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An Ambitwistor String Field Theory approach to Supergravity

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Abstract

In this thesis a covariant closed superstring field theory, equivalent to classical ten-dimensional Type II supergravity, is presented. The defining conformal field theory is the ambitwistor string worldsheet theory of Mason and Skinner [1]. This theory is known to reproduce the scattering amplitudes of Cachazo, He and Yuan [2] in which the scattering equations play an important role and our ambitwistor string field theory naturally incorporates these results. We present the operator formalism description of the ambitwsitor string and propose an action for the string field theory of the bosonic and supersymmetric theories. The correct linearised gauge symmetries and spacetime actions are explicitly reproduced and evidence is given that the action is correct to all orders. The focus is on the Neveu-Schwarz sector and the explicit description of tree level perturbation theory about flat spacetime. This thesis is fully based in our published paper [3] and so has a considerable overlap with it.

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Chapter 1

Introduction

Supergravity has long been a useful indirect tool to gain insight into string theories in non-trivial backgrounds as we can always associate it to General Relativity. The vast majority of work on supergravity has been from the perspective of the spacetime Einstein-Hilbert action or equations of motion. But the language in which the Einstein-Hilbert formulation is written can make it difficult to generalise lessons from supergravity to the full string theory. For once, the conformal invariance that plays such an important role in the worldsheet theory, is only implicit in the target space formulation and so difficult to recognise. Also, in the worldline approach we do not not have many of the features that we currently expect of string theory even thought at first look it would look as similar to the worldsheet theory.

In this thesis, which is fully based in our published paper [3], we show the initial stages of an alternative approach to ten-dimensional supergravity based on the ambitwistor worldsheet model of [1]. This ambitwistor string describes Type II supergravity in ten dimensions in terms of the chiral embedding of a worldsheet Σ into ambitwistor space. The worldsheet action for the ambitwistor string is

$$S = \int_{\Sigma} P_{\mu}\bar{\partial}X^{\mu} + \frac{1}{2}eP^2 + ..., \tag{1.1}$$

where (P_{μ}, X^{μ}) take values in the cotangent bundle of spacetime, e is a Lagrange multiplier imposing the constraint $P^2 = 0$, and the ellipsis denotes fermion and ghost contributions.

All fields are holomorphic on the worldsheet Σ.

The ambitwistor string theory (1.1) is thought to be equivalent to a perturbative description of ten-dimensional Type II supergravity [4]. Though written as a chiral worldsheet theory, with superconformal invariance very similar to that found in the conventional superstring, the spectrum of the ambitwistor theory is massless, it has the correct S-matrix, and the supergravity equations of motion are reproduced as the condition that an anomaly vanishes [4]. There are no higher derivative corrections. In this thesis we show the beginnings of a systematic study of the ambitwistor string as a covariant *String Field Theory*. In one hand, the objective is to use it as a toy model to probe into fundamental issues in string theory. On the other hand it would provide us with powerful tools from closed string field theory to help us shed light into understanding ambitwistor strings as a chiral theory and the origin of its properties as a theory a string theory. A more practical goal is to take advantage of the alternative operator formulation of the ambitwistor string to apply it to a promising, perhaps more efficient way to compute scattering amplitudes in gauge and gravity theories through the relation to the scattering equations and join in the current efforts to compute higher-loops amplitudes. The origin of the ambitwistor string lies in recent progress on the study of scattering amplitudes. In [2,5,6] Cachazo, He and Yuan (CHY) proposed remarkably compact expressions for tree-level scattering amplitudes of gravity and Yang-Mills, the key ingredient of which are the scattering equations for n momentum eigenstates with null momenta k_i

$$\sum_{j\neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0, \tag{1.2}$$

first found by Fairlie and Roberts [7] and later, in a very different context, by Gross and Mende [8]. The solutions of (1.2) determine n marked points z_i on a sphere or, in other words, they determine a point on the moduli space $\mathcal{M}_{n,0}$ of a n-punctured, genus zero, Riemann surface. Is this connection between $\mathcal{M}_{n,0}$ and tree-level scattering that suggest the link to a worldsheet theory. The ambitwistor in its worldsheet formulation give us ten-dimensional Type II supergravity amplitudes, and we understand the ambitwistor string as a critical string theory. This is the only case known of a critical string theory able to do that. The ambitwistor string also give a way to derive the CHY equations. However, although the ambitwistor string of [1] describes type II supergravity and has been generalised to other situations, these other theories [9–11] do not have the same status as the original ambitwistor string. This is because they either are not in the critical dimension or do not have a critical dimension that makes sense.

In this thesis we show the first steps in constructing a string field theory for perturbative classical Type II supergravity in flat spacetime [3]. The basic ingredient is Type II ambitwistor string theory (1.1). Following the basic structure of covariant closed bosonic string theory [12–14] and the proposed supersymmetric extension [15], we construct a superstring field theory for supergravity based on the ambitwistor string theory with action

$$S[\Psi] = \langle \Psi | c_0 Q | \Psi \rangle + \sum_{n>2} \frac{1}{n!} \{\Psi^n\}, \tag{1.3}$$

where Q is the BRST operator of the worldsheet theory, c_0 is a ghost zero mode, and $\{\Psi^n\}$ are n-point interaction terms for the string field Ψ.

The oscillator mode structure of the ambitwistor string theory and the constraints that must be imposed on the string fields are at the core of the construction of our model. The oscillator structure is different from that of the conventional string. The X^μ and P_μ fields are independent in the gauge we work in and are composed of independent, conjugate oscillators. The supersymmetric theory is equivalent to Type II supergravity and so the superstring field theory is expected to be equivalent to perturbative Type II supergravity. In support of this we study the metric as a fluctuation $h_{\mu\nu}$ about a Minkowski background, we shall show that the quadratic term gives the correct linearised action for the Type II supergravity

$$\begin{aligned} \langle \Psi | c_0 Q | \Psi \rangle \;\; = \;\; & \int \mathrm{d}^{10}x \left(\frac{1}{4} h_{\mu\nu} \Box h^{\mu\nu} + \frac{1}{2} (\partial^\nu h_{\mu\nu})^2 + \frac{1}{2} h \partial^\mu \partial^\nu h_{\mu\nu} - \frac{1}{4} h \Box h \right. \\ & \left. -4\phi\Box\phi + 2h\Box\phi - 2\phi\partial^\mu\partial^\nu h_{\mu\nu} - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right) \end{aligned} \tag{1.4}$$

We then propose a cubic interaction term and argue that it should be correct. Finally we consider the complete abstract string field theory to all orders. As we will show, the on-shell correlation functions implied by these interaction terms produce the correct on-shell scattering amplitudes and, once a gauge is fixed, the quadratic term produces a reasonable spacetime propagator.

During the course of this thesis we shall see many ways in which the ambitwsitor string field theory mirrors the conventional string field theory superficially but differs in important ways when studied in detail. It should be stressed from the outset that we are interested in a string field theory of *classical* supergravity. As such, we do not consider loops. Though the theory is fully quantum mechanical on the worldsheet, it is classical in spacetime. Loops are considered in the standard formulation of the ambitwistor string [4, 16, 17]. So, we expect that our alternative approach to the ambitwistor string presented in this thesis (using the operator formalism and ambitwistor string field theory) can be extended to loops. It would offer a maybe more effective way in making progress in the computation of amplitudes. From the point of view of computing amplitudes in gauge theories and gravity, the ambitwistor string is just a field theory (not a string theory). And in general closed string field theories the ambitwistor string 'lives' in its boundary. So, it is possible to use powerful tools only available in string field theory, and turn them into tools for the computation of amplitudes in gauge theories and gravity. Then, this thesis (and the paper [3] it's based on) are the first steps towards this objective.

This thesis is presented as follows, in the chapter 2 we give a brief overview of the standard formulation of ambitwistor string theory [1], then in chapter 3 we present its quantisation in the operator formalism and obtain the scattering equations. The operator formalism is the natural language of string field theory. We pay particular care to those aspects that will be of importance for the construction of the ambitwistor string field theory. This chapter introduces most of the key ingredients that are needed to construct the bosonic ambitwsitor string field theory which is then presented in chapter 4 for the free theory, and chapter 5 for its interactions. Chapter 6 introduces the formalism and quadratic action for the supersymmetric ambitwistor string field theory and then, in chapter 7, we discuss the interaction terms in this supersymmetric theory. We present in this thesis the first steps in a formalism that we think has a rich structure and many potential directions of development, from investigating formal and fundamental theoretical issues, to applications in the computation of quantities directly related to phenomenology.

Chapter 2

The Ambitwistor String

The gauge fixed ambitwistor string theory action is [1]

$$S[X,P] = \frac{1}{2\pi}\int_{\Sigma} P_\mu \bar{\partial} X^\mu + \frac{1}{2}\eta_{\mu\nu}\Psi_r^\mu \bar{\partial}\Psi_r^\nu + \frac{e}{2}P^2 - \chi_r P_\mu \Psi_r^\mu + b\bar{\partial}c, \tag{2.1}$$

with Σ the chiral embedding of the Riemann surface into a complexified cotangent bundle over ten-dimensional flat spacetime of coordinates (X^μ, P_μ). And the index r running as $r = 1, 2$. We will introduce in this chapter the basics of Ambitwistors space, Ambitwistor strings symmetries. Most of the differences with the conventional string arise in the bosonic part of the ambitwistor string.

2.1 Ambitwistor space

The Ambitwistor space $\mathbb{A}$, is the space of complex null geodesics in a complexified spacetime [18] [19] [20] [21] [22].

Given an analytic real spacetime $(M_{\mathbb{R}}, G_{\mathbb{R}})$, a d-dimensional manifold with a metric $G_{\mathbb{R}}$, we extend the coordinates x^μ from $\mathbb{R}^d$ to $\mathbb{C}^d$ with an holomorphic metric. The transition functions and metric have finite radius of convergence, extend to some small region, d-dimensional $\mathbb{C}^d$ manifold $\mathcal{M}$ with holomorphic metric $G_{\mu\nu}$ with $\mu = 1 \cdots d$

$\mathbb{A}$ is an holomorphic symplectic manifold, with (X^μ, P_μ) coordinates on $\mathcal{T}^*\mathcal{M}$ that

- Has symplectic potential $\theta = P_\mu \, dX^\mu$,
- symplectic form $\omega = d\theta = dP_\mu \, dX^\mu$,
- Euler vector field $\vec{V} = P_\mu \frac{\partial}{\partial P_\mu}$.
- And generates scalings $P \to \alpha P$.

Geodesics are then Hamiltonian flows for $P^2 = G^{\mu\nu} P_\mu P_\nu$, generated by the Hamiltonian vector field X_p^2. And we identify it with the Hamiltonian $X_p^2 = H$.

$$X_p^2 \;\lrcorner\; \omega + dP^2 = 0 \tag{2.2}$$

Integral curves of X_p^2 are geodesics with parallel propagated P_μ, and $G^{\mu\nu} P_\nu$ tangent to the curve.

Definition 2.1.1. *Ambitwistor space is*

$$\mathbb{A}^{2d-2} = \mathcal{T}_N^* \mathcal{M} \big/ \{X_p^2\} \tag{2.3}$$

On $P^2 = 0$, $\mathcal{T}_N^* \mathcal{M} \in \mathcal{T}^* \mathcal{M}$.

$$\mathscr{L}_{X_H} \omega = 0, \qquad X_H \lrcorner\, \omega = -dP^2 = 0 \tag{2.4}$$

$$\mathscr{L}_{X_H} \theta = 0, \qquad \left[X_H \, , \vec{V} \, \right] = X_H \tag{2.5}$$

where $\mathscr{L}$ is the Lie derivative.

So $\left(\theta, \omega, \vec{V}\right)$ *descend* to $\mathbb{A}$, i.e. $\omega = d\theta$ and $\theta = \vec{V} \lrcorner\, \omega$.

We actually use the projection in ambitwistor strings, so we take a projection

$$\mathcal{P}\mathbb{A}^{2d-3} = \mathbb{A}/\vec{V} \tag{2.6}$$

Holomorphic line bundles $\mathcal{O}(n) \to \mathcal{P}\mathbb{A}$. $\mathcal{O}(n)$ are homogeneous functions of weight n in P. Note that from $\mathbb{A} \to \mathcal{P}\mathbb{A}$, we have that the original ambitwistor space has $\mathcal{O}(-1)$.

On $\mathcal{P}\mathbb{A}$, $\theta \in \Omega^{1,0} \otimes \mathcal{O}(1)$, defines an holomorphic contact structure $\theta \wedge \big(d\theta\big)^{d-2} \neq 0$.

Remarks:

- Locally there is no information in this structure by an holomorphic version analogue of Darboux's theorem.
- It is conformally invariant, only requires that the equivalence class of the metric [G], conformal equivalence class $G\ \Omega^2 G$, $\Omega \neq 0$ function on manifold M.

However we can describe spacetime globally based on

Theorem 1. *(LeBrun 1983)*

- *The original spacetime $\big(M, [G]\big)$ can be reconstructed from the complex structure of $\mathcal{P}\mathbb{A}$.*
- *It is stable under small deformations of the complex structure of $\mathcal{P}\mathbb{A}$, but must allow $\big(M, [G], [\nabla]\big)$, null geodesics of a torsion connection.*
- *$[\nabla]$ is torsion-free if the deformation preserves existence of θ.*

The correspondences between spacetime M and space of complex null geodesics can be schematically represented as [1]):

$$\begin{array}{ccc} & \mathcal{T}^*_N\mathcal{M} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{A} & & \mathcal{M} \end{array} \qquad\qquad \begin{array}{ccc} & \mathcal{P}\mathcal{T}^*_N\mathcal{M} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{P}\mathbb{A} & & \mathcal{M} \end{array} \tag{2.7}$$

In a more hands-on approach, the Ambitwistor space may be constructed simply as a sub-bundle of the cotangent bundle T^*M of the spacetime M, which will be Minkowski spacetime for us in the cases considered in this thesis. We are interested in the complexification of M with T^*M the holomorphic tangent bundle. The default coordinates on T^*M are x^μ and p_μ, with x^μ being coordinates in M. We then define the null cotangent bundle $T^*_N M$

$$T^*_N M = \{(x,p) \in T^*M | p^2 = 0\}, \tag{2.8}$$

where p^2 is constructed using the metric of M. However, this is not exactly the space of null lines as given a point x_0^μ on a null line, the set of points $x_0^\mu + \alpha p^\mu$ all lie in the same null line for any given constant α. Shifts in the line are generated by the vector field

$$\mathscr{V} = p^\mu \frac{\partial}{\partial x^\mu}. \tag{2.9}$$

So the Ambitwistor space $\mathbb{A}$ is taken to be the quotient of $T_N^* M$ by the action of $\mathscr{V}$. Meanwhile, the projective Ambitwistor space $P\mathbb{A}$ is given by a further quotient of $\mathbb{A}$ by the action of the Euler vector field

$$\vec{V} = p_\mu \frac{\partial}{\partial p_\mu}. \tag{2.10}$$

This quotients out by the scale of p_μ, giving $P\mathbb{A}$ as the space of scaled null geodesics in M. The ambitwistor string [1] can be thought of as a sigma model describing the embedding of a worldsheet Σ into Ambitwistor space $\mathbb{A}$. The map from Σ to T^*M is realised by elevating the coordinates x^μ and p_μ to (holomorphic) worldsheet fields $P_\mu(z)$ and $X^\mu(z)$. A Lagrangian on T^*M is given by the $\beta\gamma$ system $\mathcal{L} = P_\mu \bar{\partial} X^\mu$, it is the chiral pull-back of the natural contact structure $\theta = p_\mu \, \mathrm{d}x^\mu$ on $P\mathbb{A}$ to Σ. At the level of the worldsheet, the null constraint is imposed by introducing the Lagrange multiplier field $e(z)$, a Beltrami differential, giving the Lagrangian

$$\mathcal{L} = P_\mu \bar{\partial} X^\mu + \frac{1}{2} e P^2. \tag{2.11}$$

The symmetry associated to this constraint is equivalent, at the level of the worldsheet, to the quotient by the vector field $\mathscr{V}$.

The only outstanding issue at the classical level is that of the worldsheet metric or, equivalently, the worldsheet complex structure. This is not treated explicitly and is assumed fixed by the usual Faddeev-Popov technique, resulting in the introduction of a holomorphic (b, c) ghosts system. The bosonic ambitwistor string action is taken to be

$$S = \int_\Sigma P_\mu \bar{\partial} X^\mu + \frac{1}{2} e P^2 + b \bar{\partial} c. \tag{2.12}$$

Ideally, one would gauge fix $e(z) = 0$ globally but, as discussed in [1, 16] and reviewed in section 2.2, this is not possible in general. The OPEs of the constituent fields are

$$P_\mu(z)X^\nu(\omega) = \frac{\delta^\nu_\mu}{z-\omega} + ..., \qquad b(z)c(\omega) = \frac{1}{z-\omega} + ..., \tag{2.13}$$

where the ellipsis denote terms that are non-singular in the $z \to \omega$ limit, with all other OPE's being trivial in the sense that they have no singular terms.

2.2 Symmetries of the Ambitwistor String

The fields in the ambitwistor string worldsheet transform under a holomorphic conformal transformation $z \to z + v(z)$

$$\delta(v)X^\mu = v\partial X^\mu, \qquad \delta(v)P_\mu = \partial(vP_\mu), \qquad \delta(v)e = v\partial e - e\partial v. \tag{2.14}$$

These conformal transformations are generated by the stress tensor

$$T(z) = P_\mu \partial X^\mu + T_{\text{gh}}, \tag{2.15}$$

where T_{gh} are ghost contributions. We will describe this ghosts contributions later.
For a vector field, $v(z)$ the transformation is generated by

$$\mathcal{T}(v) := \oint \mathrm{d}z\, v(z)T(z), \tag{2.16}$$

so, the action on the field $\Phi(z)$ is

$$\delta(v)\Phi(z) = [\mathcal{T}(v,)\Phi(z)], \tag{2.17}$$

where $\Phi(z)$ is a generic field of the worldsheet theory.

In addition to this conformal symmetry, which is similar to that of the conventional string, there is an additional gauge symmetry on the worldsheet that ensures the theory describes an embedding into ambitwistor space, rather than simply T^*M. The quotient by the vector field $\mathscr{V}$ is achieved in the string theory by the gauge symmetry [1]

$$\tilde{\delta}(v)X^\mu = vP^\mu, \qquad \tilde{\delta}(v)P_\mu = 0, \qquad \tilde{\delta}(v)e = \bar{\partial}v, \tag{2.18}$$

where $v(z)$ is a $(1,0)$ worldsheet vector field. As pointed out in [1], this symmetry has no counterpart in the conventional bosonic string and is a central feature of the ambitwistor string theory. This gauge symmetry is generated by $\mathcal{H}(v)$ where

$$\mathcal{H}(v) := \oint \mathrm{d}z\, v(z)H(z), \tag{2.19a}$$

$$H(z) = \frac{1}{2}P^2(z). \tag{2.19b}$$

$H(z)$ plays the role of a Hamiltonian in the ambitwistor string theory. The spacetime propagator to be discussed in section 5.3 is effectively the inverse of the zero mode of $H(z)$.

We get the following classical algebra by considering all the transformations together

$$[\mathcal{T}(v_1), \mathcal{T}(v_2)] = -\mathcal{T}\big([v_1, v_2]\big), \qquad [\mathcal{T}(v_1), \mathcal{H}(v_2)] = -\mathcal{H}\big([v_1, v_2]\big), \tag{2.20}$$

$$[\mathcal{H}(v_1), \mathcal{H}(v_2)] = 0. \tag{2.21}$$

The commutator of the worldsheet vector fields has the usual form $[v_1, v_2] = v_1\partial v_2 - v_2\partial v_1$.

The holomorphic worldsheet diffeomorphisms have been gauge-fixed in the usual way with the introduction of a (b, c) ghost system and the additional gauge transformations (2.18) are fixed by the usual Faddeev-Popov method, introducing ghosts $\tilde{b}$ and $\tilde{c}$. The constrains $T(z) = 0$ and $H(z) = 0$ are imposed in the standard way by introducing the BRST charge

$$Q = \oint \mathrm{d}z\, j(z), \tag{2.22}$$

with current

$$j(z) = c(z)\left(T(z) + \widetilde{T}_{\rm gh}(z) + \frac{1}{2}T_{\rm gh}(z)\right) + \tilde{c}(z)H(z) \tag{2.23}$$

where $T_{\rm gh}$ and $\widetilde{T}_{\rm gh}$ are stress tensors for the (b,c) ghosts and the $(\tilde{b},\tilde{c})$ ghosts respectively. We repeat the arguments of [16], which discuss the gauge-fixing of the action. The presentation closely follows that of [16] where further details may be found. We want to reproduce them here because the structure of the ghosts terms in the action we develop are important for our description later on.

The BRST operator acts within a given Dolbeault cohomology class and we cannot set $e(z) = 0$ globally. The best we can do is to set

$$e(z) = \sum_a s^a \mu^a(z), \tag{2.24}$$

where $\{\mu_a\}$ is a basis of Beltrami differentials for Σ, where $a = 1, 2, ..., n-3$. This is done by introducing the gauge-fixing fermion $F(e)$ and extending the action to

$$\widehat{S} = \int_\Sigma P_\mu\, \bar{\partial} X^\mu + b\,\bar{\partial} c + Q\ \tilde{b}\ F(e). \tag{2.25}$$

A useful choice is the anti-commutator

$$F(e) = \{e - \sum_{a=1}^{n-3} s_a \mu^a\}, \tag{2.26}$$

where $\{\mu^a\}$ is a basis for $H^{0,1}(\Sigma, T_\Sigma(-z_1 - ... - z_n))$, the z_i are points on Σ, and where the gauge transformation generated by $H(z)$ vanishes. The action of Q on the fields is $Q\tilde{b} = \pi$, $Qe = \bar{\partial}\tilde{c}$, and $Qs^a = q^a$ and so

$$Q\int_\Sigma \tilde{b} F(e) = \int_\Sigma \pi F(e) + \int_\Sigma \tilde{b}\bar{\partial}\tilde{c} - \sum_{a=1}^{n-3} q_a \int_\Sigma \tilde{b}\mu^a. \tag{2.27}$$

Integrating out the Lagrange multiplier π sets $F(e) = 0$ and so the action is

$$\widehat{S} = S - \frac{1}{2}\sum_{a=1}^{n-3} s_a \int_\Sigma \mu^a P^2 - \sum_{a=1}^{n-3}\int_\Sigma q_a \tilde{b}\mu^a, \tag{2.28}$$

where

$$S = \int_\Sigma \left(P_\mu \bar{\partial} X^\mu + b\bar{\partial}c + \tilde{b}\bar{\partial}\tilde{c} \right). \tag{2.29}$$

Integrating out the auxiliary fields s_a and q_a leads to an insertion of

$$\prod_{a=1}^{n-3} \bar{\delta}\left(\int_\Sigma \mu^a(z) H(z) \right) \int_\Sigma \mu^a(z)\tilde{b}(z) \int_\Sigma \mu^a(z) b(z), \tag{2.30}$$

into the path integral, where a indicates the modulus associated with the deformation of the worldsheet moduli corresponding to a particular Beltrami differential. An alternative perspective on the origin of these delta-function insertions will be reviewed in chapter 5. It will turn out that this alternative viewpoint is more useful in studying the ambitwistor string field theory.

2.3 Conventional Ambitwistor string and the scattering equations

In this section we show some of the details of the usual path integral treatment of the ambitwistor string [1], [18]. We call it the "conventional" ambitwistor string as opposite to our operator formalism ambitwistor string. We emphasize the details in the bosonic part, where the fundamental differences arise. And the intention is, of course, to keep this conventional formulation as reference in order to compare with our new, alternative operator formalism developed in chapter 3 for the first quantised ambitwistor string theory in operator formalism, and from chapter 4 onwards for the ambitwisor string field theory. This description will be mostly schematic, but with emphasis on the key features.

We assume an action that is gauge fixed and with the symmetries shown in section 2.2. It is then of the form

$$S=\int_{\Sigma}\left(P_{\mu}\bar{\partial}X^{\mu}+\frac{e}{2}\,P^{2}+b\bar{\partial}c\right). \tag{2.31}$$

The amplitude will be of the form

$$A_n=\int_{\Sigma}DP\,DX\,De\,e^{S}\,\prod_{i=1}^{n}\,V_i(z)\;Db\,Dc, \tag{2.32}$$

where the vertex operator is of the form

$$V_i(z)=c\,\tilde{c}\,P_{\mu}(z_i)\,P_{\nu}(z_i)\,\epsilon^{\mu\nu}\;e^{i\,k_i\cdot X(z_i)}. \tag{2.33}$$

The gauge fix

$$e(z)=\sum_{i=1}^{n-3}s_i\;\mu_i(z_i) \tag{2.34}$$

through Beltrami differentials as we showed in section 2.2, with the gauge fixing fermion $F(e)$ means we have (also was shown BRST invariance of the action $S\to S+Q\;F(e)$).

Initially we would guess we have

$$A_n=\int_{\Sigma}DP\,DX\,Db\,Dc\,D\tilde{b}\,D\tilde{c}\,e^{\int_{\Sigma}\left(P_{\mu}\bar{\partial}X^{\mu}+b\bar{\partial}c+\tilde{b}\bar{\partial}\tilde{c}\right)}\,\prod_{a=1}^{n-3}\left(\int_{\Sigma}b\,\mu_a\right)\,\left(\int_{\Sigma}\tilde{b}\,\mu_a\right)\,\bar{\delta}\left(\int_{\Sigma}P^{2}\,\mu_a\right)\,\prod_{i=1}^{n}\,V_i(z) \tag{2.35}$$

where we give the definition of $\bar{\delta}$ in appendix A.

But now, lets deduce the amplitude. Start by writing

$$\int_{\Sigma}b(z)\,\mu_i(z)=\sum_{i=1}^{n}\oint b^{i}(z)\;v_i(z)\;dz \tag{2.36}$$

where $\bar{\delta}v_i=\mu_i$. Note that in the sphere is just a transition $z\to z+v(z)$, while in general $v(z)\;=\;\sum\;\;v_n\;\;z^{-n+1}$ which is constant if $v_n\;=\;\delta_{n,1}$. So, take $v_1\;=$ constant=1 in the computation

$$\oint b^{i}(z)\;v_i(z)\;dz=\oint b^{i}(z)\;v_1\;dz=\oint b^{i}(z)\;dz=\sum_{n}\;b^{i}_{n}\;\oint\;dz\;z^{-n-2}=b^{i}_{-1} \tag{2.37}$$

using residue theorem, simple pole for $n = -1$.

Now, we initiate the computation of

$$\bar{\delta}\left(\int_{\Sigma} P^2 \, \mu_a\right) \tag{2.38}$$

by doing

$$\int_{\Sigma} P^2 \, \mu_i = \sum_{i}^{n} \oint P^2(z) \; v_i(z) \tag{2.39}$$

and we come back to it when we have the elements we need to continue.

We overlook the ghosts for a moment and consider the tree-level amplitude

$$A_n^0 = \int DP \; Dx \; e^{\int P \, \bar{\partial} X} \prod_{i=1}^{n} V_i \; b(v) \; \bar{b}(v) \; \bar{\delta}(\mathcal{H}(v)), \tag{2.40}$$

where $\mathcal{H}(v)$ was defined in 2.19. And we have V_i from 2.33. So, A_n^0 becomes

$$A_n^0 = \int DP \; DX \; e^{\left(\int P \, \bar{\partial} X + X \cdot J\right)} \prod_{i=1}^{n} P_\mu^i(z) \, P_\nu^j(z) \; \epsilon_{i,j}^{\mu\nu} \text{ (and ghosts contributions)}, \tag{2.41}$$

where we have defined the source

$$J = \sum_{i=1}^{n} k_i \; \bar{\delta}(z - z_i). \tag{2.42}$$

Now there is this crucial step, we do the X integral (by parts), (i.e. on-shell equations of motion)

$$A_n^0 = \int DP \; \delta\left[\bar{\partial} P(z) - J(z)\right] \prod_{i=1}^{n} P_\mu^i(z) \, P_\nu^j(z) \; \epsilon_{i,j}^{\mu\nu} \text{ (and ghosts contributions)}, \tag{2.43}$$

Where

$$\bar{\partial} P(z) = \sum_{i=1}^{n} k_i \; \bar{\delta}(z - z_i) \tag{2.44}$$

becomes classical, on the sphere (i.e., tree-level) is

$$P_{cl}(z) = \sum_{i=1}^{n} \frac{k_i}{z - z_i}. \tag{2.45}$$

Note that in a supersymmetric computation, we would replace P by χ and a vertex operator of the form $V = \epsilon_{\mu\nu}\, \Psi^{\mu}\, \Psi^{\nu}\, c\, \bar{c}\, e^{i\,k\,x}$.

We can now return to our computation of 2.38. For the ghost part we have only the P dependent contribution $\bar{\delta}\left(\int_{\Sigma} P^2\, \mu_a\right)$, but now we just found that P is classical, in the sphere the computation we initiated in 2.39 is

$$\frac{1}{2}\int_{\Sigma} P_{cl}^2(z)\, \mu_i(z) = \frac{1}{2}\sum_{a=i}^{n} \oint_{\partial\Sigma_a} {P_{cl}^{i}}^2(z)\; v_i(z)\; dz = \frac{1}{2}\oint P_{cl}^{a\,2}(z)\; dz, \tag{2.46}$$

where we have applied a Kronecker δ_i^a in v_i^a for the translation $z_i \to z_i + v_i^a(z)\, \delta\, t_a$ with

$$v_i^a = \begin{cases} \delta_i^a & a = 1, \ldots, n-j \\ 0 & \text{otherwise.} \end{cases} \tag{2.47}$$

So, we have that 2.48 is

$$\frac{1}{2}\int_{\Sigma} P_{cl}^2(z)\, \mu_i(z) = \frac{1}{2}\oint_{\partial\Sigma_a} \sum_{i=1}^{n}\sum_{i=1}^{n} \frac{k_i \cdot k_j}{(z - z_i)(z - z_j)} \tag{2.48}$$

and is non-zero if and only if $a = i$ or $a = j$. When $a = i = j$ we have a simple pole of order 2 so trivial solution zero. But, a non-trivial solution exist when $a = i$ and $i \neq j$ and the same solution by re-labelling is obtained with $a = j$ and $j \neq i$. Then, we take twice this case, and loose the $\frac{1}{2}$ factor to get

$$\frac{1}{2}\int_{\Sigma} P_{cl}^2(z)\, \mu_i(z) = \oint_{\partial\Sigma_a} \frac{k_i \cdot k_j}{(z - z_i)(z - z_j)} = \sum_{j \neq a}^{n} \frac{k_a \cdot k_j}{z_a - z_j} = k_i \cdot P_{cl}(z_a) \tag{2.49}$$

The above is the main statement of the CHY scattering equations.

Finally, it is worth mentioning the following. From the gauge fixing fermion $F(e)$ in the b c ghost system we have

$$\prod_{a=1}^{n-3} \bar{\delta}\left(\frac{1}{2}\int_{\Sigma} P^2\, \mu_a\right) = \prod_{a=1}^{n-3} \bar{\delta}\left(k_a \cdot P_{cl}(z)\right) = \prod_{a=1}^{n-3} \bar{\delta}\left(\sum_{j\neq a}^{n} \frac{k_a \cdot k_j}{z_a - z_j}\right). \tag{2.50}$$

Chapter 3

Ambitwistor String in Operator Formalism and the Scattering equations

3.1 The Ambitwistor string in Operator Formalism: generalities

Lets assume we have a gauge fixed ambitwistor action as described in the previous sections. And when compared with the corresponding conventional string theory action we assume the equivalent gauge fix. Now, let be $\Sigma_{g,n}$ a genus g Riemann surface with n punctures. So, for example, $\Sigma_{0,4}$ is the Riemann sphere with four punctures 3.1.

Consider first $\Sigma_{0,1}$, a single puncture in the Riemann sphere. In conventional string theory we have the following equations of motion

$$\Box X^{\mu} = J^{\mu} \tag{3.1}$$

where J^{μ} is the source which we may take as being a vertex operator inserted at the puncture.

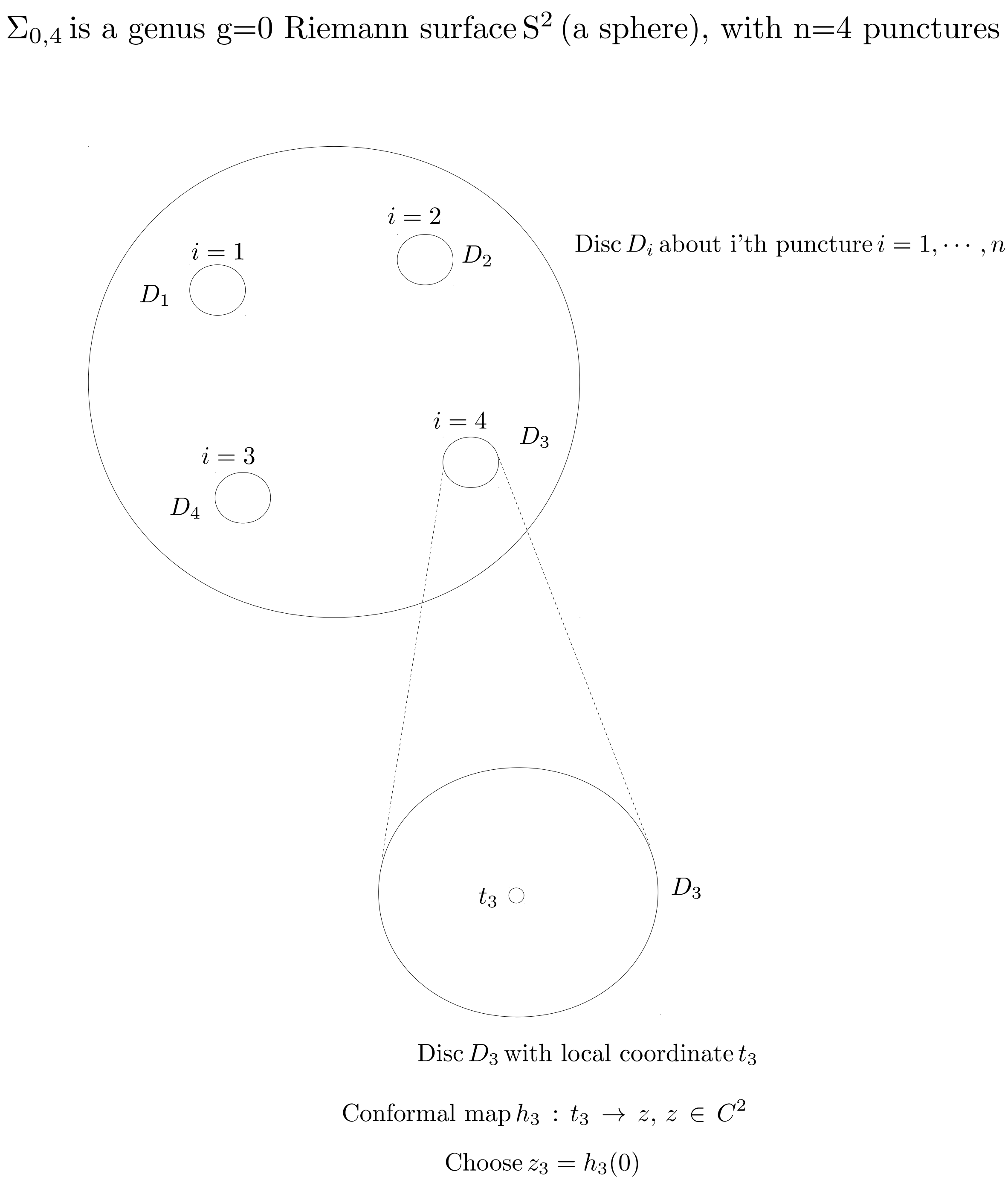

Figure 3.1: Riemann sphere with four punctures.

In this case the usual oscillator expansion is

$$X^{\mu} = x^{\mu} + p^{\mu}\ln(t) + ... \tag{3.2}$$

where the ellipsis denote oscillator modes, and we can think of t as a local coordinate around the puncture.

Note that the expansion 3.2 includes zero mode contributions. We can picture the puncture as residing in the infinite past in worldsheet time. And the relation between an operator inserted in $t = 0$ and its state being given by the usual state-operator correspondence

$$|V\rangle = \lim_{t\to 0} : V(t)|0\rangle. \tag{3.3}$$

The centre of mass-momentum p_{μ} appears as one of two zero modes in the expansion of the field X^{μ} 3.2.

By comparison, in the ambitwistor string we have crucial non-trivial differences. We have independent X and P conjugate fields

$$X^{\mu}(z), \tag{3.4}$$

$$P_{\mu}(z). \tag{3.5}$$

The $X^{\mu}(z)$ field has expansion

$$X^{\mu}(z) = x^{\mu} - \sum_{n\neq 0} \frac{\tilde{\alpha}^{\mu}_{n}}{n} z^{-n} \tag{3.6}$$

where we define

$$x^{\mu} \equiv \tilde{\alpha}^{\mu}_{0}. \tag{3.7}$$

Note that unlike the conventional string, in the ambitwistor string $X^{\mu}(z)$ has a zero conformal weight

$$h_{X,\mathbb{A}} = 0. \tag{3.8}$$

And the conjugate momentum $P_\mu(z)$ field incorporates the zero mode

$$p_\mu \tag{3.9}$$

on its expansion

$$P_\mu(z) = \sum_n \alpha_{n\mu} z^{-n-1} \tag{3.10}$$

by defining

$$p_\mu \equiv \alpha_{0\mu}. \tag{3.11}$$

As it is pointed out in [1] the zero conformal weight $h_{X,\mathbb{A}}$ of the $X^\mu(z)$ field results in a restriction to only the massless sector in the set of allowed vertex operators. And so unlike conventional string theory, we do not have a logarithmic term. This shows in the expansion of $\partial X^\mu(z)$

$$\partial X^\mu(z) = \sum_{n\neq 0} \tilde{\alpha}^\mu_n z^{-n-1} \tag{3.12}$$

which does not have x^μ i.e., a zero mode.

3.2 Operator Quantization for the Ambitwistor string

We impose canonical commutation relations for the Ambitwistor string, where we interpret them as being defined at equal $\bar{z}$. Also we note that unlike conventional string theory commutators do not depend on the spacetime metric.

$$[P_\mu(\sigma), X^\nu(\sigma')] = -i\delta^\nu_\mu \delta(\sigma - \sigma'), \tag{3.13a}$$

$$[P_\mu(\sigma), P_\nu(\sigma')] = 0, \tag{3.13b}$$

$$[X^\mu(\sigma), X^\nu(\sigma')] = 0 \tag{3.13c}$$

where we defined

$$z = e^{i\sigma} \tag{3.14}$$

and

$$z' = e^{i\sigma'}. \tag{3.15}$$

These canonical commutation relations 3.13 are satisfied when the corresponding commutation relations for the mode operators hold. They are

$$[\alpha_{n\mu}, \tilde{\alpha}^{\nu}_{m}] = -in\delta^{\nu}_{\mu}\delta_{n+m,0}, \tag{3.16a}$$

$$[\alpha_{n\mu}, \alpha_{m\nu}] = 0, \tag{3.16b}$$

$$[\tilde{\alpha}^{\mu}_{n}, \tilde{\alpha}^{\nu}_{m}] = 0, \tag{3.16c}$$

for $n \neq 0$.

And for $n = 0$

$$[\alpha_{0\mu}, \tilde{\alpha}^{\nu}_{0}] = [p_\mu, x^\nu] = -i\delta^{\nu}_{\mu} \tag{3.17}$$

holds.

We choose to quantise the vacuum state $|0\rangle$ in the following way

$$\alpha_n|0\rangle = 0 \tag{3.18}$$

for $n \geq 0$, and

$$\tilde{\alpha}_n|0\rangle = 0 \tag{3.19}$$

for $n > 0$. However, we remark that this is a choice and there are alternative choices of vacuum state for the ambitwistor string, that have their own issues to consider [23].

Notice that we do not require that the zero mode $\tilde{\alpha}^{\mu}_{0} = x^\mu$ of $X^\mu(z)$ annihilates the vacuum state in the ambitwistor string.

As we would expect we have the following OPE

$$\langle X^\mu(z)P_\nu(w)\rangle = \frac{i\ \delta^{\mu}_{\nu}}{z-w}. \tag{3.20}$$

This is easily shown as follows. Assuming the modes commutation relations 3.16 and 3.17 with our choices of vacuum 3.18 3.19, lets have the expansion of $X^\mu(z)$ 3.6 and re-write the expansion of $P_\mu(z)$ 3.10

$$X^\mu(z) = x^\mu - \sum_{n\neq 0} \frac{\tilde{\alpha}_n^\mu}{n} z^{-n} \tag{3.21}$$

and

$$P_\nu(w) = \sum_m \alpha_{m\nu} w^{-m-1} \tag{3.22}$$

where $\tilde{\alpha}_0^\mu = x^\mu$ and $\alpha_{0\nu} = p_\nu$. Then we have

$$\begin{aligned} &\langle X^\mu(z) P_\nu(w)\rangle = \langle 0|X^\mu(z)P_\nu(w)|0\rangle = \\ &\left\langle 0|x^\mu - \sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n} z^{-n} \sum_m \alpha_{m\nu} w^{-m-1}|0\right\rangle = \langle 0|\tilde{\alpha}_0^\mu\ \alpha_{0\nu}|0\rangle + \sum_{m,n>0}\langle 0|\tilde{\alpha}_n^\mu\ \alpha_{m\nu}|0\rangle \frac{(-1)}{n} z^{-n} w^{-m-1} \\ &= \sum_{m,n>0}\langle 0|\alpha_{m\nu}\ \tilde{\alpha}_n^\mu + i\ m\ \delta_\nu^\mu\ \delta_{m+n,0}|0\rangle \frac{(-z^{-n})}{n} w^{-m-1} = \sum_{m,n>0}\langle 0|\alpha_{m\nu}\ \tilde{\alpha}_n^\mu|0\rangle \frac{(-z^{-n})}{n} w^{-m-1} \\ &+ i\sum_{m,n>0}\langle 0|m\ \delta_\nu^\mu\ \delta_{m+n,0}|0\rangle \frac{(-z^{-n})}{n} w^{-m-1} = i\sum_{n>0} n\ \delta_\nu^\mu\ \frac{z^{-n}}{n} w^{-n-1} = i\frac{\delta_\nu^\mu}{z}\sum_{n=1}^{\infty}\left(\frac{w}{z}\right)^{n-1} \\ &= i\frac{\delta_\nu^\mu}{z}\sum_{k=0}^{\infty}\left(\frac{w}{z}\right)^k = i\frac{\delta_\nu^\mu}{z}\left(\frac{1}{1-\frac{w}{z}}\right) = \frac{i\ \delta_\nu^\mu}{z-w} \end{aligned} \tag{3.23}$$

as expected.

When we have more than one puncture it is useful to have local definitions and relations. Referring to figure 3.1, lets designate for the $i'th$ puncture the local coordinate t_i in a small disc D_i about the puncture and perform oscillator expansions as we have done above, but now locally for each $i'th$ puncture in terms of its local coordinate t_i. This definitions are suitable to be used in conformal maps expressions, lets them be h_i

$$h_i : t_i \to z \tag{3.24}$$

so the can be used to describe expressions in terms of coordinates z in the complex plane. In this way, the location of the $i'th$ puncture in this new coordinates can be set to be

$$z_i = h_i(0) \tag{3.25}$$

i.e., the origin of the local coordinate system. In a general case it would imply that we have a complicated expression of the oscillator expansions of the worldsheet fields when defined in this z coordinates. A simple proposal for the map would be

$$t_i = z - z_i \tag{3.26}$$

which appears in [24]. Following this source we now describe the operator formalism for the case of the conventional string. We'd like to identify the worldsheet punctures at z_i with n asymptotic states at points x_i in spacetime. For that, we insert

$$\prod_{i=1}^{n} \delta^D(X(z_i) - x_i) \tag{3.27}$$

into the path integral. The corresponding Fourier transformed insertion in momentum space gives the usual distribution that we know from standard calculations of Tachyon scattering amplitudes

$$J = \sum_{i=1}^{n} k_i \delta^2(z - z_i). \tag{3.28}$$

In this way, we get the expected equation of motion for the classical source fields

$$\Box X^\mu_{\text{cl}} = J^\mu. \tag{3.29}$$

So, now we may write the (full) field as

$$X^\mu = X^\mu_{\text{cl}} + X^\mu_{\text{q}} \tag{3.30}$$

where

$$X^{\mu}_{\mathrm{q}} \tag{3.31}$$

is a quantum fluctuation and

$$X^{\mu}_{\mathrm{cl}} \tag{3.32}$$

is the classical solution, which at genus zero is given by

$$X^{\mu}_{\mathrm{cl}}(z) = \sum_{i=1}^{n} k_i^{\mu} \ln |z - z_i|^2. \tag{3.33}$$

Now, it follows as a natural expression to write X^{μ} as a sum over the punctures

$$X^{\mu} = \sum_{i=1}^{n} X_i^{\mu} \tag{3.34}$$

where

$$X^{i} \tag{3.35}$$

is written in terms of the Hilbert space defined at $i'th$ puncture.

If we now consider the case of the ambitwistor string, its punctures will be identified with the insertions of the current

$$J = \sum_{i=1}^{n} k_i \bar{\delta}(z - z_i) \tag{3.36}$$

and the equation of motion is now for $P(z)$

$$\bar{\partial} P_{\mathrm{cl}} = J, \tag{3.37}$$

with classical solution

$$P_{\mathrm{cl}}(z) = \sum_{i=1}^{n} \frac{k_i}{z - z_i}. \tag{3.38}$$

Similar to the conventional string case, we can expand $P(z)$

$$P = \sum_{i=1}^{n} P_i \tag{3.39}$$

where each P_i is written at each puncture as an oscillator expansion after a conformal transformation on the $i'th$ puncture with corresponding Hilbert space in the local coordinates t_i of each puncture. The same simple choice $t_i = z - z_i$ that we have in the conventional string results in the expression 3.38 of the ambitwistor string for the zero modes contribution. We will give more details and discuss further on the case of the ambitwistor string in section 3.7. For the conventional string case, please refer to [24], [25], [26] and [27] for more details.

We generalise the modes commutation relations for multiple punctures as

$$[\alpha^{(i)}_{n\mu}, \tilde{\alpha}^{(j)\nu}_{m}] = -i\, n\, \delta^{ij}\, \delta^{\nu}_{\mu}\, \delta_{n+m,0}, \tag{3.40a}$$

$$[\alpha^{(i)}_{n\mu}, \alpha^{(j)}_{m\nu}] = 0, \tag{3.40b}$$

$$[\tilde{\alpha}^{(i)\mu}_{n}, \tilde{\alpha}^{(j)\nu}_{m}] = 0. \tag{3.40c}$$

$$\text{For } m, n \neq 0 \tag{3.40d}$$

and

$$[\alpha^{(i)}_{0\mu}, \tilde{\alpha}^{(j)\nu}_{0}] = [p^{(i)}_{\mu}, x^{(j)\nu}] = -i\delta^{ij}\delta^{\nu}_{\mu} \tag{3.41}$$

for $m = n = 0$. Note that the commutation relations for the modes do not depend on the background spacetime metric.

We can now use the technology developed to write the ghost insertions 2.30 in a useful way. Provided that we associate a Hilbert space with each puncture, lets define a disc $\mathscr{D}_i$ about each puncture (see figure 3.1). They are in terms of local coordinates t_i, and maybe defined as the regions $|t_i| < 1$. The Beltrami differential encodes changes in the moduli of the Riemann surface Σ. This in turn can be understood as deformations in the worldsheet in the region of a puncture. Lets take the region surrounding the disc $\mathscr{D}_i$ of the $i-th$ puncture, an infinitesimal ϵ away from it: $|t_i| < 1 + \epsilon$. And so, let the coordinates be changed to t'_i. On a region such that $|t_i| > 1 - \epsilon$ there exist a patch to which we can assign the coordinate t_i. On the overlapping section (the annulus of diameter 2ϵ) it is the boundary of the disc,

$\partial\mathscr{D}_i$. The two coordinates are related by

$$t_i' = t_i + v_i(t). \tag{3.42}$$

In this overlapping region, the Beltrami differentials may be written as

$$\mu_i = \bar{\partial} v_i. \tag{3.43}$$

And this leads to the an alternative description of the ghost insertions. This can be implemented, for example, by associating the $i'th$ puncture with an extracted disc $\mathscr{D}_i$ in such a way that

$$\partial\Sigma = \cup_{i=1}^{n} \partial\mathscr{D}_i, \tag{3.44}$$

give us

$$\int_\Sigma \mu^a(z) b(z) = \sum_{i=1}^{n} \oint_{\partial\mathscr{D}_i} \mathrm{d}z_i \, v_i^a(z_i) b^{(i)}(z) \tag{3.45}$$

where the $b^{(i)}$ are the ghosts modes associated with the Hilbert space at the $i'th$ puncture. We can write the ghost insertion term in 2.30 as

$$\prod_{a=1}^{n-3} \bar{\delta}\Big(\mathcal{H}(\vec{v}^a)\Big) \tilde{\mathbf{b}}(\vec{v}^a) \mathbf{b}(\vec{v}^a) \tag{3.46}$$

where we have

$$\mathbf{b}(\vec{v}^a) = \sum_{i=1}^{n} \oint_{\partial\mathscr{D}_i} \mathrm{d}z \, v_i^a(z) b^{(i)}(z), \tag{3.47a}$$

$$\tilde{\mathbf{b}}(\vec{v}^a) = \sum_{i=1}^{n} \oint_{\partial\mathscr{D}_i} \mathrm{d}z \, v_i^a(z) \tilde{b}^{(i)}(z), \tag{3.47b}$$

$$\mathcal{H}(\vec{v}^a) = \sum_{i=1}^{n} \oint_{\partial\mathscr{D}_i} \mathrm{d}z \, v_i^a(z) H^{(i)}(z). \tag{3.47c}$$

and where $\vec{v}^a$ means the n vector fields located at each of the $i'th$ punctures, $i = 1, \cdots, n$

$$\vec{v}^a = \big(v_1^a, v_2^a, ..., v_n^a\big). \tag{3.48}$$

We carefully choose the discs $\mathscr{D}_i$ so they do not overlap with each other, and also such that there is only one puncture in each disc. And then we can take the contour integral about the boundary $\partial\mathscr{D}_i$ of the discs $\mathscr{D}_i$. Each of these discs is centred on a point z_i. And $H^{(i)}(z)$, $\tilde{b}^{(i)}(z)$ and $b^{(i)}(z)$ are defined in the Hilbert space at the $i'th$ puncture.

3.3 The Virasoro Algebra of the Ambitwistor String

The ambitwistor string has stress tensor

$$T(z) = P_\mu \partial X^\mu \tag{3.49}$$

with the usual definition for the coefficients of its mode expansion which are related by

$$T(z) = \sum_n L_n z^{-n-2}, \tag{3.50a}$$

$$L_n = \oint \mathrm{d}z z^{n+1} T(z). \tag{3.50b}$$

As a result, the stress tensor components are quite different when compared to the conventional string

$$L_0 = \frac{1}{2}\sum_{m>0}(\alpha_{-m}\cdot\tilde{\alpha}_m + \tilde{\alpha}_{-m}\cdot\alpha_m), \tag{3.51a}$$

$$L_n = \sum_{m\neq n}\tilde{\alpha}_{n-m}\cdot\alpha_m, \tag{3.51b}$$

$$\text{where the dot means a Lorentz index contraction } \alpha\cdot\tilde{\alpha} := \alpha_\mu\tilde{\alpha}^\mu. \tag{3.51c}$$

So, we also see in the generators L_0 and L_n their independence with respect to a background spacetime metric. As expected $\tilde{\alpha}_0$ does not appear in the expansion of L_n. This follows from the mode expansion 3.12, of ∂X^μ.

The additional gauge symmetry of the Ambitwistor string, which is generated by $H(z)$

(defined in 2.19), has expansion

$$H(z) = \sum_n \widetilde{L}_n z^{-n-2}. \tag{3.52}$$

The generator $\widetilde{L}_n$ may be written as an expansion on the modes α_μ as

$$\widetilde{L}_n = \frac{1}{2}\eta^{\mu\nu} \sum_m \alpha_{m\mu}\alpha_{n-m\nu}. \tag{3.53}$$

Note that as the modes α_μ commute with each other, there is no need to define a normal ordering.

We are using a Minkowski metric $\eta^{\mu\nu}$ here because we are considering the expansion of $H(z)$ on flat backgrounds. And so for the more general case of curved spacetime backgrounds we must use the relevant metric.

If we now compute the commutators of the Ambitwistor string Virasoro algebra, we obtain

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\,\frac{D}{6}m(m^2-1), \tag{3.54a}$$

$$[L_m, \widetilde{L}_n] = (m-n)\widetilde{L}_{m+n}, \tag{3.54b}$$

$$[\widetilde{L}_m, \widetilde{L}_n] = 0. \tag{3.54c}$$

D is the dimension of spacetime, with $\mu = 1, 2, ..., D$.

The central charge coming from the free $\beta\gamma$ system is

$$c = \mp 3(2\lambda - 1)^2 \pm 1, \tag{3.55}$$

where the upper sign is for fermions and the lower sign is for bosons. And λ is the conformal weight of the highest field.

The (b, c) and $(\tilde{b}, \tilde{c})$ ghosts systems have a weight contribution of $\lambda = 2$ each. So, each contribute with -26 to the central charge, giving a critical dimension $D = 26$. This is consistent with the central charge of the single system (X, P), which has conformal weight

$+1$ for P that contributes with $c = 2$ to the central charge. Then, the total central charge is

$$c = 2D - 26 - 26 \tag{3.56}$$

which, as expected vanishes in the critical dimension.

Comparing the anomaly of the Ambitwistor string 3.54a with the anomaly of the conventional string

$$\delta_{m+n,0}\,\frac{D}{12}m(m^2-1) \tag{3.57}$$

The dimension D appears in the computation of the Virasoro algebra from the trace of the Minkowski spacetime metric $\eta_{\mu\nu}$. As a consequence it encodes the number of independent (X, P) systems considered. Because we are taking a complexified spacetime in the Ambitwistor string, D represents the complex dimension. Then, the procedure in the Ambitwistor string is to take the chiral part only (holomorphic coordinates), i.e. the $\bar{X}^\mu$ coordinates play no role.

3.4 BRST Quantisation of the Ambitwistor String

We want to impose the constraints on $T(z)$ and $H(z)$ in the usual way, from the BRST charge Q 2.22 (with current $j(z)$ 2.23), which we reproduce here for convenience

$$Q = \oint \mathrm{d}z\, j(z), \tag{3.58}$$

$$j(z) = c(z)\left(T(z) + \widetilde{T}_{\mathrm{gh}}(z) + \frac{1}{2}T_{\mathrm{gh}}(z)\right) + \tilde{c}(z)H(z). \tag{3.59}$$

So, we want

$$T(z) = 0 \tag{3.60}$$

and

$$H(z) = 0. \tag{3.61}$$

In each of the $i'th$ punctures, $i = 1, \cdots, n$, we have an associated Hilbert space. For that, lets consider the $Q^{(i)}$ BRST charge with associated $i'th$ Hilbert space, where we are defining a total BRST charge as

$$Q = \sum_{i=1}^{n} Q^{(i)}. \tag{3.62}$$

So lets consider a single Hilbert space.

We have the following usual expansions for the ghosts

$$c(z) = \sum_n c_n z^{-n+1}, \tag{3.63a}$$

$$b(z) = \sum_n b_n z^{-n-2}, \tag{3.63b}$$

$$\tilde{c}(z) = \sum_n \tilde{c}_n z^{-n+1}, \tag{3.63c}$$

$$\tilde{b}(z) = \sum_n \tilde{b}_n z^{-n-2}. \tag{3.63d}$$

And then we have the BRST charge Q in term of the generators

$$Q = \sum_n c_{-n} \left(L_n^{(m)} + L_n^{(g)} + \widetilde{L}_n^{(g)} \right) + \sum_n \tilde{c}_{-n} \widetilde{L}_n^{(m)}. \tag{3.64}$$

Note that we are re-writing the notation of the generators L_n and $\widetilde{L}_n$ that we obtained in section 3.3 as $L_n^{(m)}$ and $\widetilde{L}_n^{(m)}$ in order to distinguish them from the ghost generators, which we call $L_n^{(g)}$ and $\widetilde{L}_n^{(g)}$. These ghost generators have expansions

$$L_n^{(g)} = \sum_m (n-m) : b_{n+m} c_{-m} : -\delta_{n,0}, \tag{3.65a}$$

$$\widetilde{L}_n^{(g)} = \sum_m (n-m) : \tilde{b}_{n+m} \tilde{c}_{-m} : -\delta_{n,0}. \tag{3.65b}$$

The massless condition on the physical states of the Ambitwistor string translates into the BRST charge operator 3.64 terms as keeping only the lower oscillator modes. So, considering only these leading order terms in the BRST charge give us

$$\sum_n c_{-n} L_n^{(m)} = c_0(\alpha_{-1}\cdot\tilde{\alpha}_1 + \tilde{\alpha}_{-1}\cdot\alpha_1) + \alpha_0\cdot(c_1\tilde{\alpha}_{-1} + c_{-1}\tilde{\alpha}_1) + ... \tag{3.66}$$

and

$$\sum_n \tilde{c}_{-n} \widetilde{L}_n^{(m)} = \frac{1}{2}\tilde{c}_0\alpha_0^2 + \frac{1}{2}\tilde{c}_0\alpha_{-1}\cdot\alpha_1 + \alpha_0\cdot(\tilde{c}_{-1}\alpha_1 + \tilde{c}_1\alpha_{-1}) + ... \tag{3.67}$$

where we notice that the last generators sum is the only one that depends on the background metric (for us is the Minkowski metric, flat space). This last sum is on $\widetilde{L}^{(m)}$ and so it involves $H(z)$.

In the Ambitwistor string field theory that we will introduce in section 6 it will become an important distinction to consider the independence with respect to the metric $T(z)$. When we impose the $T(z) = 0$ constraints, they will be part of the definition of the Ambitwistor string field. And as such they will not be possible to perform a perturbative change in the background. On the other hand, the constraint imposed on $H(z)$, i.e. $H(z) = 0$ will be no other than the equations of motion. These must be modified in such a way as to introduce interaction terms that should encode the dynamical effect on the background fields.

For convenience we write the BRST charge operator Q as

$$\begin{aligned} Q &= c_0\mathcal{L}_0 + \frac{1}{2}\tilde{c}_0\alpha_0^2 + \frac{1}{2}\tilde{c}_0\alpha_{-1}\cdot\alpha_1 + \alpha_0\cdot(c_1\tilde{\alpha}_{-1} + c_{-1}\tilde{\alpha}_1 + \tilde{c}_{-1}\alpha_1 + \tilde{c}_1\alpha_{-1}) \\ &\quad -2b_0c_{-1}c_1 + 2\tilde{b}_0(c_1\tilde{c}_{-1} + \tilde{c}_1c_{-1}) + \tilde{c}_0(c_{-1}\tilde{b}_1 + c_1\tilde{b}_{-1}) + ... \end{aligned} \tag{3.68}$$

where we have taken all the terms that multiply c_0 by defining $\mathcal{L}_0$

$$\mathcal{L}_0 = (\alpha_{-1}\cdot\tilde{\alpha}_1 + \tilde{\alpha}_{-1}\cdot\alpha_1) + (b_{-1}c_1 + c_{-1}b_1 - 1) + (\tilde{b}_{-1}\tilde{c}_1 + \tilde{c}_{-1}\tilde{b}_1 - 1) + ... \tag{3.69}$$

Note that we are defining $\mathcal{L}_0$ in such a way that we are discarding higher mode terms. We will see later in the introduction of the Ambitwistor string field theory that these higher modes do not play a role.

It is interesting to notice that, if we omit the ghosts, we can write $\mathcal{L}_0$ as

$$\mathcal{L}_0 = L_0 - 2 + ... \tag{3.70}$$

As we will see, the term $c_0\mathcal{L}_0$ is crucial for the construction of the appropriate action of the Ambitwistor string field theory, when we consider the BRST charge operator Q for it. The ambitwistor string field $|\Psi\rangle$ should obey a metric-independent constraint as part of its definition

$$\mathcal{L}_0|\Psi\rangle = 0. \tag{3.71}$$

Other constraints given by the Ambitwistor BRST charge operator Q translate in the Ambitwistor string field action as gauge invariances and in the target space equations of motion. This will be shown in more detail in section 4.4. For the Ambitwistor string field (as in any string field theory), the construction of the action is built up in the operator formalism from the BRST charge operator into a quadratic string field action. The complete (nonlinear) action depends on the operator formalism description of the Ambitwistor string interactions (as a first quantised theory), as described in the following section 3.5.

3.5 The Operator Formalism for the Ambitwistor String Theory

The initial steps towards an operator formalism formulation of the Ambitwistor string where first approached in [28]. The operator formalism for the conventional string ([25] and [26]) has at its core the idea of constructing the n-punctured genus g worldsheet from the so called *surface state* $\langle\Sigma|$ which encodes all the information we need about the configuration of a particular state. The surface state may be defined as a map

$$\langle\Sigma| : \otimes_{i=1}^{n}\mathcal{H}_i \ \rightarrow \mathbb{C} \tag{3.72}$$

from the n-fold product of Hilbert spaces into the complex space. As we mentioned, each puncture in the marked Riemann surface is associated to a Hilbert space $\mathcal{H}_i$ and the surface state acts on the collection of asymptotic states $|V_i\rangle...|V_n\rangle$, where each asymptotic state is an insertion into each puncture $|V_i\rangle \in \mathcal{H}_i$.

The actual resulting function is

$$\langle\Sigma|B(\vec{\nu})|V_i\rangle...|V_n\rangle, \tag{3.73}$$

where $B(\vec{\nu})$ is the function containing the information of the ghosts insertions (described bellow). The form 3.73 is integrated over the corresponding space Γ_n of the scattering amplitude encoded in the surface state $\langle\Sigma|$. If we were considering only the case of a standard string theory, then the space Γ_n is just the moduli space of Riemann surfaces $\mathcal{M}_{n,g}$, and will become a Riemann surface with n marked points when acting on the surface state with the insertions of n (asymptotic) states $|V_i\rangle...|V_n\rangle$. However, for the ambitwistor string we will consider a different Γ_n. It is adapted from the construction presented in [9] for the ambitwistor string, and we will develop it for the ambitwistor string field theory in section 5.2. Lets just say in anticipation that in [9] the argument is that we should think of 3.73 as a top holomorphic form on the $2(2n-6)$ dimensional space $T^*\mathcal{M}_n$. And from Morse theory we can select a $2n-6$ cycle Γ_n over which we can now integrate 3.73. Further arguments from localisation theory allow to simplify the form 3.73 and so we can formally write the amplitude in terms of an integral over the $2n-6$ dimensional space $\mathcal{M}_n$.

However, at this stage of our exposition, lets continue with the standard formulation of the ambitwistor string, following [1]. So, for now, we take Γ_n to be $\mathcal{M}_{n,0}$. And lets consider only the case of classical supergravity. For that reason, we shall restrict to the tree-level case i.e., genus $g = 0$. Note then that although the theory is quantised at worldsheet level, we are only considering classical spacetime physics.

We further restrict to on-shell states only. Their connection to vertex operators $V(t)$ is easily established by

$$|V\rangle = \lim_{t\to 0} V(t)|0\rangle, \tag{3.74}$$

where t is a local coordinate that vanishes at the puncture, as illustrated in t_3 of figure 3.1. A useful example of vertex operator for us, at this point, is

$$V(z) = c(z)\tilde{c}(z)\varepsilon^{\mu\nu}P_\mu(z)P_\nu(z)e^{ik\cdot X(z)} \tag{3.75}$$

which is the massless symmetric state vertex operator. It corresponds to the state

$$|V\rangle = c_1\tilde{c}_1\varepsilon_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|k\rangle, \tag{3.76}$$

that has zero mode momentum eigenstate

$$|k\rangle = e^{ik\cdot x}|0\rangle \tag{3.77}$$

where $\varepsilon_{\mu\nu}$ is a polarisation tensor.

The resulting n-point scattering amplitude is

$$M_n = \int_{\mathcal{M}_n} \langle\Sigma|B(\vec{\nu})|V_1\rangle...|V_n\rangle. \tag{3.78}$$

The presence of the limit in 3.74 has the effect of making all dependence on the location t of the operator insertion disappear. This is characteristic of the operator formalism were the information about the localisation of the vertex insertion is encoded in the surface state $\langle\Sigma|$ instead of the states $|V_i\rangle$. We describe the surface state in the following section.

3.6 The Surface State for the Ambitwistor string

Formulation of string theory in operator formalism is an alternative, and is the only option in string field theory. At its heart is the surface state $\langle\Sigma|$ which has all the information of the conformal field theory encoded in it, in particular about the n punctures locations, so defines the moduli of a particular configuration for an amplitude on a genus g Riemann surface with n punctures. In the language used in the area of amplitudes, a Riemann surface with n marked points. These marked points (also called 'decorations' of the marked Riemann surface), have in them the insertions of the asymptotics vertex operators in each puncture. To fix ideas we refer to the simple case of a sphere, looking again to figure 3.1 the local coordinate t_i of the $i'th$ puncture is chosen for convenience to be at the centre of a small disk surrounding the puncture, so the origin of the local coordinate is $t_i = 0$ there. We establish a conformal map h_i from the puncture to the complex plane

$$h_i : t_i \to z = h_i(t_i) \in \mathbb{C}^2. \tag{3.79}$$

And as we defined for convenience t_i to be at the centre of a disk about the puncture so $t_i = 0$, then

$$z_i = h_i(0). \tag{3.80}$$

As we mentioned, the surface state $\langle\Sigma|$ has the information on the configuration of a particular amplitude, i.e. a particular set of locations for the coordinates t_i. The string states $|\Psi_i\rangle$ that will be acted on by the surface state of these punctures are defined in Hilbert spaces $\mathcal{H}_i$. The contraction of the surface state with these states will give a conformal field theory correlation function. So the surface state itself can be viewed as a map

$$\langle\Sigma| : \otimes_i \mathcal{H}_i \to \mathbb{C}. \tag{3.81}$$

Our presentation of the surface state $\langle\Sigma|$ for the ambitwistor string was developed in [28], here we give a summary and extend the results presented there.

The surface state may be written as

$$\langle\Sigma| = \int \mathrm{d}^n p \, \langle\vec{p}_n| \, \delta\left(\sum p_{(i)}\right) \, e^W \, \mathcal{Z}, \tag{3.82}$$

with the integral evaluated over all external momenta

$$\mathrm{d}^n p = \prod_{i=1}^{n} \mathrm{d}p_{(i)}. \tag{3.83}$$

The delta function

$$\delta\left(\sum p_{(i)}\right) \tag{3.84}$$

accounts for overall conservation of momentum. We will see later in 5.2 how we re-interpret this and give a less arbitrary justification of it than just imposing conservation.

The W in the exponent of 3.82 has two components

$$W = V_{X,P} + V_{\mathrm{gh}}. \tag{3.85}$$

A matter component

$$V_{X,P}(z_1, ...z_n) = \sum_{i,j} \oint_0 \mathrm{d}t_i \oint_0 \mathrm{d}t_j \frac{X(t_i) \cdot P(t_j)}{h_i(t_i) - h_j(t_j)} \tag{3.86}$$

and a ghost component

$$V_{\mathrm{gh}}(z_1, ...z_n) = \sum_{i,j} \oint_0 \mathrm{d}t_i \oint_0 \mathrm{d}t_j \frac{b(t_i)c(t_j)}{h_i(t_i) - h_j(t_j)} + \sum_{i,j} \oint_0 \mathrm{d}t_i \oint_0 \mathrm{d}t_j \frac{\tilde{b}(t_i)\tilde{c}(t_j)}{h_i(t_i) - h_j(t_j)}. \tag{3.87}$$

We define the $SL(2;\mathbb{C})$-invariant vacuum $\langle p_{(i)}; 3|$ as

$$\langle p_{(i)}; 3| \equiv \langle p_{(i)}| \otimes \langle 3_i| \otimes \langle \tilde{3}_i| \tag{3.88}$$

The ghost vacua have the standard normalisation and are given by $\langle 3| = \langle 0|c_{-1}c_0c_1$, with a similar expression for $\langle \tilde{3}|$ involving the $\tilde{c}$ ghosts and are normalised as $\langle 3|0\rangle = 1$ and similarly for $\langle \tilde{3}|$.

In the surface state equation 3.82 we apply it to define $\langle \vec{p}_n$ as

$$\langle \vec{p}_n| := \langle p_{(1)}; 3|...\langle p_{(n)}; 3|. \tag{3.89}$$

As expected the integrals in 3.82 are taking around the t_i, the location if the i'th puncture (that we chosen to be the origin for each puncture local frame, $t_i = 0$).

The $\mathcal{Z}$ in 3.82 are defined as

$$\mathcal{Z} = \prod_{r=-1}^{+1} Z_r \prod_{r=-1}^{+1} \widetilde{Z}_r, \tag{3.90}$$

where Z_r is

$$Z_r = \sum_{i=1}^{n} \sum_{m=-1}^{\infty} \mathcal{M}_{rm}(z_i)\, b^i_m. \tag{3.91}$$

with the coefficients $\mathcal{M}_{nm}(z_i)$ being

$$\mathcal{M}_{nm}(z_i) = \oint_{t_i=0} \frac{\mathrm{d}t_i}{2\pi i}\, t_i^{-m-2} \Big(h_i'(t)\Big)^{-1} \Big(h_i(t)\Big)^{n+1}, \tag{3.92}$$

and we have similar expressions for $\widetilde{Z}$, but with $b^{(i)}$ replaced by $\tilde{b}^{(i)}$. These expressions are contributions that are introduced to remove the $c_1\tilde{c}_1$ factors in three of the asymptotic string states, in effect dividing out by an $SL(2;\mathbb{R})$ symmetry factor for each of the products in $\mathcal{Z}$. For more details, please refer to [28].

We want now to write these expressions in terms of oscillator modes, as it is particularly suitable for string field theory and also for the ambitwistor string in operator formalism. We define

$$V_{X,P} = \sum_{m,n\geq 0} \sum_{i,j} \mathcal{S}^{mn}(z_i, z_j)\tilde{\alpha}^{(i)}_m \cdot \alpha^{(j)}_n, \tag{3.93}$$

$$V_{\mathrm{gh}} = \sum_{i,j} \sum_{\substack{n\geq 2\\ m\geq -1}} \mathcal{K}_{nm}(z_i, z_j)\, c^{(i)}_n\, b^{(j)}_m + \sum_{i,j} \sum_{\substack{n\geq 2\\ m\geq -1}} \mathcal{K}_{nm}(z_i, z_j)\, \tilde{c}^{(i)}_n\, \tilde{b}^{(j)}_m \tag{3.94}$$

where the functions $\mathcal{S}$ are given by

$$\mathcal{S}_{mn}(z_i, z_j) = \oint \frac{\mathrm{d}t_i}{2\pi i} \oint \frac{\mathrm{d}t_j}{2\pi i} h_i'(t_i) t_i^{-m} t_j^{-n-1} \frac{1}{h_i(t_i) - h_j(t_j)} \tag{3.95}$$

with corresponding ghosts functions given by

$$\mathcal{K}_{nm}(z_i, z_j) = -\oint \frac{\mathrm{d}t_i}{2\pi i} \oint \frac{\mathrm{d}t_j}{2\pi i}\, t_i^{-n+1} t_j^{-m-2} \Big(h_i'(t_i)\Big)^2 \Big(h_j'(t_j)\Big)^{-1} \frac{1}{h_i(t_i) - h_j(t_j)}. \tag{3.96}$$

Again, please refer to [28] for details of their derivation. However, it will be useful to let us illustrate the ideas by showing a method for calculating the functions of the punctures for the case of the conventional string, this method is developed in [28], [25], and [27].

A primary field of dimension d has a standard mode expansion

$$\phi(t) = \sum_n \phi_n \, t^{-n-d}, \tag{3.97}$$

it transforms under a conformal transformation like

$$\phi(t) \to h[\phi(t)] = (h'(t))^d \, \phi(h(t)). \tag{3.98}$$

when $t \to z = h(t)$, h' is the derivative of h with respect to t. We can re-write the field then in terms of the local t coordinates as

$$h[\phi(t)] = \sum_n h[\phi_n] \, t^{-n-d} \tag{3.99}$$

and we can get the coefficients as in any string theory with

$$h[\phi_n] = \oint_{t=0} \frac{\mathrm{d}t}{2\pi i} \, t^{n+d-1} \, h[\phi(t)]. \tag{3.100}$$

Using 3.99 we can write it in terms of the transformed field $\phi(z) = \phi(h(t))$ as

$$h[\phi_n] = \oint_{t=0} \frac{\mathrm{d}t}{2\pi i} \, t^{n+d-1} \, (h'(t))^d \, \phi(h(t)). \tag{3.101}$$

We can apply this to, for example, the dimension one field

$$\partial X^\mu(z) = \sum_n \alpha_n^\mu z^{-n-1} \tag{3.102}$$

to give

$$h[\alpha_{-n}^\mu] = \oint_{t=0} \frac{\mathrm{d}t}{2\pi i} \, t^{n+d-1} \, h'(t) \, \partial X^\mu(h(t)). \tag{3.103}$$

A useful thing to do is to write the X^μ part of the surface state $\langle\Sigma|$ in terms of an oscillator modes expansion, so $\langle\Sigma| = \langle 0|e^{V_X}$ has

$$V_X = \sum_{i,j=1}^{n} \sum_{m,n>0} \mathcal{N}_{mn}(z_i, z_j)\alpha_n^{(i)} \cdot \alpha_m^{(j)}. \tag{3.104}$$

For $m, n > 0$, which comes directly from the applicaation of commutation relations, giving

$$\mathcal{N}_{mn}(z_i, z_j) = \frac{1}{2n}\langle 0|\exp\left(\sum_{k,l}\sum_{p,q>0}\mathcal{N}_{pq}(z_i, z_j)\,\alpha_p^{(k)} \cdot \alpha_q^{(l)}\right)\ \alpha_{-m}^{(i)} \cdot \alpha_{-n}^{(j)}|0\rangle \tag{3.105}$$

has this form because the contributions of p and q only survive when are equal to -m or -n and only the i'th and j'th Fock spaces play a role.

The above expression can we summary written as

$$\langle V_2||\Phi_i\rangle|\Phi_j\rangle \tag{3.106}$$

and can be determined by the two-point function

$$\langle\partial X^{(i)}(z)\partial X^{(j)}(w)\rangle \tag{3.107}$$

By taking

$$\langle\partial X(z)\partial X(w)\rangle = -\eta^{\mu\nu}(z-w)^{-2} \tag{3.108}$$

for $m, n > 0$, we find that

$$\langle h_i[\alpha_{-n}^{\mu(i)}]\, h_j[\alpha_{-m}^{\nu(j)}]\rangle = \frac{1}{n}\oint_0 \frac{\mathrm{d}t_i}{2\pi i}\, t^{-n}\, h_i'(t_i) \oint_0 \frac{\mathrm{d}t_j}{2\pi i}\, t^{-m}\, h_j'(t_j)\, \frac{-\eta^{\mu\nu}}{(h_i(t_i) - h_j(t_j))^2}. \tag{3.109}$$

And other vertex functions are found in a very similar way.

For the ghost part, we use the ghost contraction

$$\langle b(z)c(w)\rangle = (z-w)^{-1} \tag{3.110}$$

and 3.101 it can be show in a similar way that

$$\mathcal{K}_{nm}(z_i, z_j) = -\oint \frac{\mathrm{d}t_i}{2\pi i} \oint \frac{\mathrm{d}t_j}{2\pi i}\, t_i^{-n+1} t_j^{-m-2}\, \left(h_i'(t_i)\right)^2 \left(h_j'(t_j)\right)^{-1}\, \frac{1}{h_i(t_i) - h_j(t_j)} \tag{3.111}$$

with a contribution of the c zero modes is fairly easily found from

$$\int_\Sigma \mathrm{d}^2 z \bar{\partial} b_{\mathrm{cl}} (c_{-1} z^2 + c_0 z + c_1) = \sum_{i=1}^{N} \oint_{z_i} b_{\mathrm{cl}}^{(i)} (c_{-1} z^2 + c_0 z + c_1). \tag{3.112}$$

Now, from the standard expansion

$$b^{(i)}(z) = (h_i'(t_i))^{-2} \sum_n b_n^{(i)} t_i^{-n-2} \tag{3.113}$$

and by changing the integral to local coordinates t_i results in

$$\int_\Sigma \mathrm{d}^2 z\, \bar{\partial} b_{\mathrm{cl}} (c_{-1} z^2 + c_0 z + c_1) = \sum_i \sum_n \mathcal{M}_{nm}(z_i) b_m^{(i)} \mathcal{C}^n \tag{3.114}$$

where $n = -1, 0, +1$, and $\mathcal{C} = (c_{-1}, c_0, c_1)$, with

$$\mathcal{M}_{nm}(z_i) = \oint_{t_i=0} \frac{\mathrm{d}t_i}{2\pi i}\, t_i^{-m-2} (h_i'(t))^{-1} (h_i(t))^{n+1}. \tag{3.115}$$

In a similar way, for a fermionic part, with fermions ψ^μ having

$$\langle \psi^\mu(z) \psi^\nu(w) \rangle = (z - w)^{-1} \tag{3.116}$$

and again using 3.101 results in

$$\mathcal{S}_{nm}(z_i, z_j) = -\oint \frac{\mathrm{d}t_i}{2\pi i} \oint \frac{\mathrm{d}t_j}{2\pi i}\, t_i^{-n-\frac{1}{2}} t_j^{-m-\frac{1}{2}}\, \sqrt{h_i'(t_i) h_j'(t_j)}\, \frac{1}{h_i(t_i) - h_j(t_j)} \tag{3.117}$$

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3.7 Scattering Amplitudes and the Scattering Equations from the Ambitwistor String in Operator Formalism

Given the surface state $\langle\Sigma|$, and on-shell (i.e. BRST-invariant) states $|\Psi_i\rangle \in \mathcal{H}_i$ in the i'th Hilbert space, we can construct a top form on the holomorphic cotangent bundle of the moduli space $T^*\mathcal{M}_n$ as (justification of this will be fully developed in 5.2)

$$\Omega_{|\vec{V}\rangle}(\vec{v}) = \langle\Sigma|B_{n-3}(\vec{v})|\vec{V}\rangle, \tag{3.118}$$

where

$$B_{n-3}(\vec{v}) = \prod_{a=1}^{n-3}\tilde{\mathbf{b}}(\vec{\nu}_a)\prod_{a=1}^{n-3}\mathbf{b}(\vec{\nu}_a)\prod_{a=1}^{n-3}\bar{\delta}\Big(\mathcal{H}(\vec{\nu}_a)\Big), \tag{3.119}$$

with $\mathbf{b}(\vec{\nu}_a)$, $\tilde{\mathbf{b}}(\vec{\nu}_a)$, and $\mathcal{H}(\vec{\nu}_a)$ as defined in (3.91). $|\vec{V}\rangle$ is short hand for the tensor product of asymptotic states $|V_1\rangle \otimes ... \otimes |V_n\rangle$. The forms (3.118) are motivated by similar constructions in [9, 25]. We shall argue in section 5.2 that we may actually formally evaluate this integral over moduli space $\mathcal{M}_n$. Taking the $|V\rangle$ to be on-shell states (3.76) corresponding, via the state-operator correspondence, to vertex operators (3.75). The on-shell scattering amplitude is given by integrating over the moduli space $\mathcal{M}_n$

$$\langle V(z_1), ..., V(z_n)\rangle = \int_{\mathcal{M}_n}\Omega_{|\vec{V}\rangle}(\vec{v}). \tag{3.120}$$

One point of concern might be that, since $\langle\Sigma|$ depends on a choice of local coordinates $z_i = h_i(0)$ centred on each puncture, it is not at all obvious that $\Omega_{|\vec{V}\rangle}(\vec{v})$ is well-defined on $\mathcal{M}_n$. The natural framework to describe $\Omega_{|\vec{V}\rangle}(\vec{v})$ is the bundle over $\mathcal{M}_n$ with fibres given by an independent choice of local coordinates about each puncture. However, provided that the external states $|\vec{V}\rangle$ are on-shell, in other words that they are BRST-invariant, the integrand is invariant under local reparametrisations and so the form $\Omega_{|\vec{V}\rangle}(\vec{v})$ does descend to a well-defined form on $\mathcal{M}_n$. Put another way, it does not matter which section of the bundle we choose to integrate over.

Let us consider the explicit example of the scattering of n on-shell states $|V_i\rangle$, each of the form (3.76). It is straightforward to show (see section 3.8) that

$$\langle\Sigma|\alpha_{-1}^{(i)} = \int \mathrm{d}^n p \; \langle\vec{p}_n| \; \delta\left(\sum p_{(j)}\right) \; e^W \; \sum_{j\neq i}\sum_{n\geq 0} \mathcal{S}_{1n}(z_i,z_j)\alpha_n^{(j)} \; \mathcal{Z}, \tag{3.121}$$

where $\mathcal{S}_{mn}(z_i,z_j)$ is given by (3.95). We also have that

$$\sum_{j\neq i}\sum_{n\geq 0} \mathcal{S}_{1n}(z_i,z_j)\alpha_n^{(j)}|k_j\rangle = \sum_{j\neq i} \frac{k_j}{z_i - z_j} \; |k_j\rangle, \tag{3.122}$$

where we have used the fact that $\alpha_n^{(i)}|k_i\rangle = \delta_{n,0}k_i|k_i\rangle$ for $n \geq 0$ and $\mathcal{S}_{10}(z_i,z_j) = (z_i - z_j)^{-1}$. The $\alpha_n^{(i)}$ for $n \geq 0$ commute with all operators to the right in the expression for the amplitude until they hit the $|k_i\rangle$ of the asymptotic states. Thus, in evaluating the scattering amplitude the net effect is to make the replacement

$$\alpha_{-1}^{(i)} \to \sum_{j\neq i} \frac{k_j}{z_i - z_j}. \tag{3.123}$$

This is the operator statement of the path integral result

$$P_{\mathrm{cl}}(z) = \sum_j \frac{k_j}{z - z_j}, \tag{3.124}$$

which arises when the $X^\mu(z)$ path integral is done (see [1] for details.).

Let us now focus on the ghost terms. The gauge-fixing of the worldsheet complex structure and the gauge symmetry of the Beltrami differential $e(z)$ give the ghost contribution

$$\prod_{a=1}^{n-3} \bar{\delta}\Big(\mathcal{H}(\vec{\nu}_a)\Big)\tilde{\mathbf{b}}(\vec{\nu}_a)\mathbf{b}(\vec{\nu}_a), \tag{3.125}$$

which we recognise as the $B_{n-3}(\vec{v})$ insertion in $\Omega_{|\vec{\Psi}\rangle}(\vec{v})$. The $\mathbf{b}$ and $\tilde{\mathbf{b}}$ insertions are of the standard type. The $\mathcal{H}(\vec{\nu}^a)$ contribution requires more discussion.

A similar calculation to that above (which also may be found in section 3.8) yields

$$\langle\Sigma|\alpha^{(i)}_{-m}\alpha^{(i)}_{-n} = \int \mathrm{d}^n p \, \langle\vec{p}_n| \, \delta\left(\sum p_{(j)}\right) \, e^W \, A^{(i)}_{-m}A^{(i)}_{-n} \, \mathcal{Z}, \tag{3.126}$$

where

$$A^{(i)}_{-m} \equiv \alpha^{(i)}_{-m} + \sum_{j\neq i}\sum_{n\geq 0}\mathcal{S}_{mn}(z_i, z_j)\alpha^{(j)}_n. \tag{3.127}$$

The relationship of the worldsheet vector fields $v(z)$ and the types deformations of the moduli space may be described simply. Given a disc $\mathscr{D}_i$ containing the i'th puncture, and the vector field $v_i(z)$ on the boundary of the disc we can ask if one may smoothly extend $v_i(z)$ outside the disc. Those v that cannot be extended outwards provide interesting deformations. If $v(z)$ vanishes at the puncture it describes coordinate changes that do not affect the location of the punctures. Since these do not have any effect on the moduli space $\mathcal{M}_n$, we ignore these in the on-shell theory; however, they do play a role in the off-shell theory. Those $v(z)$ that do not vanish at the puncture act to move the location of the puncture and so do have an interesting action on $\mathcal{M}_n$. At tree level, the locations of the punctures are the only moduli, so it is this class of vector fields that are of interest to us. For completeness we mention that those vector fields that cannot be extended to the full interior of the disc encode changes in the moduli of the underlying, unmarked, Riemann surface. This classification is nicely summarised in Table 1 of [14]

Let us focus then on those v_i that do not vanish at the point z_i and can be extended into the interior of the disc $\mathscr{D}_i$ surrounding the point z_i. These correspond to deformations that can move the location of the punctures

$$z_i \to z_i + v^a_i \delta\tau_a. \tag{3.128}$$

where τ_a are coordinates on the moduli space. Let us choose a basis for the $v^a_i(z)$ such that three of the punctures are kept fixed while $n-3$ are shifted by an amount given directly by a particular modulus, so that $v^a_i(z_i) = \delta^a_i$ for $i = 1, 2, ..., n-3$ and $v^a_i = 0$ for $i = n-2, n-1, n$.

Using this $v(z)$, we have

$$\mathcal{H}(\vec{\nu}^a) = \sum_{i=1}^{n} \oint \mathrm{d}z \; H^{(i)}(z) \delta_i^a = \widetilde{L}_{-1}^{(a)}. \tag{3.129}$$

Using the identity (3.126), we have

$$\begin{aligned} \langle \Sigma | \widetilde{L}_{-1}^{(a)} &= \sum_{n \geq 0} \langle \Sigma | \alpha_n^{(a)} \cdot \alpha_{-n-1}^{(a)} \\ &= \int \mathrm{d}^n p \; \langle \vec{p}_n | \; \delta \left(\sum p_{(j)} \right) \; \sum_{n \geq 0} \alpha_n^{(a)} \cdot \left(\sum_{j \neq a} \sum_{m \geq 0} \mathcal{S}_{1+n,m}(z_a, z_j) \alpha_m^{(j)} \right) \; e^W \; \mathcal{Z}. \end{aligned} \tag{3.130}$$

It is then straightforward to show that

$$\begin{aligned} \sum_{n \geq 0} \alpha_n^{(a)} \cdot \left(\sum_{j \neq a} \sum_{m \geq 0} \mathcal{S}_{1+n,m}(z_a, z_j) \alpha_m^{(j)} \right) |k_1\rangle ... |k_n\rangle &= \sum_{j \neq a} \frac{k_a \cdot k_j}{z_a - z_j} |k_1\rangle ... |k_n\rangle \\ &= k_a \cdot P_{\mathrm{cl}}(z_a) |k_1\rangle ... |k_n\rangle. \end{aligned} \tag{3.131}$$

Putting this all together gives the result

$$\int \mathrm{d}^n p \; \delta \left(\sum p_j \right) \; \langle p_1 | ... \langle p_n | \; e^{V_{X,P}} \prod_{a=1}^{n-3} \bar{\delta} \Big(\widetilde{L}_{-1}^{(a)} \Big) |k_1\rangle ... |k_n\rangle = \delta \Big(\sum k_i \Big) \prod_{a=1}^{n-3} \left(k_a \cdot P_{\mathrm{cl}}(z_a) \right), \tag{3.132}$$

where $V_{X,P}$ is given by (3.93). This is the required scattering equation and momentum conservation contributions to the amplitude. The final steps in the calculation of the scattering amplitude are straightforward and given in more detail in [28]. We present a quick overview below. The on-shell asymptotic states are given by (3.76). Substituting these into (3.120) gives the scattering amplitude

$$\int_{\mathcal{M}_n} \Omega_{|\vec{\Psi}\rangle}(\vec{v}) = \int_{\mathcal{M}_n} \langle \Sigma | \prod_{i=1}^{3} c_1^{(i)} \tilde{c}_1^{(i)} \epsilon_i^{\mu\nu} \alpha_{-1\mu}^{(i)} \alpha_{-1\nu}^{(i)} \prod_{i=4}^{n} \bar{\delta} \left(k \cdot P_{\mathrm{cl}} \right) \epsilon_i^{\mu\nu} \alpha_{-1\mu}^{(i)} \alpha_{-1\nu}^{(i)} |k_1\rangle ... |k_n\rangle, \tag{3.133}$$

where $n-3$ factors of $c(z)\tilde{c}(z)$ have been absorbed by the $\mathbf{b}(\vec{\nu}^a)\tilde{\mathbf{b}}(\vec{\nu}^a)$ insertions. Taking

$v^a(z_i) = \delta_i^a$ for i=1,...,n-3 and zero otherwise gives

$$\mathbf{b}(\vec{\nu}^a) = \oint \mathrm{d}z b^{(a)}(z) = b_{-1}^{(a)}, \tag{3.134}$$

which removes the $c_1^{(a)}$ insertion on $n-3$ of the external states.

The remaining c and $\tilde{c}$ ghosts are eliminated by the $\mathcal{Z}$ factor in $\langle\Sigma|$. It was shown above that the $\alpha_{-1}^{(i)}$ factors in the external states are converted into $P_{\rm cl}(z_i)$ factors when inserted into correlation functions involving $\langle\Sigma|$. The net result is the bosonic scattering amplitude [1]

$$M_N = \delta^D\left(\sum k_i\right)\int_{\mathcal{M}_N} \mathrm{d}^{N-3}z_i \frac{1}{\mathrm{d}\omega}\prod_{i=1}^{N}\epsilon_i^{\mu\nu}P_{\mathrm{cl}\,\mu}P_{\mathrm{cl}\,\nu}\prod_i{}' \bar{\delta}\left(k_i\cdot P_{\rm cl}\left(z_i\right)\right), \tag{3.135}$$

where

$$\prod_i{}' \bar{\delta}(k_i\cdot P_{\rm cl}(z_i)) = \frac{1}{\mathrm{d}\omega}\prod_{i=4}^{N}\bar{\delta}(k_i\cdot P_{\rm cl}(z_i)). \tag{3.136}$$

with

$$\mathrm{d}\omega = \frac{\mathrm{d}z_i\,\mathrm{d}z_j\,\mathrm{d}z_k}{(z_i-z_j)(z_j-z_k)(z_k-z_i)}. \tag{3.137}$$

As commented upon in [1], this is not the correct tree amplitude for Einstein gravity; however, Einstein *super*gravity is recovered from the $N=2$ supersymmetric extension of (2.12). The arguments above, including the emergence of the scattering equations, apply also in the supersymmetric case.

As a final comment, we note that the fact that, the form $\Omega_{|\vec{V}\rangle}(\vec{v})$ is invariant under diffeomorphisms on the worldsheet and so it is well-defined on the moduli space will become important when we consider correlation functions involving states which are not BRST-invariant. When we consider the string field theory in the next section, we shall want to generalise this discussion to off-shell quantities where the form $\Omega_{|\vec{V}\rangle}(\vec{v})$ will not be invariant under general diffeomorphisms and consequently will not be well-defined on the moduli space $\mathcal{M}_n$. In the off-shell case, a generalisation of the bundle over moduli space will be the framework we will be forced to work with.

3.8 Derivation of the Scattering equations in the Operator Formalism: further details

In this section we give further details on the derivation of the scattering amplitudes given in section 3.7. We only consider the (X, P)-dependent parts and define

$$\langle \Sigma_{X,P}| = \langle p_1|...\langle p_n| e^{V_{X,P}}, \tag{3.138}$$

where $V_{X,P}$ is given by (3.93). Consider

$$\langle \Sigma_{X,P}|\alpha^{(i)}_{-p} = \langle p_1|...\langle p_n|\Big(\alpha^{(i)}_{-p} + [V_{X,P}, \alpha^{(i)}_{-p}] + \frac{1}{2!}[V_{X,P}, [V_{X,P}, \alpha^{(i)}_{-p}]] + ...\Big) e^{V_{X,P}}. \tag{3.139}$$

The first commutator is

$$[V_{X,P}, \alpha^{(i)}_{-p}] = \sum_{j\neq i}\sum_{n\geq 0} \mathcal{S}_{pn}(z_i, z_j)\alpha^{(j)}_n, \tag{3.140}$$

Since $\mathcal{S}_{mn}(z_i, z_j) = 0$ for $m \geq 1$ and $n \geq 0$, this requires that this commutator is only non-zero if $p > 0$. We note also that $[V_{X,P}, [V_{X,P}, \alpha^{(i)}_{-p}]]$ and all higher commutators vanish, leaving

$$\langle \Sigma_{X,P}|\alpha^{(i)}_{-p} = \begin{cases} \langle p_1|...\langle p_n| \sum_{j\neq i}\sum_{n\geq 0} \mathcal{S}_{pn}(z_i, z_j)\alpha^{(j)}_n \ e^{V_{X,P}}, & p > 0, \\ \langle p_1|...\langle p_n|\alpha^{(i)}_{-p} \ e^{V_{X,P}} & p \leq 0. \end{cases} \tag{3.141}$$

Since $\alpha^{(i)}_p$ commutes with $V_{X,P}$ for $p > 0$, we then have

$$\langle \Sigma|\alpha^{(i)}_{-p} = \langle \Sigma| \sum_{j\neq i}\sum_{n\geq 0} \mathcal{S}_{pn}(z_i, z_j)\alpha^{(j)}_n, \qquad p > 0. \tag{3.142}$$

where we have replaced $\langle\Sigma_{X,P}|$ with the full surface state $\langle\Sigma|$. It is useful to note that, if we contract with a momentum eigenstate $|k_j\rangle$

$$\langle k_j|e^{V_{X,P}}\alpha^{(i)}_{-p}|k_j\rangle = \sum_{j\neq i}\mathcal{S}_{p0}(z_i,z_j)k_j \tag{3.143}$$

for $p>0$ and zero otherwise. Thus we find the result

$$\langle k_j|e^{V_{X,P}}\alpha^{(i)}_{-1}|k_j\rangle = \sum_{j\neq i}\frac{k_j}{z_i-z_j}, \tag{3.144}$$

which is simply the classical momentum $P_{\mathrm{cl}}(z)$. A computation similar to this is used to show that the α_{-1} insertions in the on-shell states of the scattering amplitude give rise to $P_{\mathrm{cl}}(z)$ in the final expression for the amplitude (3.135).

A second, related identity may be proven along the same lines:

$$\langle\Sigma_{X,P}|\alpha^{(i)}_{-p}\cdot\alpha^{(i)}_{-q} \;=\; \langle p_1|...\langle p_n|\left(\alpha^{(i)}_{-p}\cdot\alpha^{(i)}_{-q}+\alpha^{(i)}_{-q}\cdot S^{(i)}_p+\alpha^{(i)}_{-p}\cdot S^{(i)}_q+S^{(i)}_p\cdot S^{(i)}_q\right)\, e^{V_{X,P}} \tag{3.145}$$

where

$$S^{(i)}_p := \sum_{j\neq i}\sum_{n\geq 0}\mathcal{S}_{pn}(z_i,z_j)\alpha^{(j)}_n \tag{3.146}$$

Chapter 4

Bosonic Ambitwistor String Field Theory: Free Theory

From this chapter we start describing the ambitwistor string field theory for supergravity as it was developed in [3], and we will closely follow it. It is important to point out that the interacting part of it is in an initial phase of development, in particular the form of the propagator has issues yet to be resolved regarding its consistency an interpretation 5.3.

For now, lets start with the free theory. As mentioned before the fundamental differences between the conventional and ambitwistor string field theories are in the bosonic part. However, we should point out that we consider only the Neveu-Schwarz (NS) sector in the small Hilbert space approach, so in this case extending to the supersymmetric case is quite straightforward. We haven't consider the Ramond sector, but we would expect extra complications if we do so.

4.1 Closed String Field Theory: generalities

From BRST-invariance we obtained the spacetime equations of motion for ambitwistor fields on-shell, i.e. $Q|\Psi\rangle = 0$. Now, for string field theory we want to also consider off-shell fields, i.e. that do not satisfy that condition. BRST invariance will be a consequence of the classical equations of motion of the string field action $S[\Psi]$.

As it is usual in introductory descriptions of string field theory, it is easier to start with the covariant open string field [29]. A well chosen background is associated with a worldsheet conformal field theory giving a BRST operator Q. The on-shell states are solutions to $Q|\Psi\rangle = 0$. This can be derived from the action $S_2[\Psi] = \frac{1}{2}\langle\Psi|Q|\Psi\rangle$. To add interactions, we perturbate the chosen background and so include them in the string field action. In Witten's open string a cubic term constructed using the three-punctured sphere surface state $\langle\Sigma_3|$ (denoted $\{\Psi^3\} = \langle\Sigma_3||\Psi\rangle|\Psi\rangle|\Psi\rangle$), is enough to reproduce perturbation at tree level. The string field action is

$$S[\Psi] = \langle\Psi|Q|\Psi\rangle + \frac{g}{3!}\langle\Sigma_3||\Psi\rangle|\Psi\rangle|\Psi\rangle \tag{4.1}$$

We then gauge fix with the Siegel gauge $b_0|\Psi\rangle = 0$ as the standard choice. The quadratic term reduces to $\langle\Psi|Q|\Psi\rangle \to \langle\Psi|c_0L_0|\Psi\rangle$ and the propagator has the form

$$\frac{b_0}{L_0} = b_0\int_0^\infty \mathrm{d}\tau\; e^{-\tau L_0}, \tag{4.2}$$

where τ becomes a real modulus that contributes to the moduli spaces of higher point Riemann surfaces with boundary. It turns out that the open string field theory does not seems to apply to the ambitwistor case, but does the closed string field theory. In it the quadratic action has a ghost in it

$$S_2[\Psi] = \langle\Psi|c_0^-Q|\Psi\rangle, \tag{4.3}$$

where $c_0^- = c_0 - \bar{c}_0$. As a consequence we want all but the c_0^- parts of the BRST operator to annihilate $|\Psi\rangle$ and so the c_0^- part dependent BRST charge will vanish in an off-shell constraint on the string field $|\Psi\rangle$. This is the condition $(L_0 - \overline{L}_0)|\Psi\rangle = 0$, under $(b_0 - \bar{b}_0)|\Psi\rangle = 0$ level matching. For an interacting action, we add non-linear terms. Details on conventional string field theory may be found in [12–14] and also [27,30–32]. More recent advances in superstring field theory can be found in [15, 33].

4.2 Bosonic Ambitwistor String Field Theory: action and symmetries

As mentioned the ambitwistor string field theory seems to be suited to a closed string field theory treatment. This is because its perturbative structure seems to rely on closed string moduli space and so the covariant theory must be non-polynomial. We will start with a linearised model.

The target space gauge symmetry $\delta|\Psi\rangle = Q|\Lambda\rangle$ for the action and with equation of motion that asks for the condition $Q|\Psi\rangle = 0$. For that two conditions will be asked for the string field $|\Psi\rangle$. First $\mathcal{L}_0|\Psi\rangle = 0$, where $\mathcal{L}_0$ is that part of Q which multiplies c_0 (3.69). The other conditions from $Q|\Psi\rangle = 0$ are imposed by the linearised equations of motion. Then, $\mathcal{L}_0|\Psi\rangle = 0$ will turn out ot be a background independent constraint (a type of level-matching condition). While the background-dependent parts of the BRST condition will come from the equations of motion. Then non-linear interaction terms will introduce change in the equations of motion $Q|\Psi\rangle + ... = 0$ and gauge transformations $\delta|\Psi\rangle = Q|\Lambda\rangle + ...$ but not in the constraint $\mathcal{L}_0|\Psi\rangle = 0$. The string action has form of a typical closed bosonic string field theory

$$S[\Psi] = \langle\Psi|c_0 Q|\Psi\rangle + \sum_{n>2} \frac{1}{n!}\{\Psi^n\}, \tag{4.4}$$

where the interaction terms $\{\Psi^n\}$ will be explained soon. The quadratic part includes the $c_0 Q = c_0(c_0\mathcal{L}_0 + \tilde{c}_0\widetilde{L}_0 + ...)$ insertion. As c_0 is Grassmann, then the $\mathcal{L}_0$ term drops out of the quadratic term. The condition $\mathcal{L}_0|\Psi\rangle = 0$ will be part of the definition of an ambitwistor string field. And the condition $b_0|\Psi\rangle$ has a natural origin on $\{Q, b_0\} = \mathcal{L}_0$.

The action has target space gauge invariance (see [14] and [3] for more details on its relation to the so called *main identity*)

$$\delta|\Psi\rangle = Q|\Lambda\rangle + \sum_n \frac{1}{n!}|[\Lambda, \Psi_1, ..., \Psi_n]\rangle, \tag{4.5}$$

with $|\Lambda\rangle$ a gauge parameter field. The BV procedure [14] is used to treat this gauge invariance. But we will only consider the linearised symmetries, this will be described in 4.3.

Here the role of gauge-fixing is to simplify the theory and acts as a prerequisite for tree-level perturbation theory. As we are dealing with classical supergravity in ten-dimensions and so no quantisation of the target space actually exist.

A modification of Siegel gauge for the ambitwistor string field is

$$\tilde{b}_0|\Psi\rangle = 0. \tag{4.6}$$

from which we have a kinetic term for the string field theory

$$\langle\Psi|c_0 Q|\Psi\rangle = \langle\Psi|c_0\tilde{c}_0\widetilde{L}_0|\Psi\rangle, \tag{4.7}$$

where we use $\{Q, \tilde{b}_0\} = \widetilde{L}_0 + ...$, with the ellipsis denoting ghost terms that vanish on $|\Psi\rangle$. Next we have the condition $\mathcal{L}_0|\Psi\rangle = 0$, where $\mathcal{L}_0$ is that part of the BRST current that multiplies c_0, and $b_0|\Psi\rangle = 0$. In this gauge, the propagator has the form

$$\frac{\delta(\mathcal{L}_0)}{\widetilde{L}_0}\tilde{b}_0 b_0|\mathcal{R}_{LR}\rangle, \tag{4.8}$$

where $|\mathcal{R}_{LR}\rangle$ is the reflector state that relates the string fields in two Hilbert spaces and their conjugates as $\langle\Psi_L|\mathcal{R}_{LR}\rangle = |\Psi_R\rangle$. We will describe it in detail later.

We now explain why (4.4) requires to be non-polynomial. You can find a detailed description in section 5 of [14]. We want to find the Feynman rules for the ambitwistor string that means finding a minimal set of vertices that together with the propagator can be used to construct a single cover of the moduli space of punctured Riemann surfaces $\mathcal{M}_n$. If we start with the three-punctured sphere as basic building block its moduli space $\mathcal{M}_3$ is just a point. Then follow with constructing all possible four-punctured Riemann surfaces, $\mathcal{M}_4$ (the Riemann sphere), from sewing two three-point surfaces with a propagator with the two moduli coming from the propagator. IT was shown in [34–38], that it will be a fundamental region missing, $\mathcal{D}_4 \subset \mathcal{M}_4$, so it must be added in as a fundamental 4-point interaction [31]. Then the action of the closed theory must have a quadratic term, a cubic interaction $\{\Psi^3\}$ as in (4.1), and also add in a quartic interaction $\{\Psi^4\}$ encoding the missing region $\mathcal{D}_4$. This result generalises to more punctures (see [3] and references in there for details). The string vertex $\mathcal{V}_n$ is defined to be the set of Riemann surfaces in $\mathcal{D}_n$ with a choice of coordinate, up to a phase, around each puncture [14]. The fact that there are an infinite number of $\mathcal{V}_n$ that must be introduced gives rise to the non-polynomial structure of closed string field theory. See [14, 25] for more details.

4.3 The Ambitwistor string field

We now construct the ambitwistor string field theory with focus on the case of flat, empty spacetime and look for a string field that describes small, perturbative, fluctuations on that spacetime. The state operator correspondence gives the perturbative, momentum eigenstate, 'graviton' with polarisation $\varepsilon_{\mu\nu}$ as

$$|\Psi\rangle = \varepsilon_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}c_1\tilde{c}_1|k\rangle, \tag{4.9}$$

where $|k\rangle = e^{ik\cdot x}|0\rangle$. A more general state is given by a linear superposition of such states, weighted with a function $h_{\mu\nu}(k)$. Then we start with a string field

$$|\Psi\rangle = \int \mathrm{d}k \left(-\frac{1}{2}h_{\mu\nu}(k)\,\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}c_1\tilde{c}_1 + ...\right)|k\rangle, \tag{4.10}$$

where $h_{\mu\nu}$ is a function of the momentum k and $+...$ denote terms to be determined by the symmetries of the theory. The linearised gauge transformation is $\delta|\Psi\rangle = Q|\Lambda\rangle$ for a parameter field $|\Lambda\rangle$. And to have the linearised spacetime diffeomorphisms $\delta h_{\mu\nu} = \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu$ as a symmetry, we require a gauge field

$$|\Lambda\rangle = i \int \mathrm{d}k \; \lambda_\mu(k) \; \alpha^\mu_{-1} c_1 |k\rangle. \tag{4.11}$$

Using the BRST charge (3.68) we have under a linearised gauge transformation $\delta|\Psi\rangle = Q|\Lambda\rangle$ that

$$Q|\Lambda\rangle = \int \mathrm{d}k \left(\frac{i}{2} \tilde{c}_0 c_1 \, \alpha_0^2 \, \lambda_\mu(k) \, \alpha^\mu_{-1} + i\tilde{c}_1 c_1 \, \alpha_{0\mu} \lambda_\nu(k) \alpha^\mu_{-1} \alpha^\nu_{-1} + i c_{-1} c_1 \, \alpha_0^\mu \lambda_\mu(k) \right) |k\rangle. \tag{4.12}$$

This gives the correct (momentum space) variation for $h_{\mu\nu}(k)$. It can be read off from the $\alpha^\mu_{-1}\alpha^\nu_{-1} c_1 \tilde{c}_1$ coefficient. There are also terms proportional to $\tilde{c}_0 c_1$ and $c_{-1} c_1$, which have no origin in the first terms of (4.10) and so must correspond to the variation of terms denoted by $+...$ in (4.10). We introduce fields $f_\mu(k)$ and $e(k)$ to provide origins for these terms. A simple guess for the string field is

$$|\Psi\rangle = \int \mathrm{d}k \left(-\frac{1}{2} h_{\mu\nu}(k) \, \alpha^\mu_{-1} \alpha^\nu_{-1} c_1 \tilde{c}_1 + \frac{1}{2} e(k) \, c_{-1} c_1 + i f_\mu(k) \, \alpha^\mu_{-1} \tilde{c}_0 c_1 \right) |k\rangle. \tag{4.13}$$

We Identify $\alpha_{0\mu}|k\rangle = k_\mu |k\rangle$ and Fourier transforming to configuration space, the linearised gauge transformations can be found to be

$$\delta h_{\mu\nu}(x) = \partial_\mu \lambda_\nu(x) + \partial_\nu \lambda_\mu(x), \qquad \delta f_\mu(x) = -\frac{1}{2} \Box \lambda_\mu(x), \qquad \delta e(x) = 2\partial^\mu \lambda_\mu(x). \tag{4.14}$$

As $tr(\delta h_{\mu\nu}) = \delta e$, we will identify

$$e(x) = \eta^{\mu\nu} h_{\mu\nu}(x). \tag{4.15}$$

In terms of worldsheet fields, the string field may be written as an off-shell conformal field theory field

$$\Psi(z) = \int \mathrm{d}k \left(-\frac{1}{2} h_{\mu\nu}(k)\ P^\mu P^\nu c\tilde{c} + \frac{1}{2} e(k)\ \partial^2 cc + i f_\mu(k)\ P^\mu \partial \tilde{c} c \right) e^{ik\cdot X}, \tag{4.16}$$

where (4.13) and (4.16) are related by

$$|\Psi\rangle = \lim_{z\to 0} \Psi(z)|0\rangle. \tag{4.17}$$

This string field will be enough for the quadratic action defined in the next section. The gauge transformation of the worldsheet field is

$$\delta\Psi(z) = \oint_{\mathcal{C}} \mathrm{d}\omega j(\omega)\Psi(z). \tag{4.18}$$

where the contour $\mathcal{C}$ surrounds the point $\omega = z$. It can be obtained from OPEs (2.13), where the $\tilde{b}$ and $\tilde{c}$ ghosts satisfy the same OPE as the b and c ghosts reproducing the result (4.14).

4.4 The quadratic action

In closed bosonic string theory we need tp construct a quadratic term with the correct ghost number. A $\langle\Psi|Q|\Psi\rangle$ does not have the correct ghost number, but the quadratic term $\langle\Psi|c_0^- Q|\Psi\rangle$, where $c_0^\pm = c_0 \pm \bar{c}_0$, does. We must put as a condition $L_0^-|\Psi\rangle = 0$ as an additional constraint on the string field and a further condition $b_0^-|\Psi\rangle = 0$. The $L_0^- = 0$ condition is level matching. In Siegel gauge $b_0^+|\Psi\rangle = 0$ the quadratic action is $\langle\Psi|c_0^- c_0^+ L_0^+|\Psi\rangle$ and the linearised equation of motion gives $L_0^+|\Psi\rangle = 0$. In the ambitwistor string field theory a role reversal occurs, the role of c_0^- and L_0^- in the conventional closed string field theory are played by c_0 and $\mathcal{L}_0$ respectively, with $\mathcal{L}_0$ given by (3.69). In the ambitwistor string field theory then, the kinetic term is

$$S_2[\Psi] = \langle\Psi|c_0 Q|\Psi\rangle = \langle R_{LR}|c_0 Q|\Psi_L\rangle|\Psi_R\rangle. \tag{4.19}$$

under conditions

$$\mathcal{L}_0|\Psi\rangle = 0, \qquad b_0|\Psi\rangle = 0. \tag{4.20}$$

It forms part of the definition of the string field $|\Psi\rangle$. The equation of motion will depend on the spacetime metric through the $H(z)$ dependence in Q and has perturbative corrections with non-linear interaction terms.

4.5 Recovering the Fierz-Pauli action

We now want to see that the bosonic theory gives the standard Fierz-Pauli action of linearised gravity at quadratic order. At higher order the bosonic ambitwistor string field theory does not reproduce Einstein gravity as the on-shell scattering amplitudes, from which the surface states $\langle\Sigma|$ are constructed are not of Einstein gravity [1]. We take the quadratic action (4.19). We note that for the bosonic theory we don't take $\langle\Psi|$ as the usual BPZ conjugate of $|\Psi\rangle$. The standard BPZ conjugate is

$$\int \mathrm{d}k\langle\text{-}k| \left(-\frac{1}{2}h_{\mu\nu}(k)\ \alpha_1^\mu\alpha_1^\nu c_{-1}\tilde{c}_{-1} + \frac{1}{2}e(k)\ c_1c_{-1} - if_\mu(k)\ \alpha_1^\mu\tilde{c}_0c_{-1}\right). \tag{4.21}$$

For the ambitwistor case we need to define $\langle\Psi|$ as

$$\langle\Psi| = \int \mathrm{d}k\langle\text{-}k| \left(-\frac{1}{2}h_{\mu\nu}(k)\ \tilde{\alpha}_1^\mu\tilde{\alpha}_1^\nu \tilde{c}_{-1}c_{-1} + \frac{1}{2}e(k)\ \tilde{c}_1\tilde{c}_{-1} - if_\mu(k)\ \tilde{\alpha}_1^\mu\tilde{c}_0\tilde{c}_{-1}\right). \tag{4.22}$$

The reason is that the standard BPZ conjugate does not give a non-trivial quadratic action. Note that this is the only place where $\langle\Psi|$ appears in the action. WE want to find the reflector operator state. We relate these two conjugates by introducing an operator $\mathcal{O}$ that maps oscillator operators $\mathcal{O} : (\alpha_{\pm1}, c_{\pm1}, \tilde{c}_{\pm1}) \to (\tilde{\alpha}_{\pm1}, \tilde{c}_{\pm1}, c_{\pm1})$, having no effect on the $(\alpha_0, c_0, \tilde{c}_0)$. This is a non-standard inner product that is included in the bosonic theory only. The supersymmetric theory discussed later utilises the standard BPZ conjugate.

We have the c_0 term in the action so the action has the correct ghost number for a closed string field theory. It also projects out the $c_0\mathcal{L}_0|\Psi\rangle = 0$ part of the BRST constraint. The condition $\mathcal{L}_0|\Psi\rangle = 0$ is imposed separately.

By substituting (4.13) into (4.19), using the commutation relations and imposing the normalisation

$$\langle k'|\tilde{c}_{-1}\tilde{c}_0\tilde{c}_1 c_{-1}c_0c_1|k\rangle = \delta(k-k'), \tag{4.23}$$

we get

$$\begin{aligned} S_2[\Psi] \;\; &= \;\; \int \mathrm{d}k \left(-\frac{1}{4}h_{\mu\nu}(-k)k^2 h^{\mu\nu}(k) + 2ih_{\mu\nu}(-k)k^\mu f^\nu(k) + \frac{1}{8}e(-k)k^2 e(k) \right. \\ &\quad \left. -ie(-k)k^\mu f_\mu(k) - 2f_\mu(-k)f^\mu(k) \right). \end{aligned} \tag{4.24}$$

The $f_\mu(k)$ are auxiliary fields to be integrated out (have no kinetic term). In configuration space the linearised action is

$$S_2[h,e] = \int \mathrm{d}x \left(\frac{1}{4}h_{\mu\nu}\Box h^{\mu\nu} + 2h_{\mu\nu}\partial^\mu f^\nu - \frac{1}{8}e\Box e - e\partial^\mu f_\mu - 2f_\mu f^\mu \right), \tag{4.25}$$

where all fields are functions of x. The f_μ equation of motion is

$$f_\mu = -\frac{1}{2}\left(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu e \right). \tag{4.26}$$

They can be seen to make sense by looking to the gauge transformation of the components of the string field (4.14). If we now substitute (4.26) back in for f_μ in the action and integrate by parts as needed, we get

$$S_2[h] = \int \mathrm{d}x \left(\frac{1}{4}h_{\mu\nu}\Box h^{\mu\nu} + \frac{1}{2}(\partial^\nu h_{\mu\nu})(\partial_\lambda h^{\mu\lambda}) + \frac{1}{2}h\partial_\mu\partial_\nu h^{\mu\nu} - \frac{1}{4}h\Box h \right), \tag{4.27}$$

where we have imposed the identification (4.15) of $e(x)$ with the trace of the metric fluctuation $h := \eta^{\mu\nu}h_{\mu\nu}$. The action (4.27) is the Fierz-Pauli action [39] for linearised gravity. Note that we are using a background Minkowski metric $\eta_{\mu\nu}$ and its inverse to lower and raise indices.

Chapter 5

Bosonic Ambitwistor String Field Theory: Interactions

This is a crucial chapter. We describe correlation functions as forms $\Omega_{|\vec{\Psi}\rangle}(\vec{\nu})$ in a bundle over moduli space. This is the basic ingredient in the interaction terms $\{\Psi^n\}$. These forms (3.118) are constructed using on-shell asymptotic string states that play a central role in the study of the on-shell scattering amplitudes seen in section 3.7. In this chaper we generalise this objects to off-shell correlation functions as these are one of the central components in the construction of the $\{\Psi^n\}$ terms. We describe it first in the conventional bosonic string field theory [14] and then for the bosonic ambitwistor string.

5.1 Interactions in conventional bosonic string field theory

Tangent vectors to the moduli space V^a, where $a = 1, 2, ..., n-3$, are convenient for describing deformations of the worldsheet theory at the level of the moduli space. These deformations may also be considered at the level of the worldsheet. The worldsheet vector fields $v_i^a(z)$ about the i'th puncture on Σ also change the moduli. So, we can think of the $\vec{\nu}^a = (v_1^a, ..., v_n^a)$ as functions of the V^a. Given a set of deformations $\vec{\nu}^a$ corresponding to tangent vectors V^a

of the moduli space $\mathcal{M}_n$, we can define a correlation function $\Omega_{|\vec{\Psi}\rangle}$ by

$$\Omega_{|\vec{\Psi}\rangle}(\vec{\nu}) = \langle\Sigma|\mathbf{b}(\vec{\nu}_1)...\mathbf{b}(\vec{\nu}_{2n-6})|\vec{\Psi}\rangle, \tag{5.1}$$

where $|\vec{\Psi}\rangle$ means a product of n states $|\Psi_i\rangle$. The ghost insertions are

$$\mathbf{b}(\vec{\nu}^a) = \sum_{i=1}^{n}\left(\oint \mathrm{d}z\, b^{(i)}(z)v_i^a(z) + \oint \mathrm{d}\bar{z}\,\bar{b}^{(i)}(\bar{z})\bar{v}_i^a(\bar{z})\right). \tag{5.2}$$

Note that, as seen from the moduli space, $\Omega_{|\vec{\Psi}\rangle}(\vec{\nu})$ is a multilinear function of $2n-6$ tangent vectors to $\mathcal{M}_n$. Note that in v_i^a there are also complex conjugate fields $\bar{v}_i^a$. While in the ambitwistor string the $\bar{v}_i^a$ do not appear. We can think of $\Omega_{|\vec{\Psi}\rangle}(\vec{\nu})$ as a top form on the moduli space. If the states in $|\vec{\Psi}\rangle$ are on-shell, then the form $\Omega_{|\vec{\Psi}\rangle}(\vec{\nu})$ in (5.1) is a well-defined top form on the moduli space $\mathcal{M}_n$ and the integral of $\Omega_{|\vec{\Psi}\rangle}(\vec{\nu})$ over $\mathcal{M}_n$ is a well-defined object. This is the case in first quantised string theory and is a key foundation of the operator formalism for the conventional bosonic string [25]. A detailed discussion of how the form (3.118) involves a measure on moduli space, when the states are on-shell [25].

In the case when the states in $|\vec{\Psi}\rangle$ are not on-shell, $\Omega_{|\vec{\Psi}\rangle}(\vec{\nu})$ depends on the local coordinates t_i defined about the punctures where the states $|\Psi_i\rangle$ are inserted. $\Omega_{|\vec{\Psi}\rangle}$ is then not well-defined on the moduli space. However, $\Omega_{|\vec{\Psi}\rangle}$ is well defined on $\mathcal{P}_n$ i.e., the bundle over moduli space with base $\mathcal{M}_n$ and infinite-dimensional fibres given by the choice of local coordinate at each puncture. Moreover, $\Omega_{|\vec{\Psi}\rangle}$ descends to a well-defined form on the bundle $\widehat{\mathcal{P}}_n$ over $\mathcal{M}_n$ with infinite-dimensional fibres $\mathscr{T}$ given by a choice of local coordinate about each puncture up to a puncture-dependent phase $t_i \sim e^{i\theta_i}t_i$.

$$\begin{array}{ccc} \mathscr{T} & \hookrightarrow & \widehat{\mathcal{P}}_n \\ & & \downarrow \\ & & \mathcal{M}_n \end{array} \tag{5.3}$$

For more details see [14]. As $\widehat{\mathcal{P}}_n$ is infinite-dimensional, $\Omega_{|\vec{\Psi}\rangle}(\vec{\nu})$ is no longer a top form but it can be integrated over $2n-6$ dimensional regions of $\widehat{\mathcal{P}}_n$.

5.2 Ambitwistor interactions as forms, the basics

Here we will show the foundation of what was used in earlier chapters of this thesis when we described the first quantised ambitwistor string, and also serves as justification of what we said in the previous and following sections.

In [9] an alternative view on the scattering equations and how they arise in the ambitwistor string was proposed. For our ambitwistor model offers a useful framework for describing the string field interactions. We now show the main ideas and more details can be found in [9]. The observation that the algebra (2.21) relates to $T^*\mathcal{M}$, rather than simply $\mathcal{M}$, plays a central role. Following [40], [9] generalises the BRST operator in such a way that $\{Q,\mu\} = \delta\mu$ and $\{Q,e\} = \delta e$, with $\delta\mu$ and δe are anti-commuting fields. We have that μ and $\delta\mu$ depend on the coordinates of the base of $T^*\mathcal{M}$ while being independent of the fibre directions. On the other hand, e and δe vary as we move in the fibre and the base. The action is not invariant under this extended BRST transformation, but under an extension that results in having an action invariant under a generalised BRST transformation. The invariant action may be written as

$$S_0 + \{Q, \mathcal{W}\}, \tag{5.4}$$

where $S_0 = \int P_\mu \bar{\partial} X^\mu + b\bar{\partial}c + \tilde{b}\bar{\partial}\tilde{c}$. With the definitions we have used in chapter 3, $\mathcal{W} = \mathbf{b}(u) + \tilde{\mathbf{b}}(u,\tilde{u})$, where the arguments reflect the dependence of $\mathbf{b}$ and $\tilde{\mathbf{b}}$ on the base and fibre directions of $T^*\mathcal{M}$. In [9] is S_0 that is used to compute correlation functions and $\{Q,\mathcal{W}\}$ is treated as an operator insertion. The scattering amplitudes are given by

$$M_n = \int_{\Gamma\subset T^*\mathcal{M}} \widetilde{\Omega}_{|\vec{V}\rangle}(u,\tilde{u}), \tag{5.5}$$

where

$$\widetilde{\Omega}_{|\vec{V}\rangle}(u,\tilde{u}) = \left\langle e^{-\{Q,\mathbf{b}(u)+\tilde{\mathbf{b}}(u,\tilde{u})\}}\ V_1 ... V_n \right\rangle_{S_0}, \tag{5.6}$$

is a *middle-dimensional form* on $T^*\mathcal{M}$. Γ is a suitably chosen middle-dimensional cycle in $T^*\mathcal{M}$ and the V_i are vertex operators. Note that in this approach $T^*\mathcal{M}$ has the central role instead of $\mathcal{M}$. This is seen in the semi-direct product form of the worldsheet theory gauge algebra (2.21). The question is how to choose a suitable $\Gamma \subset T^*\mathcal{M}$. The correlation function is evaluated using S_0 as the classical action in the path integral. The pair of vectors u and $\tilde{u}$ generalise v and denote deformations along the base and fibre directions of $\Gamma \cap T^*\mathcal{M}$ respectively. The $e^{-\{Q,\mathcal{W}\}}$ factor in the correlation function $\widetilde{\Omega}_{|\vec{V}\rangle}$ marks standard localisation arguments that apply to evaluate the integral. Moreover, only the critical points $\tau^* \in T^*\mathcal{M}$ of an appropriate Morse function are required, not the detailed form of Γ (see B, [41] and [9] for further details). If we only talk about tree level, each critical point associates to the location of $n-3$ punctures that are solutions to the scattering equations. These critical points satisfy two conditions: the first imposes the scattering equations, the second selects a point in each of the fibres of $T^*\mathcal{M}$. By computing M_n on the critical points τ^* we get

$$M_n = \sum_{\tau^*} \left\langle \prod_{a=1}^{n-3} \mathbf{b}(v_a) \prod_{a=1}^{n-3} \tilde{\mathbf{b}}(v_a) (\det\Phi)^{-1}\ V_1 ... V_n \right\rangle_{S_0}. \tag{5.7}$$

This formal expression for the amplitude can be written as an integral over a copy of $\mathcal{M}$ in $T^*\mathcal{M}$ with coordinates τ, with delta-functions introduced in the required critical points τ^*. In [9] it is shown that the copy of $\mathcal{M}$ as the base of $T^*\mathcal{M}$ does not work. The delta functions have support exactly on the solutions to the scattering equations and so the amplitude is equivalently written in the form (3.135). The advantage of the expression (3.135) is that it is written as an integral over $\mathcal{M}$ and seems similar to the conventional string theory. Still, the (5.5) point of view is useful when we consider interactions of the associated string field theory. We'd like to see how the discussion of [9] might work in the operator formalism. In the operator formalism a surface state $\langle\Sigma|$ works such that

$$M_n = \int_{\mathcal{M}} \langle\Sigma|B_{n-3}|V_1\rangle...|V_n\rangle. \tag{5.8}$$

where B_{n-3} is given by (3.119). We can separate off the delta-functions in B_{n-3} and use them to do the integral over $\mathcal{M}$. The delta-functions have support on the scattering equations and they can be solved to give the collection of marked points on Σ, denoting them by τ^*. The amplitude becomes

$$M_n = \sum_{\tau^*} \langle\Sigma^*| \prod_{a=1}^{n-3} \mathbf{b}(v_a^*) \prod_{a=1}^{n-3} \tilde{\mathbf{b}}(v_a^*) \ (\det\Phi)^{-1}|V_1\rangle...|V_n\rangle, \tag{5.9}$$

where $\langle\Sigma^*|$ is the surface state evaluated $\tau^* \in \mathcal{M}$. Φ is the Jacobian matrix that appears when evaluating the integral on the delta-functions. The scattering equations are functions of the moduli, so there will be a Jacobian factor when evaluating the integral against the delta functions. We incorporate this Jacobian into the definition of $\langle\Sigma^*|$. We could construct a surface state from the amplitude for suitable localisation procedure described in [9]. Such a surface state $\langle\widetilde{\Sigma}|$ would satisfy

$$M_n = \int_{\Gamma\subset T^*\mathcal{M}} \langle\widetilde{\Sigma}||V_1\rangle...|V_n\rangle, \tag{5.10}$$

where $\langle\widetilde{\Sigma}|$ takes the same form as $\langle\Sigma|$ but also incorporates the $e^{-\{Q,\mathbf{b}\}}e^{-\{Q,\tilde{b}\}}$ insertion as part of its definition. The operator formalism tells us how to construct the appropriate worldsheet correlation function and then we have to choose a region where we integrate. This integration is then performed as described in [9]. Using localisation to evaluate the operator expression $\langle\widetilde{\Sigma}||V_1\rangle...|V_n\rangle$ directly is not really efficient. It is much more useful to work in terms of the worldsheet correlation function $\left\langle e^{\{Q,\mathcal{W}\}}V_1...V_n\right\rangle_{S_0}$. In any case, one could always choose to express the interaction terms in terms of worldsheet correlation functions.

Now, we want to describe how it works for the ambitwistor string, emphasizing the main features. We want to arrive to (5.1) for the ambitwistor, so we do not want to include an anti-holomorphic sector as was the case for the conventional bosonic string. For the right scattering amplitude (3.135), we need to include a string of $n-3$ $\tilde{\mathbf{b}}(\vec{\nu}^a)$ ghost insertions. We also need to include the same number of $\bar{\delta}(\mathcal{H}(\vec{\nu}^a))$ insertions. This suggests the generalisation of (5.1) to

$$\Omega_{|\vec{\Psi}\rangle}(\vec{\nu}) = \langle\Sigma|B_{n-3}(\vec{\nu})|\vec{\Psi}\rangle \tag{5.11}$$

where $B_{n-3}(\vec{\nu})$ is given by (3.119). For on-shell momentum eigenstates $|\Psi\rangle$, this is just the integrand of the on-shell scattering amplitude (3.135) which is a form on $T^*\mathcal{M}_n$. The $\tilde{\mathbf{b}}$ insertions describe the moduli associated with the Beltrami differential $e(z)$ in the worldsheet theory. These additional directions would then describe a space $\mathcal{N}_n \subset T^*\mathcal{M}_n$ which is the bundle over $\mathcal{M}_n$

$$\mathcal{N}_n \xrightarrow{\pi} \mathcal{M}_n, \tag{5.12}$$

where the fibres of $\mathcal{N}_n$ are $n-3$ dimensional and describe the moduli of $e(z)$. For on-shell $|\Psi\rangle$, the form (5.11) is well-defined on $T^*\mathcal{M}_n$. To determine the on-shell amplitude, we choose a section of $\mathcal{N}_n$ and integrate over the base $\mathcal{M}_n$. This is a formal integration.

$$\int_{\mathcal{M}_n} \Omega_{|\vec{\Psi}\rangle}(\vec{\nu}). \tag{5.13}$$

The argument is that the choice of section doesn't matter and so the integral above is well-defined. An arbitrary infinitesimal displacement in $T^*\mathcal{M}_n$, parametrised by the worldsheet vector $v(z)$ modifies the surface state as

$$\delta_{\vec{\nu}}\langle\Sigma| = \langle\Sigma|\mathcal{T}(\vec{\nu}) + \langle\Sigma|\mathcal{H}(\vec{\nu}). \tag{5.14}$$

This generalises the conventional bosonic string result [25]. $\mathcal{T}(\vec{\nu})$ generates a displacement in the base $\mathcal{M}_n$, whilst $\mathcal{H}(\vec{\nu})$ generates a displacement in the fibres of $T^*\mathcal{M}_n$. The $\delta\big(\mathcal{H}(\vec{\nu})\big)$

insertions in $\Omega_{|\vec{\Psi}\rangle}(\vec{\nu})$ cancel the $\mathcal{H}(\vec{\nu})$ component in $\delta_{\vec{\nu}}\langle\Sigma|$ resulting in

$$\delta_{\vec{\nu}}\langle\Sigma|\delta\big(\mathcal{H}(\vec{\nu})\big) = \langle\Sigma|\mathcal{T}(\vec{\nu})\delta\big(\mathcal{H}(\vec{\nu})\big). \tag{5.15}$$

So, for a general displacement in $T^*\mathcal{M}_n$, only the change in the base coordinate results in a change in $\Omega_{|\vec{\Psi}\rangle}(\vec{\nu})$. Deformations in the fibre directions seem to preserve $\Omega_{|\vec{\Psi}\rangle}(\vec{\nu})$ and so we can formally integrate over the base and (5.13) is then well-defined. Note that we are overlooking possible global issues. It hasn't yet being determined how this relates to the Morse theory and localisation results in [9] that show how the expression for the on-shell scattering amplitude as integral of a form over a half-dimensional cycle $\Gamma_n \subset T^*\mathcal{M}_n$ formally reduces to an integral over the moduli space.

It is relatively easy to extend these results to include the off-shell case. We define an infinite dimensional bundle $\mathcal{A}_n$ with base $T^*\mathcal{M}_n$ and fibres given by a choice of local coordinate about each puncture. Like in the conventional bosonic string, we impose the identification $z_i \sim e^{i\theta_i}z_i$, this reduces us from $\mathcal{A}_n$ to a bundle which we shall refer as $\widehat{\mathcal{A}}_n$.

$$\begin{array}{ccc} \mathscr{T} & \hookrightarrow & \widehat{\mathcal{A}}_n \\ & & \downarrow \\ & & T^*\mathcal{M}_n \end{array} \tag{5.16}$$

For on-shell and off-shell states the form (5.11) is well-defined on both $\mathcal{A}_n$ and, moreover on $\widehat{\mathcal{A}}_n$. When we compute, we focus on a $2n-6$ dimensional cycle in $T^*\mathcal{M}_n$ that we can formally identify as a copy of $\mathcal{M}_n$ in our expressions. So, we might formally use $\widehat{\mathcal{P}}_n$ in place of $\widehat{\mathcal{A}}_n$. The vertices would then be formally defined as in the conventional string field, in terms of the vertices $\mathcal{V}_n \subset \widehat{\mathcal{P}}_n$ by

$$\{\Psi^n\} = \int_{\mathcal{V}_n} \Omega_{|\vec{\Psi}\rangle}(\vec{\nu}), \tag{5.17}$$

where $\mathcal{V}_n \subset \widehat{\mathcal{P}}_n$.

5.3 Elements of Perturbation Theory

We will use here the proposed propagator and interaction vertices described earlier to outline the perturbative behaviour of our model. It has a parallel with conventional bosonic string but the ambitwistor string differs in many important aspects (see [4,16,17,28,42] which has comments on this issue). We focus on the bosonic fields, and also applies to the bosonic sector of the supersymmetric theory. There is some evidence showing that this construction will only work in the context of the supersymmetric theory. So, what we describe here will be later applied to the superstring field theory. Gluing lower point surfaces together is an important feature. We have a n-punctured Riemann surface Σ_n constructed from a propagator connecting two Riemann surfaces denoted by Σ_L and Σ_R. These Riemann surfaces have n_L and n_R punctures respectively, where $n-2 = n_L+n_R$. The real dimensions of the moduli spaces of these Riemann surfaces are $dim(\mathcal{M}_L) = 2n_L-6$ and $dim(\mathcal{M}_R) = 2n_R-6$. So, for the moduli space of the n-punctured Riemann surface to be correct, the propagator must carry one complex modulus. This modulus is denoted by q and appears in the gluing of local coordinates z_L and z_R in the regions of the propagator on Σ_L and Σ_R respectively as $z_L z_R = q$. In the ambitwistor string, each modulus comes with a holomorphic delta-function insertion. Then, What is the appropriate holomorphic delta-function associated with the modulus carried by the propagator arising from the propagator 4.8?. We do not have yet a complete understanding of the ambitwistor string propagator. However some recent progress has been made [9,43,44]. In this thesis we will see that the perspective of [9] provides a more natural framework in which to understand how the holomorphic delta-function associated with the propagator modulus emerges in perturbation theory. Before anything else, we fix the spacetime gauge symmetries as shown in previous sections where we've seen that there is a simple analogue of the Siegel gauge that works for the ambitwistor string, namely $\tilde{b}_0|\Psi\rangle = 0$. The kinetic term is then

$$S_2[\Psi] = \langle \mathcal{R}_{LR}|c_0\tilde{c}_0\widetilde{L}_0|\Psi_L\rangle|\Psi_R\rangle, \tag{5.18}$$

where $\langle \mathcal{R}_{LR}|$ and $|\mathcal{R}_{LR}\rangle$ are reflection states. In the calculation of the quadratic action we used string fields that satisfied the constraint $\mathcal{L}_0 = 0$. The propagator proposed is then

$$\tilde{b}_0 b_0 \frac{\delta(L_0 - 2)}{\widetilde{L}_0} |\mathcal{R}_{L,R}\rangle, \tag{5.19}$$

where $\delta(L_0 - 2)$ projects onto states for which $L_0 = 2$ and the subscripts L, R denote the Hilbert spaces to the left and right of the propagator. The possibility of more general functions is discussed in [3]. In the present description the constraint $\mathcal{L}_0|\Psi\rangle = 0$ projects out such sectors and so such terms do not appear in our Siegel gauge propagator. From [9] analysis of factorisation limits we may suggest a propagator

$$\int \mathrm{d}s\, \mathrm{d}\tilde{s}\; b_0 \tilde{b}_0\; e^{-\{Q, s b_0 + \tilde{s}\tilde{b}_0\}} |\mathcal{R}_{L,R}\rangle, \tag{5.20}$$

that can be written as

$$\frac{\tilde{b}_0 b_0}{\widetilde{L}_0 L_0} |\mathcal{R}_{L,R}\rangle. \tag{5.21}$$

When we considered the quadratic action we had $L_0 = 2$ as a condition on the string field and so, on the support of the projection $\delta(L_0 - 2)$ this propagator is equivalent to

$$\frac{\tilde{b}_0 b_0}{\widetilde{L}_0} |\mathcal{R}_{L,R}\rangle. \tag{5.22}$$

Thus, the quadratic action we have in previous sections is consistent with the propagator (5.20). We shall find that it is the form of the propagator given by (5.20) that is most useful in understanding perturbation theory in the context of the perspective of the scattering amplitudes derived using localisation in [9]. We could interpret (5.20), under a projection onto $L_0 = 2$ states, as a closed string propagator for the bundle $\mathcal{Y}$. The projection onto the base Σ is of the form (4.2). There all of the anti-holomorphic dependence has been suppressed and $q = e^{-s}$. Following [9] we want to choose a cycle $\Gamma \subset T^*\mathcal{M}$ that excludes the anti-holomorphic contributions. The additional $\tilde{s}\{Q, \tilde{b}_0\}$ deals with propagation of the fibres of $\mathcal{Y}$.

We now give a brief description of the contributions to the scattering amplitude. They come from integrating over the fundamental missing regions $\mathcal{D}_n$ of moduli space and also from contributions in regions of the moduli space constructed from $m < n$ point vertices joined by propagators. But we can consider these contributions as coming from a middle-dimensional region $\Gamma_{\mathcal{D}_n} \subset T^*\mathcal{D}_n$, so the contributions from the fundamental regions are just

$$\begin{aligned} M_n^{\Gamma_{\mathcal{D}_n}} &= \int_{\mathcal{D}_n} \prod_{a=1}^{n-3} \mathrm{d}\tau_a \langle \Sigma_n | \prod_{a=1}^{n-3} \tilde{\mathbf{b}}(\vec{\nu}^a) \mathbf{b}(\vec{\nu}^a) \bar{\delta}\Big(\mathcal{H}(\vec{\nu}^a)\Big) | \vec{\Psi}_n \rangle \\ &= \int_{\Gamma_{\mathcal{D}_n} \subset T^*\mathcal{D}_n} \Big\langle e^{-\{Q,\mathbf{b}\}} e^{-\{Q,\tilde{\mathbf{b}}\}} \vec{\Psi}_n \Big\rangle_{S_0}, \end{aligned} \tag{5.23}$$

which gives the standard scattering amplitude integrand but integrated only over the region $\mathcal{D}_n \subset \mathcal{M}_n$ instead of the full moduli space with τ^a holomorphic coordinates on $\mathcal{M}_n$. The other contributions to the scattering amplitude come from terms constructed using lower interaction terms glued together by propagators.

Next, consider the contribution given by gluing pairs of lower point vertices by a single propagator. Doing it in $T^*\mathcal{M}$, we get a propagator 5.20, giving us

$$M_n^{\mathcal{R}_1} = \sum_{\sigma,\{n_L,n_R\}} \int_{\Gamma_L \in T^*\mathcal{M}_L} \int_{\Gamma_R \in T^*\mathcal{M}_R} \int \mathrm{d}s\, \mathrm{d}\tilde{s} \langle \widetilde{\Sigma}_L | \langle \widetilde{\Sigma}_R |\, e^{-\{Q, s b_0 + \tilde{s}\tilde{b}_0\}} | \mathcal{R}_{L,R} \rangle | \vec{\Psi}_L \rangle | \vec{\Psi}_R \rangle, \tag{5.24}$$

where the sum denotes a double sum over all $\{n_L, n_R\}$ such that $n_L + n_R - 2 = n$ and $n_L, n_R \geq 3$, and σ denotes a sum over all permutations of external states. $|\vec{\Psi}_L\rangle = |\Psi_1\rangle|\Psi_2\rangle ... |\Psi_{n_L-1}\rangle$ is a product of asymptotic states located at each of the punctures of Σ_L not connected to the propagator. Similarly for $|\vec{\Psi}_R\rangle$. Inserting a complete set of states $\Phi_{L,R}$ and using $\langle \widetilde{\Sigma} | |\Psi_1\rangle ... |\Psi_n\rangle = \langle e^{-\{Q,\mathcal{W}\}} \Psi_1 ... \Psi_n \rangle_{S_0}$, we can write this in terms of correlation functions on the component Riemann surfaces as

$$\begin{aligned} M_n^{\mathcal{R}_1} = &\sum_{\sigma,\{n_L,n_R\}} \int_{\Gamma_{\mathcal{R}_1}} \sum_{\Phi_L,\Phi_R} \Big\langle e^{-\{Q,\mathbf{b}(u_L)\}} e^{-\{Q,\tilde{\mathbf{b}}(u_L,\tilde{u}_L)\}} \vec{\Psi}_L \Phi_L \Big\rangle_{S_0} \\ &\times \langle \Phi_L |\, e^{-\{Q, s b_0 + \tilde{s}\tilde{b}_0\}} | \Phi_R \rangle \Big\langle e^{-\{Q,\mathbf{b}(u_R)\}} e^{-\{Q,\tilde{\mathbf{b}}(u_R,\tilde{u}_R)\}} \Phi_R \vec{\Psi}_R \Big\rangle_{S_0}, \end{aligned} \tag{5.25}$$

where the union of the integration regions has been written as $\Gamma_{\mathcal{R}_1}$. Following [9], this may be written as the correlation function on the $n = n_L + n_R - 2$ punctured Riemann surface Σ_n as

$$M_n^{\mathcal{R}_1} = \sum_{\sigma,\{n_L,n_R\}} \int_{\Gamma_{\mathcal{R}_1}} \left\langle e^{-\{Q,\mathbf{b}(u)\}} e^{-\{Q,\tilde{\mathbf{b}}(u,\tilde{u})\}} \vec{\Psi}_{,n} \right\rangle_{S_0} \tag{5.26}$$

where $(u_L, \tilde{u}_L)$, $(u_R, \tilde{u}_R)$, and $(s, \tilde{s})$ have been combined into $(u, \tilde{u})$. The $\Gamma_{\mathcal{R}_1}$ is a half-dimensional cycle in $T^*\mathcal{M}_L \times T^*\mathcal{M}_R \times \mathbb{C}$, fixed by Morse theory. More details are commented in [3].

Chapter 6

Ambitwistor Superstring Field Theory of Supergravity: Free Theory

In [1] there's an argument establishing that the bosonic ambitwistor string does not to describe conventional Einstein gravity. However, a supersymmetric extension does describe Einstein *super*gravity. Generalisations of the ambitwistor string have been investigated [10, 11], here we consider the simplest generalisation of extending the bosonic theory to an $\mathcal{N} = 2$ (chiral) supersymmetric theory. The critical dimension on the ambitwistor string extended in this way is a positive integer only for $\mathcal{N} = 2$ and $\mathcal{N} = 4$, where the critical dimension is 10 and 2 respectively. A systematic study of the $\mathcal{N} = 4$ case seems that hasn't been considered. There's another possibility, that the dimension counting for the $\mathcal{N} = 4$ is not straightforward (cf. the conventional $\mathcal{N} = 2$ string mentioned in section 6.3). The theory has the symmetry (2.18) under which the fermions transform trivially and also a natural extension of the bosonic conformal symmetry (2.14) to a superconformal symmetry. After gauge fixing the worldsheet complex structure and the Beltrami differential $e(z)$, the $\mathcal{N} = 2$ ambitwistor superstring has action is

$$S = \int_\Sigma P_\mu \bar{\partial} X^\mu + b\bar{\partial}c + \tilde{b}\bar{\partial}\tilde{c} + \eta_{\mu\nu}\psi^\mu\bar{\partial}\psi^\nu + \eta_{\mu\nu}\widetilde{\psi}^\mu\bar{\partial}\widetilde{\psi}^\nu + \chi P_\mu\psi^\mu + \widetilde{\chi} \tag{6.1}$$

where χ and $\widetilde{\chi}$ are the worldsheet gravitini and ψ^μ and $\widetilde{\psi}^\mu$ are holomorphic worldsheet spinors. We will only consider the Neveu-Schwarz (NS) worldsheet spinors. The two gravitini χ and $\widetilde{\chi}$ can be gauge-fixed as usual so they vanish everywhere except at $n-2$ points, where we insert picture changing operators (PCOs). The usual Faddeev-Popov procedure results in the introduction of (β,γ) and $(\widetilde{\beta},\widetilde{\gamma})$ superghost systems to gauge fix the χ and $\widetilde{\chi}$ respectively. The gauge-fixed action is then

$$S = \int_\Sigma P_\mu \bar{\partial} X^\mu + \eta_{\mu\nu}\psi^\mu \bar{\partial}\psi^\nu + \eta_{\mu\nu}\widetilde{\psi}^\mu \bar{\partial}\widetilde{\psi}^\nu + b\bar{\partial}c + \tilde{b}\bar{\partial}\tilde{c} + \beta\bar{\partial}\gamma + \widetilde{\beta}\bar{\partial}\widetilde{\gamma}. \tag{6.2}$$

The non-trivial OPEs for the new fields are

$$\psi^\mu(z)\psi^\nu(\omega) = \frac{\eta^{\mu\nu}}{z-\omega} + ..., \qquad \beta(z)\gamma(\omega) = \frac{1}{z-\omega} + ..., \tag{6.3}$$

and similarly for the $(\tilde{\beta},\tilde{\gamma})$ and $\widetilde{\psi}^\mu$ fields. The bosonic part OPEs for the fields were introduced already and remain the same here.

6.1 Supersymmetries

The gravitini act as Lagrange multipliers. They impose the vanishing of the fermionic currents $G = P_\mu\psi^\mu$ and $\widetilde{G} = P_\mu\widetilde{\psi}^\mu$. These generate two worldsheet supersymmetries. Like in the bosonic case, the stress tensor $T(z)$ generates the conformal transformations with the additional transformations of the worldsheet fermions

$$\delta_v X^\mu = v\partial X^\mu, \qquad \delta_v P_\mu = \partial(vP_\mu), \qquad \delta_v\psi^\mu = \frac{1}{2}(\partial v)\psi^\mu + v\partial\psi^\mu \qquad \delta_v\widetilde{\psi}^\mu = \frac{1}{2}(\partial v)\widetilde{\psi}^\mu + v\partial\widetilde{\psi}^\mu. \tag{6.4}$$

The worldsheet spinors are invariant under the worldsheet gauge transformations generated by $H(z)$. It enforces the null condition $P^2(z) = 0$ which acts on X^μ as $\tilde{\delta}_v X^\mu = vP_\mu$. The new object is the $\mathcal{N} = 2$ worldsheet supersymmetry. The $G(z)$ supercurrent generates the transformations

$$\delta_\epsilon X^\mu = \epsilon\psi^\mu, \qquad \delta_\epsilon \psi^\mu = \epsilon P^\mu, \qquad \delta_\epsilon \widetilde{\psi}^\mu = 0, \qquad \delta_\epsilon P_\mu = 0, \tag{6.5}$$

and the $\widetilde{G}(z)$ supercurrent generates the transformations

$$\widetilde{\delta}_\epsilon X^\mu = \epsilon\widetilde{\psi}^\mu, \qquad \widetilde{\delta}_\epsilon \psi^\mu = 0, \qquad \widetilde{\delta}_\epsilon \widetilde{\psi}^\mu = \epsilon P^\mu, \qquad \widetilde{\delta}_\epsilon P_\mu = 0. \tag{6.6}$$

Similar to $\mathcal{T}(\nu)$ and $\mathcal{H}(\nu)$ of the bosonic theory, we propose the generators

$$\mathcal{G}(\varepsilon) = \oint \mathrm{d}z\, \varepsilon(z)\, G(z), \tag{6.7}$$

and similarly for $\widetilde{\mathcal{G}}(\varepsilon)$, where ε is a spin-valued worldsheet vector. The superalgebra is then

$$[\mathcal{T}(v_1), \mathcal{T}(v_2)] = -\mathcal{T}([v_1, v_2]), \qquad [\mathcal{T}(v_1), \mathcal{H}(v_2)] = -\mathcal{H}([v_1, v_2]), \tag{6.8}$$

$$[\mathcal{T}(v), \mathcal{G}(\varepsilon)] = -\mathcal{G}([v, \varepsilon]), \qquad [\mathcal{T}(v), \widetilde{\mathcal{G}}(\varepsilon)] = -\widetilde{\mathcal{G}}([v, \varepsilon]), \tag{6.9}$$

$$[\mathcal{G}(\varepsilon_1), \mathcal{G}(\varepsilon_2)] = -\mathcal{H}([\varepsilon_1, \varepsilon_2]), \qquad [\widetilde{\mathcal{G}}(\varepsilon_1), \widetilde{\mathcal{G}}(\varepsilon_2)] = -\mathcal{H}([\varepsilon_1, \varepsilon_2]), \tag{6.10}$$

with all other commutators vanishing. $\mathcal{H}$ acts as a worldsheet Hamiltonian in the $[\mathcal{G}, \mathcal{G}]$ commutator.

6.2 The SuperVirasoro Algebra

In the Neveu-Schwarz sector, in terms of modes, the fermionic fields are

$$\psi^\mu(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi^\mu_r z^{-r-\frac{1}{2}}, \qquad G(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} G_r z^{-r-\frac{3}{2}}, \tag{6.11}$$

and similarly for $\widetilde{\psi}^{\mu}(z)$ and $\widetilde{G}(z)$. The modes of the fermionic currents are

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_{n\mu}\, \psi^{\mu}_{r-n}, \qquad \widetilde{G}_r = \sum_{n\in\mathbb{Z}} \alpha_{n\mu}\, \widetilde{\psi}^{\mu}_{r-n}. \tag{6.12}$$

The superalgebra can be given in terms of these modes as

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{D}{6}m(m^2-1), \qquad [L_m, \widetilde{L}_n] = (m-n)\widetilde{L}_{m+n}, \qquad [\widetilde{L}_m, \widetilde{L}_n] = 0, \tag{6.13}$$

$$[L_m, G_r] = \frac{(m-2r)}{2}\, G_{m+r}, \qquad [L_m, \widetilde{G}_r] = \frac{(m-2r)}{2}\, \widetilde{G}_{m+r}, \tag{6.14}$$

$$\{G_r, G_s\} = 2\,\widetilde{L}_{r+s}, \qquad \{G_r, \widetilde{G}_s\} = 0, \qquad \{\widetilde{G}_r, \widetilde{G}_s\} = 2\widetilde{L}_{r+s}. \tag{6.15}$$

The matter stress tensor $T(z)$ has contributions from the fermions ψ^{μ} and $\widetilde{\psi}^{\mu}$ as well as the (X^{μ}, P_{μ}) system. The $H(z)$ is the same as in the bosonic theory. The critical dimension of the supersymmetric theory is 10 [1].

6.3 The $\mathcal{N} = 2$ String

A few comparisons [1, 23] have been made between the ambitwistor string and the holomorphic sector of the conventional type II string. Less discussed is that there are also similarities with the $\mathcal{N} = 2$ string [45–47]. After gauge-fixing this theory has action

$$S = -\frac{1}{2\pi}\int \mathrm{d}^2 z \partial_{\alpha} X^{\mu} \partial^{\alpha} \overline{X}_{\mu} - i\bar{\psi}\rho^{\alpha}\partial_{\alpha}\psi, \tag{6.16}$$

where the target space is complexified $X^{\mu} = X^{\mu}_1 + iX^{\mu}_2$ and the fermions show in complex pairs $\psi^{\mu} = \psi^{\mu}_1 + i\psi^{\nu}_2$. The critical dimension is two complex (four real) dimensions. Dimension counting of this theory is a non-trivial matter, see [47] for details. The target space has (4,0) or (2,2) signature. The oscillator algebra is

$$[\alpha^{\mu}_m, \bar{\alpha}^{\nu}_n] = m\delta_{m+n}\eta^{\mu\nu}, \qquad [\alpha^{\mu}_m, \alpha^{\nu}_n] = 0, \qquad [\bar{\alpha}^{\mu}_m, \bar{\alpha}^{\nu}_n] = 0. \tag{6.17}$$

We can compare it with the algebra of the modes of the $X^\mu(z)$ and $P_\mu(z)$ fields in the ambitwistor string. And the Super-Virasoro algebra has many similarities with the ambitwistor string case. Notorious difference is that the target space of the $\mathcal{N}=2$ string has self-dual gravity, not Einstein gravity and so a detailed comparison may not be that relevant.

6.4 The Supersymmetric BRST Operator

The fundamental objects in the supersymmetric ambitwistor string field theory are the string field $|\Psi\rangle$ and the surface state $\langle\Sigma|$ (encodes the interactions configuration), the BRST charge Q (generates the propagator of the theory). We start here with the BRST charge Q and how is used to constraint the form of the string field. In terms of the current $j(z)$ is

$$Q = \oint \mathrm{d}z\, j(z). \tag{6.18}$$

For the case we consider $\mathcal{N}=2$ of the ambitwistor string, the BRST current is

$$j(z) = c\left(T_m + T_{\beta\gamma} + \widetilde{T}_{\beta\gamma}\right) + \gamma G + \tilde{\gamma}\widetilde{G} + bc\partial c + \tilde{b}\tilde{c}\partial\tilde{c} + \frac{1}{2}\gamma^2\tilde{b} + \frac{1}{2}\tilde{\gamma}^2\tilde{b} + \tilde{c}H, \tag{6.19}$$

where $T_{\beta\gamma}$ and $\widetilde{T}_{\beta\gamma}$ are superghost stress tensors and the matter stress tensor has contributions from the worldsheet fermions $T_m(z) = P_\mu\partial X^\mu + \psi^\mu\partial\psi_\mu + \widetilde{\psi}^\mu\partial\widetilde{\psi}_\mu$. In terms of oscillator modes the relevant terms in the BRST charge are

$$\begin{aligned} Q \;=\;& c_0\mathcal{L}_0 + \frac{1}{2}\tilde{c}_0\alpha_0^2 + \frac{1}{2}\tilde{c}_0\alpha_{-1}\cdot\alpha_1 + \alpha_0\cdot(c_1\tilde{\alpha}_{-1} + c_{-1}\tilde{\alpha}_1 + \tilde{c}_{-1}\alpha_1 + \tilde{c}_1\alpha_{-1}) \\ & -2b_0c_{-1}c_1 + 2\tilde{b}_0(c_1\tilde{c}_{-1} + \tilde{c}_{-1}c_{-1}) + \tilde{c}_0(c_{-1}\tilde{b}_1 + c_1\tilde{b}_{-1}) \\ & +\gamma_{-\frac{1}{2}}\alpha_0\cdot\psi_{\frac{1}{2}} + \gamma_{\frac{1}{2}}\alpha_0\cdot\psi_{-\frac{1}{2}} + \widetilde{\gamma}_{-\frac{1}{2}}\alpha_0\cdot\widetilde{\psi}_{\frac{1}{2}} + \widetilde{\gamma}_{\frac{1}{2}}\alpha_0\cdot\widetilde{\psi}_{-\frac{1}{2}} \\ & -2\tilde{b}_0(\gamma_{-\frac{1}{2}}\gamma_{\frac{1}{2}} + \tilde{\gamma}_{-\frac{1}{2}}\tilde{\gamma}_{\frac{1}{2}}) + \ldots \end{aligned} \tag{6.20}$$

where the +... denotes terms that depend on oscillator modes that commute with all oscillators that will appear in the string field. Like in the bosonic case, the BRST charge appears in the quadratic part of the string field action multiplied by the ghost zero mode c_0. So, all terms in Q involving c_0 are projected out of the quadratic part of the action and as a consequence we have to deal with these terms separately. In the oscillator expansion of Q those terms that multiply a c_0 factor have been isolated and written as $\mathcal{L}_0$. As that part of the constraint given by $\mathcal{L}_0$ cannot be imposed on-shell by the string field equations of motion because it is projected out of the quadratic action, we must impose this constraint on the string field directly and is taken as part of the definition of the string field $|\Psi\rangle$. Then, we want a superstring field

$$\mathcal{L}_0|\Psi\rangle = 0, \qquad b_0|\Psi\rangle = 0. \tag{6.21}$$

We have used $\mathcal{L}_0$ to denote the corresponding object in the bosonic string field, but from now on $\mathcal{L}_0$ will refer to (6.22) given bellow.

We want to see that these conditions are naturally satisfied by a reasonable $|\Psi\rangle$. For this we need an explicit expression for the superstring field. We will do this in the next section. The $\mathcal{L}_0$ operator required to annihilate the string field should be

$$\begin{aligned}\mathcal{L}_0 &= (\alpha_{-1}\cdot\tilde{\alpha}_1 + \tilde{\alpha}_{-1}\cdot\alpha_1) + \frac{1}{2}(\psi_{-\frac{1}{2}}\cdot\psi_{\frac{1}{2}} + \widetilde{\psi}_{-\frac{1}{2}}\cdot\widetilde{\psi}_{\frac{1}{2}}) + (b_{-1}c_1 + c_{-1}b_1) + (\tilde{b}_{-1}\tilde{c}_1 + \tilde{c}_{-1}\tilde{b}_1)\\ &\quad -\frac{1}{2}(\gamma_{-\frac{1}{2}}\beta_{\frac{1}{2}} - \beta_{-\frac{1}{2}}\gamma_{\frac{1}{2}}) - \frac{1}{2}(\tilde{\gamma}_{-\frac{1}{2}}\tilde{\beta}_{\frac{1}{2}} - \tilde{\beta}_{-\frac{1}{2}}\tilde{\gamma}_{\frac{1}{2}}) - 1.\end{aligned} \tag{6.22}$$

6.5 Gauge Transformations and the Superstring Field

We want to derive the picture $(-1,-1)$ ambitwistor superstring field. It will be convenient to use a 'bosonised' superghosts when dealing with picture changing later. This is

$$\beta = \partial\xi e^{-\phi}, \qquad \gamma = \eta e^{\phi}, \qquad \tilde{\beta} = \partial\tilde{\xi}e^{-\tilde{\phi}}, \qquad \tilde{\gamma} = \tilde{\eta}e^{\tilde{\phi}}. \tag{6.23}$$

The superghost stress tensor contribution is $T_{\beta\gamma} = \frac{1}{2}\partial\phi\partial\phi - \partial^2\phi - \eta\partial\xi$, and similarly for $T_{\tilde{\beta}\tilde{\gamma}}$. The two sets of superghosts are independent of each other and as such we label the vacuum with two independent picture numbers $(q,\tilde{q})$. Note that in the ambitwistor string, a holomorphic thery, $(q,\tilde{q})$ labels a product of holomorphic superghost vacua. We work in the small Hilbert space description of the theory. In there the zero modes of the fields ξ and $\tilde{\xi}$ are excluded. Only derivatives of ξ and $\tilde{\xi}$ enter into the definition of the superghosts. An additional constraint on the string field $\eta_0|\Psi\rangle = \tilde{\eta}_0|\Psi\rangle = 0$ is necessary and so the set of constraints will be

$$\mathcal{L}_0|\Psi\rangle = 0, \qquad b_0|\Psi\rangle = 0, \qquad \eta_0|\Psi\rangle = 0, \qquad \tilde{\eta}_0|\Psi\rangle = 0. \tag{6.24}$$

They will be part of the definition of $|\Psi\rangle$. Similar to the case of the bosonic string field in section 4.3, we propose a linearised transformation $\delta|\Psi\rangle = Q|\Lambda\rangle$ corresponding to linearised gauge transformations in spacetime. We know that the on-shell correlation functions involving the string fields must reduce to the integrand of the on-shell scattering amplitude, then looking at the vertex operators involved, a good guess for the picture $(-1,-1)$ string field is

$$\Psi(z) = \int \mathrm{d}k \left(E_{\mu\nu}(k)\, \psi^\mu\, \widetilde{\psi}^\nu\, e^{-\phi-\tilde{\phi}} c\tilde{c} + ... \right)\, e^{ik\cdot X}, \tag{6.25}$$

where $E_{\mu\nu}(k)$ is a momentum space field, made of a sum of symmetric and antisymmetric parts in the μ and ν indices. As we know that we want to get back the linearised target space of diffeomorphisms from $Q|\Lambda\rangle$, we take the gauge parameter field as

$$\Lambda(z) = -\int \mathrm{d}k \left(i\lambda_\mu(k)\, \psi^\mu\, \partial\tilde{\xi}\, e^{-2\tilde{\phi}-\phi} - i\widetilde{\lambda}_\mu(k)\, \widetilde{\psi}^\mu\, \partial\xi\, e^{-2\phi-\tilde{\phi}} + \Omega(k)\, \partial\tilde{c}\, \partial\xi\, \partial\tilde{\xi}\, e^{-2\phi-2\tilde{\phi}} \right) c\tilde{c}\, e^{ik\cdot X}, \tag{6.26}$$

where λ, $\widetilde{\lambda}$ and Ω are momentum-dependent parameters. The gauge transformation of the string field $\Psi(z)$ to linear order is

$$\delta\Psi(z) = \oint_z \mathrm{d}\omega\, j(\omega)\, \Lambda(z), \tag{6.27}$$

where $j(\omega)$ is the BRST current. From the OPEs obtained in (2.13)

$$\xi(z)\eta(\omega) = \frac{1}{z-\omega} + ..., \qquad e^{\ell_1\phi(z)} e^{\ell_2\phi(\omega)} = (z-\omega)^{-\ell_1\ell_2} e^{(\ell_1+\ell_2)\phi(\omega)} + ... \tag{6.28}$$

Similar expressions exist for the fields $(\tilde{\phi}, \tilde{\eta}, \tilde{\xi})$ of the transformation given by $\oint_z \mathrm{d}\omega\; j(\omega)\,\Lambda(z)$, and they can be computed directly. The $Q\Lambda(z)$ has terms that cannot be interpreted as target space transformations of $E_{\mu\nu}(k)$ in the limited ansatz (6.25) above. So, we need to generalise the ansatz (6.25) to include additional terms. In a similar way as the bosonic string field case we get a minimal ansatz for the superstring field

$$\begin{aligned}\Psi(z) \;\; = \;\; & \int \mathrm{d}k \Big(E_{\mu\nu}(k)\; \psi^{\mu}\, \widetilde{\psi}^{\nu}\; e^{-\phi-\tilde{\phi}} + 2e(k)\; \eta\, \partial\tilde{\xi}\, e^{-2\tilde{\phi}} + 2\tilde{e}(k)\; \tilde{\eta}\, \partial\xi\, e^{-2\phi} \\ & + i f_{\mu}(k)\; \psi^{\mu}\, \partial\tilde{\xi}\; e^{-2\tilde{\phi}-\phi}\, \partial\tilde{c} + i \tilde{f}_{\mu}(k)\; \widetilde{\psi}^{\mu}\, \partial\xi\; e^{-2\phi-\tilde{\phi}}\, \partial\tilde{c}\Big)\, c\tilde{c}\, e^{ik\cdot X}. \quad (6.29)\end{aligned}$$

The linearised transformations of the momentum space component fields are

$$\delta E_{\mu\nu}(k) = ik_{\mu}\widetilde{\lambda}_{\nu}(k) + ik_{\nu}\lambda_{\mu}(k) \qquad \delta e(k) = -\frac{i}{2}k^{\mu}\lambda_{\mu}(k) + \Omega(k), \qquad \delta\tilde{e}(k) = \frac{i}{2}k^{\mu}\widetilde{\lambda}_{\mu}(k) + \Omega(k) \quad (6.30)$$

$$\delta f_{\mu}(k) = \frac{1}{2}k^2\lambda_{\mu}(k) + ik_{\mu}\Omega(k), \qquad \delta\tilde{f}_{\mu}(k) = -\frac{1}{2}k^2\widetilde{\lambda}_{\mu}(k) + ik_{\mu}\Omega(k), \quad (6.31)$$

where $k^2 = \eta^{\mu\nu}k_{\mu}k_{\nu}$. We Fourier transform the linearised transformations to configuration space

$$\delta E_{\mu\nu}(x) = \partial_{\mu}\widetilde{\lambda}_{\nu}(x) + \partial_{\nu}\lambda_{\mu}(x) \qquad \delta e(x) = -\frac{1}{2}\partial^{\mu}\lambda_{\mu}(x) + \Omega(x), \qquad \delta\tilde{e}(x) = \frac{1}{2}\partial^{\mu}\widetilde{\lambda}_{\mu}(x) + \Omega(x) \quad (6.32)$$

$$\delta f_{\mu}(x) = -\frac{1}{2}\Box\lambda_{\mu}(x) + \partial_{\mu}\Omega(x), \qquad \delta\tilde{f}_{\mu}(x) = \frac{1}{2}\Box\widetilde{\lambda}_{\mu}(x) + \partial_{\mu}\Omega(x), \quad (6.33)$$

where $\Box = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}$.

The relations

$$\delta\tilde{f}_{\mu}(x) = \frac{1}{2}\partial^{\nu}\Big(\delta E_{\nu\mu}(x)\Big) + \partial_{\mu}\Big(\delta e(x)\Big), \qquad \delta f_{\mu}(x) = -\frac{1}{2}\partial^{\nu}\Big(\delta E_{\mu\nu}(x)\Big) + \partial_{\mu}\Big(\delta\tilde{e}(x)\Big), \quad (6.34)$$

suggest that the associated fields should be identified. We will do this in the next section. Note that like in the case of the bosonic string field, this superstring field is complete in the linearised theory, but other terms might have a role when we consider the interaction terms.

6.6 The Ambitwistor Superstring Field Quadratic Action

Working in the picture $(-1,-1)$ we now can obtain a specific quadratic ambitwistor superstring action. For the quadratic action, the state $|\Psi\rangle$ in terms of the superghosts (β,γ) will be used. In terms of mode oscillators the picture $(-1,-1)$ superstring field (see [3])

$$\begin{aligned} |\Psi\rangle &= \int \mathrm{d}k \Big(E_{\mu\nu}(k)\, \psi^{\mu}_{-\frac{1}{2}} \widetilde{\psi}^{\nu}_{-\frac{1}{2}} + 2e(k)\, \gamma_{-\frac{1}{2}} \widetilde{\beta}_{-\frac{1}{2}} + 2\tilde{e}(k)\, \widetilde{\gamma}_{-\frac{1}{2}} \beta_{-\frac{1}{2}} \\ &\quad + i f_{\mu}(k)\, \psi^{\mu}_{-\frac{1}{2}} \widetilde{\beta}_{-\frac{1}{2}} \tilde{c}_0 + i \tilde{f}_{\mu}(k)\, \widetilde{\psi}^{\mu}_{-\frac{1}{2}} \beta_{-\frac{1}{2}} \tilde{c}_0 \Big)\, c_1 \tilde{c}_1\, |\text{-1,-1}, k\rangle. \end{aligned} \tag{6.35}$$

And the conjugate string field is

$$\begin{aligned} \langle\Psi| &= \int \mathrm{d}k\, \langle\text{-1,-1,-}k|\, c_{-1}\tilde{c}_{-1} \Big(E_{\mu\nu}(k)\, \psi^{\mu}_{\frac{1}{2}} \widetilde{\psi}^{\nu}_{\frac{1}{2}} + 2e(k)\, \gamma_{\frac{1}{2}} \widetilde{\beta}_{\frac{1}{2}} + 2\tilde{e}(k)\, \widetilde{\gamma}_{\frac{1}{2}} \beta_{-\frac{1}{2}} \\ &\quad + i f_{\mu}(k)\, \psi^{\mu}_{\frac{1}{2}} \widetilde{\beta}_{\frac{1}{2}} \tilde{c}_0 + i \tilde{f}_{\mu}(k)\, \widetilde{\psi}^{\mu}_{\frac{1}{2}} \beta_{\frac{1}{2}} \tilde{c}_0 \Big). \end{aligned} \tag{6.36}$$

Unlike the bosonic case, in this superymmetric case the conjugation is the standard BPZ conjugation (the $\alpha_{n\mu}$ and $\tilde{\alpha}^{\mu}_{n}$ mode operators do not appear in (6.29)). The reflector state that appears in the propagator will be closer to the one that appears in the conventional superstring. For the same reasons we obtained the quadratic action for the bosonic ambitwistor string field, we can justify the supersymmetric case, obtaining

$$S_2[\Psi] = \frac{1}{2}\langle\Psi|c_0 Q|\Psi\rangle. \tag{6.37}$$

Substituting the string fields (6.35), (6.36) and BRST operator (6.20) into the quadratic action (6.37) gives

$$S_2[\Psi] = \int \mathrm{d}k\, \mathrm{d}k' \langle\text{-1,-1,-}k'|\, c_{-1}\tilde{c}_{-1} c_0 c_1 \tilde{c}_1\, \mathcal{F}\, |\text{-1,-1}, k\rangle, \tag{6.38}$$

where $\mathcal{F} = \mathcal{F}(\tilde{c}_0, \tilde{b}_0, \alpha_0, \psi_{\pm\frac{1}{2}}, \gamma_{\pm\frac{1}{2}}, \beta_{\pm\frac{1}{2}})$ is a function that is not annihilated by the $c_{\pm 1}$, $\tilde{c}_{\pm 1}$ or c_0 ghosts. The vacuum is normalised to

$$\langle\text{-}1,\text{-}1,\text{-}k'|c_{-1}\tilde{c}_{-1}c_0\tilde{c}_0c_1\tilde{c}_1|\text{-}1,\text{-}1,k\rangle = \delta(k+k'), \tag{6.39}$$

and so the only contributions that come from the $\mathcal{F}$ function are proportional to $\tilde{c}_0$. It is easy to then derive the action

$$\begin{aligned} S_2[\Psi] \;=\; \int \mathrm{d}k \biggl(& -\frac{1}{4}E_{\mu\nu}(\text{-}k)\,k^2\,E^{\mu\nu}(k) - 2\tilde{e}(\text{-}k)\,p^2\,e(k) - if^{\mu}(\text{-}k)\,k^{\nu}\,E_{\mu\nu}(k) + i\tilde{f}^{\nu}(\text{-}k)\,k^{\mu}\,E_{\mu\nu}(k) \\ & +2if^{\mu}(\text{-}k)\,k_{\mu}\,\tilde{e}(k) + 2i\tilde{f}^{\mu}(\text{-}k)\,k_{\mu}\,e(k) - f_{\mu}(\text{-}k)\,f^{\mu}(k) - \tilde{f}^{\mu}(\text{-}k)\,\tilde{f}_{\mu}(k)\biggr). \end{aligned} \tag{6.40}$$

Fourier transforming this action to configuration space gives

$$\begin{aligned} S_2[\Psi] \;=\; \int \mathrm{d}x \biggl(& \frac{1}{4}E_{\mu\nu}(x)\Box E^{\mu\nu}(x) + 2\tilde{e}(x)\Box e(x) - f_{\mu}(x)f^{\mu}(x) - \tilde{f}^{\mu}(x)\tilde{f}_{\mu}(x) \\ & -f^{\mu}(x)\Bigl[\partial^{\nu}E_{\mu\nu}(x) - 2\partial_{\mu}\tilde{e}(x)\Bigr] + \tilde{f}^{\nu}(x)\Bigl[\partial^{\mu}E_{\mu\nu}(x) + \partial_{\nu}e(x)\Bigr]\biggr). \end{aligned} \tag{6.41}$$

As expected there are no kinetic terms for the $f(x)$ and $\tilde{f}(x)$ fields. They are actually auxiliary fields that are integrated out. The equations of motion for these auxiliary fields are

$$f_{\mu}(x) = -\frac{1}{2}\Bigl(\partial^{\nu}E_{\mu\nu}(x) - 2\partial_{\mu}\tilde{e}(x)\Bigr), \qquad \tilde{f}_{\mu}(x) = \frac{1}{2}\Bigl(\partial^{\nu}E_{\nu\mu}(x) + 2\partial_{\mu}e(x)\Bigr). \tag{6.42}$$

We put back these $f(x)$ and $\tilde{f}(x)$ into the action, giving

$$S_2[E,e,\tilde{e}] = \int \mathrm{d}x\,\left(\frac{1}{4}E_{\mu\nu}(x)\Box E^{\mu\nu}(x) + 2\tilde{e}(x)\Box e(x) + f_{\mu}(x)f^{\mu}(x) + \tilde{f}^{\mu}(x)\tilde{f}_{\mu}(x)\right), \tag{6.43}$$

where the fields $f_{\mu}(x)$ and $\tilde{f}_{\mu}(x)$ are taken as defined by (6.42). Note that if we attempt to quantise, these auxiliary functions may be integrated out in the configuration space path integral. We also owuld like to remark that $E_{\mu\nu}(x)$ does not have definite symmetry so there is more than just the graviton in the spectrum of the theory. This is what we expect from the massless NS sector of Type II supergravity. However, to see the connection with the linearisation of the standard Type II supergravity action further analysis is needed.

6.7 Field Redefinitions

We take inspiration in a construction from conventional *bosonic* string field theory on toroidal backgrounds in the derivation of Double field theory [48]. It describes the physics of the massless NS sector of the bosonic string. So, we will follow [48] to get the massless NS sector from the action (6.43) to show that the ambitwistor string field gives rise to the correct supergravity limit. We define

$$\vartheta^{\pm} = \frac{1}{2}\big(e \pm \tilde{e}\big). \tag{6.44}$$

where ϑ^{+} and ϑ^{-} transform as

$$\delta\vartheta^{+} = \frac{1}{2}\partial^{\mu}\epsilon_{\mu} + \Omega, \qquad \delta\vartheta^{-} = -\frac{1}{2}\partial^{\mu}\zeta_{\mu}, \tag{6.45}$$

with the definitions

$$\zeta_{\mu} = \frac{1}{2}(\lambda_{\mu} + \widetilde{\lambda}_{\mu}), \qquad \epsilon_{\mu} = -\frac{1}{2}(\lambda_{\mu} - \widetilde{\lambda}_{\mu}). \tag{6.46}$$

The ϑ^{+} is a Stuckelberg field and we can fix the Ω transformation under $\vartheta^{+} = 0$. After integrations by parts $e = -\tilde{e}$ the action becomes

$$S_2 = \int \mathrm{d}x \left(\frac{1}{4}E_{\mu\nu}\Box E^{\mu\nu} - 4\vartheta^{-}\Box\vartheta^{-} + \frac{1}{4}(\partial^{\nu}E_{\mu\nu})^2 - 2\vartheta^{-}(\partial^{\mu}\partial^{\nu}E_{\mu\nu}) + \frac{1}{4}(\partial^{\nu}E_{\nu\mu})^2\right). \tag{6.47}$$

We can write the $E_{\mu\nu}$ field with symmetric and antisymmetric parts $h_{\mu\nu}$ and $b_{\mu\nu}$ giving

$$S_2 = \int \mathrm{d}x \left(\frac{1}{4}h_{\mu\nu}\Box h^{\mu\nu} + \frac{1}{2}(\partial^{\nu}h_{\mu\nu})^2 - 2\vartheta^{-}(\partial^{\mu}\partial^{\nu}h_{\mu\nu}) - 4\vartheta^{-}\Box\vartheta^{-} + \frac{1}{4}b_{\mu\nu}\Box b^{\mu\nu} + \frac{1}{2}(\partial^{\nu}b_{\mu\nu})^2\right). \tag{6.48}$$

This is the linearised action for a metric, B-field and scalar field ϑ^-. A more natural field choice will include the dilaton ϕ

$$\phi = \vartheta^- + \frac{1}{4}h, \tag{6.49}$$

where $h = \eta^{\mu\nu}h_{\mu\nu} = \eta^{\mu\nu}E_{\mu\nu}$ is the trace of the graviton. The motivation for the field redefinition is that this dilaton is invariant under the linearised gauge transformations. Integrating the $b_{\mu\nu}$ terms by parts, we can rewrite the action as

$$\frac{1}{4}b_{\mu\nu}\Box b^{\mu\nu} + \frac{1}{2}(\partial^\nu b_{\mu\nu})^2 \approx -\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}, \tag{6.50}$$

where $\approx$ means equality up to total derivatives and the Kalb-Ramond field strength takes the usual form $H_{\mu\nu\lambda} = \partial_{[\mu}b_{\nu\lambda]}$. Then, the linearised action is

$$\begin{aligned} S_2[h, b, \phi] \;\; = \;\; & \int \mathrm{d}x \left(\frac{1}{4}h_{\mu\nu}\Box h^{\mu\nu} + \frac{1}{2}(\partial^\nu h_{\mu\nu})^2 + \frac{1}{2}h\partial^\mu\partial^\nu h_{\mu\nu} - \frac{1}{4}h\Box h \right. \\ & \left. -4\phi\Box\phi + 2h\Box\phi - 2\phi\partial^\mu\partial^\nu h_{\mu\nu} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right). \end{aligned} \tag{6.51}$$

With $b_{\mu\nu} = 0$ and $\phi = 0$ we get back the Fierz-Pauli action (4.27) for the graviton $h_{\mu\nu}$. The linearised Ricci scalar is $R = \partial^\mu\partial^\nu h_{\mu\nu} - \Box h$ gives the standard dilaton coupling. The action (6.51) is the linearised action for the NS sector of the Type II supergravity as we wanted. The full non-linear action up to Weyl rescaling is

$$S = \int e^{-\phi}\left(R * 1 - \frac{1}{2}H_{(3)} \wedge *H_{(3)} + *\,\mathrm{d}\phi \wedge \,\mathrm{d}\phi\right). \tag{6.52}$$

Note that the gauge transformations of the component fields are

$$\delta h_{\mu\nu} = \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu, \qquad \delta b_{\mu\nu} = \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu, \qquad \delta\phi = 0, \tag{6.53}$$

which are the standard linearised gauge transformations of the graviton, Kalb-Ramond field and dilaton.

The quadratic term ansatz (6.37) produce the correct linearised theory, transforms under the correct linearised gauge transformations (diffeomorphisms and antisymmetric tensor transformations). If we gauge fix and compute the propagator for the theory under a Siegel gauge $\tilde{b}_0|\Psi\rangle = 0$, we would get the quadratic action

$$S_2[\Psi] = \frac{1}{2}\langle\Psi|c_0\tilde{c}_0\widetilde{L}_0|\Psi\rangle, \tag{6.54}$$

like in the bosonic case. The only difference is that the string fields are of the supersymmetric theory and $\langle\Psi|$ is related to $|\Psi\rangle$ by standard BPZ conjugation. So, the propagator has the same form as (4.8), but with the reflector state for the supersymmetric theory. A GSO projector must be inserted to ensure that only GSO projected states propagate.

Chapter 7

Ambitwistor Superstring Field theory of Supergravity: Interactions

The ambitwistor superstring field theory has a non-polynomial action of the form (4.4) with $|\Psi\rangle$ now given by the $(-1,-1)$ picture NS superstring field (6.29) and the BRST charge (6.20). In this chapter we consider the interaction terms $\{\Psi^n\}$ of the superstring field theory. The new ingredients have been somehow being introduced in the bosonic case. Its supersymmetric generalisation has as fundamental object that generalises the 3.6 and will be described in 7.2. The ghost insertions are the same as the ones of the bosonic case but, gauge fixing the gravitini will result now in superghost insertions. These that are treated in terms of picture changing operators [49, 50]. The picture changing operators, PCOs, are the main new ingredient for developing the structure of the bosonic interaction terms (5.11). We consider the PCOs in the next section and explicitly derive superstring fields in the $(0,0)$, $(-1,0)$ and $(0,-1)$ pictures. If we were considering going beyond tree level, it might have been worth a treatment in terms of super-Riemann surfaces [40]. But at tree level we do not have to worry about obstructions to a PCO approach at higher genus [51]. Type II supergravity does not exist as a spacetime quantum theory. However, there have been interesting developments relating the form of one-loop ambitwistor string amplitudes to results obtained directly from supergravity [17, 42].

7.1 Picture Changing Operators

The BRST current is

$$j(z) = c(z)T(z) + \gamma(z)S(z) + \tilde{\gamma}(z)\widetilde{S}(z) + \tilde{c}(z)H(z), \tag{7.1}$$

where the supercurrents are $S(z)$ and $\widetilde{S}(z)$. We can choose to set the gravitini to zero everywhere except at $n-2$ points. On a super Riemann surface of sufficiently low genus the integrating out of the odd moduli results in the insertion of picture changing operators $\mathcal{X}(z)$ and $\widetilde{\mathcal{X}}(z)$ at $n-2$ points. These can be written in terms of the supercurrents as $\mathcal{X}(z) = \delta(\beta(z))S(z)$ and $\widetilde{\mathcal{X}}(z) = \delta(\widetilde{\beta}(z))\widetilde{S}(z)$.

The insertion of picture changing operators in the string field theory has some extra subtle technical issues to consider. We could choose a picture number in which to represent the string field and then incorporate PCO insertions at $n-2$ points into the definition of the forms $\Omega_{|\vec{\Psi}\rangle}$ [33]. The standard choice would be to use superstring fields in the $(-1,-1)$ picture. A computed correlation function will have a total picture number $(-2,-2)$. Then, using string fields of picture $(-1,-1)$ means inserting $n-2$ $\mathcal{X}$ PCOs and $n-2$ $\widetilde{\mathcal{X}}$ PCOs to compute a correlation function of n string fields. But, it is not as easy as that. There are issues with this approach if PCOs are allowed to collide [52]. Possible solution have been proposed [53, 54]. As we will be considering the current technology of string field theory applied to the ambitwistor string, these issues will not affect us, at least at tree level. Picture changing operators come from the supergeometry, and in our case is $\Sigma_{2|2}$, with picture changing operators $\mathcal{X}$ *and* $\widetilde{\mathcal{X}}$. One part comes from the gauge fixing of each of the two gravitini χ and $\widetilde{\chi}$. As opposed to the action being given by integrating over a middle-dimensional cycle of $\Sigma_{2|1} \times \widetilde{\Sigma}_{2|1}$ in the conventional string [40]. So, in the ambitwistor string we have two sets of *holomorphic* picture changing operators

$$\mathcal{X}(z) = \oint_z \mathrm{d}\omega\ j(\omega)\xi(z), \qquad \widetilde{\mathcal{X}}(z) = \oint_z \mathrm{d}\omega\ j(\omega)\tilde{\xi}(z), \tag{7.2}$$

where $j(z)$ is the BRST current. Working with the OPEs obtained in (2.13) and (6.28), we get

$$\mathcal{X}(z) = c\partial\xi + e^{\phi}P_\mu\psi^\mu + \frac{1}{2}\partial\eta\, e^{2\phi}\tilde{b} + \frac{1}{2}\partial\left(\eta\, e^{2\phi}\tilde{b}\right), \tag{7.3}$$

and

$$\widetilde{\mathcal{X}}(z) = c\partial\tilde{\xi} + e^{\tilde{\phi}}P_\mu\widetilde{\psi}^\mu + \frac{1}{2}\partial\tilde{\eta}\, e^{2\tilde{\phi}}\tilde{b} + \frac{1}{2}\partial\left(\tilde{\eta}\, e^{2\tilde{\phi}}\tilde{b}\right). \tag{7.4}$$

We integrate the PCOs around the relevant punctures, with corresponding insertions

$$\mathcal{X}_0 = \int_{\mathcal{C}} \frac{\mathrm{d}z}{z}\mathcal{X}(z), \tag{7.5}$$

where $\mathcal{C}$ is a contour around the puncture where the picture-changed string field is inserted. The picture $(-1,0)$ string field is then

$$\Psi^{(-1,0)}(z) = \widetilde{\mathcal{X}}_0\Psi^{(-1,-1)}(z) = \oint_z \frac{\mathrm{d}\omega}{\omega - z}\widetilde{\mathcal{X}}(\omega)\Psi^{(-1,-1)}(z). \tag{7.6}$$

with fully developed expression

$$\begin{aligned}
\Psi^{(-1,0)}(z) \quad = \quad & \int \mathrm{d}k\Big(- e(k)\,\eta - \Big(\tilde{e}(k)\,\partial\tilde{\xi}\partial^2 c\,\tilde{c} + i\tilde{f}_\mu(k)\,\widetilde{\Pi}^\mu\,\partial\xi\,\partial\tilde{c}\,\tilde{c}\Big)\,e^{-2\phi} \\
& + 2\tilde{e}(k)\,(P\cdot\widetilde{\psi} + ik\cdot\partial\widetilde{\psi})\tilde{\eta}\,\partial\xi\,\tilde{c}\,e^{\tilde{\phi}-2\phi} + \left(E_{\mu\nu}(k)\,\widetilde{\Pi}^\nu\psi^\mu\tilde{c} + \frac{i}{2}f_\mu(k)\,\psi^\mu\partial\tilde{c}\right)e^{-\phi} \\
& + \frac{1}{2}E_{\mu\nu}(k)\,\tilde{\eta}\psi^\mu\widetilde{\psi}^\nu\,e^{-\phi-\tilde{\phi}} - \tilde{e}(k)\,\left(2\partial\tilde{\eta}\tilde{b}\,\tilde{c} + \frac{3}{2}\partial^2\tilde{\eta}\right)\tilde{\eta}\,\partial\xi\,e^{-2\phi+2\tilde{\phi}} \\
& + \frac{i}{2}\tilde{f}_\mu(k)\,\widetilde{\psi}^\mu(\tilde{\eta}\partial\tilde{c} - 2\partial\tilde{\eta})\partial\xi\,e^{-2\phi+\tilde{\phi}}\Big)c\,e^{ik\cdot X}
\end{aligned} \tag{7.7}$$

We can infer the picture $(0,-1)$ string field expression by looking at the above result (7.7). Finally, the picture $(0,0)$ string field is

$$\Psi^{(0,0)}(z) = \mathcal{X}_0\widetilde{\mathcal{X}}_0\Psi^{(-1,-1)}(z) := \oint_z \frac{\mathrm{d}\omega}{\omega - z}\oint_z \frac{\mathrm{d}\omega'}{\omega' - z}\mathcal{X}(\omega)\mathcal{X}(\omega')\Psi^{(-1,-1)}(z). \tag{7.8}$$

Applying $\mathcal{X}_0$ to (7.7) gives

$$\begin{aligned}
\Psi^{(0,0)}(z) \;=\; & \int \mathrm{d}k \Bigg(E_{\mu\nu}(k)\, \Pi^{\mu}\widetilde{\Pi}^{\nu}\tilde{c} + \frac{1}{2}e(k)\, \partial^2 c + \frac{1}{2}\tilde{e}(k)\, \partial^2 c + \frac{i}{2}f_{\mu}(k)\, \Pi^{\mu}\partial\tilde{c} + \frac{i}{2}\tilde{f}_{\mu}(k)\, \widetilde{\Pi}^{\mu}\partial\tilde{c} \\
& - \left(e(k)\, \left(P\cdot\psi + ik\cdot\partial\psi\right)\eta - \frac{1}{2}E_{\mu\nu}(k)\, \eta\widetilde{\Pi}^{\nu}\psi^{\mu} + \frac{i}{2}f_{\mu}(k)\, \psi^{\mu}\partial\eta \right) e^{\phi} \\
& - \left(\tilde{e}(k)\, \left(P\cdot\widetilde{\psi} + ik\cdot\partial\widetilde{\psi}\right)\eta - \frac{1}{2}E_{\mu\nu}(k)\, \Pi^{\mu}\tilde{\eta}\widetilde{\psi}^{\nu} + \frac{i}{2}\tilde{f}_{\mu}(k)\, \widetilde{\psi}^{\nu}\partial\tilde{\eta} \right) e^{\tilde{\phi}} \\
& - e(k)\, \partial\eta\, \tilde{b}\, \eta\, e^{2\phi} - \tilde{e}(k)\, \partial\tilde{\eta}\, \tilde{b}\, \tilde{\eta}\, e^{2\tilde{\phi}} \Bigg)\, c\, e^{ik\cdot X}, \qquad (7.9)
\end{aligned}$$

where

$$\Pi^{\mu} = P^{\mu} + (k\cdot\psi)\psi^{\mu}, \qquad \widetilde{\Pi}^{\mu} = P^{\mu} + (k\cdot\widetilde{\psi})\widetilde{\psi}^{\mu}. \qquad (7.10)$$

Note that the leading term is what we would expect from the picture $(0,0)$ vertex operator $V(z) = c\tilde{c}\,\varepsilon^{\mu\nu}\,\Pi_{\mu}\widetilde{\Pi}_{\nu}e^{ik\cdot X}$ as found in [1].

7.2 The Surface Superstate

We want to consider interaction terms to cubic order. The main object is the supersymmetric generalisation of the surface state constructed in 3.6. The surface state for the supersymmetric theory was constructed in [28] and it was constrained by requiring to give the right scattering amplitude when contracted with asymptotic states with appropriate ghost and PCO insertions.

The surface state for the supersymmetric ambitwistor theory [28] is an extension of the one developed for the bosonic theory to include the fermionic sector and the superghosts. We denote the supersymmetric surface state also as $\langle\Sigma|$

$$\langle\Sigma| = \int \prod_{i=1}^{n} \mathrm{d}p_{(i)}\, \delta\left(\sum p_{(i)}\right)\, \langle q_1; p_{(1)}|...\langle q_n; p_{(n)}|\, \exp(V_m + V_{\mathrm{gh}} + \widetilde{V}_{\mathrm{gh}})\mathcal{Z}, \qquad (7.11)$$

where the matter contribution to V is

$$V_m = \sum_{m,n}\sum_{i,j}\left(\mathcal{S}^{mn}(z_i,z_j)\tilde{\alpha}^{(i)}_m\cdot\alpha^{(j)}_n + \mathcal{K}^{rs}(z_i,z_j)\psi^{(i)}_r\cdot\psi^{(j)}_s + \mathcal{K}^{rs}(z_i,z_j)\widetilde{\psi}^{(i)}_r\cdot\widetilde{\psi}^{(j)}_s\right), \qquad (7.12)$$

with $\mathcal{S}^{mn}(z_i, z_j)$ is identical to that given in the bosonic theory. The function $\mathcal{K}^{rs}(z_i, z_j)$ is

$$\mathcal{K}^{rs}(z_i, z_j) = \oint_{t_i=0} \mathrm{d}t_i \oint_{t_j=0} \mathrm{d}t_j \, t_i^{-m-\frac{1}{2}} t_j^{-n-\frac{1}{2}} \sqrt{h_i' h_j'} \, \frac{1}{h_i(t_i) - h_j(t_j)}. \tag{7.13}$$

The ghosts contribute to the V_{gh} term, as given in (3.87) but now extended by a similar expression involving the superghosts. The superghost term is identical to that in ordinary superstring [55]. And $\mathcal{Z}$ as in the bosonic theory serves as to take away the $c(z)$ and $\tilde{c}(z)$ ghosts respectively of three of the n string fields that contract with $\langle\Sigma|$. The q in the $\langle q_i; p_{(i)}|$ denote the picture number of the vaccuum being used. It is standard to set $q = -1$, where picture changing operators are inserted to make sure that the overall picture number is -2 at tree level.

7.3 Interactions: The Action to cubic order

Considering what was described in the previous section, the 3-point interaction term can be of the form

$$\{\Psi^3\} = \langle\Sigma||\Psi^{(-1,-1)}\rangle|\Psi^{(-1,-1)}\rangle|\Psi^{(0,0)}\rangle, \tag{7.14}$$

where the picture $(-1,-1)$ states are given by (6.35) and the picture $(0,0)$ state can be derived by substituting (7.9) into (3.74). There are possible alternatives, e.g write it in terms of three $(-1,-1)$ picture string fields with a single pair of $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ PCOs inserted. We could substitute these expressions into the (7.14) and derive a cubic correction to the linearised action (6.51). But it would be cumbersome, long to compute and would take a complicated form. Meanwhile using the string fields (6.29) and (7.9) and evaluating (7.14) as an off-shell correlation function in the worldsheet conformal field theory would be best. In the past it was thought that there was no off-shell extension to on-shell amplitudes in conformal field theory [56], but then a new approach was developed for the bosonic [57] and supersymmetric [58] string theories. There are various subtleties involved in the computation of conformal field theory correlation functions off-shell. These issues have been studied in [57] and, for the most part, are due to the fact that the formalism is no longer conformally invariant and so many of the tools that are usefully employed in conformal field theory are no longer possible. The off-shell amplitudes computed by conformal field theory methods are the same as those computed by string field theory. It easier to deal with the string field interactions as off-shell correlation functions in the conformal field theory, using the conformal field theory description of the string field (6.29) instead of (6.35). The cubic interaction term (7.14) can be written as the off-shell correlation function

$$\{\Psi^3\} = \langle \Psi^{(-1,-1)}(z_1)\Psi^{(-1,-1)}(z_2)\Psi^{(0,0)}(z_3)\rangle. \tag{7.15}$$

The computation is involved and some aspects have been worked out in detail and found to be consistent with the expected action of type II supergravity to cubic order

$$\begin{aligned} S_3 \;=\; & \int \mathrm{d}x\Big(-\frac{1}{8}E_{\mu\nu}\Big(-(\partial_\lambda E^{\lambda\nu})(\partial_\rho E^{\mu\rho}) - (\partial_\lambda E^{\lambda\rho})(\partial_\rho E^{\mu\nu}) - 2(\partial^\mu E_{\lambda\rho})(\partial^\nu E^{\lambda\rho}) \\ & +2(\partial^\mu E_{\lambda\rho})(\partial^\rho E^{\lambda\nu}) + 2(\partial^\lambda E^{\mu\lambda})(\partial^\nu E_{\lambda\rho})\Big) + \frac{1}{2}E_{\mu\nu}f^\mu\tilde{f}^\nu - \frac{1}{2}f^\mu f_\mu\tilde{e} + \frac{1}{2}\tilde{f}^\mu\tilde{f}_\mu e \\ & -\frac{1}{8}E_{\mu\nu}\Big((\partial^\mu\partial^\nu e)\tilde{e} - (\partial^\mu e)(\partial^\nu\tilde{e}) - (\partial^\nu e)(\partial^\mu\tilde{e}) + e\partial^\mu\partial^\nu\tilde{e}\Big) \\ & -\frac{1}{4}f^\mu\Big(E_{\mu\nu}\partial^\nu\tilde{e} + \partial^\nu(E_{\mu\nu}\tilde{e})\Big) + \frac{1}{4}f^\mu\Big((\partial_\mu e)\tilde{e} - e\partial_\mu\tilde{e})\Big) \\ & -\frac{1}{4}\tilde{f}^\nu\Big(E_{\mu\nu}\partial^\mu e + \partial^\mu(E_{\mu\nu}e)\Big) + \frac{1}{4}\tilde{f}^\nu\Big((\partial_\nu e)\tilde{e} - e\partial_\nu\tilde{e})\Big)\Big), \end{aligned} \tag{7.16}$$

where we would also need to include all the $\tilde{\alpha}$ modes part to obtain the full expression.

Adapting the steps given in [48] we can show that, once the auxiliary fields f_μ and $\tilde{f}_\mu$ are eliminated, the correct cubic actions for the NS sector of the Type II string is recovered. We expect (7.16) to be reproduced by the ambitwistor string field theory interaction (7.15). The detailed computation is lengthy and we have not checked it in full. The terms cubic in $E_{\mu\nu}$ in (7.16) follow from the fact that the operator formalism must reproduce the correct three-point on-shell scattering amplitude and so are very easy to check. The string field Ψ was found to be suitable for the linearised theory. But, an issue remains at to whether or not additional contributions to Ψ are necessary in the non-linear theory. A more detailed study of the cubic action is needed. Finally, lets mention that in our paper [3] we give a brief outline of a proposal for the treatment of the action beyond cubic order.

7.4 Discussion

In this thesis, we have presented a model of the ambitwistor string theory of [1] that can be used to give an ambitwistor string field theory description of Type II supergravity. Some details remain to be worked out. In particular in the supersymmetric sector and the details of the perturbation theory but the general structure is clear.

An important issue to resolve is to understand the propagator in full, even in the first quantised ambitwistor string. The treatment in [9, 43, 44], does not yet provide a good geometric understanding of the propagator. Further insights into similarities and differences between the conventional and the ambitwistor string are needed.

It would also be interesting to extend the theory to loop level. Even though the supergravity that this ambitwistor superstring field theory describes does not exist as a quantum theory, the study of the loop integrands in such theories has provided striking proposals for simplifying loop calculations which may be applicable to gauge, gravity and other theories by further symmetries that relate to the CHY equations. It would be interesting to see how the operator formalism could offer an alternative, maybe more efficient way of dealing with these developments.

We haven't consider the Ramond sector of the theory at all. Recently, Sen demonstrated how a kinetic term for the Ramond sector may be introduced, giving a full BV master action for the type II and Heterotic string field theories [15] and also [65]. We would expect that the construction may be extended fully to our ambitwistor superstring field theory.

Of particular interest are the similarities and differences with the conventional string field theory. An important difference is, as noted in [1], that the $X(z)X(\omega)$ OPE is trivial in the ambitwistor theory and so we can consider functions of X, including metrics on curved spacetimes. We note that the generalisation of the ambitwistor worldsheet theory to general curved NS backgrounds is surprisingly straightforward [4].

Finally, lets mention the possibility of constructing a string field theory explicitly in ambitwistor space itself. In this thesis we have worked in cotangent bundle variables X^μ and P_μ and constructed a string field theory in terms of the zero modes of these variables, i.e. in spacetime. It would be interesting to see if by working explicitly in terms of ambitwistor coordinates on $P\mathbb{A}$, we can formulate a version of supergravity in terms of the natural language of ambitwistors.

Appendices

Appendix A

About the definition of bar-delta

What does $\bar{\delta}$ means?

From

$$P(z) = dz \sum_i \frac{k_i}{z - z_i} \tag{A.1}$$

we have

$$\bar{\partial} P(z) = d\bar{z} \wedge dz \sum_i k_i \frac{\partial}{\partial \bar{z}} \Big(\frac{1}{z - z_i} \Big) \tag{A.2}$$

But definition in equation (2.25) of [66] is, for a (0,1) form with $z = x + iy$

$$\delta^1 := \delta(x)\delta(y) d\bar{z} = \frac{1}{2\pi} d\bar{z} \frac{\partial}{\partial z} \frac{1}{z} \tag{A.3}$$

So, that means for us

$$\bar{\delta}(z - z_i) = d\bar{z} \frac{\partial}{\partial z} \frac{1}{z - z_i} \tag{A.4}$$

and

$$\bar{\partial} P(z) = -dz \sum_i k_i \bar{\delta}(z - z_i) = d\bar{z} \wedge dz \sum_i k_i \, \delta^{(2)}(z - z_i) \tag{A.5}$$

where $\delta^{(2)}(z - z_i) = \delta(\mathbb{R}e(z - z_i))\delta(\mathbb{I}m(z - z_i))$

Appendix B

On Morse theory

Morse theory [67] [68] is effective if there are enough smooth functions which have all their critical points nondegenerate. Such functions are called **Morse functions**.
For that, we want to use Sards theorem. From it we prove that there are plenty of Morse functions on any given manifold. We establish the theorem without proof, but lets just say that to porve it we make use of the theory of Lebesgue measure.

Theorem 2. ***Sards Theorem:*** *Let $f : X \in \mathbb{R}^n$ be a $\mathcal{C}^\infty$ map. Then the image $f(\mathcal{C}_f)$ of the set of critical points of f is of measure zero in $\mathbb{R}^n$.*

Definition B.0.1. *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a smooth map with $x \in U$, x be a critical point of f.*
*We say x is a **nondegenerate critical point** if the **Hessian matrix***

$$H_f(x) = \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right) \tag{B.1}$$

is nonsingular.

Lemma B.0.1. *Non-degenerate critical points are isolated.*
Proof:

Consider the map $g = \nabla f : U \to \mathbb{R}^n$, i.e.,

$$g(x) = \left(\frac{\partial f(x)}{\partial x_1}, \cdots, \frac{\partial f(x)}{\partial x_n} \right) \tag{B.2}$$

Then the critical points of f are the zeros of g. We look at the derivative of g at a point x, but this exactly the linear map $\mathbb{R}^n \to \mathbb{R}^n$ given by the Hessian matrix (8.1).
Assuming that this is nonsingular at $x = x_0$, we can apply the Inverse Function Theorem to conclude that g is a local diffeomorphism at x. Therefore, $g(x) = 0$ has a unique solution in a neighbourhood of x_0, which means that there is no other critical points of f in a neighborhood of x_0 .

Note: As the non-degeneracy of critical points is preserved under diffeomorphisms, we can define nondegenerate critical points of a smoot function on a manifold.

Definition B.0.2. *A smooth function on a manifold is called a* ***Morse function*** *if all its critical points are non-degenerate.*

We now want to see that there are plenty of Morse functions. For that we need the following

Lemma B.0.2. *Let X be an n-dimensional smooth submanifold of $\mathbb{R}^N$. Given any point $x \in X$, there exist n coordinate projections $x_{i_1}, \cdots, x_{i_n}$, being such that restricted to a neighbourhood of $x \in X$, they form a coordinate system for X.*

Proof:
It follows directly from the corresponding linear algebra result: If L is any n-dimensional vector subspace of $\mathbb{R}^N$, then some n of the coordinate functions (treated as linear functionals) restricted to L are linearly independent. Now, choose $L = T_xX$ and observe that the derivative of x_i restricted to X is x_i restricted to T_xX. Now we choose $x_{i_1}, \cdots, x_{i_n}$ so that on L they are independent and define $\phi_k = x_{i_k}, 1kn$. Then $\phi : X \to \mathbb{R}^n$ is such that $D(\phi)_x$ is an isomorphism. So from the inverse function theorem, we get the proof.

We can also now establish the following theorem without proof. But lets mention the strategy. We cover X with a countable family of open subsets $\{U_i\}$ such that the statement is true for each U_i in place of X. And then the measure of a countable union of Lebesgue measure zero sets, which is of course of zero Lebesgue measure.

Theorem 3. *Let $X \to \mathbb{R}^N$ be a smooth manifold and $f : X \to \mathbb{R}$ be any smooth function. Then for almost all vectors $u \in \mathbb{R}^N$, the mapping f_u defined by*

$$f_u(x) = f(x) + \langle x, u \rangle$$

is a Morse function on X.

We can now then establish a useful theorem that we leave also without proof

Theorem 4. *Let $X \subset \mathbb{R}^N$ be any smooth closed manifold. Then for almost all points $z \in \mathbb{R}^N$, the square of the distance function from y restricted to X is a proper Morse function.*

We are now finally ready to introduce Morse Lemma

Theorem 5. ***Morse Lemma:***

Let $f : X \to \mathbb{R}$ be a smooth map and $p \in X$ be a non-degenerate critical point of f. Then there exists a chart (U, ϕ) for X near p such that $\phi(p) = 0$ and

$$f \circ \phi^{-1}(x) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2 \tag{B.3}$$

for all $x \in \phi(U)$ and for some k.

Proof:

to be shown later.

Finally we give a useful definition and an important lemma that follows from Morse Lemma.

Definition B.0.3. *At a non-degenerate critical point p of a smooth function f, the Hessian $H_f(p)$ is non-singular and the number k that occurs in B.3 is the number of negative eigenvalues of $H_f(p)$. This number is called the* ***index of f at p****.*

Remark B.0.1. *If the index is zero, then it is clear that f(p) is a local minimum and if the index is equal to n then f(p) is a local maximum. For any other value of the index, f(p) fails to be either a minimum or a maximum, i.e. they are saddle points.*

and

Remark B.0.2. *Consider the vector field grad(f) where f is a Morse function. By definition, the critical points of f are precisely the zeros of this vector field. Moreover, the local index of the vector field is precisely equal to $(1)^k$ where k is the index of the critical point as will be seen below. If ν_k denotes the number of critical points of f of index k, let us denote the alternate sum*

$$e(f) := \sum_k (-1)^k \nu_k.$$

And now follows the important lemma

Lemma B.0.3. *The number $e(f)$ is equal to the index of the vector grad(f) and hence is equal to the self-intersection number of the manifold X.*

Proof:

If p is a critical point of f of index k we will show that $ind_p(grad(f)) = (1)^k$, from which the lemma would follow.

By Morse Lemma, we can choose coordinates at p for X so that

$$f(\mathbf{x}) = f(p) - \sum^{k} x_i^2 + \sum_{i=k+1} x_i^2$$

Therefore, in this neighborhood of p, gradf has the form

$$\mathbf{x} \to (-2x_1, \cdots, -2x_k, 2x_{k+1}, \cdots, 2x_n).$$

And so the local index of grad(f) around this point is $(1)^k$ as required.

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