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# The strategy of conflict and cooperation[*]

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## Abstract

In this paper, I introduce (i) a novel and unified framework, called *cooperative extensive form games*, for the study of strategic competition and cooperation, which have been studied in specific contexts, and (ii) a novel solution concept, called *cooperative equilibrium system*. I show that non-cooperative extensive form games are a special case of cooperative extensive form games, in which players can strategically cooperate (e.g., by writing a possibly costly contract) or act non-cooperatively. To the best of my knowledge, I propose the first solution to the long-standing open problem of "strategic cooperation" first identified by von Neumann (1928). I have one main result to report: I prove that cooperative equilibrium system always exists in finite $n$-person cooperative strategic games with possibly imperfect information. The proof is constructive in the case of perfect information games.

*Keywords*: strategic cooperation, non-cooperative games, cooperative games, backward induction, extensive form games

# 1    Introduction

> I hope we can go back to von Neumann and Morgenstern's vision of cooperative
> games and resurrect it. What is missing from their cooperative game theory
> was that it ignored externalities created by coalition forming which affect
> other players in strategic situations e.g. competition among big firms. (Eric
> Maskin's paraphrased response to a question on the open problems in game
> theory in Nobel Symposium "One Hundred Years of Game Theory: Future
> Applications and Challenges.")

Conflict, competition and cooperation are widely studied not only in economics but
also computer science, biology, and political science. There are many fields within eco-
nomics in which competition and cooperation are often explored, including industrial orga-
nization, mechanism design, contract theory, finance, market design, matching, networks,
public economics, social choice, and elections and voting. Wars, airline alliances, trade,
oligopolistic cartels, vote trading, transportation, corruption, the evolution of species and
genes, scientific collaboration, and team sports are only some of the examples of strategic
situations that involve both competition and cooperation.[1] However, despite the substan-
tial progress that has been made in the past century in non-cooperative and cooperative
games, we still lack a unified framework to systematically study these topics.

The aim of this paper is to provide a unified framework, called *cooperative exten-
sive form games*, for the study of strategic competition and cooperation, and introduce
a novel solution, called *cooperative equilibrium system*. This novel framework captures
both payoff-based and strategic externalities due to coalition forming. I show that non-
cooperative extensive form games are a special case of cooperative extensive form games,
in which players can strategically cooperate—e.g., by writing a possibly costly contract—
or act non-cooperatively. I have one main result to report: I prove that every cooperative
extensive form game with possibly imperfect information possesses a cooperative equi-
librium system (Theorem 2). In particular, there is a cooperative equilibrium system
which includes only pure strategies in every perfect information game (Theorem 1). This
existence proof is constructive and given by the *Recursive Backward Induction* (RBI) al-
gorithm, which is a novel generalization of the well-known Backward Induction algorithm.

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[1]For competition and cooperation among freight carriers, see, e.g., Krajewska et al. (2008); for more
examples in multi-agent systems in computer science, Doran et al. (1997); for more applications of game
theory, Binmore (2007); for coalitional manipulations in mechanism design, Green and Laffont (1979) and
Crémer (1996); for collusion in industrial organization, Tirole (1992); for regulation in political economy,
Laffont and Tirole (1991); for coalition formation in non-democracies, Acemoglu et al. (2008).

However, in imperfect information games the existence proof does rely on non-constructive steps in the RBI.

To the best of my knowledge, I propose the first solution to the long-standing open problem of "strategic cooperation" in $n$-person games first identified by von Neumann (1928). Prior literature includes, but not limited to, von Neumann and Morgenstern (1944), Aumann (1959), Harsanyi (1974), Rubinstein (1980), Bernheim, Peleg, and Whinston (1987), Aumann and Myerson (1988), Bloch (1996), Ray and Vohra (1997), Brams, Jones, and Kilgour (2005), and Chander and Wooders (2020); the authors defined key concepts and discussed important issues in this literature.[2]

It is well-known that a Nash equilibrium is in general vulnerable to deviations by coalitions with two or more players. In their seminal contribution, Bernheim et al. (1987) addressed this problem and defined a concept in which any coalitional deviation is required to be immune to further deviations by subcoalitions. However, as recognized by Bernheim et al. (1987, p. 7), deviating subcoalitions do not consider forming a coalition with non-deviating players in their coalition-proof Nash equilibrium: "This rules out the possibility that some member of the deviating coalition might form a pact to deviate further with someone not included in this coalition." In cooperative strategic games, deviating players can form a coalition with other players, not just with the members of their subcoalition, and every player 'rationally' takes into account regrouping of others in a cooperative equilibrium system.

In section 5, I also provide theoretical applications of cooperative strategic games to oligopolistic competition with mergers, dynamic logrolling (i.e., trading votes) à la Casella and Palfrey (2019), and "corruption"—i.e., buying someone's cooperation to induce them to choose a particular action. I introduce a general logrolling model and show that it always possesses a cooperative equilibrium system. This existence result extends Casella and Palfrey's (2019) result to logrolling games with (i) 'dynamically rational' voters, (ii) any type of voting rule, (iii) any type of preferences, and (iv) imperfect information.

**Formal definition of the framework**: A cooperative strategic game is denoted by $\Gamma = (P, X, I, u, S, H)$, which is an extensive form game with an addition of coalitional utility function $u_C$ for each feasible coalition $C$. $P$ denotes the set of players, $X$ the game tree, $I$ the player function, $H$ the set of all information sets, and $S$ the set of all mixed strategy profiles. For every feasible (possibly singleton) coalition $C$, $u_{C|P'}$ denotes the von Neumann-Morgenstern utility of player $C$ given a partition of players $P'$, which captures the possibility of positive or negative externalities when a player joins a coalition.

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[2]The list is by no means comprehensive; for further discussion, see section 2.

Note that I use the word "cooperation" in the following sense: "the fact of doing something together or of working together towards a shared aim" (Oxford Advanced Learner's Dictionary).[3] In this framework, the "aim" of a coalition is to maximize coalition's utility function, just like the aim of an individual player is to maximize their own utility function.[4]

**Interpretation of the framework**: First, note that cooperative strategic games generalize non-cooperative extensive form games in the sense that every non-cooperative extensive form game is a cooperative extensive form game but not vice versa. Put differently, a cooperative extensive form game $\Gamma = (P, X, I, (u_i)_{i \in P}, S, H)$ reduces to a non-cooperative extensive form game, where the only feasible coalitions are the singleton coalitions.

Unlike in non-cooperative games, the set of players is endogenous—i.e., it may evolve throughout the game according to the following rule: If some players, say $i$ and $j$, form a coalition $C = \{i, j\}$, then each of them becomes an *agent* of player $C$.[5] The expectation would be that once a coalition $C$ forms the agents choose their strategy according to player $C$'s utility, $u_C$. Coalition forming may be costly. At every information set, any coalition can form and any available action can be played—the framework is very general. Once a coalition forms, agents are free to leave their coalition whether they have written a possibly costly and binding contract or not; though, doing so may be costly as well. So, players in general can do whatever they want even if their choice is "irrational." Neither coalitions nor the strategies the players will pick would necessarily be revealed at each information set. Of course, if players are "rational," then they can infer what coalitions might be formed and what strategies might be chosen, but that concerns the "solution concept" and not the framework.

In cooperative strategic games, players are free to act independently or form coalitions, which could be via formal or informal institutions. But the right to form a coalition may certainly be restricted, and it might be impossible to coordinate actions under certain reasonable situations. If some players cannot form a coalition for any reason, then this is part of the cooperative strategic game so that all players rationally take this into account.[6]

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[3]This is distinct from labeling a strategy "cooperate" in non-cooperative games. For more details, see section 2.1.

[4]Note that coalitions do not "receive" utility, nor do the individuals. Utility function is a convenient way to represent preferences. And, the difference between the individual utility and coalitional utility comes from the fact that not every individual's preferences may be fully aligned with the preferences of the coalition which the individual belongs.

[5]A related but somewhat opposite approach called "player splitting" is studied by Perea y Monsuwé et al. (2000) and Mertens (1989), and it was used in the refinement of Nash equilibrium.

[6]For an informative discussion on the enforcement issues in cooperative and non-cooperative games, see Serrano (2004).
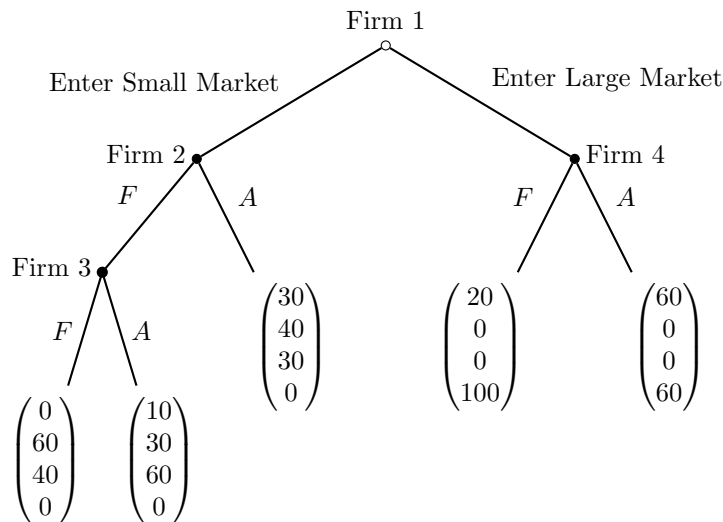
Figure 1: An international market entry game

**Informal definition of the solution concept**: A *system* is a family of collection of strategy profiles and a family of collection of coalitions. (Note that a solution concept in non-cooperative games consists of only one strategy profile.) A *subtree* of a game tree is like a subgame but any information set can be the root of the subtree. A *supertree* of a game is a clone of the game such that the player who acts at the root of the game joins a coalition with some other player(s). (These notions are formally defined in section 3.)

A *cooperative equilibrium system* is a system that is both subtree and supertree "perfect." Put differently, at every subtree and every supertree the followings must be satisfied: Given the appropriate "counterfactual," singleton players do not have any individual incentive to deviate, coalitional players do not have any joint incentives to deviate (based on coalitional utility function), and each coalition is stable in the sense that its agents prefer their coalition to any other coalition or to being a singleton player.

## 1.1 Illustrative example

Figure 1 presents a stylized international market entry game. Firm 1 chooses to enter the market either in a small (S) country or a large (L) country. In the small country, there is a leader (Firm 2) and a follower (Firm 3); in the large country, there is a monopolist (Firm 4). Firms in markets S and L can choose to fight (F) or accommodate (A). The total size of the "cake" is 100 units in market S and 120 units in market L, which are distributed as shown in the figure. Pre-entry profits are normalized to zero. If Firm 1 enters an economy in a country, then by reciprocity the existing firm(s) in that country gain access

to Firm 1's market. Thus, everything else being equal Firm 1's entry is beneficial for local firms, though the distribution of these extra gains depends on the strategic choices of the firms. If the leader (Firm 2) chooses $F$, then the follower (Firm 3) prefers to choose $A$. Anticipating this, the leader's best response is to choose $A$. However, if they collude by both choosing $F$, then they would both be better off and essentially drive Firm 1 out of the market. The monopolist in market L prefers to choose $F$ over $A$.

The cooperative equilibrium system solution is based on a recursive procedure called Recursive Backward Induction (RBI), which is formally defined in section 3. To illustrate the steps that lead to the cooperative equilibrium system in a simple setting, I make the following assumptions in the game presented in Figure 1. I assume that there is no cost to forming a coalition; though, even if it is costly (but not 'too' costly), the firms would still be willing to form coalitions in this example. For each (non-empty) coalition $C \subseteq \{1, 2, 3, 4\}$, let the coalitional utility function be defined as $u_C(\cdot) := \sum_{i \in C} u_i(\cdot)$.[7]

Figure 2 illustrates the cooperative equilibrium system of this market entry game in four steps. The first step (A) starts from the standard non-cooperative subgame perfect equilibrium solution, where every player acts independently and non-cooperatively. The outcome of this solution is (30, 40, 30, 0). Given the subgame perfect equilibrium, step B shows that both Firm 2 and Firm 3 benefit from colluding in which both of them choose F, which leads to the outcome (0, 60, 40, 0), where $u_{2,3}(0, 60, 40, 0) = 100$ which maximizes the coalition's utility compared to $u_{2,3}(30, 40, 30, 0) = 70$ and $u_{2,3}(10, 30, 60, 0) = 90$. (Notation "2, 3" signifies the cooperation between Firm 2 and 3.) Anticipating this collusion, Firm 1 chooses to enter market L, which leads to the outcome (20, 0, 0, 100). Step C shows that both Firm 1 and Firm 2 collude by best responding to the solution in step B. Firm 1 enters market S, and Firm 2 accommodates, which leads to the outcome (30, 40, 30, 0), which is strictly better for both than the outcome in step B. Other firms best respond to Firm 1 and Firm 2; Firm 3 chooses A and Firm 4 chooses F. Step D shows that Firm 4 anticipates the collusion between Firm 1 and 2 and offers Firm 1 to collude; Firm 1 and Firm 4 join a coalition $C' = \{1, 4\}$. One of the best responses of player $C'$ to the solution in step C is that agent Firm 1 enters market L and agent Firm 4 accommodates, which leads to the mutually beneficial outcome of (60, 0, 0, 60).[8] This is the outcome of the cooperative equilibrium system illustrated in Figure 2.

Notice that the reason Firm 4 colludes with Firm 1 in step D is the credible threat of

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[7]Note that the cooperative strategic games introduced in section 3 allow for any type of von Neumann-Morgenstern coalitional preferences.

[8]Note that the other best response of player $C'$ is to choose L and F but this would not be individiually rational for Firm 1 compared to step C.
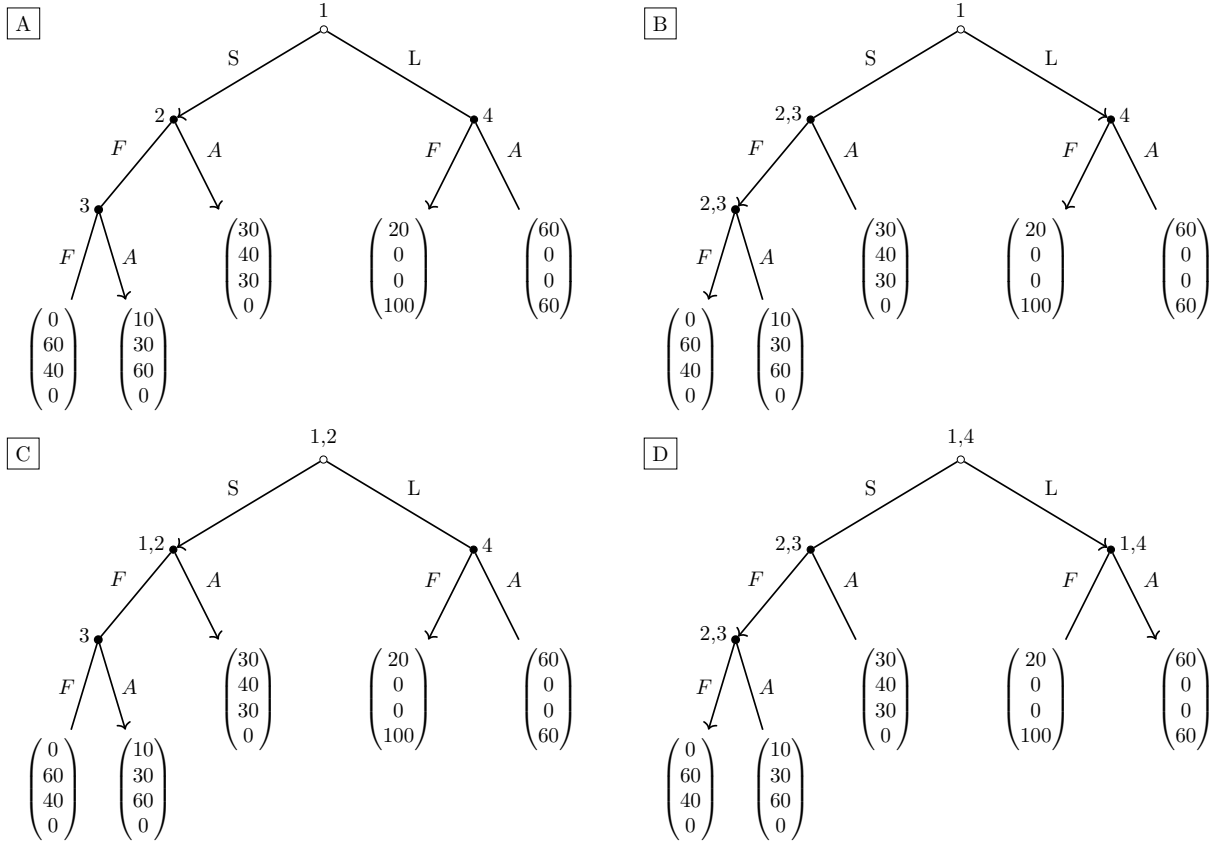
Figure 2: Equilibrium system solution of the international market entry game in four steps A–D. Assume that $u_C(\,\cdot\,) := \sum_{i \in C} u_i(\,\cdot\,)$ for every coalition $C \subseteq \{1,2,3,4\}$. Step A: The standard backward induction leads to the outcome (30, 40, 30, 0). Step B: Given the solution in step A, both Firm 2 and Firm 3 benefit from colluding in which both of them choose F, which leads to the outcome (0, 60, 40, 0). Firm 1 anticipates this collusion, so enters market L, which leads to the outcome (20, 0, 0, 100). Step C: Both Firm 1 and Firm 2 can do strictly better than the outcome in step B by colluding in which Firm 1 enters market S, and Firm 2 accommodates, leading to the outcome (30, 40, 30, 0). Step D: Firm 4 anticipates the collusion between Firm 1 and 2, so proposes Firm 1 to cooperate by choosing A provided that Firm 1 enters market L, which is mutually beneficial compared to (30, 40, 30, 0). Accordingly, the cooperative equilibrium system outcome is (60, 0, 0, 60). A step-by-step cooperative equilibrium system solution of this game is available at https://youtu.be/I1hIO7CrLnM.

collusion between Firm 1 and Firm 2 in step C, which would lead to a very bad outcome for Firm 4. In turn, the reason Firm 2 colludes with Firm 1 in step C is the credible threat of Firm 1 to choose L, in which case Firm 2's payoff would be 0.

At the outset, the collusion between Firm 1 and Firm 4 to obtain (60, 0, 0, 60) might seem obvious; though, the next example illustrates that the collusion heavily relies on 'off-path' threats. Everything else being equal, suppose that the outcome (20, 0, 0, 100) is changed to (40, 0, 0, 80). Then, in the unique cooperative equilibrium system Firm 1 would enter market L, and Firm 4 would choose F rather than colluding with Firm 1. This is because Firm 1 now would *not* benefit from cooperation with Firm 2, so such a threat would *not* be credible. This variation shows that off-path credible coalitional threats play a critical role in finding and sustaining the cooperative equilibrium system. This is akin to the crucial role of the credibility of threats in non-cooperative games (Selten (1965), Schelling (1980), Brams (1994)).

# 2   The problem of strategic cooperation and related literature

The connections between non-cooperative games and cooperative games, which abstracts away from strategic interaction, have been studied since von Neumann (1928), who came up with the maximin solution in a three-person zero-sum game in the lesser known part of his famous paper (see, p. 311). However, von Neumann noticed that any two players can strategically cooperate in that three-person game for mutual gain, which can destablize the maximin solution. Von Neumann also notice that as the number of players increase in a game, the strategic cooperation possibilities increase too. He concludes "aber eine befriedigende allgemeine Theorie fehlt zur Zeit noch," which is translated by Sonya Bargmann as

> "*a satisfactory general theory is as yet lacking.*"

Von Neumann (1928) appears to be the first person to propose the strategic cooperation issue as an open problem. Later, von Neumann mentions about this problem to mathematician Kurt Gödel, which we know from Gödel's personal notes that are recently published by von Plato (2021). Gödel notes that "with games of more than two players, there does not always exist any reasonable statistical system of play".

To the best of my knowledge, in this paper I propose the first solution to this long-standing problem via the cooperative equilibrium system in cooperative strategic games.

Prior research in game theory—mainly repeated games and the lesser-known farsighted approach to cooperative games—has made many contributions to the study of cooperation from various perspectives. Next, I discuss how the current paper's contribution fits into this existing literature. (Note that the literature is huge and fragmented, and I do not attempt to give a comprehensive survey of the literature in this paper.)

## 2.1 Cooperation in repeated games and the Nash program

The approach in this paper differs from the repeated games and the Nash program approach to cooperation in the following ways. The theory of repeated games mainly asks whether cooperative outcomes in a one-shot game can be sustained as Nash equilibrium or subgame perfect equilibrium when the game is repeated sufficiently many times. Similarly, under the Nash program the main question is whether it is possible to construct a non-cooperative game whose equilibrium outcome coincides with a cooperative solution. Thus, they are not concerned about cooperative behavior of players *per se*, rather they study non-cooperative behavior that gives rise to cooperative *outcomes*. Seminal works in this literature include Nash (1953) and Rubinstein (1982).

To give an example, in repeated games to sustain mutually beneficial outcomes as a subgame perfect equilibrium, players use credible threats, but this is done completely non-cooperatively—i.e., players choose their actions individually and independently at each stage. By contrast, in a cooperative strategic game credible threats are utilized by not only individuals but also group of players who strategically collaborate and coordinate their actions.

## 2.2 Cooperative approaches in non-cooperative games

A closely related strand of literature is the study of coalition-proofness in non-cooperative games such as strong Nash equilibrium by Aumann (1959), strong perfect equilibrium by Rubinstein (1980), and coalition-proof Nash equilibrium by Bernheim et al. (1987).[9] Roughly speaking, strong Nash equilibrium is a Nash equilibrium in which there is no coalitional deviation that can benefit all of its members. Coalition-proof Nash equilibrium is a weaker notion than strong Nash equilibrium, and it additionally requires that coalitional deviations—fixing the strategies of the other players—should be internally consistent in the sense that subcoalitions should not have incentives to further deviate. The

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[9]Note that Rubinstein's (1980) supergame notion refers to a repetition of a game, which differs from the supergame/supertree notion used in this paper (see section 3).

most obvious difference between these solutions and the cooperative equilibrium system solution is that they refine the set of Nash equilibria (to the extent that they do not always exists), whereas the cooperative equilibrium system is neither a refinement nor a coarsening of Nash equilibrium, but it always exists (see Theorem 2).

Of note, as Bernheim et al. (1987, p. 7) themselves point out, coalitional deviations in coalition-proof Nash equilibrium is restrictive in the sense that deviating subcoalitions do not consider forming a coalition with non-deviating players: "This rules out the possibility that some member of the deviating coalition might form a pact to deviate further with someone not included in this coalition. Such arrangements are clearly much more complex than those made entirely by members of the coalition itself... Clearly, further is required to resolve these issues in a fully satisfactory way." These complexities of strategic formation of coalitions are addressed in cooperative strategic games: Every coalition structure is considered and may potentially emerge as part of the cooperative equilibrium system solution in cooperative strategic games.

Finally, another major difference is that the concepts in the coalition-proofness literature predict some *strategy profiles* that are coalition-proof according to some notion; whereas, the cooperative equilibrium system is formally a family of collection of strategy profiles and coalitions in which the prediction includes a set of coalitions that is "stable."

## 2.3 Coalition formation, and farsighted (non-cooperative) approaches to cooperative games

Harsanyi's (1974) seminal work in cooperative games led to a recently a burgeoning body of literature that incorporates elements from non-cooperative games into cooperative games such as farsightedness and backward induction. This integration has greatly improved our understanding of both frameworks and their interrelations. There is a vast literature on the study of coalition formation and farsighted cooperative approaches in different contexts; see, e.g., Ichiishi (1981), Moulin and Peleg (1982), Greenberg (1990), Zhao (1992), Chwe (1994), Perry and Reny (1994), Moldovanu and Winter (1995), Bloch (1996), Ray and Vohra (1999), Maskin (2003), Brams, Jones, and Kilgour (2005), Bachrach et al. (2009), Jackson and Dutta (2001), Dutta, Ghosal, and Ray (2005), Herings, Predtetchinski, and Perea (2006), Acemoglu, Egorov, and Sonin (2008), Herings, Mauleon, and Vannetelbosch (2009), Jackson (2008), Jackson and Wolinsky (1996), Karos and Robles (2021), and Kimya (2020). For an informative and extensive review of the literature in economics and computer science, see, e.g., Ray (2007), Ray and Vohra (2015), Chalki-

adakis, Elkind, and Wooldridge (2011), Dafoe et al. (2020), Elkind, Rahwan, and Jennings (2013) and Aziz and Savani (2016), and the references therein.

Similar to Bernheim et al. (1987), Ray and Vohra (1997, p. 33) point out that they also study only "internally consistent" deviations: "We must state at the outset that our treatment is limited by the assumption that agreements can be written only between members of an existing coalition; once a coalition breaks away from a larger coalition it cannot forge an agreement with any member of its complement. Thus, deviations can only serve to make an existing coalition structure finer—never coarser. This is also the assumption in the definition of a coalition proof Nash equilibrium. It must be emphasized that an extension of these notions to the case of arbitrary blocking is far from trivial." Diamantoudi and Xue (2007) extend Ray and Vohra's (1997) equilibrium binding agreement notion by characterizing it with vNM stable sets in an appropriate cooperative game and then allow for any internal or external deviations. As is well known, vNM stable sets need not exists in general, and this also applies to the extended equilibrium binding agreement notion of Diamantoudi and Xue (2007). In addition, Ray and Vohra (1997), and Diamantoudi and Xue (2007) study a consistency notion that is not based on extensive form representation and that the contracts are written to be costless in their models. They also show that inefficiency issues are not resolved by coalition formation. Nor do cooperative strategic games resolve these issues. For more details and for an extension of these models, I refer the interested reader to "equilibrium process of coalition formation" in Ray and Vohra (2015), who provide an extensive survey of the recent literature.

Starting from a standard extensive form game with perfect information, Chander and Wooders (2020) construct a cooperative game and introduce the notion of subgame perfect core; as with the standard core, it may be empty. Their main finding is a contribution to the Nash program: If the subgame-perfect core of a perfect information game is non-empty, then it can be implemented as an SPNE payoff of a relevant non-cooperative game. As discussed with respect to Bernheim et al.'s (1987) and Ray and Vohra's (1997) concepts, Chander and Wooders's (2020) concept assumes that once a coalition forms, the remaining players stay as independent players and do not 'regroup'; though, Chander and Wooders do briefly discuss a special case in which the remaining players form a single coalition as discussed in Maskin (2003). In addition, Chander and Wooders discuss extending their analysis to more general extensive form games as well. They define the notion of subgame perfect strong Nash equilibrium in general extensive form games. A subgame perfect strong Nash equilibrium is a strong Nash equilibrium of every subgame in the game, which is akin to strong perfect equilibrium of Rubinstein (1980).

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Aumann and Myerson (1988) were first to introduce endogenous network formation games in a strategic setting. Given a cooperative game, they construct an auxiliary link formation game, which is a non-cooperative extensive form game with perfect information, in which pairs of players are offered to form links in a sequential order.[10] Subgame perfect equilibrium links in this non-cooperative extensive form game with perfect information give a "cooperation structure" à la Myerson (1977), and players receive their Myerson value in the cooperative game restricted to this cooperation structure. Because such link formation games with perfect information are a subset of non-cooperative extensive form games, they are also a subset of cooperative extensive form games. To see this, notice that any non-cooperative link formation game is non-cooperative in nature—i.e., players choose their actions independently. The labels of the actions—whether they are called "links" between two players, or they are named alphabetically—do not matter in terms of the predictions. An interesting direction for future research would be to extend non-cooperative link formation games to more general cooperative strategic games and explore whether the cooperative equilibrium system predictions coincide with cooperative solutions such as the Shapley value and the Myerson value. The network formation literature is not restricted to non-cooperative link formation games. There is a huge general network formation games literature, which is distinct from but closely related to the coalition formation literature.[11]

The current paper differs from the above literature in mainly two respects: the framework and the solution concept. First, various setups used in this literature are directly comparable to neither standard extensive form games nor cooperative strategic games. This is in part due to the 'cyclic' behavior in coalition formation frameworks and the fact that more than one player or coalition can choose an 'action' at a given decision node, which cannot occur under extensive form games.[12] Second, in cases in which a coalition formation setup such as Kimya's (2020) is comparable to an extensive form game with perfect information, the solution concept in question generally coincides with standard non-cooperative concepts such as the backward induction. This is not surprising because the main idea is to incorporate non-cooperative notions into cooperative games as first proposed by Harsanyi (1974).

While a cooperative equilibrium system is neither a refinement nor a generalization of a non-cooperative equilibrium concept, cooperative strategic games generalize extensive

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[10]For more details of this model, see e.g. van den Nouweland (2005)

[11]For more details, see the references provided above in this subsection; e.g., Jackson (2008).

[12]Note that this is not unusual because the frameworks in this literature emerged from cooperative games.

form games with possibly imperfect information. As introduced in section 3, cooperative equilibrium system is based on a unique procedure that combines backward induction with some elements of forward induction reasoning in which players rationally cooperate or act non-cooperatively.[13] I next compare and contrast cooperative equilibrium system solution concept with non-cooperative concepts.

## 2.4 The differences between cooperative strategic games and non-cooperative games

Cooperative strategic games differ from non-cooperative games in three main dimensions: (i) Philosophical/conceptual dimension, (ii) the solution concept, and (iii) the mathematical framework.

First, as Nash (1951, p. 286) points out: "Our theory, in contradistinction, is based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others." By contrast, cooperative strategic games assume that players can form coalitions to act together in an interactive strategic situation unless otherwise specified. If non-cooperation can be enforced among some of the players, then this would be part of the coalitional strategic form. Second, the cooperative equilibrium system concept incorporates backward induction reasoning as well as some elements of forward induction reasoning, and it builds on the equilibrium ideas of Cournot (1838), von Neumann (1928), von Neumann and Morgenstern (1944), Nash (1951), and Selten (1965). However, it is neither a refinement nor a coarsening of strategic equilibrium. This is not a surprise because it is well known that strategic equilibrium is in general not immune to coalitional deviations. Third, the framework of cooperative strategic games include extensive form games as a special case.

## 2.5 Computational aspects

Regarding the computation of a cooperative equilibrium system in $n$-person cooperative extensive form games, note that the number of subgames and supergames that must be solved rises exponentially as the number of players $n$ rises. This is because supergames, every subgame of supergames, and every supergame of subgames of supergames etc. must

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[13]Backward induction reasoning is based on the assumption that at any point in the game players make rational choices taking into account the future only, so they do not draw any conclusions from past choices. Forward induction reasoning generally assumes that past choices affect future behavior in a rational way. Unlike backward induction, forward induction does not have a unique definition in the literature. For more information, see, e.g., Perea (2012).

be solved to compute a cooperative equilibrium system of the full game (for details, see section 3). In the most general case in which every coalition is feasible and all utilities depend on the partition of players, it is clear that computing the cooperative equilibrium system is at least as "hard" as computing a non-cooperative subgame perfect Nash equilibrium. Even confirming whether a system is a cooperative equilibrium system is not "easy" because, unlike a Nash equilibrium, a system is a family of collection of strategy profiles and a family of collection of coalitions. (Note that a solution concept in non-cooperative games consists of only one strategy profile.) In the special case in which no coalitions (except the singletons) are feasible, standard hardness results in non-cooperative games apply; see e.g. Gilboa and Zemel (1989), Papadimitriou (1994), Daskalakis, Goldberg, and Papadimitriou (2009), and Etessami and Yannakakis (2010). This is because cooperative extensive form games reduce to standard non-cooperative games in that case.

# 3  The setup and the solution concept

## 3.1  The setup

Let $\Gamma = (P, X, I, u, S, H)$ denote a *cooperative extensive form game* with perfect recall and with possibly imperfect information, which is a standard extensive form game with an addition of coalitional utility function for each feasible coalition as explained below. $\Gamma$ will also be referred to as a cooperative strategic game or coalitional extensive form game.[14]

**Players:** Let $P$ be a finite set of players, which may be equal to $N = \{1, 2, ..., n\}$ or a partition of $N$. Each element of $P$ is called a *coalition* denoted by $C \in P$ or *player* denoted by $i \in P$. Each $i \in N$ has a finite set of pure actions $A_i$. With a slight abuse of notation singleton coalition $\{i\}$ is represented as $i$. The set of players may evolve throughout the game as defined next.

**Forming a coalition:** If, for some $k$, players $i_1, i_2, ..., i_k$ form a coalition $C = \{i_1, i_2, ..., i_k\}$, then each individual $i_j$ becomes an "agent" of coalition $C$, which we treat as player $C$.[15] The agents of player $C$ choose their strategy guided by player $C$'s utility function as defined next.

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[14]I refer to standard textbooks such as Fudenberg and Tirole (1991), whose notation I mostly adapt, for details about extensive form games.

[15]If a player does not want to join a coalition, then the coalition cannot form—i.e., any member can veto the formation of the coalition they belong.

*Remark* 1. Cooperative extensive form games generalize non-cooperative extensive form games in the sense that every non-cooperative extensive form game is a cooperative extensive form game but not vice versa. If no coalition (with two or more person) is feasible, then a cooperative strategic game would reduce to a non-cooperative extensive form game.

*Remark* 2. Any coalition may form to play any course of action, and players may break their coalitions even if they write a contract; though, doing so may be costly. But, as mentioned earlier, players are free to choose "irrational" actions. The framework does not put any specific restrictions on the behavior of players, but the solution concept does.

*Remark* 3 (Incomplete information games). Not to complicate the notation even further, I do not give separately the notation of (Bayesian) cooperative strategic games with incomplete information, which can be treated as cooperative strategic games with complete but imperfect information á la Harsanyi (1967), where nature chooses the types at the beginning of the game.

**Utility functions:** Let $X$ denote a game tree, $x \in X$ a node in the tree, $|X|$ the cardinality of $X$, $x_0$ the root of the game tree where Nature moves (if any), and $z \in Z$ a terminal node, which is a node that is not a predecessor of any other node. Let $u$ denote the profile of utility functions for each feasible coalition including each singleton player. Let $u_{C|P'} : Z \to \mathbb{R}$ denote the payoff function of coalition $C \subseteq N$ where for each terminal node $z$ and each (feasible) partition of players $P'$, $u_C(z|P')$ denotes the von Neumann-Morgenstern utility of player $C$, which may be singleton, if $z$ is reached, given the partition of players $P'$.

Every coalition $C$ for which $u_C$ is given is called *feasible*. If a coalition is not feasible, then there is no utility function for that coalition.[16] For $i \in P$, player $i$'s individual (von Neumann-Morgenstern) utility given a player partition $P'$ is denoted by $u_i(\cdot|P')$. Note that $u_i(\cdot|P)$ does not necessarily equal to $u_i(\cdot|P')$, where $P' \neq P$, because of the possibility of externalities (positive or negative) when a player joins a coalition. For example, a player $i$ may personally get a different utility from an outcome when $i$ forms a coalition with some player $j$ compared to the case in which $i$ joins a coalition with $j' \neq j$. (More generally, a coalition may also affect the utility of the players outside the coalition.) This distinction will be useful in determining which coalitions are "individually rational." When a player joins a coalition $C$, the strategic decisions are made based on the utility function $u_C$ of the coalition $C$, whereas player $i$ makes the decision to join a coalition based on the individual utility function $u_i$ given this coalition. Finally, the psychological

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[16]This is to save notation. Alternatively, a set of feasible coalitions may be additionally defined, or the individual utility from an infeasible coalition could be defined as minus infinity.

and monetary costs, if any, incurred due to coalition-forming can be incorporated into the individual utility functions.

*Remark* 4. While a coalitional utility function may be any vNM utility function in general, in some situations one could imagine that coalitional utility may be coming from some cooperative game.

**Strategies:** Let $I : X \to \mathbb{N}$ denote the player function, where $I(x)$ gives the "active" player who moves at node $x$, and $A(x)$ the set of pure actions at node $x$. Let $h \in H$ denote an information set, $h(x)$ the information set at node $x$ where there is possibly another node $x' \neq x$ such that $x' \in h(x)$, and the active player at $h(x)$ does not know whether she is at $x$ or at $x'$. If $h(x)$ is a singleton, then, with a slight abuse of notation, $h(x) = x$. Moreover, if $x \in h(x)$, then $A(x) = A(x')$. Let $A(h)$ denote the set of pure actions at $h$. Let $A_i = \bigcup_{h_i \in H_i} A(h_i)$ denote player $i$'s set of all pure actions where $H_i$ is player $i$'s set of all information sets. Let $S_i' = \bigtimes_{h_i \in H_i} A(h_i)$ denote the set of all pure strategies of $i$ where a pure strategy $s_i' \in S_i'$ is a function $s_i' : H_i \to A_i$ satisfying $s_i'(h_i) \in A(h_i)$ for all $h_i \in H_i$. Let $s' \in S'$ denote a pure strategy profile and $u_C(s')$ its von Neumann-Morgenstern (expected) utility for player $C$. Let $\Delta(A(h_i))$ denote the set of probability distributions over $A(h_i)$, $b_i \in \bigtimes_{h_i \in H_i} \Delta(A(h_i))$ a behavior strategy of $i$, and $b$ a profile of behavior strategies. Let $s \in S$ denote the mixed strategy profile that is equivalent to behavior strategy profile $b$ in the sense of Kuhn (1953). I assume that $\Gamma$ is common knowledge.

*Remark* 5. When two or more agents form a coalition, $C$, their informations sets merge. In incomplete information games, their types are also revelead to each other.

**Subgames and subtrees:** A *subgame* $G$ of a game $\Gamma$ is the game $\Gamma$ restricted to an information set with a singleton node and all of its successors in $\Gamma$. The largest subgame at a node $x$ is the subgame that is not a subgame of any other subgame (except itself) starting at $x$. At the root of a game, the largest subgame is the game itself. Let $G$ be a subgame and $H_i'$ be the set of $i$'s information sets in the subgame. Then, for every $h_i \in H_i'$ function $s_i(\,\cdot\,|h_i)$ denotes the restriction of strategy $s_i$ to $h_i$ in the subgame $G$. If all information sets are singletons, then $\Gamma$ is called a game of perfect information.

Let $\bar{x} = |X|$ denote the number of nodes in $X$, and $succ(x)$ and $Succ(x)$ be the set of immediate successors and all successors of a node $x$ (excluding $x$ in both definitions), respectively. Let $root(h)$ denote the node that is the root of the subgame containing information set $h$ such that there is no other subgame starting at an information set between $root(h)$ and $h$—i.e., $root(h)$ is the closest singleton "ancestor" of $h$. Note that in

perfect information games $x = root(x)$ for all nodes $x \in X$. $Succ(h)$ denotes the set of all successor information sets of information set $h$. A *subtree* $T$ of a game $\Gamma$ is the game tree of $\Gamma$ restricted to an information set (not necessarily singleton) and all of its successors in $\Gamma$. So the root of a subtree may be a non-singleton information set.[17] For an information set $h$, $T(h)$ denotes the largest subtree whose root is $h$. A subgame is a subtree whoose root is singleton.

**Supergames and supertrees:** Let $\Gamma(x)$ be the largest subgame starting at $x$ with $|N|$ players. For a coalition $C$, define $P_C = \{\{j\} \subseteq N | j \in N, j \notin C\} \cup C$—i.e., the partition of $N$ in which only the agents in $C$ form a coalition. For a player $i$, define $F_i = \{C' \in 2^N | i \in C'$ where $C'$ is feasible$\}$—i.e., the set of all feasible coalitions in $N$ that include $i$.

A *supergame* of $\Gamma(h)$, denoted by $\Gamma_{P_C}$, is the game $\Gamma$ in which the set of players $N$ is replaced with the set of players given by the partition $P_C$ where $C \in F_i$ and $i \in I(h)$. The set of all supergames is defined as follows:

$$super(\Gamma(h)) = \{\Gamma_{P_C} | C \in F_i, i \in I(h)\}.$$

Note that if the original game is an $n$-player game, then its supergame $\Gamma_{P_C}$ is an $(n - |C| + 1)$-player game. Note that each $j \in C$ acts as an agent of a single player $C$ in $\Gamma_{P_C}$. For example, if $\Gamma$ is a six-player game where $N = \{1, 2, 3, 4, 5, 6\}$, then $\Gamma_{\{1,2,4\}}$ is the game in which the players are $\{1, 2, 4\}, 3, 5$, and $6$. $1, 2$, and $4$ are the agents of coalitional player $\{1, 2, 4\}$.

Let $\Gamma(h)$ be the largest subgame starting at $root(h)$ and $T(h)$ be a subtree of game $\Gamma(h)$. A *supertree* of $T(h)$, denoted by $T_{P_C}$, is the subtree of $\Gamma(h)$ in which the set of players $N$ is replaced with the set of players given by the partition $P_C$ where $C \in F_i$ and $i \in I(h)$. The set of all supertrees is defined as follows:

$$super(T(h)) = \{\Gamma_{P_C} | C \in F_i, i \in I(h)\}.$$

**Systems:** A *system* or a "solution profile" is a pair $(\sigma, \pi)$ which is defined as follows. Let $G(h)$ be the set of all subgames of the largest subgame starting at $root(h)$ for some information set $h$. For example, if $h$ is the root of game $\Gamma$, then $G(h)$ is the set of all subgames of $\Gamma$, including itself. For a given $h$, let $\sigma(h, g)$ denote a strategy profile in some subgame $g \in G(h)$. Note that $g$ depends on $h$. For simplicity, subscripted notation $g_h$

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[17]Note that the root of an information set $h$ is different than the root of a subtree $T(h)$.

will not be used. Note that function $\sigma(h, \cdot)$ gives for each information set a collection of (possibly different) strategy profiles, one for each successor subgame: $\sigma(h, \cdot) := (s_g)_{g \in G(h)}$. More compactly, $\sigma := (s_{h,g})_{h \in H, g \in G(h)}$. In plain words, $\sigma$ is a family of collection of strategy profiles.

*Remark* 6. Suppose that $g'$ is a subgame of $g$. In general, $\sigma(h, g|g') \neq \sigma(h, g')$. In other words, $\sigma(h, g')$ is not necessarily equal to the restriction of the strategy profile $\sigma(h, g)$ to the subgame starting at node $g'$. This is because a different coalition may form in subgame $g'$ than in subgame $g$, so the strategy profile chosen in $g'$ can be different than the restriction of the strategy profile chosen in $g$.

Similarly, given some $h$ let $\pi(h, g)$ denote a (feasible) partition of players in subgame $g \in G(h)$. Let $\pi(h, \cdot) := (\pi_g)_{g \in G(h)}$ and $\pi := (\pi_{h,g})_{h \in H, g \in G(h)}$. Function $\pi(h, \cdot)$ gives for each information set a collection of (possibly different) partition of players, one for each successor subgame. In plain words, $\pi$ is a family of collection of partition of players.

The interpretation of a pair $(\sigma(h, g), \pi(h, g))$ is as follows. Let $C$ be some coalition in $\pi(h, g)$. Each agent $i$ of player $C$ chooses their actions in $g$ based on the strategy profile $\sigma(h, g)$. Their actions are guided by the coalitional utility function $u_C(\cdot | \pi(h, g))$ in the subgame $g$. In summary, a system is a complete description of what actions will be played by which coalitions at every subtree and every supertree, and every subtree of every supertree and so on.

**Non-cooperative solution concepts:** Let $\Gamma = (P, X, I, u, S, H)$ be a cooperative strategic game. A mixed strategy profile $s$ is called a Nash equilibrium if for every player $j \in P$, $s_j \in \arg\max_{s'_j \in S'_j} u_j(s'_j, s_{-j})$. A mixed strategy profile $s$ is a called a subgame-perfect Nash equilibrium (SPNE) if for every subgame $G$ of $\Gamma$ the restriction of $s$ to subgame $G$ is a Nash equilibrium in $G$.

## 3.2 The Recursive Backward Induction algorithm: Perfect information games

Let $\Gamma$ be a cooperative strategic game with perfect information without a chance move. I define *Recursive Backward Induction* (RBI) algorithm by a recursive induction procedure. RBI algorithm outputs a system on each subgame $G(x)$ starting at some non-terminal node $x$ by inducting on the number of players ($n$) and number of nodes ($\bar{x}$) in the successor nodes of $x$ including $x$ itself (and excluding the terminal nodes). For example, if $x$ is a penultimate node, then $(n, \bar{x}) = (1, 1)$. Let $i = I(x)$ be the active player at $x$. The

solution of the game in which Nature moves at the root of the game is simply the profile of the solutions of the subgames starting at every immediate successor of the root.

1. **Base case**: Let $(n, \bar{x}) = (1, 1)$. Equilibrium system at $G(x)$ is defined by the pair $(\sigma^*, \pi^*)$ such that $\sigma^*(x) \in \arg\max_{s \in S} u_i(s)$ and $\pi^* = \{i\}$.

2. **Induction step**: Assume that cooperative equilibrium system is defined for all subgames with parameters $(m, \bar{y})$ satisfying $1 \leq m \leq n$, $1 \leq \bar{y} \leq \bar{x}$ such that $(n, \bar{x}) \neq (1, 1)$ and $(m, \bar{y}) \neq (n, \bar{x})$.

   By assumption, $[\sigma^*(x'), \pi^*(x')]$ is defined for all $x' \in Succ(x)$ where $x$ is the root of subgame $G(x)$. The solution is extended to subgame $G(x)$ with parameters $(n, \bar{x})$ as follows. I first define "reference points" to compare the solutions of all supergames with the non-cooperative choice of $i$ at $x$.

   (a) The "autarky" reference point at $x$, $r_0(x)$: Let $b_i^*(x)$ be a utility-maximizing behavior strategy of $i$ at $x$ given $[\sigma^*(x'), \pi^*(x')]$:

   $$b_i^*(x) \in \arg\max_{b_i'(x) \in \Delta A_i(x)} u_i(b_i'(x) | \sigma^*(x'), \pi^*(x')).$$

   Let $r_0(x) := [\tau^{\{i\}}(x), \pi^{\{i\}}(x)]$ which is defined as the extension of $[\sigma^*(x'), \pi^*(x')]$ with $x' \in succ(x)$ to the largest subgame starting at node $x$ where $i$ chooses the behavior strategy $b_i^*(x)$, and $\pi^{\{i\}}(x)$ is defined as the partition in which $i$ and other players in the subgame at $x$ act non-cooperatively.[18]

   (b) Reference points: For any $(n, \bar{x})$ consider supergame $\Gamma_{P_C}$ where $C \ni i$ is a feasible non-singleton coalition. Because $\Gamma_{P_C}$ is an $(n - |C| + 1)$-person subgame with $\bar{x}$ nodes, its cooperative equilibrium system, $[\tau^C(x), \pi^C(x)]$, is defined by assumption.

   Let $(r_j(x))_{j=1}^{\bar{j}}$ be a sequence for $i$ where $r_j(x) := [\tau^{C_j}(x), \pi^{C_j}(x)]$ satisfying (i) for every two indices $j$ and $j'$ with $j < j'$, we have $u_i(\tau^{C_j}(x) | \pi^{C_j}(x)) \leq u_i(\tau^{C_{j'}}(x) | \pi^{C_{j'}}(x))$ (i.e., nondecreasing sequence for $i$), and (ii) if $u_i(\tau^{C_j}(x) | \pi^{C_j}(x)) = u_i(\tau^{C_{j'}}(x) | \pi^{C_{j'}}(x))$ and $C_j \subsetneq C_{j'}$, then $r_j$ precedes $r_{j'}$ for supergame coalitions $C_j$ and $C_{j'}$ containing $i$.[19]

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[18]Put differently, $\tau^{\{i\}}(x, x)$ is defined as the strategy profile in which $i$ chooses behavior action $b_i^*(x)$, and $\pi^{\{i\}}(x)$ is defined as the partition in which $i$ and other players in the subgame at $x$ act non-cooperatively. And, the rest follows the strategy profile and partitions given in system $[\sigma^*(x'), \pi^*(x')]$ for all successors $x' \in Succ(x)$. Note that $\tau^{\{i\}}(x) = \tau^{\{i\}}(x, x)$ for all $x' \in Succ(x)$—i.e., it gives a strategy profile for every subgame.

[19]Ties are broken arbitrarily.

(c) Individually rational reference points: Given the autarky reference point $r_0^*(x)$, the following *individually rational* (IR) reference points are defined inductively as follows.

Assume that $r_j(x)$ is IR, denoted as $r_j^*(x)$, for some $j \geq 0$. Let $j' > j$ be the smallest number such that $r_{j'}(x)$ is IR with respect to $r_j^*(x)$—i.e., $u_{i'}(\tau^{C_{j'}}(x)|\pi^{C_{j'}}(x)) > u_{i'}(\tau^{C_j}(x)|\pi^{C_j}(x))$ for every agent $i' \in \bar{C}_{j'}$ where $C_{j'} \subseteq \bar{C}_{j'}$ and $\bar{C}_{j'} \in \pi^{C_{j'}}(x)$.[20]

Let $r_{\bar{l}}^*(x)$ be the greatest IR reference point that maximizes $i$'s utility, and call it *supergame perfect* at $x$. Note that $r_{\bar{l}}^*(x) = [\tau^{C_{\bar{l}}}(x), \pi^{C_{\bar{l}}}(x)]$. It may be that $r_0^*(x)$ where $\bar{l} = 0$ is the only IR reference point.

Equilibrium system of $\Gamma(x)$ is defined as system $[\sigma^*, \pi^*]$ where $[\sigma^*(x), \pi^*(x)] = r_{\bar{l}}^*(x)$, which extends $[\sigma^*(x'), \pi^*(x')]$ where $x' \in Succ(x)$. When $x$ is the root of the game tree of $\Gamma$, $[\sigma^*, \pi^*]$ is said to be cooperative equilibrium system of $\Gamma$.

To summarize, a cooperative equilibrium system is the one that is both subgame and supergame "perfect" at every subgame and supergame. Notice that the RBI procedure incorporates backward induction reasoning as well as some elements of forward induction reasoning in the sense that the coalitions formed in the 'past' rationally affect the behavior of players in the future. The next theorem shows that a cooperative equilibrium system always exists.

**Theorem 1** (Existence). *There exists a cooperative equilibrium system in pure strategies in every finite n-person cooperative strategic game with perfect information.*

The proof of the theorem is given in the Appendix, section 6.1.

Theorem 1 shows that in finite cooperative strategic games with perfect information, there is always a cooperative equilibrium system where all the strategies that make up the solution are pure strategies. I call coalitions *stable* if they survive the RBI and players and agents *dynamically rational* if they utilize the RBI in cooperative strategic games.

## 3.3   Imperfect information games

Let $\Gamma$ be an extensive form game with imperfect information. I next define Recursive Induction (RBI) algorithm which outputs a system on each subtree $T(h)$ starting at some

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[20]In other words, $\bar{C}_{j'}$ contains coalition $C_{j'}$ such that $\bar{C}_{j'}$ is part of the solution of the supergame $\Gamma_{P_{C_{j'}}}$.

information set $h$ by inducting on the number of players ($n$) and number of information sets ($\bar{h}$) in the successor information sets of $h$ including $h$ itself (and excluding the terminal nodes). For example, if $h$ is a penultimate information set, then $(n, \bar{h}) = (1, 1)$. Let $i = I(h)$ be the active player at information set $h$.

1. **Base case**: Let $(n, \bar{h}) = (1, 1)$ and $g_{P'}$ be the largest subgame at $root(h)$ where $P'$ is the set of players in the subgame.[21] Equilibrium system at $T(h)$ is defined by the pair $(\sigma^*(h), \pi^*(h)) := (\sigma^*, \pi^*)$ such that $\sigma^*(h)$ is a subgame-perfect equilibrium in $g_{P'}$ and $\pi^* = P'$ in which no player forms a coalition.[22]

2. **Induction step**:

   Assume that cooperative equilibrium system is defined for all subtrees with parameters $(m, \bar{y})$ satisfying $1 \leq m \leq n$, $1 \leq \bar{y} \leq \bar{h}$ such that $(n, \bar{h}) \neq (1, 1)$ and $(m, \bar{y}) \neq (n, \bar{h})$. By assumption, $[\sigma^*(h'), \pi^*(h')]$ is defined for all $h' \in Succ(h)$ where $h$ is the root of subtree $T(h)$. The solution is extended to subtree $T(h)$ with parameters $(n, \bar{h})$ as follows. I first define reference points to iteratively compare the solutions of all supertrees with the non-cooperative choice of $i$ at $h$.

   (a) The autarky reference point at $h$, $r_0(h)$: Let $r_0(h) := [\tau^{\{i\}}(h), \pi^{\{i\}}(h)]$ which is defined as the extension of $[\sigma^*(h'), \pi^*(h')]$ with $h' \in succ(h)$ to the largest subgame starting at node $root(h)$ such that $[\tau^{\{i\}}(h), \pi^{\{i\}}(h)]$ is an SPNE in this subgame where all players acting between $root(h)$ and $h$ inclusive choose their strategies non-cooperatively.[23]

   (b) Reference points: For any $(n, \bar{h})$ consider supertree $T_{P_C}(h)$ where $C \ni i$ is a feasible non-singleton coalition. Because $T_{P_C}(h)$ is an $(n - |C| + 1)$-person subtree with $\bar{h}$ information sets, its cooperative equilibrium system, $[\tau^C(h), \pi^C(h)]$, is defined by assumption.

   Let $(r_j(h))_{j=1}^{\bar{j}}$ be a sequence for $i$ where $r_j(h) := [\tau^{C_j}(h), \pi^{C_j}(h)]$ satisfying (i) for every two indices $j$ and $j'$ with $j < j'$, we have $u_i(\tau^{C_j}(h)|\pi^{C_j}(h)) \leq u_i(\tau^{C_{j'}}(h)|\pi^{C_{j'}}(h))$ (i.e., nondecreasing sequence for $i$), and (ii) if $u_i(\tau^{C_j}(h)|\pi^{C_j}(h)) = u_i(\tau^{C_{j'}}(h)|\pi^{C_{j'}}(h))$ and $C_j \subsetneq C_{j'}$, then $r_j$ precedes $r_{j'}$ for supertree coalitions $C_j$ and $C_{j'}$ containing $i$.[24]

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[21]Note that $g_{P'}$ is the only element in $G(h)$.

[22]Note that $\sigma^*(h)$ is a strategy profile in $g_{P'}$.

[23]Note that SPNE is defined with respect to players, which may be coalitions, which are given by partition $\pi^*(h')$.

[24]Ties are broken arbitrarily.

(c) IR reference points: The autarky reference point $r_0(h)$ is individually rational by definition. The following IR reference points are defined inductively as follows.

Assume that $r_j(h)$ is IR, denoted as $r_j^*(h)$, for some $j \geq 0$. Let $j' > j$ be the smallest number such that $r_{j'}(h)$ is IR with respect to $r_j^*(h)$—i.e., $u_{i'}(\tau^{C_{j'}}(h)|\pi^{C_{j'}}(h)) > u_{i'}(\tau^{C_j}(h)|\pi^{C_j}(h))$ for every $i' \in \bar{C}_{j'}$ where $C_{j'} \subseteq \bar{C}_{j'}$ and $\bar{C}_{j'} \in \pi^{C_{j'}}(h)$.

Let $r_{\bar{l}}^*(h)$ be the greatest IR reference point, which maximizes $i$'s utility. Note that $r_{\bar{l}}^*(h) = [\tau^{C_{\bar{l}}}(h), \pi^{C_{\bar{l}}}(h)]$. It may be that $r_0^*(h)$ where $\bar{l} = 0$ is the only IR reference point.

Equilibrium system of $T(h)$ is defined as $[\sigma^*, \pi^*]$ where $[\sigma^*(h), \pi^*(h)] = r_{\bar{l}}^*(h)$, which extends $[\sigma^*(h'), \pi^*(h')]$ where $h' \in Succ(h)$. When $h$ is the root of the game tree of $\Gamma$, $[\sigma^*, \pi^*]$ is said to be cooperative equilibrium system of $\Gamma$.

In other words, a cooperative equilibrium system is the one that is both subtree and supertree "perfect" for every subtree and supertree. If the cooperative extensive form structure of a normal form game is not given, then a cooperative equilibrium system of the game is defined as a cooperative equilibrium system of a cooperative extensive form game whose reduced normal form is the given normal form game.[25] As in perfect information games, coalitions are called *stable* if they survive the RBI and players and agents are called *dynamically rational* if they utilize the RBI in cooperative strategic games. Next theorem states the existence of the cooperative equilibrium system in imperfect information games.

**Theorem 2** (Existence). *There exists a cooperative equilibrium system in possibly mixed strategies in every finite n-person cooperative strategic game.*

The proof of the theorem is given in the Appendix (see section 6.2).

Next, I discuss some of the possibilities of modifying the framework of cooperative strategic games, and how this might affect the definition of the cooperative equilibrium system.

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[25]In incomplete information games, subgame perfection in the RBI algorithm is changed to perfect Bayesian equilibrium. Cooperative equilibrium system can be extended from finite to infinite horizon games in an analogous way to the extension of Backward Induction idea to infinite horizon games via subgame perfect equilibrium.

## 3.4 Modifications

**Partial, fixed, and mixed cooperation**: Suppose that supergame of a game consists not only of the games in which two or more players form a coalition but also the games in which coalitions are subgame-dependent. For example, at an information set $h$, $i$ cooperates with $j$ but at information set $h'$ $i$ cooperates with $j'$, in which case the active players at $h$ and $h'$ would be defined as $\{i, j\}$ and $\{i, j'\}$, respectively. Since utilities for both $\{i, j\}$ and $\{i, j'\}$ are well defined, the RBI algorithm would be well defined as well—i.e., a cooperative equilibrium system would be the one that is both subtree and supertree perfect according to this more general supertree definition. In "fixed cooperation" variation, it is possible to incorporate into cooperative strategic game structure the option that two (or more) players, say $i$ and $j$, act together at an information set $h$, while acting separately at other nodes. In this case, supertree coalitions at $h$ must include both $i$ and $j$ and the IR reference points at $h$ must be calculated with respect to $u_{\{i,j\}}$. In another variation, it may be that player $i$ at an information set $h$ "mixes" between two or more supertrees, e.g., between coalitions $\{i, j\}$ and $\{i, j'\}$, to achieve a better individually rational reference point at $h$. Any of these variations can be incorporated into the cooperative strategic games and, mutatis mutandis, the cooperative equilibrium system would be well-defined and would exist (i.e., Theorem 2 applies).

**Refinements, coarsenings, and other solutions**: The cooperative equilibrium system does not include threats that are non-credible, so it is possible to generalize this solution by allowing non-credible threats in the spirit of Nash equilibrium versus SPNE. Moreover, a generalized version of cooperative equilibrium system can be defined via the RBI, in which a refinement or a coarsening of SPNE is used in the RBI process such as sequential equilibrium by Kreps and Wilson (1982). Define *cooperative system* as a system in which every individual player, every coalitional player, and every agent in a coalition do their best compared to the respective counterfactuals; though, these counterfactuals need not be credible—i.e., they do not necessarily satisfy the steps described in the RBI. It is clear that every cooperative equilibrium system is a cooperative system, but not vice versa.

Another variation could be as follows. Under the current setting, agents form coalitions provided that they have strict incentives to do so, weakening this assumption is certainly reasonable, though this would also potentially increase the set of solutions. One can also consider concepts such as maximin strategy and maximin equilibrium (optimin criterion) (Ismail, 2014) in the framework of cooperative strategic games. Maximin or "optimin" cooperative system would be a system in which players simulatenously maximize their

minimum utility under profitable deviations.

# 4 A fully worked out example

Figure 3 (A) illustrates a three-player cooperative strategic game in which Player 1 (P1) starts by choosing $L$ or $R$. For simplicity, I make the following assumptions. First, coalition forming is not costly; though, as mentioned earlier even if it is costly, the analysis would remain valid as long as the associated costs are not 'too high'. Second, I assume that $u_C(\cdot) := \min_{i \in C} u_i(\cdot)$ for every non-empty $C \subseteq \{1, 2, 3\}$. That is, one outcome is preferred to the other if the minimum utility a member of the coalition receives from the former outcome is greater than the minimum utility a member receives from the latter. For example, $u_C(R, d, l) > u_C(R, d, k)$ for the coalition of $C = \{1, 3\}$, because $u_C(R, d, l) = 5$ and $u_C(R, d, k) = 1$. (Note that in general a coalitional utility can be any von Neumann-Morgenstern utility function.)

I next solve this game's cooperative equilibrium system following the main steps in the RBI algorithm. (For a more detailed step-by-step solution, see Figure 6.) The best-response of P3 at the penultimate nodes $x_1, x_2, x_3$ and $x_4$ are shown in Figure 3 (B) by the lines with arrows, which corresponds to Step 1 in the RBI.

Consider the left subgame starting at node $x_5$ in which it is P2 to make a choice. The non-cooperative best response of P2 at this decision node is to choose $a$, which leads to outcome (5, 5, 3) given P3's choice of $e$. Thus, (5, 5, 3) is the outcome of the autarky reference point $r_0$ at that node (step 2.a in the RBI). The only supergame at this node is the one in which $\{2, 3\}$ forms a coalition, which is illustrated in Figure 4 (C). Note that $\{2, 3\}$'s utility is maximized at outcome (1, 6, 4), which is the outcome of the cooperative equilibrium system in this supergame (Step 2.b in the RBI). This supergame is individually rational with respect to the autarky reference point because both P2 and P3 prefer (1, 6, 4) to (5, 5, 3) (Step 2.c in the RBI).[26] This is also the greatest IR reference point because there is no other supergame. Figure 4 (C) illustrates (1, 6, 4) as the outcome of the RBI in this step.

Now, consider the right subgame starting at node $x_6$ in which player 2 acts after P1's choice of R. The non-cooperative best response of P2 at this decision node is to choose $c$. Thus, the autarky reference point outcome in this node is (2, 2, 6) as is illustrated in Figure 4 (D). In the supergame in which $\{2, 3\}$ form a coalition, $u_{\{2,3\}}$ is maximized

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[26]Note that (4, 4, 5) also maximizes $\{2, 3\}$'s utility but it is not individually rational for P2 with respect to (5, 5, 3).
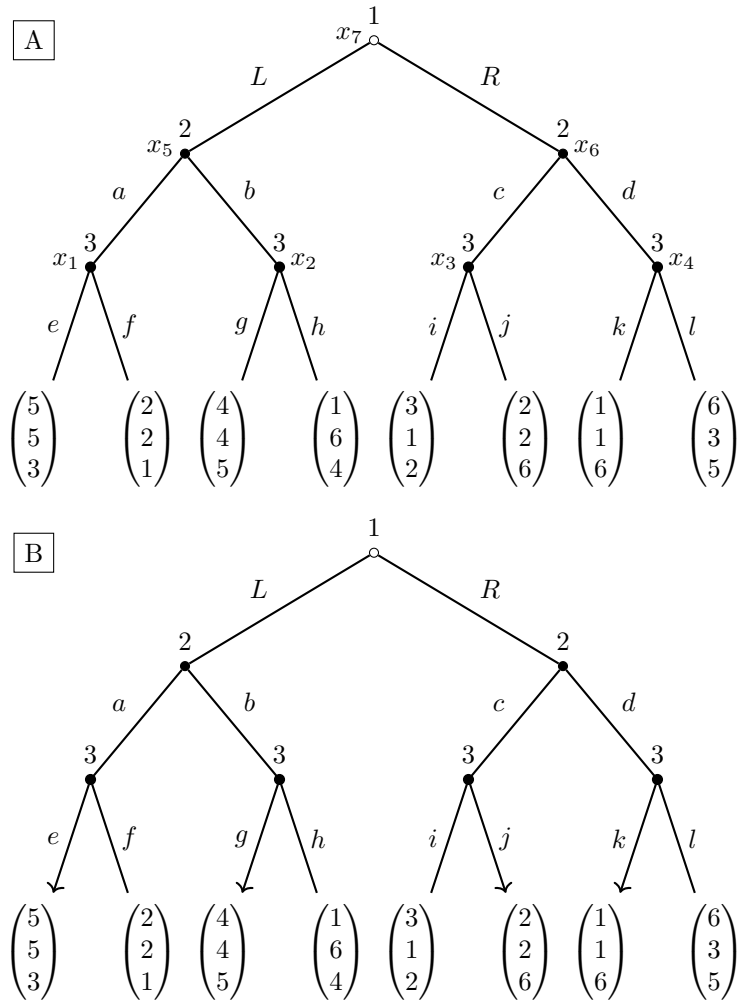
Figure 3: A: A three-player cooperative strategic game. B: The best-response of P3 at the penultimate nodes are shown by the lines with arrows.
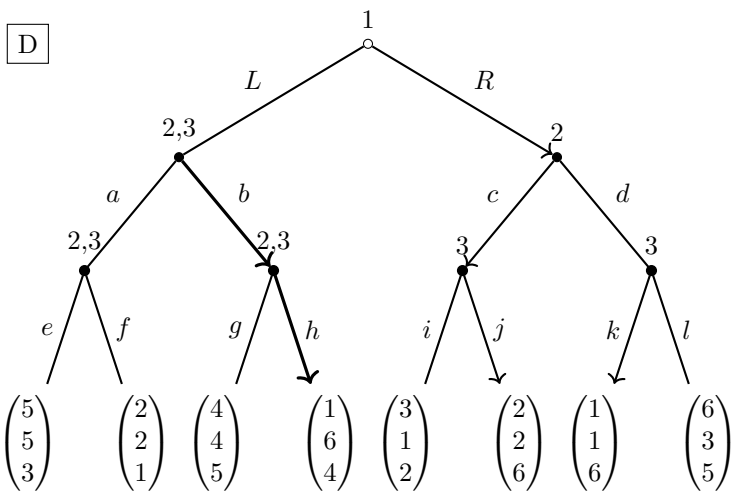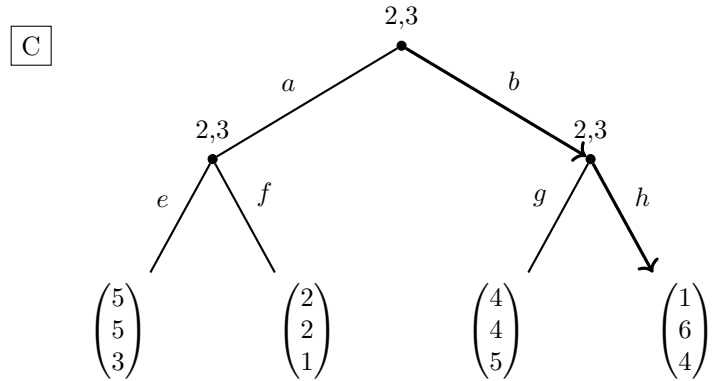
Figure 4: Equilibrium system solution steps (C) and (D). The lines with arrows represent best responses, which could be non-cooperative or coalitional. C: The first coalition forms: Looking forward, P2 forms a coalition with P3 to play $b$ and $h$. This overrides the backward induction outcome, so the IR reference point is $(1, 6, 4)$. D: P1 anticipates the coalition of P2 and P3 and hence chooses to $R$, because P1 prefers $(2, 2, 6)$ to $(1, 6, 4)$. The autarky reference point $r_0$ outcome at the root is $(2, 2, 6)$.

at (6, 3, 5), but this is clearly not individually rational for P3 with respect to (2, 2, 6). Therefore, the greatest IR reference point at this subgame is the autarky reference point.

Finally, consider the root of the game at $x_7$ where P1 makes a choice. Figure 4 (D) illustrates P1's non-cooperative best response of R at this node, given the solutions found in the previous steps of the RBI. Thus, the autarky reference point outcome at the root is (2, 2, 6). To find the other reference points, I analyze the supergames at node $x_7$ which are given by the coalitions $\{1, 2\}, \{1, 3\}$ and $\{1, 2, 3\}$.

First, in supergame $\Gamma_{\{1,2\}}$, the cooperative equilibrium system outcome is (5, 5, 3), which is illustrated in Figure 4 (E). This is because coalition $\{1, 2\}$ maximizes its coalitional utility by choosing $L$ and $a$, given the best responses of P3. Second, Figure 4 (F) illustrates the cooperative equilibrium system outcome (6, 3, 5) in supergame $\Gamma_{\{1,3\}}$. Notice that coalition $\{1, 3\}$ maximizes its coalitional utility by choosing $R$ and $l$, given the best responses of P2. Third, the supergame of the grand coalition $\{1, 2, 3\}$ maximizes its utility at (4, 4, 5) by choosing $L, b$ and $g$. Thus, the cooperative equilibrium system outcome of this supergame is (4, 4, 5).

Next, by Step 2.b in the RBI, the reference point outcomes that come from supergames $\{1, 2, 3\}, \{1, 2\}$, and $\{1, 3\}$ are (4, 4, 5), (5, 5, 3), and (6, 3, 5), respectively. Notice that these outcomes are ordered as an increasing sequence with respect to P1's payoffs. The following step is to the find individually rational reference points. By Step 2.c, the autarky reference point is IR by definition, and its outcome is (2, 2, 6). Given the autarky reference point, the reference point of the supergame $\Gamma_{\{1,2,3\}}$ is *not* IR because P3 prefers (2, 2, 6) to (4, 4, 5). However, the reference point of supergame $\Gamma_{\{1,2\}}$ is IR because both P1 and P2 prefer (5, 5, 3) to (2, 2, 6). Given this IR reference point, the reference point of supergame $\Gamma_{\{1,3\}}$ is also IR because both P1 and P3 prefer (6, 3, 5) to (5, 5, 3). As a result, (6, 3, 5) is the greatest IR reference point outcome for P1 at the root of the game.

A complete cooperative equilibrium system solution of the game in Figure 3 (A) is presented in Figure 6. But a non-complete ("on-path") description of this game can be summarized by a list of players, stable coalitions, and their strategies:

$$[\{R\}, \{a, d\}, \{e, g, j, l\}; \{1, 3\}, 2],$$

in which P1 and P3 form a coalition (so each of them becomes an agent of player $\{1, 3\}$), agent P1 chooses $R$, P2 chooses $a$ and $d$, and agent P3 chooses $e$, $g$, $j$, and $l$, from left to right. The outcome of this solution is (6, 3, 5). It is notable that during the solution process we have seen that all coalitions—$\{1, 2\}$, $\{2, 3\}$, and $\{1, 3\}$—except the grand coalition were IR at some point, though the only stable coalition turned out to be
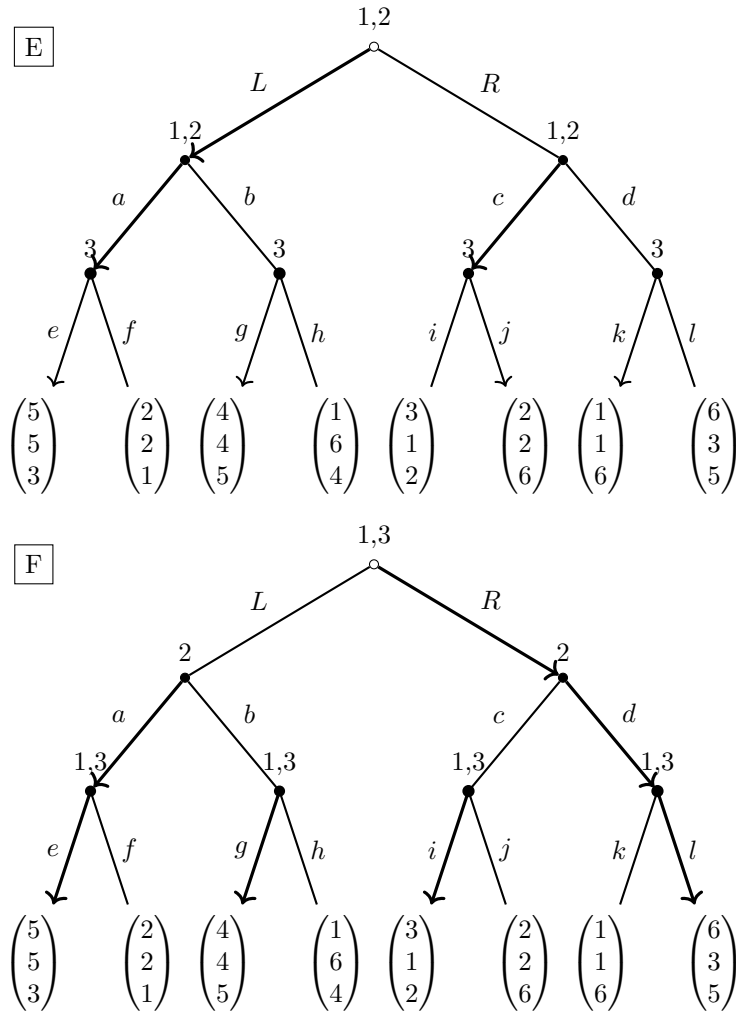
Figure 5: E: The coalition between P2 and P3 "breaks down," and a new coalition forms: P1 and P2 cooperate to play $L$ and $a$ to receive 5 each, which is better than 2 each at the previous reference point. The new reference point is, once again, (5, 5, 3). F: Another coalition, namely $\{1, 2\}$, breaks down. P1's forming a coalition with P2 is a credible threat to P3. Thus, P3 forms a coalition with P1: Agent P1 plays $R$ and agent P3 plays $l$ in response to P2's $d$. As a result, (6, 3, 5) is the cooperative equilibrium system outcome of this game because this is the outcome of the greatest IR reference point at the root of the game.

the one between P1 and P3.[27]

At the outset, it might be tempting to conclude—without running the RBI algorithm—that P1 and P3 will obviously form a coalition to obtain (6, 3, 5). However, this conclusion would be false. To give an example, consider the game in Figure 3 (A) in which, all else being equal, outcome (3, 1, 2) is replaced with outcome (3, 4, 3). This change seems to be irrelevant because P1 and P3 can still form a coalition to obtain (6, 3, 5). However, the outcome of the new game based on the same RBI procedure would be (5, 5, 3), which is significantly different than the previous outcome—why is this? This is because P2 has now a credible threat against the coalition of P1 and P3. Notice that if P1 plays $R$, then P2 would respond by $c$, knowing that the coalition would choose $i$ that leads to the more egalitarian outcome, (3, 4, 3).[28] Because the IR reference point in Figure 5 (E) is (5, 5, 3), it would not be individually rational for P1 to collaborate with P3, given P2's credible threat of choosing $c$ instead of $d$. As a result, "on-path" cooperative equilibrium system of this game can be summarized as

$$[\{L\}, \{a, c\}, \{e, g, j, k\}; \{1, 2\}, 3],$$

in which P1 and P2 form a coalition, agent P1 chooses $L$, agent P2 chooses $a$ and $c$, and P3 chooses $e$, $g$, $j$, and $k$, from left to right. The outcome of this solution is (5, 5, 3). In this three-person game, a credible threat by P2 prevents P3 destabilizing the coalition of P1 and P2—i.e., the supergame coalition of $\{1, 3\}$ is not individually rational with respect to the supergame coalition of $\{1, 2\}$.

# 5 Applications

## 5.1 Oligopolistic competition with mergers

Let $\Theta(\mathcal{P}, c, r)$ denote a coalitional oligopolistic competition with the possibility of forming coalitions such as mergers and cartels, which is defined as a cooperative strategic game $\Gamma = (P, X, I, u, \Sigma, H)$ with the following parameters and interpretation. Each player $i \in P$ is called a firm. $\mathcal{P}$ denotes the inverse demand function, $c$ the cost function, and $r$ the sharing rule in case of a coalition among two or more firms. Note that this is a very general model of (dynamic) oligopolistic competition that includes the standard $n$-player

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[27]A natural interpretation could be that the RBI process first occurs in the minds of the players, and then they form coalitions.

[28]In this example, for simplicity, I assumed that payoff transfers are not possible, but even if they were, one could construct a similar example.

1. Node $x_7$: The solution at the root of the game:

   (a) Node $x_7$:
   $$[\{R\}, \{a, d\}, \{e, g, j, l\}; \{1, 3\}, 2].$$

   (b) Node $x_6$ (the subgame after action $R$):

      i. Node $x_6$:
      $$[\{d\}, \{i, l\}; 2, \{1, 3\}].$$

      ii. Nodes $x_3$ to $x_4$:
      $$[\{i, l\}; \{1, 3\}].$$

   (c) Node $x_5$ (the subgame after action $L$):

      i. Node $x_5$:
      $$[\{a\}, \{e, g\}; 2, \{1, 3\}].$$

      ii. Nodes $x_1$ to $x_2$:
      $$[\{e, g\}; \{1, 3\}].$$

   (d) Nodes $x_1$ to $x_4$ (the penultimate nodes):

   $$[\{e, g, i, l\}; \{1, 3\}].$$

2. Node $x_6$: The solution at the subgame after action $R$:

   (a) Node $x_6$:
   $$[\{c\}, \{j, k\}; 2, 3].$$

   (b) Nodes $x_3$ to $x_4$:
   $$[\{j, k\}; 3].$$

3. Node $x_5$: The solution at the subgame after action $L$:

   (a) Node $x_5$ (the subgame after action $L$):

   $$[\{b\}, \{h\}; \{2, 3\}].$$

   (b) Nodes $x_1$ to $x_2$:
   $$[\{e, g\}; \{3\}].$$

4. Nodes $x_1$ to $x_4$: The solution at the penultimate nodes:

   (a) Nodes $x_1$ to $x_4$:
   $$[\{e, g, j, k\}; 3].$$

Figure 6: A cooperative equilibrium system of the game in Figure 3 (A).

Cournot, Bertrand, and Stackelberg competitions. The following corollary illustrates that a cooperative equilibrium system exists under "standard" assumptions in the models of oligopolistic competition.[29]

**Corollary 1.** *Under standard assumptions on the inverse demand, cost, and profit (including coalitional profit) functions, every coalitional oligopolistic competition possesses a cooperative equilibrium system.*

This corollary directly follows from Theorem 2.

Note that there are many ways of modelling $n$-firm strategic interaction via cooperative strategic games based on how payoffs are defined. Next, I provide a simple example of Stackelberg competition in which there are a leader and 2 followers who observe the quantity choice of the leader and then choose their quantities simultaneously. The purpose of the example is to illustrate a cooperative equilibrium system in a simple but non-trivial coalitional oligopolistic competition.

For ease of comparison with the Cournot competition and the standard Stackelberg competition, assume that the cost functions are symmetric. In this Stackelberg model, it is clear that the grand coalition (i.e., monopoly) payoff is greater than the total payoff of the firms when they act independently. Thus, there are incentives for the firms to form the grand coalition.

Formally, let player (firm) 1 be the leader and player $j \in \{2, 3\}$ be a follower. Player 1 chooses a quantity $q_1 \geq 1$, and after observing $q_1$ each player $j$ simultaneously chooses a quantity $q_j \geq 0$. Let $Q = \sum_{i=1}^{3} q_i$ be the total quantity. The inverse demand function is given by $\mathcal{P}(Q) = a - bQ$ where $a > 0$ and $b > 0$. The cost function of player $i$ is given by $c(q_i) = cq_i$ for some constant $c > 0$. The profit function of firm $i$ is defined as $\Pi_i(Q) = \mathcal{P}(Q)q_i - c(q_i)$. If two or more firms form a merger $C$ (i.e., collude), then the profit of the merger $C$ is defined as $\Pi_C(Q) = \mathcal{P}(Q)q_C - c(q_C)$, where $q_C = \sum_{i \in C} q_i$, and individual profit of each "agent" firm in $C$ is defined as $\Pi_{i|C}(Q) = \Pi_i^S(Q) + \frac{\Pi_C(Q) - \sum_{i \in C} \Pi_i^S(Q)}{|C|}$, where $\Pi_i^S(Q)$ denotes the standard Stakelberg payoff of firm $i$ when all firms act independently and non-cooperatively. In simple words, firms that are part of a merger share equally the gains (or losses) from collusion.[30]

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[29]The standard assumptions are the ones that guarantee the existence of equilibrium in non-cooperative models of quantity or price competition; see, e.g., Roberts and Sonnenschein (1976), Novshek (1985), and Shapiro (1989).

[30]As mentioned earlier, one could consider any vNM utility function for the firms in general, so this is definitely not the only way to share the gains. However, this is not an unreasonable assumption because the firms are symmetric and the leader makes more profit than the followers in the non-cooperative Stackelberg model. So, according to this sharing rule the leader would end up with more profit than the others if it decides to collude with some of the followers.

**Proposition 1** (Three-firm Stackelberg competition). *In the Stackelberg model with a leader and two followers as described above, there is a unique cooperative equilibrium system outcome in which the leader receives a payoff of $\frac{(a-c)^2}{8b}$ and the followers each receive $\frac{(a-c)^2}{32b}$. An on-path cooperative equilibrium system of this game is summarized as*

$$[(\frac{a-c}{2b}, \frac{a-c}{8b}, \frac{a-c}{8b}); 1, \{2,3\}],$$

*where the leader does not collude with any other firm and produce $\frac{a-c}{2b}$ whereas the two followers all collude and produce each $\frac{a-c}{8b}$.*

*Proof.* First, note that under full collusion the payoff of the monopoly would be $\frac{(a-c)^2}{4b}$. In the two-firm non-cooperative Stackelberg model the payoffs of the leader and the follower are $\frac{(a-c)^2}{8b}$ and $\frac{(a-c)^2}{16b}$, respectively.

Second, in the three-player Stackelberg model described above the non-cooperative subgame perfect equilibrium payoffs of the leader and each of the followers are $\frac{(a-c)^2}{12b}$ and $\frac{(a-c)^2}{36b}$, respectively. To see this, notice that the reaction function of player 2 and player 3 are given by $\frac{a+c-b(q_1+q_3)}{2b}$ and $\frac{a-b(q_1+q_2)+c}{2b}$, respectively. Solving them simultaneously for any given $q_1$, we obtain $\frac{a-c-bq_1}{3b}$, which is the Stackelberg best response function for each follower. As a result, the leader's (non-cooperative) Stackelberg equilibrium production level is $\frac{a-c}{2b}$ and the followers each produce $\frac{a-c}{6b}$.

Notice that full collusion maximizes total profits (e.g., $\frac{(a-c)^2}{12b} + \frac{2(a-c)^2}{36b} < \frac{(a-c)^2}{4b}$). It is also clear that the followers would collude given the production of the leader. This is because each follower receives $\frac{(a-c)^2}{36b}$ in the three-player non-cooperative Stackelberg model whereas if the followers collude then they would each receive $\frac{1}{2}\frac{(a-c)^2}{16b} = \frac{(a-c)^2}{32b} > \frac{(a-c)^2}{36b}$. The main question left to answer is whether the leader can be better off by not colluding with the followers, anticipating that the followers would collude anyway. If all three firms collude, then the leader would receive $\frac{13(a-c)^2}{108b}$ since we have

$$\frac{(a-c)^2}{12b} + \frac{1}{3}\left(\frac{(a-c)^2}{4b} - \frac{(a-c)^2}{12b} - \frac{2(a-c)^2}{36b}\right) = \frac{13(a-c)^2}{108b}.^{31} \tag{1}$$

Note that $\frac{13(a-c)^2}{108b} < \frac{(a-c)^2}{8b}$, which is the leader's payoff in the case in which only the followers collude. This is because the case in which the followers collude is equivalent to the standard two-firm Stackelberg competition in which the leader receives $\frac{(a-c)^2}{8b}$ and the follower receives $\frac{(a-c)^2}{16b}$. Thus, the autarky reference point of the leader, in which only

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[31]The leader would receive even less payoff in the case in which the leader colludes with only with one of the followers.
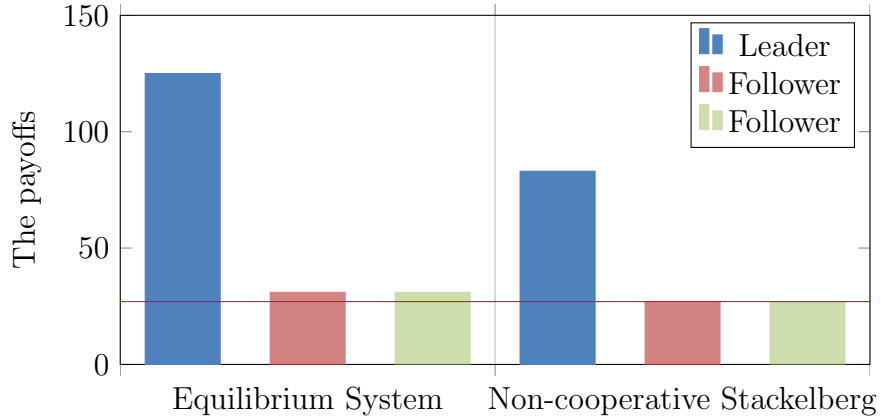
Figure 7: An illustration of cooperative equilibrium system and non-cooperative Stackelberg equilibrium payoffs in a three-firm competition, where $\frac{(a-c)^2}{b} = 1000$.

the followers collude, is the greatest IR reference point. As a result, in the three-player Stackelberg model there is a unique cooperative equilibrium system outcome in which the leader receives $\frac{13(a-c)^2}{108b}$ and the followers each receive $\frac{(a-c)^2}{32b}$. Note that when the followers collude there are many ways to produce $q_2 + q_3$, so there are multiple cooperative equilibrium systems; though, in all of them the followers collude against the leader and they all receive the same payoffs. $\square$

Surprisingly, Proposition 1 shows that the unique equilibrium system outcome in the three-firm cooperative Stackelberg competition coincides with the outcome of the standard two-firm non-cooperative Stackelberg competition despite the fact that the total profits are maximized when all firms collude. This is because the leader rationally anticipates the collusion between the followers and benefits more from this positive externality than joining their coalition to form a monopoly. This apparent puzzle is not unique to the above game. It has been studied since long time; see, e.g., Salant et al. (1983). For some specific parameters, Figure 7 illustrates the payoffs of the three firms under the cooperative equilibrium system and non-cooperative Stackelberg equilibrium in this three-player game.

## 5.2   Logrolling

Early literature on logrolling includes seminal works of Buchanan and Tullock (1962), and Riker and Brams (1973), whose vote trading paradox shows that individually advantageous vote trading may lead to Pareto inferior outcomes for everyone including the

traders.[32] More recently, Casella and Palfrey (2019) propose a vote trading model in which voters have separable preferences over proposals and each proposal is decided by majority voting. Their finding is quite general and striking. They show that for any finite number of voters (who are assumed to be myopic), proposals, and any separable preferences and initial vote profile there exists a sequence of strictly payoff-improving trades that leads to a stable vote profile in the sense that no further strictly payoff-improving trade is possible.

Let $L(K, v, r)$ denote a logrolling game, which is defined as a cooperative extensive form game with possibly imperfect information $\Gamma = (P, X, I, u, \Sigma, H)$ with the following parameters and interpretation. Each player $i \in P$ is called a voter, $K = \{1, 2, ..., \bar{k}\}$ denotes the number of proposals to vote, and $r$ denotes the voting rule to determine the acceptance or rejection of a proposal such as the majority rule. Each voter $i$ has $v_i(k)$ votes to vote for or against proposal $k$. Let $v = (v_1, ..., v_n)$ denote the initial vote profile. Given the voting rule, a pure strategy profile leads to an outcome denoted by $o = (o_1, o_2, ..., o_{\bar{k}})$ in which proposal $k$ is either accepted ($o_k = 1$) or rejected ($o_k = 0$). With a slight abuse of notation, $u_i(o)$ denotes $u_i(s')$ where pure strategy profile $s'$ leads to outcome $o$. Forming a coalition can be interpreted as the agents that make up the coalition 'exchange' votes either physically à la Casella and Palfrey (2019) or by any other agreement.

Next, I state below an existence result for this very general logrolling game. The following corollary directly follows from Theorem 1 and Theorem 2.

**Corollary 2.** *Every logrolling game with possibly imperfect information possesses a cooperative equilibrium system. If the game is of perfect information, then there is a cooperative equilibrium system that includes only pure strategies.*

In other words, for any finite number of voters and proposals, any initial vote profiles, any voting rule, any type of preferences (seperable or not), there exists a cooperative equilibrium system in every logrolling game. Corollary 2 extends Casella and Palfrey's (2019) existence result to logrolling games with (i) dynamically rational voters, (ii) any type of voting rule, (iii) any type of preferences, and (iv) imperfect information. In the logrolling game, the dynamic rationality of the voters is arguably an important extension in part because otherwise some voters might engage in myopically payoff-improving trade, which may eventually decrease their utility.

In their working paper, Casella and Palfrey (2018) extend their analysis to farsighted vote trading as well. They find that farsighted vote trading may not lead to a stable

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[32]For more discussion, I refer the interested reader to Brams (2003).

outcome. There are two main differences between their finding and Corollary 2: the frameworks and the concepts. Their farsightedness notion is similar to the one used in the literature reviewed in section 2.3, which differs from the dynamic rationality notion in cooperative extensive form games.

## 5.3 Monetary transfers and "corruption"

In this subsection, I show how to study "corruption" by incorporating wealth into coalitonal extensive form games. Let $\Gamma = (P, X, I, u, \Sigma, H, W)$ be a coalitonal extensive form game with wealth in which $W = (W_1, W_2, ..., W_n)$ denotes a wealth profile where $W_i \in \mathbb{R}_+$ denotes the wealth of each player $i$. Players can make non-individually rational supertree coalitions individually rational by transferring wealth to some agents who otherwise would not rationally accept to cooperate at an information set. Obviously, wealth goes into the utility function, and there is a trade-off between transferring wealth and obtaining a more desirable solution. Via this way, one could capture the payments that are made to individuals in order to "buy" their cooperation in strategic situations. As a result, coalitonal extensive form games with wealth would enable social scientists to study the strategic effects of corruption (e.g., buying someone's cooperation to induce them to choose a particular action, which otherwise would not be chosen). Mutatis mutandis, the existence results (Theorem 1 and Theorem 2) would analogously remain valid under this extension.

To given an example, consider Figure 8, which is a slight modified version of the market entry game presented in Figure 1. Note that in the cooperative equilibrium system of the original game Firm 1 and Firm 4 colludes, so Firm 1 enters the market L and Firm 4 chooses A. The cooperative equilibrium system outcome is (60, 0, 0, 60). Suppose that the initial wealth profile is $W = (0, 100, 0, 0)$ and that the utility is linear in money for the sake of simplicity. Notice that Firm 2 would now be willing to transfer 31 units of money to Firm 1 to incentivize cooperation.[33] This transfer of money would make Firm 2 better off because otherwise Firm 2 receives a utility of 0 at the cooperative equilibrium system outcome, and it clearly makes Firm 1 better off. As a result, coalition $\{1, 2\}$ would form, where Firm 1 chooses to enter market S and Firm 2 chooses A under the RBI with the given initial wealth structure. The outcome of the new cooperative equilibrium system would be $(30, 40, 30, 0)$ and the final wealth distribution would be $W = (31, 69, 0, 0)$.

Figure 8 presents a simple game that illustrates how money transfers from one player to another can affect the cooperative equilibrium system of a game; though, there are

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[33]I assume that only integer unit transfers are possible.

Firm 1

Enter Market S          Enter Market L

Firm 2                              Firm 4

F      A                          F      A

Firm 3

F      A

$$\begin{pmatrix} 0 \\ 50 \\ 40 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 10 \\ 30 \\ 60 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 30 \\ 40 \\ 30 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 20 \\ 0 \\ 0 \\ 100 \end{pmatrix} \quad \begin{pmatrix} 60 \\ 0 \\ 0 \\ 60 \end{pmatrix}$$
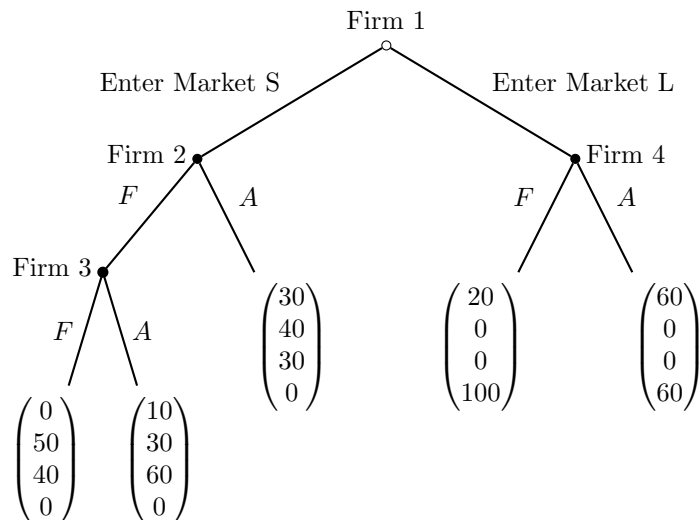
Figure 8: International market entry game with an initial wealth distribution $W = (0, 100, 0, 0)$

certainly more complex cooperative strategic games with different initial wealth distributions. Clearly, the possibility of money transfer between two players may trigger other players to offer "counter-bribes." For example, if Firm 4 had an initial wealth of say $W_4 = 40$, then Firm 4 would be able to "match" Firm 2's offer, hence incentivize Firm 1 to enter market L.

# 6   Conclusions

Cooperative strategic games include wars, airline alliances, and scientific publication. I propose a solution in such games, which is based on a unique procedure in which players act non-cooperatively or cooperate in a rational way. A cooperative equilibrium system is a system which includes a family of collection of strategy profiles and stable coalitions such that given the counterfactual independent players do not have any incentive to deviate unilaterally and coalitions are individually rational and stable in the sense that their members prefer to be in their coalition than be out of the coalition.

Because any non-cooperative extensive form game is a cooperative strategic game, it is trivial to show that a cooperative equilibrium system outcome may be Pareto inefficient. Even though we restrict attention to games in which every player can form a coalition, a cooperative equilibrium system outcome may be still inefficient in part because of coalitional utility functions, which capture positive or negative externalities. Is there a more

specific class of coalitional utility functions, which may come from an appropriate cooperative (characteristic function form) game, such that cooperative equilibrium system outcomes are always Pareto efficient?

A number of fields may benefit from applications of the framework and the solution concept proposed in this paper. A political scientist may apply the model to conflict and cooperation among countries, a computer scientist to multi-agent systems, a biologist to evolution of species and genes, an operations researcher to freight carriers, and an economist to examples such as airline alliances and oligopolistic cartels, among others.[34]

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[34]The applied researcher may find it useful to check Martin Antonov's implementation of the RBI for three-player cooperative strategic games on the Game Theory Explorer (Savani and von Stengel, 2015).

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# Appendix

## 6.1 Proof of Theorem 1

The idea of the proof is that at every step of the RBI algorithm the proposed maximizations exist, the RBI algorithm does not contain indefinite "cycles," and it outputs a cooperative equilibrium system after finitely many steps.

1. **Base case**: Let $(n, \bar{x}) = (1, 1)$ and $(\sigma^*, \pi^*)$ be an equilibrium system at $G(x)$ where $\pi^* = \{i\}$ and $\sigma^*(x) \in \arg\max_{s \in S} u_i(s)$. Note that $\arg\max_{s \in S} u_i(s)$ is non-empty because there is at least a pure action (out of finitely many pure actions) of player $i$ that maximizes player $i$'s utility at $x$.

2. **Induction step**: In this step, it is assumed that cooperative equilibrium system is defined for all subgames with parameters $(m, \bar{y})$ satisfying $1 \leq m \leq n$, $1 \leq \bar{y} \leq \bar{x}$ such that $(n, \bar{x}) \neq (1, 1)$ and $(m, \bar{y}) \neq (n, \bar{x})$. This is a sound assumption because all these parameters are finite.

The next steps shows the existence of a cooperative equilibrium system of subgame $G(x)$ with parameters $(n, \bar{x})$.

(a) Note that given $[\sigma^*(x'), \pi^*(x')]$, $b_i^*(x)$ exists because

$$\max_{b_i'(x) \in \Delta A_i(x)} u_i(b_i'(x) | \sigma^*(x'), \pi^*(x'))$$

$$= \max_{b_i'(x) \in A_i(x)} u_i(b_i'(x) | \sigma^*(x'), \pi^*(x')),$$

where $A_i(x)$ is a finite set; though, this does not preclude the existence of a (completely) mixed behavior strategy maximizer. Thus, $r_0(x) := [\tau^{\{i\}}(x), \pi^{\{i\}}(x)]$ is well-defined.

(b) The non-autarky reference points are cooperative equilibrium systems of the relevant supergames at $x$. Notice that each supergame $\Gamma_{P_C}$ at $x$ has strictly fewer players than the number of players at $G(x)$. Supergame $\Gamma_{P_C}$'s each subgame (excluding itself) has strictly fewer nodes than $\Gamma_{P_C}$, and each supergame of $\Gamma_{P_C}$'s subgame has strictly fewer players than $\Gamma_{P_C}$'s subgame, and so on.[35] Because there are finitely many supergames and subgames of a game, every cooperative equilibrium system of supergames and subgames at $x$ is well-defined. Note that the induction hypothesis assumes that every cooperative equilibrium system of $\Gamma_{P_C}$ is defined; and, we have just shown that this assumption is sound.

As a result, $(r_j(x))_{j=1}^{\bar{j}}$ where $r_j(x) := [\tau^{C_j}(x), \pi^{C_j}(x)]$ is a finite, well-defined, and nondecreasing sequence from the perspective of $i$'s payoff.

(c) Autarky reference point $r_j^*(x)$ where $j = 0$ is assumed to be individually rational because if player $i$ does not form a coalition with any other player, then $i$ will choose their action non-cooperatively at $x$. The following individually rational reference points are defined inductively starting from $r_0^*(x)$.

Assume that $r_j(x)$ is IR, denoted as $r_j^*(x)$, for some $j \geq 0$. Let $j' > j$ be the smallest number such that $r_{j'}(x)$ is IR with respect to $r_j^*(x)$—i.e.,

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[35]Eventually, the game in consideration would be the one described in the base case.

$u_{i'}(\tau^{C_{j'}}(x)|\pi^{C_{j'}}(x)) > u_{i'}(\tau^{C_j}(x)|\pi^{C_j}(x))$ for every agent $i' \in \bar{C}_{j'}$ where $C_{j'} \subseteq \bar{C}_{j'}$ and $\bar{C}_{j'} \in \pi^{C_{j'}}(x)$.

In plain words, $r_{j'}^*(x)$ is IR with respect to $r_j^*(x)$ means that every agent in coalition $C_{j'}$ strictly prefers cooperative equilibrium system of supergame $\Gamma_{P_{C_{j'}}}$ to cooperative equilibrium system of supergame $\Gamma_{P_{C_j}}$. In addition, $j'$ is chosen such that there is no IR reference point between $r_{j'}^*(x)$ and $r_j^*(x)$.

Because $(r_j(x))_{j=1}^{\bar{j}}$ is a finite nondecreasing sequence for $i$ and the sequence of IR reference points is a strictly increasing subseqeunce of it, there exists the greatest IR reference point, denoted by $r_{\bar{l}}^*(x) = [\tau^{C_{\bar{l}}}(x), \pi^{C_{\bar{l}}}(x)]$, that maximizes $i$'s utility. Note that $r_{\bar{l}}^*(x)$ may be the autarky reference point $r_0^*(x)$.

Next, I show that the RBI algorithm does not contain any indefinite cycles, which would prevent it from halting even if it contains finitely many steps. Base case is the standard utility maximization over finitely many actions, so there is no cycles. Induction step needs some elaboration regarding whether cycles exist or not. Note that this step is recursive in that it assumes that cooperative equilibrium system of "smaller" games has been defined. Then, at every node the procedure compares the solution of each supergame starting with the autarky reference point (Step 2.a) and calculates the greatest individually rational reference point of a supergame inductively (Steps 2.b and 2.c). This procedure does not include any cycles and will end after finitely many iterations because (i) there are finitely many players and actions in each supergame, and a supergame of a subgame has strictly fewer players than the players in the subgame due to forming coalitions; and (ii) the greatest individually rational reference point for the respective player exists because there are finitely many such reference points and the sequence of individually rational reference points is a strictly increasing sequence as shown above. Thus, the induction step does not contain any cycles either.

Finally, I show that RBI outputs a cooperative equilibrium system after finitely many steps. Note that a cooperative equilibrium system of $\Gamma(x)$ is defined as system $[\sigma^*, \pi^*]$ where $[\sigma^*(x), \pi^*(x)] = r_{\bar{l}}^*(x)$, which extends $[\sigma^*(x'), \pi^*(x')]$ where $x' \in Succ(x)$. A cooperative equilibrium system, $[\sigma^*, \pi^*]$ exists because (i) we have just shown that the greatest IR reference point $r_{\bar{l}}^*(x)$ exists for every $x$ including the root of $\Gamma$, and (ii) that there are finitely many nodes in $\Gamma$.

## 6.2 Proof of Theorem 2

The RBI in imperfect information games is similar to the RBI in perfect information games, so are their proofs. Thus, I will only give below the parts of the proof that differ from the proof of Theorem 1. The idea of the proof is the same as before: Show that at every step of the RBI the proposed solutions exist, that the RBI does not contain indefinite cycles, and that it outputs a cooperative equilibrium system after finitely many steps.

1. **Base case**: Note that in this case $g_{P'}$ is the largest subgame at $root(h)$ where $P'$ is the set of players in the subgame. A cooperative equilibrium system $(\sigma^*, \pi^*)$ at subtree $T(h)$ exists because no player in $P'$ joins a coalition and there are finitely many players in $g_{P'}$, so a subgame-perfect equilibrium $\sigma^*(h)$ in game $g_{P'}$.

2. **Induction step**: By induction hypothesis, it is assumed that cooperative equilibrium system is defined for all subtrees with parameters $(m, \bar{y})$ satisfying $1 \leq m \leq n$, $1 \leq \bar{y} \leq \bar{h}$ such that $(n, \bar{h}) \neq (1, 1)$ and $(m, \bar{y}) \neq (n, \bar{h})$.

   (a) The autarky reference point at $h$, $r_0(h) := [\tau^{\{i\}}(h), \pi^{\{i\}}(h)]$: As in the perfect information case, the autarky reference point includes a non-cooperative choice of $i$ at $h$, which exists because the largest subgame starting at node $root(h)$ has a subgame-perfect equilibrium.

   (b) Reference points: As in the perfect information case, by the induction hypothesis a cooperative equilibrium system of every supertree $T_{P_C}(h)$ is well-defined. Thus, the nondecreasing sequence of $(r_j(h))_{j=1}^{\bar{j}}$ for player $i$ is also well-defined.

   (c) IR reference points are defined inductively and are strictly increasing with respect to player $i$'s payoff as in the perfect information case. Because there are finitely many IR reference points, the greatest IR reference point $r_{\bar{l}}^*(h)$ exists.

The argument why the RBI does not contain any indefinite cycles is the same as in the perfect information case. In summary, neither the base case, nor any case in the induction step leads to a cycle because (i) there are finitely many players and actions in each supertree, and a supertree of a subtree has strictly fewer players than the players in the subtree because some coalitions form in the supertree; and (ii) the sequence of individually rational reference points is a strictly increasing sequence for player $i$ and it is finite so the greatest individually rational reference point for $i$ exists.

Finally, it is left to show that RBI outputs a cooperative equilibrium system after finitely many steps. As in section 6.1, a cooperative equilibrium system, $[\sigma^*, \pi^*]$ exists because (i) the greatest IR reference point $r_i^*(h)$ exists for every information set $h$ including the root of $\Gamma$, and (ii) that there are finitely many information sets in $\Gamma$.