equiv \mathbf{0}$, letting $\mathbf{x}(0) = \mathbf{m}^{(\beta)}$ be the
 395 initial state vector of system (1), where $\mathbf{m}^{(\beta)}$ is the unit vector of the β -axis of \mathbf{x} , it follows that $\dot{\mathbf{x}}(0) = \sum_{i=1}^p w_i(\mathbf{x}(0))\mathbf{A}_{i0}(\mathbf{x}(0))\mathbf{m}^{(\beta)} =$ the β -th column of $\sum_{i=1}^p w_i(\mathbf{x}(0))\mathbf{A}_i(\mathbf{x}(0))$. Because the trajectory of a positive system cannot leave the positive orthant, so that $\dot{\mathbf{x}}^{(\alpha)}(0) \succ 0$ for $\forall \alpha \neq \beta$ where $\dot{\mathbf{x}}^{(\alpha)}$ is the α -th element of $\dot{\mathbf{x}}$. Therefore, the off-diagonal elements of $\sum_{i=1}^p w_i(\mathbf{x}(t))\mathbf{A}_i(\mathbf{x}(t))$ must
 400 be nonnegative, i.e., $\sum_{i=1}^p w_i(\mathbf{x}(t))\mathbf{A}_i(\mathbf{x}(t))$ must be a Metzler matrix.

For the case that system input $\mathbf{u} \neq \mathbf{0}$, letting $\mathbf{x}(0) = \mathbf{0}$, positivity implies $\dot{\mathbf{x}}(0) = \sum_{i=1}^p w_i(\mathbf{x}(0))\mathbf{B}_i(\mathbf{x}(0))\mathbf{u}(0) \succ 0$ for every $\mathbf{u}(0) \succ 0$, that is, $\sum_{i=1}^p w_i(\mathbf{x}(t))\mathbf{B}_i(\mathbf{x}(t)) \succ 0$.

Sufficiency: In order to show that $\mathbf{x}(t) \succ 0$, it is sufficient to check that
 405 the vector $\dot{\mathbf{x}}$ does not point toward the outside of positive orthant whenever

Table 1: $a, b, \lambda, \mathbf{D}_{j\ell}$ of Theorem 1 to 3 and Corollary 1

	λ	$\mathbf{D}_{j\ell}$
Theorem1 $a = 16$ $b = 0.4$	[0.2866; 0.7444]	$\mathbf{D}_{11}(x_1) = [-0.0701x_1^2 - 4.6595 \times 10^{-12}x_1 - 13.2969, 0.3906x_1^2 + 0.3003x_1 - 7.5660]$ $\mathbf{D}_{21}(x_1) = [-0.0701x_1^2 - 5.4887 \times 10^{-13}x_1 - 13.2969, 0.3906x_1^2 + 0.3003x_1 - 7.5660]$ $\mathbf{D}_{31}(x_1) = [-0.0701x_1^2 - 3.7538 \times 10^{-12}x_1 - 13.2969, 0.3906x_1^2 + 0.3003x_1 - 7.5660]$
Theorem2 $a = 34$ $b = 0$	[4.7005; 4.6041]	$\mathbf{D}_{11}(x_1) = [-4.7878x_1^2 + 15.5363x_1 - 183.7030, -2.8995x_1^2 + 7.5012x_1 - 137.4550]$ $\mathbf{D}_{21}(x_1) = [-7.5239x_1^2 + 24.4027x_1 - 190.0470, -9.8691x_1^2 + 47.9237x_1 - 168.3621]$ $\mathbf{D}_{31}(x_1) = [-63.4531x_1^2 + 23.5314x_1 - 223.3872, -6.0523x_1^2 + 48.1855x_1 - 179.4600]$
Corollary1 $a = 38$ $b = 0.8$	[12.9377; 12.4232]	$\mathbf{D}_{11}(x_1) = [-7.8902x_1^2 + 23.7111x_1 - 530.5932, -4.0795x_1^2 + 13.8278x_1 - 378.2983]$ $\mathbf{D}_{21}(x_1) = [-32.0914x_1^2 + 222.2634x_1 - 1017.8435, -13.4412x_1^2 + 74.2394x_1 - 413.5052]$ $\mathbf{D}_{31}(x_1) = [-42.7542x_1^2 + 120.1953x_1 - 1951.8239, -10.4494x_1^2 + 58.9231x_1 - 401.9825]$
Theorem3 $a = 74$ $b = 1$	[10.9280; 10.6704]	$\mathbf{D}_{11}(x_1) = [3.3600x_1^2 + 150.6442x_1 - 237.2883, -4.6218x_1^2 + 14.6757x_1 - 319.5539]$ $\mathbf{D}_{21}(x_1) = [-101.2932x_1^2 + 394.9653x_1 - 888.4543, -15.9847x_1^2 + 110.4992x_1 - 371.7873]$ $\mathbf{D}_{31}(x_1) = [-79.7588x_1^2 + 115.8095x_1 - 2480.0106, -16.7703x_1^2 + 81.9643x_1 - 345.6084]$

\mathbf{x} is on the boundary of positive orthant. This is equivalent to verify that the vector of $\dot{\mathbf{x}}(t) = \sum_i^p w_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t))$ corresponding to the zero components of $\mathbf{x} \succ 0$ are nonnegative, the set of indices of such

components is denoted by \mathcal{I} , i.e., $\mathbf{x}^{(\alpha)} = 0$ for $\alpha \in \mathcal{I}$, we can write that

410 $\dot{\mathbf{x}}^{(\alpha)}(t) = \sum_i^p w_i(\mathbf{x}(t)) \left(\sum_{(\beta) \notin \mathcal{I}} \mathbf{A}_i^{(\alpha, \beta)}(\mathbf{x}(t)) \mathbf{x}^\beta(t) + \mathbf{B}_i^{(\alpha)}(\mathbf{x}(t)) \mathbf{u}(t) \right)$ for $\alpha \in \mathcal{I}$, where

$\sum_i^p w_i(\mathbf{x}(t)) \mathbf{A}_i^{(\alpha, \beta)}(\mathbf{x}(t))$ is the α -th row, β -th column element of $\sum_i^p w_i(\mathbf{x}(t)) \mathbf{A}_i(\mathbf{x}(t))$,

$\sum_i^p w_i(\mathbf{x}(t)) \mathbf{B}_i^{(\alpha)}(\mathbf{x}(t))$ is the α -th row element of $\sum_i^p w_i(\mathbf{x}(t)) \mathbf{B}_i(\mathbf{x}(t))$. So, from

the nonnegativity of $\sum_i^p w_i(\mathbf{x}(t)) \mathbf{B}_i^{(\alpha)}(\mathbf{x}(t))$ and $\sum_i^p w_i(\mathbf{x}(t)) \mathbf{A}_i^{(\alpha, \beta)}(\mathbf{x}(t))$ with $\alpha \neq$

β , it follows that $\dot{\mathbf{x}}^{(\alpha)}(t) \succ 0$. Assume that \mathbf{x} is at the origin of the coordi-

415 nates and $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$, then $\alpha \in \{1, 2, \dots, n\}$, so we can

obtain the positivity conditions that $\sum_i^p w_i(\mathbf{x}(t)) \mathbf{A}_i(\mathbf{x}(t))$ is Metzler matrix and

$\sum_i^p w_i(\mathbf{x}(t)) \mathbf{B}_i(\mathbf{x}(t)) \succ 0$. The proof is completed.

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