A conformal approach to the existence and asymptotic properties of solutions to the Einstein field equations

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Declaration

I, Marica Minucci, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaborations and publications:

This thesis incorporates work in collaboration with Dr. Rodrigo Panosso Macedo and Dr. Juan Antonio Valiente Kroon which can be found in the following papers:

- M. Minucci & J. A. Valiente Kroon, *A conformal approach to the stability of Einstein spaces with spatial sections of negative scalar curvature*, Class. Quantum Grav. **38**, 145026 (2021).

- M. Minucci & J. A. Valiente Kroon, *On the non-linear stability of the Cosmological region of the Schwarzschild-de Sitter spacetime*, arXiv:2302.04004 [gr-qc] (2023).
- M. Minucci, R. Panosso Macedo, & J. A. Valiente Kroon, *The Maxwell-scalar field system near spatial infinity*, J. Math. Phys. **63**, 082501 (2022).

Abstract

The study of the initial value problem in General Relativity by means of conformal methods was initiated by H. Friedrich in 1986. In this seminal work, the standard conformal Einstein field equations are used to prove the non-linear stability of the de Sitter spacetime. These equations constitute the main technical tool of this thesis. In the first part of this thesis, a technique based on a more general formulation of these equations, the extended conformal Einstein field equations, and a conformal Gaussian gauge is used to establish the non-linear stability of de Sitter-like spacetimes. The gauge freedom associated to the field equations is fixed using the properties of the conformal geodesics. The conformal Gaussian gauge system allows recasting the evolution equations as a symmetric hyperbolic system, which enables the use of standard Cauchy stability results. The same strategy is used to study the non-linear stability of the Cosmological region of the Schwarzschild-de Sitter spacetime. The key observation is that this region can be covered by a non-intersecting congruence of conformal geodesics. Thus, the future domain of dependence of suitable spacelike hypersurfaces can be expressed in terms of a conformal Gaussian gauge. A perturbative argument allows then to prove existence and stability results close to the conformal boundary, excluding the asymptotic points where the Cosmological horizon intersects the conformal boundary. In the second part of this thesis, the asymptotic properties of the Maxwell-scalar field system on a flat spacetime are studied by means of the framework of the cylinder at spatial infinity. The analysis is aimed to understand the effects of the non-linearities of this system on the regularity of solutions and polyhomogeneous expansions at the critical sets. The main result is that the non-linear interaction causes both fields to be more singular at the conformal boundary than when the fields are non-interacting.

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All theses, no matter how modest they may appear, have to begin somewhere. This particular thesis begins here, with the people without whom this PhD would have not been the same.

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To all of you, I would like to dedicate this poem:

'Hope' is the thing with feathers -
That perches in the soul -
And sings the tune without the words -
And never stops - at all -
And sweetest - in the Gale - is heard -
And sore must be the storm -
That could abash the little Bird
That kept so many warm -
I've heard it in the chillest land -
And on the strangest Sea -
Yet - never - in Extremity,
It asked a crumb - of Me.

Emily Dickinson - 'Hope' is the thing with feathers

To Davide,

who always believed in me.

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Chapter 1

Introduction

> “*The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.*”
>
> Hermann Minkowski, *1908*

In 1911, Einstein formulated the *equivalence principle*, thus drawing attention to gravitation for the first time since Newton. This principle postulates that the mechanical phenomena, but also the optical and electromagnetic ones, in a gravitational field and a field produced by an accelerated observer are equivalent. Einstein deduced the redshift of the spectral lines of the Sun and the deflection of the light rays around a star during a total eclipse. However in the latter case, he predicted only the partial deviation since his work still relies on *Newton's law of gravitation*. The equivalence principle laid the foundation for a general theory of gravity not only restricted to *uniform motions*, as it indicated a way to counter the objections raised in the past against such an extension since Newton's time. At first, this approach seemed to suggest that it was impossible to dispute the criticism until one removes the possibility of complications associated with gravitation. Even though Einstein had overcome the major hurdles in developing his theory in 1913, this was published in its complete form in 1916.

Einstein's Theory of General Relativity is the most successful theory of gravity. Since its formulation, its predictions have been confirmed via several observational tests. The most recent one represented by the first image of the *supermassive black hole* at the centre of our galaxy — known as *Sagittarius* A^* — has been published by the Event Horizon Telescope (EHT) Collaboration in May 2022. This image shows a dark central region — the black hole *shadow* — surrounded by a bright ring-like structure generated by glowing gas. Thus, it describes the light bent by the powerful gravity of the black hole, which is four million times more massive than the Sun.

One of the main differences with Newton's theory of gravity, is that in Einstein's gravity the gravitational field acquires its own dynamical properties. Its evolution is complicated even in the absence of matter. In stark contrast with Newton's theory, in which the field equation —i.e. the Poisson equation — combines with the boundary condition that the field vanishes at infinity so that the gravitational field vanishes when there is no matter. In Einstein's theory of General Relativity, the equations governing the gravitational field, known as *Eintein field equations*, allow an idealised situation representing gravitational waves in an otherwise empty universe without any matter source. This reflects the different mathematical nature of the equations involved in these two cases. The Poisson equation is elliptic, whereas the Einstein equations are 'essentially' hyperbolic. More precisely, the latter are *gauge hyperbolic*, meaning they are hyperbolic by imposing suitable *gauge conditions*. Solutions of hyperbolic equations can be determined uniquely by their values on a suitable initial hypersurface.

The *Cauchy problem* is the task of establishing a one-to-one correspondence between solutions and initial data and studying the properties of this correspondence. The solution determined by particular initial data may be *global* —i.e. defined on the whole space where the equations are defined, or *local* —i.e. only defined on a neighbourhood of the initial hypersurface. For linear hyperbolic equations it is, in general, possible to solve the Cauchy problem globally. For non-linear hyperbolic equations this is much more difficult and whether it can be done or not must be decided on a case-by-case basis.

The previous discussion shows that studying the Einstein field equations corresponds to analysing a system of non-linear hyperbolic equations. For these equations, one must expect to encounter the problem that generically solutions of the Cauchy problem for non-linear hyperbolic equations do not exist globally. This issue is complicated by the fact that the distinction between local and global solutions made above does not apply: to define the notions of local and global we used the concept of *background space* —i.e. the space where the equations are defined. As will be seen in the following, in the case of the Einstein field equations there is no background space; the spacetime manifold is part of the solution. Thus, one talks about the local and global properties of the solutions and not about local and global solutions. On the other hand, the lack of background space is also responsible for the existence of solutions of the Einstein field equations with global features such as the formation of black holes.

1.1 The Cauchy problem in General Relativity

The study of the Cauchy problem in General Relativity started in the decade 1950 with the work of Fourès-Bruhat [18]. In this work, it was shown that if the gauge is fixed appropriately, the equations governing General Relativity split into constraint equations and evolution equations. In more detail, the equations representing the core of Einstein's theory of gravitation —i.e. the *Einstein field equations*— in the vacuum case with vanishing cosmological constant are

$$\tilde{R}_{ab} = 0. \tag{1.1}$$

In terms of the local coordinates x^μ satisfying the condition

$$\tilde{\nabla}_\nu \tilde{\nabla}^\nu x^\mu = 0,$$

known as the *harmonicity condition*, these equations can be recasted as a system of equations for the metric coefficients $\tilde{\boldsymbol{g}}$ of the form

$$\tilde{g}^{\lambda\rho}\partial_\lambda\partial_\rho\tilde{g}_{\mu\nu} = Q_{\mu\nu}(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{\Gamma}}), \tag{1.2}$$

where the right-hand side depends on these coefficients and the *Christoffel symbols* $\tilde{\boldsymbol{\Gamma}}$, which are given by

$$\tilde{\Gamma}_\mu{}^\nu{}_\lambda = \frac{1}{2}\tilde{g}^{\nu\rho}(\partial_\mu\tilde{g}_{\rho\lambda} + \partial_\lambda\tilde{g}_{\mu\rho} - \partial_\rho\tilde{g}_{\mu\lambda}).$$

Now, by defining

$$\tilde{H}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{S}_{\mu\nu},$$

where

$$\tilde{S}_{\mu\nu} \equiv \tilde{\Gamma}_\alpha{}^\mu{}_\beta\partial_\mu\tilde{g}^{\alpha\beta} + \tilde{g}^{\alpha\beta}\partial_\mu\tilde{\Gamma}_\alpha{}^\mu{}_\beta + \tilde{\Gamma}_\alpha{}^\nu{}_\beta\partial_\nu\tilde{g}^{\alpha\beta} + \tilde{g}^{\alpha\beta}\partial_\nu\tilde{\Gamma}_\alpha{}^\nu{}_\beta,$$

one is led to consider a Cauchy problem for the *reduced Einstein field equations*

$$\tilde{H}_{\mu\nu} = 0, \tag{1.3}$$

which constitute a system of wave equations for the metric components $\tilde{g}_{\mu\nu}$.

The spacetime is foliated by a family of 3-dimensional spacelike Cauchy hypersurfaces $\tilde{\mathcal{S}}$. Since the spacetime metric $\tilde{\boldsymbol{g}}$ is constrained by the Einstein equations, one expects that the induced metric $\tilde{\boldsymbol{h}}$ and extrinsic curvature $\tilde{\boldsymbol{K}}$ are also constrained. In fact, let $\tilde{\mathcal{S}}$ denote a 3-dimensional spacelike hypersurface with normal $\tilde{n}^\mu$. By projecting the Einstein field equations along the normal direction to $\tilde{\mathcal{S}}$, we obtain the so-called *Hamiltonian and momentum constraints*. Moreover, by projecting the Einstein field equations on $\tilde{\mathcal{S}}$, one obtains a set of evolution equations for the data $\tilde{\boldsymbol{h}}$ and $\tilde{\boldsymbol{K}}$. Thus, one can formulate an initial value problem for the equations (1.3) supplemented with data

$$\tilde{g}_{\mu\nu}|_{\tilde{\mathcal{S}}} = \tilde{h}_{\mu\nu}, \qquad \tilde{n}^\alpha\partial_\alpha\tilde{g}_{\mu\nu}|_{\tilde{\mathcal{S}}} = 2\tilde{K}_{\mu\nu},$$

satisfying

$$\tilde{g}^{\alpha\beta}\tilde{\Gamma}_\alpha{}^\mu{}_\beta|_{\tilde{\mathcal{S}}} = 0, \qquad \tilde{n}^\nu\partial_\nu(\tilde{g}^{\alpha\beta}\tilde{\Gamma}_\alpha{}^\mu{}_\beta)|_{\tilde{\mathcal{S}}} = 0,$$

where $\tilde{h}_{\mu\nu}$ is a 3-dimensional Riemannian metric and $\tilde{K}_{\mu\nu}$ is a symmetric tensor on $\tilde{\mathcal{S}}$. In this formulation of General Relativity, the initial data corresponds to a triple $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$.

Conversely, one can ask whether an initial data set satisfying the constraint equations gives rise to a unique spacetime upon evolution. A fundamental result in this regard was proven by Choquet-Bruhat and Geroch in [11] where it was shown that associated to each triple $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ satisfying the constraint equations in vacuum, there exists a unique maximal globally hyperbolic development $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. The adjective hyperbolic refers to the fact that the evolution equations obtained in this formulation of the Einstein field equations are hyperbolic. This property is fundamental as it allows us to formulate and establish relativistic causality within General Relativity.

1.2 Penrose's proposal

In order to have a better understanding of Einstein's theory of General Relativity, it is necessary to understand the causal structure in the context of the Einstein field equations. In any investigation of the problem, the consequences of these equations must be taken into account. However, gaining control of the evolution process defined by these equations remains a significant challenge. Along with the question of how to combine the analysis of the Einstein field equations with that of the causal structure, which are closely related in Einstein's theory. The causal structure is determined by the null cones of the metric due to the local causality requirement. On the other hand, it can be used to reconstruct the null cone structure. Moreover, the null hypersurfaces defined by the solutions determine the physical characteristics of the field equations that govern the evolution process. These relationships between causal structure, null cone structure, and physical characteristics offer opportunities for the desired analysis but also contribute to the *quasi-linearity* of the equations—see [29].

The work of Bondi, van der Burg and Metzner [7], Sachs [62, 63], Newman and Penrose [58] paved the way towards the formulation of the geometric concept of *asymptotic simplicity* based on the aforementioned relationships. In 1963, Penrose introduced this concept which suggests analysing the asymptotic behaviour of gravitational fields in terms of the smooth extensibility of the conformal structure through null infinity [59, 60].

The basic model is provided by Minkowski spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{\eta}})$, with $\tilde{\mathcal{M}} = \mathbb{R}^4$ and line element $\tilde{\boldsymbol{\eta}}$ written in Cartesian coordinates $(\tilde{x}^\mu) = (\tilde{t}, \tilde{x}^\alpha)$ as

$$\tilde{\boldsymbol{\eta}} = \eta_{\mu\nu}\mathbf{d}\tilde{x}^\mu \otimes \mathbf{d}\tilde{x}^\nu,$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. By introducing spatial spherical coordinates $(\tilde{\rho}, \theta, \phi)$ defined by $\tilde{\rho}^2 \equiv \delta_{\alpha\beta}\tilde{x}^\alpha\tilde{x}^\beta$ where $\delta_{\alpha\beta} = \mathrm{diag}(1,1,1)$, and an arbitrary choice of (θ, ϕ) on $\mathbb{S}^2$, the metric $\tilde{\boldsymbol{\eta}}$ can be written as

$$\tilde{\boldsymbol{\eta}} = -\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2\boldsymbol{\sigma}, \tag{1.4}$$

with $\tilde{t} \in (-\infty, \infty)$, $\tilde{\rho} \in [0, \infty)$ and where $\boldsymbol{\sigma}$ denotes the standard metric on $\mathbb{S}^2$. By introducing the coordinate transformation

$$\tilde{t}(\tau, \chi) = \frac{\sin\tau}{\cos\tau + \cos\chi}, \qquad \tilde{\rho}(\tau, \chi) = \frac{\sin\chi}{\cos\tau + \cos\chi},$$

and introducing a *conformal rescaling*

$$\boldsymbol{\eta} = \Theta^2\tilde{\boldsymbol{\eta}} \tag{1.5}$$

with conformal factor $\Theta = \cos\tau + \cos\chi$, one obtains the *conformal metric* $\boldsymbol{\eta}$ in the form

$$\boldsymbol{\eta} = -\mathbf{d}\tau \otimes \mathbf{d}\tau + \mathbf{d}\chi \otimes \mathbf{d}\chi + \sin^2\chi\boldsymbol{\sigma}. \tag{1.6}$$

This metric is locally identical to the metric $\boldsymbol{g}_{\mathcal{E}}$ of a spherically symmetric spacetime $(\mathcal{M}_{\mathcal{E}}, \boldsymbol{g}_{\mathcal{E}})$ with $\mathcal{M}_{\mathcal{E}} \equiv \mathbb{R} \times \mathbb{S}^3$ known as *Einstein static universe*—see Chapter 5 of [44]. Consequently, the rescaling procedure compactifies Minkowski spacetime into a region of the Einstein cylinder corresponding to the domain

$$\tilde{\mathcal{M}} = \{p \in \mathcal{M}_{\mathcal{E}} \mid |\tau \pm \chi| < \pi,\ \chi \geq 0\}.$$

One can analytically extend (1.6) to the whole of the Einstein static universe, where $-\infty < \tau < \infty$ and χ, θ, ϕ are regarded as coordinates on $\mathbb{S}^3$. Even though these coordinates are singular at $\chi = 0$, $\chi = \pi$ and $\theta = 0$, $\theta = \pi$, these singularities can be removed by transforming to other local coordinates in the neighbourhood where (1.6) is singular. Thus, the conformal metric, the conformal factor and the underlying manifold can be smoothly extended to yield the *conformally compactified Minkowski spacetime* with manifold

$$\mathcal{M} = \{p \in \mathcal{M}_{\mathcal{E}} \mid |\tau \pm \chi| \leq \pi,\ \chi \geq 0\} = \tilde{\mathcal{M}} \cup \mathscr{I} \cup i^0 \cup i^+ \cup i^-.$$

The boundary of $\mathcal{M}$ may be thought of as representing the conformal structure of infinity of Minkowski spacetime. The set $\mathscr{I}$ represents the *conformal boundary* which is the set of points where $\Theta = 0$ and $\mathbf{d}\Theta \neq 0$. The two components

$$\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^- = \{p \in \mathcal{M}_{\mathcal{E}} \mid |\tau \pm \chi| = \pi,\ \chi > 0\}$$

represent *future and past null infinity.* They are generated by the future and past endpoints, respectively, acquired by the null geodesics. Moreover, since $\boldsymbol{\eta}(\mathbf{d}\Theta, \mathbf{d}\Theta)|_{\mathscr{I}^\pm} = 0$ these are null hypersurfaces with respect to the conformal metric $\boldsymbol{\eta}$.

The two points

$$i^+ = \{p \in \mathcal{M}_{\mathcal{E}} \mid \tau = \pi,\ \chi = 0\}, \qquad i^- = \{p \in \mathcal{M}_{\mathcal{E}} \mid \tau = -\pi,\ \chi = 0\},$$

where $\Theta = 0$, $\mathbf{d}\Theta = 0$ and $\mathrm{Hess}_{\boldsymbol{\eta}}\Theta = -\boldsymbol{\eta}$ correspond to the future and past endpoints of the timelike geodesics and thus represent *future and past timelike infinity.*

Finally, the point

$$i^0 = \{p \in \mathcal{M}_{\mathcal{E}} \mid \tau = 0,\ \chi = \pi\},$$

where $\Theta = 0$, $\mathbf{d}\Theta = 0$ and $\mathrm{Hess}_{\boldsymbol{\eta}}\Theta = \boldsymbol{\eta}$ corresponds to the point where spacelike geodesics run in both directions and thus represents *spacelike infinity.* By including this point, the Cauchy hypersurface corresponding to $t = 0$ of the Minkowski spacetime with metric induced by $\boldsymbol{\eta}$ conformally extends to the sphere $\mathbb{S}^3$ endowed with its standard metric.

The conformal structure of Minkowski spacetime just described is regarded as the 'normal' behaviour of a spacetime at infinity. One can obtain spacetimes locally identical to $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{\eta}})$ but with different topological properties. Moreover, it is possible to obtain alternative conformal representations of Minkowski spacetime —see Chapter 6.

The process of extending the *differential structure* and the *conformal structure* of a Minkowski spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{\eta}})$ to obtain a smooth conformal extension $(\mathcal{M}, \boldsymbol{\eta})$ was largely generalised by Penrose in [59]. In this work, it is suggested that this construction applies to many solutions to the Einstein field equations, allowing us to relate the null cone structure to the structure of the field equations and the large-scale behaviour of their solutions. In the case of the solution to the vacuum Einstein field equations near $\mathscr{I}$, the latter is a null hypersurface for the conformal metric $\boldsymbol{g}$ representing future (past) null infinity. This structure allows us to obtain the precise fall-off behaviour required in asymptotic analysis. In case $\mathscr{I}$ is sufficiently smooth, local differential geometry can be used to simplify the analysis.

1.3 The conformal Einstein field equations

Although the study of the initial value problem in General Relativity started in the decade 1950 with the work of Fourès-Bruhat, the global non-linear stability of generic solutions to the Einstein field equations is still an open problem. The first global non-linear stability results in General Relativity appeared in the decade of 1980 and are due to the work of Friedrich [23, 24]. One of the main features in these works is the use of the so-called *conformal Einstein field equations* to pose an initial value problem. The central concept of the conformal Einstein field equations is that of a *conformal transformation*.

The relevance of the construction introduced by Penrose in 1963 goes beyond the study of asymptotics and isolated gravitational systems. In his proposal, one considers a *physical spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, where $\tilde{\mathcal{M}}$ is a 4-dimensional manifold and $\tilde{\boldsymbol{g}}$ is a Lorentzian metric, which is a solution to the Einstein field equations

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} + \lambda\tilde{g}_{ab} = \kappa T_{ab}, \tag{1.7}$$

where $\tilde{R}_{ab}$ and $\tilde{R}$ are the Ricci tensor and the Ricci scalar of $\tilde{\boldsymbol{g}}$ respectively, λ is the Cosmological constant and $\tilde{T}_{ab}$ is the energy momentum tensor. Then, one conformally embeds the *physical spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ into the *unphysical spacetime* $(\mathcal{M}, \boldsymbol{g})$ via a *conformal rescaling* with *conformal factor* Ξ. Since the *unphysical metric* $\boldsymbol{g}$ contains the same causal information as the *physical metric* $\tilde{\boldsymbol{g}}$, a natural question is *how the Einstein field equations* *behave under a conformal transformation of the metric.* The theory proposed by Einstein gives no reason to expect that the field equations would be particularly well-behaved under conformal rescalings. These equations being chosen to determine isometry classes of solutions are not *conformally covariant*. This can be seen from the transformation law of the Ricci tensor

$$R_{ab} - \tilde{R}_{ab} = -\frac{2}{\Xi}\nabla_a\nabla_b\Xi - g_{ab}g^{cd}\left(\frac{1}{\Xi}\nabla_c\nabla_d\Xi - \frac{3}{\Xi^2}\nabla_c\Xi\nabla_d\Xi\right), \tag{1.8}$$

where the righthand side $\boldsymbol{\nabla}$ denotes the connection associated to $\boldsymbol{g}$. If the Einstein field equations (1.7) are assumed in this relation and the conformal factor Ξ is considered as a given function on the solution manifold, one obtains an equation for the conformal metric $\boldsymbol{g}$. The resulting equation has two deficiencies. One is the occurrence of factors Ξ^{-1} which makes the equation singular when $\Xi = 0$. The second one is due to the function Ξ and the manifold underlying the solution not being given a priori. They are related to the global geometry of the solution and must be determined jointly with the metric.

Nevertheless, the Einstein equations allow us to resolve these problems. The resulting equations are known as *conformal Einstein field equations*. These equations are *conformally regular* in the sense that there exists a conformal representation of these equations which do not contain factors Ξ^{-1}. This is in stark contrast with the *conformally singular* equations like the massive Klein-Gordon equations and *conformally invariant* equations like the Maxwell or Yang-Mills equations. Furthermore, the conformal Einstein field equations constitute a system of differential conditions on the curvature tensors with respect to the Levi-Civita connection of $\boldsymbol{g}$ and the conformal factor Ξ. The original formulation of these equations requires the introduction of so-called *gauge source functions* to construct evolution equations —see e.g. [22]. An alternative approach to gauge fixing is to adapt the analysis to a congruence of curves. A natural candidate for congruence is given by *conformal geodesics* —a conformally invariant generalisation of the standard notion of geodesics. Using these curves to fix the gauge allows us to define a *conformal Gaussian gauge system*. To combine this gauge choice with the conformal Einstein field equations it is necessary to make use of a more general version of the latter —the *extended conformal Einstein field equations*.

In this thesis, we discuss the origin, the properties and some applications of the *conformal Einstein field equations*.

1.4 Main results in this thesis

One of the problems one encounters whilst analysing the conformal Einstein field equations is the issue of gauge freedom. In the classical treatment of the Cauchy problem in General Relativity, a suitable choice of coordinates allows us to reduce the equations to a system of wave equations for the metric components. In the original treatment of the conformal Einstein field equations, the hyperbolic reduction strategy used led to a first-order system of equations. In the case of the extended conformal Einstein field equations, the gauge fixing is performed by exploiting a congruence of curves with special conformal properties: *conformal geodesics*. This hyperbolic reduction strategy leads to a first-order system of symmetric hyperbolic equations. The latter approach is used to discuss the non-linear stability of the de Sitter-like spacetime in Chapter 3. This spacetime is a solution to the vacuum Einstein field equations with positive Cosmological constant with spatial sections of constant negative scalar curvature. In this chapter, perturbations of exact initial data corresponding to the de Sitter-like spacetime are considered. Then the theory of symmetric hyperbolic systems with compact spatial sections is used to obtain a non-linear stability result for small perturbation of the exact solution.

A common feature that is exploited in the analysis of constant curvature spacetimes using conformal methods is that they can be conformally embedded in a cylinder. The latter is convenient as an explicit solution to the conformal Einstein field equations can be identified. In other words, most of the existence and stability results using the conformal Einstein field equations have been restricted to the analysis of perturbations of conformally flat spacetimes. Therefore, an interesting question is whether the conformal Einstein field equations can be exploited in the analysis of global properties of non-conformally flat spacetimes and, in particular, in the analysis of the stability of black hole spacetimes. On the other hand, from a physical point of view, observations have established that the

Universe is expanding. Therefore, spacetimes describing isolated systems embedded in a de Sitter universe constitute a class of physically relevant spacetimes to be analysed. Given these remarks, in Chapter 4 the Schwarzschild-de Sitter spacetime is analysed using the extended conformal Einstein field equations. The presence of a Cosmological constant with a de Sitter-like value is of importance as it implies that the conformal boundary is spacelike. The insight gained from the analysis of the evolution of the exact initial data corresponding to the Schwarzschild-de Sitter spacetime is used to discuss non-linear perturbations of this exact data by exploiting the theory of symmetric hyperbolic systems. The spacetimes constructed in this way can be regarded as perturbations of the Schwarzschild-de Sitter spacetime. In view of the domain of dependence properties of the solutions to the Einstein field equations, the stability of the asymptotic region can be analysed independently of the black hole exterior region. Hence, in the discussion in Chapter 4, the domain of dependence of the initial data is contained in the region corresponding to the Cosmological region of the Schwarzschild-de Sitter spacetime.

One of the main difficulties in establishing a global result for the stability of the Minkowski spacetime using conformal methods lies in the fact that the initial data for the conformal Einstein field equations are not smooth at spatial infinity i^0. In the case of the *problem of spatial infinity*, a milestone in the resolution of this problem is the construction of a new representation of spatial infinity known as the *cylinder at spatial infinity*. With this motivation in mind, in Chapter 6, this framework is used to study the behaviour of the Maxwell-scalar field system near spatial infinity. In particular, it is shown that unless the initial data is fine-tuned, this system exhibits a singular behaviour at the critical sets where null infinity meets spatial infinity.

Collectively, these results show how the conformal Einstein field equations and, more generally, conformal methods can be employed to analyse perturbations of spacetimes of interest and extract information about their conformal structure.

1.5 Outline of the thesis

This thesis consists of two parts. In Chapter 2, the various formulations of the conformal Einstein field equations are introduced along with a discussion concerning hyperbolic reduction strategies. This chapter includes the mathematical preliminaries of the two parts. The first part of this thesis is concerned with the question of the non-linear stability of Cosmological spacetimes. In Chapter 3, we discuss the non-linear stability of de Sitter-like spacetimes with spatial sections of negative scalar curvature, which is the first result of this thesis. In Chapter 4, the same approach is used to discuss the non-linear stability of the Cosmological region of the Schwarzschild-de Sitter spacetime. The second part of this thesis is devoted to the problem of spatial infinity. In Chapter 5, the space-spinor formalism is introduced to obtain a hyperbolic reduction. In Chapter 6, this procedure is paired with Friedrich's representation of spatial infinity to study the asymptotic properties of the Maxwell-scalar field system propagating on Minkowski spacetime. Altogether, these results show how conformal methods can be used to study the non-linear perturbations of spacetimes to obtain information about the global regularity of these solutions and their asymptotic properties.

1.6 Notations and Conventions

The signature convention for Lorentzian spacetime metrics will be $(-,+,+,+)$. In the rest of this thesis $\{{}_{a},{}_{b},{}_{c},...\}$ denote spacetime abstract tensor indices and $\{{}_{\boldsymbol{a}},{}_{\boldsymbol{b}},{}_{\boldsymbol{c}},...\}$ will be used as spacetime frame indices taking the values $0,...,3$. In this way, given a basis $\{\boldsymbol{e_a}\}$ a generic tensor is denoted by T_{ab} while its components in the given basis are denoted by $T_{\boldsymbol{ab}} \equiv T_{ab}\boldsymbol{e_a}^{a}\boldsymbol{e_b}^{b}$. The Greek indices ${}_{\mu}, {}_{\nu}, \ldots$ denote spacetime coordinate indices while the indices ${}_{\alpha}, {}_{\beta}, \ldots$ denote spatial coordinate indices. In addition to the index notation described above, when convenient, it is also used an index-free notation. Given a 1-form $\boldsymbol{\omega}$ and a vector $\boldsymbol{v}$, the action of $\boldsymbol{\omega}$ on $\boldsymbol{v}$ is denoted by $\langle \boldsymbol{\omega}, \boldsymbol{v} \rangle$. The *musical isomorphisms* $^{\sharp}$ and $^{\flat}$ are used to denote the contravariant version $\boldsymbol{\omega}^{\sharp}$ of $\boldsymbol{\omega}$ and the covariant version $\boldsymbol{v}^{\flat}$ of

$\boldsymbol{v}$ with respect to a given Lorentzian metric $\boldsymbol{g}$. This notation can be extended to tensors of higher rank.

Part of the analysis will require the use of spinors. In this respect, the notation and conventions of Penrose & Rindler [61] will be followed. In particular, capital Latin indices $\{_A, _B, _C, ...\}$ will denote abstract spinor indices while boldface capital Latin indices $\{_{\boldsymbol{A}}, _{\boldsymbol{B}}, _{\boldsymbol{C}}, ...\}$ will denote frame spinorial indices with respect to a specified spin dyad $\{\epsilon_{\boldsymbol{A}}{}^A\}$.

The conventions for the curvature tensors are fixed by the relation

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) v^c = R^c{}_{dab} v^d.$$

Chapter 2

Methods of conformal geometry

2.1 Introduction

Given Penrose's characterisation of the asymptotic behaviour of gravitational fields in terms of the extensibility of the conformal structure across null infinity [59, 60], it is possible to deduce the global structure of spacetimes from an analysis of the behaviour under conformal rescalings of the Einstein field equations. This requires a suitable conformal representation consisting of a system of equations for all the conformal fields which is regular at the conformal boundary and whose solutions imply solutions to the Einstein field equations. The resulting equations are the so-called *conformal Einstein field equations* and were originally introduced by Friedrich in 1981 [19].

This chapter will discuss two different versions of the conformal Einstein field equations: the *standard conformal Einstein field equations* and a more general version of these equations, the *extended conformal Einstein field equations*. Particular emphasis is given to the relation between the frame formulations of these two versions. Since all the applications of the conformal Einstein field equations discussed in this thesis are concerned with the vacuum case, we only consider the conformal formulations equivalent to the vacuum Einstein field equations. The rest of the chapter provides a brief overview of the tools of conformal geometry inspired by [81], which serve as mathematical preliminaries for the analysis in the following chapters.

2.2 Conformal transformation relations

This section presents the formalism necessary to introduce the *conformal Einstein field equations*. These equations provide a conformal equivalent of the vacuum Einstein field equations—see [27, 81].

2.2.1 Conformal rescalings

Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote a *physical spacetime*. In order to construct an *unphysical spacetime* $(\mathcal{M}, \boldsymbol{g})$ whose manifold $\mathcal{M}$ has a boundary $\mathscr{I}$ and a metric $\boldsymbol{g}$, one introduces a *conformal rescaling*

$$\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}, \tag{2.1}$$

where Ξ is the so-called *conformal factor*. This relation is such that $\tilde{\mathcal{M}}$ is conformal to the interior of $\mathcal{M}$ and maps points belonging to the infinity of $\tilde{\mathcal{M}}$ to a finite position in $\mathscr{I}$ —see Figure 2.1. The function Ξ on $\mathcal{M}$ is a *boundary defining function*. In particular, the *conformal boundary* $\mathscr{I}$ is defined as the set of all points characterised by a vanishing Ξ. Accordingly, the *unphysical spacetime* can be defined as the union of the physical spacetime with the conformal boundary

$$\mathcal{M} \equiv \tilde{\mathcal{M}} \cup \mathscr{I},$$

where $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$ is the union of *future null infinity* and *past null infinity*. These spacetimes have the same causal structure — i.e. a trajectory which is timelike, spacelike or lightlike with respect to $\tilde{\boldsymbol{g}}$ is so also with respect to $\boldsymbol{g}$.

The conformal rescaling gives rise to an equivalence class of metrics conformally related to $\tilde{\boldsymbol{g}}$ on $\tilde{\mathcal{M}}$, known as the *conformal structure* $[\tilde{\boldsymbol{g}}]$.

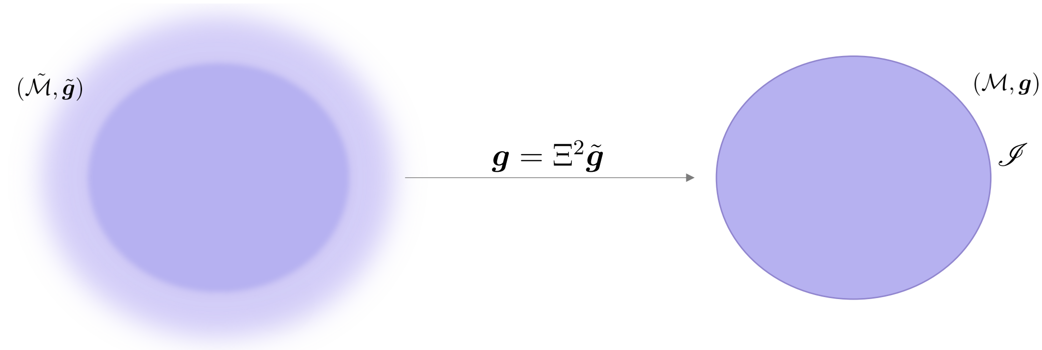

Figure 2.1: The infinite physical spacetime manifold $\tilde{\mathcal{M}}$ is mapped into the unphysical conformally equivalent manifold $\mathcal{M}$ with boundary $\mathscr{I}$. This picture is adapted from [81].

2.2.2 The change of connection

Let $\boldsymbol{\nabla}$ and $\tilde{\boldsymbol{\nabla}}$ denote the Levi-Civita connections of the metrics $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$ related to each other by Equation (2.1). The action of $(\boldsymbol{\nabla} - \tilde{\boldsymbol{\nabla}})$ on a function f is defined by

$$(\nabla_a - \tilde{\nabla}_a) f = 0.$$

Moreover, since

$$(\nabla_a - \tilde{\nabla}_a)(f v^b) = f(\nabla_a - \tilde{\nabla}_a) v^b,$$

one can define the *transition tensor* $Q_a{}^c{}_b$ so that

$$(\nabla_a - \tilde{\nabla}_a) v^b = Q_a{}^b{}_c v^c. \tag{2.2}$$

From these relations, by setting $f = \omega_b v^b$, one finds that the action of $(\boldsymbol{\nabla} - \tilde{\boldsymbol{\nabla}})$ on a covector $\boldsymbol{\omega}$ is given by

$$(\nabla_a - \tilde{\nabla}_a)\omega_b = -Q_a{}^c{}_b \omega_c. \tag{2.3}$$

Hence, given the connection $\tilde{\boldsymbol{\nabla}}$ one can obtain the connection $\boldsymbol{\nabla}$ by finding the specific form of the transition tensor $Q_a{}^c{}_b$. Since these connections are torsion-free, one can show that $Q_a{}^c{}_b = Q_{(a}{}^c{}_{b)}$ —i.e. the transition tensor is a symmetric tensor.

2.2.3 Transformation formulae

Transformation formula for the connection

In order to find the explicit form of the transition tensor $Q_a{}^c{}_b$, one considers the action of $(\boldsymbol{\nabla} - \tilde{\boldsymbol{\nabla}})$ on the metric $\boldsymbol{g}$ obtained from Equation (2.3) as follows

$$(\nabla_a - \tilde{\nabla}_a) g_{bc} = -Q_a{}^d{}_b g_{dc} - Q_a{}^d{}_c g_{bd}. \tag{2.4}$$

Now, since $\nabla_a g_{bc} = 0$, $\tilde{\nabla}_a \tilde{g}_{bc} = 0$ and $g_{bc} = \Xi^2 \tilde{g}_{bc}$, one has that

$$2(\Xi \tilde{\nabla}_a \Xi) \tilde{g}_{bc} = Q_a{}^d{}_b g_{dc} + Q_a{}^d{}_c g_{bd}, \tag{2.5}$$

from which

$$2(\Xi^{-1} \nabla_a \Xi) g_{bc} = Q_a{}^d{}_b g_{dc} + Q_a{}^d{}_c g_{bd}. \tag{2.6}$$

This equation can be paired with two equations obtained from Equation (2.6) by means of cyclic permutations of the indices $_{abc}$

$$2(\Xi^{-1} \nabla_b \Xi) g_{ca} = Q_b{}^d{}_c g_{da} + Q_c{}^d{}_b g_{ad}, \tag{2.7a}$$

$$2(\Xi^{-1} \nabla_c \Xi) g_{ab} = Q_c{}^d{}_a g_{db} + Q_b{}^d{}_a g_{cd}. \tag{2.7b}$$

By adding Equations (2.7a)-(2.7b), subtracting Equation (2.6) and by using the symmetry properties of g_{ab} and $Q_a{}^c{}_b$, it follows that

$$(\Xi^{-1} \nabla_b \Xi) g_{ca} + (\Xi^{-1} \nabla_c \Xi) g_{ab} - (\Xi^{-1} \nabla_a \Xi) g_{bc} = Q_b{}^d{}_c g_{da}.$$

Thus, by solving for $Q_b{}^d{}_c$ and upon defining

$$S_{bc}{}^{da} \equiv \delta_b{}^a \delta_c{}^d + \delta_b{}^d \delta_c{}^a - g^{ad} g_{bc}, \qquad \Upsilon_a \equiv \Xi^{-1} \nabla_a \Xi, \qquad \Upsilon_b{}^d{}_c \equiv S_{bc}{}^{da} \Upsilon_a, \tag{2.8}$$

one has that

$$Q_b{}^d{}_c = S_{bc}{}^{da} (\Upsilon_a). \tag{2.9}$$

In conclusion, one has that $\boldsymbol{\nabla}$ and $\tilde{\boldsymbol{\nabla}}$ are related to each other via

$$\nabla_a - \tilde{\nabla}_a = S_{ab}{}^{cd} (\Upsilon_d). \tag{2.10}$$

It is worth noticing that the tensor $S_{bc}{}^{da}$ is conformally invariant since

$$\delta_b{}^a \delta_c{}^d + \delta_b{}^d \delta_c{}^a - g^{ad} g_{bc} = \delta_b{}^a \delta_c{}^d + \delta_b{}^d \delta_c{}^a - \tilde{g}^{ad} \tilde{g}_{bc}.$$

Transformation formulae for the curvature

The relation between the Levi-Civita connections $\boldsymbol{\nabla}$ and $\tilde{\boldsymbol{\nabla}}$ provided by Equation (2.10) can be used to obtain the transformation formulae relating the curvature tensors of these connections. More precisely, from the definition the Riemann tensor $\tilde{\boldsymbol{R}}$ of the connection $\tilde{\boldsymbol{\nabla}}$, by recalling that this connection is torsion-free and by using Equation (2.9), one has that the relation between the Riemann tensor of the connections $\boldsymbol{\nabla}$ and $\tilde{\boldsymbol{\nabla}}$ is given by

$$\tilde{R}^{c}{}_{dab} - R^{c}{}_{dab} = 2\nabla_{[a}\Upsilon_{b]}{}^{c}{}_{d} + 2\Upsilon_{[a}{}^{c}{}_{|e|}\Upsilon_{b]}{}^{e}{}_{d}. \tag{2.11}$$

This transformation formula can be used to obtain the relation between the Ricci tensor of the connections $\tilde{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ as given by Equation (1.8), as well as the following relation between the Ricci scalar of these connections

$$R - \Xi^{-2}\tilde{R} = -\frac{6}{\Xi}\nabla_c\nabla^c\Xi + \frac{12}{\Xi^2}\nabla_c\Xi\nabla^c\Xi. \tag{2.12}$$

The presence of terms containing negative powers of Ξ on the right-hand side of Equations (1.8) and (2.12) makes these quantities singular at the conformal boundary.

Transformation formulae for the concomitants

The Riemann tensor $\tilde{R}^{c}{}_{dab}$ admits an irreducible decomposition in terms of the Schouten tensor $\tilde{L}_{ab}$ and the Weyl tensor $C^{c}{}_{dab}$ as

$$\tilde{R}^{c}{}_{dab} = C^{c}{}_{dab} + 2S_{d[a}{}^{ce}\tilde{L}_{b]e}. \tag{2.13}$$

The Schouten tensor $\tilde{L}_{ab}$ of the connection $\tilde{\boldsymbol{\nabla}}$ is defined as

$$\tilde{L}_{ab} = \frac{1}{2}\left(\tilde{R}_{ab} - \frac{1}{6}\tilde{R}\tilde{g}_{ab}\right). \tag{2.14}$$

The transformation formula for the Schouten tensor of the connections $\tilde{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$

$$L_{ab} - \tilde{L}_{ab} = \frac{1}{2}\left(R_{ab} - \tilde{R}_{ab}\right) - \frac{1}{12}\left(R - \Xi^{-2}\tilde{R}\right)g_{ab}$$

is given by means of the transformation formulae provided by Equations (1.8)-(2.12) as

$$L_{ab} - \tilde{L}_{ab} = -\frac{1}{\Xi}\nabla_a\nabla_b\Xi + \frac{1}{2\Xi^2}\nabla_c\Xi\nabla^c\Xi g_{ab}. \tag{2.15}$$

The transformation formula for the Weyl tensor $C^{c}{}_{dab}$ is obtained from the irreducible decomposition of the Riemann tensor (2.13) and the transformation formulae for the Riemann $R^{c}{}_{dab}$ and Schouten L_{ab} tensors, (2.11) and (2.15). From this, one finds that the Weyl tensor is an invariant of the conformal class $[\tilde{\boldsymbol{g}}]$ —i.e. $C^{c}{}_{dab} = \tilde{C}^{c}{}_{dab}$. This invariance along with Equation (2.10) gives the following identity for this tensor

$$\nabla_a(\Xi^{-1}C^{a}{}_{bcd}) = \Xi^{-1}\tilde{\nabla}_a C^{a}{}_{bcd}.$$

2.3 The standard conformal Einstein field equations

The first conformal formulation of the Einstein field equations goes back to the seminal work of Friedrich [20] and is known as the *conformal Einstein field equations*, also known as *standard conformal Einstein field equations*. This formulation consists of a system of equations for the conformal fields appearing in the Einstein field equations which are written in terms of the Levi-Civita connection of the conformally rescaled spacetime and are regular up to the conformal boundary. A solution to these equations implies, under suitable conditions, a solution to the Einstein field equations.

2.3.1 The vacuum Einstein field equations

The discussion of the following chapters assumes no matter content. As a result, this analysis can be restricted to the vacuum case for the Einstein field equations (1.7). The vacuum Einstein field equations are given by

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} + \lambda\tilde{g}_{ab} = 0, \tag{2.16}$$

with

$$\tilde{R} = 4\lambda, \tag{2.17}$$

where $\tilde{R}_{ab}$ and $\tilde{R}$ represent, respectively the Ricci tensor and Ricci scalar of the metric $\tilde{g}_{ab}$. The definition of the physical Schouten tensor (2.14) can be used to rewrite the Einstein field equations in terms of $\tilde{L}_{ab}$ as

$$\tilde{L}_{ab} = \frac{1}{6}\lambda\tilde{g}_{ab}. \tag{2.18}$$

2.3.2 Derivation of the conformal Einstein field equations

Equation for the conformal factor

Since the transformation law (1.8) provides a Ricci tensor which is singular at the conformal boundary, this suggests starting from the transformation law for the Schouten tensor, Equation (2.15). This equation contains two singular terms on the right-hand side that need to be regularised. This can be done by using Equation (2.12) to replace $\Xi^{-2}\nabla_c\Xi\nabla^c\Xi$ along with Equation (2.14) for $\tilde{L}_{ab}$ and by introducing the *Friedrich scalar*

$$s \equiv \frac{1}{4}\nabla_a\nabla^a\Xi + \frac{1}{24}R\Xi, \tag{2.19}$$

where R is the Ricci scalar of the metric $\boldsymbol{g}$. Nonetheless, one obtains an expression for L_{ab} still containing formally singular terms. This problem can be solved by looking at the resulting equation as an equation for

$$\nabla_a\nabla_b\Xi = -\Xi L_{ab} + sg_{ab}. \tag{2.20}$$

This change of perspective is such that in Equation (2.20) the Friedrich scalar s and the Schouten tensor L_{ab} are to be considered as unknowns. Hence, suitable equations for these fields need to be constructed.

Equation for the Friedrich scalar

To obtain a suitable equation for the Friedrich scalar s, one applies ∇_c to Equation (2.20), commutes the covariant derivatives and then contracts the indices $_b$ and $_c$ so that

$$\nabla_a(\nabla^c\nabla_c\Xi) + R_{ca}\nabla^c\Xi = -L_{ca}\nabla^c\Xi - \Xi\nabla^c L_{ac} + \nabla_c s. \tag{2.21}$$

Then by using the definition of the Schouten tensor (2.14) and the Friedrich scalar (2.19) in Equation (2.21) one has

$$3\nabla_a s - \frac{1}{6}\Xi\nabla_a R = -3L_{ac}\nabla^c\Xi - \Xi\nabla^c L_{ac}. \tag{2.22}$$

Now, upon introducing the Einstein tensor of the metric $\boldsymbol{g}$

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab},$$

one recalls that $\nabla^a G_{ab} = 0$ which can be written in terms of the Schouten tensor L_{ab} and replaced in Equation (2.22) so that one has

$$\nabla_a s = -L_{ac} \nabla^c \Xi, \tag{2.23}$$

providing an equation for s.

Equation for the Schouten tensor

Since Equation (2.20) contains the physical Schouten tensor $\tilde{L}_{ab}$ one needs to provide a differential condition on $\tilde{L}_{ab}$. In order to achieve this one introduces the second Bianchi identity for the Riemann tensor of the metric $\tilde{\boldsymbol{g}}$

$$\tilde{\nabla}_{[e} \tilde{R}^a{}_{|b|cd]} = 0. \tag{2.24}$$

Then by replacing the irreducible decomposition of the Riemann Tensor (2.13) in (2.24) one has

$$\tilde{\nabla}_c \tilde{L}_{db} - \tilde{\nabla}_d \tilde{L}_{cd} = \tilde{\nabla}_a C^a{}_{bcd}. \tag{2.25}$$

Since the Riemann tensor of the metric $\boldsymbol{g}$ satisfies equations analogous to (2.13) and (2.24) it follows that by the computation just described one obtains

$$\nabla_c L_{db} - \nabla_d L_{cd} = \nabla_a C^a{}_{bcd}. \tag{2.26}$$

Now, to find an expression for this equation in terms of undifferentiated fields one introduces the physical *Cotton tensor* $\tilde{Y}_{cdb}$ defined as

$$\tilde{Y}_{cdb} = \tilde{\nabla}_c \tilde{L}_{db} - \tilde{\nabla}_d \tilde{L}_{cd}, \tag{2.27}$$

and makes use of the identity

$$\nabla_a (\Xi^{-1} C^a{}_{bcd}) = \Xi^{-1} \tilde{\nabla}_a C^a{}_{bcd}$$

so that Equation (2.25) reads as

$$\Xi^{-1}\tilde{Y}_{cdb} = \nabla_a(\Xi^{-1}C^a{}_{bcd}). \tag{2.28}$$

This equation is singular at $\mathscr{I}$ due to the presence of Ξ^{-1} terms. To regularise this equation it is convenient to define the *rescaled Weyl tensor* $d^c{}_{dab}$ via the relation

$$d^c{}_{dab} \equiv \Xi^{-1}C^c{}_{dab}, \tag{2.29}$$

and the *rescaled Cotton tensor* via

$$T_{cdb} \equiv \Xi^{-1}\tilde{Y}_{cdb} \tag{2.30}$$

to be replaced in (2.28) so that one has

$$T_{cdb} = \nabla_a d^a{}_{bcd}, \tag{2.31}$$

which is formally regular. Since the rescaled Cotton tensor T_{cdb} provides the coupling with the matter fields, in vacuum one has that $T_{cdb} = 0$—see Chapter 10 of [81]. Thus, it follows that in a vacuum one has

$$\nabla_a d^a{}_{bcd} = 0. \tag{2.32}$$

Eventually, by using the definition of the rescaled Weyl Tensor (2.29) and Equation (2.31) in Equation (2.26) one obtains the so-called *Cotton equation*

$$\nabla_c L_{db} - \nabla_d L_{cd} = d^a{}_{bcd}\nabla_a\Xi, \tag{2.33}$$

which is regular for $\Xi = 0$.

Equation of propagation for the Cosmological constant

To relate the solution of the conformal Einstein field equations to solutions of the Einstein field equations, one needs to regularise the transformation relation for the Ricci scalar (1.8). To achieve this, one multiplies Equation (1.8) by Ξ^2, uses the equation for the physical Ricci scalar (2.17) and the definition of the Friedrich scalar (2.19) so that one has

$$\lambda = 6\Xi s - 3\nabla_c\Xi\nabla^c\Xi. \tag{2.34}$$

This equation plays the role of a constraint which is preserved during the evolution by virtue of the other conformal field equations—see Chapter 8 of [81]. In particular, one has the following lemma:

Lemma 1 (**Propagation of the Cosmological constant**). *If Equations* (2.20) *and* (2.23) *are satisfied on* $\mathcal{M}$ *and Equation* (2.34) *holds at a point* $p \in \mathcal{M}$*, then this equation is also satisfied on* $\mathcal{M}$*.*

2.3.3 The metric formulation of the conformal Einstein field equations

The discussion of the previous sections leads to the definition of the *conformal Einstein field equations* as

$$\nabla_a \nabla_b \Xi = -\Xi L_{ab} + s g_{ab}, \tag{2.35a}$$

$$\nabla_a s = -L_{ac} \nabla^c \Xi, \tag{2.35b}$$

$$\nabla_c L_{db} - \nabla_d L_{cb} = d^a{}_{bcd} \nabla_a \Xi, \tag{2.35c}$$

$$\nabla_a d^a{}_{bcd} = 0, \tag{2.35d}$$

$$6\Xi s - 3\nabla_c \Xi \nabla^c \Xi = \lambda, \tag{2.35e}$$

complemented by the irreducible decomposition of the Riemann tensor (2.13).

In order to formulate these equations in a compact form it is useful to introduce the so-called *zero-quantities*

$$Z_{ab} \equiv \nabla_a \nabla_b \Xi + \Xi L_{ab} - s g_{ab},$$

$$Z_a \equiv \nabla_a s + L_{ac} \nabla^c \Xi,$$

$$\Delta_{bcd} \equiv \nabla_c L_{db} - \nabla_d L_{cb} - d^a{}_{bcd} \nabla_a \Xi,$$

$$\Lambda_{bcd} \equiv \nabla_a d^a{}_{bcd},$$

$$Z \equiv 6\Xi s - 3\nabla_c \Xi \nabla^c \Xi - \lambda,$$

so that the *conformal Einstein field equations* can be written as

$$Z_{ab} = 0, \qquad Z_a = 0, \qquad \Delta_{bcd} = 0, \qquad \Lambda_{bcd} = 0, \qquad Z = 0. \tag{2.37}$$

A solution to the *conformal Einstein field equations* is a collection of fields $\{g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd}\}$ satisfying (2.37). This solution is in turn a solution to the vacuum Einstein field equations as shown in Chapter 8 of [81] and summarised by the following proposition

Proposition 1 (Solutions to the conformal Einstein field equations as solutions to the Einstein field equations). *Let*

$$\{g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd}\}$$

denote a solution to the Equations (2.35a)-(2.35d) *such that* $\Xi \neq 0$ *on an open set* $\mathcal{U} \subset \mathcal{M}$*. If, in addition, Equation* (2.35e) *is satisfied at a point* $p \in \mathcal{U}$*, then the metric*

$$\tilde{g}_{ab} = \Xi^{-2} g_{ab}$$

is a solution to the Einstein field equations on $\mathcal{U}$*.*

2.3.4 The frame formulation of the conformal Einstein field equations

This derivation of the frame version of Equations (2.35a)-(2.35e) requires the introduction of a frame. Let $\{\boldsymbol{e_a}\}$ be a frame on $\mathcal{M}$ with $\{\boldsymbol{\omega^a}\}$ associated coframe so that $\langle \boldsymbol{\omega^a}, \boldsymbol{e_b} \rangle = \delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}$. Accordingly, one defines the *frame metric* as the frame

$$g_{\boldsymbol{ab}} \equiv \boldsymbol{e_a}{}^a \boldsymbol{e_b}{}^b g_{ab} = \boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b}).$$

Upon choosing the frame $\{\boldsymbol{e_a}\}$ to be orthonormal with respect to the metric $\boldsymbol{g}$ so that

$$\boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b}) = \mathrm{diag}(-1, 1, 1, 1).$$

Thus, the metric $\boldsymbol{g_{ab}}$ is expressed in terms of the coframe $\{\boldsymbol{\omega^a}\}$ as

$$\boldsymbol{g^{ab}} = \boldsymbol{g}^{ab} \omega^{\boldsymbol{a}}{}_a \omega^{\boldsymbol{b}}{}_b = \boldsymbol{g}^\sharp(\boldsymbol{\omega^a}, \boldsymbol{\omega^b}).$$

In terms of the frame $\{\boldsymbol{e_a}\}$ the connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ of Levi-Civita connection $\boldsymbol{\nabla}$ are defined via

$$\nabla_{\boldsymbol{a}}\boldsymbol{e_b} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c},$$

with $\nabla_{\boldsymbol{a}} \equiv \boldsymbol{e_a}{}^a\nabla_a$ denotes the covariant directional derivative in the direction of $\boldsymbol{e_a}$. The torsion $\boldsymbol{\Sigma}$ of $\boldsymbol{\nabla}$ can be expressed in terms of the frame $\{\boldsymbol{e_a}\}$ and the connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ via

$$\Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c} = [\boldsymbol{e_a}, \boldsymbol{e_b}] - (\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})\boldsymbol{e_c}.$$

The components of the Riemann tensor $R^a{}_{bcd}$ of the Levi-Civita connection $\boldsymbol{\nabla}$ are written in terms of the connection coefficients $\Gamma_a{}^c{}_b$ as

$$R^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv \partial_{\boldsymbol{a}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \partial_{\boldsymbol{b}}(\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{a}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}) + \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}}$$

and will be referred as the *geometric curvature* $R^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ of $\boldsymbol{\nabla}$. The expression of the irreducible decomposition of the Riemann tensor is

$$\rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv \Xi d^{\boldsymbol{c}}{}_{\boldsymbol{dab}} + 2S_{\boldsymbol{d}[\boldsymbol{a}}{}^{\boldsymbol{ce}}L_{\boldsymbol{b}]\boldsymbol{e}}$$

and will be referred as the *algebraic curvature* $\rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$. In the latter expression, $L_{\boldsymbol{ab}}$ is the Schouten tensor of the metric $\boldsymbol{g}$ and $d^a{}_{bcd}$ is the rescaled Weyl tensor.

Using these definitions, the *frame formulation of the conformal Einstein field Equations* is provided by the following set of equations

$$\begin{aligned}
&\nabla_{\boldsymbol{a}}\nabla_{\boldsymbol{b}}\Xi = -\Xi L_{ab} + s\eta_{\boldsymbol{ab}},\\
&\nabla_{\boldsymbol{a}}s = -L_{\boldsymbol{ac}}\nabla^c\Xi,\\
&\nabla_c L_{db} - \nabla_d L_{cb} = d^a{}_{bcd}\nabla_a\Xi,\\
&\nabla_a d^a{}_{bcd} = 0,\\
&6\Xi s - 3\nabla_c\Xi\nabla^c\Xi = \lambda.
\end{aligned}$$

and is complemented by the *structure equations*

$$\begin{aligned}
&\Sigma_{\boldsymbol{c}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0,\\
&R^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = \rho^{\boldsymbol{c}}_{\boldsymbol{dab}},
\end{aligned}$$

which expresses the fact that for the Levi-Civita connection $\boldsymbol{\nabla}$, its torsion must vanish and its geometric and algebraic curvature must coincide.

To formulate these equations in a compact form it is useful to introduce the following set of **zero-quantities**:

$$\Sigma_{\boldsymbol{ab}} \equiv \Sigma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \boldsymbol{e}_{\boldsymbol{c}}, \tag{2.40a}$$

$$\Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv R^{\boldsymbol{c}}{}_{\boldsymbol{dab}} - \rho^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \tag{2.40b}$$

$$Z_{\boldsymbol{ab}} \equiv \nabla_{\boldsymbol{a}} \nabla_{\boldsymbol{b}} \Xi + \Xi L_{\boldsymbol{ab}} - s\eta_{\boldsymbol{ab}}, \tag{2.40c}$$

$$Z_{\boldsymbol{a}} \equiv \nabla_{\boldsymbol{a}} s + L_{\boldsymbol{ac}} \nabla^{\boldsymbol{c}} \Xi, \tag{2.40d}$$

$$\Delta_{\boldsymbol{cdb}} \equiv \nabla_{\boldsymbol{c}} L_{\boldsymbol{db}} - \nabla_{\boldsymbol{d}} L_{\boldsymbol{cb}} - d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} \nabla_{\boldsymbol{a}} \Xi, \tag{2.40e}$$

$$\Lambda_{\boldsymbol{bcd}} \equiv \nabla_{\boldsymbol{a}} d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \tag{2.40f}$$

$$Z \equiv 6\Xi s - 3\nabla_{\boldsymbol{c}} \Xi \nabla^{\boldsymbol{c}} \Xi - \lambda. \tag{2.40g}$$

In terms of these zero quantities, the *frame version of the conformal Einstein field equations* can be written as

$$\Sigma_{\boldsymbol{ab}} = 0, \qquad \Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0, \qquad Z_{\boldsymbol{ab}} = 0, \qquad Z_{\boldsymbol{a}} = 0, \tag{2.41a}$$

$$\Delta_{\boldsymbol{cdb}} = 0, \qquad \Lambda_{\boldsymbol{bcd}} = 0, \qquad Z = 0. \tag{2.41b}$$

Accordingly, a solution to the frame conformal Einstein field equations is a collection $\{\boldsymbol{e}_{\boldsymbol{a}}, \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \Xi, s, L_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}\}$ satisfying the previous set of equations. The equations associated to the zero quantities $\Sigma_{\boldsymbol{ab}}$ and $\Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ provide differential conditions for the components of the frame vectors $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ and for the coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$. The role of the equations associated to the zero quantities $Z_{\boldsymbol{ab}}$, $Z_{\boldsymbol{a}}$, $\Delta_{\boldsymbol{cdb}}$, $\Lambda_{\boldsymbol{bcd}}$, Z and $M_{\boldsymbol{a}}$ is similar to that of their metric counterparts.

Considering a frame version of the conformal field equations introduces further gauge freedom in the system. This gauge freedom corresponds to the Lorentz transformations preserving the $\boldsymbol{g}$-orthonormality of the frame vectors $\{\boldsymbol{e}_{\boldsymbol{a}}\}$. In this case, one speaks of a *frame gauge freedom*.

The relation between the solution to the frame conformal Einstein field equations and the solution to the Einstein field equations is provided by the following lemma —see [81]

Lemma 2 (Solutions to the frame conformal Einstein field equations as solutions to the Einstein field equations). *Let*

$$\{\boldsymbol{e_a}, \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \Xi, s, L_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}\} \tag{2.42}$$

denote a solution to the Equations (2.41) *with* $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ *satisfying the metric compatibility condition*

$$\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}}\eta_{\boldsymbol{dc}} + \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}}\eta_{\boldsymbol{bd}} = 0$$

and such that

$$\Xi \neq 0, \qquad \det(\eta^{ab}\boldsymbol{e}_a \otimes \boldsymbol{e}_b) \neq 0,$$

in an open set $\mathcal{U} \subset \mathcal{M}$*. Then the metric*

$$\tilde{\boldsymbol{g}} = \Xi^{-2}\eta_{\boldsymbol{ab}}\boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}}$$

is a solution to the Einstein field Equations on $\mathcal{U}$*.*

The proof of this Lemma exploits the geometrical significance that the conformal Einstein field equations encode. In particular, if $\Sigma_{\boldsymbol{c}}{}^{\boldsymbol{c}}{}_b = 0$ then $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}}$ correspond to the connection coefficients with respect of $\{\boldsymbol{e_a}\}$ of the Levi-Civita connection of $\boldsymbol{g} = \eta_{\boldsymbol{ab}}\boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}}$. Equations $\Xi^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0$, $\Delta_{cdb} = 0$ and $\Lambda_{\boldsymbol{bcd}} = 0$ ensure that $L_{\boldsymbol{ab}}$ and $C^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$ are the components of the Schouten and Weyl tensors of $\boldsymbol{\nabla}$ with respect to the frame $\{\boldsymbol{e_a}\}$. Finally, equations $Z_{\boldsymbol{ab}} = 0$ and $Z_{\boldsymbol{a}} = 0$ imply that $\tilde{g}_{ab} = \Xi^{-2}g_{ab}$ satisfy the Einstein field equations expressed as Equation (2.18).

2.4 The extended conformal Einstein field equations

This section presents the formalism necessary to introduce the *extended conformal Einstein field equations*. These equations provide a more general formulation of the conformal Einstein field equations—see [27, 28, 81]. More precisely, whereas the latter are conformal field equations formulated in terms of the Levi-Civita connection of the unphysical metric $\boldsymbol{g}$, the extended conformal Einstein field equations is a system of equations providing a conformal representation of the Einstein field equations written in terms of *Weyl connections*. The use of Weyl connections introduces further freedom in the equations that can be exploited to incorporate conformal gauges.

2.4.1 Weyl connections

A Weyl connection is a torsion-free connection $\hat{\boldsymbol{\nabla}}$ such that

$$\hat{\nabla}_a g_{bc} = -2 f_a g_{bc}, \tag{2.43}$$

where f_a is a fixed smooth covector. It follows from the above that the connections ∇_a and $\hat{\nabla}_a$ are related to each other by

$$\hat{\nabla}_a v^b - \nabla_a v^b = S_{ac}{}^{bd} f_d v^c, \qquad S_{ac}{}^{bd} \equiv \delta_a{}^b \delta_c{}^d + \delta_a{}^d \delta_c{}^b - g_{ac} g^{bd}, \tag{2.44}$$

where v^a is an arbitrary vector. Given that

$$\nabla_a v^b - \tilde{\nabla}_a v^b = S_{ac}{}^{bd} (\Xi^{-1} \nabla_a \Xi) v^c,$$

one has that

$$\hat{\nabla}_a v^b - \tilde{\nabla}_a v^b = S_{ac}{}^{bd} \beta_d v^c, \qquad \beta_d \equiv f_d + \Xi^{-1} \nabla_d \Xi.$$

In the following, it will be convenient to define

$$d_a \equiv \Xi f_a + \nabla_a \Xi, \tag{2.45}$$

so that $d_a = \Xi \beta$.

2.4.2 The core equations

Let $\hat{R}^c{}_{dab}$ and $\hat{L}_{ab}$ denote, respectively, the Riemann and Schouten tensors of the Weyl connection $\hat{\nabla}_a$. Since the Weyl connection $\hat{\boldsymbol{\nabla}}$ is torsion-free, the Riemann tensor $\hat{R}^c{}_{dab}$ can be decomposed in terms of the Schouten tensor $\hat{L}_{ab}$ and the conformally invariant *Weyl tensor* $C^c{}_{dab}$. Furthermore, for a generic Weyl connection, one has that $\hat{L}_{ab} \neq \hat{L}_{ba}$. Thus, one has the decomposition

$$\hat{R}^c{}_{dab} = 2 S_{d[a}{}^{ce} \hat{L}_{b]e} + C^c{}_{dab}.$$

The (vanishing) torsion of $\hat{\nabla}_{\boldsymbol{a}}$ is denoted by $\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$. The Schouten tensors of the connections $\tilde{\nabla}$, ∇ and $\hat{\nabla}$ are related to each other via

$$\hat{L}_{ab} - L_{ab} = \nabla_a f_b - \frac{1}{2} S_{ab}{}^{cd} f_c f_d, \tag{2.46a}$$

$$\hat{L}_{ab} - \tilde{L}_{ab} = \hat{\nabla}_a \beta_b - \frac{1}{2} S_{ab}{}^{cd} \beta_c \beta_d, \tag{2.46b}$$

$$L_{ab} - \tilde{L}_{ab} = \nabla_a(\Xi^{-1}\nabla_b \Xi) + \frac{1}{2}\Xi^{-2} S_{ab}{}^{cd} \nabla_c \Xi \nabla_d \Xi. \tag{2.46c}$$

Using the relations above, the relation between the covariant derivatives $\hat{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$

$$\hat{\nabla}_a \hat{L}_{bc} = \nabla_a \hat{L}_{bc} - S_{ab}{}^{ef} f_e \hat{L}_{fc} - S_{ac}{}^{ef} f_e \hat{L}_{bf}$$

and $\hat{\nabla}_a S_{bc}{}^{de} = 0$, one has

$$\begin{aligned} \hat{\nabla}_a \hat{L}_{bc} - \hat{\nabla}_b \hat{L}_{ac} = {} & \nabla_a L_{bc} - \nabla_b L_{ac} + (\nabla_b \nabla_a - \nabla_a \nabla_b) f_c \\ & + S_{bc}{}^{ef} f_e (\nabla_a f_f - L_{af}) - S_{ac}{}^{ef} f_e (\nabla_b f_f - L_{bf}). \end{aligned} \tag{2.47}$$

Using Equation (2.46a) and the properties of $S_{ab}{}^{cd}$, one has that

$$\begin{aligned} S_{bc}{}^{ef} f_e (\nabla_a f_f - L_{af}) - S_{ac}{}^{ef} f_e (\nabla_b f_f - L_{bf}) &= S_{ac}{}^{ef} f_e (\hat{L}_{bf}) - S_{bc}{}^{ef} f_e (\hat{L}_{af}) \\ &= 2 S_{c[a}{}^{ef} \hat{L}_{b]f} f_e. \end{aligned}$$

Eventually, by recalling the splitting of the Riemann tensor

$$\hat{R}^c{}_{dab} = C^c{}_{dab} + 2 S_{d[a}{}^{ce} \hat{L}_{b]e},$$

one has that

$$\hat{\nabla}_a \hat{L}_{bc} - \hat{\nabla}_b \hat{L}_{ac} = \nabla_a L_{bc} - \nabla_b L_{ac} - C^e{}_{cab} f_e.$$

Hence, by recalling Equation (2.26) and the definition (2.45), it follows that the Weyl connection version of the Cotton equation (2.35c) is given by

$$\hat{\nabla}_a \hat{L}_{bc} - \hat{\nabla}_b \hat{L}_{ac} = d_e d^e{}_{cab}. \tag{2.48}$$

To obtain the Weyl connection equivalent of the Bianchi equation

$$\nabla_a d^a{}_{bcd} = 0, \tag{2.49}$$

one considers

$$
\begin{aligned}
\hat{\nabla}_a d^a{}_{bcd} &= \nabla_a d^a{}_{bcd} - S_{ae}{}^{fa} f_f d^e{}_{bcd} + S_{ab}{}^{fe} f_f d^a{}_{ecd} \\
&\quad + S_{ac}{}^{fe} f_f d^a{}_{bed} + S_{ad}{}^{fe} f_f d^a{}_{bce} \\
&= \nabla_a d^a{}_{bcd} - 4 f_a d^a{}_{bcd} + \delta_b{}^e f_a d^a{}_{ecd} + \delta_a{}^e f_b d^a{}_{ecd} - g_{ab} g^{fe} f_f d^a{}_{ecd} \\
&\quad + \delta_c{}^e f_a d^a{}_{bed} + \delta_a{}^e f_c d^a{}_{bed} - g_{ac} g^{fe} f_f d^a{}_{bed} + \delta_d{}^e f_a d^a{}_{bce} \\
&\quad + \delta_a{}^e f_d d^a{}_{bce} - g_{ad} g^{fe} f_f d^a{}_{bce} \\
&= \nabla_a d^a{}_{bcd} - f_a d^a{}_{dcb} + f_a d^a{}_{cdb} \\
&= \nabla_a d^a{}_{bcd} - f_a d^a{}_{bcd},
\end{aligned} \tag{2.50}
$$

where we used the Bianchi identity

$$
d^a{}_{bcd} + d^a{}_{cdb} + d^a{}_{dbc} = 0.
$$

Eventually, the Bianchi Equation (2.49) expressed in terms of the Weyl connection $\hat{\boldsymbol{\nabla}}$ reads as

$$
\hat{\nabla}_a d^a{}_{bcd} = f_a d^a{}_{bcd}.
$$

In conclusion, one has a system of *core equations*

$$
\hat{\nabla}_a \hat{L}_{bc} - \hat{\nabla}_b \hat{L}_{ac} = d_e d^e{}_{cab}, \tag{2.51a}
$$

$$
\hat{\nabla}_a d^a{}_{bcd} = f_a d^a{}_{bcd}, \tag{2.51b}
$$

providing differential conditions on the Schouten tensor of the Weyl connection $\hat{L}_{ab}$ and the rescaled Weyl tensor $d^a{}_{bcd}$. These equations need to be supplemented by a set of equations providing information about the metric g_{ab} and which allows determining the covector f_a defining the Weyl connection $\hat{\boldsymbol{\nabla}}$. The most convenient way of doing this is by providing a *frame formulation* of the extended conformal Einstein field equations.

2.4.3 The frame formulation of the extended conformal Einstein field equations

Let $\{\boldsymbol{e_a}\}$, $\boldsymbol{a} = \boldsymbol{0}, \ldots, \boldsymbol{3}$ denote a $\boldsymbol{g}$-orthogonal frame with associated coframe $\{\boldsymbol{\omega^a}\}$. Thus, one has that

$$\boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b}) = \eta_{\boldsymbol{ab}}, \qquad \langle \boldsymbol{\omega^a}, \boldsymbol{e_b} \rangle = \delta_{\boldsymbol{b}}{}^{\boldsymbol{a}}.$$

Given a vector $\boldsymbol{v}$, its components with respect to the frame $\{\boldsymbol{e_a}\}$ are denoted by $v^{\boldsymbol{a}}$.

Let $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ and $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ denote, respectively, the connection coefficients of $\nabla_{\boldsymbol{a}}$ and $\hat{\nabla}_a$ with respect to the frame $\{\boldsymbol{e_a}\}$. It follows then from equation (2.44) that

$$\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} + S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}} f_{\boldsymbol{d}}.$$

In particular, one has that

$$f_{\boldsymbol{a}} = \frac{1}{4}\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{b}}.$$

In order to formulate the frame version of the extended conformal Einstein field equations it is convenient to introduce the *geometric curvature* $\hat{R}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ and the *algebraic curvature* $\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ given, respectively, by

$$\hat{R}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv \partial_{\boldsymbol{a}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \partial_{\boldsymbol{b}}(\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{a}} - \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}) + \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}}, \tag{2.52a}$$

$$\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv \Xi d^{\boldsymbol{c}}{}_{\boldsymbol{dab}} + 2S_{\boldsymbol{d}[\boldsymbol{a}}{}^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{b}]\boldsymbol{e}} \tag{2.52b}$$

and define the following *zero-quantities*:

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e_c} \equiv [\boldsymbol{e_a}, \boldsymbol{e_b}] - (\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})e_{\boldsymbol{c}}, \tag{2.53a}$$

$$\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv \hat{R}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} - \hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \tag{2.53b}$$

$$\hat{\Delta}_{\boldsymbol{cdb}} \equiv \hat{\nabla}_{\boldsymbol{c}}\hat{L}_{\boldsymbol{db}} - \hat{\nabla}_{\boldsymbol{d}}\hat{L}_{\boldsymbol{cb}} - d_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \tag{2.53c}$$

$$\hat{\Lambda}_{\boldsymbol{bcd}} \equiv \hat{\nabla}_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} - f_{\boldsymbol{a}}d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}, \tag{2.53d}$$

where $\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$, $\hat{L}_{\boldsymbol{ab}}$ and $d^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ denote the components of the torsion, of the Schouten tensor of $\hat{\nabla}_a$ and the rescaled Weyl tensor with respect to the frame $\{\boldsymbol{e_a}\}$. In terms of these quantities, one can write the *the frame version of the extended conformal Einstein field equations* as given by the conditions

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e}_{\boldsymbol{c}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0, \qquad \hat{\Delta}_{\boldsymbol{cdb}} = 0, \qquad \hat{\Lambda}_{\boldsymbol{bcd}} = 0. \tag{2.54}$$

In the above equations the fields Ξ and $d_{\boldsymbol{a}}$ —cfr. (2.45)— are regarded as *conformal gauge fields* which are determined by supplementary conditions. In order to account for this it is convenient to define

$$\delta_{\boldsymbol{a}} \equiv d_{\boldsymbol{a}} - \Xi f_{\boldsymbol{a}} - \hat{\nabla}_{\boldsymbol{a}}\Xi, \tag{2.55a}$$

$$\gamma_{\boldsymbol{ab}} \equiv \hat{L}_{\boldsymbol{ab}} - \hat{\nabla}_{\boldsymbol{a}}(\Xi^{-1}d_{\boldsymbol{b}}) - \Xi^{-2}S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}}d_{\boldsymbol{c}}d_{\boldsymbol{d}} + \frac{1}{6}\lambda\Xi^{-2}\eta_{\boldsymbol{ab}}, \tag{2.55b}$$

$$\varsigma_{\boldsymbol{ab}} \equiv \hat{L}_{[\boldsymbol{ab}]} - \hat{\nabla}_{[\boldsymbol{a}}f_{\boldsymbol{b}]}. \tag{2.55c}$$

The conditions

$$\delta_{\boldsymbol{a}} = 0, \qquad \gamma_{\boldsymbol{ab}} = 0, \qquad \varsigma_{\boldsymbol{ab}} = 0, \tag{2.56}$$

will be called the *supplementary conditions*. Equation (2.55a) provides the relation between the covectors $\boldsymbol{d}$, $\boldsymbol{f}$ and the conformal factor Ξ. Equation (2.55b) provides the relation between the Schouten tensor $\hat{L}_{ab}$ of the Weyl connection and the Schouten tensor $\tilde{L}_{ab}$ determined by the Einstein field equations —i.e. an analogue to the standard conformal equation $Z_{\boldsymbol{ab}} = 0$. Eventually, Equation (2.55c) encodes the relation between the antisymmetry of the Schouten tensor $\hat{L}_{ab}$ and the derivatives of the covector f_a. Altogether these equations play a role in relating the extended conformal Einstein field equations to the Einstein field equations and also in the propagation of the constraints.

The correspondence between the extended conformal Einstein field equations and the Einstein field equations is given by the following —see Proposition 8.3 in [81] and [31]:

Proposition 2 (**Solutions to the frame extended conformal Einstein field equations as solutions to the Einstein field equations**). *Let*

$$\{\boldsymbol{e}_{\boldsymbol{a}}, \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \hat{L}_{\boldsymbol{ab}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}\}$$

denote a solution to the extended conformal Einstein field equations (2.54) *for some choice of the conformal gauge fields* $\{\Xi,\, d_{\boldsymbol{a}}\}$ *satisfying the supplementary conditions* (2.56)*. Furthermore, suppose that*

$$\Xi \neq 0 \qquad \text{and} \qquad \det(\eta^{\boldsymbol{ab}}\boldsymbol{e}_{\boldsymbol{a}} \otimes \boldsymbol{e}_{\boldsymbol{b}}) \neq 0$$

on an open subset $\mathcal{U}$*. Then the metric*

$$\tilde{\boldsymbol{g}} = \Xi^{-2}\eta_{\boldsymbol{ab}}\boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}}$$

is a solution to the Einstein field equations (4.1) *on* $\mathcal{U}$*.*

2.5 Conformal geodesics and conformal Gaussian gauge systems

In this section, the notion of *conformal geodesic* is introduced along with the definition of a *conformal Gaussian gauge system*. Moreover, since the extended conformal Einstein field equations are naturally suited to the use of a gauge based on conformal geodesics, it will be discussed how to use the *conformal geodesic equations* to fix this gauge. Then, it is also discussed how to obtain a system of evolution equations from the extended conformal Einstein field equations. The last part of this section is devoted to the discussion of the propagation of the constraints.

2.5.1 Conformal Geodesics

A *conformal geodesic* on a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is a pair $(x(\tau), \boldsymbol{\beta}(\tau))$ consisting of a curve $x(\tau)$ on $\tilde{\mathcal{M}}$, with parameter $\tau \in I \subset \mathbb{R}$, tangent $\dot{\boldsymbol{x}}(\tau)$ and a covector $\boldsymbol{\beta}(\tau)$ along $x(\tau)$ satisfying the equations

$$\tilde{\nabla}_{\dot{x}}\dot{\boldsymbol{x}} = -2\langle \boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle \dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\boldsymbol{\beta}^{\sharp}, \tag{2.57a}$$

$$\tilde{\nabla}_{\dot{x}}\boldsymbol{\beta} = \langle \boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle \boldsymbol{\beta} - \frac{1}{2}\tilde{\boldsymbol{g}}^{\sharp}(\boldsymbol{\beta}, \boldsymbol{\beta})\dot{\boldsymbol{x}}^{\flat} + \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot), \tag{2.57b}$$

where $\tilde{\boldsymbol{L}}$ denotes the Schouten tensor of the Levi-Civita connection $\tilde{\boldsymbol{\nabla}}$. Associated to a conformal geodesic, it is natural to consider a frame $\{\boldsymbol{e_a}\}$ which is *Weyl propagated* along $\boldsymbol{x}(\tau)$.

A frame $\{\boldsymbol{e_a}\}$ on $\tilde{\mathcal{M}}$ is said to be Weyl propagated along the conformal geodesic $(\boldsymbol{x}(\tau), \boldsymbol{\beta}(\tau))$ if it satisfies

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{e_a} = -\langle\boldsymbol{\beta}, \boldsymbol{e_a}\rangle\dot{\boldsymbol{x}} - \langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\boldsymbol{e_a} + \tilde{\boldsymbol{g}}(\boldsymbol{e_a}, \dot{\boldsymbol{x}})\boldsymbol{\beta}^\sharp, \tag{2.58}$$

the so-called *Weyl propagation equation.*

The motivation for considering these curves is understood when one observes their behaviour under conformal transformations and transition to Weyl connections. Given the 1-form $\boldsymbol{f}$ defining the Weyl connection as in Equation (2.43). If one defines

$$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} - \boldsymbol{f},$$

the pair $(\boldsymbol{x}(\tau), \hat{\boldsymbol{\beta}}(\tau))$ will satisfy the equations

$$\hat{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = -2\langle\hat{\boldsymbol{\beta}}, \dot{\boldsymbol{x}}\rangle\dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\hat{\boldsymbol{\beta}}^\sharp, \tag{2.59a}$$

$$\hat{\nabla}_{\dot{\boldsymbol{x}}}\hat{\boldsymbol{\beta}} = \langle\hat{\boldsymbol{\beta}}, \dot{\boldsymbol{x}}\rangle\hat{\boldsymbol{\beta}} - \frac{1}{2}\tilde{\boldsymbol{g}}^\sharp(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\beta}})\dot{\boldsymbol{x}}^\flat + \hat{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot), \tag{2.59b}$$

where $\hat{\boldsymbol{L}}$ denotes the Schouten tensor of the Weyl connection $\hat{\boldsymbol{\nabla}}$. Notice that if one chooses this connection so that $\boldsymbol{f} = \boldsymbol{\beta}$, then the conformal geodesic equations reduce to

$$\hat{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = 0, \qquad \hat{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot) = 0 \tag{2.60}$$

and the Weyl propagation equation reduces to the usual propagation equation

$$\hat{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{e_a} = 0. \tag{2.61}$$

2.5.2 Reparametrisations of conformal geodesics

Given two solutions to the conformal geodesic Equations (2.57a)-(2.57b), $(x(\tau), \boldsymbol{\beta}(\tau))$ and $(\bar{x}(\bar{\tau}), \bar{\boldsymbol{\beta}}(\bar{\tau}))$, it is natural to ask under which conditions $x(\tau)$ and $\bar{x}(\bar{\tau})$ coincide locally as sets of points so that $\tau = \tau(\bar{\tau})$ and $x(\tau(\bar{\tau})) = \bar{x}(\bar{\tau})$. Let

$$\dot{\boldsymbol{x}} \equiv \frac{d\boldsymbol{x}}{d\tau}, \qquad \bar{\boldsymbol{x}}' \equiv \frac{d\bar{\boldsymbol{x}}}{d\bar{\tau}} \tag{2.62}$$

denote the corresponding tangent vectors and assume that these curves do not describe null geodesics —i.e. $\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}}) \neq 0$ and $\tilde{\boldsymbol{g}}(\bar{\boldsymbol{x}}',\bar{\boldsymbol{x}}') \neq 0$. By definition, the tangent vector $\bar{\boldsymbol{x}}'$ satisfies

$$\tilde{\nabla}_{\bar{\boldsymbol{x}}'}\bar{\boldsymbol{x}}' = -2\langle \bar{\boldsymbol{\beta}}, \bar{\boldsymbol{x}}'\rangle \bar{\boldsymbol{x}}' + \tilde{\boldsymbol{g}}(\bar{\boldsymbol{x}}',\bar{\boldsymbol{x}}')\bar{\boldsymbol{\beta}}^{\sharp}, \tag{2.63a}$$

$$\tilde{\nabla}_{\bar{\boldsymbol{x}}'}\bar{\boldsymbol{\beta}} = \langle \bar{\boldsymbol{\beta}}, \bar{\boldsymbol{x}}'\rangle \bar{\boldsymbol{\beta}} - \frac{1}{2}\tilde{\boldsymbol{g}}^{\sharp}(\bar{\boldsymbol{\beta}},\bar{\boldsymbol{\beta}})\bar{\boldsymbol{x}}'^{\flat} + \tilde{\boldsymbol{L}}(\bar{\boldsymbol{x}}',\cdot). \tag{2.63b}$$

Now, by letting $\tau' \equiv \frac{d\tau}{d\bar{\tau}}$ one has that

$$\bar{\boldsymbol{x}}' = \frac{d\bar{\boldsymbol{x}}}{d\bar{\tau}} = \frac{d\boldsymbol{x}(\tau(\bar{\tau}))}{d\bar{\tau}} = \frac{d\tau}{d\bar{\tau}}\frac{d\boldsymbol{x}}{d\tau} = \tau'\dot{\boldsymbol{x}},$$

which, in turn, implies

$$\tilde{\nabla}_{\bar{\boldsymbol{x}}'}\bar{\boldsymbol{x}}' = \tilde{\nabla}_{\tau'\dot{\boldsymbol{x}}}(\tau'\dot{\boldsymbol{x}}) = \tau''\dot{\boldsymbol{x}} + \tau'^2\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}}.$$

Then by using the Equation (2.57a) for $\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}}$ one has

$$\tilde{\nabla}_{\bar{\boldsymbol{x}}'}\bar{\boldsymbol{x}}' = \tau''\dot{\boldsymbol{x}} + \tau'^2(-2\langle\boldsymbol{\beta},\dot{\boldsymbol{x}}\rangle\dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}})\boldsymbol{\beta}^{\sharp}).$$

By replacing the latter into Equation (2.63a) and using $\bar{\boldsymbol{x}}' = \tau'\dot{\boldsymbol{x}}$, one has

$$\tau''\dot{\boldsymbol{x}} + \tau'^2(-2\langle\boldsymbol{\beta},\dot{\boldsymbol{x}}\rangle\dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}})\boldsymbol{\beta}^{\sharp}) = -2\langle\bar{\boldsymbol{\beta}},\tau'\dot{\boldsymbol{x}}\rangle\tau'\dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\tau'\dot{\boldsymbol{x}},\tau'\dot{\boldsymbol{x}})\bar{\boldsymbol{\beta}}^{\sharp},$$

so that

$$\tau''\dot{\boldsymbol{x}} + 2\tau'^2\langle\bar{\boldsymbol{\beta}}-\boldsymbol{\beta},\dot{\boldsymbol{x}}\rangle\dot{\boldsymbol{x}} - \tau'^2\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}})(\bar{\boldsymbol{\beta}}^{\sharp}-\boldsymbol{\beta}^{\sharp}) = 0. \tag{2.64}$$

Thus, the difference $\bar{\boldsymbol{\beta}}^{\sharp}-\boldsymbol{\beta}^{\sharp}$ has components only along $\dot{\boldsymbol{x}}$ and one can write

$$\bar{\boldsymbol{\beta}} - \boldsymbol{\beta} = \alpha\dot{\boldsymbol{x}}^{\flat} \tag{2.65}$$

with $\alpha \in \mathbb{R}$. By replacing this into Equation (2.64) one obtains the following differential equation

$$\tau''\dot{\boldsymbol{x}} + \alpha\tau'^2\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}})\dot{\boldsymbol{x}} = 0. \tag{2.66}$$

Now, by considering

$$\tilde{\nabla}_{\bar{\boldsymbol{x}}'}\bar{\boldsymbol{\beta}} = \langle \bar{\boldsymbol{\beta}}, \bar{\boldsymbol{x}}'\rangle \bar{\boldsymbol{\beta}} - \frac{1}{2}\tilde{\boldsymbol{g}}^{\sharp}(\bar{\boldsymbol{\beta}},\bar{\boldsymbol{\beta}})\bar{\boldsymbol{x}}'^{\flat} + \tilde{\boldsymbol{L}}(\bar{\boldsymbol{x}}',\cdot)$$

by replacing $\bar{\boldsymbol{x}}' = \tau'\dot{\boldsymbol{x}}$ and using Equation (2.57b), one has

$$\tau'\tilde{\nabla}_{\dot{\boldsymbol{x}}}\bar{\boldsymbol{\beta}} = \tau'\langle\bar{\boldsymbol{\beta}}, \dot{\boldsymbol{x}}\rangle\bar{\boldsymbol{\beta}} - \frac{1}{2}\tau'\tilde{\boldsymbol{g}}^\sharp(\bar{\boldsymbol{\beta}}, \bar{\boldsymbol{\beta}})\dot{\boldsymbol{x}}^\flat + \tau'\tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot).$$

This equation can be simplified so that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\bar{\boldsymbol{\beta}} = \langle\bar{\boldsymbol{\beta}}, \dot{\boldsymbol{x}}\rangle\bar{\boldsymbol{\beta}} - \frac{1}{2}\tilde{\boldsymbol{g}}^\sharp(\bar{\boldsymbol{\beta}}, \bar{\boldsymbol{\beta}})\dot{\boldsymbol{x}}^\flat + \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot).$$

By subtracting the Equation (2.57b) for $\boldsymbol{\beta}$

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}(\bar{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \langle\bar{\boldsymbol{\beta}} - \boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle(\bar{\boldsymbol{\beta}} - \boldsymbol{\beta}) - \frac{1}{2}(\tilde{\boldsymbol{g}}^\sharp(\bar{\boldsymbol{\beta}}, \bar{\boldsymbol{\beta}}) - \tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\beta}, \boldsymbol{\beta}))\dot{\boldsymbol{x}}^\flat,$$

and using the Equation (2.65) for α one obtains

$$\dot{\alpha}\dot{\boldsymbol{x}}^\flat + \alpha\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}}^\flat = -\frac{1}{2}\alpha^2\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\dot{\boldsymbol{x}}^\flat.$$

Eventually, by using Equation (2.57a) for $\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}}$ one has

$$\dot{\alpha} = 2\langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\alpha + \frac{1}{2}\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\alpha^2. \tag{2.67}$$

This Equation, along with Equations (2.65) and (2.66), encodes the requirement that the curves $x(\tau)$ and $\bar{x}(\bar{\tau})$ coincide as sets.

Now, let us consider the following

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}(\alpha\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})) = \alpha\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\dot{\alpha}.$$

By means of Equation (2.67) one has that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}(\alpha\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})) = \alpha\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\Big(2\langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\alpha + \frac{1}{2}\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\alpha^2\Big). \tag{2.68}$$

A direct computation using Equations (2.57a) and (2.57b) shows that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = -2\langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}),$$

which can be replaced into Equation (2.68) so that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}(\alpha\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})) = \frac{1}{2}(\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\alpha)^2.$$

This equation can be solved to obtain

$$\alpha\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}}) = \frac{2\alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}_\star,\dot{\boldsymbol{x}}_\star)}{1-\alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}_\star,\dot{\boldsymbol{x}}_\star)(\tau-\tau_\star)}$$

where $\alpha_\star \equiv \alpha(\tau_\star)$, $\dot{\boldsymbol{x}}_\star \equiv \dot{\boldsymbol{x}}(\tau_\star)$ and $\tau_\star$ denotes a fiduciary value of the parameter τ. This value can be replaced into Equations (2.65) and (2.66) so that one has

$$\bar{\boldsymbol{x}}' = \frac{4\mathcal{X}}{1+2\mathcal{X}\alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}_\star,\dot{\boldsymbol{x}}_\star)(\tau-\tau_\star)}\dot{\boldsymbol{x}}, \tag{2.69a}$$

$$\bar{\boldsymbol{\beta}} = \beta + \frac{2\alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}_\star,\dot{\boldsymbol{x}}_\star)}{(1-\alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}_\star,\dot{\boldsymbol{x}}_\star)(\tau-\tau_\star))\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}})}\dot{\boldsymbol{x}}^\flat, \tag{2.69b}$$

$$\tau = \tau_\star + \frac{4\mathcal{X}(\bar{\tau}-\bar{\tau}_\star)}{1+2\mathcal{X}\alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}_\star,\dot{\boldsymbol{x}}_\star)(\bar{\tau}-\bar{\tau}_\star)}, \tag{2.69c}$$

where $\mathcal{X}$ is a non-zero real constant. This discussion is summarised in the following lemma:

Lemma 3. *The admissible reparametrisations of non-null conformal geodesics to non-null conformal geodesics are given by transformations of the form*

$$\tau \to \frac{a\tau+b}{c\tau+d} \tag{2.70}$$

with a, b, c, $d \in \mathbb{R}$.

For further discussion on this topic see [31, 81].

2.5.3 Geodesics as conformal geodesics

It is natural to ask what is the relation between conformal geodesics and metric geodesics. For null conformal geodesics, this relation can be readily established. If $(x(\tau), \boldsymbol{\beta}(\tau))$ denotes a null conformal geodesic, it follows from Equation (2.57a) that

$$\tilde{\nabla}_{\boldsymbol{x}'}\boldsymbol{x}' = -2\langle\boldsymbol{\beta},\boldsymbol{x}'\rangle\boldsymbol{x}' + \tilde{\boldsymbol{g}}(\boldsymbol{x}',\boldsymbol{x}')\boldsymbol{\beta}^\sharp = -2\langle\boldsymbol{\beta},\boldsymbol{x}'\rangle\boldsymbol{x}'.$$

Using the same argument as the previous section one finds that null conformal geodesics are, up to a reparametrisation, null geodesics.

The situation for non-null conformal geodesics is more complicated and requires restrictions of the Schouten tensor of the spacetime. In particular, one has the following result

Lemma 4. *Any non-null $\tilde{\boldsymbol{g}}$-geodesic in an Einstein spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is, up to a reparametrisation, a non-null conformal geodesic.*

The proof of this lemma can be found in [37, 81].

2.5.4 Conformal Gaussian gauge systems

The main reason to introduce conformal geodesics in the analysis of spacetimes by means of the extended conformal Einstein field equations is that they provide a way for fixing the gauge fields $(\Xi, \boldsymbol{d})$. Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ be a solution to the vacuum Einstein field equations and $\mathcal{U}$ being an open set on $\tilde{\mathcal{M}}$. By assuming that this set is covered by a non-intersecting congruence of conformal geodesics and by identifying the timelike leg of the frame $\{\boldsymbol{e_a}\}$ with $\boldsymbol{e}_0 = \dot{\boldsymbol{x}}$, one can single out a metric $\boldsymbol{g} \in [\tilde{\boldsymbol{g}}]$ by means of a *canonical* conformal factor Θ such that

$$\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = -1, \qquad \boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}. \tag{2.71}$$

From the above conditions, it follows that

$$\dot{\Theta} = \langle \boldsymbol{\beta}, \dot{\boldsymbol{x}} \rangle \Theta.$$

Taking further derivatives with respect to τ and using the conformal geodesic equations (2.57a)-(2.57b) together with the Einstein field equations written in terms of the Schouten tensor (2.18) leads to the relation

$$\dddot{\Theta} = 0.$$

From the latter one has the following result:

Proposition 3. *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote a vacuum spacetime with positive Cosmological constant. Suppose that $(\boldsymbol{x}(\tau), \boldsymbol{\beta}(\tau))$ is a solution to the conformal geodesic equations* (2.57a)-(2.57b) *and that $\{\boldsymbol{e_a}\}$ is a $\boldsymbol{g}$-orthonormal frame which is Weyl propagated along the curve $\boldsymbol{x}(\tau)$. If Θ satisfies* (2.71) *and $\tau_\star \in \mathbb{R}$ is an arbitrary constant defining the value of τ at a fiduciary point of the conformal geodesic, then one has that*

$$\Theta(\tau) = \Theta_\star + \dot{\Theta}_\star(\tau - \tau_\star) + \frac{1}{2}\ddot{\Theta}_\star(\tau - \tau_\star)^2, \tag{2.72}$$

where the coefficients

$$\Theta_\star \equiv \Theta(\tau_\star), \qquad \dot{\Theta}_\star \equiv \dot{\Theta}(\tau_\star) \qquad \ddot{\Theta}_\star \equiv \ddot{\Theta}_\star(\tau_\star)$$

are constant along the conformal geodesic and are subject to the constraints

$$\dot{\Theta}_\star = \langle \boldsymbol{\beta}_\star, \dot{\boldsymbol{x}}_\star \rangle \Theta_\star, \qquad \Theta_\star \ddot{\Theta}_\star = \frac{1}{2}\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\beta}_\star, \boldsymbol{\beta}_\star) - \frac{1}{6}\lambda.$$

Moreover, along each conformal geodesic, one has that

$$\Theta \beta_{\boldsymbol{0}} = \dot{\Theta}, \qquad \Theta \beta_{\boldsymbol{i}} = \Theta_\star \beta_{\boldsymbol{i}\star},$$

where $\beta_{\boldsymbol{a}} \equiv \langle \boldsymbol{\beta}, \boldsymbol{e}_{\boldsymbol{a}} \rangle$.

A proof of the above result can be found in [81], Proposition 5.1 in Section 5.5.5.

Thus, if a spacetime can be covered by a non-intersecting congruence of conformal geodesics, then the location of the conformal boundary is known *a priori* in terms of data at a fiduciary initial hypersurface $\mathcal{S}_\star$.

These curves can be used to specify the gauge field $\boldsymbol{d}$ via $\boldsymbol{d} \equiv \Theta\boldsymbol{\beta}$. The constraints for the initial data for Θ can then be written in terms of $\boldsymbol{d}$ as

$$\dot{\Theta}_\star = \langle \boldsymbol{d}_\star, \dot{\boldsymbol{x}}_\star \rangle, \qquad \Theta_\star \ddot{\Theta}_\star = \frac{1}{2}\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{d}_\star, \boldsymbol{d}_\star) - \frac{1}{6}\lambda.$$

Remark 1. The conformal factor is canonical in the sense that if $\boldsymbol{g} = \Theta^2\tilde{\boldsymbol{g}}$ with $\tilde{\boldsymbol{g}}$ as a solution to the vacuum Einstein field equations. Thus, requiring the normalisation $\boldsymbol{g}(\dot{\boldsymbol{x}}(\tau), \dot{\boldsymbol{x}}(\tau)) = -1$ fixes the form of the conformal factor to be a quadratic function of τ.

Conformal Gaussian gauges

Now, assume as before that $\mathcal{U}$ is a region of the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ which is covered by a non-intersecting congruence of conformal geodesics $(\boldsymbol{x}(\tau), \boldsymbol{\beta}(\tau))$. Moreover, suppose that $\dot{\boldsymbol{x}} \equiv \dot{\boldsymbol{x}}(\tau_\star)$ is orthogonal to a fiduciary spacelike hypersurface $\mathcal{S}_\star \subset \mathcal{U}$ determined by the condition $\tau = \tau_\star$ so that $\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = -1$. From Proposition 3, it follows that this requirement singles out a *canonical representative* $\boldsymbol{g}$ of the conformal class $[\tilde{\boldsymbol{g}}]$. In particular, the conformal factor is explicitly known as given by formula (2.72) once the initial data $\Theta_\star$, $\dot{\Theta}_\star$ and $\ddot{\Theta}_\star$ are specified on $\mathcal{S}_\star$. The construction of a *conformal Gaussian system* requires the introduction of a $\boldsymbol{g}$-orthonormal frame $\{\boldsymbol{e_a}\}$ which is Weyl propagated along the conformal geodesics and whose time leg is given by $\boldsymbol{e_0} = \dot{\boldsymbol{x}}$. Since one can choose a Weyl connection $\hat{\boldsymbol{\nabla}}$ so that the 1-form f_a coincides with the 1-form β_a of the conformal geodesics. It follows that for this connection one has

$$\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = 0, \qquad f_{\boldsymbol{0}} = 0, \qquad \hat{L}_{\boldsymbol{0a}} = 0.$$

This gauge choice can be supplemented by choosing the parameter τ of the conformal geodesics as the time coordinate so that

$$\boldsymbol{e_0} = \boldsymbol{\partial}_\tau.$$

Furthermore, one can construct a spacetime system of coordinates by choosing some local spatial coordinates $\underline{x} = (x^\alpha)$ on $\mathcal{S}_\star$. Since the congruence of conformal geodesics is non-intersecting, one can extend the coordinates $\underline{x}$ off $\mathcal{S}_\star$ by requiring them to remain constant along the conformal geodesic which intersects $\mathcal{S}_\star$ at the point p on $\mathcal{S}_\star$ with coordinates $\underline{x}$. The spacetime coordinates $\overline{x} = (\tau, x^\alpha)$ obtained in this way are known as *conformal Gaussian coordinates*. More generally, the collection of conformal factor Θ, Weyl propagated frame $\{\boldsymbol{e_a}\}$ and coordinates (τ, x^α) obtained by the procedure outlined in the previous paragraph is known as a *conformal Gaussian gauge system*. This choice of gauge leads to a natural $1+3$ decomposition of the field equations. More details on this construction can be found in [81], Section 13.4.1.

2.5.5 The $\tilde{\boldsymbol{g}}$-adapted conformal geodesic equations

In the last section, it has been shown that as a consequence of the normalisation condition (2.71), the parameter τ is the $\boldsymbol{g}$-proper time of the curve $x(\tau)$. In some computations it is more convenient to consider a parametrisation in terms of a $\tilde{\boldsymbol{g}}$-proper time $\tilde{\tau}$ as it allows to

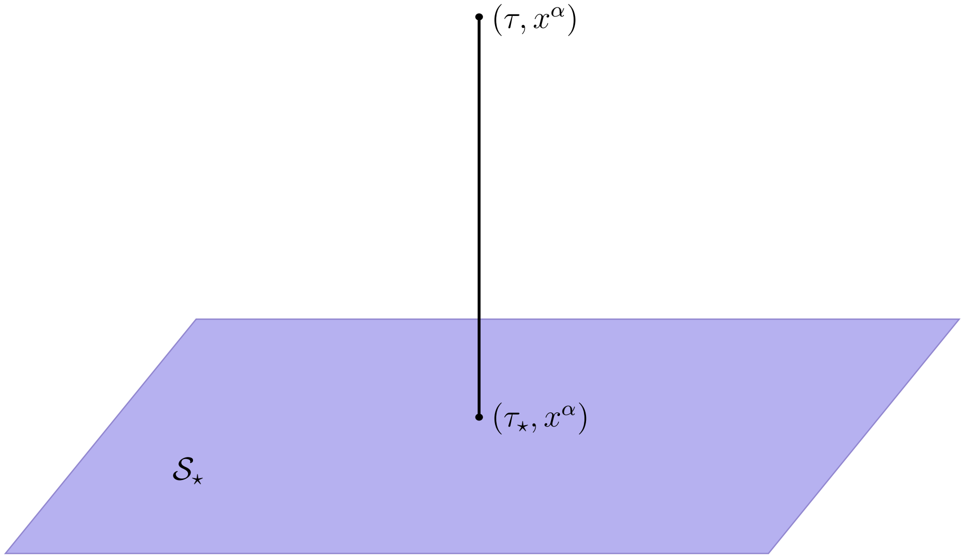

Figure 2.2: Schematic depiction of the construction of conformal Gaussian coordinates. The hypersurface in purple represents the initial hypersurface $\mathcal{S}_\star$. The black line represents a conformal geodesic leaving the initial hypersurface at $\tau = \tau_\star$. The coordinates $\underline{x} = x^\alpha$ of a point $p \in \mathcal{S}_\star$ are propagated off the hypersurface along the conformal geodesic. This picture is adapted from [81].

work directly with the physical metric. To this end, consider the parameter transformation $\tilde{\tau} = \tilde{\tau}(\tau)$ given by

$$\frac{\mathrm{d}\tau}{\mathrm{d}\tilde{\tau}} = \Theta, \qquad \text{so that} \qquad \tilde{\tau} = \tilde{\tau}_\star + \int_{\tau_\star}^{\tau} \frac{\mathrm{ds}}{\Theta(\mathrm{s})}, \tag{2.73}$$

with inverse $\tau = \tau(\tilde{\tau})$. In what follows, let $\tilde{x}(\tilde{\tau}) \equiv x(\tau(\tilde{\tau}))$. It can then be verified that

$$\tilde{\boldsymbol{x}}' \equiv \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{\tau}} = \frac{\mathrm{d}\tau}{\mathrm{d}\tilde{\tau}}\frac{\mathrm{d}x}{\mathrm{d}\tau} = \Theta\dot{\boldsymbol{x}}, \tag{2.74}$$

so that

$$\tilde{\boldsymbol{g}}(\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{x}}') = -1.$$

Hence, $\tilde{\tau}$ is, indeed, the $\tilde{\boldsymbol{g}}$-proper time of the curve $\tilde{x}(\tilde{\tau})$. In order to write the equation for the curve $\tilde{x}(\tilde{\tau})$ in a convenient way, one considers the split

$$\boldsymbol{\beta} = \tilde{\boldsymbol{\beta}} + \varpi\dot{\boldsymbol{x}}^\flat \qquad \text{with} \qquad \varpi \equiv \frac{\langle \boldsymbol{\beta}, \dot{\boldsymbol{x}} \rangle}{\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})} \tag{2.75}$$

and where the covector $\tilde{\boldsymbol{\beta}}$ satisfies

$$\langle\tilde{\boldsymbol{\beta}},\dot{\boldsymbol{x}}\rangle = 0, \qquad \boldsymbol{g}^\sharp(\boldsymbol{\beta},\boldsymbol{\beta}) = \langle\boldsymbol{\beta},\dot{\boldsymbol{x}}\rangle^2 + \boldsymbol{g}^\sharp(\tilde{\boldsymbol{\beta}},\tilde{\boldsymbol{\beta}}). \tag{2.76}$$

It can be readily verified that

$$\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}}) = -\Theta^{-2}, \qquad \langle\boldsymbol{\beta},\dot{\boldsymbol{x}}\rangle = \Theta^{-1}\dot{\Theta}, \qquad \varpi = \Theta\dot{\Theta}. \tag{2.77}$$

Using the split (2.75) in equations (2.57a)-(2.57b) and taking into account the relations in (2.74), (2.76) and (2.77) one obtains the following ***$\tilde{\boldsymbol{g}}$-adapted equations for the conformal geodesics***:

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{\beta}}^\sharp, \tag{2.78a}$$

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\boldsymbol{\beta}} = \tilde{\beta}^2\tilde{\boldsymbol{x}}'^\flat + \tilde{\boldsymbol{L}}(\tilde{\boldsymbol{x}}',\cdot) - \tilde{\boldsymbol{L}}(\tilde{\boldsymbol{x}}',\tilde{\boldsymbol{x}}')\tilde{\boldsymbol{x}}'^\flat, \tag{2.78b}$$

where $\tilde{\boldsymbol{L}}$ is given by Equation (2.18) and $\tilde{\beta}^2 \equiv \tilde{\boldsymbol{g}}^\sharp(\tilde{\boldsymbol{\beta}},\tilde{\boldsymbol{\beta}})$. The latter is a consequence of (2.76) as the covector $\tilde{\boldsymbol{\beta}}$ is spacelike and, thus, the definition of $\tilde{\beta}^2$ makes sense.

The Weyl propagation equation (2.58) can also be cast in a $\tilde{\boldsymbol{g}}$-adapted form. A calculation shows that

$$\tilde{\nabla}_{\tilde{x}'}(\Theta\boldsymbol{v}) = -\langle\tilde{\boldsymbol{\beta}},\Theta\boldsymbol{v}\rangle\tilde{\boldsymbol{x}}' + \tilde{\boldsymbol{g}}(\Theta\boldsymbol{v},\tilde{\boldsymbol{x}}')\tilde{\boldsymbol{\beta}}^\sharp. \tag{2.79}$$

2.6 Hyperbolic reduction procedure

The tensorial nature of the conformal Einstein field equations requires the derivation of a suitable symmetric hyperbolic evolution system from them to discuss the existence and the asymptotic properties of their solutions. This procedure is known as *hyperbolic reduction*. The starting point of this procedure consists of the specification of the gauge. The extended conformal Einstein field equations, being expressed in terms of Weyl connections, contain a bigger gauge freedom than the standard conformal Einstein field equations. In this case, the hyperbolic reduction procedure consists of adapting the gauge to a congruence of conformal geodesics to construct *conformal Gaussian gauge systems*. As discussed before, one of the advantages of this procedure is that, in a vacuum, the properties of conformal geodesics single out a conformal factor from the conformal class $[\tilde{\boldsymbol{g}}]$ — see Proposition 3. This means that one gains an a priori knowledge of the location of the conformal boundary. Moreover, the connection coefficients and components of the Schouten tensor concerning a Weyl propagated $\tilde{\boldsymbol{g}}$-orthonormal frame satisfy certain relations which lead to a particularly simple system of evolution equations. The evolution of all the geometric unknowns, with an exception made for the components of the rescaled Weyl tensor, are either fixed by the gauge or given by transport equations along the congruence. A direct study of the conformal Einstein field equations shows that these are overdetermined. There are more equations than unknown, even by taking into account all the possible symmetries of the tensorial fields. Thus, the process of hyperbolic reduction for the conformal field equations necessarily requires discarding some of the equations. The discarded equations are treated as constraints. These constraints will satisfy in turn a system of evolution equations, the so-called *subsidiary evolution system*. From this system, it will follow that the constraint equations will be satisfied if they hold at some initial hypersurface and the evolution equations are imposed. This construction is called the *propagation of the constraints*. The solution of the evolution system together with the propagation of the constraints yields the required solution of the conformal Einstein field equations.

2.6.1 The main evolution system

One of the main advantages of writing the extended conformal field equations in terms of zero-quantities and using a frame formalism

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e}_{\boldsymbol{c}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0, \qquad \hat{\Delta}_{\boldsymbol{cdb}} = 0, \qquad \hat{\Lambda}_{\boldsymbol{bcd}} = 0$$

is that the various evolution equations can be readily identified as certain components of the zero-quantities.

The required evolution equations for the frame components, the connection coefficients, the components of the Schouten tensor and the electric and magnetic part of the rescaled Weyl tensor are obtained from the conditions

$$\hat{\Sigma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e}_{\boldsymbol{c}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = 0, \qquad \hat{\Delta}_{\boldsymbol{0bc}} = 0, \qquad \hat{\Lambda}_{(\boldsymbol{a}|\boldsymbol{0}|\boldsymbol{b})} = 0, \qquad \hat{\Lambda}^{*}_{(\boldsymbol{a}|\boldsymbol{0}|\boldsymbol{b})} = 0,$$

where $\hat{\Lambda}^{*}_{\boldsymbol{bcd}} = \frac{1}{2}\epsilon_{\boldsymbol{cd}}{}^{\boldsymbol{ef}}\hat{\Lambda}_{\boldsymbol{bef}}$. In particular, the evolution equation for components of the covector f_a defining the Weyl connection is given by

$$\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{c0b}} = 0.$$

Using the definitions of the zero-quantities given in Equations (2.53a)-(2.53d) and making use of the gauge conditions

$$\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = 0, \qquad \hat{L}_{\boldsymbol{0a}} = 0, \qquad f_{\boldsymbol{0}} = 0 \qquad \text{and} \qquad \boldsymbol{e}_0 = \boldsymbol{\partial}_\tau,$$

one obtains the evolution equations

$$\partial_{\boldsymbol{0}} e_{\boldsymbol{b}}{}^{\nu} = -\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}} e_{\boldsymbol{c}}{}^{\nu}, \tag{2.80a}$$

$$\partial_{\boldsymbol{0}} \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} = -\hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}} + 2\eta_{\boldsymbol{d0}}\eta^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{be}} - 2\delta_{\boldsymbol{d}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{b0}} - 2\delta_{\boldsymbol{0}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{bd}} - \Theta d^{\boldsymbol{c}}{}_{\boldsymbol{d0b}}, \tag{2.80b}$$

$$\partial_{\boldsymbol{0}} \hat{L}_{\boldsymbol{bc}} = \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} \hat{L}_{\boldsymbol{dc}} + \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} \hat{L}_{\boldsymbol{bd}} + d_{\boldsymbol{a}} d^{\boldsymbol{a}}{}_{\boldsymbol{c0b}}. \tag{2.80c}$$

These equations constitute a system of *transport equations* along the conformal geodesics. The evolution equations for the components of the rescaled Weyl tensor are obtained by using the following expressions

$$d_{\boldsymbol{abcd}} = 2(h_{\boldsymbol{b}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{a}} - h_{\boldsymbol{a}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{b}}) - 2(\tau_{[\boldsymbol{c}} d^{*}{}_{\boldsymbol{d}]\boldsymbol{e}} \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{ab}} + \tau_{[\boldsymbol{a}} d^{*}{}_{\boldsymbol{b}]\boldsymbol{e}} \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{cd}}) \tag{2.81}$$

and

$$d^{*}{}_{\boldsymbol{abcd}} = 2(h_{\boldsymbol{b}[\boldsymbol{c}} d^{*}{}_{\boldsymbol{d}]\boldsymbol{a}} - h_{\boldsymbol{f}[\boldsymbol{c}} d^{*}{}_{\boldsymbol{d}]\boldsymbol{b}}) + 2(\tau_{[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{e}} \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{ab}} + \tau_{[\boldsymbol{a}} d_{\boldsymbol{b}]\boldsymbol{e}} \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{cd}}), \tag{2.82}$$

for the decomposition of the Weyl candidate in its electric and magnetic parts in the Equations

$$\nabla^{\boldsymbol{a}} d_{\boldsymbol{abcd}} = 0 \qquad \text{and} \qquad \nabla^{\boldsymbol{a}} d^{*}{}_{\boldsymbol{abcd}} = 0,$$

from which one obtains

$$\begin{aligned}
\hat{\Lambda}_{\boldsymbol{b0d}} &= \boldsymbol{e_0} d^{*}{}_{\boldsymbol{bd}} + \mathcal{D}^{\boldsymbol{f}} d_{\boldsymbol{fbd}} - a^{\boldsymbol{f}} d_{\boldsymbol{fbd}} - a^{\boldsymbol{c}} d_{\boldsymbol{bcd}} - 2\chi^{\boldsymbol{fc}}(h_{\boldsymbol{b}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{f}} - h_{\boldsymbol{f}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{b}}) \\
&\quad + 2\chi^{\boldsymbol{fc}}(\tau_{[\boldsymbol{c}} d^{*}{}_{\boldsymbol{d}]\boldsymbol{e}} \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{fb}} + \tau_{[\boldsymbol{f}} d^{*}{}_{\boldsymbol{b}]\boldsymbol{e}} \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{cd}}), \\
\hat{\Lambda}^{*}{}_{\boldsymbol{b0d}} &= \boldsymbol{e_0} d_{\boldsymbol{bd}} - \mathcal{D}^{\boldsymbol{f}} d^{*}{}_{\boldsymbol{fbd}} + a^{\boldsymbol{f}} d^{*}{}_{\boldsymbol{fbd}} + a^{\boldsymbol{c}} d^{*}{}_{\boldsymbol{bcd}} + 2\chi^{\boldsymbol{fc}}(h_{\boldsymbol{b}[\boldsymbol{c}} d^{*}{}_{\boldsymbol{d}]\boldsymbol{f}} - h_{\boldsymbol{f}[\boldsymbol{c}} d^{*}{}_{\boldsymbol{d}]\boldsymbol{b}}) \\
&\quad - 2\chi^{\boldsymbol{fc}}(\tau_{[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{e}} \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{fb}} + \tau_{[\boldsymbol{f}} d_{\boldsymbol{b}]\boldsymbol{e}} \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{cd}}),
\end{aligned}$$

where

$$h_{\boldsymbol{ab}} \equiv g_{\boldsymbol{ab}} - \tau_{\boldsymbol{a}}\tau_{\boldsymbol{b}}, \qquad \chi_{\boldsymbol{ab}} = h_{\boldsymbol{a}}{}^{\boldsymbol{c}}\nabla_{\boldsymbol{c}}\tau_{\boldsymbol{b}}, \qquad \chi = h^{\boldsymbol{ab}}\chi_{\boldsymbol{ab}}, \qquad a_{\boldsymbol{a}} \equiv \tau^{\boldsymbol{b}}\nabla_{\boldsymbol{b}}\tau_{\boldsymbol{a}}.$$

These equations are completely general. In the particular case of a conformal Gaussian gauge system, one has $\boldsymbol{e}_{\boldsymbol{0}} = \boldsymbol{\partial}_\tau$.

2.6.2 The subsidiary evolution system

This section addresses the construction of a system of subsidiary equations for the evolution equations discussed in the previous section. The particular problem consists of constructing evolution equations for the zero-quantities

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \qquad \hat{\Delta}_{\boldsymbol{cdb}}, \qquad \hat{\Lambda}_{\boldsymbol{bcd}}$$

encoding the extended conformal Einstein field equations. In addition, in the present hyperbolic reduction procedure, one also needs to construct evolution equations for the additional zero-quantities

$$\delta_{\boldsymbol{a}}, \qquad \gamma_{\boldsymbol{ab}}, \qquad \varsigma_{\boldsymbol{ab}},$$

which play the role of constraints of the conformal equations. The necessity of the extra zero quantities can be traced back to Proposition 2. These subsidiary equations need to be homogeneous in zero-quantities so that the vanishing of the latter on an initial hypersurface readily implies a unique vanishing solution. The basic assumption in the construction of the subsidiary system is that the evolution equations associated with the extended conformal field equations are satisfied. That is, one assumes that

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e}_{\boldsymbol{c}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0, \qquad \hat{\Delta}_{\boldsymbol{cdb}} = 0$$

hold, together with the standard system for the components of the Weyl spinor

$$\hat{\Lambda}_{\boldsymbol{bcd}} = 0, \qquad \hat{\Lambda}^*{}_{\boldsymbol{bcd}} = 0.$$

These evolution equations have been constructed using the gauge conditions

$$\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{b}} = 0, \qquad f_{\boldsymbol{0}} = 0, \qquad \hat{L}_{\boldsymbol{0a}} = 0,$$

which, therefore can also be used in the construction of the subsidiary system. Moreover, in the present gauge

$$d_{\boldsymbol{0}} = \Theta \boldsymbol{\beta}_{\boldsymbol{0}} = \nabla_{\boldsymbol{0}}\Theta$$

so that one has

$$\delta_{\boldsymbol{0}} = 0.$$

Similarly, by virtue of the gauge conditions and the evolution equation for $\beta_{\boldsymbol{a}}$

$$\hat{\nabla}_{\boldsymbol{0}}\beta_{\boldsymbol{a}} + \beta_{\boldsymbol{0}}\beta_{\boldsymbol{a}} - \frac{1}{2}\eta_{\boldsymbol{0a}}(\beta_{\boldsymbol{e}}\beta^{\boldsymbol{e}} - 2\lambda\Theta^{-2}) = 0,$$

one has

$$\gamma_{\boldsymbol{0b}} = \hat{L}_{\boldsymbol{0b}} - \hat{\nabla}_{\boldsymbol{0}}\beta_{\boldsymbol{b}} - \frac{1}{2}S_{\boldsymbol{0b}}{}^{\boldsymbol{ef}}\beta_{\boldsymbol{e}}\beta_{\boldsymbol{f}} + \lambda\Theta^{-2}\eta_{\boldsymbol{0b}} = 0.$$

Finally, as a result of the evolution equation for the covector $\boldsymbol{f}$ one has

$$\varsigma_{\boldsymbol{0b}} = -\hat{L}_{\boldsymbol{0b}} - \hat{\nabla}_{\boldsymbol{0}}f_{\boldsymbol{b}} + -\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}f_{\boldsymbol{e}} = 0.$$

The construction of subsidiary equations is similar to the one discussed for the main evolution system. There are however certain differences. The most conspicuous one is the fact that one is now working with a non-metric connection. A detailed derivation of the subsidiary system and a comprehensive discussion can be found in Chapter 3 — see also [27, 28, 81].

2.7 The conformal constraint equations

The *conformal constraint Einstein equations* are a set of intrinsic equations implied by the standard conformal Einstein field equations on spacelike hypersurfaces $\mathcal{S}$ of the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$. A derivation of these equations in their frame form can be found in [81], Section 11.4.

Let $\mathcal{S}$ denote a spacelike hypersurface in the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$. The metric $\boldsymbol{g}$ induces a 3-dimensional metric

$$\boldsymbol{h} = \varphi^*\boldsymbol{g} \qquad \text{on} \quad \mathcal{S},$$

via the embedding $\varphi : \mathcal{S} \to \mathcal{M}$. Since one considers the setting where the 1-form $\boldsymbol{f}$ vanishes on $\mathcal{S}$, the initial data for the extended conformal evolution equations and those implied by the hyperbolic reduction of the standard conformal Einstein field equations are the same. Now, let $\{\boldsymbol{e_a}\}$ denote a $\boldsymbol{g}$-orthonormal frame adapted to $\mathcal{S}$. That is, the vector $\boldsymbol{e_0}$ is chosen to coincide with the unit normal vector to the hypersurface and while the spatial vectors $\{\boldsymbol{e_i}\}$, $\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$ are intrinsic to $\mathcal{S}$. In our signature conventions, we have that $\boldsymbol{g}(\boldsymbol{e_0}, \boldsymbol{e_0}) = -1$. The extrinsic curvature is described by the components $\chi_{\boldsymbol{ij}}$ of the Weingarten tensor $\chi_{\boldsymbol{ab}}$. One has that $\chi_{\boldsymbol{ij}} = \chi_{\boldsymbol{ji}}$ and, moreover

$$\chi_{\boldsymbol{ij}} = -\Gamma_{\boldsymbol{i}}{}^{\mathbf{0}}{}_{\boldsymbol{j}}.$$

We denote by Ω the restriction of the spacetime conformal factor Ξ to $\mathcal{S}$ and by Σ the normal component of the gradient of Ω. The field $l_{\boldsymbol{ij}}$ denote the components of the Schouten tensor of the induced metric h_{ij} on $\mathcal{S}$. With the above conventions, the conformal constraint equations in the vacuum case are given by —see [81]:

$$D_{\boldsymbol{i}}D_{\boldsymbol{j}}\Omega = \Sigma\chi_{\boldsymbol{ij}} - \Omega L_{\boldsymbol{ij}} + s h_{\boldsymbol{ij}}, \tag{2.84a}$$

$$D_{\boldsymbol{i}}\Sigma = \chi_{\boldsymbol{i}}{}^{\boldsymbol{k}} D_{\boldsymbol{k}}\Omega - \Omega L_{\boldsymbol{i}}, \tag{2.84b}$$

$$D_{\boldsymbol{i}}s = L_{\boldsymbol{i}}\Sigma - L_{\boldsymbol{ik}}D^{\boldsymbol{k}}\Omega, \tag{2.84c}$$

$$D_{\boldsymbol{i}}L_{\boldsymbol{jk}} - D_{\boldsymbol{j}}L_{\boldsymbol{ik}} = \Sigma d_{\boldsymbol{kij}} + D^{\boldsymbol{l}}\Omega d_{\boldsymbol{lkij}} - (\chi_{\boldsymbol{ik}}L_{\boldsymbol{j}} - \chi_{\boldsymbol{jk}}L_{\boldsymbol{i}}), \tag{2.84d}$$

$$D_{\boldsymbol{i}}L_{\boldsymbol{j}} - D_{\boldsymbol{j}}L_{\boldsymbol{i}} = d_{\boldsymbol{lij}}D^{\boldsymbol{l}}\Omega + \chi_{\boldsymbol{i}}{}^{\boldsymbol{k}}L_{\boldsymbol{jk}} - \chi_{\boldsymbol{j}}{}^{\boldsymbol{k}}L_{\boldsymbol{ik}}, \tag{2.84e}$$

$$D^{\boldsymbol{k}}d_{\boldsymbol{kij}} = -(\chi^{\boldsymbol{k}}{}_{\boldsymbol{i}}d_{\boldsymbol{jk}} - \chi^{\boldsymbol{k}}{}_{\boldsymbol{j}}d_{\boldsymbol{ik}}), \tag{2.84f}$$

$$D^{\boldsymbol{i}}d_{\boldsymbol{ij}} = \chi^{\boldsymbol{ik}}d_{\boldsymbol{ijk}}, \tag{2.84g}$$

$$\lambda = 6\Omega s + 3\Sigma^2 - 3D_{\boldsymbol{k}}\Omega D^{\boldsymbol{k}}\Omega, \tag{2.84h}$$

$$D_{\boldsymbol{j}}\chi_{\boldsymbol{ki}} - D_{\boldsymbol{k}}\chi_{\boldsymbol{ji}} = \Omega d_{\boldsymbol{ijk}} + h_{\boldsymbol{ij}}L_{\boldsymbol{k}} - h_{\boldsymbol{ik}}L_{\boldsymbol{j}}, \tag{2.84i}$$

$$l_{\boldsymbol{ij}} = \Omega d_{\boldsymbol{ij}} + L_{\boldsymbol{ij}} - \chi(\chi_{\boldsymbol{ij}} - \frac{1}{4}\chi h_{\boldsymbol{ij}}) + \chi_{\boldsymbol{ki}}\chi_{\boldsymbol{j}}{}^{\boldsymbol{k}} - \frac{1}{4}\chi_{\boldsymbol{kl}}\chi^{\boldsymbol{kl}}h_{\boldsymbol{ij}}, \tag{2.84j}$$

with the understanding that

$$h_{\boldsymbol{ij}} \equiv g_{\boldsymbol{ij}} = \delta_{\boldsymbol{ij}}$$

and where we have defined

$$L_{\boldsymbol{i}} \equiv L_{\mathbf{0}\boldsymbol{i}}, \qquad d_{\boldsymbol{ij}} \equiv d_{\mathbf{0}\boldsymbol{i}\mathbf{0}\boldsymbol{j}}, \qquad d_{\boldsymbol{ijk}} \equiv d_{\boldsymbol{i}\mathbf{0}\boldsymbol{jk}}.$$

The fields $d_{\boldsymbol{ij}}$ and $d_{\boldsymbol{ijk}}$ correspond, respectively, to the electric and magnetic parts of the rescaled Weyl tensor and the scalar s denotes the *Friedrich scalar* defined as in Equation (2.19). Finally, $L_{\boldsymbol{ij}}$ denote the spatial components of the Schouten tensor of $\boldsymbol{g}$.

In the derivation of the equations (2.84a)-(2.84j) it has been assumed that the connection $\boldsymbol{D}$ is the Levi-Civita connection of the intrinsic metric $\boldsymbol{h}$. Thus, by analogy to the full conformal field equations, one also has the relations

$$\sigma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} = 0, \qquad \Pi^{\boldsymbol{k}}{}_{\boldsymbol{lij}} = \pi^{\boldsymbol{k}}{}_{\boldsymbol{lij}}, \tag{2.85}$$

where $\sigma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$, $\Pi^{\boldsymbol{k}}{}_{\boldsymbol{lij}}$ and $\pi^{\boldsymbol{k}}{}_{\boldsymbol{lij}}$ are given by

$$\sigma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} e_{\boldsymbol{k}} \equiv [e_{\boldsymbol{i}}, e_{\boldsymbol{j}}] - (\gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} - \gamma_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{i}}) e_{\boldsymbol{k}}, \tag{2.86a}$$

$$\Pi^{\boldsymbol{k}}{}_{\boldsymbol{lij}} \equiv e_{\boldsymbol{i}}(\gamma_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{l}}) - e_{\boldsymbol{j}}(\gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{l}}) + \gamma_{\boldsymbol{m}}{}^{\boldsymbol{k}}{}_{\boldsymbol{l}}(\gamma_{\boldsymbol{j}}{}^{\boldsymbol{m}}{}_{\boldsymbol{i}} - \gamma_{\boldsymbol{i}}{}^{\boldsymbol{m}}{}_{\boldsymbol{j}}) + \gamma_{\boldsymbol{j}}{}^{\boldsymbol{m}}{}_{\boldsymbol{l}}\gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{m}} - \gamma_{\boldsymbol{i}}{}^{\boldsymbol{m}}{}_{\boldsymbol{l}}\gamma_{\boldsymbol{j}}{}^{\boldsymbol{k}}{}_{\boldsymbol{m}}, \tag{2.86b}$$

$$\pi_{\boldsymbol{klij}} \equiv h_{\boldsymbol{ik}} l_{\boldsymbol{lj}} - h_{\boldsymbol{il}} l_{\boldsymbol{kl}} + h_{\boldsymbol{jl}} l_{\boldsymbol{ki}} - h_{\boldsymbol{jk}} l_{\boldsymbol{li}} \tag{2.86c}$$

and denote, respectively, the components of the *torsion*, the *geometric curvature* and the *algebraic curvature* of the connection $\boldsymbol{D}$.

Proposition 4. *A solution to the conformal constraint equations on $\mathcal{S}$ is a collection*

$$\boldsymbol{u}_\star \equiv \{\Omega, \Sigma, s, \boldsymbol{e}_i, \gamma_i{}^k{}_j, \chi_{ij}, L_{ij}, L_i, d_{ij}, d_{ijk}\}$$

satisfying (2.84a)-(2.84j) *together with the supplementary conditions* (2.85).

2.7.1 The Hamiltonian and momentum constraints

An alternative way of discussing the conformal constraint equations is to start with the usual Hamiltonian and momentum constraints in the physical space $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}})$

$$\tilde{r} + \tilde{\chi}^2 - \tilde{\chi}_{ab}\tilde{\chi}^{ab} = 2\lambda, \tag{2.87a}$$

$$\tilde{D}^b \tilde{\chi}_{ab} - \tilde{D}_a \tilde{\chi} = 0, \tag{2.87b}$$

where $\tilde{r}$ is the Ricci scalar of $\tilde{\boldsymbol{h}}$, λ is the Cosmological constant. Given the conformal transformation

$$\boldsymbol{h} = \Omega^2 \tilde{\boldsymbol{h}},$$

and the frame $\{\boldsymbol{e_i}\}$ introduced in the previous section, a direct computations of (2.87a)-(2.87b) gives the so-called *Hamiltonian and momentum constraints*:

$$2\Omega D_i D^i \Omega - 3 D_i \Omega D^i \Omega + \frac{1}{2}\Omega^2 r + 3\Sigma^2 - \frac{1}{2}\Omega^2(\chi^2 - \chi_{ij}\chi^{ij}) - 2\Omega\Sigma\chi = \lambda, \tag{2.88a}$$

$$\Omega^3 D^i(\Omega^{-2}\chi_{ik}) - \Omega(D_k\chi - 2\Omega^{-1}D_k\Sigma) = 0, \tag{2.88b}$$

where r is the Ricci scalar of $\boldsymbol{h}$. Those equations containing terms involving Ω^{-1} and Ω^{-2} are not formally regular at $\Omega = 0$.

The relation between the conformal Hamiltonian and momentum constraint equations (2.88a)-(2.88b) and the conformal constraint equations (2.84a)-(2.84j) is the content of the following lemma

Lemma 5. *A solution $\{\mathcal{S}, \boldsymbol{u}_\star\}$ to the conformal constraint equations* (2.84a)-(2.84j) *implies a solution to the conformal Hamiltonian and momentum constraint* (2.88a)-(2.88b)*. Conversely, a solution $\{\mathcal{S}, \boldsymbol{h}, \boldsymbol{\chi}, \Omega, \Sigma\}$ of* (2.88a)-(2.88b) *give rise to a solution to* (2.84a)-(2.84j) *on the points of $\mathcal{S}$ for which $\Omega \neq 0$.*

The proof can be found in Chapter 11 of [81] – see also [21]. It follows from this lemma that the formulation of a Cauchy problem for the conformal field equations, by prescribing initial data on a 3-dimensional manifold $\mathcal{S}$ in which $\Omega = 0$ requires using equations (2.84a)-(2.84j) to determine initial data for the conformal evolution equations.

Chapter 3

The non-linear stability of de Sitter-like spacetimes with spatial sections of negative scalar curvature

3.1 Introduction

In the Mathematical Relativity literature, a *Cosmological spacetime* is usually understood as a spacetime with compact spatial sections. Understanding the long-time evolution of generic examples of these spacetimes in, say the *vacuum case*, is one of the open challenges in the area. Although generic initial data is expected to form singularities towards the future, it is nevertheless essential to address the stability of those solutions which are known to be geodesically complete. The fundamental example of a geodesically complete Cosmological spacetime is given by the *de Sitter spacetime*. Its non-linear stability was analysed in the seminal work by Friedrich [23, 24]. A central aspect of this result is the use of conformal methods to transform the question of the global existence of solutions to a finite existence problem. An alternative approach to the study of the non-linear stability of vacuum Cosmological solutions to the Einstein field equations by means of so-called *CMC foliations* has been used by Andersson & Moncrief [3, 4] to prove the non-linear stability of 4-dimensional Friedmann-Lemaître-Robinson-Walker (FLRW) vacuum solutions. Using similar methods, in [17] Fajman & Kröncke studied the non-linear stability of large classes of Cosmological solutions to the vacuum Einstein field equations with a positive Cosmological constant in arbitrary dimensions. These solutions are characterised by having spatial sections with constant scalar curvature which can be either positive or negative.

The purpose of this chapter is to show that, in four dimensions, the stability results for spacetimes with spatial sections of constant negative scalar curvature given in [17] can be addressed via a generalisation of the conformal methods developed by Friedrich [23, 26, 27, 34] —see also [81]. This discussion exploits the hyperbolic reduction procedure discussed in Chapter 2 and is based on:

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3.1.1 De Sitter-like spacetimes

In what follows, for a *de Sitter-like spacetime* is understood a vacuum spacetime with a positive value of the Cosmological constant and compact spatial sections of negative scalar curvature. General results on conformal geometry show that *if* these spacetimes admit a conformal compactification *à la* Penrose then the conformal boundary of the spacetime must be spacelike —see e.g. [81], Theorem 10.1. Following the standard usage, we refer to the conformal extension of a de Sitter-like spacetime as the *unphysical spacetime*. The usefulness of this conformal extension lies in the fact that points representing the infinity of the physical spacetime (e.g. the endpoints of timelike geodesics) are mapped to a finite location in the unphysical spacetime.

In this particular case, we consider de Sitter-like spacetimes which can be conformally embedded into a portion of a cylinder whose sections have negative scalar curvature. The conformal embedding is realised by means of a conformal factor Θ which depends quadratically on the affine parameter τ of *conformal geodesics*, and the affine parameter is used as a time coordinate for the physical metric.

Key in the conformal approach is that the unphysical metric provides a solution to the *conformal Einstein field equations* [22, 81]. In this chapter we make use of a more general version of these equations, the *extended conformal Einstein field equations*, allowing the use of *conformal Gaussian coordinate systems* in which coordinates are propagated along conformal geodesics.

As already mentioned, the appeal of conformal methods in the study of solutions to the Einstein field equations lies in the observation that local results for the unphysical spacetime can, in principle, be translated into global results for the physical spacetime. In the original formulation of the conformal Einstein field equations the conformal factor realising the conformal embedding of the physical spacetime in a compact manifold is an unknown of the problem. However, remarkably, the use of conformal Gaussian coordinates systems provide a natural conformal factor which singles out a representative of the conformal class of the spacetime. Accordingly, the location of the conformal boundary is known *a priori*, thus simplifying further the analysis of the evolution equations. The extended conformal Einstein field equations expressed in terms of a conformal Gaussian system can be shown to imply a conformal evolution system which takes the form of a symmetric hyperbolic system —i.e. a class of evolution systems for which there exists a well-developed existence, uniqueness and stability theory [49].

3.1.2 The main result

The analysis of the conformal properties of de Sitter-like spacetimes with compact spatial sections allows us to formulate a result concerning the existence of solutions to the initial value problem for the Einstein field equations.

Our main result can be stated as:

Main Result 1. *Given smooth initial data* $(\boldsymbol{h}, \boldsymbol{K})$ *for the Einstein field equations on* $\mathcal{S}$ *which is suitably close (as measured by a suitable Sobolev norm) to the data implied by the metric* $\mathring{\tilde{\boldsymbol{g}}}$ *of de Sitter-like spacetime, there exists a smooth metric* $\tilde{\boldsymbol{g}}$ *defined over* $[0, \infty) \times \mathcal{S}$

which is close to $\mathring{\tilde{\boldsymbol{g}}}$ *(again, in the sense of Sobolev norms) and solves the vacuum Einstein field equations with Cosmological constant* $\lambda = 3$*. The spacetime* $([0,\infty) \times \mathcal{S}, \tilde{\boldsymbol{g}})$ *is future geodesically complete.*

Remark 2. A precise formulation of this result is given in Theorem 1 in Section 3.7. The construction of the initial data required in the above result has been analysed in [82].

This analysis is part of the general programme in Mathematical Relativity to understand the endpoint of the evolution of "Cosmological spacetimes" (i.e. spacetimes with compact sections) under the Einstein field equations, the so-called *Einstein flow*. In particular, it identifies a class of spacetimes for which it is possible to show non-linear stability and the existence of a regular conformal representation. These special properties are not shared by generic Cosmological solutions. Thus, it is important to identify the situations in which this is the case.

3.2 The background solution

In the following let $(\tilde{\mathcal{M}}, \mathring{\tilde{\boldsymbol{g}}})$ denote the solution to the vacuum Einstein field equations with positive Cosmological constant

$$\tilde{R}_{ab} = 3\tilde{g}_{ab}, \tag{3.1}$$

given by $\tilde{\mathcal{M}} = \mathbb{R} \times \mathcal{S}$ and

$$\mathring{\tilde{\boldsymbol{g}}} = -\mathbf{d}t \otimes \mathbf{d}t + \sinh^2 t\, \mathring{\boldsymbol{\gamma}}, \tag{3.2}$$

where $\mathring{\boldsymbol{\gamma}}$ is a positive definite Riemannian metric of constant negative curvature over a compact manifold $\mathcal{S}$ such that

$$r[\mathring{\boldsymbol{\gamma}}] = -6.$$

The spacetime $(\tilde{\mathcal{M}}, \mathring{\tilde{\boldsymbol{g}}})$ is future geodesically complete —see Appendix A.2.

Remark 3. The value $\lambda = 3$ for the Cosmological constant is conventional and set for convenience. This analysis can be carried out for any other positive value. Indeed, given

$\lambda > 0$ define the metric $\bar{g}_{ab}$ via the relation

$$\bar{g}_{ab} = \frac{3}{\lambda} g_{ab}.$$

As this is a constant conformal rescaling, the Ricci tensor is invariant —i.e. $\bar{R}_{ab} = R_{ab}$; see e.g. equation (5.6a) on page 116 in [81]. It follows then that equation (4.1) implies

$$\bar{R}_{ab} = \lambda \bar{g}_{ab}.$$

Remark 4. The existence of compact 3-manifolds with constant negative scalar curvature has been analysed in the mathematical literature —see [48]. These 3-manifolds are *locally isometric* to quotients of the hyperbolic space $\mathbb{H}^3$. The admissible topologies are discussed in [6]. This class of manifolds is sometimes called *conformally rigid hyperbolic manifolds* as, despite being conformally flat, they do not admit globally defined conformal Killing vectors nor non-trivial trace-free Codazzi tensors. These properties play a crucial role in the perturbative construction of initial data for the conformal evolution system as discussed in Section 3.4.

The Riemann curvature tensor $r^i{}_{jkl}[\mathring{\boldsymbol{\gamma}}]$ of the metric $\mathring{\boldsymbol{\gamma}}$ is given by

$$r_{ijkl}[\mathring{\gamma}] = \mathring{\gamma}_{il}\mathring{\gamma}_{jk} - \mathring{\gamma}_{ik}\mathring{\gamma}_{jl}.$$

From the above expressions it follows that

$$\tilde{R} = 12,$$

so that

$$\tilde{L}_{ab} = \frac{1}{2}\tilde{g}_{ab}. \tag{3.3}$$

A spacetime of the form given by $(\tilde{\mathcal{M}}, \mathring{\tilde{\boldsymbol{g}}})$ is known as a *background solution*. In the rest of this section, we analyse this class of solutions to the Einstein field equations from the point of view of conformal geometry. In particular, we use the conformal geodesics to provide a *canonical* conformal extension —see Proposition 3.

3.2.1 A class of conformal geodesics

Let $x(s)$ be metric geodesics on $(\tilde{\mathcal{M}}, \mathring{\tilde{\boldsymbol{g}}})$ whose tangent vector is proportional to $\boldsymbol{\partial}_t$ —i.e. $\dot{\boldsymbol{x}} = \alpha\boldsymbol{\partial}_t$ for some proportionality function α and where the overdot denotes differentiation with respect to the affine parameter $s \in \mathbb{R}$. The geodesic equation

$$\tilde{\nabla}_{\dot{x}}\dot{\boldsymbol{x}} = 0$$

implies that

$$\begin{aligned}\tilde{\nabla}_{\boldsymbol{\partial}_t}(\alpha\boldsymbol{\partial}_t) &= \tilde{\nabla}_t\alpha + \tilde{\nabla}_{\boldsymbol{\partial}_t}\boldsymbol{\partial}_t \\ &= \tilde{\nabla}_t\alpha + \Gamma_t{}^{\mu}{}_t\boldsymbol{\partial}_{\mu}.\end{aligned}$$

A direct calculation for the metric (3.2) shows that $\Gamma_t{}^{\mu}{}_t = 0$ so that one concludes that $\partial_t\alpha = 0$ —that is, α is constant along the integral curves of $\boldsymbol{\partial}_t$. Without loss of generality, we then set $\alpha = 1$ so that $\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = -1$. In summary, we have that the curves

$$x(t) = (t, \underline{x}_\star), \qquad \underline{x}_\star \in \mathcal{S},$$

are non-intersecting timelike $\tilde{\boldsymbol{g}}$-geodesics over $\tilde{\mathcal{M}}$. In a slight abuse of notation, the coordinate t has been used as a parameter of the curve.

Reparametrisation as conformal geodesic

In the following, we use the methods in the proof of Lemma 4 to recast the family of geodesics discussed in Subsection 3.2.1 as conformal geodesics —see also [81]. Accordingly, we consider a reparametrisation of the form

$$\tau \mapsto t(\tau),$$

while we look for a 1-form $\tilde{\boldsymbol{\beta}}$ given by the Ansatz

$$\tilde{\boldsymbol{\beta}} = \alpha(\tau)\boldsymbol{x}'^{\flat} = \alpha(t)\mathbf{d}t,$$

where $'$ denotes derivatives with respect to t. From the chain rule, it follows that

$$\dot{\boldsymbol{x}} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{t}\boldsymbol{x}', \qquad \dot{t} \equiv \frac{\mathrm{d}t}{\mathrm{d}\tau}.$$

In particular, one readily has that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = \dot{t}^2\tilde{\nabla}_{\boldsymbol{x}'}\boldsymbol{x}' + \ddot{t}\boldsymbol{x}'.$$

Substituting the previous expressions into equations (2.57a) and (2.57b), taking into account expression (3.3) for the components of the Schouten tensor one obtains the system of ordinary differential equations

$$\ddot{t} + \alpha\dot{t}^2 = 0, \tag{3.4a}$$

$$\dot{\alpha} = \frac{1}{2}\dot{t}(\alpha^2 - 1). \tag{3.4b}$$

The general solution to the above system can be found to be

$$\alpha(\tau) = c_1\tau + c_2,$$
$$t(\tau) = -2\mathrm{arctanh}(c_1\tau + c_2) + c_3,$$

with c_1, c_2, $c_3 \in \mathbb{R}$ constants. For simplicity, one can, e.g. set $c_1 = -1$, $c_2 = c_3 = 0$ to get the simpler expressions

$$\alpha(\tau) = -\tau,$$
$$t(\tau) = 2\mathrm{arctanh}\,\tau.$$

Thus, observing that

$$\sinh\Big(2\mathrm{arctanh}\,\tau\Big) = \frac{2\tau}{1-\tau^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(2\mathrm{arctanh}\,\tau\Big) = \frac{2}{1-\tau^2},$$

it follows that the pair $(x(\tau), \tilde{\boldsymbol{\beta}}(\tau))$, $\tau \in (-1,1)$ with

$$x(\tau) = (2\,\mathrm{arctanh}\,\tau, \underline{x}_\star), \qquad \tilde{\boldsymbol{\beta}}(\tau) = -\frac{2\tau}{1-\tau^2}\mathbf{d}\tau,$$

give rise to a congruence of non-intersecting conformal geodesics on the background spacetime $(\tilde{\mathcal{M}}, \mathring{\tilde{\boldsymbol{g}}})$. Using the parameter τ as a new coordinate in the metric (3.2) one concludes that

$$\mathring{\tilde{\boldsymbol{g}}} = \frac{4}{(1-\tau^2)^2}\bigg(-\mathbf{d}\tau\otimes\mathbf{d}\tau + \tau^2\mathring{\boldsymbol{\gamma}}\bigg). \tag{3.5}$$

Notice that the metric is singular at $\tau = \pm 1$.

The canonical factor associated to the congruence of conformal geodesics

The line element (3.5) readily suggest the conformal factor

$$\Theta \equiv \frac{1}{2}(1-\tau^2).$$

Remark 5. Alternatively, we can make use of the equation

$$\dot{\Theta} = \langle \tilde{\boldsymbol{\beta}}, \dot{\boldsymbol{x}} \rangle \Theta, \qquad \langle \tilde{\boldsymbol{\beta}}, \dot{\boldsymbol{x}} \rangle = \alpha \dot{t} = -\frac{2\tau}{1-\tau^2},$$

implied by the condition $\Theta^2 \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = -1$. Integrating one readily finds that

$$\frac{\Theta}{\Theta_\star} = \frac{1-\tau^2}{1-\tau_\star^2},$$

where $\Theta_\star$ is the value of the conformal factor at a fiduciary time $\tau_\star$. Observe, also, that

$$\begin{aligned} \tilde{\boldsymbol{\beta}}(\tau) &= -\frac{2\tau}{1-\tau^2}\mathbf{d}\tau, \\ &= \mathbf{d}\Big(\ln \Theta(\tau)\Big). \end{aligned} \tag{3.6}$$

Following expression (3.5) we introduce a new unphysical metric $\mathring{\boldsymbol{g}}$ via the relation

$$\mathring{\boldsymbol{g}} = \Theta^2 \mathring{\tilde{\boldsymbol{g}}}, \qquad \Theta \equiv \frac{1}{2}(1-\tau^2),$$

so as to ensure that $\Theta \geq 0$ for $|\tau| \leq 1$. It follows then that

$$\mathring{\boldsymbol{g}} = -\mathbf{d}\tau \otimes \mathbf{d}\tau + \tau^2 \mathring{\boldsymbol{\gamma}} \tag{3.7}$$

is well defined for $\tau \in [\tau_\star, 1]$ with $\tau_\star > 0$. For future use, we define the *spatial metric* $\mathring{\boldsymbol{h}}$

$$\mathring{\boldsymbol{h}} \equiv \tau^2 \mathring{\boldsymbol{\gamma}},$$

with associated Levi-Civita connection to be denoted by $\mathring{D}$. Also, denote by $\mathring{\mathfrak{D}}$ the Levi-Civita connection of the metric $\mathring{\boldsymbol{\gamma}}$. A Penrose diagram of the conformal representation of the background solution described by the metric (3.7) is given in Figure 3.1.

Remark 6. Observe that as the metrics $\mathring{\boldsymbol{\gamma}}$ and $\mathring{\boldsymbol{h}}$ are conformally related via a conformal factor independent of the spatial coordinates, it follows then that expressed in terms of local (spatial) coordinates one has that

$$\mathring{D}_\alpha = \mathring{\mathfrak{D}}_\alpha.$$

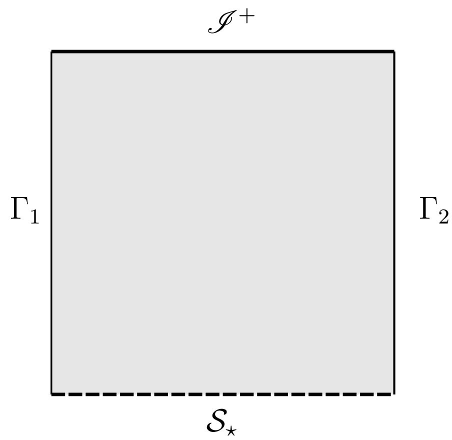

Figure 3.1: Penrose diagram of the background solution. The conformal representation discussed in the main text has compact sections of negative scalar curvature. The vertical lines Γ_1 and Γ_2 correspond to axes of symmetry. The solution has a singularity in the past and a spacelike future conformal boundary. Hence, in our discussion, we only consider the future evolution of the initial hypersurface $\mathcal{S}_\star$.

Remark 7. A computation readily shows that the integral curves of the vector field $\boldsymbol{\partial}_\tau$ are geodesics of the metric $\mathring{\boldsymbol{g}}$ given by equation (3.7) —that is, one has that

$$\nabla_{\boldsymbol{\partial}_\tau}\boldsymbol{\partial}_\tau = 0.$$

Remark 8. Taking into account the expression (3.6), the conformal transformation law for conformal geodesics gives that

$$\boldsymbol{\beta} = \tilde{\boldsymbol{\beta}} - \mathbf{d}\Big(\ln\Theta(\tau)\Big) = 0.$$

To any (non-singular) congruence of conformal geodesics one can associate a Weyl connection $\hat{\boldsymbol{\nabla}}$ via the rule

$$\hat{\boldsymbol{\nabla}} - \tilde{\boldsymbol{\nabla}} = \boldsymbol{S}(\tilde{\boldsymbol{\beta}}).$$

In the present case, $\tilde{\boldsymbol{\beta}}$ is a closed 1-form and, thus, the Weyl connection is, in fact, a Levi-Civita connection which coincides with $\boldsymbol{\nabla}$.

3.2.2 The background spacetime as a solution to the conformal Einstein field equations

In this subsection, we show how to recast the *unphysical spacetime* $(\mathcal{M}, \mathring{\boldsymbol{g}})$ with $\mathcal{M} = [\tau_\star, \infty) \times \mathcal{S}$ as a solution to the conformal Einstein field equations. This construction is conveniently done using an adapted frame formalism.

The frame

Let $\{\mathring{\boldsymbol{c}}_{\boldsymbol{i}}\}$, $\boldsymbol{i} = 1, 2, 3$, denote a $\mathring{\boldsymbol{\gamma}}$-orthonormal frame over $\mathcal{S}$ with associated cobasis $\{\mathring{\boldsymbol{\alpha}}^{\boldsymbol{i}}\}$. Accordingly, one has that

$$\mathring{\boldsymbol{\gamma}}(\mathring{\boldsymbol{c}}_i, \mathring{\boldsymbol{c}}_j) = \delta_{\boldsymbol{ij}}, \qquad \langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{j}}, \mathring{\boldsymbol{c}}_{\boldsymbol{i}} \rangle = \delta_{\boldsymbol{i}}{}^{\boldsymbol{j}},$$

so that

$$\mathring{\boldsymbol{\gamma}} = \delta_{\boldsymbol{ij}} \mathring{\boldsymbol{\alpha}}^{\boldsymbol{i}} \otimes \mathring{\boldsymbol{\alpha}}^{\boldsymbol{j}}.$$

The above frame is used to introduce a $\mathring{\boldsymbol{g}}$-orthonormal frame $\{\mathring{\boldsymbol{e}}_{\boldsymbol{a}}\}$ with associated cobasis $\{\mathring{\boldsymbol{\omega}}^{\boldsymbol{b}}\}$ so that $\langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{b}}, \mathring{\boldsymbol{e}}_{\boldsymbol{a}} \rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$. We do this by setting

$$\begin{aligned} \mathring{\boldsymbol{e}}_0 &\equiv \boldsymbol{\partial}_\tau, \qquad & \mathring{\boldsymbol{e}}_{\boldsymbol{i}} &\equiv \frac{1}{\tau} \mathring{\boldsymbol{c}}_{\boldsymbol{i}}, \\ \mathring{\boldsymbol{\omega}}^0 &\equiv \mathbf{d}\tau, \qquad & \mathring{\boldsymbol{\omega}}^{\boldsymbol{i}} &= \tau \mathring{\boldsymbol{\alpha}}^{\boldsymbol{i}}, \end{aligned}$$

so that

$$\mathring{\boldsymbol{g}} = \eta_{\boldsymbol{ab}} \mathring{\boldsymbol{\omega}}^{\boldsymbol{a}} \otimes \mathring{\boldsymbol{\omega}}^{\boldsymbol{b}}.$$

Remark 9. It follows that all the coefficients of the frame are smooth (C^∞) over $[\tau_\star, \infty) \times \mathcal{S}$, $\tau_\star > 0$.

The connection coefficients

The connection coefficients $\mathring{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$ of the Levi-Civita connection $\mathring{D}$ with respect to the frame $\{\mathring{\boldsymbol{c}}_{\boldsymbol{i}}\}$ are defined through the relations

$$\mathring{D}_{\boldsymbol{i}} \mathring{\boldsymbol{c}}_{\boldsymbol{j}} = \mathring{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} \mathring{\boldsymbol{c}}_{\boldsymbol{k}}, \qquad \gamma_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} \equiv \langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{k}}, \mathring{D}_{\boldsymbol{i}} \mathring{\boldsymbol{c}}_{\boldsymbol{j}} \rangle.$$

Similarly, for the connection coefficients $\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$ of the Levi-Civita connection $\mathring{\nabla}$ with respect to the frame $\{\mathring{\boldsymbol{e}}_{\boldsymbol{a}}\}$ one has that

$$\mathring{\nabla}_{\boldsymbol{a}}\mathring{\boldsymbol{e}}_{\boldsymbol{b}} = \mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\mathring{\boldsymbol{e}}_{\boldsymbol{c}}, \qquad \mathring{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \equiv \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{c}}, \mathring{\nabla}_{\boldsymbol{a}}\mathring{\boldsymbol{e}}_{\boldsymbol{b}}\rangle.$$

We now proceed to compute the various connection coefficients.

The coefficients $\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}$. Recalling the definition of the connection coefficients and the basis fields $\{\mathring{\boldsymbol{e}}_i\}$ and $\{\mathring{\boldsymbol{\omega}}^j\}$ one has that

$$\begin{aligned}
\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}} &= \langle \boldsymbol{\omega}^k, \mathring{\nabla}_{\boldsymbol{i}}\mathring{\boldsymbol{e}}_{\boldsymbol{j}}\rangle = \langle \boldsymbol{\omega}^{\boldsymbol{k}}, \mathring{e}_{\boldsymbol{i}}{}^{\alpha}\mathring{\nabla}_{\alpha}\mathring{\boldsymbol{e}}_{\boldsymbol{j}}\rangle \\
&= \frac{1}{\tau}\langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{k}}, \mathring{c}_{\boldsymbol{i}}{}^{\alpha}\mathring{\nabla}_{\alpha}\mathring{\boldsymbol{c}}_{\boldsymbol{j}}\rangle = \frac{1}{\tau}\langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{k}}, \mathring{c}_{\boldsymbol{i}}{}^{\alpha}\mathring{D}_{\alpha}\mathring{\boldsymbol{c}}_{\boldsymbol{j}}\rangle = \frac{1}{\tau}\langle \mathring{\boldsymbol{\alpha}}^{\boldsymbol{k}}, \mathring{D}_{\boldsymbol{i}}\mathring{\boldsymbol{c}}_{\boldsymbol{j}}\rangle \\
&= \frac{1}{\tau}\mathring{\gamma}_{\boldsymbol{i}}{}^{\boldsymbol{k}}{}_{\boldsymbol{j}}.
\end{aligned}$$

The coefficients $\mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{0}}$. Recall that $\mathring{\boldsymbol{e}}_{\boldsymbol{0}} = \boldsymbol{\partial}_\tau$ is tangent to geodesics —see Remark 7. Thus,

$$\mathring{\nabla}_{\boldsymbol{0}}\mathring{\boldsymbol{e}}_{\boldsymbol{0}} = \mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}}\mathring{\boldsymbol{e}}_{\boldsymbol{c}},$$

from where it follows that

$$\mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{a}}{}_{\boldsymbol{0}} = 0.$$

The coefficients $\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}}$ and $\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}}$. In this case, we have that

$$\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}} = \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{j}}, \mathring{\nabla}_{\boldsymbol{i}}\mathring{\boldsymbol{e}}_{\boldsymbol{0}}\rangle = \mathring{\chi}_{\boldsymbol{i}}{}^{\boldsymbol{j}},$$

where $\chi_{\boldsymbol{i}}{}^{\boldsymbol{j}}$ denote the components of the *Weingarten tensor*. Defining $\mathring{\chi}_{\boldsymbol{ij}} \equiv \eta_{\boldsymbol{jk}}\mathring{\chi}_{\boldsymbol{i}}{}^{\boldsymbol{k}}$, one has that $\mathring{\chi}_{\boldsymbol{ij}} = \mathring{\chi}_{(\boldsymbol{ij})}$ as the congruence defined by $\boldsymbol{\partial}_\tau$ can readily be verified to be hypersurface orthogonal. Thus, in this case $\mathring{\chi}_{\boldsymbol{ij}}$ coincides with the components of the extrinsic curvature of the hypersurfaces of constant τ. To compute $\mathring{\chi}_{\boldsymbol{ij}}$ recall that

$$\chi_{ab} = -\frac{1}{2}\mathcal{L}_{\partial_\tau}h_{ab},$$

where $\mathcal{L}_{\partial_\tau}$ denotes the Lie derivative along the direction of $\boldsymbol{\partial}_t$. As

$$\mathcal{L}_{\partial_\tau}\mathring{\boldsymbol{h}} = \mathcal{L}_{\partial_\tau}\left(\tau^2\mathring{\boldsymbol{\gamma}}\right) = 2\tau\mathring{\boldsymbol{\gamma}} = \frac{2}{\tau}\mathring{\boldsymbol{h}},$$

one concludes that

$$\mathring{\chi}_{\boldsymbol{ij}} = -\frac{1}{\tau}\delta_{\boldsymbol{ij}}.$$

Exploiting the metricity of the connection $\mathring{\nabla}$ one finds that, moreover,

$$\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{j}} = -\mathring{\chi}_{\boldsymbol{ij}} = \frac{1}{\tau}\delta_{\boldsymbol{ij}}.$$

The coefficients $\mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{j}}{}_{\boldsymbol{i}}$. In this case one readily finds that

$$\begin{aligned}
\mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{j}}{}_{\boldsymbol{i}} &= \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{j}}, \mathring{\nabla}_{\boldsymbol{0}}\mathring{\boldsymbol{e}}_{\boldsymbol{i}}\rangle = \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{j}}, \mathring{\nabla}_{\boldsymbol{0}}\left(\frac{1}{\tau}\mathring{\boldsymbol{c}}_{\boldsymbol{i}}\right)\rangle \\
&= -\frac{1}{\tau^2}\langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{j}}, \mathring{\boldsymbol{c}}_{\boldsymbol{i}}\rangle = -\frac{1}{\tau^2}\langle \tau\mathring{\boldsymbol{\alpha}}^{\boldsymbol{j}}, \mathring{\boldsymbol{c}}_{\boldsymbol{i}}\rangle \\
&= -\frac{1}{\tau}\delta_{\boldsymbol{i}}{}^{\boldsymbol{j}}.
\end{aligned}$$

The coefficients $\mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}}$. In this case, one readily finds that

$$\mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{0}}{}_{\boldsymbol{i}} = \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{0}}, \mathring{\nabla}_{\boldsymbol{0}}\mathring{\boldsymbol{e}}_{\boldsymbol{i}}\rangle = \langle \mathring{\boldsymbol{\omega}}^{\boldsymbol{0}}, \mathring{\nabla}_{\boldsymbol{0}}\left(\frac{1}{\tau}\mathring{\boldsymbol{c}}_{\boldsymbol{i}}\right)\rangle = -\frac{1}{\tau^2}\langle \mathbf{d}\tau, \mathring{\boldsymbol{c}}_{\boldsymbol{i}}\rangle = 0.$$

The coefficients $\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{0}}$. Observing that $[\mathring{\boldsymbol{e}}_{i}, \mathring{\boldsymbol{e}}_{\boldsymbol{0}}] = 0$ and recalling that in the absence of torsion one has that

$$[\mathring{\boldsymbol{e}}_{\boldsymbol{i}}, \mathring{\boldsymbol{e}}_{\boldsymbol{0}}] = \left(\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}} - \mathring{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{i}}\right)\boldsymbol{e}_{\boldsymbol{c}},$$

it follows from the previous results that

$$\mathring{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{0}}{}_{\boldsymbol{0}} = 0.$$

Remark 10. It follows that all the coefficients of the connection are smooth (C^∞) over $[\tau_\star, \infty) \times \mathcal{S}$.

Remark 11. For later use it is observed that the extrinsic curvature (Weingarten tensor) can be written in abstract index notation as

$$\mathring{\chi}_{ij} = -\frac{1}{\tau}\mathring{h}_{ij}. \tag{3.8}$$

Conformal fields

The next step is the computation of the components of the conformal fields appearing in the extended conformal Einstein field equations. To this end, we make use of the conformal Einstein constraints discussed in Section 2.7.

We make use of an adapted frame with $\boldsymbol{e}_0 = \boldsymbol{\partial}_\tau$ and make the identification $\Omega \mapsto \Theta$ in equations (2.84a)-(2.84j). Observe that one has that

$$\mathring{D}_i \Omega = 0.$$

The scalars Σ and s. By definition one has that

$$\mathring{\Sigma} \equiv \boldsymbol{n}(\Theta) = -\partial_\tau \Theta = \tau.$$

The minus sign arises from the fact that in our conventions $(\mathbf{d}\tau)^\sharp = -\boldsymbol{\partial}_\tau$. Using the latter in the conformal equation (2.84h) with $\lambda = 3$ one readily concludes

$$\mathring{s} = 1.$$

Components of the Schouten tensor. The constraint (2.84b) readily yields for $\Theta \geq 0$ that

$$\mathring{L}_i = 0.$$

The spatial components, $\mathring{L}_{ij}$, are computed using the constraint (2.84a). Observing that in our case $\mathring{D}_i \mathring{D}_j \Theta = 0$ one readily concludes that

$$\mathring{L}_{ij} = 0.$$

Thus, all the components of the Schouten tensor, except for its trace, vanish. This trace is proportional to the Ricci scalar of the metric (3.7).

Components of the rescaled Weyl tensor. The constraint (2.84i) offers an easy way of computing the magnetic part of the rescaled Weyl tensor. As $\mathring{D}_j \mathring{\chi}_{ki} = 0$ and we already know that $\mathring{L}_i = 0$, it follows then that $\mathring{d}_{ijk} = 0$ so that, in fact,

$$\mathring{d}^*{}_{ij} = 0.$$

To compute the electric part of the rescaled Weyl tensor we make use of the constraint equation (2.84j). This equation requires knowing the value of the Schouten tensor $\mathring{l}_{ij}$ of the metric $\mathring{\boldsymbol{h}}$. From the definition of the 3-dimensional Schouten tensor one readily finds that if $r[\mathring{\boldsymbol{\gamma}}] = -6$, then

$$\boldsymbol{Schouten}[\mathring{\boldsymbol{\gamma}}] = -\frac{1}{2}\mathring{\boldsymbol{\gamma}}.$$

Now, we have that $\mathring{\boldsymbol{h}} = \tau^2\mathring{\boldsymbol{\gamma}}$ so that $\mathring{\boldsymbol{h}}$ and $\mathring{\boldsymbol{\gamma}}$ are conformally related. However, the conformal factor does not depend on the spatial coordinates. It follows then, from the conformal transformation rule of the Schouten tensor that

$$\boldsymbol{Schouten}[\mathring{\boldsymbol{\gamma}}] = \boldsymbol{Schouten}[\mathring{\boldsymbol{h}}].$$

Hence, one has that

$$\mathring{l}_{ij} = -\frac{1}{2}\mathring{\gamma}_{ij} = -\frac{1}{2\tau^2}\mathring{h}_{ij}.$$

Now, a calculation using equation (3.8) reveals that

$$\mathring{l}_{ij} = -\mathring{\chi}\Big(\mathring{\chi}_{ij} - \frac{1}{4}\mathring{\chi}\mathring{h}_{ij}\Big) + \mathring{\chi}_{ki}\mathring{\chi}_j{}^k - \frac{1}{4}\mathring{\chi}_{kl}\mathring{\chi}^{kl}\mathring{h}_{ij}$$

so that

$$\mathring{d}_{ij} = 0.$$

Remark 12. In summary, one has that the metric (3.7) is conformally flat.

Ricci scalar. Finally, although it does not appear as an unknown in the extended conformal Einstein equations, it is of interest to compute the Ricci scalar of the metric. To do this we observe that from the definition of the Friedrich scalar one has that

$$\mathring{R}\Theta = 24\Big(s - \frac{1}{4}\mathring{\nabla}_c\mathring{\nabla}^c\Theta\Big).$$

A computation readily yields $\mathring{\nabla}_c\mathring{\nabla}^c\Theta = -2$ so that one concludes that

$$\mathring{R} = \frac{72}{1-\tau^2}.$$

That is, the Ricci scalar is singular at $\tau = 1$.

Remark 13. Although the Ricci scalar of the background solution is singular, this will not pose any difficulty in our subsequent analysis as the Ricci scalar does not appear as an unknown in the extended conformal Einstein field equations.

3.3 Evolution equations

In this section, we discuss the evolution system associated to the extended conformal Einstein equations (2.54) written in terms of a conformal Gaussian system. This evolution system is central in the discussion of the stability of the background spacetime. In addition, we also discuss the subsidiary evolution system satisfied by the zero-quantities associated to the field equations (2.53a)-(2.53d) and the supplementary zero-quantities (2.55a)-(2.55c). The subsidiary system is key in the analysis of the so-called *propagation of the constraints* which allows to establish the relation between a solution to the extended conformal Einstein equations (2.54) and the Einstein field equations (4.1).

3.3.1 The conformal Gaussian gauge

In order to obtain suitable evolution equations for the conformal fields, we make use of a *conformal Gaussian gauge*. More precisely, we assume that we are working on a region $\mathcal{U} \subset \mathcal{M}$ which can be covered by a congruence of non-intersecting conformal geodesics. Then Proposition 3 gives the conformal factor associated to the curves of the congruence

$$\Theta(\tau) = \frac{1}{2}\Big(1 - (\tau - \tau_\star)^2\Big), \tag{3.9}$$

by choosing

$$\Theta_\star = \frac{1}{2}, \qquad \dot{\Theta}_\star = 0, \qquad \ddot{\Theta}_\star = -\frac{1}{2},$$

for $\tau = \tau_\star$, $\tau_\star \in (0,1)$. This choice of initial data for the the conformal factor is associated to a congruence that leaves orthogonally a fiduciary initial hypersurface $\mathcal{S}_\star$ with $\tau = \tau_\star$ — notice, however, that the congruence of conformal geodesics is, in general, not hypersurface orthogonal.

Remark 14. Since the conformal factor Θ given by equation (3.9) does not depend on the initial data for the evolution equations it can be regarded as universal —i.e. valid not only for the background solution but also for perturbations thereof. Similarly, a consequence of Proposition 3, it follows that the components $d_{\boldsymbol{a}}$ of the the covector $\boldsymbol{d}$ are, in the same sense, universal.

Along the congruence of conformal geodesics, one considers a $\boldsymbol{g}$-orthogonal frame $\{\boldsymbol{e_0}\}$ which is Weyl-propagated and such that $\boldsymbol{\tau} = \boldsymbol{e_0}$. The Weyl connection $\hat{\nabla}_a$ associated to the congruence then satisfies

$$\hat{\nabla}_{\boldsymbol{\tau}} \boldsymbol{e_a} = 0, \qquad \hat{\boldsymbol{L}}(\tau, \cdot) = 0,$$

which is equivalent to

$$\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = 0, \qquad f_{\boldsymbol{0}} = 0, \qquad \hat{L}_{\boldsymbol{0a}} = 0, \tag{3.10}$$

—see e.g. [81], Section 13.4, page 366. By choosing the parameter, τ, of the conformal geodesics as time coordinate one gets the additional gauge condition

$$\boldsymbol{e_0} = \boldsymbol{\partial}_\tau, \qquad e_{\boldsymbol{0}}{}^\mu = \delta_0{}^\mu.$$

On $\mathcal{S}_\star$ we choose some local coordinates $\underline{x} = (x^\alpha)$. Assuming that each curve of the congruence intersects $\mathcal{S}_\star$ only once, one can extend the coordinates off the initial hypersurface by requiring them to be constant along the conformal geodesic which intersects $\mathcal{S}_\star$ at the point with coordinates $\underline{x}$. The coordinates $\overline{x} = (\tau, \underline{x})$ thus obtained are known as *conformal Gaussian coordinates.*

3.3.2 The main evolution system

The required evolution equations for the frame components, connection coefficients and components of the Schouten tensor are obtained from the conditions

$$\hat{\Sigma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \boldsymbol{e_c} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = 0, \qquad \hat{\Delta}_{\boldsymbol{0bc}} = 0. \tag{3.11}$$

In particular, the evolution equation for components of the covector f_a defining the Weyl connection is given by

$$\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{c0b}} = 0.$$

In the following, we analyse each of these equations in more detail.

Evolution equations for the components of the frame

Now, starting from equation (2.53a)

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e}_{\boldsymbol{c}} \equiv [\boldsymbol{e}_{\boldsymbol{a}}, \boldsymbol{e}_{\boldsymbol{b}}] - (\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})\boldsymbol{e}_{\boldsymbol{c}}$$

and writing $\boldsymbol{e}_{\boldsymbol{a}} = e_{\boldsymbol{a}}{}^{\mu}\boldsymbol{\partial}_{\mu}$, it follows that the condition $\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\boldsymbol{e}_{\boldsymbol{c}} = 0$ implies

$$(\partial_{\boldsymbol{a}} e_{\boldsymbol{b}}{}^{\nu} - \partial_{\boldsymbol{b}} e_{\boldsymbol{a}}{}^{\nu}) = (\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}})e_{\boldsymbol{c}}{}^{\nu}, \qquad \partial_{\boldsymbol{a}} \equiv e_{\boldsymbol{a}}{}^{\mu}\partial_{\mu}.$$

Setting $\boldsymbol{a} = \boldsymbol{0}$ it follows that the evolution equation for the components of the frame takes the form

$$\partial_{\boldsymbol{0}} e_{\boldsymbol{b}}{}^{\nu} = -\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}} e_{\boldsymbol{c}}{}^{\nu}. \tag{3.12}$$

Evolution equations for the components of the connection

In order to obtain the evolution equation for the components of the frame not determined by the gauge conditions one considers the condition $\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = 0$.

Now, since

$$\hat{R}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = \partial_{\boldsymbol{0}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \partial_{\boldsymbol{b}}(\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + (\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}}) + \hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}} - \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}),$$

then using the gauge condition $\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} = 0$ one has that

$$\hat{R}^{\boldsymbol{c}}_{\boldsymbol{d0b}} = \boldsymbol{e}_{\boldsymbol{0}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}.$$

In addition, observing that

$$S_{\boldsymbol{d}[\boldsymbol{0}}{}^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{b}]\boldsymbol{e}} = \delta_{\boldsymbol{d}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{b0}} + \delta_{\boldsymbol{0}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{bd}} - g_{\boldsymbol{d0}}g^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{be}} - \delta_{\boldsymbol{d}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{0b}} - \delta_{\boldsymbol{b}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{b0}} + g_{\boldsymbol{db}}g^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{0e}},$$

together with the gauge condition $\hat{L}_{\boldsymbol{0a}} = 0$, it follows that

$$\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = \Theta d^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} + 2\delta_{\boldsymbol{d}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{b0}} + 2\delta_{\boldsymbol{0}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{bd}} - 2\eta_{\boldsymbol{d0}}\eta^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{be}},$$

where it has been used that $g_{\boldsymbol{d0}}g^{\boldsymbol{ce}} = \eta_{\boldsymbol{d0}}\eta^{\boldsymbol{ce}}$. It follows that the evolution equation for the coefficients of the connection not determined by the gauge is given by

$$\partial_{\boldsymbol{0}}(\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}} = 2\eta_{\boldsymbol{d0}}\eta^{\boldsymbol{ce}}\hat{L}_{\boldsymbol{be}} - 2\delta_{\boldsymbol{d}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{b0}} - 2\delta_{\boldsymbol{0}}{}^{\boldsymbol{c}}\hat{L}_{\boldsymbol{bd}} - \Theta d^{\boldsymbol{c}}{}_{\boldsymbol{d0b}}.$$

The above expression can be written in terms of the Levi-Civita connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ and the 1-form $f_{\boldsymbol{a}}$ through the relation

$$\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} + S_{\boldsymbol{ab}}{}^{\boldsymbol{cd}} f_{\boldsymbol{d}}.$$

In particular, since

$$f_{\boldsymbol{a}} = \frac{1}{4}\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{b}},$$

it follows from the gauge condition $f_{\boldsymbol{0}} = 0$ and $\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{c0b}} = 0$ that

$$\partial_{\boldsymbol{0}} f_{\boldsymbol{i}} = -f_{\boldsymbol{j}}\hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}} + \hat{L}_{\boldsymbol{i0}}. \tag{3.13}$$

Evolution equations for the components of the Schouten tensor

The evolution equations for the components of the Schouten tensor not determined by the gauge are obtained from the condition $\hat{\Delta}_{\boldsymbol{0db}} = 0$. It follows then that

$$\nabla_{\boldsymbol{0}}\hat{L}_{\boldsymbol{db}} - \nabla_{\boldsymbol{d}}\hat{L}_{\boldsymbol{0b}} - d_{\boldsymbol{a}} d^{\boldsymbol{a}}{}_{\boldsymbol{b0d}} = 0.$$

However, in the conformal Gaussian gauge one has that $\hat{L}_{\boldsymbol{0b}} = 0$ so that the evolution equation for the components of the Schouten tensor can be simplified to

$$\partial_{\boldsymbol{0}}\hat{L}_{\boldsymbol{db}} = \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}\hat{L}_{\boldsymbol{cb}} + \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}\hat{L}_{\boldsymbol{dc}} + d_{\boldsymbol{a}} d^{\boldsymbol{a}}{}_{\boldsymbol{b0d}} = 0,$$

as $\hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} = 0$.

Evolution equations for the components of the rescaled Weyl tensor

The evolution equations for the components of the Weyl tensor are extracted from the decomposition of the zero-quantity $\hat{\Lambda}_{\boldsymbol{bcd}}$. As this zero-quantity contains a contracted derivative, the decomposition is more involved than for the other zero-quantities. As in the case of the conformal constraint equations, this analysis is best done using the decomposition of the rescaled Weyl tensor in its electric and magnetic parts with respect to the tangent to the congruence of conformal geodesics on which our gauge is based.

In the following, let ${h_a}^b$ denote the projector to the hyperplanes orthogonal to the tangent vector field τ^a to the congruence of conformal geodesics. One has that

$$ {h_a}^b = {\delta_a}^b - \tau_a\tau^b, $$

so that

$$ \begin{aligned} \hat{\Lambda}_{\boldsymbol{bcd}} &= \nabla^{\boldsymbol{a}}({\delta_{\boldsymbol{a}}}^{\boldsymbol{f}} d_{\boldsymbol{fbcd}}) = {\delta_{\boldsymbol{a}}}^{\boldsymbol{f}} \nabla^{\boldsymbol{a}} d_{\boldsymbol{fbcd}} \\ &= \tau^{\boldsymbol{f}}\tau_{\boldsymbol{a}}\nabla^{\boldsymbol{a}} d_{\boldsymbol{fbcd}} + {h_{\boldsymbol{a}}}^{\boldsymbol{f}}\nabla^{\boldsymbol{a}} d_{\boldsymbol{fbcd}} \\ &= \tau^{\boldsymbol{f}}\mathcal{D} d_{\boldsymbol{fbcd}} + \mathcal{D}^{\boldsymbol{f}} d_{\boldsymbol{fbcd}}, \end{aligned} $$

where $\mathcal{D}_a \equiv {h_a}^b\nabla_b$ and $\mathcal{D} \equiv \tau^a\nabla_a$ denote, respectively, the Sen and Fermi covariant derivatives associated to the congruence. Now, observing that the acceleration and Weingarten tensor of the congruence is given, respectively by

$$ \begin{aligned} a_a &\equiv \tau^b\nabla_b\tau_a = \mathcal{D}\tau_a, \\ \chi_{ab} &\equiv {h_a}^c\nabla_c\tau_b = \mathcal{D}_a\tau_b, \end{aligned} $$

it follows that

$$ \begin{aligned} \hat{\Lambda}_{\boldsymbol{bcd}}\tau^{\boldsymbol{c}} = \hat{\Lambda}_{\boldsymbol{b0d}} &= \tau^{\boldsymbol{c}}\mathcal{D}(\tau^{\boldsymbol{f}} d_{\boldsymbol{fbcd}}) + \tau^{\boldsymbol{c}}\mathcal{D}^{\boldsymbol{f}} d_{\boldsymbol{fbcd}} - a^{\boldsymbol{f}}\tau^{\boldsymbol{c}} d_{\boldsymbol{fbcd}} \\ &= \mathcal{D}(\tau^{\boldsymbol{f}}\tau^{\boldsymbol{c}} d_{\boldsymbol{fbcd}}) + \mathcal{D}^{\boldsymbol{f}} d_{\boldsymbol{fb0d}} - a^{\boldsymbol{f}} d_{\boldsymbol{fb0d}} - a^{\boldsymbol{c}} d_{\boldsymbol{0bcd}} - \chi^{\boldsymbol{fc}} d_{\boldsymbol{fbcd}}, \end{aligned} $$

so that

$$ \hat{\Lambda}_{\boldsymbol{b0d}} = \mathcal{D} d_{\boldsymbol{0b0d}} + \mathcal{D}^{\boldsymbol{f}} d_{\boldsymbol{fb0d}} - a^{\boldsymbol{f}} d_{\boldsymbol{fb0d}} - a^{\boldsymbol{c}} d_{\boldsymbol{0bcd}} - \chi^{\boldsymbol{fc}} d_{\boldsymbol{fbcd}}. $$

To further simplify we make use of the decomposition

$$ d_{\boldsymbol{abcd}} = 2(l_{\boldsymbol{b}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{a}} - l_{\boldsymbol{a}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{b}}) - 2(\tau_{[\boldsymbol{c}} {d^*}_{\boldsymbol{d}]\boldsymbol{e}} {\epsilon^{\boldsymbol{e}}}_{\boldsymbol{ab}} + \tau_{[\boldsymbol{a}} {d^*}_{\boldsymbol{b}]\boldsymbol{e}} {\epsilon^{\boldsymbol{e}}}_{\boldsymbol{cd}}), $$

of the rescaled Weyl tensor in terms of its electric part $d_{\boldsymbol{ab}}$ and magnetic part $d^*_{\boldsymbol{ab}}$ with respect to the vector field τ^a where $l_{ab} = h_{ab} - \tau_a\tau_b$ to obtain

$$ \begin{aligned} \hat{\Lambda}_{\boldsymbol{b0d}} =& \mathcal{D} d_{\boldsymbol{bd}} + \mathcal{D}^{\boldsymbol{f}} d_{\boldsymbol{fbd}} - a^{\boldsymbol{f}} d_{\boldsymbol{fbd}} - a^{\boldsymbol{c}} d_{\boldsymbol{bcd}} - 2\chi^{\boldsymbol{fc}}(l_{\boldsymbol{b}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{f}} - l_{\boldsymbol{f}[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{b}}) \\ &+ 2\chi^{\boldsymbol{fc}}(\tau_{[\boldsymbol{c}} {d^*}_{\boldsymbol{d}]\boldsymbol{e}} {\epsilon^{\boldsymbol{e}}}_{\boldsymbol{fb}} + \tau_{[\boldsymbol{f}} {d^*}_{\boldsymbol{b}]\boldsymbol{e}} {\epsilon^{\boldsymbol{e}}}_{\boldsymbol{cd}}). \end{aligned} $$

To finally extract the required evolution equation we consider $\hat{\Lambda}_{(\boldsymbol{b}|\boldsymbol{0}|\boldsymbol{d})}$. Observing that all the involved tensors are spatial one obtains, after some simplification, that

$$\hat{\Lambda}_{(\boldsymbol{i}|\boldsymbol{0}|\boldsymbol{j})} = \partial_{\boldsymbol{0}} d_{\boldsymbol{ij}} + \epsilon^{\boldsymbol{kl}}{}_{(\boldsymbol{i}} D_{|\boldsymbol{l}|} d^*{}_{\boldsymbol{j})\boldsymbol{k}} - 2a_{\boldsymbol{l}} \epsilon^{\boldsymbol{kl}}{}_{(\boldsymbol{i}} d^*{}_{\boldsymbol{j})\boldsymbol{k}} + \chi d_{\boldsymbol{ij}} - 2\chi^{\boldsymbol{k}}{}_{(\boldsymbol{i}} d_{\boldsymbol{j})\boldsymbol{k}} = 0. \tag{3.14}$$

To complete the system of evolution equations for the components of the Weyl tensor one carries out a completely analogous calculation with the zero-quantity

$$\hat{\Lambda}^*_{\boldsymbol{bcd}} \equiv \nabla^{\boldsymbol{a}} d^*{}_{\boldsymbol{abcd}}$$

and the decomposition

$$d^*_{\boldsymbol{abcd}} = 2(l_{\boldsymbol{b}[\boldsymbol{c}} d^*{}_{\boldsymbol{d}]\boldsymbol{a}} - l_{\boldsymbol{f}[\boldsymbol{c}} d^*{}_{\boldsymbol{d}]\boldsymbol{b}}) + 2(\tau_{[\boldsymbol{c}} d_{\boldsymbol{d}]\boldsymbol{e}} \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{ab}} + \tau_{[\boldsymbol{a}} d_{\boldsymbol{b}]\boldsymbol{e}} \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{cd}}),$$

where the Hodge dual of the rescaled Weyl tensor is defined as

$$d^*{}_{abcd} \equiv \frac{1}{2} \epsilon_{ab}{}^{ef} d_{cdef}.$$

More precisely, the decomposition

$$\hat{\Lambda}^*{}_{\boldsymbol{bcd}} = \tau^{\boldsymbol{a}} \mathcal{D} d^*{}_{\boldsymbol{abcd}} + \mathcal{D}^{\boldsymbol{a}} d^*{}_{\boldsymbol{abcd}},$$

leads, after a lengthy computation, to the evolution equation

$$\hat{\Lambda}^*{}_{(\boldsymbol{i}|\boldsymbol{0}|\boldsymbol{j})} = \partial_{\boldsymbol{0}} d^*{}_{\boldsymbol{ij}} - \epsilon^{\boldsymbol{k}}{}_{\boldsymbol{l}(\boldsymbol{i}} D^{\boldsymbol{l}} d_{\boldsymbol{j})\boldsymbol{k}} - 2a^{\boldsymbol{l}} \epsilon_{\boldsymbol{l}(\boldsymbol{i}}{}^{\boldsymbol{k}} d_{\boldsymbol{j})\boldsymbol{k}} + \chi d^*{}_{\boldsymbol{ij}} - 2\chi^{\boldsymbol{k}}{}_{(\boldsymbol{i}} d^*{}_{\boldsymbol{j})\boldsymbol{k}} = 0, \tag{3.15}$$

in which all the fields are spatial.

Remark 15. The zero-quantities $\hat{\Lambda}_{\boldsymbol{bcd}}$ and $\hat{\Lambda}^*{}_{\boldsymbol{bcd}}$ are not independent. In fact, $\hat{\Lambda}_{\boldsymbol{bcd}} = 0$ if and only if $\hat{\Lambda}^*{}_{\boldsymbol{bcd}} = 0$.

Remark 16. Equations (3.14) and (3.15) imply a symmetric hyperbolic evolution system for the (ten) independent components of the fields $E_{\boldsymbol{ab}}$ and $B_{\boldsymbol{ab}}$ —see e.g. [2] for explicit expressions of the associated matrices.

3.3.3 The subsidiary evolution system

The analysis of the relation between the solutions to the evolution equations and actual solutions to the full conformal Einstein field equations, the so-called *propagation of the constraints*, requires the construction of a system of *subsidiary evolution equations for the zero-quantities* associated to the conformal equations (2.53a)-(2.53d) and the gauge conditions (2.55a)-(2.55c). For the standard argument of the propagation of the constraints to follow through, the subsidiary system is required to be homogeneous in the zero-quantities. If this is the case, then it follows from the uniqueness of solutions to symmetric hyperbolic systems that if the zero-quantities vanish initially, then they will vanish for all later times as the vanishing (zero) solution is always a solution of a homogeneous evolution equation.

General remarks

The basic assumption in the construction of the subsidiary evolution system is that the evolution equations associated to the extended conformal field equations are satisfied. Hence, we assume that

$$\hat{\Sigma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = 0, \qquad \hat{\Delta}_{\boldsymbol{0bc}} = 0,$$

together with

$$\hat{\Lambda}_{(\boldsymbol{i}|\boldsymbol{0}|\boldsymbol{j})} = 0, \qquad \hat{\Lambda}^{*}{}_{(\boldsymbol{i}|\boldsymbol{0}|\boldsymbol{j})} = 0.$$

These evolution equations have been constructed using the gauge conditions

$$f_{\boldsymbol{0}} = 0, \qquad \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = 0, \qquad \hat{L}_{\boldsymbol{0b}} = 0.$$

These gauge conditions will also be used in the construction of the subsidiary evolution system. Accordingly, the construction requires the evolution equations for the additional zero-quantities δ_a, γ_{ab} and ς_{ab} which are associated to the gauge. In our gauge $d_0 = 0$ so that

$$\delta_{\boldsymbol{0}} = 0.$$

Since $\hat{L}_{\boldsymbol{0b}} = 0$, by virtue of the definition (2.8) and the evolution equation for the covector β_a, namely,

$$\hat{\nabla}_{\boldsymbol{0}}\beta_{\boldsymbol{a}} + \beta_{\boldsymbol{0}}\beta_{\boldsymbol{a}} - \frac{1}{2}\eta_{\boldsymbol{0a}}(\beta_{\boldsymbol{e}}\beta^{\boldsymbol{e}} - 2\lambda\Theta^{-2}) = 0,$$

it follows that

$$\gamma_{\boldsymbol{0b}} = \hat{L}_{\boldsymbol{0b}} - \hat{\nabla}_{\boldsymbol{0}}\beta_{\boldsymbol{b}} - \frac{1}{2}S_{\boldsymbol{0b}}{}^{\boldsymbol{ef}}\beta_{\boldsymbol{e}}\beta_{\boldsymbol{f}} + \lambda\Theta^{-2}\eta_{\boldsymbol{0b}} = 0.$$

As a result of the Θ^{-2} in the last term of this equation, it can only be used away from the conformal boundary —this is, however, not a problem in our analysis as the propagation of the constraints only need to be considered in the regions where $\Theta \neq 0$. Moreover, by virtue of the gauge conditions (3.10) and the evolution equation (3.13), we have

$$\varsigma_{\boldsymbol{0b}} = -\hat{L}_{\boldsymbol{b0}} - \hat{\nabla}_{\boldsymbol{0}}f_{\boldsymbol{b}} + \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}f_{\boldsymbol{e}} = 0.$$

The subsidiary equation for the torsion

To obtain the subsidiary equation for the no-torsion condition we consider the totally antisymmetric covariant derivative $\hat{\nabla}_{[\boldsymbol{a}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]}$ and observe that

$$3\hat{\nabla}_{[\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} = \hat{\nabla}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{c}}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}} - \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}}. \tag{3.16}$$

On the other hand, from the first Bianchi identity

$$\hat{R}^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} + \hat{\nabla}_{[\boldsymbol{a}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} + \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\Sigma}_{\boldsymbol{c}]}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}} = 0,$$

and the definition of $\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{cab}}$ one obtains

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}]} = -\hat{\Xi}^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} - \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\Sigma}_{\boldsymbol{c}]}{}^{\boldsymbol{d}}{}_{\boldsymbol{e}}, \tag{3.17}$$

where it has been used that, by construction, $\hat{\rho}^{\boldsymbol{d}}{}_{[\boldsymbol{cab}]} = 0$. The desired evolution equation is obtained combining equations (3.16) and (3.17) to yield

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} = -\frac{1}{3}\hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} - \frac{1}{3}\hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} - \hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{0bc}}. \tag{3.18}$$

This evolution equation is homogeneous in the various zero-quantities.

The subsidiary equation for the Ricci identity

To obtain a subsidiary equation for the Ricci identity, we consider the totally symmetrised covariant derivative $\hat{\nabla}_{[\boldsymbol{a}}\hat{\Xi}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]}$ and observe that

$$3\hat{\nabla}_{[\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \hat{\nabla}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ecf}} - \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{efb}}. \tag{3.19}$$

Using the second Bianchi identity

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{R}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} + \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}\hat{R}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{c}]\boldsymbol{f}} = 0$$

and the definition (2.53c) it follows that

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\Xi}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = -\hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}\hat{R}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{c}]\boldsymbol{f}} - \hat{\nabla}_{[\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]}. \tag{3.20}$$

The first term on the right-hand side is already of the required form. The second one needs to be analysed in more detail. For this, one makes use of the definition (2.52b) so that

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \hat{\nabla}_{[\boldsymbol{a}}C^{\boldsymbol{d}}_{|\boldsymbol{e}|\boldsymbol{bc}]} + 2S_{\boldsymbol{e}[\boldsymbol{b}}{}^{\boldsymbol{df}}\hat{\nabla}_{\boldsymbol{a}}\hat{L}_{\boldsymbol{c}]\boldsymbol{f}}.$$

To further expand this expression we consider the combination $\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\hat{\nabla}_{\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}}$. A direct computation shows that

$$\hat{\nabla}_{[\boldsymbol{a}}C^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \nabla_{[\boldsymbol{a}}C^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} + \delta_{[\boldsymbol{a}}{}^{\boldsymbol{d}}f_{|\boldsymbol{f}}C^{\boldsymbol{f}}{}_{\boldsymbol{e}|\boldsymbol{bc}]} + \eta_{\boldsymbol{e}[\boldsymbol{a}}f^{\boldsymbol{f}}C^{\boldsymbol{d}}{}_{|\boldsymbol{f}|\boldsymbol{bc}]}.$$

Moreover, one has

$$\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{\boldsymbol{a}}C^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} = -\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\nabla_{\boldsymbol{a}}C^{\boldsymbol{a}}{}_{\boldsymbol{fgh}}.$$

Thus, by using that $C^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = \Theta d^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ and the definition (2.53d) it follows that

$$\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\hat{\nabla}_{\boldsymbol{a}}C^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} = \Theta\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\hat{\Lambda}_{\boldsymbol{fgh}} + 2\nabla^{\boldsymbol{g}}\Theta d^{*\boldsymbol{d}}{}_{\boldsymbol{efg}} + 2\Theta f^{\boldsymbol{g}}d^{*}{}_{\boldsymbol{gef}}{}^{\boldsymbol{d}} + 2\Theta f^{\boldsymbol{g}}d^{*\boldsymbol{d}}{}_{\boldsymbol{gfe}}.$$

A similar computation using the definition (2.53c) yields

$$2\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}S_{\boldsymbol{eb}}{}^{\boldsymbol{dg}}\hat{\Delta}_{\boldsymbol{acg}} = 2\Theta\beta_{\boldsymbol{g}}d^{*\boldsymbol{g}}{}_{\boldsymbol{ef}}{}^{\boldsymbol{d}} - 2\Theta\beta_{\boldsymbol{g}}d^{*\boldsymbol{gd}}{}_{\boldsymbol{fe}}.$$

Thus, using the symmetries of $d^{*}{}_{\boldsymbol{cdab}}$ and the definition (2.55a) one concludes that

$$\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\hat{\nabla}_{\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} = \Theta\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\hat{\Lambda}_{\boldsymbol{fgh}} - 2\Theta\delta^{\boldsymbol{g}}d^{*\boldsymbol{d}}{}_{\boldsymbol{efg}} + \epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}S_{\boldsymbol{eb}}{}^{\boldsymbol{dg}}\hat{\Delta}_{\boldsymbol{acg}}.$$

Alternatively, using the properties of the generalised Hodge duals we can write

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\rho}^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \frac{1}{6}\Theta\epsilon^{\boldsymbol{f}}{}_{\boldsymbol{abc}}\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\hat{\Lambda}_{\boldsymbol{fgh}} - \frac{1}{3}\Theta\epsilon^{\boldsymbol{f}}{}_{\boldsymbol{abc}}\delta^{\boldsymbol{g}}d^{*\boldsymbol{d}}{}_{\boldsymbol{efg}} - S_{\boldsymbol{e}[\boldsymbol{b}}{}^{\boldsymbol{dg}}\hat{\Delta}_{\boldsymbol{ac}]\boldsymbol{g}}.$$

Combining the expressions, we obtain the following evolution equation

$$\begin{aligned}\hat{\nabla}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} =& \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ecf}} + \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{efb}} - \hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{c}}\hat{R}^{\boldsymbol{d}}{}_{\boldsymbol{e0f}} - \frac{1}{2}\Theta\epsilon^{\boldsymbol{f}}{}_{\boldsymbol{0bc}}\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\hat{\Lambda}_{\boldsymbol{fgh}} \\ &+ \epsilon^{\boldsymbol{f}}{}_{\boldsymbol{0bc}}\delta^{\boldsymbol{g}}d^{*\boldsymbol{d}}{}_{\boldsymbol{efg}} + 3S_{\boldsymbol{e0}}{}^{\boldsymbol{dg}}\hat{\Delta}_{\boldsymbol{cbg}},\end{aligned} \tag{3.21}$$

which is homogeneous in the zero-quantities.

Subsidiary equation for the Cotton equation

Now, to compute the subsidiary equation for the Cotton equation we consider $\hat{\nabla}_{[\boldsymbol{a}}\hat{\Delta}_{\boldsymbol{bc}]\boldsymbol{d}}$. On the one hand, a direct computation yields

$$3\hat{\nabla}_{[\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{bc}]\boldsymbol{d}} = \hat{\nabla}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{bcd}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ced}} - \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ebd}}.$$

On the other hand, using the definition of $\hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{cab}}$ and the symmetries of $\hat{\rho}^{\boldsymbol{e}}{}_{\boldsymbol{cab}}$ one obtains

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\Delta}_{\boldsymbol{bc}]\boldsymbol{d}} = -\hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{cab}]}\hat{L}_{\boldsymbol{ed}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{d}[\boldsymbol{ab}}\hat{L}_{\boldsymbol{c}]\boldsymbol{e}} - \hat{\rho}^{\boldsymbol{e}}{}_{\boldsymbol{d}[\boldsymbol{ab}}\hat{L}_{\boldsymbol{c}]\boldsymbol{e}} + \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{|\boldsymbol{e}|}\hat{L}_{\boldsymbol{c}]\boldsymbol{d}} - \hat{\nabla}_{[\boldsymbol{a}}d_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - d_{\boldsymbol{e}}\hat{\nabla}_{[\boldsymbol{a}}d^{\boldsymbol{e}}{}_{|\boldsymbol{d}|\boldsymbol{bc}]}.$$

Using the definition of δ_{a} and γ_{ab} one finds that

$$\hat{\nabla}_{[\boldsymbol{a}}d_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} = -\Theta\delta_{[\boldsymbol{a}}\beta_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - \Theta\gamma_{[\boldsymbol{a}|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - \Theta f_{[\boldsymbol{a}}\beta_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} + \Theta\hat{L}_{[\boldsymbol{a}|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]}.$$

Finally, a calculation shows that $\epsilon_{\boldsymbol{f}}{}^{\boldsymbol{abc}}\nabla_{\boldsymbol{a}}d^{\boldsymbol{e}}{}_{\boldsymbol{dbc}} = \epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\nabla_{\boldsymbol{a}}d^{\boldsymbol{e}}{}_{\boldsymbol{fgh}}$, so that using

$$\hat{\nabla}_{[\boldsymbol{a}}C^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} = \nabla_{[\boldsymbol{a}}C^{\boldsymbol{d}}{}_{|\boldsymbol{e}|\boldsymbol{bc}]} + \delta_{[\boldsymbol{a}}{}^{\boldsymbol{d}}f_{|\boldsymbol{f}}C^{\boldsymbol{f}}{}_{\boldsymbol{e}|\boldsymbol{bc}]} + \eta_{\boldsymbol{e}[\boldsymbol{a}}f^{\boldsymbol{f}}C^{\boldsymbol{d}}{}_{|\boldsymbol{f}|\boldsymbol{bc}]},$$

and the properties of the generalised duals we find that

$$\hat{\nabla}_{[\boldsymbol{a}}d^{\boldsymbol{e}}{}_{|\boldsymbol{d}|\boldsymbol{bc}]} = \frac{1}{6}\epsilon_{\boldsymbol{abc}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\hat{\Lambda}_{\boldsymbol{fgh}} + \delta_{[\boldsymbol{a}}{}^{\boldsymbol{e}}f_{|\boldsymbol{f}}d^{\boldsymbol{f}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} + \eta_{\boldsymbol{d}[\boldsymbol{a}}f^{\boldsymbol{f}}d^{\boldsymbol{e}}{}_{|\boldsymbol{f}|\boldsymbol{bc}]}.$$

Combining the above expressions and using the properties of the decomposition of $\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ we obtain the expression

$$\hat{\nabla}_{[\boldsymbol{a}}\hat{\Delta}_{\boldsymbol{bc}]\boldsymbol{d}} = -\hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{cab}]}\hat{L}_{\boldsymbol{ed}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{d}[\boldsymbol{ab}}\hat{L}_{\boldsymbol{c}]\boldsymbol{e}} + \hat{\Sigma}_{[\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{b}}\hat{\nabla}_{|\boldsymbol{e}|}\hat{L}_{\boldsymbol{c}]\boldsymbol{d}} + \Theta\delta_{[\boldsymbol{a}}\beta_{|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} + \Theta\gamma_{[\boldsymbol{a}|\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d}|\boldsymbol{bc}]} - \frac{1}{6}\epsilon_{\boldsymbol{abc}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\hat{\Lambda}_{\boldsymbol{fgh}}\beta_{\boldsymbol{e}}$$

and, eventually, the evolution equation

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{bcd}} = \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ced}} + \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ebd}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{0bc}}\hat{L}_{\boldsymbol{ed}} + \delta_{\boldsymbol{b}}d_{\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{dc0}} + \delta_{\boldsymbol{c}}d_{\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d0b}}$$
$$+ \Theta\gamma_{\boldsymbol{be}}d^{\boldsymbol{e}}{}_{\boldsymbol{dc0}} + \Theta\gamma_{\boldsymbol{ce}}d^{\boldsymbol{e}}{}_{\boldsymbol{d0b}} - \frac{1}{2}\epsilon_{\boldsymbol{0bc}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\hat{\Lambda}_{\boldsymbol{fgh}}\beta_{\boldsymbol{e}},$$

which is homogeneous in zero quantities as required.

Subsidiary equations for the Bianchi identity

Finally, we are left to show the propagation of the physical Bianchi identity. In view of the contracted derivative appearing in this equation, the construction of suitable subsidiary equations is more involved.

Since $h_a{}^b = \delta_a{}^b + \tau_a\tau^b$, it follows then that

$$\hat{\Lambda}_{abc} = \delta_a{}^d\hat{\Lambda}_{dbc} = (h_a{}^d - \tau_a\tau^d)\hat{\Lambda}_{dbc} = h_a{}^d\hat{\Lambda}_{dbc} - \tau_a\tau^d\hat{\Lambda}_{dbc}. \tag{3.22}$$

Now, let

$$\Omega_{abc} \equiv h_a{}^d\hat{\Lambda}_{dbc}, \qquad \Omega_{bc} \equiv \tau^d\hat{\Lambda}_{dbc}.$$

By construction, the tensor Ω_{bc} is antisymmetric, hence it admits a decomposition in *electric* and *magnetic parts*. That is, one can write

$$\Omega_{bc} = \Omega_{[bc]} = \Omega^*_e\epsilon^e{}_{bc} - 2\Omega_{[b}\tau_{c]},$$

where

$$\Omega_a \equiv \Omega_{cb}\tau^b h_a{}^c, \qquad \Omega^*_a \equiv \Omega^*_{cb}\tau^b h_a{}^c.$$

Furthermore, one also has that

$$\Omega_{dbc} = \Omega_{d[bc]} = H^*_{de}\epsilon^e{}_{dc} - 2H_{d[b}\tau_{c]},$$

where

$$H_{da} \equiv \Omega_{dcb}\tau^b h_a{}^c, \qquad H^*{}_{da} \equiv \Omega^*_{dcb}\tau^b h_a{}^c.$$

Substituting the above expressions for Ω_{bc} and Ω_{dbc} into equation (3.22) it follows then that

$$\hat{\Lambda}_{abc} = h_a{}^d(H^*_{de}\epsilon^e{}_{dc} - 2H_{d[b}n_{c]}) - n_a(\Omega^*_e\epsilon^e{}_{bc} - 2\Omega_{[b}n_{c]}). \tag{3.23}$$

Crucially, one can verify that if the evolution equations (3.14) and (3.15) for the electric and magnetic part of the rescaled Weyl tensor are satisfied then

$$H_{da} = 0, \qquad H^{*}{}_{da} = 0.$$

If the above holds, then equation (3.23) reduces to

$$\hat{\Lambda}_{abc} = n_a(2\Omega_{[b}n_{c]} - \Omega^{*}{}_{e}\epsilon^{e}{}_{bc}) = n_a\Omega_{bc}.$$

Remark 17. The tensors Ω_a and Ω^{*}_{a} encode, respectively, the *Gauss constraints* for the electric and magnetic parts of the Weyl tensor —that is, the equations

$$\mathcal{D}^a d_{ab} = 0, \qquad \mathcal{D}^a d^{*}_{ab} = 0.$$

To conclude the computation, it remains to compute $\nabla^a\hat{\Lambda}_{abc}$. A direct calculation gives

$$\nabla^a\hat{\Lambda}_{abc} = \nabla^a\tau_a\Omega_{bc} + \tau_a\nabla^a\Omega_{bc} = \nabla^a\tau_a\Omega_{bc} + \partial_\tau\Omega_{bc}. \tag{3.24}$$

An alternative computation of $\nabla^a\hat{\Lambda}_{abc}$ using the commutator of the covariant derivative ∇ gives

$$2\nabla^b\hat{\Lambda}_{bcd} = 2\nabla^{[b}\nabla^{a]}d_{abcd} = 2R^{e}{}_{[c}{}^{ba}d_{d]eab} - 2R^{e}{}_{a}{}^{ba}d_{ebcd} + \Sigma_b{}^{e}{}_{a}\nabla_e d^{ab}{}_{cd}.$$

Observing that $\hat{\Sigma}_a{}^c{}_b = \Sigma_a{}^c{}_b$ as $\hat{\nabla} - \nabla = S(f)$, it follows that the equation

$$\hat{R}^a{}_{bcd} - R^a{}_{bcd} = 2(\delta^a{}_{[c}\hat{\nabla}_{d]}\hat{f}_b + \hat{\nabla}_{[c}\hat{f}^a\hat{g}_{d]b} - \delta^a{}_b\hat{\nabla}_{[c}\hat{f}_{d]} - \delta^a{}_{[c}\hat{f}_{d]}\hat{f}_b + \hat{g}_{b[c}\hat{f}_{d]}\hat{f}^a + \delta^a{}_{[c}\hat{g}_{d]b}\hat{f}_e\hat{f}^e)$$

together with the definitions of the zero quantities $\hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ and $\varsigma_{\boldsymbol{ab}}$ and the symmetries of $d_{\boldsymbol{abcd}}$ so that after projecting the equations with respect to the frame one obtains

$$\nabla^{\boldsymbol{b}}\hat{\Lambda}_{\boldsymbol{bcd}} = \hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{c}}{}^{\boldsymbol{ba}}d_{\boldsymbol{d}]\boldsymbol{eab}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{a}}{}^{\boldsymbol{ba}}d_{\boldsymbol{ebcd}} + \frac{1}{2}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{a}}\nabla_{\boldsymbol{e}}d^{\boldsymbol{ab}}{}_{\boldsymbol{cd}} + \varsigma^{\boldsymbol{ab}}d_{\boldsymbol{abcd}}, \tag{3.25}$$

which is homogeneous in zero quantities. Hence, combining equations (3.24) and (3.25), we obtain the following equation for the components of Ω_{ab}:

$$\partial_{\boldsymbol{0}}\Omega_{\boldsymbol{bc}} = \hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{b}}{}^{\boldsymbol{af}}d_{\boldsymbol{c}]\boldsymbol{efa}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{f}}{}^{\boldsymbol{af}}d_{\boldsymbol{eabc}} + \frac{1}{2}\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{f}}\nabla_{\boldsymbol{e}}d^{\boldsymbol{fa}}{}_{\boldsymbol{bc}} + \varsigma^{\boldsymbol{fa}}d_{\boldsymbol{fabc}} - \chi\Omega_{\boldsymbol{bc}}.$$

Subsidiary equations for the gauge conditions

To conclude our discussion of the subsidiary equations, we are left with the task of providing evolution equations for the zero-quantities associated to the gauge. In order to do so we expand $\hat{\nabla}_{[\mathbf{0}}\delta_{\mathbf{b}]}$, $\hat{\nabla}_{[\mathbf{0}}\gamma_{\mathbf{b}]\mathbf{c}}$ and $\hat{\nabla}_{[\mathbf{0}}\varsigma_{\mathbf{bc}]}$ to get

$$
\begin{aligned}
2\hat{\nabla}_{[\mathbf{0}}\delta_{\mathbf{b}]} &= \hat{\nabla}_{\mathbf{0}}\delta_{\mathbf{b}} + \hat{\Gamma}_{\mathbf{b}}{}^{\mathbf{e}}{}_{\mathbf{e}}\delta_{\mathbf{e}},\\
2\hat{\nabla}_{[\mathbf{0}}\gamma_{\mathbf{b}]\mathbf{c}} &= \hat{\nabla}_{\mathbf{0}}\gamma_{\mathbf{bc}} + \hat{\Gamma}_{\mathbf{b}}{}^{\mathbf{e}}{}_{\mathbf{0}}\gamma_{\mathbf{ec}},\\
2\hat{\nabla}_{[\mathbf{0}}\varsigma_{\mathbf{bc}]} &= \hat{\nabla}_{\mathbf{0}}\varsigma_{\mathbf{bc}} - \hat{\Gamma}_{\mathbf{b}}{}^{\mathbf{e}}{}_{\mathbf{0}}\varsigma_{\mathbf{ce}} - \hat{\Gamma}_{\mathbf{c}}{}^{\mathbf{e}}{}_{\mathbf{0}}\varsigma_{\mathbf{eb}}.
\end{aligned}
$$

We then compute $\hat{\nabla}_{[a}\delta_{b]}$, $\hat{\nabla}_{[a}\gamma_{b]c}$ and $\hat{\nabla}_{[a}\varsigma_{bc]}$ explicitly making use of the definitions of the zero-quantities and re-expressing the result in terms of zero-quantities so as to obtain

$$
\begin{aligned}
2\hat{\nabla}_{[\mathbf{a}}\delta_{\mathbf{b}]} &= -\gamma_{[\mathbf{ab}]} + \varsigma_{\mathbf{ab}} - \frac{1}{2}\Theta^{-1}\Sigma_{\mathbf{a}}{}^{\mathbf{e}}{}_{\mathbf{b}}\hat{\nabla}_{\mathbf{e}}\Theta,\\
2\hat{\nabla}_{[\mathbf{a}}\gamma_{\mathbf{b}]\mathbf{c}} &= \hat{\Delta}_{\mathbf{abc}} + \beta_{\mathbf{e}}\hat{\Xi}^{\mathbf{e}}{}_{\mathbf{cab}} - \hat{\Sigma}_{\mathbf{a}}{}^{\mathbf{e}}{}_{\mathbf{b}}\hat{\nabla}_{\mathbf{e}}\beta_{\mathbf{c}} + 2\beta_{\mathbf{c}}\gamma_{[\mathbf{ab}]} - 2\beta_{[\mathbf{a}}\gamma_{\mathbf{b}]\mathbf{c}}\\
&\qquad +\eta_{\mathbf{c}[\mathbf{a}}\beta^{\mathbf{e}}\gamma_{\mathbf{b}]\mathbf{e}} + 2\lambda\Theta^{-2}\delta_{[\mathbf{a}}\eta_{\mathbf{b}]\mathbf{c}} + \beta_{[\mathbf{a}}\eta_{\mathbf{b}]\mathbf{c}}\beta_{\mathbf{e}}\beta^{\mathbf{e}} - 2\lambda\Theta^{-2}\eta_{\mathbf{c}[\mathbf{a}}\beta_{\mathbf{b}]},\\
\hat{\nabla}_{[\mathbf{a}}\varsigma_{\mathbf{bc}]} &= \frac{1}{2}\hat{\Delta}_{[\mathbf{abc}]} + \frac{1}{2}\hat{\Xi}^{\mathbf{e}}{}_{[\mathbf{cab}]}f_{\mathbf{e}} - \frac{1}{2}\hat{\Sigma}_{[\mathbf{a}}{}^{\mathbf{e}}{}_{\mathbf{b}}\hat{\nabla}_{|\mathbf{e}|}f_{\mathbf{c}]}.
\end{aligned}
$$

From the above expressions, it follows that the evolution equations for δ_a, γ_{ab} and ς_{ab} are given by

$$\hat{\nabla}_{\mathbf{0}}\delta_{\mathbf{i}} = \gamma_{\mathbf{i0}} - \hat{\Gamma}_{\mathbf{i}}{}^{\mathbf{e}}{}_{\mathbf{0}}\delta_{\mathbf{e}}, \tag{3.26a}$$

$$\hat{\nabla}_{\mathbf{0}}\gamma_{\mathbf{ic}} = -\gamma_{\mathbf{jc}}\hat{\Gamma}_{\mathbf{i}}{}^{\mathbf{j}}{}_{\mathbf{0}} - \beta_{\mathbf{0}}\gamma_{\mathbf{ic}} - \beta_{\mathbf{c}}\gamma_{\mathbf{i0}} + \eta_{\mathbf{0c}}(\beta^{\mathbf{e}}\gamma_{\mathbf{ie}} - 2\lambda\Theta^{-2}\delta_{\mathbf{i}}), \tag{3.26b}$$

$$\hat{\nabla}_{\mathbf{0}}\varsigma_{\mathbf{jk}} = \hat{\Gamma}_{\mathbf{j}}{}^{\mathbf{e}}{}_{\mathbf{0}}\varsigma_{\mathbf{ke}} + \Gamma_{\mathbf{k}}{}^{\mathbf{e}}{}_{\mathbf{0}}\varsigma_{\mathbf{ej}} + \frac{1}{2}\hat{\Delta}_{\mathbf{jk0}} + \frac{1}{2}\hat{\Xi}^{\mathbf{e}}{}_{\mathbf{0jk}}f_{\mathbf{e}} + \frac{1}{2}\hat{\Sigma}_{\mathbf{j}}{}^{\mathbf{e}}{}_{\mathbf{k}}\hat{\Gamma}_{\mathbf{e}}{}^{\mathbf{f}}{}_{\mathbf{0}}f_{\mathbf{f}}, \tag{3.26c}$$

where, in particular, the evolution equation for the covector β_a,

$$\hat{\nabla}_{\mathbf{0}}\beta_{\mathbf{a}} + \beta_{\mathbf{0}}\beta_{\mathbf{a}} - \frac{1}{2}\eta_{\mathbf{0a}}(\beta_{\mathbf{e}}\beta^{\mathbf{e}} - 2\lambda\Theta^{-2}) = 0,$$

has been used in the derivation of equation (3.26b). Again, as required, the equations (3.26a)-(3.26c) are homogeneous in various zero-quantities.

Remark 18. Observe that equation (3.26b) contains the potentially singular term $\lambda\Theta^{-2}\delta_{\mathbf{i}}$. As such, this equation can only be used away from the conformal boundary where $\Theta \neq 0$.

This is a consequence of the use of a conformal Gaussian gauge hinged on a standard Cauchy hypersurface. This singular behaviour is of no consequence in our analysis as one is only interested on solutions to the subsidiary equations away from the conformal boundary.

3.3.4 Summary: structural properties of the evolution and subsidiary equations

As a conclusion of the long computations in this section, we now provide a summary of the conformal evolution equations, the associated subsidiary system and the structural properties of these systems which will be required in the reminder of our analysis.

The computations discussed in the previous subsections show that, in a conformal Gaussian gauge, the various fields associated to the extended vacuum conformal Einstein field equations satisfy the evolution equations

$$\partial_\tau e_{\boldsymbol{b}}{}^{\nu} = -\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{0}} e_{\boldsymbol{c}}{}^{\nu}, \tag{3.27a}$$

$$\partial_\tau \hat{L}_{\boldsymbol{db}} = \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} \hat{L}_{\boldsymbol{cb}} + \hat{\Gamma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} \hat{L}_{\boldsymbol{dc}} + d_{\boldsymbol{a}} \hat{d}^{\boldsymbol{a}}{}_{\boldsymbol{b0d}}, \tag{3.27b}$$

$$\partial_\tau f_{\boldsymbol{i}} = -f_{\boldsymbol{j}} \hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}} + \hat{L}_{\boldsymbol{i0}}, \tag{3.27c}$$

$$\partial_\tau (\hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) = -\hat{\Gamma}_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}} - \Xi \hat{d}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} - 2\delta_{\boldsymbol{d}}{}^{\boldsymbol{c}} \hat{L}_{\boldsymbol{b0}} - 2\delta_{\boldsymbol{0}}{}^{c} \hat{L}_{\boldsymbol{bd}} + 2 g_{\boldsymbol{d0}} g^{\boldsymbol{ce}} \hat{L}_{\boldsymbol{be}}, \tag{3.27d}$$

$$\partial_\tau d_{\boldsymbol{bd}} + \epsilon^{\boldsymbol{ef}}{}_{(\boldsymbol{d}} D_{\boldsymbol{f}} d^*_{\boldsymbol{b})e} = 2 a_{\boldsymbol{f}} \epsilon^{\boldsymbol{ef}}_{(\boldsymbol{d}} d^*_{\boldsymbol{b})e} - \chi d_{\boldsymbol{bd}} + 2\chi^{\boldsymbol{f}}{}_{(\boldsymbol{b}} d_{\boldsymbol{d})\boldsymbol{f}}, \tag{3.27e}$$

$$\partial_\tau d^*_{\boldsymbol{bd}} - \epsilon^{\boldsymbol{e}}{}_{\boldsymbol{f}(\boldsymbol{d}} D^{\boldsymbol{f}} d_{\boldsymbol{b})e} = 2 a^{\boldsymbol{f}} \epsilon_{\boldsymbol{f}(\boldsymbol{d}}{}^{\boldsymbol{e}} d_{\boldsymbol{b})e} - \chi d^*_{\boldsymbol{bd}} + 2\chi^{\boldsymbol{f}}_{(\boldsymbol{b}} d^*_{\boldsymbol{d})\boldsymbol{f}}. \tag{3.27f}$$

Letting $\boldsymbol{e}$, $\boldsymbol{\Gamma}$, $\hat{\boldsymbol{L}}$ and $\boldsymbol{\phi}$ denote, respectively, the independent components of the coefficients of the frame, the connection coefficients, the Schouten tensor of the Weyl connection and the rescaled Weyl tensor and setting, for convenience, $\hat{\mathbf{u}} \equiv (\hat{\boldsymbol{\upsilon}}, \hat{\boldsymbol{\phi}})$, $\hat{\boldsymbol{\upsilon}} \equiv (\boldsymbol{e}, \hat{\boldsymbol{\Gamma}}, \hat{\boldsymbol{L}})$, one has that equations (3.27a)-(3.27f) can be written, schematically, in the form

$$\partial_\tau \hat{\boldsymbol{\upsilon}} = \mathbf{K}\hat{\boldsymbol{\upsilon}} + \mathbf{Q}(\hat{\boldsymbol{\upsilon}}, \hat{\boldsymbol{\upsilon}}) + \mathbf{L}(\bar{x})\hat{\boldsymbol{\phi}}, \tag{3.28a}$$

$$(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e}))\partial_\tau \hat{\boldsymbol{\phi}} + \mathbf{A}^\alpha(\boldsymbol{e})\partial_\alpha \hat{\boldsymbol{\phi}} = \mathbf{B}(\hat{\boldsymbol{\Gamma}})\hat{\boldsymbol{\phi}}, \tag{3.28b}$$

where $\mathbf{K}$ and $\mathbf{Q}$ denote, respectively, a matrix and a quadratic form, both with constant coefficients while $\mathbf{L}$ is a matrix with coefficients depending smoothly on the coordinates. Moreover, $\mathbf{A}^{\mu}(\boldsymbol{e})$ denote, for $\mu = 0, \ldots, 3$ Hermitian matrix-valued functions depending smoothly on $\boldsymbol{e}$. In particular $\mathbf{I} + \mathbf{A}^{0}(\boldsymbol{e})$ is positive definite for $\boldsymbol{e}$ suitably close to the background solution —with closeness understood in the sense of Sobolev norms. Finally, $\mathbf{B}(\hat{\mathbf{\Gamma}})$ denotes a smooth matrix-value function of the component of the connection.

Remark 19. Altogether, the conformal evolution system described by equations (3.28a)-(3.28b) constitutes a quasilinear symmetric hyperbolic system for which a well-posedness theory is available —see [49], also [81] for an abridged version. This theory will be used in the remaining sections of this article to establish the stability of the solution to the Einstein field equations given by the metric (3.2).

Remark 20. A remarkable structural property of the conformal evolution system (3.28a)-(3.28b) is that the equations in (3.28a) are, in fact, mere transport equations along conformal geodesics. The true hyperbolic content of the system is contained in the *Bianchi subsystem* (3.28b). This property does not play any particular role in our analysis, but it may prove key in, for example, the analysis of the formation of singularities.

Regarding the subsidiary evolution system, the key conclusion from the system

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} = -\frac{1}{3}\hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} - \frac{1}{3}\hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Sigma}_{\boldsymbol{e}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} - \hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{0bc}}, \tag{3.29a}$$

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ebc}} = \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{ecf}} + \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}\hat{\Xi}^{\boldsymbol{d}}{}_{\boldsymbol{efb}} - \hat{\Sigma}_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{c}}\hat{R}^{\boldsymbol{d}}{}_{\boldsymbol{e0f}} - \frac{1}{2}\Theta\epsilon^{\boldsymbol{f}}{}_{\boldsymbol{0bc}}\epsilon_{\boldsymbol{e}}{}^{\boldsymbol{dgh}}\Lambda_{\boldsymbol{fgh}} \tag{3.29b}$$

$$+\epsilon^{\boldsymbol{f}}{}_{\boldsymbol{0bc}}\delta^{\boldsymbol{g}}d^{*\boldsymbol{d}}{}_{\boldsymbol{efg}} + 3S_{\boldsymbol{e0}}{}^{\boldsymbol{dg}}\hat{\Delta}_{\boldsymbol{cbg}}, \tag{3.29c}$$

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{bcd}} = \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ced}} + \hat{\Gamma}_{\boldsymbol{c}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\hat{\Delta}_{\boldsymbol{ebd}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{0bc}}\hat{L}_{\boldsymbol{ed}} + \delta_{\boldsymbol{b}}d_{\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{dc0}} + \delta_{\boldsymbol{c}}d_{\boldsymbol{e}}d^{\boldsymbol{e}}{}_{\boldsymbol{d0b}} \tag{3.29d}$$

$$+\Theta\gamma_{\boldsymbol{be}}d^{\boldsymbol{e}}{}_{\boldsymbol{dc0}} + \Theta\gamma_{\boldsymbol{ce}}d^{\boldsymbol{e}}{}_{\boldsymbol{d0b}} - \frac{1}{2}\epsilon_{\boldsymbol{0bc}}{}^{\boldsymbol{f}}\epsilon_{\boldsymbol{d}}{}^{\boldsymbol{egh}}\Lambda_{\boldsymbol{fgh}}\beta_{\boldsymbol{e}}, \tag{3.29e}$$

$$\hat{\nabla}_{\boldsymbol{0}}\hat{\Omega}_{\boldsymbol{bc}} = \hat{\Xi}^{\boldsymbol{e}}{}_{[\boldsymbol{b}}{}^{\boldsymbol{af}}d_{\boldsymbol{c}]\boldsymbol{efa}} - \hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{f}}{}^{\boldsymbol{af}}d_{\boldsymbol{eabc}} + \frac{1}{2}\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{e}}{}_{\boldsymbol{f}}\nabla_{\boldsymbol{e}}d^{\boldsymbol{fa}}{}_{\boldsymbol{bc}} + \varsigma^{\boldsymbol{fa}}d_{\boldsymbol{fabc}} - \chi\Omega_{\boldsymbol{bc}}, \tag{3.29f}$$

$$\hat{\nabla}_{\boldsymbol{0}}\delta_{\boldsymbol{i}} = \gamma_{\boldsymbol{i0}} - \hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\delta_{\boldsymbol{e}}; \tag{3.29g}$$

$$\hat{\nabla}_{\boldsymbol{0}}\gamma_{\boldsymbol{ic}} = -\gamma_{\boldsymbol{jc}}\hat{\Gamma}_{\boldsymbol{i}}{}^{\boldsymbol{j}}{}_{\boldsymbol{0}} - \beta_{\boldsymbol{0}}\gamma_{\boldsymbol{ic}} - \beta_{\boldsymbol{c}}\gamma_{\boldsymbol{i0}} + \eta_{\boldsymbol{0c}}(\beta^{\boldsymbol{e}}\gamma_{\boldsymbol{ie}} - 2\lambda\Theta^{-2}\delta_{\boldsymbol{i}}), \tag{3.29h}$$

$$\hat{\nabla}_{\boldsymbol{0}}\varsigma_{\boldsymbol{jk}} = \hat{\Gamma}_{\boldsymbol{j}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\varsigma_{\boldsymbol{ke}} + \Gamma_{\boldsymbol{k}}{}^{\boldsymbol{e}}{}_{\boldsymbol{0}}\varsigma_{\boldsymbol{ej}} + \frac{1}{2}\hat{\Delta}_{\boldsymbol{jk0}} + \frac{1}{2}\hat{\Xi}^{\boldsymbol{e}}{}_{\boldsymbol{0jk}}f_{\boldsymbol{e}} + \frac{1}{2}\hat{\Sigma}_{\boldsymbol{j}}{}^{\boldsymbol{e}}{}_{\boldsymbol{k}}\hat{\Gamma}_{\boldsymbol{e}}{}^{\boldsymbol{f}}{}_{\boldsymbol{0}}f_{\boldsymbol{f}}, \tag{3.29i}$$

is that the zero-quantities $\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$, $\hat{\Xi}^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$, $\hat{\Delta}_{\boldsymbol{abc}}$, $\hat{\Lambda}_{\boldsymbol{abc}}$, $\delta_{\boldsymbol{ab}}$, $\gamma_{\boldsymbol{ab}}$ and $\varsigma_{\boldsymbol{ab}}$ satisfy, if the conformal evolution equations (3.27a)-(3.27e) hold, a symmetric hyperbolic system which is homogeneous in the zero-quantities —accordingly, the particular situation in which all the zero quantities vanish identically giving rise to the subsidiary evolution system. The subsidiary system is regular away from the conformal boundary —i.e. the sets for which the conformal factor vanishes.

3.4 Initial data for the evolution equations

Given a solution $(\mathcal{S}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ to the Einstein constraint equations (i.e the Hamiltonian and the momentum constraints), there exists an algebraic procedure to compute initial data for the conformal evolution equations —see e.g. [81], Lemma 11.1, page 265. Now, a suitable perturbative existence theorem which covers perturbations of the initial data implied by the metric (3.2) on the hypersurfaces of constant t has been provided in [82] —see Theorem 1. From this result one can deduce the following assertion:

Proposition 5. *Let $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ with $\mathcal{S}$ compact, $\mathring{\mathring{\boldsymbol{h}}}$ a smooth Riemannian metric of constant negative curvature and $\mathring{\boldsymbol{K}} = \varkappa\mathring{\boldsymbol{h}}$ with $\varkappa$ a constant, denote an initial data set for the vacuum Einstein field equations with positive Cosmological constant. Then for each pair of sufficiently small (in the sense of suitable Sobolev norms) tensors T_{ij} and $\bar{T}_{ij}$ over $\mathcal{S}$, transverse-tracefree with respect to $\mathring{\boldsymbol{h}}$, and each sufficiently small scalar field Φ over $\mathcal{S}$, there exists a solution of the Einstein constraint equations $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ with positive Cosmological constant which is suitably close to $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$ and such that $tr_{\mathring{\boldsymbol{h}}}(\boldsymbol{K} - \mathring{\boldsymbol{K}}) = \Phi$ and for which the electric and magnetic parts of the Weyl tensor (restricted to $\mathcal{S}$) of the resulting spacetime development take the form*

$$d_{ij} = \mathring{L}(\boldsymbol{X})_{ij} + T_{ij} - \frac{1}{3} tr_{\boldsymbol{h}}(\mathring{L}(\boldsymbol{X}) + \boldsymbol{T})h_{ij},$$
$$d^*_{ij} = \mathring{L}(\bar{\boldsymbol{X}})_{ij} + \bar{T}_{ij} - \frac{1}{3} tr_{\boldsymbol{h}}(\mathring{L}(\bar{\boldsymbol{X}}) + \bar{\boldsymbol{T}})h_{ij},$$

for some covectors $\boldsymbol{X}$, $\bar{\boldsymbol{X}}$ over $\mathcal{S}$ and where $\mathring{L}$ denotes the conformal Killing operator with respect to $\mathring{\boldsymbol{h}}$.

Remark 21. Thus, choosing the free data T_{ij}, $\bar{T}_{ij}$ and Φ suitably small one can ensure that the *perturbed data* $(\mathcal{S}, \boldsymbol{h}, \boldsymbol{K})$ is close to $(\mathcal{S}, \mathring{\boldsymbol{h}}, \mathring{\boldsymbol{K}})$. Accordingly, the associated initial data for the conformal evolution equations will be close to initial data for the background solution.

Remark 22. Theorem 1 in [82] applies to the broader class of *conformally rigid hyperbolic* compact manifolds —that is, Einstein manifolds with negative Ricci scalar which do not admit a non-trivial Codazzi tensor; see the discussion in Section 3.4.3 of this reference. The precise statement of the result also excludes values of $\varkappa$ which are related in a specific manner to the eigenvalues of the Laplacian of $\mathring{\boldsymbol{h}}$ —however, we do not require this level of detail in the subsequent discussion.

3.5 Analysis of the existence and stability of solutions

In this section, we develop the theory of the existence, uniqueness and stability of solutions to the Einstein field equations which can be regarded as perturbations of the background solution. The argument proceeds in several steps: first, the Cauchy stability of solutions to symmetric hyperbolic systems is used to conclude the existence of solutions to the conformal evolution system (3.27a)-(3.27f); in a second step the uniqueness of solutions to the subsidiary system (3.29a)-(3.29i) to argue the propagation of constraints; finally general theory of the conformal Einstein field equations is invoked to establish the connection between solutions to the conformal equations and actual solutions to the Einstein field equations.

3.5.1 A symmetric hyperbolic evolution system

In the following, we look for solutions to the system (3.28a)-(3.28b) of the form

$$\hat{\mathbf{u}} = \mathring{\mathbf{u}} + \breve{\mathbf{u}},$$

where $\mathring{\mathbf{u}}$ is the solution to the conformal evolution equations (3.27a)-(3.27f) implied by a background solution, while $\breve{\boldsymbol{u}}$ denotes a small perturbation. Accordingly, making use of the schematic notation of equations (3.28a)-(3.28b) one can set

$$\hat{\boldsymbol{\upsilon}} = \mathring{\boldsymbol{\upsilon}} + \breve{\boldsymbol{\upsilon}}, \qquad \hat{\boldsymbol{\phi}} = \breve{\boldsymbol{\phi}}, \tag{3.30a}$$

$$\hat{\boldsymbol{e}} = \mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}, \qquad \hat{\boldsymbol{\Gamma}} = \mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}}. \tag{3.30b}$$

Now, we have found that on the initial surface $\mathcal{S}_\star$ described by the condition $\tau = \tau_\star$ one can write $\mathring{\mathbf{u}}_\star = (\mathring{\boldsymbol{\upsilon}}_\star, \mathring{\boldsymbol{\phi}}_\star) = (\mathring{\boldsymbol{\upsilon}}_\star, 0)$. As the conformal factor Θ and the covector $\boldsymbol{d}$ are universal, it follows that

$$\partial_\tau \mathring{\boldsymbol{\upsilon}} = \mathbf{K}\mathring{\boldsymbol{\upsilon}} + \mathbf{Q}(\mathring{\boldsymbol{\upsilon}}, \mathring{\boldsymbol{\upsilon}}).$$

Substituting (3.30a) and (3.30b) into equations (3.28a) and (3.28b) yields evolution equations for $\breve{\mathbf{u}} = (\breve{\boldsymbol{\upsilon}}, \breve{\boldsymbol{\phi}})$ which, schematically, take the form

$$\partial_\tau \breve{\boldsymbol{\upsilon}} = \mathbf{K}\breve{\boldsymbol{\upsilon}} + \mathbf{Q}(\mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\upsilon}} + \mathbf{Q}(\breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\upsilon}} + \mathbf{L}(\bar{x})\breve{\boldsymbol{\phi}}, \tag{3.31a}$$

$$(\mathbf{I} + \mathbf{A}^0(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}))\partial_\tau \breve{\boldsymbol{\phi}} + \mathbf{A}^\alpha(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}})\partial_\alpha \breve{\boldsymbol{\phi}} = \mathbf{B}(\mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\phi}}. \tag{3.31b}$$

Now, in the following, it is convenient to define

$$\bar{\mathbf{A}}^0(\tau, \underline{x}, \breve{\mathbf{u}}) \equiv \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} + \mathbf{A}^0(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}) \end{pmatrix}, \qquad \bar{\mathbf{A}}^\alpha(\tau, \underline{x}, \breve{\mathbf{u}}) \equiv \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{A}^\alpha(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}) \end{pmatrix}$$

and

$$\bar{\mathbf{B}}(\tau, \underline{x}, \breve{\mathbf{u}}) \equiv \breve{\mathbf{u}}\bar{\mathbf{Q}}\breve{\mathbf{u}} + \bar{\mathbf{L}}(\bar{x})\breve{\mathbf{u}} + \bar{\mathbf{K}}\breve{\mathbf{u}},$$

where

$$\breve{\mathbf{u}}\bar{\mathbf{Q}}\breve{\mathbf{u}} \equiv \begin{pmatrix} \breve{\boldsymbol{\upsilon}}\mathbf{Q}\breve{\boldsymbol{\upsilon}} & 0 \\ 0 & \mathbf{B}(\breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\phi}} \end{pmatrix}, \qquad \bar{\mathbf{L}}(\bar{x})\breve{\mathbf{u}} \equiv \begin{pmatrix} \mathring{\boldsymbol{\upsilon}}\mathbf{Q}\breve{\boldsymbol{\upsilon}} + \mathbf{Q}(\breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\upsilon}} & \mathbf{L}(\bar{x})\breve{\boldsymbol{\phi}} \\ 0 & 0 \end{pmatrix},$$

$$\bar{\mathbf{K}}\breve{\mathbf{u}} \equiv \begin{pmatrix} \mathbf{K}\breve{\boldsymbol{\upsilon}} & 0 \\ 0 & \mathbf{B}(\mathring{\Gamma})\breve{\boldsymbol{\phi}} \end{pmatrix},$$

denote, respectively, quadratic, linear and constant terms in the unknowns. In terms of the latter it is possible to rewrite the system (3.31a) and (3.31b) in the form

$$\bar{\mathbf{A}}^0(\tau, \underline{x}, \breve{\mathbf{u}})\partial_\tau \breve{\mathbf{u}} + \bar{\mathbf{A}}^\alpha(\tau, \underline{x}, \breve{\mathbf{u}})\partial_\alpha \breve{\mathbf{u}} = \bar{\mathbf{B}}(\tau, \underline{x}, \breve{\mathbf{u}}). \tag{3.32}$$

From the discussion in the previous sections, it follows that the system described by (4.30) is a symmetric hyperbolic system for which the theory of [49] can be applied. The natural domain of the solutions to this system is of the form

$$\mathcal{M} = [\tau_\star, \tau_\bullet) \times \mathcal{S}, \qquad \tau_\star \in (0,1), \qquad \tau_\bullet \geq 1.$$

3.5.2 The existence, uniqueness and Cauchy stability of the solution

The existence of de Sitter-like solutions to the conformal evolution system (4.30) is given by the following proposition:

Proposition 6 (***existence and uniqueness of the solutions to the perturbed de Sitter-like evolution equations***)**.** *Given $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ and $m \geq 4$, one has that:*

(i) There exists $\varepsilon > 0$ such that if

$$||\breve{\mathbf{u}}_\star||_{\mathcal{S},m} < \varepsilon, \tag{3.33}$$

then there exists a unique solution $\breve{\mathbf{u}} \in C^{m-2}(\left[\tau_\star, \tfrac{3}{2}\right) \times \mathcal{S}, \mathbb{R}^N)$ to the Cauchy problem for the conformal evolution equations (4.30) with initial data $\mathbf{u}(\tau_\star, \underline{x}) = \breve{\mathbf{u}}_\star$, $\tau_\star > 0$ and with N denoting the dimension of the vector $\mathbf{u}$.

(ii) Given a sequence of initial data $\breve{\mathbf{u}}^{(n)}_\star$ such that

$$||\breve{\mathbf{u}}^{(n)}{}_\star||_{\mathcal{S},m} < \varepsilon, \qquad \text{and} \qquad ||\breve{\mathbf{u}}^{(n)}{}_\star||_{\mathcal{S},m} \xrightarrow{n\to\infty} 0,$$

then for the corresponding solutions $\breve{\mathbf{u}}^{(n)} \in C^{m-2}\left(\left[\tau_\star, \tfrac{3}{2}\right) \times \mathcal{S}, \mathbb{R}^N\right)$, one has $||\breve{\mathbf{u}}^{(n)}||_{\mathcal{S},m} \to 0$ uniformly in $\tau \in \left[\tau_\star, \tfrac{3}{2}\right)$ as $n \to \infty$.

Remark 23. In the above proposition $||\breve{\mathbf{u}}_\star||_{\mathcal{S},m}$ denotes the standard L^2-Sobolev norm over $\mathcal{S}$ of order $m \geq 4$ of the independent components of the vector $\breve{\mathbf{u}}_\star$.

Proof. The proof is an application of the existence and stability results for symmetric hyperbolic systems with compact spatial sections —see e.g. [81], Section 12.3 which, in turn, follows from Kato's theory for symmetric hyperbolic systems over $\mathbb{R}^n$ [49]. More precisely, since the 3-dimensional manifold $\mathcal{S}$ is compact, there exists a finite cover consisting of open sets $\mathcal{R}_1, \ldots, \mathcal{R}_M \subset \mathcal{S}$ such that $\cup_{i=1}{}^M \mathcal{R}_i = \mathcal{S}$. On each of the open sets $\mathcal{R}_i$ it is possible to introduce coordinates $\underline{x}_i \equiv (x^\alpha{}_i)$ which allow one to identify $\mathcal{R}_i$ with open subsets $\mathcal{B}_i \subset \mathbb{R}^3$. As $\mathcal{S}$ is assumed to be a smooth manifold, the coordinate patches can be chosen so that the change of coordinates on intersecting sets is smooth. The initial data $\breve{\mathbf{u}}_\star : \mathcal{S} \to \mathbb{R}^N$ is a smooth function on $\mathcal{S}$ and can be restricted to a particular open set $\mathcal{R}_i$. The restriction $\breve{\mathbf{u}}_{i\star}$, in local coordinates x_i can be regarded as a function $\breve{\mathbf{u}}_{i\star} : \mathcal{B}_i \to \mathbb{R}^N$. Now, assuming that $\mathcal{R} \subset \mathbb{R}^3$ is bounded with smooth boundary $\partial\mathcal{R}$, it is possible to extend $\breve{\mathbf{u}}_{i\star}$ to a function $\mathcal{E}\breve{\mathbf{u}}_{i\star} : \mathbb{R}^3 \to \mathbb{R}^N$ —see e.g. Proposition 12.2 in [81]. Using these extensions it is possible to define the Sobolev norm

$$||\breve{\mathbf{u}}_\star||_{\mathcal{S},m} \equiv \sum_{i=1}^{M} ||\breve{\mathbf{u}}_{i\star}||_{\mathbb{R}^3,m}.$$

Now, for each of the $\mathcal{E}\breve{\mathbf{u}}_{i\star}$ one can formulate an initial value problem of the form

$$\begin{aligned} &\bar{\mathbf{A}}^0(\tau, \underline{x}, \breve{\mathbf{u}})\partial_\tau \breve{\mathbf{u}} + \bar{\mathbf{A}}^\alpha(\tau, \underline{x}, \breve{\mathbf{u}})\partial_\alpha \breve{\mathbf{u}} = \mathcal{B}(\tau, \underline{x}, \breve{\mathbf{u}}), \\ &\breve{\mathbf{u}}(\tau_\star, \underline{x}) = \mathcal{E}\breve{\mathbf{u}}_{i\star}(\underline{x}) \in H^m(\mathcal{S}, \mathbb{R}^N) \quad \text{for } m \geq 4. \end{aligned}$$

For this initial value problem, it is observed that:

(a) The matrices $\bar{\mathbf{A}}^\mu(\tau, \underline{x}, \mathcal{E}\breve{\mathbf{u}}_{i\star})$ are positive definite and depend linearly on the solution $\breve{\mathbf{u}}_i$ with coefficients which are constant.

(b) The dependence of $\mathcal{B}$ on $\breve{\mathbf{u}}_i$ is at most quadratic: there are linear and quadratic terms for the connection coefficients; linear terms for the components of the Schouten tensor. The explicit dependence on $(\tau, \underline{x})$ comes from the conformal factor and the covector d_a —this dependence is smooth.

(c) The connection coefficients and the components of the Schouten tensor of the background solution are smooth functions (C^∞) of $(\tau, \underline{x})$.

(d) The dependence of the frame coefficients of the background solution is smooth (C^∞) on τ for $\tau \in [\tau_\star, \frac{3}{2}]$ with $\tau_\star > 0$.

It follows from the above observations that our system satisfies the conditions of Kato's theorems —see Appendix A.1. This theory implies existence, uniqueness and stability — i.e. points (i) and (ii) in the theorem. Notice, however, that strictly speaking, this theorem only applies to settings in which the spatial sections are diffeomorphic to $\mathbb{R}^3$. To address this one makes use of the following strategy: standard results on causality theory imply that

$$D^+(\mathcal{R}_i) \cap I^+(\mathcal{S} \setminus \mathcal{R}_i) = \emptyset,$$

where $D^+(\mathcal{R}_i)$ denotes the causal future of $\mathcal{R}_i$ —see e.g. [81], Theorem 14.1. Accordingly, the value of $\breve{\mathbf{u}}$ on $\mathcal{D}_i \equiv D^+(\mathcal{R}_i)$ is determined only by the data on $\mathcal{R}_i$. Then the solution on $\mathcal{D}_i$ is independent of the particular extension $\mathcal{E}\breve{\mathbf{u}}_{i\star}$ being used. Hence, one can speak of a solution $\breve{\mathbf{u}}_i$ on a domain $\mathcal{D}_i \subset [\tau_\star, \tau_i] \times \mathcal{R}_i$. Since the manifold is smooth and as a consequence of uniqueness, it follows that given two solutions $\breve{\mathbf{u}}_i$ and $\breve{\mathbf{u}}_j$ defined, respectively, on intersecting domains $\mathcal{D}_i$ and $\mathcal{D}_j$ they must coincide on $\mathcal{D}_i \bigcap \mathcal{D}_j$. Proceeding in the same manner over the whole finite cover of $\mathcal{S}$ and since the compactness of $\mathcal{S}$ ensures the existence of a minimum non-zero existence time for the whole of the domains $\mathcal{D}_i$, then there is a unique solution $\breve{\mathbf{u}}$ on $[\tau_\star, \frac{3}{2}] \times \mathcal{S}$ with $\frac{3}{2} = \min_{i=1,\ldots,M}\{\tau_i\}$ which is constructed by patching together the localised solutions $\breve{\mathbf{u}}_1, \ldots, \breve{\mathbf{u}}_M$ defined, respectively on the domains $\mathcal{D}_i, \ldots, \mathcal{D}_M$. The existence interval $[\tau_\star, \frac{3}{2})$ follows from the fact that the background solution $\mathring{\mathbf{u}}$ has this existence interval.

□

Remark 24. The existence and Cauchy stability of the solution to the initial value problem for the original conformal evolution problem

$$\begin{aligned}
&\mathbf{A}^0(\tau, \underline{x}, \hat{\mathbf{u}})\partial_\tau \hat{\mathbf{u}} + \mathbf{A}^\alpha(\tau, \underline{x}, \hat{\mathbf{u}})\partial_\alpha \hat{\mathbf{u}} = \mathbf{B}(\tau, \underline{x}, \hat{\mathbf{u}}),\\
&\hat{\mathbf{u}}|_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star \in H^m(\mathcal{S}, \mathbb{R}^N) \quad \text{for} \quad m \geq 4
\end{aligned}$$

follows from the fact that $\hat{\mathbf{u}}$ satisfies the same properties as $\breve{\mathbf{u}}$ in Proposition 6 and then it exists in the same solution manifold and with the same regularity properties, existence and uniqueness.

3.5.3 Propagation of the constraints

In this section, we discuss the so-called *propagation of the constraints*. This argument is essential to establish the connection between solutions to the conformal evolution systems and actual solutions to the Einstein field equations. More precisely, one has the following:

Proposition 7 (***propagation of the constraints***)**.** *Let* $\hat{\mathbf{u}}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ *denote initial data for the conformal evolution equations on a* 3*-manifold* $\mathcal{S}_\star \approx \mathcal{S}$ *such that*

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}|_{\mathcal{S}_\star} = 0, \quad \hat{\Xi}^{c}{}_{\boldsymbol{dab}}|_{\mathcal{S}_\star} = 0, \quad \hat{\Delta}_{\boldsymbol{abc}}|_{\mathcal{S}_\star} = 0, \quad \hat{\Lambda}_{\boldsymbol{abc}}|_{\mathcal{S}_\star} = 0,$$

and

$$\delta_{\boldsymbol{a}}|_{\mathcal{S}_\star} = 0, \quad \gamma_{\boldsymbol{ab}}|_{\mathcal{S}_\star} = 0, \quad \varsigma_{\boldsymbol{ab}}|_{\mathcal{S}_\star} = 0,$$

then the solution $\breve{\mathbf{u}}$ *to the conformal evolution equations given by Proposition 6 implies a* C^{m-2} *solution* $\hat{\mathbf{u}} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$ *to the extended conformal field equations on* $\left[\tau_\star, 1\right) \times \mathcal{S}$*.*

Proof. The proof follows from the properties of the subsidiary evolution system. First, it is observed that by assumption

$$\hat{\Sigma}_{\boldsymbol{0}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = 0, \qquad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{d0b}} = 0, \qquad \hat{\Delta}_{\boldsymbol{0bc}} = 0,$$

hold —cfr. the equations in (3.11). Moreover, the associated evolution equations are expressed in terms of a conformal Gaussian gauge system and the independent components of the rescaled Weyl tensor satisfy either the evolution system (3.14) and (3.15). Now, following the discussion of Section 3.3.3, the independent components of the zero-quantities

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \quad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \quad \hat{\Delta}_{\boldsymbol{abc}}, \quad \hat{\Lambda}_{\boldsymbol{abc}}, \quad \delta_{\boldsymbol{a}}, \quad \gamma_{\boldsymbol{ab}}, \quad \varsigma_{\boldsymbol{ab}},$$

which are not determined by either the evolution equations or gauge conditions satisfy a symmetric hyperbolic system which is homogeneous in the zero-quantities. More precisely, defining $\hat{\mathbf{X}} \equiv \{\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \hat{\Delta}_{\boldsymbol{abc}}, \hat{\Lambda}_{\boldsymbol{abc}}, \delta_{\boldsymbol{a}}, \gamma_{\boldsymbol{ab}}, \varsigma_{\boldsymbol{ab}}\}$, these equations can be recast as a symmetric hyperbolic system of the form

$$\partial_\tau \hat{\mathbf{X}} = \mathbf{H}(\hat{\mathbf{X}}),$$

where $\mathbf{H}$ is a homogeneous function of its arguments —i.e. $\mathbf{H}(\mathbf{0}) = \mathbf{0}$. It follows then that a solution to the initial value problem

$$\partial_\tau \hat{\mathbf{X}} = \mathbf{H}(\hat{\mathbf{X}}),$$
$$\hat{\mathbf{X}}_\star = 0.$$

is given (trivially) by $\hat{\mathbf{X}} = 0$. Moreover, following Kato's theorem it follows this is the unique solution. Thus, the zero-quantities must vanish on $\Big[\tau_\star, 1\Big) \times \mathcal{S}$. That is, the solution $\breve{\mathbf{u}}$ to the conformal evolution equations implies a solution to the extended conformal Einstein field equations over the latter domain. □

From the above statement, making use of the relation between the extended conformal Einstein field equations and the actual Einstein field equations —see Proposition 8.3 in [81] it follows the corollary:

Corollary 1. *The metric*

$$\boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}$$

obtained from the solution to the conformal evolution equations given in Proposition 6 implies a solution $\tilde{\boldsymbol{g}}$ to the vacuum Einstein field equations with $\lambda = 3$.

3.6 Future geodesic completeness

In this section, we discuss the future geodesic completeness of the spacetimes obtained in the previous section. Our analysis distinguishes two cases: null geodesics and timelike geodesics.

3.6.1 Null geodesics

As a consequence of the compactness of the unphysical manifold

$$\mathcal{M} = \Big\{ (\tau, \underline{x}) \in \mathbb{R} \times \mathcal{S} \mid \tau_\bullet \leq \tau \leq 1 \Big\},$$

null geodesics in the unphysical manifold starting at the initial hypersurface $\mathcal{S}_\star$, reach the conformal boundary in a finite amount of affine parameter. Furthermore, null geodesics with respect to the unphysical metric $\boldsymbol{g}$ coincide, up to a reparametrisation, with null geodesics with respect to the physical metric $\tilde{\boldsymbol{g}}$ on $\tilde{\mathcal{M}}$. More precisely, let γ be a null geodesic in $(\mathcal{M}, \boldsymbol{g})$ with affine parameter v such that $v = 0$ on $\partial\tilde{\mathcal{M}}$. The equations for γ are

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}v^2} + \Gamma^\mu{}_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}v}\frac{\mathrm{d}x^\lambda}{\mathrm{d}v} = 0.$$

Let $\tilde{\gamma}$ denote the corresponding geodesics in $\tilde{\mathcal{M}}$. Using a different parameter $\tilde{v} = \tilde{v}(v)$ and the relation between the Christoffel symbols $\Gamma^\mu{}_{\nu\lambda}$ and $\tilde{\Gamma}^\mu{}_{\nu\lambda}$ it follows that

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tilde{v}^2} + \tilde{\Gamma}^\mu{}_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tilde{v}}\frac{\mathrm{d}x^\lambda}{d\tilde{v}} = -\frac{1}{\tilde{v}'}\left(\frac{\tilde{v}''}{\tilde{v}'} + 2\frac{\Theta'}{\Theta}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{v}}.$$

By requiring that $\tilde{v}$ to be an affine parameter the right-hand side must vanish. This implies $\tilde{v}' = \mathrm{const}/\Theta^2$, and absorbing the constant into $\tilde{v}$ we obtain

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}v} = \frac{1}{\Theta^2}.$$

Furthermore, at $\mathscr{I}^+$, $\Theta = 0$ and $\mathbf{d}\Theta \neq 0$, and we may choose v so that near $\partial\tilde{M}$, $v \sim -\Theta$. Thus $\tilde{v} \sim -1/v$ becomes unbounded —i.e. the physical affine parameter for the physical geodesic must blow up as $\Theta \to 0$. Thus, $\tilde{\gamma}$ never reaches $\partial\tilde{\mathcal{M}}$ and the null geodesic must be complete —see also the discussion in [70], Chapter 3.

3.6.2 Timelike geodesics

The argument used for null geodesics cannot readily be applied to the discussion of timelike geodesics as these are not conformally invariant. Instead, we make use of timelike conformal geodesics.

Every timelike metric geodesic on the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ can be recast, after a reparametrisation, as a conformal geodesics $(x, \tilde{\boldsymbol{\beta}})$ —see Chapter 2 and [37, 81]. Under the rescaling $\boldsymbol{g} = \Theta^2\tilde{\boldsymbol{g}}$, the conformal geodesic $(x, \tilde{\boldsymbol{\beta}})$ transforms into a geodesic $(x, \boldsymbol{\beta})$ in the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$. Now, it is known that any $\boldsymbol{g}$-conformal geodesic that leaves $\mathscr{I}^+$ orthogonally into the past, is up to a reparametrisation, a timelike future complete geodesic for the physical metric $\tilde{\boldsymbol{g}}$ —see e.g. [35, 37]. Moreover, a conformal geodesic through a point of $\mathscr{I}^+$ which is not orthogonal to the conformal boundary cannot represent a geodesic in the physical spacetime.

Now, from the $\tilde{\boldsymbol{g}}$-future geodesic completeness of the background solution (see Appendix A.2) it follows that every conformal geodesic in the background spacetime starting orthogonal to the initial hypersurface $\mathcal{S}_\star$ must reach the conformal boundary $\mathscr{I}^+$. Hence, every timelike $\tilde{\boldsymbol{g}}$-geodesic is, up to a reparametrisation, a timelike conformal curve reaching $\mathscr{I}^+$ orthogonally. Moreover, let us consider a pair $(x(\tau), \tilde{\boldsymbol{\beta}}(\tau))$ with parameter $\tau \in \mathbb{R}$. Furthermore, let us suppose that this geodesic starts at $\tau = \tau_\star$, i.e. the initial hypersurface $\mathcal{S}$, and it reaches the conformal boundary $\mathscr{I}^+$ at $\tau = 1$. Now, consider a small perturbation of the quantities $(x, \tilde{\boldsymbol{\beta}})$ so that

$$
\begin{aligned}
\hat{x} &= x + \breve{x}, \\
\hat{\boldsymbol{\beta}} &= \tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}},
\end{aligned}
$$

where $\breve{x}$ and $\breve{\boldsymbol{\beta}}$ are small perturbations. In this case, the perturbed conformal geodesic equations read

$$
\begin{aligned}
&\tilde{\nabla}_{\boldsymbol{x}'}(\boldsymbol{x}' + \breve{\boldsymbol{x}}') = -2\langle(\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}), (\boldsymbol{x}' + \breve{\boldsymbol{x}}')\rangle(\boldsymbol{x}' + \breve{\boldsymbol{x}}') + \tilde{\boldsymbol{g}}((\boldsymbol{x}' + \breve{\boldsymbol{x}}'), (\boldsymbol{x}' + \breve{\boldsymbol{x}}'))(\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}})^\sharp, \\
&\tilde{\nabla}_{\boldsymbol{x}'}(\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}) = \langle(\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}), (\boldsymbol{x}' + \breve{\boldsymbol{x}}')\rangle(\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}) - \frac{1}{2}\boldsymbol{g}^\sharp((\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}), (\tilde{\boldsymbol{\beta}} + \breve{\boldsymbol{\beta}}))(\boldsymbol{x} + \breve{\boldsymbol{x}})'^\flat + \tilde{\boldsymbol{L}}((\boldsymbol{x}' + \breve{\boldsymbol{x}}'), \cdot),
\end{aligned}
$$

where the metric, covariant derivative and Schouten tensor are those obtained from the solution to the Einstein field equations given in Corollary 1. These equations can be read as a system of ordinary differential equations for the fields $\breve{\boldsymbol{x}}$ and $\breve{\boldsymbol{\beta}}$. Because of the smoothness of the perturbed spacetime it follows that one can make use of the stability theory for ordinary differential equations —see e.g. [43], Theorem 2.1 in page 94 and Corollary 4.1 on page 101. In particular, these conformal geodesics will have the same existence interval as those in the background spacetime. Accordingly, it follows that $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is future $\tilde{\boldsymbol{g}}$-geodesically complete.

Remark 25. An alternative way of concluding the future geodesic completeness of the solutions to the Einstein field equations provided by Corollary 1 is to make use of the theory in [10] —see also Appendix A.2. By choosing the $\varepsilon > 0$ in condition (4.31) of Proposition 6 sufficiently small, it can be shown that the physical metric $\tilde{\boldsymbol{g}}$ satisfies the bounds required to show geodesic completeness.

3.7 The main result

We summarise the discussion of the preceding sections with a more detailed formulation of the main result of this chapter:

Theorem 1. *Let $\hat{\mathbf{u}}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ denote smooth initial data for the conformal evolution equations satisfying the conformal constraint equations on a hypersurface $\mathcal{S}_\star$. Then, there exists $\varepsilon > 0$ such that if*

$$||\breve{\mathbf{u}}_\star||_{\mathcal{S}_\star,m} < \varepsilon, \qquad m \geq 4$$

then there exists a unique C^{m-2} solution $\tilde{\boldsymbol{g}}$ to the vacuum Einstein field equation with positive Cosmological constant over $[\tau_\star, \infty) \times \mathcal{S}_\star$ for $\tau_\star > 0$ which is future geodesically complete and whose restriction to $\mathcal{S}_\star$ implies the initial data $\hat{\mathbf{u}}_\star$. Moreover, the solution $\hat{\mathbf{u}}$ remains suitably close (in the Sobolev norm $\|\cdot\|_{\mathcal{S},m}$) to the background solution $\mathring{\mathbf{u}}$.

Remark 26. It follows from Proposition 5 that there exists an open set of initial data for the Einstein field equations satisfying the hypothesis of the above theorem.

Chapter 4

The non-linear stability of the Cosmological region of the Schwarzschild-de Sitter spacetime

4.1 Introduction

One of the key problems in Mathematical Relativity is the non-linear stability of black hole spacetimes. This problem is challenging for its mathematical and physical features. Most efforts to establish the non-linear stability of black hole spacetimes in both the asymptotically flat and Cosmological settings have, so far, relied on the use of vector field methods —see e.g. [16]. The results in [23, 26, 81] show that the *conformal Einstein field equations* are a powerful tool for the analysis of the stability of the vacuum asymptotically simple spacetimes.

In view of the success of conformal methods to analyse the global properties of asymptotically simple spacetimes, it is natural to ask whether a similar strategy can be used to study the non-linear stability of black hole spacetimes. The discussion in this chapter is based on

M. Minucci and J. A. Valiente Kroon, *On the non-linear stability of the Cosmological* *region of the Schwarzschild-de Sitter spacetime*, ArXiv e-prints (2023), arXiv:2302.04004 [gr-qc].

which provides a first step in this direction by analysing certain aspects of the conformal structure of the sub-extremal Schwarzschild-de Sitter spacetime which can be used, in turn, to adapt techniques from the asymptotically simple setting to the black hole case.

4.1.1 The Schwarzschild-de Sitter spacetime

The Schwarzschild-de Sitter spacetime is a spherically symmetric solution to the vacuum Einstein field equations with positive cosmological constant. This spacetime depends on the de Sitter-like value of the cosmological constant λ and on the mass m of the black hole. Assuming spherical symmetry almost completely singles out the Schwarzschild-de Sitter spacetimes among the vacuum solutions to the Einstein field equations with de Sitter-like cosmological constraint. The other admissible solution is the Nariai spacetime —see e.g. [67]. In the Schwarzschild-de Sitter spacetime the relation between the mass and the Cosmological constant determines the position of the *Cosmological* and *black hole horizons* —see e.g. [42]. In this analysis, we restrict our attention to a choice of the parameters λ and m for which the exact solution is sub-extremal —see Section 4.2 for a definition of this notion. The sub-extremal Schwarzschild-de Sitter spacetime has three horizons. Of particular interest for our analysis is the Cosmological horizon which bounds a region (*the Cosmological region*) of the spacetime in which the roles of the coordinates t and r reversed. This spacetime can be studied by means of the extended conformal Einstein field equations —see [39]. In analogy to the de Sitter spacetime, the Cosmological region has an asymptotic region admitting a smooth conformal extension towards the future (or past) also known as future asymptotically de Sitter. Since the Cosmological constant takes a de Sitter-like value, the conformal boundary of the spacetime is spacelike and moreover, there exists a conformal representation in which the induced 3-metric on the conformal boundary $\mathscr{I}$ is homogeneous. Thus, it is possible to integrate the extended conformal field equations along single conformal geodesics —see [38].

In this chapter, we analyse the sub-extremal Schwarzschild-de Sitter spacetime as a solution to the extended conformal Einstein field equations and use the insights to prove existence and stability results. The starting point for this discussion is the analysis of conformal geodesic equations leaving spacelike hypersurfaces in the Cosmological region of the spacetime. The results of this analysis can be used to rewrite the spacetime in the conformal Gaussian gauge associated to these curves. Nevertheless, even though the conformal geodesic equations for spherically symmetric spacetimes can be written in quadratures, in general, the integral involved cannot be solved analytically. In view of this difficulty, the properties of the sub-extremal Schwarzschild-de Sitter spacetime are analysed by means of an initial value problem for the extended conformal Einstein field equations. Accordingly, initial data implied by the Schwarschild-de Sitter spacetime on a fiduciary spacelike hypersurface $\mathcal{S}_\star$ are used to analyse the behaviour of the conformal evolution equations. A perturbative argument then allows us to prove existence and stability results close to the conformal boundary and away from the asymptotic points where the Cosmological horizon intersects the conformal boundary. In particular, we show that small enough perturbations of initial data for the sub-extremal Schwarzschild-de Sitter spacetime give rise to a solution to the Einstein field equations which is regular at the conformal boundary. This analysis can be regarded as a first step towards a stability argument for perturbation data on the Cosmological horizons.

4.1.2 The main result

The analysis of the conformal properties of the Schwarzschild-de Sitter spacetime allows us to formulate a result concerning the existence of solutions to the initial value problem for the Einstein field equations with de Sitter-like Cosmological constant which can be regarded as perturbations of portions of the initial hypersurface at $\mathcal{S}_\star \equiv \{r = r_\star\}$ in the Cosmological region of the spacetime. In this region these hypersurfaces are spacelike and the coordinate t is spatial. In the following, let $\mathcal{R}_\bullet$ denote finite cylinder within $\mathcal{S}_\star$ for which $|t| < t_\bullet$ for some suitable positive constant $t_\bullet$. Let $D^+(\mathcal{R}_\bullet)$ denote the future domain of dependence of $\mathcal{R}_\bullet$. For the Schwarzschild-de Sitter spacetime such a region is unbounded towards the future and admits a smooth conformal extension with a spacelike conformal boundary.

Our main result can be stated as:

Main Result 2. *Given smooth initial data* $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ *for the vacuum Einstein field equations on* $\mathcal{R}_\bullet \subset \mathcal{S}_\star$ *which is suitably close (as measured by a suitable Sobolev norm) to the data implied by the Schwarzschild-de Sitter metric* $\mathring{\tilde{\boldsymbol{g}}}$ *in the Cosmological region of the spacetime, there exists a smooth metric* $\tilde{\boldsymbol{g}}$ *defined over the whole of* $D^+(\mathcal{R}_\bullet)$ *which is close to* $\mathring{\tilde{\boldsymbol{g}}}$*, solves the vacuum Einstein field equations with positive Cosmological constant and whose restriction to* $\mathcal{R}_\bullet$ *implies the initial data* $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$*. The metric* $\tilde{\boldsymbol{g}}$ *admits a smooth conformal extension which includes a spacelike conformal boundary.*

Remark 27. A detailed version of this theorem will be given in Section 4.5.

Observe that the above result is restricted to the future domain of dependence of a suitable portion $\mathcal{R}_\bullet$ of the spacelike hypersurface $\mathcal{S}_\star$. The reason for this restriction is the degeneracy of the conformal structure at the asymptotic points of the Schwarzschild-de Sitter spacetime where the conformal boundary, the Cosmological horizon and the singularity seem to "meet" —see [39]. In particular, at these points the background solution experiences a divergence of the Weyl curvature. This singularity is remarkably similar to that produced by the ADM mass at spatial infinity in asymptotically flat spacetimes —see e.g. [81], chapter 20. It is thus conceivable that an approach analogous to that used in the analysis of the problem of spatial infinity in [28] may be of help to deal with this singular behaviours of the conformal structure.

The ultimate aim of the programme started in this analysis is to obtain a proof of the stability of the Schwarzschild-de Sitter spacetime for data prescribed on the Cosmological horizon. Key to this end is the observation that the hypersurfaces of constant coordinate r, $\mathcal{S}_\star$, can be chosen to be arbitrarily close to the horizon. As such, an adaptation of the *optimal* local existence results for the characteristic initial value problem developed in [53] —see also [45]— should allow to evolve from the Cosmological horizon to a hypersurface $\mathcal{S}_\star$. These ideas will be developed in future work.

4.1.3 Related results

The non-linear stability of the Schwarzschild-de Sitter spacetime has been studied by means of the *vector field methods* that have proven successful in the analysis of asymptotically flat black holes —see e.g. [64, 65, 66]. An alternative approach has made use of *methods of microlocal analysis* in the steps of Melrose's school of geometric scattering —see [47, 46]. This type of analysis requires to be careful when discussing the behaviour of the solution at the horizons. In the initial value problem discussed in this chapter, the future domain of dependence of the solution is contained in the Cosmological region of the spacetime away from the asymptotic points. The methods developed in this work aim at providing a complementary approach to the non-linear stability of this Cosmological black hole spacetime. The interrelation between the results obtained in this chapter and those obtained by vector field methods and microlocal analysis will be discussed elsewhere.

4.2 The sub-extremal Schwarzschild-de Sitter spacetime

The purpose of this section is to discuss the key properties of the sub-extremal Schwarzschild-de Sitter spacetime that will be used in our argument on the stability of the Cosmological region of this exact solution.

4.2.1 Basic properties

The *Schwarzschild-de Sitter spacetime*, $(\tilde{\mathcal{M}}, \mathring{\tilde{\boldsymbol{g}}})$, is a spherically symmetric solution to the vacuum Einstein field equations with positive Cosmological constant

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}, \qquad \lambda > 0 \tag{4.1}$$

with $\tilde{\mathcal{M}} = \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2$ and line element given in *standard coordinates* (t, r, θ, φ) by

$$\mathring{\tilde{\boldsymbol{g}}} = -\left(1 - \frac{2m}{r} - \frac{\lambda}{3}r^2\right)\mathbf{d}t \otimes \mathbf{d}t + \left(1 - \frac{2m}{r} - \frac{\lambda}{3}r^2\right)^{-1}\mathbf{d}r \otimes \mathbf{d}r + r^2\boldsymbol{\sigma}, \tag{4.2}$$

where

$$\boldsymbol{\sigma} \equiv \mathbf{d}\theta \otimes \mathbf{d}\theta + \sin^2\theta \mathbf{d}\varphi \otimes \mathbf{d}\varphi,$$

denotes the standard metric on $\mathbb{S}^2$. The coordinates (t, r, θ, φ) take the range

$$t \in (-\infty, \infty), \qquad r \in (0, \infty), \qquad \theta \in (0, \pi), \qquad \varphi \in [0, 2\pi).$$

This line element can be rescaled so that

$$\mathring{\boldsymbol{g}} = -D(r)\mathbf{d}t \otimes \mathbf{d}t + \frac{1}{D(r)}\mathbf{d}r \otimes \mathbf{d}r + r^2\boldsymbol{\sigma}, \tag{4.3}$$

where

$$M \equiv 2m\sqrt{\frac{\lambda}{3}}$$

and

$$D(r) \equiv 1 - \frac{M}{r} - r^2.$$

In our conventions M, r and λ are dimensionless quantities.

4.2.2 Horizons and global structure

The location of the horizons of the Schwarzschild-de Sitter spacetime follows from the analysis of the zeros of the function $D(r)$ in the line element (4.3).

Since $\lambda > 0$, then the function $D(r)$ can be factorised as

$$D(r) = -\frac{1}{r}(r - r_b)(r - r_c)(r - r_-),$$

where r_b and r_c are, in general, distinct positive roots of $D(r)$ and r_- is a negative root. Moreover, one has that

$$0 < r_b < r_c, \qquad r_c + r_b + r_- = 0.$$

The root r_b corresponds to a black hole-type of horizon and r_c to a Cosmological de Sitter-like type of horizon. One can verify that

$$D(r) > 0 \qquad \text{for} \qquad r_b < r < r_c,$$

$$D(r) < 0 \qquad \text{for} \qquad 0 < r < r_b \qquad \text{and} \qquad r > r_c.$$

Accordingly, $\mathring{\boldsymbol{g}}$ is static in the region $r_b < r < r_c$ between the horizons. There are no other static regions outside this range.

Using Cardano's formula for cubic equations, we have

$$r_- = -\frac{2}{\sqrt{3}}\cos\left(\frac{\phi}{3}\right), \tag{4.4a}$$

$$r_b = \frac{1}{\sqrt{3}}\left(\cos\left(\frac{\phi}{3}\right) - \sqrt{3}\sin\left(\frac{\phi}{3}\right)\right), \tag{4.4b}$$

$$r_c = \frac{1}{\sqrt{3}}\left(\cos\left(\frac{\phi}{3}\right) + \sqrt{3}\sin\left(\frac{\phi}{3}\right)\right). \tag{4.4c}$$

where the parameter ϕ is defined through the relation

$$M = \frac{2\cos\phi}{3\sqrt{3}}, \qquad \phi \in \left(0, \frac{\pi}{2}\right). \tag{4.5}$$

In the sub-extremal case we have that $0 < M < 2/3\sqrt{3}$ and $\phi \in (0, \pi/2)$. This describes a black hole in a Cosmological setting. The extremal case corresponds to the value $\phi = 0$ for which $M = 2/3\sqrt{3}$ —in this case the Cosmological and black hole horizons coincide. Finally, the hyper-extremal case is characterised by the condition $M > 2/3\sqrt{3}$ —in this case the spacetime contains no horizons.

The Penrose diagram of the Schwarzschild-de Sitter is well known —see Figure 4.1. Details of its construction can be found in e.g. [42, 81].

4.2.3 Other coordinate systems

In our analysis, we will also make use of *retarded and advanced Eddington-Finkelstein null coordinates* defined by

$$u \equiv t - r^*, \qquad v \equiv t + r^*, \tag{4.6}$$

where r^* is the *tortoise coordinate* given by

$$r^*(t) \equiv \int \frac{\mathrm{d}r}{D(r)}, \qquad \lim_{r\to\infty} r^*(r) = 0. \tag{4.7}$$

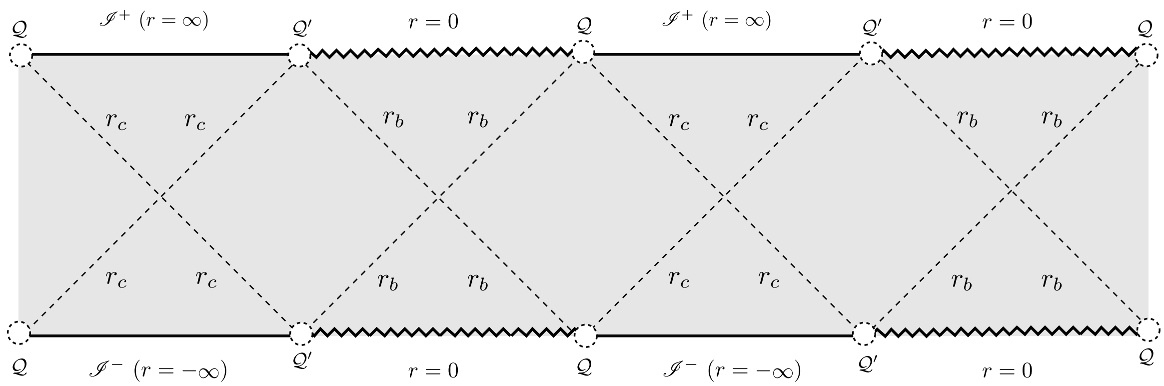

Figure 4.1: Penrose diagram of the sub-extremal Schwarzschild-de Sitter spacetime. The serrated line denotes the location of the singularity; the continuous black line denotes the conformal boundary; the dashed line shows the location of the black hole and Cosmological horizons denoted by $\mathcal{H}_b$ and $\mathcal{H}_c$ respectively. As described in the main text, these horizons are located at $r = r_b$ and $r = r_c$. The excluded points $\mathcal{Q}$ and $\mathcal{Q}'$ where the singularity seems to meet the conformal boundary correspond to asymptotic regions of the spacetime that does not belong to the singularity nor the conformal boundary.

It follows that u, $v \in \mathbb{R}$. In terms of these coordinates the metric $\mathring{\boldsymbol{\tilde{g}}}$ takes, respectively, the forms

$$\mathring{\boldsymbol{\tilde{g}}} = -D(r)\mathbf{d}u \otimes \mathbf{d}u + (\mathbf{d}u \otimes \mathbf{d}r + \mathbf{d}r \otimes \mathbf{d}u) + r^2\boldsymbol{\sigma},$$
$$\mathring{\boldsymbol{\tilde{g}}} = -D(r)\mathbf{d}v \otimes \mathbf{d}v + (\mathbf{d}v \otimes \mathbf{d}r + \mathbf{d}r \otimes \mathbf{d}v) + r^2\boldsymbol{\sigma}.$$

In order to compute the Penrose diagrams, Figures 4.2 and 4.3, we make use of *Kruskal coordinates* defined via

$$U \equiv \frac{1}{2}\exp(bu), \qquad V \equiv \frac{1}{2}\exp(bv)$$

where u and v are the Eddington-Finkelstein coordinates as defined in (4.6) and b is a constant which can be freely chosen. A further change of coordinates is provided by

$$T \equiv U + V, \qquad \Psi \equiv U - V.$$

These coordinates are related to r and t via

$$T(r,t) = \cosh(bt)\exp(br^*(r)), \qquad \Psi(r,t) = \sinh(bt)\exp(br^*(r)).$$

Then by recalling that

$$r_- < 0 < r_b < r_c \qquad \text{and} \qquad r_- + r_b + r_c = 0,$$

the equation of $r^*(r)$ as defined by (4.7) renders

$$r^*(r) = -\frac{r_b \ln(r - r_b)}{(r_b - r_c)(2r_b + r_c)} + \frac{r_c \ln(r - r_c)}{r_b^2 + r_b r_c - 2r_c^2} + \frac{(r_b + r_c)\ln(r + r_b + r_c)}{(2r_b + r_c)(r_b + 2r_c)}.$$

Hence, in order to have coordinates which are regular down to the Cosmological horizon, the constant b must be given by

$$b = \frac{r_b^2 + r_b r_c - 2r_c^2}{2r_c}.$$

4.3 Construction of a conformal Gaussian gauge in the Cosmological region

The hyperbolic reduction of the extended conformal Einstein field equations makes use of a conformal Gaussian gauge system —i.e. coordinates and frame are propagated along a suitable congruence of conformal geodesics. This congruence provides, in turn, a canonical representative of the conformal class of a solution to the Einstein field equations —see Proposition 3.

A class of non-intersecting conformal geodesics which cover the whole maximal extension of the sub-extremal Schwarzschild-de Sitter spacetime has been studied in [38]. The main outcome of the analysis in that reference is that the resulting congruence covers the whole maximal analytic extension of the spacetime and, accordingly, provides a global system of coordinates —modulo the usual difficulties with the prescription of coordinates on $\mathbb{S}^2$. This congruence is prescribed in terms of data prescribed on a Cauchy hypersurface of the spacetime. In this analysis, we are interested in the evolution of perturbations of the Schwarzschild-de Sitter spacetime from data prescribed on hypersurfaces of constant coordinate r in the Cosmological region of the spacetime. Thus, the congruence of conformal geodesics constructed in [38] is of no direct use to us. Consequently, in this section,

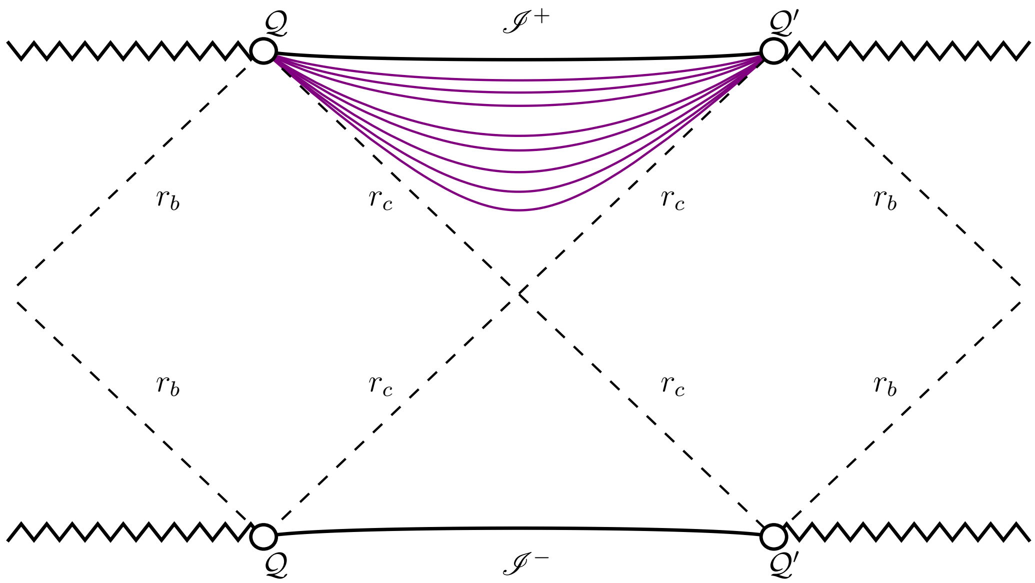

Figure 4.2: Hypersurfaces with constant r are plotted on the Penrose diagram of the Cosmological region of the sub-extremal Schwarzschild-de Sitter spacetime.

we study a class of conformal geodesics of the Schwarzschild-de Sitter spacetime which is prescribed in terms of data on hypersurfaces of constant r in the Cosmological region. These curves turn out to be geodesics of the physical metric $\tilde{\boldsymbol{g}}$ and intersect the conformal boundary orthogonally.

4.3.1 Basic setup

In the following, it is assumed that

$$r_c < r < \infty$$

corresponding to the Cosmological region of the Schwarzschild-de Sitter spacetime. Given a fixed $r = r_\star$ we denote by $\mathcal{S}_{r_\star}$ (or $\mathcal{S}_\star$ for short) the spacelike hypersurfaces of constant $r = r_\star$ in this region —see Figure 4.2. Points on $\mathcal{S}_\star$ can be described in terms of the coordinates (t, θ, φ).

Initial data for the congruence

In order to prescribe the congruence of conformal geodesics, we follow the general strategy outlined in Chapter 2 —see also [31, 38]. This requires prescribing the value of a conformal factor $\Theta_\star$ over $\mathcal{S}_\star$. We will only be interested on prescribing the data on compact subsets of $\mathcal{S}_\star$ so it is natural to require that

$$\Theta_\star = 1, \qquad \dot{\Theta}_\star = 0.$$

The second condition implies that the resulting conformal factor will have a time reflection symmetry with respect to $\mathcal{S}_\star$. Now, following [31, 38] we require that

$$\tilde{\boldsymbol{x}}'_\star \perp \mathcal{S}_\star, \qquad \tilde{\beta}_\star = \Theta_\star^{-1}\mathrm{d}\Theta_\star.$$

The latter, in turn, implies that

$$t = t_\star \qquad {t'}_\star = \frac{1}{\sqrt{D_\star}}, \qquad {r'}_\star = 0, \qquad \tilde{\beta}_{t\star} = 0, \qquad \tilde{\beta}_{r\star} = 0, \tag{4.8}$$

where $t_\star \in (-t_\bullet, t_\bullet)$ for some $t_\bullet \in \mathbb{R}^+$. Notice that the tangent vector $\tilde{x}'$ coincides with the future unit normal to $\tilde{\mathcal{S}}$.

Given a sufficiently large constant $t_\bullet$ we define

$$\mathcal{R}_\bullet = \{p \in \mathcal{S}_\star \mid t(p) \in (-t_\bullet, t_\bullet)\}.$$

The constant $t_\bullet$ will be assumed to be large enough so that $D^+(\mathcal{R}_\bullet) \cap \mathscr{I}^+ \neq \varnothing$.

The starting point of the curves on $\mathcal{S}_\star$ is prescribed in terms of the coordinates $(t, \theta, \varphi) = (t_\star, \theta_\star, \varphi_\star)$ The conditions (4.8) gives rise to a congruence of conformal geodesics which has a trivial behaviour in the angular coordinates —that is, it is spherically symmetric. In other words, this corresponds to effectively analysing the curves on a 2-dimensional manifold $\tilde{\mathcal{M}}/\mathrm{SO}(3)$ with quotient metric $\tilde{\boldsymbol{\ell}}$ given by

$$\tilde{\boldsymbol{\ell}} = -D(r)\mathbf{d}t \otimes \mathbf{d}t + D^{-1}(r)\mathbf{d}r \otimes \mathbf{d}r, \tag{4.9}$$

obtained upon re-writing the metric $\mathring{\tilde{\boldsymbol{g}}}$ as a warped product. Accordingly, the only non-trivial parameter characterising each curve of the congruence is $t_\star$.

The geodesic equations

It follows that for the initial data conditions (4.8) one has $\beta^2 = 0$ so that the resulting congruence of conformal geodesics is, after reparametrisation, a congruence of metric geodesics. This last observation simplifies the subsequent discussion. The $\tilde{\boldsymbol{g}}$-geodesic equations then imply that

$$r' = \sqrt{\gamma^2 - D(r)}, \qquad D(r)t'^2 - \frac{1}{D(r)}r'^2 = 1, \tag{4.10}$$

where γ is a constant. Evaluating at $\mathcal{S}_\star$ one readily finds that

$$t'_\star = \frac{|\gamma|}{|D_\star|}.$$

Observe that since we are in the Cosmological region of the spacetime we have that $D_\star < 0$. Moreover, the unit normal to $\mathcal{S}_\star$ is given by

$$\boldsymbol{n} = \left(\frac{1}{\sqrt{|D_\star|}}\right)\mathbf{d}r$$

while

$$\tilde{\boldsymbol{x}}'_\star = \tilde{r}'_\star \boldsymbol{\partial}_r + t'_\star \boldsymbol{\partial}_t.$$

So, it follows that $\tilde{\boldsymbol{x}}'_\star$ and $\boldsymbol{n}^\sharp$ are parallel if and only if $\gamma = 0$.

The conformal factor

In order to obtain simpler expressions we set $\lambda = 3$ and $\tau_\star = 0$. It follows then from formula (2.72) that one gets an explicit expression for the conformal factor. Namely, one has that

$$\Theta(\tau) = 1 - \frac{1}{4}\tau^2. \tag{4.11}$$

The roots of $\Theta(\tau)$ are given by

$$\tau_+ \equiv 2, \qquad \tau_- \equiv -2.$$

In the following, we concentrate on the root τ_+ corresponding to the location of the future conformal boundary $\mathscr{I}^+$. The relation with the physical proper time $\tilde{\tau}$ is obtained from equation (2.73), so that

$$\tilde{\tau} = 2\mathrm{arctanh}\left(\frac{\tau}{2}\right), \qquad \tau = 2\tanh\left(\frac{\tilde{\tau}}{2}\right). \tag{4.12}$$

From these expressions, we deduce that

$$\tau \to \tau_{\pm} = \pm 2, \qquad \text{as} \quad \tilde{\tau} \to \infty.$$

Moreover, the conformal factor Θ can be rewritten in terms of the $\tilde{\boldsymbol{g}}$-proper time $\tilde{\tau}$ as

$$\Theta(\tilde{\tau}) = \mathrm{sech}^2\left(\frac{\tilde{\tau}}{2}\right).$$

Remark 28. In [37] it has been shown that conformal geodesics in Einstein space will reach the conformal boundary orthogonally if and only if they are, up to a reparametrisation standard (metric) geodesics. In the present case, this property can be directly verified using equations (4.10).

4.3.2 Qualitative analysis of the behaviour of the curves

Having, in the previous subsection, set up the initial data for the congruence of conformal geodesics, in this subsection we analyse the qualitative behaviour of the curves. In particular, we show that the curves reach the conformal boundary in a finite amount of (conformal) proper time. Moreover, we also show that the curves do not intersect in the future of the initial hypersurface $\mathcal{S}_{\star}$.

Behaviour towards the conformal boundary

Recalling that

$$r' = \sqrt{|D(r)|} \tag{4.13}$$

and observing that $D(r) < 0$, it follows that if $r'_{\star} \neq 0$ then, in fact $r' > 0$. Moreover, one can show that $r''_{\star} > 0$ and that $r''_{\star} \neq 0$ for $r \in [r_{\star}, \infty)$. Thus, the curves escape to the conformal boundary.

Now, we show that the congruence of conformal geodesics reaches the conformal boundary in an infinite amount of the physical proper time. In order to see this, we observe that $D(r) < 0$, consequently from equation

$$r' = \pm\sqrt{|D(r)|}$$

it follows that $r(\tilde{\tau})$ is a monotonic function. Moreover, using equations

$$D(r) = -\frac{1}{r}(r - r_b)(r - r_-)(r - r_c)$$

and

$$t' = \frac{|\gamma + \beta r|}{|D(r)|} = 0$$

we find that

$$\tilde{\tau} = \int_{r_\star}^{r} \sqrt{\frac{\bar{r}}{(\bar{r} - r_b)(\bar{r} - r_c)(\bar{r} - r_-)}}\mathrm{d}\bar{r}.$$

It is possible to rewrite this integral in terms of elliptic functions —see e.g. [51]. More precisely, one has that

$$\tilde{\tau} = \frac{2r_\star}{\alpha^2\sqrt{r_\star(\alpha_+ - \alpha_-)}}\bigg(\kappa^2\mathrm{w} + (\alpha^2 - \kappa^2)\Pi[\phi, \alpha^2, \kappa]\bigg), \tag{4.14}$$

where $\Pi[\phi, \alpha^2, \kappa]$ is the incomplete elliptic integral of the third kind and

$$\mathrm{sn}^2\mathrm{w} = \left(\frac{r_c - r_-}{r_b - r_-}\right)\left(\frac{r - r_b}{r - r_c}\right), \qquad \alpha^2 \equiv \frac{r_b - r_-}{r_c - r_-},$$

$$\kappa^2 \equiv \frac{r_c(r_b - r_-)}{r_\star(r_c - r_-)}, \qquad \phi \equiv \arcsin(\mathrm{snw}),$$

with sn denotes the Jacobian elliptic function. From the previous expressions and the general theory of elliptic functions it follows that $\tilde{\tau}(r, r_\star)$ as defined by Equation (4.14) is an analytic function of its arguments. Moreover, it can be verified that

$$\tilde{\tau} \to \infty \qquad \text{as} \quad r \to \infty.$$

Accordingly, as expected, the curves escape to infinity in an infinite amount of physical proper time $\tilde{\tau}$. Using the reparametrisation formulae (4.12) the latter corresponds to a finite amount of unphysical proper time τ.

The deviation equations

To analyse whether the congruence of conformal geodesics develops conjugate points, one introduces a family of conformal geodesics, denoted by $x(\tau,\sigma)$ and $\boldsymbol{\beta}(\tau,\sigma)$, depending smoothly on a parameter σ and with tangent vector field $\partial_\tau x = \dot{x}$. Then, let

$$\boldsymbol{z} \equiv \partial_\sigma \boldsymbol{x}, \qquad \boldsymbol{\omega} \equiv \tilde{\nabla}_{\boldsymbol{z}} \boldsymbol{\beta}.$$

The fields $\boldsymbol{z}$ and $\boldsymbol{\omega}$ denote, respectively, the *deviation vector field* and the *deviation 1-form*—see [31]. The *conformal Jacobi equation* and the *1-form deviation equation* are given by

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{z} = \tilde{\boldsymbol{R}}(\dot{\boldsymbol{x}},\boldsymbol{z})\dot{\boldsymbol{x}} - \boldsymbol{S}(\boldsymbol{\omega};\dot{\boldsymbol{x}},\dot{\boldsymbol{x}}) - 2\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{z}), \tag{4.15a}$$

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{\omega} = -\boldsymbol{\beta}\cdot\tilde{\boldsymbol{R}}(\dot{\boldsymbol{x}},\boldsymbol{z}) + \frac{1}{2}\bigg(\boldsymbol{\omega}\cdot\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\cdot) + \boldsymbol{\beta}\cdot\boldsymbol{S}(\boldsymbol{\omega};\dot{\boldsymbol{x}},\cdot) + \boldsymbol{\beta}\cdot\boldsymbol{S}(\boldsymbol{\beta};\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{z},\cdot)\bigg), \tag{4.15b}$$

where $\tilde{\boldsymbol{R}}(\cdot,\cdot)$ denotes the Riemann tensor of the metric $\tilde{\boldsymbol{g}}$ and

$$\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\boldsymbol{y}) \equiv \langle\boldsymbol{\beta},\dot{\boldsymbol{x}}\rangle\boldsymbol{y} + \langle\boldsymbol{\beta},\boldsymbol{y}\rangle\dot{\boldsymbol{x}} - \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}},\boldsymbol{y})\boldsymbol{\beta}^\sharp,$$
$$\boldsymbol{\omega}\cdot\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\cdot) \equiv \langle\boldsymbol{\omega},\dot{\boldsymbol{x}}\rangle\boldsymbol{\beta} + \langle\boldsymbol{\beta},\dot{\boldsymbol{x}}\rangle\boldsymbol{\omega} - \tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\omega},\boldsymbol{\beta})\dot{\boldsymbol{x}}^\flat.$$

To compute the $\tilde{\boldsymbol{g}}$-adapted version of the conformal geodesics one introduces a reparametrisation $\tilde{\boldsymbol{x}} \equiv \boldsymbol{x}(\tilde{\tau},\sigma)$ of $\boldsymbol{x}(\tau,\sigma)$ in terms of the physical proper time $\tilde{\tau}$. Furthermore, let

$$\tilde{\boldsymbol{z}} \equiv \partial_\sigma \tilde{\boldsymbol{x}}, \qquad \tilde{\boldsymbol{\omega}} \equiv \tilde{\nabla}_{\tilde{\boldsymbol{z}}}\tilde{\boldsymbol{\beta}}.$$

In terms of this new variables one has that Equations (4.15a) and (4.15b) read as

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\boldsymbol{z}} = \tilde{R}(\tilde{\boldsymbol{x}}',\tilde{\boldsymbol{z}})\tilde{\boldsymbol{x}}' + \tilde{\boldsymbol{\omega}}^\sharp, \tag{4.17a}$$

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\boldsymbol{\omega}} = -\tilde{\boldsymbol{\beta}}\cdot\tilde{R}(\tilde{\boldsymbol{x}}',\tilde{\boldsymbol{z}}) + \tilde{\boldsymbol{x}}'^\flat\tilde{\nabla}_{\tilde{\boldsymbol{z}}}\boldsymbol{\beta}^2 + \boldsymbol{\beta}^2\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\boldsymbol{z}}^\beta. \tag{4.17b}$$

Moreover, it is possible to show that $\tilde{\nabla}_{\tilde{\boldsymbol{z}}}\boldsymbol{\beta}^2$ is constant along a given conformal geodesic.

Now, by considering the 2-dimensional metric $\tilde{\boldsymbol{\ell}}$ as given by (4.9), one has that the $\tilde{\boldsymbol{g}}$-adapted deviation equations (4.17a) and (4.17b) are equivalent to each other and to equation

$$\not{D}_{\tilde{x}'}\not{D}_{\tilde{x}'}\tilde{\boldsymbol{z}} = \frac{1}{2}R[\tilde{\boldsymbol{\ell}}]\epsilon_{\tilde{\boldsymbol{\ell}}}(\tilde{\boldsymbol{x}}',\tilde{\boldsymbol{z}})\epsilon_{\tilde{\boldsymbol{\ell}}}(\tilde{\boldsymbol{x}}',\cdot)^\sharp \pm (\not{D}_{\tilde{z}}\boldsymbol{\beta}\epsilon_{\tilde{\boldsymbol{\ell}}}(\tilde{\boldsymbol{x}}',\cdot)^\sharp + \boldsymbol{\beta}\epsilon_{\tilde{\boldsymbol{\ell}}}(\not{D}_{\tilde{x}'}\tilde{\boldsymbol{z}},\cdot)), \tag{4.18}$$

where $\not{D}$ denotes the Levi-Civita covariant derivative of $\tilde{\boldsymbol{\ell}}$, $\epsilon_{\tilde{\boldsymbol{\ell}}}$ is the volume form of $\tilde{\boldsymbol{\ell}}$ and $R[\tilde{\boldsymbol{\ell}}]$ denotes the Ricci scalar of $\tilde{\boldsymbol{\ell}}$. For conformal curves satisfying the initial conditions (4.8) the question of whether the deviation vector field $\tilde{\boldsymbol{z}}$ is non-vanishing can be rephrased in terms of a similar question for the scalar

$$\tilde{\omega} \equiv \epsilon_{\tilde{\boldsymbol{\ell}}}(\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{z}}). \tag{4.19}$$

Then, by replacing this definition in Equation (4.18), one has

$$\not{D}_{\tilde{x}'} \not{D}_{\tilde{x}'}\tilde{\omega} = \left(\boldsymbol{\beta}^2 + \frac{1}{2}R[\tilde{\boldsymbol{\ell}}]\right)\tilde{\omega} + \not{D}_{\tilde{z}}\boldsymbol{\beta}. \tag{4.20}$$

Analysis of the behaviour of the conformal deviation equation

In the previous section, it has been shown that for congruences of conformal geodesics in this spherically symmetric spacetime, the behaviour of the deviation vector of the congruence can be understood by considering the evolution of the scalar $\tilde{\omega}$ —see also [31, 38]. If this scalar does not vanish, then the congruence is non-intersecting. Since in the present case one has $\boldsymbol{\beta} = 0$ and $R[\tilde{\boldsymbol{\ell}}] = -\partial_r{}^2 D(r)$, it follows that the evolution equation (4.20) takes the form

$$\frac{\mathrm{d}^2\tilde{\omega}}{\mathrm{d}\tilde{\tau}^2} = \left(1 + \frac{M}{r^3}\right)\tilde{\omega}, \qquad r \equiv r(\tilde{\tau}, r_\star).$$

Since in our setting $r \geq r_\star > r_c$, it follows that

$$1 + \frac{M}{r^3} > 1,$$

from where, in turn, one obtains the inequality

$$\frac{\mathrm{d}^2\tilde{\omega}}{\mathrm{d}\tilde{\tau}^2} > \tilde{\omega}.$$

Accordingly, the scalars $\tilde{\omega}$ and $\omega \equiv \Theta\tilde{\omega}$ satisfy the inequalities

$$\tilde{\omega} \geq \bar{\omega}, \qquad \omega \geq \Theta\bar{\omega},$$

where $\bar{\omega}$ is the solution of

$$\frac{\mathrm{d}^2\bar{\omega}}{\mathrm{d}\tilde{\tau}^2} = \bar{\omega}, \qquad \bar{\omega}(0, \rho_\star) = \frac{r_\star}{\rho_\star}, \qquad \bar{\omega}'(0, \rho_\star) = 0.$$

The solution to this last differential equation is given by

$$\bar{\omega} = (r_\star/\rho_\star)\cosh\tilde{\tau}.$$

Using equations (4.11) and (4.12) we get the inequality

$$\omega \geq \left(1 - \frac{\tau^2}{4}\right)\frac{r_\star}{\rho_\star}\cosh\left(2\mathrm{arctanh}\left(\frac{\tau}{2}\right)\right) = \frac{r_\star}{\rho_\star}\left(1 + \frac{\tau^2}{4}\right) > 0.$$

Consequently, we get the limit

$$\lim_{\tau \to \pm 2} \omega \geq \frac{2r_\star}{\rho_\star} > 0.$$

Hence, we conclude that the geodesics with $r_\star > r_\bullet$ which go to the conformal boundary $\mathscr{I}^+$ located at $\tau = 2$ do not develop any caustics.

The discussion of the previous paragraphs can be summarised in the following:

Proposition 8. *The congruence of conformal geodesics given by the initial conditions* (4.8) *leaving the initial hypersurface $\mathcal{S}_\star$ reach the conformal boundary $\mathscr{I}^+$ without developing caustics.*

The content of this Proposition can be visualised in Figure 4.3.

4.3.3 Estimating the size of $D^+(\mathcal{R}_\bullet)$

Up to this point the size of the domain $\mathcal{R}_\bullet \subset \mathcal{S}_\star$ (or more precisely, the value of the constant $t_\bullet$ has remained unspecified). An inspection of the Penrose diagram of the Schwarzschild-de Sitter spacetime shows that if the value of $t_\bullet$ is too small, it could happen that the future domain of dependence $D^+(\mathcal{R}_\bullet)$ is bounded and, accordingly, will not reach the spacelike conformal boundary $\mathscr{I}^+$ —see e.g. Figure 4.4. Given our interest in constructing perturbations of the Schwarzschild-de Sitter spacetime which contain as much as possible of the conformal boundary it is then necessary to ensure that $t_\bullet$ is sufficiently large. In this subsection given a fiduciary hypersurface $\mathcal{S}_\star$ in the Cosmological region of the spacetime, we provide an estimate of how large should $t_\bullet$ for $D^+(\mathcal{R}_\bullet)$ to be unbounded. In order to obtain

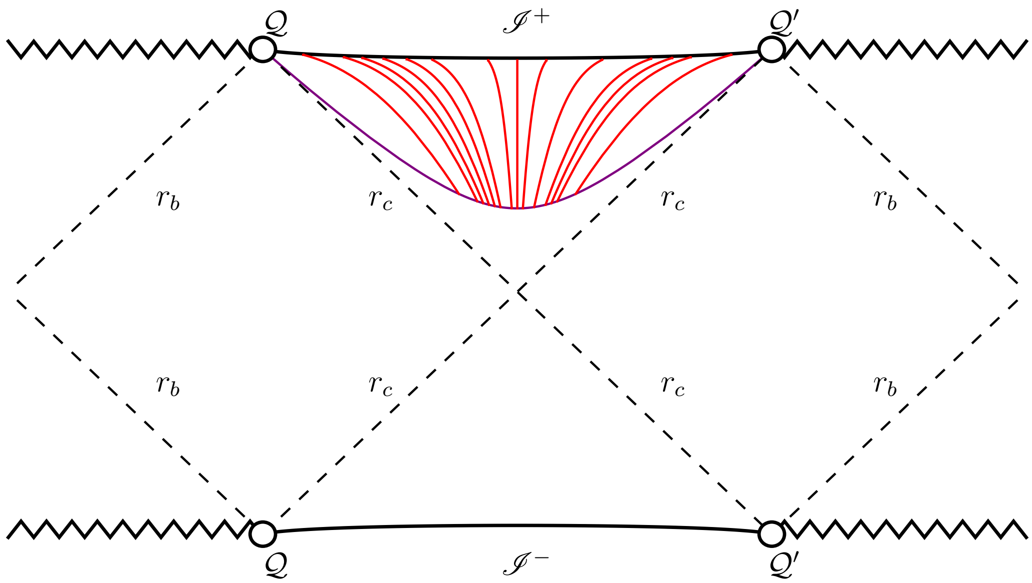

Figure 4.3: The conformal geodesics are plotted on the Penrose diagram of the Cosmological region of the sub-extremal Schwarzschild-de Sitter spacetime. The purple line represents the initial hypersurface $\mathcal{S}_\star$ corresponding to $r = r_\star$. The red lines represent conformal geodesics with constant time leaving this initial hypersurface. The curves are computed by setting $\lambda = 3$ and $\phi = \frac{\pi}{4}$.

this estimate we consider the future-oriented inward-pointing null geodesics emanating from the end-points of $\mathcal{R}_\bullet$ and look at where these curves intersect the conformal boundary.

In order to carry out the analysis in this subsection it is convenient to consider the coordinate $z \equiv 1/r$. In terms of this new coordinate, the line element (4.3) takes the form

$$\mathring{\tilde{\boldsymbol{g}}} = \frac{1}{z^2}\bigg(- F(z)\mathbf{d}t \otimes \mathbf{d}t + \frac{1}{F(z)}\mathbf{d}z \otimes \mathbf{d}z + \boldsymbol{\sigma}\bigg),$$

where

$$F(z) \equiv z^2 D(1/z).$$

The above expression suggest defining an *unphysical metric* $\bar{\boldsymbol{g}}$ via

$$\bar{\boldsymbol{g}} = \Xi^2 \mathring{\tilde{\boldsymbol{g}}}, \qquad \Xi \equiv z.$$

More precisely, one has

$$\bar{\boldsymbol{g}} = -F(z)\mathbf{d}t \otimes \mathbf{d}t + \frac{1}{F(z)}\mathbf{d}z \otimes \mathbf{d}z + \boldsymbol{\sigma}. \tag{4.21}$$

In order to study the null geodesics we consider the Lagrangian

$$\mathcal{L} = -F(z)\dot{t}^2 + \frac{1}{F(z)}\dot{z}^2,$$

where $\cdot \equiv \frac{d}{ds}$. In the case of null conformal conformal geodesics $\mathcal{L} = 0$ so that

$$\dot{t} = \pm\frac{1}{F(z)}\dot{z}.$$

This, in turn, means that

$$\frac{dt}{dz}\dot{z} = \pm\frac{1}{F(z)}\dot{z}.$$

By integrating both sides it follows that

$$\int_{t_\bullet}^{t_+} dt = \pm\int_{z_\star}^{0} \frac{1}{F(z)}dz,$$

where t_+ denotes the value of the (spacelike) coordinate t at which the null geodesic reaches $\mathscr{I}^+$. Accordingly for the inward-pointing light rays emanating from the points on $\mathcal{S}_\star$ defined by the condition $t = t_\bullet$ one has that

$$t_+ = t_\bullet - \int_0^{z_\star} \frac{1}{F(z)}dz. \tag{4.22}$$

An analogous condition holds for the inward-pointing light rays emanating from the points with $t = -t_\bullet$. Since in the Cosmological region $F(z) > 0$ it follows that

$$\int_0^{z_\star} \frac{1}{F(z)}dz > 0.$$

The key observation in the analysis in this subsection is the following: $D^+(\mathcal{R}_\bullet)$ is unbounded (so that it intersects the conformal boundary) if t_+ as given by Equation (4.22) satisfies $t_+ > 0$. As $t_\bullet > 0$, having $t_+ < 0$ would mean that the light rays emanating from the points with $t = t_\bullet$ and $t = -t_\bullet$ intersect before reaching $\mathscr{I}^+$. Now, the condition $t_+ > 0$ implies, in turn, that

$$t_\bullet > \int_0^{z_\star} \frac{1}{F(z)}dz.$$

As the integral in the right-hand side of the above inequality is not easy to compute we provide, instead, a lower bound. One has then that

$$t_\bullet > \frac{z_\star}{F_\circledast},$$

where $F_{\circledast}$ denotes the maximum of

$$F(z) = z^2 - Mz^3 + 1$$

over the interval $[0, z_\star]$. Thus, $F'(z)$ vanishes if $z = 0$ or $z = z_\odot \equiv 2/3M$. Also, notice that $F'(z) > 0$ for $z \approx 0$. It can be readily verified that $F''(0) > 0$ while $F''(2/3M) < 0$ so that an inflexion point occurs in the interval $(0, z_\odot)$ and there are no other inflexion points in $[0, z_\star]$. Now, looking at the definition of M, equation (4.4c), and the expression for r_c as given by equation (4.5) one concludes that $z_\odot > z_c \equiv 1/r_c$. As $z_\odot$ is independent of $z_\star$, it is not possible to decide whether $z_\odot$ lies in $[0, z_\star]$ or not. In any case, one has that

$$F(z_\odot) = 1 + \frac{4}{27M^2} \geq F_{\circledast},$$

so that

$$t_\bullet > \frac{27M^2 z_\star}{27M^2 + 4}. \tag{4.23}$$

One can summarise the discussion in this subsection as follows:

Lemma 6. *If condition* (4.23) *holds then* $D^+(\mathcal{R}_\bullet)$ *is unbounded.*

Remark 29. In the rest of this analysis it is assumed that condition (4.23) always holds.

4.3.4 Conformal Gaussian coordinates in the sub-extremal Schwarzschild-de Sitter spacetime

We now combine the results of the previous subsections to show that the congruence of conformal geodesics defined by the initial conditions (4.8) can be used to construct a *conformal Gaussian coordinate system* in a domain in the chronological future of $\mathcal{R}_\bullet \subset \mathcal{S}_\star$, $J^+(\mathcal{R}_\bullet \subset \mathcal{S}_\star)$, containing a portion of the conformal boundary $\mathscr{I}^+$.

In the following let $\widetilde{SdS}_I$ denote the Cosmological region of the Schwarzschild-de Sitter spacetime —that is

$$\widetilde{SdS}_I = \{p \in \tilde{\mathcal{M}} \mid r(p) > r_c\}.$$

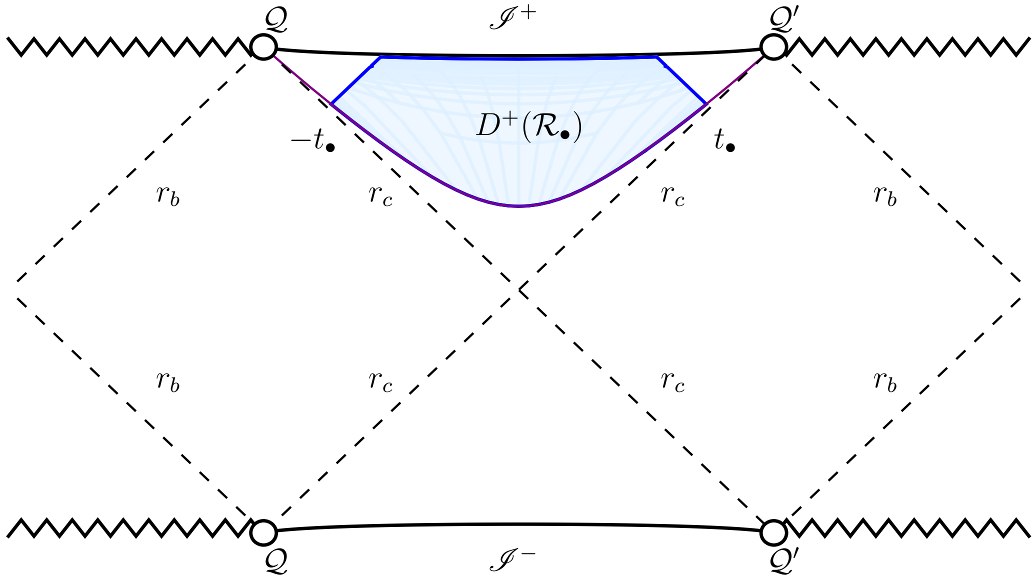

Figure 4.4: The plotted future domain of dependence of the solution $D^+(\mathcal{R}_\bullet)$ on the Penrose diagram of the Cosmological region of the sub-extremal Schwarzschild-de Sitter spacetime. The value of $t_\bullet$ can be chosen as close as possible to the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ so as to satisfy condition (4.23).

Moreover, denote by SdS_I the conformal representation of $\widetilde{SdS}_I$ defined by the conformal factor Θ defined by the non-singular congruence of conformal geodesics given by Proposition 8. For $r > r_c$ let $z \equiv 1/r$ —cfr the line element (4.21). In terms of these coordinates, one has that

$$SdS_I = \{p \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^2 \mid 0 \leq z(p) \leq z_\star\} \tag{4.24}$$

where $z_\star \equiv 1/r_\star$ with $r_\star > r_c$. In particular, the conformal boundary, $\mathscr{I}^+$, corresponds to the set of points for which $z = 0$.

The analysis of the previous subsections shows that the conformal geodesics defined by the initial conditions (4.8) can be thought of as curves on SdS_I of the form

$$(\tau, t_\star) \mapsto \Big(t(\tau, t_\star), z(\tau, t_\star), \theta_\star, \varphi_\star\Big).$$

Thus, in particular, the congruence of curves defines a map

$$\psi : [0, 2] \times [-t_\bullet, t_\bullet] \to [0, z_\star] \times [-t_\bullet, t_\bullet].$$

This map is analytic in the parameters $(\tau, t_\star)$. Moreover, the fact that the congruence of conformal geodesics is non-intersecting implies that the map is, in fact, invertible —the analysis of the conformal geodesic deviation equation implies that the Jacobian of the transformation is non-zero for the given value of the parameters. In particular, it can be readily verified that the function $\Theta\tilde{\omega}$ coincides with the Jacobian of the transformation. Accordingly, the inverse map ψ^{-1}

$$\psi^{-1} : [0, z_\star] \times [-t_\bullet, t_\bullet] \to [0, 2] \times [-t_\bullet, t_\bullet], \qquad (t, z) \mapsto \big(\tau(t, z), t_\star(t, z)\big)$$

is well-defined. Thus, ψ^{-1} gives the transformation from the *standard Schwarzschild coordinates* (t, z, θ, φ) into the *conformal Gaussian coordinates* $(\tau, t_\star, \theta, \varphi)$. In the following let

$$\mathcal{M}_\bullet \equiv [0, 2] \times [-t_\bullet, t_\bullet].$$

As the conformal geodesics of our congruence are timelike, we have that

$$\mathcal{M}_\bullet \subset J^+(\mathcal{R}_\bullet).$$

All throughout we assume, as discussed in Subsections 4.3.1 and 4.3.3, that $t_\bullet$ is sufficiently large to ensure that $D^+(\mathcal{R}_\bullet)$ contains a portion of $\mathscr{I}^+$ —cfr Lemma 6.

Proposition 9. *The congruence of conformal geodesics on SdS_I defined by the initial conditions on $\mathcal{S}_\star$ given by* (4.8) *induce a conformal Gaussian coordinate system over $D^+(\mathcal{R}_\bullet)$ which is related to the standard coordinates (t, r) via an analytic map.*

4.4 The Schwarzschild-de Sitter spacetime in the conformal Gaussian system

In the previous section, we have established the existence of conformal Gaussian coordinates in the domain $\mathcal{M}_\bullet \subset SdS_I$ of the Schwarzschild-de Sitter spacetime. In this section, we proceed to analyse the properties of this exact solution in these coordinates. This analysis is focused on the structural properties relevant for the analysis of stability in the latter parts of this article.

Remark 30. The metric coefficients implied by the line element (4.21) are analytic functions of the coordinates in the region $\mathcal{M}_\bullet$ —barring the usual degeneracy of spherical coordinates.

4.4.1 Weyl propagated frames

The ultimate aim of this section is to cast the Schwarzschild-de Sitter spacetime in the region $\mathcal{M}_\bullet$ as a solution to the extended conformal Einstein field equations introduced in Section 2.4.3. A key step in this construction is the use of a Weyl propagated frame. In this section, we discuss a class of these frames in $\mathcal{M}_\bullet$.

Since the congruence of conformal geodesics implied by the initial data (4.8) satisfies $\tilde{\boldsymbol{\beta}} = 0$, the Weyl propagation equation (2.79) reduces to the usual parallel propagation equation —that is,

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}(\Theta\tilde{\boldsymbol{e}}_{\boldsymbol{a}}) = \tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\boldsymbol{e}_{\boldsymbol{a}} = 0. \tag{4.25}$$

The subsequent computations can be simplified by noticing that the line element (4.3) is in warped-product form. Given the spherical symmetry of the Schwarzschild-de Sitter spacetime, most of the discussion of a frame adapted to the symmetry of the spacetime can be carried out by considering the 2-dimensional Lorentzian metric

$$\begin{aligned}\boldsymbol{\ell} &= \ell_{AB}\mathbf{d}x^A \otimes \mathbf{d}x^B \\ &= -D(r)\mathbf{d}t \otimes \mathbf{d}t + \frac{1}{D(r)}\mathbf{d}r \otimes \mathbf{d}r.\end{aligned}$$

In the spirit of a conformal Gaussian system, we begin by setting the *time leg* of the frame as $\boldsymbol{e}_{\boldsymbol{0}} = \dot{\boldsymbol{x}}$. Then since

$$\dot{\boldsymbol{x}} = \Theta^{-1}\tilde{\boldsymbol{x}}',$$

it follows that

$$\boldsymbol{e}_{\boldsymbol{0}} = \Theta^{-1}\tilde{\boldsymbol{x}}'.$$

Now, recall that

$$\tilde{\boldsymbol{x}}' = \tilde{t}'\boldsymbol{\partial}_t + \tilde{r}'\boldsymbol{\partial}_r, \qquad \tilde{t} = t(\tilde{\tau}), \quad \tilde{r} = r(\tilde{\tau}),$$

and let

$$\boldsymbol{\omega} \equiv \epsilon_{\boldsymbol{\ell}}(\tilde{\boldsymbol{x}}', \cdot).$$

It follows then that $\langle \boldsymbol{\omega}, \tilde{\boldsymbol{x}}' \rangle = 0$ so that it is natural to consider a *radial leg* of the frame, $\boldsymbol{e_1}$, which is proportional to $\boldsymbol{\omega}^\sharp$. By using the condition $\boldsymbol{\ell}(\boldsymbol{e_1}, \boldsymbol{e_1}) = 1$ one readily finds that

$$\boldsymbol{e_1} = \Theta \boldsymbol{\omega}^\sharp.$$

It can be readily verified by a direct computation that the vector $\boldsymbol{e_1}$ as defined above satisfies the propagation equation (4.25).

Finally, the vectors $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$ are chosen in such a way that they span the tangent space of the 2-spheres associated to the orbits of the spherical symmetry. Accordingly, by setting

$$\boldsymbol{e_2} = e_{\boldsymbol{2}}{}^{\mathcal{A}} \boldsymbol{\partial}_{\mathcal{A}}, \qquad \boldsymbol{e_3} = e_{\boldsymbol{3}}{}^{\mathcal{A}} \boldsymbol{\partial}_{\mathcal{A}}, \qquad \mathcal{A} = 2,\, 3,$$

it follows readily from the warped-product structure of the metric that

$$\tilde{x}'^{A}(\partial_A e_{\boldsymbol{2}}{}^{\mathcal{A}}) = \tilde{x}'^{A}(\partial_A e_{\boldsymbol{3}}{}^{\mathcal{A}}) = 0.$$

In other words, one has that the frame coefficients $e_{\boldsymbol{2}}{}^{\mathcal{A}}$ and $e_{\boldsymbol{3}}{}^{\mathcal{A}}$ are constant along the conformal geodesics. Thus, in order to complete the Weyl propagate frame $\{\boldsymbol{e_a}\}$ we choose *two arbitrary orthonormal vectors* $\tilde{\boldsymbol{e}}_{\boldsymbol{2}\star}$ and $\tilde{\boldsymbol{e}}_{\boldsymbol{3}\star}$ spanning the tangent space of $\mathbb{S}^2$ and define vectors $\{\boldsymbol{e_2}, \boldsymbol{e_3}\}$ on $\mathcal{M}_{\bullet}$ by extending (constantly) the value of the associated coefficients $\left(e_{\boldsymbol{2}}{}^{\mathcal{A}}\right)_\star$ and $\left(e_{\boldsymbol{3}}{}^{\mathcal{A}}\right)_\star$ along the conformal geodesic.

The analysis of this subsection can be summarised in the following:

Proposition 10. *Let $\tilde{\boldsymbol{x}}'$ denote the vector tangent to the conformal geodesics defined by the initial data* (4.8) *and let $\{\boldsymbol{e}_{\boldsymbol{2}\star},\, \boldsymbol{e}_{\boldsymbol{3}\star}\}$ be an arbitrary orthonormal pair of vectors spanning the tangent bundle of $\mathbb{S}^2$. Then the frame $\{\boldsymbol{e_0},\, \boldsymbol{e_1},\, \boldsymbol{e_2},\, \boldsymbol{e_3}\}$ obtained by the procedure described in the previous paragraphs is a $\boldsymbol{g}$-orthonormal Weyl propagated frame. The frame depends analytically on the unphysical proper time τ and the initial position $t_\star$ of the curve.*

Remark 31. In the previous proposition we ignore the usual complications due to the non-existence of a globally defined basis of $T\mathbb{S}^2$. The key observation is that any local choice works well.

4.4.2 The Weyl connection

The connection coefficients associated to a conformal Gaussian gauge consists of two pieces: the 1-form defining the Weyl connection and the Levi-Civita connection of the metric $\bar{\boldsymbol{g}}$. We analyse these two pieces in turn.

The 1-form associated to the Weyl connection

We start by recalling that in Section 4.3 a congruence of conformal geodesics with data prescribed on the hypersurface $\mathcal{S}_\star$ was considered. This congruence was analysed using the $\tilde{\boldsymbol{g}}$-adapted conformal geodesic equations. The initial data for this congruence was chosen so that the curves with tangent given by $\tilde{\boldsymbol{x}}'$ satisfy the standard (affine) geodesic equation. Consequently, the (spatial) 1-form $\tilde{\boldsymbol{\beta}}$ vanishes. Thus, the 1-form $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta} = -\dot{\Theta}\tilde{\boldsymbol{x}}'^{\flat},$$

—cfr. equation (2.75). Now, recalling that $\tilde{\boldsymbol{x}}' = r'\boldsymbol{\partial}_r$ and observing equation (4.13) one concludes that

$$\tilde{\boldsymbol{x}}'^{\flat} = \frac{1}{|\sqrt{D(r)}|}\mathbf{d}r.$$

Rewritten in terms of z, the latter gives

$$\tilde{\boldsymbol{x}}'^{\flat} = -\frac{1}{z\sqrt{|F(z)|}}\mathbf{d}z.$$

As $F(0) = 1$, and $\dot{\Theta}|_{\mathscr{I}^+} = -1$ (cfr. equation (4.11)), it then follows that

$$\boldsymbol{\beta} \approx -\frac{1}{z}\mathbf{d}z \qquad \text{for} \quad z \approx 0.$$

That is, $\boldsymbol{\beta}$ is singular at the conformal boundary. However, in the subsequent analysis the key object is not $\boldsymbol{\beta}$ but $\bar{\boldsymbol{\beta}}$, the 1-form associated to the conformal geodesics equations written in terms of the connection $\bar{\boldsymbol{\nabla}}$. Now, from the conformal transformation rule $\bar{\boldsymbol{\beta}} = \boldsymbol{\beta} + \Xi^{-1}\mathbf{d}\Xi$ and by recalling that $\Xi = z$, it follows that

$$\bar{\boldsymbol{\beta}} = \frac{\dot{\Theta}}{z\sqrt{|F(z)|}}\mathbf{d}z + \frac{1}{z}\mathbf{d}z.$$

Thus, from the preceding discussion it follows that $\bar{\boldsymbol{\beta}}$ is smooth at $\mathscr{I}^+$ and, moreover, $\bar{\boldsymbol{\beta}}|_{\mathscr{I}^+} = 0$. Notice, however, that $\bar{\boldsymbol{\beta}} \neq 0$ away from the conformal boundary.

Computation of the connection coefficients

The 1-form $\boldsymbol{\beta}$ defines in a natural way a Weyl connection $\hat{\boldsymbol{\nabla}}$ via the relation

$$\hat{\boldsymbol{\nabla}} - \tilde{\boldsymbol{\nabla}} = \mathbf{S}(\boldsymbol{\beta})$$

where $\mathbf{S}$ corresponds to the tensor $S_{ab}{}^{cd}$ as defined in (2.44). As the coordinates and connection coefficients associated to the physical connection $\tilde{\boldsymbol{\nabla}}$ are not well adapted to a discussion near the conformal boundary we resort to the unphysical Levi-Civita connection $\bar{\boldsymbol{\nabla}}$ to compute $\hat{\boldsymbol{\nabla}}$. From the discussion in the previous subsections, we have that

$$\bar{\boldsymbol{\nabla}} - \tilde{\boldsymbol{\nabla}} = \mathbf{S}(z^{-1}\mathbf{d}z).$$

It thus follows that

$$\hat{\boldsymbol{\nabla}} - \bar{\boldsymbol{\nabla}} = \mathbf{S}(\bar{\boldsymbol{\beta}}).$$

Now let $\{\boldsymbol{e_a}\}$ denote the Weyl propagated frame as given by Proposition 10. The connection coefficients $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ are define through the relation

$$\hat{\nabla}_{\boldsymbol{a}}\boldsymbol{e_c} = \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}\boldsymbol{e_b}.$$

Now, writing $\boldsymbol{e_a} = e_{\boldsymbol{a}}{}^{\mu}\boldsymbol{\partial}_{\mu}$ one has that

$$\hat{\nabla}_{\boldsymbol{a}}\boldsymbol{e_c} = \Big(\hat{\nabla}_{\mu}e_{\boldsymbol{c}}{}^{\nu}\Big)e_{\boldsymbol{a}}{}^{\mu}\boldsymbol{\partial}_{\nu},$$

where

$$\begin{aligned} \hat{\nabla}_{\mu}e_{\boldsymbol{c}}{}^{\nu} &= \bar{\nabla}_{\mu}e_{\boldsymbol{c}}{}^{\nu} + S_{\mu\lambda}{}^{\nu\rho}\bar{\beta}_{\rho}e_{\boldsymbol{c}}{}^{\lambda}, \\ &= \partial_{\mu}e_{\boldsymbol{c}}{}^{\nu} + \bar{\Gamma}_{\mu}{}^{\nu}{}_{\lambda}e_{\boldsymbol{c}}{}^{\lambda} + S_{\mu\lambda}{}^{\mu\rho}\bar{\beta}_{\rho}e_{\boldsymbol{c}}{}^{\lambda}. \end{aligned} \tag{4.26}$$

Now, a direct computation shows that the only non-vanishing Christoffel symbols of the metric (4.21), $\bar{\Gamma}_{\mu}{}^{\nu}{}_{\lambda}$ are given by

$$\begin{aligned} &\bar{\Gamma}_{t}{}^{t}{}_{z} = -\bar{\Gamma}_{z}{}^{z}{}_{z} = \frac{z(\frac{3}{2}Mz-1)}{1+z^2(Mz-1)}, \\ &\bar{\Gamma}_{t}{}^{z}{}_{t} = z(\tfrac{3}{2}Mz-1)\Big(1+z^2(Mz-1)\Big), \\ &\bar{\Gamma}_{\varphi}{}^{\theta}{}_{\varphi} = -\cos\theta\sin\theta, \qquad \bar{\Gamma}_{\theta}{}^{\varphi}{}_{\varphi} = \cot\theta. \end{aligned}$$

Observe that the coefficients $\bar{\Gamma}_{t}{}^{t}{}_{z}$, $\bar{\Gamma}_{z}{}^{z}{}_{z}$ and $\bar{\Gamma}_{t}{}^{z}{}_{t}$ are analytic at $z=0$.

Remark 32. The connection coefficients $\bar{\Gamma}_{\varphi}{}^{\theta}{}_{\varphi}$, $\bar{\Gamma}_{\theta}{}^{\varphi}{}_{\varphi}$ correspond to the connection of the round metric over $\mathbb{S}^2$. In the rest of this section, we ignore this coordinate singularity due to the use of spherical coordinates.

It follows from the discussion in the previous paragraphs and Proposition 10 that each of the terms in the righthand side of (4.26) is a regular function of the coordinate z and, in particular, analytic at $z = 0$. Contraction with the coefficients of the frame does not change this. Accordingly, it follows that the Weyl connection coefficients $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ are smooth functions of the coordinates used in the conformal Gaussian gauge on the future of the fiduciary initial hypersurface $\mathcal{S}_\star$ up to and beyond the conformal boundary.

4.4.3 The components of the curvature

In this section, we discuss the behaviour of the various components of the curvature of the Schwarzschild-de Sitter spacetime in the domain $\mathcal{M}_\bullet$. We are particularly interested in the behaviour of the curvature at the conformal boundary.

The subsequent discussion is best done in terms of the conformal metric $\bar{\boldsymbol{g}}$ as given by (4.21). Consider also the vector $\bar{\boldsymbol{e}}_{\boldsymbol{0}}$ given by

$$\bar{\boldsymbol{e}}_{\boldsymbol{0}} = \sqrt{|F(z)|}\boldsymbol{\partial}_z, \qquad F(z) = z^2 - Mz^3 - 1.$$

This vector is orthogonal to the conformal boundary $\mathscr{I}^+$ which, in these coordinates is given by the condition $z = 0$.

The rescaled Weyl tensor

Given a timelike vector, the components of the rescaled Weyl tensor d_{abcd} can be conveniently encoded in the electric and magnetic parts relative to the given vector. For the vector $\bar{\boldsymbol{e}}_{\boldsymbol{0}}$ these are given by

$$d_{ac} = d_{abcd}\bar{e}_{\boldsymbol{0}}{}^b\bar{e}_{\boldsymbol{0}}{}^d, \qquad d^*{}_{ac} = d^*{}_{abcd}\bar{e}_{\boldsymbol{0}}{}^b\bar{e}_{\boldsymbol{0}}{}^d,$$

where ${d^*}_{abcd}$ denotes the Hodge dual of d_{abcd}. A computation using the package `xAct` for `Mathematica` readily gives that the only non-zero components of the electric part are given by

$$
\begin{aligned}
d_{tt} &= -M\Big(z^2(1-Mz)-1\Big),\\
d_{\theta\theta} &= -\frac{M}{2},\\
d_{\varphi\varphi} &= -\frac{M}{2}\sin^2\theta,
\end{aligned}
$$

while the magnetic part vanishes identically. Observe, in particular, that the above expressions are regular at $z=0$ —again, disregarding the coordinate singularity due to the use of spherical coordinates. The smoothness of the components of the Weyl tensor is retained when re-expressed in terms of the Weyl propagated frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ as given in Proposition 10.

The Schouten tensor

A similar computer algebra calculation shows that the non-zero components of the Schouten tensor of the metric $\bar{\boldsymbol{g}}$ are given by

$$
\begin{aligned}
\bar{L}_{tt} &= \frac{1}{2}(2Mz-1)(1+z^2(Mz-1)),\\
\bar{L}_{zz} &= -\frac{1}{2}\frac{(2Mz-1)}{1+z^2(Mz-1)},\\
\bar{L}_{\theta\theta} &= -\frac{1}{2}(Mz-1),\\
\bar{L}_{\varphi\varphi} &= -\frac{1}{2}\sin^2\theta(Mz-1).
\end{aligned}
$$

Again, disregarding the coordinate singularity on the angular components, the above expressions are analytic on $\mathcal{M}_\bullet$ —in particular at $z=0$. To obtain the components of the Schouten tensor associated to the Weyl connection $\hat{\nabla}$ we make use of the transformation rule

$$
\bar{L}_{ab}-\hat{L}_{ab}=\bar{\nabla}_a\bar{\beta}_b-\frac{1}{2}S_{ab}{}^{cd}\bar{\beta}_c\bar{\beta}_d.
$$

The smoothness of $\bar{\beta}_a$ has already been established in Subsection 4.4.2. It follows then that the components of $\hat{L}_{ab}$ with respect to the Weyl propagated frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ are regular on $\mathcal{M}_\bullet$.

4.4.4 Summary

The analysis of the preceeding subsections is summarised in the following:

Proposition 11. *Given* $t_\bullet > 0$ *and the Weyl propagated frame* $\{\boldsymbol{e_a}\}$ *as given by Proposition 10, the connection coefficients of the Weyl connection associated to the congruence of conformal geodesics, the components of the rescaled Weyl tensor and the components of the Schouten tensor of the Weyl connection are smooth on* $\mathcal{M}_\bullet$ *and in particular at the conformal boundary.*

Remark 33. In other words, the sub-extremal Schwarzschild-de Sitter spacetime expressed in terms of a conformal Gaussian gauge system gives rise to a solution to the extended conformal Einstein field equations on the region $\mathcal{M}_\bullet \subset D^+(\mathcal{R}_\bullet)$ where $\mathcal{R}_\bullet \subset \mathcal{S}_\star$.

4.4.5 Construction of a background solution with compact spatial sections

The region $\mathcal{R}_\bullet \subset \mathcal{S}_\star$ has the topology of $I \times \mathbb{S}^2$ where $I \subset \mathbb{R}$ is an open interval. Accordingly, the spacetime arising from $\mathcal{R}_\bullet$ will have spatial sections with the same topology. As part of the perturbative argument given in Section 4.5 based on the general theory of symmetric hyperbolic systems as given in [50] it is convenient to consider solutions with compact spatial sections. We briefly discuss how the (conformal) Schwarzschild-de Sitter spacetime in the conformal Gaussian system over $\mathcal{M}_\bullet$ can be recast as a solution to the extended conformal Einstein field equations with compact spatial sections.

The key observation on this construction is that the Killing vector $\boldsymbol{\xi} = \boldsymbol{\partial}_t$ in the Cosmological region of the spacetime is spacelike. Thus, given a fixed $z_\circ < z_c$, we have that the hypersurface $\mathcal{S}_{z_\circ}$ defined by the condition $z = z_\circ$ has a translational invariance —that is, the intrinsic metric $\boldsymbol{h}$ and the extrinsic curvature $\boldsymbol{K}$ are invariant under the replacement $t \mapsto t + \varkappa$ for $\varkappa \in \mathbb{R}$. Moreover, the congruence of conformal geodesics given by Proposition 11 are such that the value of the coordinate t is constant along a given curve.

Consider now, the timelike hypersurfaces $\mathcal{T}_{-2t_\bullet}$ and $\mathcal{T}_{2t_\bullet}$ in $D^+(\mathcal{S}_\star)$ generated, respectively, by the future-directed geodesics emanating from $\mathcal{S}_\star$ at the points with $t = -2t_\bullet$ and $t = 2t_\bullet$. From the discussion in the previous paragraph, one can identify $\mathcal{T}_{-2t_\bullet}$ and $\mathcal{T}_{2t_\bullet}$ to obtain a smooth spacetime manifold $\bar{\mathcal{M}}_\bullet$ with compact spatial sections —see Figure 4.5. A natural foliation of $\bar{\mathcal{M}}_\bullet$ is given by the hypersurfaces $\bar{\mathcal{S}}_z$ of constant z with $0 \leq z \leq z_\star$ having the topology of a 3-handle —that is, $\mathcal{H}_z \approx \mathbb{S}^1 \times \mathbb{S}^2$.

The metric $\bar{\boldsymbol{g}}$ on SdS_I, cfr (4.24), induces a metric on $\bar{\mathcal{M}}_\bullet$ which, by an abuse of notation, we denote again by $\bar{\boldsymbol{g}}$. As the initial conditions defining the congruence of conformal geodesics of Proposition 8 have translational invariance, it follow that the resulting curves also have this property. Accordingly, the congruence of conformal geodesics on SdS_I given by Proposition 8 induces a non-intersecting congruence of conformal geodesics on $\bar{\mathcal{M}}_\bullet$ —recall that each of the curves in the congruence has constant coordinate t.

In summary, it follows from the discussion in the preceding paragraphs that the solution to the extended conformal Einstein field equations in a conformal Gaussian gauge as given by Proposition 11 implies a similar solution over the manifold $\bar{\mathcal{M}}_\bullet$. In the following we will denote this solution by $\mathring{\mathbf{u}}$. The initial data induced by $\mathring{\mathbf{u}}$ on $\bar{\mathcal{S}}_\star$ will be denoted by $\mathring{\mathbf{u}}_\star$.

4.5 The construction of non-linear perturbations

In this section, we bring together the analysis carried out in the previous sections to construct non-linear perturbations of the Schwarzschild-de Sitter spacetime on a suitable portion of the Cosmological region.

4.5.1 Initial data for the evolution equations

Given a solution $(\mathcal{S}_\star, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ to the Einstein constraint equations, there exists an algebraic procedure to compute initial data for the conformal evolution equations —see [81], Lemma 11.1. In the following, it will be assumed that we have at our disposal a family of initial data sets for the vacuum Einstein field equations corresponding to perturbations of initial

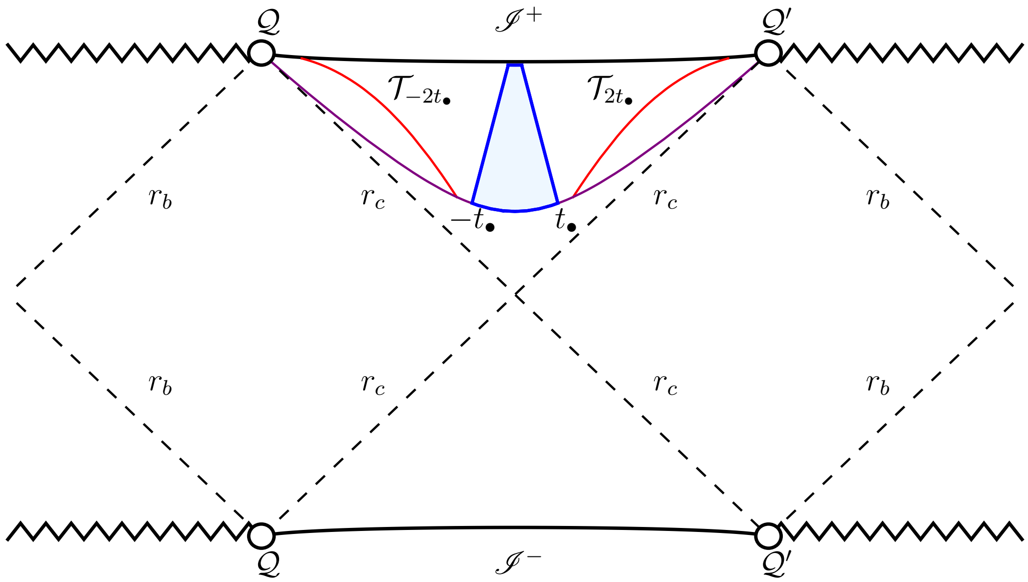

Figure 4.5: The red curves identify the timelike hypersurfaces $\mathcal{T}_{-2t_\bullet}$ and $\mathcal{T}_{2t_\bullet}$. The resulting spacetime manifold $\bar{M}_\bullet$ has compact spatial sections, $\bar{\mathcal{S}}_z$,with the topology of $\mathbb{S}^1 \times \mathbb{S}^2$.

data for the Schwarzschild-de Sitter spacetime on hypersurfaces of constant coordinate r in the Cosmological region. Initial data for the conformal evolution equations can then be constructed out of these basic initial data sets. *Assumptions of this type are standard in the analysis of non-linear stability.*

Given a compact hypersurface $\bar{\mathcal{S}}_z \approx \mathbb{S}^1 \times \mathbb{S}^2$ and a function $\mathbf{u} : \bar{\mathcal{S}}_z \to \mathbb{R}^N$ let $||\mathbf{u}||_{\bar{\mathcal{S}}_z,m}$ for $m \geq 0$ denote the standard L^2-Sobolev norm of order m of $\mathbf{u}$. Moreover, denote by $H^m(\bar{\mathcal{S}}_z, \mathbb{R}^N)$ the associated Sobolev space —i.e. the completion of the functions $\mathbf{w} \in C^\infty(\bar{\mathcal{S}}_z, \mathbb{R}^N)$ under the norm $||\ \ ||_{\bar{\mathcal{S}}_z,m}$.

In the following, consider some initial data set for the conformal evolution equations $\mathbf{u}_\star$ on $\mathcal{R}_\bullet \approx [-t_\bullet, t_\bullet] \times \mathbb{S}^2$ which is a small perturbation of exact data $\mathring{\mathbf{u}}_\star$ for the Schwarzschild-de Sitter spacetime in the sense that

$$\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star, \qquad ||\breve{\mathbf{u}}_\star||_{\mathcal{R}_\bullet,m} < \varepsilon$$

for $m \geq 4$ and some suitably small $\varepsilon > 0$. Making use of a smooth cut-off function over $\bar{\mathcal{S}}_{z_\star} \approx \mathbb{S}^1 \times \mathbb{S}^2$ the perturbation data $\breve{\mathbf{u}}_\star$ over $\mathcal{R}_\bullet$ can be matched to vanishing data $\mathbf{0}$ on $[-2t_\bullet, -\frac{3}{2}t_\bullet] \times \mathbb{S}^2 \cup [\frac{3}{2}t_\bullet, 2t_\bullet] \times \mathbb{S}^2$ with a smooth transition region, say, $[-\frac{3}{2}t_\bullet, -t_\bullet] \times \mathbb{S}^2 \cup$

$[t_\bullet, \frac{3}{2}t_\bullet] \times \mathbb{S}^2$. In this way one can obtain a vector-valued function $\breve{\bar{\mathbf{u}}}_\star$ over $\bar{\mathcal{S}}_\star \approx \mathbb{S}^1 \times \mathbb{S}^2$ whose size is controlled by the perturbation data $\breve{\mathbf{u}}_\star$ on $\mathcal{R}_\bullet$. In a slight abuse of notation, in order to ease the reading, we write $\breve{\mathbf{u}}_\star$ rather than $\breve{\bar{\mathbf{u}}}$.

4.5.2 Structural properties of the evolution equations

In this section, we briefly review the key structural properties of the evolution system associated to the extended conformal Einstein equations (2.54) written in terms of a conformal Gaussian system. This evolution system is central in the discussion of the stability of the background spacetime. In addition, we also discuss the subsidiary evolution system satisfied by the zero-quantities associated to the field equations, (2.53a)-(2.53d), and the supplementary zero-quantities (2.55a)-(2.55c). The subsidiary system is key in the analysis of the so-called *propagation of the constraints* which allows to establish the relation between a solution to the extended conformal Einstein equations (2.54) and the Einstein field equations (4.1). One of the advantages of the hyperbolic reduction of the extended conformal Einstein field equations by means of conformal Gaussian systems is that it provides a priori knowledge of the location of the conformal boundary of the solutions to the conformal field equations.

Conformal Gaussian gauge systems lead to a *hyperbolic reduction* of the extended conformal Einstein field equation (2.54). The particular form of the resulting evolution equations will not be required in the analysis, only general structural properties. In order to describe these denote by $\boldsymbol{\upsilon}$ the independent components of the coefficients of the frame $e_{\boldsymbol{a}}{}^{\mu}$, the connection coefficients $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ and the Weyl connection Schouten tensor $\hat{L}_{\boldsymbol{ab}}$ and by $\boldsymbol{\phi}$ the independent components of the rescaled Weyl tensor $d_{\boldsymbol{abcd}}$, expressible in terms of its electric and magnetic parts with respect to the timelike vector $\boldsymbol{e_0}$. Also, let $\boldsymbol{e}$ and $\boldsymbol{\Gamma}$ denote, respectively, the independent components of the frame and connection. In terms of these objects one has the following:

Lemma 7. *The extended conformal Einstein field equations* (2.54) *expressed in terms of a conformal Gaussian gauge imply a symmetric hyperbolic system for the components* $(\boldsymbol{\upsilon}, \boldsymbol{\phi})$

of the form

$$\partial \boldsymbol{\upsilon} = \mathbf{K}\boldsymbol{\upsilon} + \mathbf{Q}(\boldsymbol{\Gamma})\boldsymbol{\upsilon} + \mathbf{L}(\bar{x})\boldsymbol{\phi}, \tag{4.27a}$$

$$\Big(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e})\Big)\partial_\tau \boldsymbol{\phi} + \mathbf{A}^\alpha(\boldsymbol{e})\partial_\alpha \boldsymbol{\phi} = \mathbf{B}(\boldsymbol{\Gamma})\boldsymbol{\phi}, \tag{4.27b}$$

where $\mathbf{I}$ *is the unit matrix,* $\mathbf{K}$ *is a constant matrix* $\mathbf{Q}(\boldsymbol{\Gamma})$ *is a smooth matrix-valued function,* $\mathbf{L}(\bar{x})$ *is a smooth matrix-valued function of the coordinates,* $\mathbf{A}^\mu(\boldsymbol{e})$ *are Hermitian matrices depending smoothly on the frame coefficients and* $\mathbf{B}(\boldsymbol{\Gamma})$ *is a smooth matrix-valued function of the connection coefficients.*

Remark 34. In this analysis we will be concerned with situations in which the matrix-valued function $\mathbf{I}+\mathbf{A}^0(\boldsymbol{e})$ is positive definite. This is the case, for example, in perturbations of a background solution.

Remark 35. Explicit expressions of the evolution equations and further discussion on their derivation can be found in [56] —see also [81], Section 13.4 for a spinorial version of the equations.

For the evolution system (4.27a)-(4.27b) one has the following *propagation of the constraints* result [56]:

Lemma 8. *Assume that the evolution equations* (4.27a)-(4.27b) *hold. Then the independent components of the zero-quantities*

$$\hat{\Sigma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \quad \hat{\Xi}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}, \quad \hat{\Delta}_{\boldsymbol{abc}}, \quad \hat{\Lambda}_{\boldsymbol{abc}}, \quad \delta_{\boldsymbol{a}}, \quad \gamma_{\boldsymbol{ab}}, \quad \varsigma_{\boldsymbol{ab}},$$

not determined by either the evolution equations or the gauge conditions satisfy a symmetric hyperbolic system which is homogeneous in the zero-quantities. As a result, if the zero-quantities vanish on a fiduciary spacelike hypersurface $\mathcal{S}_\star$*, then they also vanish on the domain of dependence.*

Remark 36. It follows from Lemmas 7, 8 and Proposition 2 that a solution to the conformal evolution equations (4.27a)-(4.27b) with data on $\mathcal{S}_\star$ satisfying the conformal constraints implies a solution to the Einstein field equations away from the conformal boundary.

4.5.3 Setting up the perturbative existence argument

In the spirit of the schematic notation used in the previous section, we set $\mathbf{u} \equiv (\boldsymbol{v}, \boldsymbol{\phi})$. Moreover, consistent with this notation let $\mathring{\mathbf{u}}$ denotes a solution to the evolution equations (4.27a) and (4.27b) arising from some data $\mathring{\mathbf{u}}_\star$ prescribed on a hypersurface at $r = r_\star$. We refer to $\mathring{\mathbf{u}}$ as the *background solution*. We will construct solutions to (4.27a) and (4.27b) which can be regarded as a perturbation of the background solution in the sense that

$$\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}.$$

This means, in particular, that one can write

$$\boldsymbol{e} = \mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}, \qquad \boldsymbol{\Gamma} = \mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}}, \qquad \phi = \mathring{\phi} + \breve{\phi}. \tag{4.28}$$

The components of $\breve{\boldsymbol{e}}$, $\breve{\boldsymbol{\Gamma}}$ and $\breve{\phi}$ are our unknowns. Making use of the decomposition (4.28) and exploiting that $\mathring{\boldsymbol{u}}$ is a solution to the conformal evolution equations one obtains the equations

$$\partial_\tau \breve{\boldsymbol{v}} = \mathbf{K}\breve{\boldsymbol{v}} + \mathbf{Q}(\mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{v}} + \mathbf{Q}(\breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{v}} + \mathbf{L}(\bar{x})\breve{\boldsymbol{\phi}} + \mathbf{L}(\bar{x})\mathring{\boldsymbol{\phi}}, \tag{4.29a}$$

$$(\mathbf{I} + \mathbf{A}^0(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}))\partial_\tau \breve{\boldsymbol{\phi}} + \mathbf{A}^\alpha(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}})\partial_\alpha \breve{\boldsymbol{\phi}} = \mathbf{B}(\mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\phi}} + \mathbf{B}(\mathring{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\phi}}. \tag{4.29b}$$

Now, it is convenient to define

$$\bar{\mathbf{A}}^0(\tau, \underline{x}, \breve{\mathbf{u}}) \equiv \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} + \mathbf{A}^0(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}) \end{pmatrix}, \qquad \bar{\mathbf{A}}^\alpha(\tau, \underline{x}, \breve{\mathbf{u}}) \equiv \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{A}^\alpha(\mathring{\boldsymbol{e}} + \breve{\boldsymbol{e}}) \end{pmatrix},$$

and

$$\bar{\mathbf{B}}(\tau, \underline{x}, \breve{\mathbf{u}}) \equiv \breve{\mathbf{u}}\bar{\mathbf{Q}}\breve{\mathbf{u}} + \bar{\mathbf{L}}(\bar{x})\breve{\mathbf{u}} + \bar{\mathbf{K}}\breve{\mathbf{u}},$$

where

$$\breve{\mathbf{u}}\bar{\mathbf{Q}}\breve{\mathbf{u}} \equiv \begin{pmatrix} \breve{\boldsymbol{v}}\mathbf{Q}\breve{\boldsymbol{v}} & 0 \\ 0 & \mathbf{B}(\breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\phi}} + \mathbf{B}(\breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{\phi}} \end{pmatrix}, \qquad \bar{\mathbf{L}}(\bar{x})\breve{\mathbf{u}} \equiv \begin{pmatrix} \mathring{\boldsymbol{v}}\mathbf{Q}\breve{\boldsymbol{v}} + \mathbf{Q}(\breve{\boldsymbol{\Gamma}})\mathring{\boldsymbol{v}} & \mathbf{L}(\bar{x})\breve{\boldsymbol{\phi}} + \mathbf{L}(\bar{x})\mathring{\boldsymbol{\phi}} \\ 0 & 0 \end{pmatrix},$$

$$\bar{\mathbf{K}}\breve{\mathbf{u}} \equiv \begin{pmatrix} \mathbf{K}\breve{\boldsymbol{v}} & 0 \\ 0 & \mathbf{B}(\mathring{\Gamma})\breve{\boldsymbol{\phi}} + \mathbf{B}(\mathring{\Gamma})\mathring{\boldsymbol{\phi}} \end{pmatrix},$$

denote, respectively, expressions which are quadratic, linear and constant terms in the unknowns.

In terms of the above expressions it is possible to rewrite the system (4.29a)-(4.29b) in the more concise form

$$\bar{\mathbf{A}}^0(\tau, \underline{x}, \breve{\mathbf{u}})\partial_\tau \breve{\mathbf{u}} + \bar{\mathbf{A}}^\alpha(\tau, \underline{x}, \breve{\mathbf{u}})\partial_\alpha \breve{\mathbf{u}} = \bar{\mathbf{B}}(\tau, \underline{x}, \breve{\mathbf{u}}). \tag{4.30}$$

These equations are in a form where the theory of first-order symmetric hyperbolic systems can be applied to obtain a existence and stability result for small perturbations of the initial data $\mathring{\mathbf{u}}_\star$. This requires, however, the introduction of the appropriate norms measuring the size of the perturbed initial data $\breve{\mathbf{u}}_\star$.

Remark 37. In the following it will be assumed that the background solution $\mathring{\mathbf{u}}$ is given by the Schwarzschild-de Sitter background solution written in a conformal Gaussian gauge system as described in Proposition 11. It follows that the entries of $\mathring{\mathbf{u}}$ are smooth functions on $\bar{\mathcal{M}}_\bullet \equiv [0,2] \times \bar{\mathcal{S}}_\star \approx [0,2] \times \mathbb{S}^1 \times \mathbb{S}^2$.

Theorem 2 (***existence and uniqueness of the solutions to the conformal evolution equations***)**.** *Given $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ satisfying the conformal constraint equations on $\bar{\mathcal{S}}_\star$ and $m \geq 4$, one has that:*

(i) There exists $\varepsilon > 0$ such that if

$$||\breve{\mathbf{u}}_\star||_{\bar{\mathcal{S}}_\star, m} < \varepsilon, \tag{4.31}$$

then there exists a unique solution $\breve{\mathbf{u}} \in C^{m-2}([0,2] \times \bar{\mathcal{S}}_\star, \mathbb{R}^N)$ to the Cauchy problem for the conformal evolution equations (4.30) *with initial data $\mathbf{u}(0, \underline{x}) = \breve{\mathbf{u}}_\star$ and with N denoting the dimension of the vector $\mathbf{u}$.*

(ii) Given a sequence of initial data $\breve{\mathbf{u}}_\star^{(n)}$ such that

$$||\breve{\mathbf{u}}_\star^{(n)}||_{\bar{\mathcal{S}}_\star, m} < \varepsilon, \qquad \text{and} \qquad ||\breve{\mathbf{u}}_\star^{(n)}||_{\bar{\mathcal{S}}_\star, m} \xrightarrow{n \to \infty} 0,$$

then for the corresponding solutions $\breve{\mathbf{u}}^{(n)} \in C^{m-2}([0,2] \times \bar{\mathcal{S}}_\star, \mathbb{R}^N)$, one has $||\breve{\mathbf{u}}^{(n)}||_{\bar{\mathcal{S}}_\star, m} \to 0$ uniformly in $\tau \in \left[\tau_\star, \frac{5}{2}\right)$ as $n \to \infty$.

Proof. The proof is a direct application of Kato's existence, uniqueness and stability theory for symmetric hyperbolic systems [50] to developments with compact spatial sections —see Theorem 12.4 in [81]; see also [56]. □

Remark 38. In view of the localisation properties of hyperbolic equations the matching of the perturbation data on $\mathcal{R}_\bullet$ does not influence the solution $\mathbf{u}$ on $D^+(\mathcal{R}_\bullet)$. Accordingly, in the subsequent discussion we discard the solution $\mathbf{u}$ on the region $\bar{\mathcal{M}}_\bullet \setminus D^+(\mathcal{R}_\bullet)$ as this has no physical relevance.

Moreover, given the *propagation of the constraints*, Lemma 8, and the relation between the extended conformal Einstein field equations and the vacuum Einstein field equations, Lemma 2, one has the following:

Corollary 2. *The metric*

$$\boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}$$

obtained from the solution to the conformal evolution equations given in Theorem 2 implies a solution $\tilde{\boldsymbol{g}}$ to the vacuum Einstein field equations with positive Cosmological constant on $\tilde{\mathcal{M}} \equiv D^+(\mathcal{R}_\bullet)$. This solution admits a smooth conformal extension with a spacelike conformal boundary. In particular, the timelike geodesics fully contained in $\tilde{\mathcal{M}}$ are complete.

Remark 39. The resulting spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is a non-linear perturbation of the sub-extremal Schwarzschild-de Sitter spacetime on a portion of the Cosmological region of the background solution which contains a portion of the asymptotic region.

Remark 40. *As $\mathcal{R}_\bullet$ is not compact, its development has a Cauchy horizon $H^+(\mathcal{R}_\bullet)$.*

4.6 The main result

We summarise the discussion of the preceding sections with a more detailed formulation of the main result of this chapter:

Theorem 3. *Let* $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ *denote smooth initial data for the conformal evolution equations satisfying the conformal constraint equations on a hypersurface* $\overline{\mathcal{S}}_\star$*. Then, there exists* $\varepsilon > 0$ *such that if*

$$||\breve{\mathbf{u}}_\star||_{\overline{\mathcal{S}}_\star,m} < \varepsilon, \qquad m \geq 4$$

then there exists a unique C^{m-2} *solution* $\tilde{\boldsymbol{g}}$ *to the vacuum Einstein field equation with positive Cosmological constant over* $[\tilde{\tau}_\star, \infty) \times \overline{\mathcal{S}}_\star$ *for* $\tilde{\tau}_\star > 0$ *whose restriction to* $\overline{\mathcal{S}}_\star$ *implies the initial data* $\mathbf{u}_\star$*. Moreover, the solution* $\mathbf{u}$ *remains suitably close (in the Sobolev norm* $\|\cdot\|_{\mathcal{S},m}$*) to the background solution* $\mathring{\mathbf{u}}$*.*

Chapter 5

Spinors and spacetime

5.1 Introduction

One of the most important areas of application of spinorial methods is the study of asymptotic properties in General Relativity. These methods are particularly powerful when combined with a technique which employs conformal rescalings for the analysis of the structure of the Einstein field equations and their solutions [59, 60]. The purpose of this chapter is to develop the formalism of spinors in spacetime. The discussion builds up from the basic features of the so-called *spacetime spinors* with the main aim to introduce the essential features of the *space-spinor* formalism. This is a framework in which spinors are endowed with a *Hermitian inner product* and will be used in Chapter 6 to analyse the asymptotic behaviour of the Maxwell-scalar field system on a fixed background. In particular, it provides a systematic approach to the construction of evolution equations which can be regarded as the spinorial equivalent of the $1+3$ decomposition for tensors.

It is important to notice that throughout this chapter the signature convention for Lorentzian spacetime metrics will change to $(+,-,-,-)$. This will be the preferred convention for the rest of this thesis.

5.2 Spacetime spinors

In this section, the formalism of *spacetime spinors*, also known as 2-*spinor formalism*, is discussed. In particular, the basic features of 2-spinors in spacetime is presented. Given a spacetime $(\mathcal{M}, \boldsymbol{g})$, at any given point $p \in \mathcal{M}$ it is possible to associate a spinorial structure. This structure is closely related to the representation theory of the group $\mathrm{SL}(2, \mathbb{C})$. This group has two inequivalent representations in terms of two-dimensional complex vector spaces which are complex conjugates of each other. This discussion then begins with the definition of a symplectic vector space following [81].

5.2.1 2-Spinors algebra

Let $\mathfrak{S}$ denote a complex vector space, a 2-dimensional symplectic vector space is defined as follows:

Definition. *A symplectic vector space is a 2-dimensional vector space $\mathfrak{S}$ endowed with a 2-form*

$$[\cdot, \cdot] : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathbb{C}$$

which is

(i) ***Skew-symmetric****: given $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{S}$*

$$[\boldsymbol{\xi}, \boldsymbol{\eta}] = -[\boldsymbol{\eta}, \boldsymbol{\xi}].$$

(ii) ***Bilinear****: given $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathfrak{S}$ and $z \in \mathbb{C}$ one has*

$$[\boldsymbol{\xi} + z\boldsymbol{\zeta}, \boldsymbol{\eta}] = [\boldsymbol{\xi}, \boldsymbol{\eta}] + z[\boldsymbol{\zeta}, \boldsymbol{\eta}], \qquad [\boldsymbol{\xi}, \boldsymbol{\eta} + z\boldsymbol{\zeta}] = [\boldsymbol{\xi}, \boldsymbol{\eta}] + z[\boldsymbol{\xi}, \boldsymbol{\zeta}].$$

(iii) ***Non-degenerate****:*

$$\text{if} \quad [\boldsymbol{\xi}, \boldsymbol{\eta}] = 0 \qquad \text{for all} \quad \boldsymbol{\eta} \in \mathfrak{S} \quad \text{then} \quad \boldsymbol{\xi} = 0.$$

Let $\mathfrak{S}^*$ be the dual space of $\mathfrak{S}$. We build the tensor algebra over $\mathfrak{S}$ and $\mathfrak{S}^*$ in the usual way. Given bases of $\mathfrak{S}$ and $\mathfrak{S}^*$ we may define a non-natural isomorphism between the two spaces: two vectors are regarded as being the "same" if their components with respect to the two bases are identical. However, the skew-scalar product defines a natural isomorphism: to the element, $\boldsymbol{\xi} \in \mathfrak{S}$ we associate $[\boldsymbol{\xi}, \cdot] \in \mathfrak{S}^*$ which is a linear map

$$\boldsymbol{\xi} \in \mathfrak{S} \longrightarrow \boldsymbol{\xi}^\flat \equiv [\boldsymbol{\xi}, \cdot] \in \mathfrak{S}^*.$$

5.2.2 Spin bases

The definition of the 2-dimensional symplectic vector space $\mathfrak{S}$ implies that the space of vectors orthogonal to a non-zero vector $\boldsymbol{\xi}$ consists of the vectors proportional to $\boldsymbol{\xi}$. In other words, given $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{S}$, they are linearly dependent if $[\boldsymbol{\xi}, \boldsymbol{\eta}] = 0$ so that $\boldsymbol{\eta} = z\boldsymbol{\xi}$, with $z \in \mathbb{C}$ and $z \neq 0$. This is a consequence of the skew-symmetric scalar product for which every vector is self-orthogonal. This property can be used to construct a *spin basis*.

Definition (Spin basis). *Given two vectors $\boldsymbol{o}, \boldsymbol{\iota} \in \mathfrak{S}$. If $\boldsymbol{o}$ is non-zero and $\boldsymbol{\iota}$ is such that $[\boldsymbol{o}, \boldsymbol{\iota}] = 1$, then $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ is a* ***spin basis*** *for $\mathfrak{S}$.*

Remark 41. From this definition it is clear that $\boldsymbol{\iota}$ is *not unique*, since it is possible to add an arbitrary multiple of $\boldsymbol{o}$ to it preserving the normalisation.

Given a vector $\boldsymbol{\xi} \in \mathfrak{S}$, the *components of* $\boldsymbol{\xi}$ with respect to the spin basis $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ are denoted as $\boldsymbol{\xi}^A$, with $A = 0, 1$, where

$$\boldsymbol{\xi} = \boldsymbol{\xi}^0 \boldsymbol{o} + \boldsymbol{\xi}^1 \boldsymbol{\iota},$$

with

$$\boldsymbol{o}^A = (1, 0), \qquad \boldsymbol{\iota}^A = (0, 1).$$

It follows that

$$\boldsymbol{\xi}^0 \equiv [\boldsymbol{\xi}, \boldsymbol{\iota}], \qquad \boldsymbol{\xi}^1 \equiv -[\boldsymbol{\xi}, \boldsymbol{o}].$$

Similarly, due to the natural isomorphism between $\mathfrak{S}$ and $\mathfrak{S}^*$, the components of the dual vector $\boldsymbol{\xi}^\flat \in \mathfrak{S}^*$ are denoted by $\boldsymbol{\xi}_A$, with $A = 0, 1$, where

$$\boldsymbol{\xi} = \boldsymbol{\xi}_0 \boldsymbol{o} + \boldsymbol{\xi}_1 \boldsymbol{\iota},$$

with

$$\boldsymbol{\xi}_0 \equiv -[\boldsymbol{\iota}, \boldsymbol{\xi}], \qquad \boldsymbol{\xi}_1 \equiv [\boldsymbol{o}, \boldsymbol{\xi}].$$

The discussion in this thesis uses a combination of index-free and abstract index notation. A complete rigorous discussion is given in Penrose and Rindler [61].

5.2.3 The spinor ϵ_{AB}

Since the skew-symmetric 2-form $[\cdot,\cdot]$ is a function

$$[\cdot,\cdot]: \qquad \mathfrak{S} \otimes \mathfrak{S} \longrightarrow \mathbb{C},$$

there exists a 2-spinor $\epsilon_{AB} \in \mathfrak{S}_{AB}$ such that

$$[\boldsymbol{\xi}, \boldsymbol{\eta}] = \epsilon_{AB} \xi^A \eta^B.$$

The spinor ϵ_{AB} is called the $\boldsymbol{\epsilon}$-spinor. Since

$$[\boldsymbol{\xi}, \boldsymbol{\eta}] = -[\boldsymbol{\eta}, \boldsymbol{\xi}]$$

it follows that ϵ_{AB} is skew-symmetric, i.e. $\epsilon_{AB} = -\epsilon_{BA}$. The condition that $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ is a spin basis for $\mathfrak{S}$ translates into

$$\epsilon_{AB} o^A o^B = \epsilon_{AB} \iota^A \iota^B = 0, \qquad \epsilon_{AB} o^A \iota^B = 1.$$

In components with respect to this basis, we have

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This is a non-singular matrix and its inverse is defined, up to a conventional factor -1, as

$$\epsilon^{AB} = -(\epsilon^{-1})^{AB}.$$

The convention on kernel letter of dual elements means that ϵ_{AB} can be regarded as an index-lowering operator. For

$$[\boldsymbol{\xi}, \boldsymbol{\eta}] = \epsilon_{AB}\xi^A\eta^B = (\epsilon_{AB}\xi^A)\eta^B,$$

one has that $[\boldsymbol{\xi}, \cdot]$ is the dual of ξ^B and so

$$\xi_B = (\epsilon_{AB}\xi^A).$$

This equation has an important concomitant. Since ϵ_{AB} is non-singular we have

$$(\epsilon^{-1})^{BC}\xi_B = \xi^A\epsilon_{AB}(\epsilon^{-1})^{BC} = \xi^A\delta_A{}^C = \xi^C,$$

where $\delta_A{}^C$ is the *spinor Kronecker's delta* and define

$$\epsilon^{BC} = -(\epsilon^{-1})^{BC}$$

so that

$$\xi^A = -\epsilon^{CA}\xi_C.$$

Thus, the spinor ϵ_{AB} provides a convenient way to express the duality between the spaces $\mathfrak{S}$ and $\mathfrak{S}^*$. Now, given a linear transformation applied to a spin basis

$$\tilde{o}^A = \alpha o^A + \beta\iota^A, \qquad \tilde{\iota}^A = \gamma o^A + \delta\iota^A,$$

since the same transformation hold with indices lowered

$$\tilde{o}_A = \alpha o_A + \beta\iota_A, \qquad \tilde{\iota}_A = \gamma o_A + \delta\iota_A,$$

one has that $\{\tilde{o}, \tilde{\iota}\}$ form a spin basis if and only if

$$[\tilde{o}, \tilde{\iota}] = \alpha\delta - \beta\gamma = 1.$$

Thus, the transformation matrix is in $\mathrm{SL}(2,\mathbb{C})$. Since the condition that a linear transformation preserves spin bases is the condition that it be a *symplectomorphism*, hence $\mathrm{Sp}(2)$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})$.

5.2.4 Decompositions in irreducible components

Since $\mathfrak{S}$ is a 2-dimensional space, for any spinor ζ one has

$$\zeta_{\cdots[ABC]\cdots} = 0$$

for at least two of the bracketed indices must be equal. In particular, we have the *Jacobi identity*

$$\epsilon_{A[B}\epsilon_{CD]} = 0 = \epsilon_{AB}\epsilon_{CD} + \epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{BC}.$$

This is used in the form given by the following lemma

Lemma 9. *Let $\zeta_{\ldots AB\ldots}$ be a multivalent spinor. Then*

$$\zeta_{\ldots AB\ldots} = \zeta_{\ldots(AB)\ldots} + \frac{1}{2}\epsilon_{AB}\zeta_{\ldots C\ldots}{}^{C}.$$

The proof of this lemma can be found in [70, 81]. This is a special case of a more general result:

Theorem 4. *Any spinor $\zeta_{\ldots AB\ldots}$ can be decomposed as the sum of the totally symmetric spinor $\zeta_{(A\ldots F)}$ and products of $\boldsymbol{\epsilon}$-spinors with totally symmetric spinors of lower valence.*

The proof of this theorem can be found in [61]. The type of spinorial decompositions provided by this theorem will be used systematically in the rest of this thesis. In particular, we will make use of a *decomposition in irreducible components* defined as

$$\begin{aligned}\chi_{ABCD} =& \chi_{(ABCD)} + \frac{1}{2}\chi_{(AB)P}{}^{P}\epsilon_{CD} + \frac{1}{2}\chi_{P}{}^{P}{}_{(CD)}\epsilon_{AB} + \frac{1}{4}\chi_{P}{}^{P}{}_{Q}{}^{Q}\epsilon_{AB}\epsilon_{CD} \\ &+ \frac{1}{2}\epsilon_{A(C}\chi_{D)B} + \frac{1}{2}\epsilon_{B(C}\chi_{D)A} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}\chi,\end{aligned} \tag{5.1}$$

where

$$\chi_{AB} \equiv \chi_{Q(AB)}{}^{Q}, \qquad \chi \equiv \chi_{PQ}{}^{PQ}.$$

From definition (5.1), it follows that

$$\chi_{ABCD} = 0$$

if and only if

$$\chi_{(ABCD)} = 0, \qquad \chi_{(AB)P}{}^{P} = 0, \qquad \ldots \qquad \chi = 0.$$

5.2.5 Components with respect to a basis

In order to discuss spinors in terms of a specific basis it is convenient to introduce bold indices $\boldsymbol{A}$, $\boldsymbol{B}$, ... ranging over $\mathbf{0}$, $\mathbf{1}$. Thus, $\xi^{\boldsymbol{A}}$ and $\eta_{\boldsymbol{A}}$ represent the components of ξ^A and η_B with respect to a specific basis.

Given a spin basis $\{\boldsymbol{o}, \boldsymbol{\iota}\}$, one introduces the symbol $\epsilon_{\boldsymbol{A}}{}^A$ where

$$\epsilon_{\mathbf{0}}{}^A \equiv o^A, \qquad \epsilon_{\mathbf{1}}{}^A \equiv \iota^A$$

and dual cobasis $\epsilon^{\boldsymbol{A}}{}_A$ so that one has

$$\epsilon_{\boldsymbol{A}}{}^A \epsilon^{\boldsymbol{B}}{}_A = \delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}.$$

Using this notation, one has that two spinors $\xi^{\boldsymbol{A}}$ and $\eta_{\boldsymbol{B}}$ can be written as

$$\xi^A = \xi^{\boldsymbol{A}} \epsilon_{\boldsymbol{A}}{}^A, \qquad \eta_B = \eta_{\boldsymbol{B}} \epsilon^{\boldsymbol{B}}{}_B$$

where the components $\xi^{\boldsymbol{A}}$ and $\eta_{\boldsymbol{B}}$ are given by

$$\xi^{\boldsymbol{A}} \equiv \xi^A \epsilon^{\boldsymbol{A}}{}_A, \qquad \eta_{\boldsymbol{B}} \equiv \eta_B \epsilon_{\boldsymbol{B}}{}^B.$$

The components of the skew-symmetric spinor ϵ_{AB} with respect to the basis $\epsilon_{\boldsymbol{A}}{}^A$ are given by

$$\epsilon_{\boldsymbol{AB}} \equiv \epsilon_{AB} \epsilon_{\boldsymbol{A}}{}^A \epsilon_{\boldsymbol{B}}{}^B = \begin{pmatrix} o_A o^A & o_A \iota^A \\ \iota_A o^A & \iota_A \iota^A \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

From the definition of ϵ^{AB} it follows that

$$\epsilon^{\boldsymbol{AB}} \equiv \epsilon^{AB} \epsilon^{\boldsymbol{A}}{}_A \epsilon^{\boldsymbol{B}}{}_B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

5.2.6 Complex conjugation of spinors

To relate spinors with tensors, one needs to consider the operation of complex conjugation. In our conventions given a spinor $\zeta^A \in \mathfrak{S}^A$ the operation of complex conjugation corresponds to

$$\overline{\zeta^A} = \bar{\zeta}^{A'},$$

where $\bar{\zeta}^{A'} \in \mathfrak{S}^{A'}$. A spinor $\zeta^{A\ldots CS'\ldots U'}{}_{D\ldots EW'\ldots Y'}$ with m unprimed contravariant indices, n primed contravariant indices, l unprimed covariant indices and p primed covariant indices describes the most general type of spinor. It is obtained from the $\mathfrak{S}$-linear map

$$\zeta : \mathfrak{S}_A \times \cdots \times \mathfrak{S}_C \times \mathfrak{S}_{S'} \times \cdots \times \mathfrak{S}_{U'} \times \mathfrak{S}^D \times \cdots \times \mathfrak{S}^E \times \mathfrak{S}^{W'} \times \cdots \times \mathfrak{S}^{V'} \longrightarrow \mathbb{C}.$$

The algebra $\mathfrak{S}^\bullet$ is then extended to include this more general type of spinors with primed and unprimed indices. Since $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ are not isomorphic, one can write

$$\zeta_{AA'} = \zeta_{A'A},$$

so that the relative position of primed and unprimed indices is irrelevant. Conversely, reordering groups of primed or unprimed indices is only allowed in the case of spinors with special symmetries. The rules for the raising and lowering of the indices of valence 1 spinors are naturally extended to higher valence spinors. Primed indices are raised and lowered using the spinors $\epsilon^{A'B'}$ and $\epsilon_{A'B'}$ which are related to ϵ^{AB} and ϵ_{AB} by

$$\bar{\epsilon}^{A'B'} \equiv \overline{\epsilon^{AB}}, \qquad \bar{\epsilon}_{A'B'} \equiv \overline{\epsilon_{AB}},$$

with the convention of setting $\bar{\epsilon}^{A'B'} = \epsilon^{A'B'}$ and $\bar{\epsilon}_{A'B'} = \epsilon_{A'B'}$.

The discussion concerning the irreducible decomposition of spinors, in particular Lemma 9 and Theorem 4, can be easily extended to the case of spinors with primed indices or combinations of primed and unprimed indices.

5.3 The relation between spinors and world tensors

This section explores the relationship between spinors and World tensors. Spinors provide a simple representation of several tensorial operations. Although every four-dimensional world tensor can be represented in terms of spinors, the converse is not true. Some spinors admit no discussion in terms of World tensors. This observation is based on the fact that 2-spinors are related to representations of $\mathrm{SL}(2,\mathbb{C})$, while world tensors are related to the Lorentz group, $\mathrm{O}(1,3)$. These groups are not isomorphic to each other. The correspondence between 2-spinors and world tensors is established via the *Hermicity* property.

5.3.1 Hermitian spinors

A spinor $\boldsymbol{\xi} \in \mathfrak{S}^{\bullet}$ is said to be *Hermitian* if it is equal to its complex conjugate, that is

$$\boldsymbol{\xi} = \bar{\boldsymbol{\xi}}.$$

This implies that $\boldsymbol{\xi}$ needs to have the same number of primed and unprimed indices. Hence, if $\boldsymbol{\xi}$ is a spinor with the same number of primed and unprimed indices $\xi_{AA'\ldots DD'}{}^{EE'\ldots HH'}$, the Hermicity condition reads as

$$\xi_{AA'\ldots DD'}{}^{EE'\ldots HH'} = \bar{\xi}_{AA'\ldots DD'}{}^{EE'\ldots HH'}.$$

Now, given two bases $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ and $\{\bar{\boldsymbol{o}}, \bar{\boldsymbol{\iota}}\}$ of $\mathfrak{S}$ and $\bar{\mathfrak{S}}$, respectively. A spinor $\xi^{AA'} \in \mathfrak{S}^{AA'}$ can be written in terms of these bases as

$$\xi^{AA'} = a o^A \bar{o}^{A'} + b\iota^A \bar{\iota}^{A'} + c o^A \bar{\iota}^{A'} + d\iota^A \bar{o}^{A'} \tag{5.2}$$

for a, b, c, $d \in \mathbb{C}$. In the case of the spinor $\xi^{AA'}$ being Hermitian it follows that a, $b \in \mathbb{R}$ and $c = \bar{d}$. It follows that one can think of the Hermitian spinor $\xi^{AA'} \in \mathfrak{S}^{AA'}$ as describing a four-dimensional *World vector* ξ^a.

This discussion can be extended to higher valence Hermitian spinors so that one can regard each pair of unprimed-primed indices as associated with tensorial indices. From this discussion, it follows that the metric tensor g_{ab} has spinorial counterpart $g_{AA'BB'}$ where

$$g_{AA'BB'} \equiv \epsilon_{AB}\epsilon_{A'B'} \tag{5.3}$$

with the following properties

$$g^{AA'BB'} = \epsilon^{AB}\epsilon^{A'B'}, \tag{5.4a}$$

$$g_{AA'BB'}g^{BB'CC'} = g_{AA'}{}^{CC'} = \delta_A{}^C\delta_{A'}{}^{C'}, \tag{5.4b}$$

$$g_{AA'BB'}g^{AA'BB'} = 4, \tag{5.4c}$$

$$g_{AA'BB'} = g_{BB'AA'}. \tag{5.4d}$$

Moreover, given a vector $v_{AA'} \in \mathfrak{S}_{AA'}$ it follows that

$$v^{BB'} = v_{AA'}g^{AA'BB'}, \qquad v_{BB'} = v^{AA'}g_{AA'BB'}.$$

5.3.2 The Infeld-van der Waerden symbols

In order to describe the correspondence between spinors and World tensors at a point $p \in \mathcal{M}$, one can consider a basis $\{\boldsymbol{e_a}\} \subset T|_p(\mathcal{M})$ and its dual basis $\{\boldsymbol{\omega^a}\} \subset T^*|_p(\mathcal{M})$ so that $\langle \boldsymbol{\omega^b}, \boldsymbol{e_a}\rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$. Moreover, let

$$\boldsymbol{g_{ab}} \equiv \boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b})$$

denote the components of the metric $\boldsymbol{g}$ with respect to the $\{\boldsymbol{\omega^a}\}$. This basis is $\boldsymbol{g}$-orthogonal —i.e. $\boldsymbol{g_{ab}} = \boldsymbol{\eta_{ab}}$. Finally, let $\{\epsilon_{\boldsymbol{A}}\} \subset \mathfrak{S}$ denote a spin basis, and let $\epsilon_{\boldsymbol{AB}}$ denote the components of the spinor ϵ_{AB} with respect to this basis. The scalars $\boldsymbol{g_{ab}}$ and $\epsilon_{\boldsymbol{AB}}$ can be put in correspondence with each other via an equation of the form

$$\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}} = \sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}\sigma^{\boldsymbol{b}}{}_{\boldsymbol{BB'}}\boldsymbol{\eta_{ab}}, \tag{5.5}$$

where $\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}$ are the so-called *Infeld-van der Waerden symbols*. These can be regarded as the entries of four (2×2) matrices, $\boldsymbol{a} = \boldsymbol{0}, \ldots, \boldsymbol{3}$. Given $\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}$, one defines the inverse symbol $\sigma_{\boldsymbol{b}}{}^{\boldsymbol{BB'}}$ via the relations

$$\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}\sigma^{\boldsymbol{b}}{}_{\boldsymbol{AA'}} = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}, \qquad \sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}\sigma^{\boldsymbol{a}}{}_{\boldsymbol{BB'}} = \delta_{\boldsymbol{B}}{}^{\boldsymbol{A}}\delta_{\boldsymbol{B'}}{}^{\boldsymbol{A'}}. \tag{5.6}$$

Using these expressions it follows that equation (5.5) can be inverted so that

$$\boldsymbol{\eta_{ab}} = \sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}\sigma_{\boldsymbol{b}}{}^{\boldsymbol{BB'}}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}. \tag{5.7}$$

This equation together with the observation that $\boldsymbol{\eta_{ab}} = \overline{\boldsymbol{\eta_{ab}}}$ leads to

$$\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}} = \overline{\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}}. \tag{5.8}$$

Thus, $\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}$ and $\sigma_{\boldsymbol{b}}{}^{\boldsymbol{BB'}}$ describe *Hermitian matrices*. These matrices satisfy the relations (5.5), (5.6), (5.7) and (5.8) and can be explicitly written as

$$\sigma_{\boldsymbol{0}}{}^{\boldsymbol{AA'}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_{\boldsymbol{1}}{}^{\boldsymbol{AA'}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_{\boldsymbol{2}}{}^{\boldsymbol{AA'}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_{\boldsymbol{3}}{}^{\boldsymbol{AA'}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence, corresponding, up to a normalisation factor, to the so-called *Pauli matrices*. Now, consider an arbitrary vector $\boldsymbol{v} \in T|_p(\mathcal{M})$ and a covector $\boldsymbol{u} \in T^*|_p(\mathcal{M})$. In terms of the bases $\{\boldsymbol{e_a}\}$ and $\{\boldsymbol{\omega^a}\}$, these can be written as

$$\boldsymbol{v} = v^{\boldsymbol{a}}\boldsymbol{e_a}, \qquad v^{\boldsymbol{a}} \equiv \langle \boldsymbol{\omega^a}, \boldsymbol{v}\rangle,$$

$$\boldsymbol{u} = u_{\boldsymbol{a}}\boldsymbol{\omega^a}, \qquad u_{\boldsymbol{a}} \equiv \langle \boldsymbol{u}, \boldsymbol{e_a}\rangle.$$

The components $v^{\boldsymbol{a}}$ and $u_{\boldsymbol{a}}$ can be put in correspondence with the Hermitian spinors $v^{\boldsymbol{AA'}}$ and $u_{\boldsymbol{AA'}}$ using the Infeld-van der Waerden symbols via the rules

$$v^{\boldsymbol{AA'}} = v^{\boldsymbol{a}}\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}},$$

$$u_{\boldsymbol{AA'}} = u_{\boldsymbol{a}}\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}.$$

These correspondences can be extended to tensors of arbitrary rank. For example, given the tensor $T_{ab}{}^c$ its components with respect to $\{\boldsymbol{e_a}\}$ and $\{\boldsymbol{\omega^a}\}$, denoted by $T_{\boldsymbol{ab}}{}^{\boldsymbol{c}}$, are in correspondence with the spinor $T_{\boldsymbol{AA'BB'}}{}^{\boldsymbol{CC'}}$ via the following

$$T_{\boldsymbol{AA'BB'}}{}^{\boldsymbol{CC'}} \equiv T_{\boldsymbol{ab}}{}^{\boldsymbol{c}}\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}\sigma^{\boldsymbol{b}}{}_{\boldsymbol{BB'}}\sigma_{\boldsymbol{c}}{}^{\boldsymbol{CC'}}.$$

The spinor $T_{\boldsymbol{AA'BB'}}{}^{\boldsymbol{CC'}}$ is called the *spinorial counterpart* of the tensor components $T_{\boldsymbol{ab}}{}^{\boldsymbol{c}}$.

5.3.3 Null tetrads

Given a Hermitian spinor $\xi^{AA'} \in \mathfrak{S}^{AA'}$, in terms of the spin bases $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ and $\{\bar{\boldsymbol{o}}, \bar{\boldsymbol{\iota}}\}$ it can be written according to Equation (5.2). Hence every spin basis $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ gives rise to an associated vector basis consisting of null vectors. This null tetrad has the peculiarity of consisting of two real null vectors and two complex null vectors which are the complex conjugate of each other. Hence, let

$$l^{AA'} \equiv o^A\bar{o}^{A'}, \qquad n^{AA'} \equiv \iota^A\bar{\iota}^{A'}, \qquad m^{AA'} \equiv o^A\bar{\iota}^{A'}, \qquad \bar{m}^{AA'} \equiv \iota^A\bar{o}^{A'} \tag{5.9}$$

and let l^a, n^a, m^a and $\bar{m}^a$ denote the tensorial counterparts of the above spinors. One has that

$$l_a n^a = -m_a\bar{m}^a = 1, \tag{5.10}$$

and all the remaining contractions vanish. This null tetrad $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ can be used to construct an orthonormal tetrad $\{\boldsymbol{e_a}\}$ as follows

$$\boldsymbol{e_0} = \frac{1}{\sqrt{2}}(\boldsymbol{l} + \boldsymbol{n}) \tag{5.11a}$$

$$\boldsymbol{e_1} = \frac{1}{\sqrt{2}}(\boldsymbol{m} + \bar{\boldsymbol{m}}) \tag{5.11b}$$

$$\boldsymbol{e_2} = \frac{i}{\sqrt{2}}(\boldsymbol{m} - \bar{\boldsymbol{m}}) \tag{5.11c}$$

$$\boldsymbol{e_3} = \frac{1}{\sqrt{2}}(\boldsymbol{l} - \boldsymbol{n}). \tag{5.11d}$$

The relations (5.10) can be used to show that $\{\boldsymbol{e_a}\}$ is an orthonormal tetrad with $\boldsymbol{e_0}$ being timelike, while $\boldsymbol{e_1}$, $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$ are spacelike. Moreover, since a right-handed phase change in the spin basis of the form

$$o^A \longmapsto e^{i\theta} o^A, \qquad \iota^A \longmapsto e^{-i\theta} \iota^A$$

gives *right-handed rotations*

$$\boldsymbol{e_1} \longmapsto \cos 2\theta \boldsymbol{e_1} + \sin 2\theta \boldsymbol{e_2}, \tag{5.12a}$$

$$\boldsymbol{e_2} \longmapsto -\sin 2\theta \boldsymbol{e_1} + \cos 2\theta \boldsymbol{e_2} \tag{5.12b}$$

with $\boldsymbol{e_0}$ and $\boldsymbol{e_3}$ unchanged, the triad $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ as defined by Equations (5.11b)-(5.11d) is said to be *right-handed*.

5.4 The spinorial structure of the spacetime manifold

The discussion of the previous section has been restricted to spinors at a given point of the spacetime manifold $\mathcal{M}$. A spinorial structure on the whole spacetime manifold $\mathcal{M}$ is called a *spin bundle* and is denoted by $\mathfrak{S}(\mathcal{M})$. In order to relate spinors defined at different points of the spacetime manifold, it is necessary to introduce the concept of *connection* and its associated *covariant derivative*. Hence, the notion of connection needs to be extended so that it applies to *spinorial fields*. A spinorial field is a smooth assignment of a spinor $\xi_{A\ldots CD'\ldots F'}{}^{G\ldots LP'\ldots N}$ to each point of the spacetime manifold. The sets of spinorial fields over $\mathcal{M}$ will be denoted similarly to the sets of spinors at a point, that is, $\mathfrak{S}^\bullet(\mathcal{M})$, $\mathfrak{S}_A(\mathcal{M})$, $\mathfrak{S}^A(\mathcal{M})$ and so on.

5.4.1 The spinorial covariant derivative

A *spinorial covariant derivative* is a map

$$\nabla_{AA'} : \mathfrak{S}^{B\ldots C'}{}_{D\ldots E'}(\mathcal{M}) \longrightarrow \mathfrak{S}^{B\ldots C'}{}_{AD\ldots A'E'}(\mathcal{M}).$$

satisfying the following properties:

(i) Linearity: Given $\xi^{B\ldots C'}{}_{D\ldots E'}, \eta^{B\ldots C'}{}_{D\ldots E'} \in \mathfrak{S}^{B\ldots C'}{}_{D\ldots E'}(\mathcal{M})$,

$$\nabla_{AA'}(\xi^{B\ldots C'}{}_{D\ldots E'} + \eta^{B\ldots C'}{}_{D\ldots E'}) = \nabla_{AA'}\xi^{B\ldots C'}{}_{D\ldots E'} + \nabla_{AA'}\eta^{B\ldots C'}{}_{D\ldots E'}.$$

(ii) Leibnitz rule: Given the fields $\xi^{B\ldots C'}{}_{D\ldots E'} \in \mathfrak{S}^{B\ldots C'}{}_{D\ldots E'}(\mathcal{M})$ and $\eta^{F\ldots G'}{}_{H\ldots I'} \in \mathfrak{S}^{F\ldots G'}{}_{H\ldots I'}(\mathcal{M})$

$$\nabla_{AA'}(\xi^{B\ldots C'}{}_{D\ldots E'}\eta^{F\ldots G'}{}_{H\ldots I'}) = \eta^{F\ldots G'}{}_{H\ldots I'}\nabla_{AA'}\xi^{B\ldots C'}{}_{D\ldots E'} + \xi^{B\ldots C'}{}_{D\ldots E'}\nabla_{AA'}\eta^{F\ldots G'}{}_{H\ldots I'}.$$

(iii) Hermiticity: Given the field $\xi^{B\ldots C'}{}_{D\ldots E'} \in \mathfrak{S}^{B\ldots C'}{}_{D\ldots E'}(\mathcal{M})$ one has that

$$\overline{\nabla_{AA'}\xi^{B\ldots C'}{}_{D\ldots E'}} = \nabla_{AA'}\bar{\xi}^{B\ldots C'}{}_{D\ldots E'}.$$

(iv) Action on scalars: Given a scalar ϕ, then $\nabla_{AA'}\phi$ is the spinorial counterpart of $\nabla_a\phi$.

(v) Representation of derivations: Given a derivation $\mathcal{D}$ on the spinor fields, there exists a spinor $\xi^{AA'}$ such that

$$\mathcal{D}\eta^{B\ldots C'}{}_{D\ldots E'} = \xi^{AA'}\nabla_{AA'}\eta^{B\ldots C'}{}_{D\ldots E'},$$

for all $\eta^{B\ldots C'}{}_{D\ldots E'} \in \mathfrak{S}^\bullet(\mathcal{M})$.

For more details about this construction see [61, 70, 81].

5.4.2 Spin connection coefficients

The *spinorial counterparts of the connection coefficients* $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ are given after suitable contraction with the Infeld-van der Waerden symbols by the spinor components

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}} \equiv \omega^{\boldsymbol{CC'}}{}_{CC'}\nabla_{\boldsymbol{AA'}}e_{\boldsymbol{BB'}}{}^{CC'},$$

where $\nabla_{\boldsymbol{AA'}} \equiv e_{\boldsymbol{AA'}}{}^{AA'}\nabla_{AA'}$ is the *directional covariant derivative* in the direction of $e_{\boldsymbol{AA'}}$. Now, since

$$\omega^{\boldsymbol{CC'}}{}_{CC'} = \epsilon^{\boldsymbol{C}}{}_{C}\bar{\epsilon}^{\boldsymbol{C'}}{}_{C'}, \qquad e_{\boldsymbol{BB'}}{}^{BB'} = \epsilon_{\boldsymbol{B}}{}^{B}\bar{\epsilon}_{\boldsymbol{B'}}{}^{B'}.$$

It follows that

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}} = \epsilon^{\boldsymbol{C}}{}_{C}\delta_{\boldsymbol{B'}}{}^{\boldsymbol{C'}}\nabla_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{B}}{}^{C} + \bar{\epsilon}^{\boldsymbol{C'}}{}_{C'}\delta_{\boldsymbol{B}}{}^{\boldsymbol{C}}\nabla_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{B'}}{}^{C'},$$

so that upon defining the *spin connection coefficients*

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{B}} \equiv \epsilon^{\boldsymbol{C}}{}_{C}\nabla_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{B}}{}^{C}, \tag{5.13}$$

one has

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{BB'}} = \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{B}}\delta_{\boldsymbol{B'}}{}^{\boldsymbol{C'}} + \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{C'}}{}_{\boldsymbol{B'}}\delta_{\boldsymbol{B}}{}^{\boldsymbol{C}}.$$

Furthermore, since

$$\delta_{\boldsymbol{B}}{}^{\boldsymbol{C}} = \epsilon_{\boldsymbol{B}}{}^{Q}\epsilon^{\boldsymbol{C}}{}_{Q}$$

and by requiring that

$$\nabla_{\boldsymbol{AA'}}\delta_{\boldsymbol{B}}{}^{\boldsymbol{C}} = 0,$$

one has that

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{B}} = -\epsilon_{\boldsymbol{B}}{}^{Q}\nabla_{\boldsymbol{AA'}}\epsilon^{\boldsymbol{C}}{}_{Q}.$$

From this relation, it follows that the action of the spinor covariant derivative $\nabla_{\boldsymbol{AA'}}$ on a spinor $\xi_{\boldsymbol{B'}}{}^{\boldsymbol{CC'}}$ is given by

$$\nabla_{\boldsymbol{AA'}}\xi_{\boldsymbol{B'}}{}^{\boldsymbol{CC'}} = e_{\boldsymbol{AA'}}(\xi_{\boldsymbol{B'}}{}^{\boldsymbol{CC'}}) - \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{M'}}{}_{\boldsymbol{B'}}\xi_{\boldsymbol{M'}}{}^{\boldsymbol{CC'}} + \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{N}}\xi_{\boldsymbol{B'}}{}^{\boldsymbol{NC'}} + \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{C'}}{}_{\boldsymbol{N'}}\xi_{\boldsymbol{B'}}{}^{\boldsymbol{CN'}}.$$

This relation can be generalised to spinors of arbitrary valence and several primed and unprimed indices.

5.4.3 Metric and Levi-Civita spin connection coefficients

The *spinorial counterpart of the metric compatibility condition*

$$\nabla_a g_{bc} = 0$$

is given by

$$\nabla_{AA'} g_{BB'CC'} = \nabla_{AA'}(\epsilon_{BC}\epsilon_{B'C'}) = \epsilon_{B'C'}\nabla_{AA'}\epsilon_{BC} + \epsilon_{BC}\nabla_{AA'}\epsilon_{B'C'} = 0,$$

from which one has

$$\nabla_{AA'}\epsilon_{BC} = 0, \qquad \nabla_{AA'}\epsilon_{B'C'} = 0.$$

These relations can be written in an explicit form so that

$$\nabla_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{BC}} = e_{\boldsymbol{AA'}}(\epsilon_{\boldsymbol{BC}}) - \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{P}}{}_{\boldsymbol{B}}\epsilon_{\boldsymbol{PC}} - \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{C}}\epsilon_{\boldsymbol{BQ}} = 0,$$

$$\nabla_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{B'C'}} = e_{\boldsymbol{AA'}}(\epsilon_{\boldsymbol{B'C'}}) - \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{P'}}{}_{\boldsymbol{B'}}\epsilon_{\boldsymbol{P'C'}} - \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{C'}}\epsilon_{\boldsymbol{B'Q'}} = 0.$$

Since $e_{\boldsymbol{AA'}}(\epsilon_{\boldsymbol{BC}}) = 0$, $e_{\boldsymbol{AA'}}(\epsilon_{\boldsymbol{B'C'}}) = 0$ and $\epsilon_{\boldsymbol{BC}}$, $\epsilon_{\boldsymbol{B'C'}}$ are constants, it follows that

$$\Gamma_{\boldsymbol{AA'BC}} = \Gamma_{\boldsymbol{AA'(BC)}}, \qquad \bar{\Gamma}_{\boldsymbol{AA'B'C'}} = \bar{\Gamma}_{\boldsymbol{AA'(B'C')}}.$$

5.4.4 The spinorial curvature

The spinorial counterpart of the curvature tensors can be introduced naturally by looking at the commutator of spinorial covariant derivatives. One can write

$$[\nabla_{AA'}, \nabla_{BB'}]\xi^{CC'} = R^{CC'}{}_{PP'AA'BB'}\xi^{PP'}$$

where $R^{CC'}{}_{DD'AA'BB'}$ is the *spinorial counterpart of the Riemann curvature tensor* $R^c{}_{dab}$ and

$$[\nabla_{AA'}, \nabla_{BB'}] = \nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'} - \Sigma_{AA'}{}^{PP'}{}_{BB'}\nabla_{PP'}$$

with $\Sigma_{AA'}{}^{CC'}{}_{BB'}$ being the *spinorial counterpart of the torsion tensor* $\Sigma_a{}^c{}_b$ of $\boldsymbol{\nabla}$.

In order to express the Riemann curvature spinor $R^{CC'}{}_{DD'AA'BB'}$ in terms of spin connection coefficients one makes use of the expression of the geometric curvature

$$R^{\boldsymbol{c}}{}_{\boldsymbol{dab}} \equiv \partial_{\boldsymbol{a}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) - \partial_{\boldsymbol{b}}(\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}) + \Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{a}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}) + \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{d}}\Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{f}} - \Sigma_{\boldsymbol{a}}{}^{\boldsymbol{f}}{}_{\boldsymbol{b}}\Gamma_{\boldsymbol{f}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}$$

contracted with the Infeld-van der Waerden symbols, so that one has

$$\begin{aligned} R^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} \equiv & \boldsymbol{e}_{\boldsymbol{AA'}}(\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}) - \boldsymbol{e}_{\boldsymbol{BB'}}(\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}) + \Gamma_{\boldsymbol{FF'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}(\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{AA'}} - \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{BB'}}) \\ & + \Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{FF'}} - \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{FF'}} - \Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{BB'}}\Gamma_{\boldsymbol{FF'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}. \end{aligned}$$

Now, using the definition of spin connection coefficients (5.13), one can define

$$\begin{aligned} R^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}} \equiv & \boldsymbol{e}_{\boldsymbol{AA'}}(\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}) - \boldsymbol{e}_{\boldsymbol{BB'}}(\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}) - \Gamma_{\boldsymbol{FB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{B}} \\ & - \Gamma_{\boldsymbol{BF'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{F'}}{}_{\boldsymbol{B'}} + \Gamma_{\boldsymbol{FA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{A}} + \Gamma_{\boldsymbol{AF'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\Gamma}_{\boldsymbol{BB'}}{}^{\boldsymbol{F'}}{}_{\boldsymbol{A'}} \\ & + \Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{F}} - \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{F}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{F}} - \Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{BB'}}\Gamma_{\boldsymbol{FF'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}} \end{aligned}$$

so that

$$R^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} = R^{\boldsymbol{C}}{}_{\boldsymbol{DAA'BB'}}\delta_{\boldsymbol{D'}}{}^{\boldsymbol{C'}} + \bar{R}^{\boldsymbol{C'}}{}_{\boldsymbol{D'AA'BB'}}\delta_{\boldsymbol{D}}{}^{\boldsymbol{C}} \tag{5.15}$$

which can be regarded as the *first Cartan structure equation*.

5.4.5 The spinorial Ricci tensor and Ricci scalar

To introduce the spinorial counterpart of the Ricci tensor and Ricci scalar, we consider Equation (5.15) contracted with the spinors ϵ_{FC} and $\epsilon_{F'C'}$ so that

$$R_{CC'DD'AA'BB'} = -R_{CDAA'BB'}\epsilon_{C'D'} - \bar{R}_{C'D'AA'BB'}\epsilon_{CD}. \tag{5.16}$$

One then uses the following skew-symmetry

$$R_{CDAA'BB'} = -R_{CDBB'AA'}$$

along with the split

$$R_{CDAA'BB'} = R_{CDAB}\epsilon_{A'B'} + \bar{R}_{CDA'B'}\epsilon_{AB}$$

so that one has

$$R_{CDAA'BB'} = \frac{1}{2}R_{CDAQ'B}{}^{Q'}\epsilon_{A'B'} + \frac{1}{2}R_{CDA'QB'}{}^{Q}\epsilon_{AB} \tag{5.17}$$

The spinorial counterparts of the Ricci tensor and scalar, $R_{AA'BB'}$ and R, are obtained from Equations (5.16) and (5.17) as

$$R_{AA'BB'} = -\frac{1}{2}R_{PA}{}^{P}{}_{BQ'}{}^{Q'}\epsilon_{A'B'} - \frac{1}{2}\bar{R}_{P'A'}{}^{P'}{}_{B'Q}{}^{Q}\epsilon_{AB} + R_{ABA'B'Q}{}^{Q},$$

$$R = -2R_{PQE'}{}^{PQE'}.$$

The spinorial counterpart of the symmetric trace-free part of the Ricci tensor Φ_{ab} is defined as

$$\Phi_{ABA'B'} \equiv \frac{1}{2}R_{AA'BB'} - \frac{1}{8}R\epsilon_{AB}\epsilon_{A'B'}$$

and satisfies the following symmetries

$$\Phi_{ABA'B'} = \Phi_{BAA'B'} = \Phi_{ABB'A'} = \Phi_{BAB'A'}. \tag{5.19}$$

5.5 Space-spinor formalism

In the remaining part of this chapter, it is discussed the *space-spinor* formalism. This constitutes a framework for spinors in which a further structure is introduced — the so-called *Hermitian inner product* and it can be used to describe foliations of spacetimes. In particular, it provides a suitable description of the geometry of three-dimensional Riemannian manifolds. The notion of space spinors provides a systematic approach to the construction of evolution equations which can be regarded as the spinorial equivalent of the $1 + 3$ decomposition for tensors.

5.5.1 The Hermitian inner product

Let $(\mathcal{M}, \boldsymbol{g})$ denote a four-dimensional Lorentzian manifold. At each point $p \in \mathcal{M}$ is associated a two-dimensional symplectic vector space $\mathfrak{S}|_p(\mathcal{M})$.

A *Hermitian inner product* on a symplectic two-dimensional vector space $\mathfrak{S}$ is a function

$$\langle \cdot, \cdot \rangle : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathbb{C}$$

which is

(i) Hermitian: given $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{S}$

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \overline{\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle}.$$

(ii) Linear in the second entry: given $\boldsymbol{\xi}, \boldsymbol{\eta}, \zeta \in \mathfrak{S}$ and $z \in \mathbb{C}$ one has

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} + z\boldsymbol{\zeta} \rangle = \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle + z\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle.$$

(iii) Positive definite: given $\boldsymbol{\xi} \in \mathfrak{S}$ one has

$$\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle \geq 0$$

and

$$\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = 0 \qquad \text{if and only if} \quad \boldsymbol{\xi} = 0.$$

As a consequence of (i) and (ii) it follows that given $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathfrak{S}$ and $z \in \mathbb{C}$ one has

$$\langle \boldsymbol{\xi} + z\boldsymbol{\zeta}, \boldsymbol{\eta} \rangle = \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle + \bar{z}\langle \boldsymbol{\zeta}, \boldsymbol{\eta} \rangle$$

i.e. the Hermitian inner product is *antilinear* in the first entry.

5.5.2 Hermitian conjugation

The Hermitian inner product can be expressed using a Hermitian spinor $\varpi_{AA'} \in \mathfrak{S}_{AA'}(\mathcal{M})$ such that

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \varpi_{AA'}\bar{\xi}^{A'}\eta^{A}. \tag{5.20}$$

Given a spinor basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$, the components of $\varpi_{AA'}$ with respect to the basis are given by

$$\varpi_{\boldsymbol{AA'}} \equiv \varpi_{AA'}\epsilon_{\boldsymbol{A}}{}^{A}\epsilon_{\boldsymbol{A'}}{}^{A'}.$$

These components $\varpi_{AA'}$ are the entries of a diagonalisable (2×2) matrix whose eigenvalues are positive, in accordance with the *positivity condition*. Furthermore, the scaling of the basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ can be fixed so that the matrix $(\varpi_{AA'})$ reduces

$$(\varpi_{AA'}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

—i.e. the identity matrix. As a consequence of this normalisation condition, one has

$$\varpi_{AA'} = o_{A}\bar{o}_{A'} + \iota_{A}\bar{\iota}_{A'} = \epsilon^{\boldsymbol{1}}{}_{A}\bar{\epsilon}^{\boldsymbol{1'}}{}_{A'} + \epsilon^{\boldsymbol{0}}{}_{A}\bar{\epsilon}^{\boldsymbol{0'}}{}_{A'}, \tag{5.21a}$$

$$\varpi_A{}^{A'} = o_A\bar{o}^{A'} + \iota_A\bar{\iota}^{A'} = \epsilon^{\mathbf{1}}{}_A\bar{\epsilon}_{\mathbf{0}'}{}^{A'} - \epsilon^{\mathbf{0}}{}_A\bar{\epsilon}_{\mathbf{1}'}{}^{A'}, \tag{5.21b}$$

$$\varpi^{AA'} = o^A\bar{o}^{A'} + \iota^A\bar{\iota}^{A'} = \epsilon_{\mathbf{0}}{}^A\bar{\epsilon}_{\mathbf{0}'}{}^{A'} + \epsilon_{\mathbf{1}}{}^A\bar{\epsilon}_{\mathbf{1}'}{}^{A'}. \tag{5.21c}$$

From this, it follows that

$$\varpi_{AA'}\varpi^{A'B} = \delta_A{}^B. \tag{5.22}$$

In particular, $\varpi_{AA'}\varpi^{AA'} = 2$.

The *operation of Hermitian conjugation* † is a map

$$^\dagger : \mathfrak{S}^\bullet(\mathcal{M}) \longrightarrow \mathfrak{S}^\bullet(\mathcal{M})$$

such that given $\mu_A \in \mathfrak{S}(\mathcal{M})$, its *Hermitian conjugate* $\mu^\dagger{}_A$ is defined as

$$\mu^\dagger{}_A \equiv \varpi_A{}^{A'}\bar{\mu}_{A'}. \tag{5.23}$$

From this definition, it follows that one can write

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \varpi_{AA'}\bar{\xi}^{A'}\eta^A = -\bar{\xi}^\dagger{}_A\eta^A = \eta_A\bar{\xi}^{\dagger A}. \tag{5.24}$$

This operation can be extended to higher valence spinors by requiring that given $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathfrak{S}^\bullet(\mathcal{M})$ one has

$$(\boldsymbol{\mu}\boldsymbol{\lambda})^\dagger = \boldsymbol{\mu}^\dagger\boldsymbol{\lambda}^\dagger.$$

Since $\varpi_{AA'}$ is the identity matrix, it follows that

$$\mu^{\dagger\dagger}{}_{A_1\ldots A_k} = (-1)^k\mu_{A_1\ldots A_k}.$$

Furthermore, due to the *positive definite condition*, one has

$$\mu^{\dagger A}\mu_A = 0 \qquad \text{if and only if } \mu_A = 0.$$

The Hermitian conjugate of the $\boldsymbol{\epsilon}$-spinor is provided by

$$\epsilon^\dagger{}_{AB} = \varpi_A{}^{A'}\varpi_B{}^{B'}\epsilon_{A'B'} = \epsilon_{AB}.$$

5.5.3 Timelike congruences

Let $\mathcal{S}$ be a spacelike hypersurface of the spacetime $(\mathcal{M}, \boldsymbol{g})$. It is possible to postulate the existence of a spinorial structure on this hypersurface. A sufficient condition is that the vacuum Einstein field equations can be solved on $\mathcal{S}$. Hence, since this spacetime is *globally hyperbolic*, it admits a spinorial structure. Furthermore, the $\boldsymbol{g}$-normal to $\mathcal{S}$ induces the operation of Hermitian conjugation. The resulting spinorial structure on $\mathcal{S}$ is denoted by $\mathfrak{S}(\mathcal{S})$ endowed with the operation of Hermitian conjugation †. Let $\boldsymbol{\tau}$ be a *future-directed timelike vector* of the spacetime $(\mathcal{M}, \boldsymbol{g})$. Let $\mathcal{S}_\tau$ be the hyperplanes generated by $\boldsymbol{\tau}$. Since this vector is not necessarily hypersurface orthogonal, these hyperplanes do not necessarily coincide with the tangent bundles to the leaves of the foliation of $\mathcal{M}$. Let $\tau^{AA'}$ be the spinorial counterpart of $\boldsymbol{\tau}$, normalised so that

$$\boldsymbol{g}(\boldsymbol{\tau}, \boldsymbol{\tau}) = 2.$$

It is possible to identify this spinor with $\tau^{AA'} = \varpi^{AA'}$ expressing the Hermitian inner product. More precisely, the spinor $\tau^{AA'}$ induces a Hermitian product on $\mathcal{S}$, so that given $\xi^A, \eta^A \in \mathfrak{S}(\mathcal{S})$ one has

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \tau_{AA'} \bar{\xi}^{A'} \eta^A. \tag{5.25}$$

This is due to $\tau^{AA'}$ being the spinorial counterpart of a spacetime vector, so that

$$\overline{\tau_{AA'} \bar{\xi}^{A'} \eta^A} = \tau_{AA'} \bar{\eta}^{A'} \xi^A \tag{5.26}$$

—i.e. $\tau^{AA'}$ is a Hermitian spinor. Moreover, since $\tau^{AA'}$ is timelike future-directed and the vector $\xi^A \bar{\xi}^{A'}$ describes a future-directed null vector, it follows that

$$\tau_{AA'} \xi^A \bar{\xi}^{A'} \geq 0,$$

justifying the choice $\tau^{AA'} = \varpi^{AA'}$. Hence, there exists a spin basis $\{\epsilon_{\boldsymbol{A}}{}^A\}$ such that

$$\tau^{AA'} = \epsilon_{\boldsymbol{0}}{}^A \epsilon_{\boldsymbol{0}'}{}^{A'} + \epsilon_{\boldsymbol{1}}{}^A \epsilon_{\boldsymbol{1}'}{}^{A'}$$

and

$$\tau_{AA'} \tau^{BA'} = \epsilon_A{}^B. \tag{5.27}$$

The tensorial counterpart of a spinor $\mu_{A_1 A'_1 \dots A_k A'_k}$ can be expanded in terms of the spatial frame $\{\boldsymbol{\epsilon_{AB}}\}$ if and only if the following conditions hold

$$\tau^{A_1 A'_1}\mu_{A_1 A'_1 \dots A_k A'_k} = 0, \qquad \dots \qquad \tau^{A_k A'_k}\mu_{A_1 A'_1 \dots A_k A'_k} = 0$$

—i.e. the spinor $\mu_{A_1 A'_1 \dots A_k A'_k}$ is *spatial* with respect to $\boldsymbol{\tau}$. Hence, the space spinor counterpart is given by

$$\mu_{A_1 B_1 \dots A_k B_k} = \tau_{B_1}{}^{A'_1} \dots \tau_{B_k}{}^{A'_k}\mu_{A_1 A'_1 \dots A_k A'_k} = \mu_{(A_1 B_1)\dots(A_k B_k)}.$$

To obtain the space spinor counterpart of tensors that are not spatial, one introduces the *projector*

$$h^{BB'}{}_{AA'} \equiv \epsilon_A{}^B \epsilon_{A'}{}^{B'} - \frac{1}{2}\tau_{AA'}\tau^{BB'},$$

so that given a non-spatial spinor $\xi_{A_1 A'_1 \dots A_k A'_k}$ its projection is the spatial spinor

$$\xi_{A_1 A'_1 \dots A_k A'_k} h^{A_1 A'_1}{}_{B_1 B'_1} \dots h^{A_k A'_k}{}_{B_k B'_k}.$$

Then, the space spinor version of the above spatial spinor is obtained by contracting the primed indices with $\tau_C{}^{B'}$

$$\xi_{(A_1 C_1)\dots(A_k C_k)} = \tau_{C_1}{}^{B'_1} \dots \tau_{C_k}{}^{B'_k} h^{A_1 A'_1}{}_{B_1 B'_1} \dots h^{A_k A'_k}{}_{B_k B'_k} \xi_{A_1 A'_1 \dots A_k A'_k}.$$

The pure time components of $\xi_{A_1 A'_1 \dots A_k A'_k}$ can be obtained by contraction of each primed-unprimed pair of indices with $\tau^{AA'}$ as

$$\xi_{A_1}{}^{A_1} \dots {}_{A_K}{}^{A_K} = \tau^{A_1 A'_1} \dots \tau^{A_k A'_k} \xi_{A_1 A'_1 \dots A_k A'_k}.$$

Eventually, the mixed time-spatial components of $\xi_{A_1 A'_1 \dots A_k A'_k}$ are obtained by suitable contractions with $\tau^{AA'}$ and $\tau_B{}^{B'}$.

A useful example for this discussion is provided by a Hermitian spinor $v_{AA'} \in \mathfrak{S}^\bullet(\mathcal{M})$ for which one has the following space-spinor decomposition

$$v_{AA'} = \frac{1}{2}\tau_{AA'} v_{PP'} \tau^{PP'} - \tau^B{}_{A'} v_{(BA)} = \frac{1}{2}\tau_{AA'} v - \tau^Q{}_{A'} v_{(QA)}$$

where $v \equiv v_{PP'}\tau^{PP'}$. Upon defining $v_{AB} \equiv \tau_B{}^{A'} v_{AA'}$ and observing that $v = v_Q{}^Q$ one can rewrite the previous equation as

$$v_{AB} = \frac{1}{2}\epsilon_{AB} v + v_{(AB)}.$$

5.5.4 The Sen connection and the acceleration vector

The space spinor counterpart of the spacetime spinorial covariant derivative $\nabla_{AA'}$ is obtained by contraction with the timelike spinor $\tau^{AA'}$ as

$$\nabla_{AB} \equiv \tau_B{}^{A'}\nabla_{AA'}$$

which in turn can be written as

$$\nabla_{AB} = \frac{1}{2}\epsilon_{AB}\mathcal{D} + \mathcal{D}_{AB}, \tag{5.28}$$

where the operator $\mathcal{D}$ is the *directional derivative* of the connection $\boldsymbol{\nabla}$ in the direction of $\boldsymbol{\tau}$ defined by

$$\mathcal{D} \equiv \tau^{AA'}\nabla_{AA'},$$

whereas the operator $\mathcal{D}_{AB}$ is the *Sen connection* of $\boldsymbol{\nabla}$ associated to $\boldsymbol{\tau}$, defined as

$$\mathcal{D}_{AB} \equiv \tau_{(B}{}^{A'}\nabla_{A)A'}.$$

According to these definitions, one has

$$\nabla_{AA'} = \frac{1}{2}\tau_{AA'}\mathcal{D} - \tau_{A'}{}^{Q}\mathcal{D}_{AQ}. \tag{5.29}$$

Remark 42. The timelike vector $\boldsymbol{\tau}$ is not necessarily hypersurface orthogonal, so the Sen connection has non-vanishing torsion which can be expressed in terms of the covariant derivative of $\tau^{AA'}$. Furthermore, even in case of $\boldsymbol{\tau}$ being hypersurface orthogonal, $\mathcal{D}_{AB}$ doesn't coincide with the Levi-Civita connection $\boldsymbol{D}$ of the intrinsic 3-metric of the hypersurfaces $\mathcal{S}_\tau$ orthogonal to $\boldsymbol{\tau}$.

Given the spinor $\boldsymbol{\chi}$ defined as

$$\chi_{ABCD} \equiv \frac{1}{\sqrt{2}}\tau_D{}^{C'}\nabla_{AB}\tau_{CC'}. \tag{5.30}$$

Using the decomposition of ∇_{AB} given by Equation (5.28), one has

$$\chi_{ABCD} = \frac{1}{2}\epsilon_{AB}\chi_{CD} + \chi_{(AB)CD}$$

where

$$\chi_{AB} \equiv \frac{1}{\sqrt{2}}\tau_B{}^{A'}\mathcal{D}\tau_{AA'}, \qquad \chi_{(AB)CD} \equiv \frac{1}{\sqrt{2}}\tau_D{}^{C'}\mathcal{D}_{AB}\tau_{CC'}.$$

The spinor χ_{AB} corresponds to the *acceleration vector* of $\boldsymbol{\tau}$, whereas $\chi_{(AB)CD}$ is related to the *Weingarten tensor* of the distribution defined by $\boldsymbol{\tau}$.

In the *hypersurface orthogonal case* the spinor $\chi_{(AB)CD}$ corresponds to the extrinsic curvature of the orthogonal hypersurfaces $\mathcal{S}_\tau$, the covariant derivative D_{AB} acts on a given spinor ξ_C as

$$D_{AB}\xi_C \equiv \mathcal{D}_{AB}\xi_C + \frac{1}{\sqrt{2}}\chi_{(AB)C}{}^Q\xi_Q \tag{5.31}$$

and it is torsion-free. Since

$$\mathcal{D}_{AB}\epsilon_{CD} = 0 \tag{5.32}$$

and using the symmetry $\chi_{ABCD} = \chi_{AB(CD)}$, one has

$$D_{AB}\epsilon_{CD} = 0. \tag{5.33}$$

Hence, D_{AB} coincides with the spinorial counterpart of the Levi-Civita connection of the leaves of the foliation defined by $\boldsymbol{\tau}$. Furthermore, where $\mathcal{D}_{AB}$ is not a real differential operator, D_{AB} satisfies

$$(D_{AB}\xi_C)^\dagger = -D_{AB}\xi^\dagger{}_C.$$

Hence, D_{AB} is a real differential operator.

The space-spinor formalism is used in Chapter 6 to exploit the symmetry properties of the Maxwell-scalar field system to contruct a suitable system of evolution equations.

Chapter 6

The Maxwell-scalar field system near spatial infinity

6.1 Introduction

Among the main open questions in Mathematical Relativity, there is the so-called *problem of spatial infinity* —see e.g. [36]. This problem concerns the understanding of the consequences of the degeneracy of the conformal structure of the spacetime at spatial infinity. A systematical method to tackle this problem goes back to the seminal work of Friedrich [28]. The key idea of this work is the development of a representation of spatial infinity, the so-called *F-gauge*, which allows the formulation of a regular Cauchy problem in a neighbourhood of spatial infinity for the *conformal Einstein field equations*. In this setting, it is possible to show that, unless the initial data is fine-tuned, the solutions to the conformal Einstein field equations develop two types of logarithmic singularities at the *critical sets* $\mathcal{I}^{\pm}$ where null infinity meets spatial infinity. There are logarithmic singularities associated to the linear part of the equations and the ones associated to the non-linear equations which appear at higher order in the expansion. In the particular case of *time-symmetric initial data sets* for the Einstein field equations which admit a point compactification at infinity for which the resulting *conformal metric* is analytic, it is shown in [25] that a certain subset of the logarithmic singularities can be avoided if the conformal metric $\boldsymbol{h}$ satisfies the conformally invariant condition

$$D_{\{i_p \ldots i_1} b_{jk\}} = 0, \qquad p = 0, 1, 2, \ldots,$$

where b_{jk} denotes the Cotton-Bach tensor of the metric $\boldsymbol{h}$ and $\{\ldots\}$ denotes the operation of computing the symmetric trace-free part, in particular, if $\boldsymbol{h}$ is conformally flat then $b_{jk} = 0$. Although this condition is necessary to avoid logarithmic singularities at the critical sets it is not sufficient. It has been shown that static solutions to the Einstein field equations are logarithmic-free at the critical points of Friedrich's representation of spatial infinity. Moreover, the analysis in [79, 80] strongly suggests the conjecture that, among the class of time-symmetric initial data sets, only those which are static in a neighbourhood of infinity will give rise to developments which are free of logarithmic singularities at the critical sets. The gluing techniques developed in e.g. [14, 12] allow the construction of large classes of initial data sets with this property.

In general, linearised fields propagating on the Minkowski spacetime also develop logarithmic singularities at the critical sets —see e.g. [77, 78]. In particular, the Maxwell field system provides useful insights to study the linearised gravitational field and as a model for the Bianchi equations satisfied by the components of the Weyl tensor. Looking beyond linear model problems for the Einstein field equations, it is natural to look for systems which can be used to understand the effects of the non-linear interactions on the regularity of solutions at the conformal boundary.

In this chapter, we consider the possibility of using the *Maxwell-scalar field system on the Minkowski spacetime* for this purpose. More precisely, we study the asymptotic properties of the Maxwell-scalar field system near spatial infinity. The content of this chapter is based on:

M. Minucci, R. Panosso Macedo, & J. A. Valiente Kroon, *The Maxwell-scalar field system near spatial infinity*, J. Math. Phys. **63**, 082501 (2022).

6.1.1 The Maxwell-scalar field system

The Maxwell-scalar field system is described by the Maxwell equations with sources coupled with a conformally invariant wave equation. The coupling is realised by means of the covariant derivative. The main considerations in this chapter allow us to formulate a regular finite initial value problem for this system near spatial infinity. More precisely, we develop a theory for the solutions to these equations in a neighbourhood of spatial infinity —in particular, the *solution jets* at the cylinder at spatial infinity. This can be done by studying their asymptotic expansions near spatial infinity with a technique that goes under the name of *F-expansions*. This construction exploits the fact that the cylinder at spatial infinity, $\mathcal{I}$, is a *total characteristic* of the evolution equations associated with the Maxwell-scalar field system. Accordingly, the evolution equations reduce to an interior system (*transport equations*) upon evaluation on the cylinder $\mathcal{I}$. These transport equations allows us to relate the properties of the initial data, as defined on a fiduciary initial hypersurface $\mathcal{S}_\star$, with radiative properties of the solution which are defined at null infinity $\mathscr{I}^\pm$ and fully determine the solution jets on the cylinder at spatial infinity.

6.1.2 The main result

The main outcome of this analysis is contained in the following theorem:

Main Result 3. *For generic analytic data for the Maxwell-scalar field system with finite energy, the solution jets on the cylinder at spatial infinity $\mathcal{I}$ develop logarithmic singularities at the critical sets $\mathcal{I}^\pm$.*

In other words, generic solutions to the Maxwell-scalar field system are singular at the critical sets $\mathcal{I}^\pm$. Under the further assumption that these singularities propagate along null infinity, it is possible to analyse the consequences of these singularities on the *peeling* properties of the Maxwell and scalar fields. One has the following corollary:

Corollary 3. *If the solution jets give rise to a solution to the Maxwell-scalar field system near $\mathcal{I}$, then the Maxwell-scalar field system generically has logarithmic singularities which* *spread along the conformal boundary destroying the smoothness of the Faraday tensor and scalar field tensor along the conformal boundary. In particular, there is no classical peeling behaviour at null infinity.*

Although the content of the Main Result is analogous to what it is obtained in the case of the Einstein field equations, *the detailed analysis leading to the result shows that, in fact, the Maxwell-scalar system is not a good model problem as the elements of the solution jets are more singular at the critical sets than what a direct extrapolation from the vacuum conformal Einstein field equations would suggest.* This new singular behaviour can be traced back to the cubic coupling between the Maxwell and scalar fields. The latter is the most important insight obtained from our analysis.

6.2 The cylinder at spatial infinity and the F-gauge

The purpose of this section is to provide a succinct discussion of Friedrich's representation of the neighbourhood of spatial infinity for the Minkowski spacetime. This conformal representation, known as *the cylinder at spatial infinity*, is well suited to analyse the behaviour of fields near spatial infinity. In this representation, spatial infinity i^0 which corresponds to a point in the standard representation of the Minkowski spacetime is blown up to a 2-sphere. Further details on this construction can be found in [28, 74, 40].

6.2.1 Conformal extensions of the Minkowski spacetime

We start with the Minkowski metric $\tilde{\boldsymbol{\eta}}$ in spatial spherical coordinates $(\tilde{t}, \tilde{\rho}, \theta, \phi)$ as given by Equation (1.4) with $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, $\tilde{t} \in (-\infty, \infty)$, $\tilde{\rho} \in [0, \infty)$ and where $\boldsymbol{\sigma}$ denotes the standard metric on $\mathbb{S}^2$. A strategy to construct a conformal representation of the Minkowski spacetime close to i^0 is to make use of *inversion coordinates* $(x^\alpha) = (t, x^i)$ defined by —see [70]—

$$x^\mu = -\tilde{x}^\mu / \tilde{X}^2, \qquad \tilde{X}^2 \equiv \tilde{\eta}_{\mu\nu}\tilde{x}^\mu \tilde{x}^\nu,$$

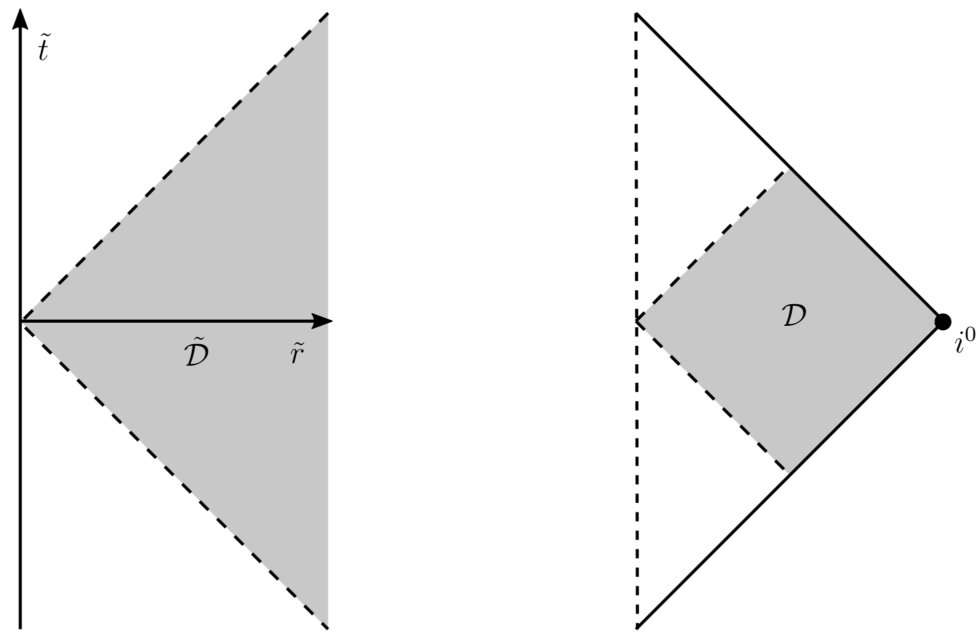

Figure 6.1: Left: The region $\tilde{\mathcal{D}}$, the complement of the light cone through the origin, in the physical Minkowski spacetime. Intuitively, this region contains spatial infinity. Right: the corresponding region $\mathcal{D}$ in the Penrose diagram of the Minkowski spacetime.

which is valid in the domain

$$\tilde{\mathcal{D}} \equiv \{p \in \mathbb{R}^4 \mid \eta_{\mu\nu}\tilde{x}^\mu(p)\tilde{x}^\nu(p) < 0\},$$

representing the complement of the light cone through the origin.

The inverse transformation is given by

$$\tilde{x}^\mu = -x^\mu/X^2, \qquad X^2 = \eta_{\mu\nu}x^\mu x^\nu.$$

Observe, in particular, that $X^2 = 1/\tilde{X}^2$. Using these coordinates one identifies a conformal representation of the Minkowski spacetime with *unphysical metric* given by

$$\bar{\boldsymbol{\eta}} = \Xi^2\tilde{\boldsymbol{\eta}}, \qquad \Xi \equiv X^2,$$

where

$$\bar{\boldsymbol{\eta}} = \eta_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu.$$

Thus, one has a conformal representation of Minkowski spacetime which is also flat.

The introduction of an *unphysical radial coordinate* via the relation $\rho^2 \equiv \delta_{\alpha\beta}x^\alpha x^\beta$, allows us to write the metric $\bar{\boldsymbol{\eta}}$ as

$$\bar{\boldsymbol{\eta}} = \mathbf{d}t \otimes \mathbf{d}t - \mathbf{d}\rho \otimes \mathbf{d}\rho - \rho^2 \boldsymbol{\sigma}, \qquad \Xi = t^2 - \rho^2,$$

with $t \in (-\infty, \infty)$ and $\rho \in (0, \infty)$. In this conformal representation, spatial infinity i^0 corresponds to the origin of the domain

$$\mathcal{D} \equiv \{p \in \mathbb{R}^4 \mid \eta_{\mu\nu}x^\mu(p)x^\nu(p) < 0\}.$$

This region contains the asymptotic region of the Minkowski spacetime around spatial infinity. The relation between the two representations of spatial infinity is illustrated in Figure 6.1. Observe that $(\tilde{t}, \tilde{\rho})$ are related to (t, ρ) via

$$\tilde{t} = -\frac{t}{t^2 - \rho^2}, \qquad \tilde{\rho} = -\frac{\rho}{t^2 - \rho^2}.$$

Finally, introducing a time coordinate τ through the relation $t = \rho\tau$ one finds that the metric $\bar{\boldsymbol{\eta}}$ can be written as

$$\bar{\boldsymbol{\eta}} = \rho^2 \mathbf{d}\tau \otimes \mathbf{d}\tau - (1-\tau^2)\mathbf{d}\rho \otimes \mathbf{d}\rho + \rho\tau \mathbf{d}\rho \otimes \mathbf{d}\tau + \rho\tau \mathbf{d}\tau \otimes \mathbf{d}\rho - \rho^2 \boldsymbol{\sigma}.$$

6.2.2 The cylinder at spatial infinity

The conformal representation containing the *cylinder at spatial infinity* is obtained by considering the rescaled metric

$$\boldsymbol{\eta} \equiv \frac{1}{\rho^2}\bar{\boldsymbol{\eta}}. \tag{6.1}$$

Introducing the coordinate $\varrho \equiv -\ln\rho$ the metric $\boldsymbol{\eta}$ can be reexpressed as

$$\boldsymbol{\eta} = \mathbf{d}\tau \otimes \mathbf{d}\tau - (1-\tau^2)\mathbf{d}\varrho \otimes \mathbf{d}\varrho - \tau(\mathbf{d}\tau \otimes \mathbf{d}\varrho + \mathbf{d}\varrho \otimes \mathbf{d}\tau) - \boldsymbol{\sigma}.$$

Observe that spatial infinity i^0, which is at infinity with respect to the metric $\boldsymbol{\eta}$, corresponds to a set which has the topology of $\mathbb{R} \times \mathbb{S}^2$ —see [28, 1]. Following the previous discussion, one considers the conformal extension $(\mathcal{M}, \boldsymbol{\eta})$ where

$$\boldsymbol{\eta} = \Theta^2 \tilde{\boldsymbol{\eta}}, \qquad \Theta \equiv \rho(1-\tau^2),$$

and

$$\mathcal{M} \equiv \Big\{ p \in \mathbb{R}^4 \mid -1 \leq \tau \leq 1, \;\; \rho(p) \geq 0 \Big\}.$$

In this representation future and past null infinity are described by the sets

$$\mathscr{I}^+ \equiv \Big\{ p \in \mathcal{M} \mid \tau(p) = 1 \Big\}, \qquad \mathscr{I}^- \equiv \Big\{ p \in \mathcal{M} \mid \tau(p) = -1 \Big\},$$

while the physical Minkowski spacetime can be identified with the set

$$\tilde{\mathcal{M}} \equiv \Big\{ p \in \mathcal{M} \mid -1 < \tau(p) < 1, \;\; \rho(p) > 0 \Big\}.$$

In addition, the following sets play a role in our discussion:

$$\mathcal{I} \equiv \Big\{ p \in \mathcal{M} \mid \; |\tau(p)| < 1, \; \rho(p) = 0 \Big\},$$

corresponding to the *cylinder at spatial infinity*, and

$$\mathcal{I}^+ \equiv \Big\{ p \in \mathcal{M} \mid \tau(p) = 1, \; \rho(p) = 0 \Big\}, \qquad \mathcal{I}^- \equiv \Big\{ p \in \mathcal{M} \mid \tau(p) = -1, \; \rho(p) = 0 \Big\},$$

which describe the *critical sets* where null infinity touches spatial infinity. Additionally, let

$$\tilde{\mathcal{S}}_\star = \{ p \in \mathbb{R}^4 \mid \tilde{t}(p) = 0 \}, \qquad \mathcal{S}_\star = \{ p \in \mathcal{M} \mid \tau(p) = 0 \},$$

describing the time-symmetric hypersurface of the Minkowski spacetime. The region where $\mathcal{S}_\star$ intersect $\mathcal{I}$ is denoted with $\mathcal{I}^0$. A schematic representation of these sets is shown in Figure 6.2.

6.3 The Maxwell-scalar field system

In this section, we provide a brief account of the Maxwell-scalar field system with particular attention to its conformal properties and formulation in terms of spinors.

6.3.1 Equations in the physical spacetime

In the following let $\tilde{F}_{ab}$ denote an antisymmetric tensor (*the Faraday tensor*) over a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and let $\tilde{\nabla}$ be the Levi-Civita connection of the metric $\tilde{\boldsymbol{g}}$. The Maxwell equations with source are given by

$$\tilde{\nabla}^a \tilde{F}_{ab} = \tilde{J}_b, \tag{6.2a}$$

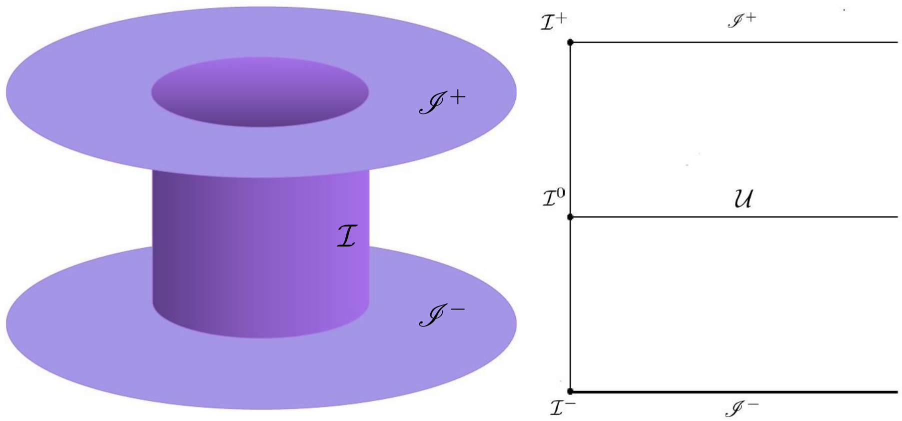

Figure 6.2: Left: schematic representation of the cylinder at spatial infinity of the Minkowski spacetime in the so-called *F-gauge* where null infinity corresponds to the locus of points with $\tau = \pm 1$. The cylinder $\mathcal{I}$ is a total characteristic of Maxwell-scalar field equations. Right: longitudinal section in which the angular dependence has been suppressed. Here $\mathcal{U}$ denotes an open set in a neighbourhood of i and $\mathcal{M}(\mathcal{U})$ its development; $\mathcal{I}^{\pm}$ are the critical sets where the cylinder meets spatial infinity and $\mathcal{I}^0$ is the intersection of the cylinder with the initial hypersurface. These figures are coordinated rather than conformal diagrams —in particular, conformal geodesics correspond to vertical lines. This picture is inspired by [81].

$$\tilde{\nabla}_{[a}\tilde{F}_{bc]} = 0. \tag{6.2b}$$

The homogeneous equation (6.2b) is automatically satisfied if one sets

$$\tilde{F}_{ab} = \tilde{\nabla}_a\tilde{A}_b - \tilde{\nabla}_b\tilde{A}_a,$$

where $\tilde{A}_a$ denotes the *4-vector gauge potential*. Coupled to the above, we consider the *conformally invariant wave equation*

$$\tilde{\mathfrak{D}}_a\tilde{\mathfrak{D}}^a\tilde{\phi} - \frac{1}{6}\tilde{R}\tilde{\phi} = 0, \tag{6.3}$$

where $\tilde{\phi}$ denotes a complex scalar field. The coupling between the Maxwell field and the scalar field is encoded in the covariant derivative

$$\tilde{\mathfrak{D}}_a = \tilde{\nabla}_a - \mathrm{i}\mathfrak{q}\tilde{A}_a,$$

where $\mathfrak{q}$ is a coupling constant (the *charge*). The current $\tilde{J}_a$ in the inhomogeneous equation (6.2a) is given by

$$\tilde{J}_a = \mathrm{i}\mathfrak{q}\left(\bar{\tilde{\phi}}\tilde{\mathfrak{D}}_a\tilde{\phi} - \tilde{\phi}\overline{(\tilde{\mathfrak{D}}_a\tilde{\phi})}\right)$$

Gauge invariance

The Maxwell-scalar field system (6.2a)-(6.2b) and (6.3) is invariant under the *gauge transformation*

$$\tilde{\phi} \rightarrow \tilde{\phi}' = e^{\mathrm{i}\chi}\tilde{\phi}, \qquad \tilde{A}_a \rightarrow \tilde{A}'_a = \tilde{A}_a + \nabla_a\chi, \tag{6.4}$$

in the sense that $\tilde{F}_{ab}$ and $\tilde{J}_a$ are not affected by the transformation. Moreover, the *Lorenz gauge condition*

$$\tilde{\nabla}^a\tilde{A}_a = 0, \tag{6.5}$$

is preserved by the transformation (6.4) for any χ such that

$$\square\chi = 0.$$

Even with the Lorenz gauge condition imposed, there is some residual gauge freedom left. This residual gauge freedom can be fixed at the level of the initial conditions —in particular, there is a natural choice which allows to control the initial value of the components of A_a and its derivatives by the energy of the system —see Section 6.5.

Conformal transformation properties

Consider a conformal rescaling of the form

$$g_{ab} = \Xi^2\tilde{g}_{ab}.$$

Associated with the latter we define the *unphysical Faraday tensor*, *unphysical vector potential* and the *unphysical scalar field* via

$$F_{ab} \equiv \tilde{F}_{ab}, \qquad A_a \equiv \tilde{A}_a, \qquad \phi \equiv \Xi^{-1}\tilde{\phi},$$

so that a computation using the standard conformal transformation formulae (see e.g. [81]) shows that

$$\nabla^a F_{ab} = J_b, \tag{6.6a}$$

$$\nabla_{[a} F_{bc]} = 0, \tag{6.6b}$$

$$F_{ab} = \nabla_a A_b - \nabla_b A_a, \tag{6.6c}$$

$$\mathfrak{D}_a \mathfrak{D}^a \phi - \frac{1}{6} R\phi = 0, \tag{6.6d}$$

where

$$\mathfrak{D}_a \equiv \nabla_a - \mathrm{i}\mathfrak{q} A_a$$

and

$$J_a = \mathrm{i}\mathfrak{q}\left(\overline{\phi}\mathfrak{D}_a\phi - \phi\overline{(\mathfrak{D}_a\phi)}\right). \tag{6.7}$$

In particular, it follows that

$$\tilde{J}_a = \Xi^2 J_a.$$

Moreover, one can verify that

$$\nabla^a J_a = 0.$$

Introducing the Hodge dual F^*_{ab} of the Faraday tensor in the usual way via

$$F^*_{ab} \equiv \frac{1}{2}\epsilon_{ab}{}^{cd} F_{cd},$$

the Maxwell equation (6.6b) can be rewritten as

$$\nabla^a F^*_{ab} = 0. \tag{6.8}$$

6.3.2 Spinorial expressions

In this subsection, we provide the spinorial version of the equations in the unphysical spacetime.

Let $F_{AA'BB'}$ denote the spinorial counterpart of the Faraday tensor F_{ab}. It satisfies the well-known decomposition

$$F_{AA'BB'} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB},$$

where $\phi_{AB} = \phi_{(AB)}$ is the so-called *Maxwell spinor* —see e.g. [70, 81]. A calculation with this expression shows that equations (6.6a) and (6.8) are equivalent to

$$\nabla^{B}{}_{A'}\phi_{AB} = J_{AA'}, \tag{6.9}$$

where

$$J_{AA'} \equiv \mathfrak{i}\mathfrak{q}\left(\bar{\phi}\nabla_{AA'}\phi - \phi\nabla_{AA'}\bar{\phi}\right) + 2\mathfrak{q}^2|\phi|^2 A_{AA'},$$

is the spinorial counterpart of the current J_a and $A_{AA'}$ is the spinorial counterpart of the vector potential A_a. Observe that both $A_{AA'}$ and $J_{AA'}$ are Hermitian spinors. In view of its symmetries, equation (6.6c) can be rewritten as

$$\phi_{AB} = \nabla_{A'(A}A_{B)}{}^{A'}. \tag{6.10}$$

The wave equation for the vector potential and the generalised Lorenz gauge

It is well known that in the Lorenz gauge, the vector potential satisfies a wave equation. In light of the Lorenz gauge condition in spinorial form

$$\nabla^{AA'}A_{AA'} = 0, \tag{6.11}$$

it is possible to remove the symmetrisation in the equation (6.10) so as to obtain

$$\nabla_{AA'}A_{B}{}^{A'} = \phi_{AB}. \tag{6.12}$$

Applying $\nabla^{A}{}_{B'}$, using the spinorial Maxwell equation (6.9) and making use of the commutator of the covariant derivative $\nabla_{AA'}$ one obtains

$$\square A_{BB'} + 2\Phi_{AA'BB'}A^{AA'} = J_{BB'}.$$

Now, since

$$J_{BB'} = 2\mathfrak{q}^2|\phi|^2 A_{BB'} + \mathfrak{i}\mathfrak{q}\bar{\phi}\nabla_{BB'}\phi - \mathfrak{i}\mathfrak{q}\phi\nabla_{BB'}\bar{\phi}$$

the wave equation for the vector potential reads as

$$\square A_{BB'} + 2\Phi_{AA'BB'}A^{AA'} = 2\mathfrak{q}^2|\phi|^2 A_{BB'} + \mathfrak{i}\mathfrak{q}\bar{\phi}\nabla_{BB'}\phi - \mathfrak{i}\mathfrak{q}\phi\nabla_{BB'}\bar{\phi}.$$

The wave equation for the Maxwell spinor

The unphysical charged wave equation is given by

$$g^{ab}\mathfrak{D}_a\mathfrak{D}_b\phi - \frac{R\phi}{6} = 0.$$

This equation can be recast in spinor formalism by replacing

$$\mathfrak{D}_a = \nabla_a - \mathrm{i}\mathfrak{q}A_a$$

and then by separating the soldering forms so that we have

$$\Box\phi = \mathfrak{q}^2\phi A_{AA'}A^{AA'} + 2\mathrm{i}\mathfrak{q}A^{AA'}\nabla_{AA'}\phi + \mathrm{i}\mathfrak{q}\phi\nabla_{AA'}A^{AA'}.$$

Hence, by using the Lorenz gauge condition (6.11) we have

$$\Box\phi = \mathfrak{q}^2\phi A_{AA'}A^{AA'} + 2\mathrm{i}\mathfrak{q}A^{AA'}\nabla_{AA'}\phi.$$

Summary

In summary, the study of the Maxwell-scalar field system can be reduced, making use of the generalised Lorenz gauge condition (6.11), to the system of wave equations

$$\Box\phi = \mathfrak{q}^2\phi A_{AA'}A^{AA'} + 2\mathrm{i}\mathfrak{q}A^{AA'}\nabla_{AA'}\phi, \tag{6.13a}$$

$$\Box A_{AA'} + 2\Phi_{ABA'B'}A^{BB'} = 2\mathfrak{q}|\phi|^2 A_{AA'} + \mathrm{i}\mathfrak{q}\bar{\phi}\nabla_{AA'}\phi - \mathrm{i}\mathfrak{q}\phi\nabla_{AA'}\bar{\phi}. \tag{6.13b}$$

These equations are supplemented by initial conditions for the values of ϕ and $A_{AA'}$ and of their normal derivatives. This will be discussed in more detail in Section 6.5.

6.3.3 Decomposition of the equations in the space-spinor formalism

Before providing a detailed decomposition of the equations (6.13a)-(6.13b), it is convenient to provide a rougher decomposition which brings to the foreground the structural properties of the evolution system and its relation to the Maxwell constraint equations. This decomposition is done using the *space-spinor formalism* as described in e.g. [81] —see also [5, 69].

Basic relations

Let $\tau^{AA'}$ denote the spinorial counterpart of a timelike vector field τ^a tangent to a congruence of curves. The Hermitian spinor $\tau^{AA'}$ is chosen to have the normalisation

$$\tau_{AA'}\tau^{AA'} = 2.$$

Consistent with the latter, we consider a spin dyad $\{o^A,\, \iota^A\}$ chosen so that

$$\tau^{AA'} = o^A\bar{o}^{A'} + \iota^A\bar{\iota}^{A'}.$$

It follows then that

$$\tau_{AA'}\tau^{BA'} = \delta_A{}^B.$$

The above relations induce a Hermitian conjugation operation via the relation

$$\mu^\dagger_A \equiv \tau_A{}^{A'}\bar{\mu}_{A'},$$

with the obvious extension to higher valence spinors. In particular, one has that $\iota_A = o^\dagger_A$. The space-spinor formalism allows working only with spinors with unprimed indices. In this spirit, one has the following decompositions for the spinorial counterpart of the current vector and the vector potential:

$$J_{AA'} = \tfrac{1}{2}j\tau_{AA'} - j_{AB}\tau^B{}_{A'}, \tag{6.14a}$$

$$A_{AA'} = \tfrac{1}{2}\alpha\tau_{AA'} - \alpha_{AB}\tau^B{}_{A'}, \tag{6.14b}$$

with j_{AB} and α_{AB} symmetric spinors.

Decomposition of the covariant derivative. The spinor $\tau^{AA'}$ also induces a decomposition of the spinorial covariant derivatives $\nabla_{AA'}$. For this, one introduces

$$\mathcal{D} \equiv \tau^{AA'}\nabla_{AA'}, \qquad \mathcal{D}_{AB} \equiv \tau_{(A}{}^{A'}\nabla_{B)A'},$$

the *Fermi* and *Sen connections* associated to the congruence defined by τ^a—see Chapter 5.

The Maxwell equations in space-spinor form. Some manipulations show that the spinorial Maxwell equation (6.9) can be decomposed as

$$\mathcal{D}^{AB}\phi_{AB} = \tfrac{1}{2}j,$$
$$\mathcal{D}\phi_{AB} - 2\mathcal{D}^{Q}{}_{(A}\phi_{B)Q} = -j_{AB}.$$

The former equation is to be interpreted as a *constraint* while the latter as *evolution equations* —in fact, it can be shown to be (up to some numerical factors) a symmetric hyperbolic system for the independent components of ϕ_{AB}, see [81]. Similarly, from the equation (6.12) one obtains the system

$$\mathcal{D}\alpha + 2\mathcal{D}^{AC}\alpha_{AC} + \sqrt{2}\alpha\chi^{AC}{}_{AC} - 2\sqrt{2}\alpha^{AC}\chi_{A}{}^{B}{}_{CB} - \chi^{AC}\alpha_{AC} = 2\mathcal{A}(x),$$
$$\mathcal{D}\alpha_{CD} - \mathcal{D}_{CD}\alpha - \sqrt{2}\alpha\chi_{(C}{}^{A}{}_{D)A} - 2\mathcal{D}_{(C}{}^{A}\alpha_{D)A} - \chi_{(C}{}^{A}\alpha_{D)A}$$
$$+2\sqrt{2}\alpha^{AB}\chi_{(C|A|D)B} + \tfrac{1}{2}\alpha\chi_{CD} = 2\hat{\phi}_{CD},$$

where χ_{AB} and χ_{ABCD} are, respectively, the *acceleration* and *Weingarten spinors* defined by the relation

$$\nabla_{AA'}\tau_{CC'} = -\tfrac{1}{2}\chi_{CD}\tau_{AA'}\tau^{D}{}_{C'} + \sqrt{2}\chi_{ABCD}\tau^{B}{}_{A'}\tau^{D}{}_{C'}.$$

and where

$$\hat{\phi}_{AB} \equiv \tau_{A}{}^{C'}\tau_{B}{}^{D'}\bar{\phi}_{C'D'}$$

is the *Hermitian conjugate* of ϕ_{AB}.

The scalar field. It is also illustrative to express equation (6.13a) in terms of the Fermi and the Sen connections $\mathcal{D}$ and $\mathcal{D}_{AB}$. Making use of the decomposition (5.29), a calculation gives that

$$\mathcal{D}^2\phi + 2\mathcal{D}_{AB}\mathcal{D}^{AB}\phi = -\sqrt{2}\chi^{AB}{}_{AB}\mathcal{D}\phi + \chi^{AB}\mathcal{D}_{AB}\phi + 2\sqrt{2}\chi_{A}{}^{Q}{}_{BC}\mathcal{D}^{AB}\phi.$$

Remark 43. The equations presented in this section are completely general and make no assumption on the background spacetime. When evaluated on the conformal representation of the Minkowski spacetime discussed in Section 6.2.2 they acquire a much simpler form.

Detailed decomposition in conformal Minkowski

In this section, we consider the decomposition of the various fields in the case of the conformal representation of Minkowski spacetime discussed in Section 6.2.2.

Frame choice. Following [30] we consider a *Newman-Penrose (NP) frame* satisfying

$$\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{A}\boldsymbol{A}'}, \boldsymbol{e}_{\boldsymbol{B}\boldsymbol{B}'}) = \epsilon_{\boldsymbol{A}\boldsymbol{B}}\epsilon_{\boldsymbol{A}'\boldsymbol{B}'},$$

of the form

$$\begin{aligned}
\boldsymbol{e}_{\boldsymbol{0}\boldsymbol{0}'} &= \frac{1}{\sqrt{2}}\Big((1-\tau)\boldsymbol{\partial}_\tau + \rho\boldsymbol{\partial}_\rho\Big),\\
\boldsymbol{e}_{\boldsymbol{1}\boldsymbol{1}'} &= \frac{1}{\sqrt{2}}\Big((1+\tau)\boldsymbol{\partial}_\tau - \rho\boldsymbol{\partial}_\rho\Big),\\
\boldsymbol{e}_{\boldsymbol{0}\boldsymbol{1}'} &= -\frac{1}{\sqrt{2}}\boldsymbol{X}_+,\\
\boldsymbol{e}_{\boldsymbol{1}\boldsymbol{0}'} &= -\frac{1}{\sqrt{2}}\boldsymbol{X}_-,
\end{aligned}$$

where $\boldsymbol{X}_+$ and $\boldsymbol{X}_-$ are complex vectors spanning the tangent space of $\mathbb{S}^2$ with dual covectors $\boldsymbol{\alpha}_+$ and $\boldsymbol{\alpha}_-$ such that metric of the 2-sphere is given by

$$\boldsymbol{\sigma} = 2(\boldsymbol{\alpha}^+ \otimes \boldsymbol{\alpha}^- + \boldsymbol{\alpha}^- \otimes \boldsymbol{\alpha}^+).$$

The vector τ^a giving rise to the space-spinor decomposition of the Maxwell-scalar field introduced in Section 6.3.3 is chosen as

$$\tau^a = e_{\boldsymbol{0}\boldsymbol{0}'}{}^a + e_{\boldsymbol{1}\boldsymbol{1}'}{}^a = \sqrt{2}(\boldsymbol{\partial}_\tau)^a.$$

A peculiarity of the conformal representation introduced in the equation (6.1) is that the Ricci scalar vanishes —that is,

$$R[\boldsymbol{g}] = 0.$$

The reduced wave equations. As the expression of the wave operator in the F-gauge acting on a spin-weighted scalar will be used repeatedly, it is convenient to define the *F-reduced wave operator* $\blacksquare$ acting on a scalar ζ as

$$\blacksquare\zeta \equiv (1-\tau^2)\ddot{\zeta} + 2\tau\rho\dot{\zeta}' - \rho^2\zeta'' - 2\tau\dot{\zeta} - \frac{1}{2}\eth\bar{\eth}\zeta - \frac{1}{2}\bar{\eth}\eth\zeta, \tag{6.15}$$

where for simplicity of the presentation we have used the notation

$$\dot{} \equiv \partial_\tau, \qquad {}' \equiv \partial_\rho,$$

and $\eth$ and $\bar{\eth}$ denote the NP *eth* and *eth bar* operators —see e.g. [61, 70]. In particular, the operator $\frac{1}{2}(\eth\bar{\eth} + \bar{\eth}\eth)$ corresponds to the Laplacian on $\mathbb{S}^2$.

After some lengthy computations, best carried out using the suite `xAct` for tensorial and spinorial manipulations in the `Wolfram programming language` [54], the wave equations (6.13a)-(6.13b) can be seen to be equivalent to the scalar system:

$$\begin{aligned}
&\blacksquare\phi = \mathfrak{q}^2\phi\left(\frac{1}{2}\alpha^2 - 2\alpha_1^2 + 2\alpha_0\alpha_2\right) + \mathrm{i}\sqrt{2}\mathfrak{q}\Big(2\alpha_1\rho\phi' + \alpha\dot{\phi} - 2\tau\alpha_1\dot{\phi} + \alpha_0\eth\phi - \alpha_2\bar{\eth}\phi\Big),\\
&\blacksquare\alpha - 4\dot{\alpha}_1 = 2\mathfrak{q}^2\alpha|\phi|^2 + \mathrm{i}\sqrt{2}\mathfrak{q}\Big(\dot{\phi}\bar{\phi} - \dot{\bar{\phi}}\phi\Big),\\
&\blacksquare\alpha_0 + \alpha_0 = 2\mathfrak{q}^2\alpha_0|\phi|^2 + \frac{\mathrm{i}\mathfrak{q}}{\sqrt{2}}(\phi\bar{\eth}\bar{\phi} - \bar{\phi}\bar{\eth}\phi),\\
&\blacksquare\alpha_1 - \dot{\alpha} = 2\mathfrak{q}^2\alpha_1|\phi|^2 + \frac{\mathrm{i}\mathfrak{q}\rho}{\sqrt{2}}(\phi\bar{\phi}' - \bar{\phi}\phi') + \frac{\mathrm{i}\mathfrak{q}\tau}{\sqrt{2}}(\bar{\phi}\dot{\phi} - \phi\dot{\bar{\phi}}),\\
&\blacksquare\alpha_2 + \alpha_2 = 2\mathfrak{q}^2\alpha_2|\phi|^2 + \frac{\mathrm{i}\mathfrak{q}}{\sqrt{2}}(\bar{\phi}\eth\phi - \phi\eth\bar{\phi}).
\end{aligned}$$

The scalars α, α_0, α_1 and α_2 have spin weight 0, 1, 0, -1, respectively. In particular, α denotes the *time component* of the Hermitian spinor $A_{AA'}$ while α_0, α_1, α_2 are the independent components of its *spatial part* —cfr. the decomposition in equation (6.14b).

It is observed that the righthand sides of the above wave equations can be decoupled if one defines

$$\alpha_\pm \equiv \alpha \pm \alpha_1, \qquad j_\pm \equiv j \pm j_1,$$

all of them of spin-weight 0. In terms of these new variables, the system of wave equations can be rewritten as

$$\begin{aligned}
&\blacksquare\phi = s,\\
&\blacksquare\alpha_+ - 2\dot{\alpha}_+ = j_+,\\
&\blacksquare\alpha_- + 2\dot{\alpha}_- = j_-,\\
&\blacksquare\alpha_0 + \alpha_0 = j_0,
\end{aligned}$$

$$\blacksquare\alpha_2 + \alpha_2 = j_2,$$

with the obvious definitions. For future use, it is observed that the source terms

$$s = s(\bar{x}, \phi, \alpha_0, \alpha_2, \alpha_\pm),$$
$$j_0 = j_0(\bar{x}, \phi, \alpha_0, \alpha_2, \alpha_\pm), \qquad j_2 = j_2(\bar{x}, \phi, \alpha_0, \alpha_2, \alpha_\pm), \qquad j_\pm = j_\pm(\bar{x}, \phi, \alpha_0, \alpha_2, \alpha_\pm),$$

are at most cubic polynomial expressions in the unknowns ϕ, α_0, α_2, $\alpha_\pm$.

6.4 General structural properties and expansions near spatial infinity

In this section, we discuss general structural properties of the evolution system, equations (6.24a)-(6.24e), associated to the Maxwell-scalar field system. In particular, we study a type of asymptotic expansions near spatial infinity which was first introduced, for the conformal Einstein field equations, by H. Friedrich in [28]. In the following, we refer to these expansions as *F-expansions*. This construction exploits the fact that the cylinder at spatial infinity, $\mathcal{I}$, introduced in Section 6.2.2 is a *total characteristic* of the evolution equations associated to the Maxwell-scalar system. Accordingly, the evolution equations reduce to an interior system (*transport equations*) upon evaluation on the cylinder $\mathcal{I}$. This can clearly be seen from the form of the reduced wave operator $\blacksquare$ as given by equation (6.15) —all the ∂_ρ derivatives disappear from the equation if one sets $\rho = 0$. These transport equations allow to relate properties of the initial data, as defined on a fiduciary initial hypersurface $\mathcal{S}_\star$, with radiative properties of the solution which are defined at null infinity $\mathscr{I}^\pm$.

For the convenience of the subsequent discussion we define

$$\boldsymbol{\alpha} \equiv \begin{pmatrix} \alpha_+ \\ \alpha_- \\ \alpha_0 \\ \alpha_2 \end{pmatrix}, \qquad \boldsymbol{j} \equiv \begin{pmatrix} j_+ \\ j_- \\ j_0 \\ j_2 \end{pmatrix}, \qquad \mathbf{A} \equiv \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{B} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In terms of the above vectors and matrices the system (6.24a)-(6.24e) can be rewritten as

$$\blacksquare\phi = s, \tag{6.16a}$$

$$\blacksquare\boldsymbol{\alpha} + \mathbf{A}\dot{\boldsymbol{\alpha}} + \mathbf{B}\boldsymbol{\alpha} = \boldsymbol{j}. \tag{6.16b}$$

The source terms can, in turn, be written as

$$s = \phi\boldsymbol{\alpha}^{\dagger}\mathbf{Q}\boldsymbol{\alpha} + \boldsymbol{\alpha}^{\dagger}\mathbf{K}(x)\boldsymbol{\partial}\phi,$$

$$\boldsymbol{j} = 2\mathfrak{q}|\phi|^2\boldsymbol{\alpha} + \phi\mathbf{L}\boldsymbol{\partial}\bar{\phi} + \bar{\phi}\mathbf{L}\boldsymbol{\partial}\phi,$$

where † denotes the Hermitian transpose of a vector (i.e. transposition plus complex conjugation), $\mathbf{Q}$ is a constant matrix, while $\mathbf{K}(x)$ and $\mathbf{L}(x)$ are coordinate-dependent matrices. Observe that if $\mathfrak{q} = 0$ then $\mathbf{K}(x) = \mathbf{L}(x) = 0$.

6.4.1 Transport equations on the cylinder at spatial infinity

The key observation in our analysis is that the F-reduced wave operator $\blacksquare$, as defined by (6.15), reduces to an operator intrinsic to $\mathcal{I}$. Defining the *F-reduced wave operator on* $\mathcal{I}$, $\blacktriangle \equiv \blacksquare|_{\mathcal{I}}$, acting on a scalar ζ as

$$\blacktriangle\zeta \equiv (1-\tau^2)\ddot{\zeta} - 2\tau\dot{\zeta} - \frac{1}{2}(\eth\bar{\eth} + \bar{\eth}\eth)\zeta.$$

one readily observes that this operator is intrinsic to $\mathcal{I}$. An alternative way of expressing this observation is that the cylinder at spatial infinity is a total characteristic of the evolution system (6.16a)-(6.16b).

Remark 44. The intrinsic operator $\blacktriangle$ is clearly hyperbolic for $|\tau| < 1$. However, at $\tau = \pm 1$ it degenerates. To investigate the effect of this degeneracy at the critical sets $\mathcal{I}^{\pm}$ it is convenient to study the transport equations implied by evolution equations.

The leading order transport equations

Evaluating equations (6.16a)-(6.16b) on $\mathcal{I}$ (i.e. at $\rho = 0$) one obtains the interior system of wave equations:

$$\blacktriangle\phi^{(0)} = \phi^{(0)}\boldsymbol{\alpha}^{(0)\dagger}\mathbf{Q}\boldsymbol{\alpha}^{(0)} + \boldsymbol{\alpha}^{(0)\dagger}\mathbf{K}\boldsymbol{\partial}\phi^{(0)}, \tag{6.17a}$$

$$\blacktriangle\boldsymbol{\alpha}^{(0)} + \mathbf{A}\dot{\boldsymbol{\alpha}}^{(0)} + \mathbf{B}\boldsymbol{\alpha}^{(0)} = 2\mathfrak{q}|\phi^{(0)}|^2\boldsymbol{\alpha}^{(0)} + \phi^{(0)}\mathbf{L}\boldsymbol{\partial}\bar{\phi}^{(0)} + \bar{\phi}^{(0)}\mathbf{L}\boldsymbol{\partial}\phi^{(0)}, \qquad (6.17b)$$

with

$$\phi^{(0)} \equiv \phi|_{\mathcal{I}}, \qquad \boldsymbol{\alpha}^{(0)} \equiv \boldsymbol{\alpha}|_{\mathcal{I}}.$$

Initial data for the system (6.17a)-(6.17b) is provided by the restriction of the initial data for the fields ϕ, $\dot{\phi}$, $\boldsymbol{\alpha}$ and $\dot{\boldsymbol{\alpha}}$ on $\mathcal{S}_\star$ to $\mathcal{I}$.

Remark 45. The interior system (6.17a)-(6.17b) is, in principle, coupled and non-linear. However, certain classes of initial data allow for a decoupling of the system. This feature is discussed in Section 6.6.

Higher order transport equations

Making use of the structural properties of the evolution system (6.16a)-(6.16b) it is possible to consider higher-order generalisations of the transport equations introduced the previous subsection. To this end, one considers the commutator

$$\partial_\rho^p\blacksquare\zeta - \blacksquare\partial_\rho^p\zeta = 2\tau p\partial_\rho^p\dot{\zeta} - 2p\rho\partial_\rho^p\zeta' - p(p-1)\partial_\rho^p\zeta,$$

with ∂_ρ^p denoting p applications of the derivative ∂_ρ. Now, applying the operator ∂_ρ to equations (6.16a)-(6.16b) a total number of p times and restricting to $\mathcal{I}$ one finds that

$$\blacktriangle\phi^{(p)} + 2\tau p\dot{\phi}^{(p)} - p(p-1)\phi^{(p)} = s^{(p)},$$
$$\blacktriangle\boldsymbol{\alpha}^{(p)} + \mathbf{A}\dot{\boldsymbol{\alpha}}^{(p)} + 2\tau p\dot{\boldsymbol{\alpha}}^{(p)} + \mathbf{B}\boldsymbol{\alpha}^{(p)} - p(p-1)\boldsymbol{\alpha}^{(p)} = \mathbf{R}^{(p)},$$

where $s^{(p)}$ and $\mathbf{R}^{(p)}$ denote source terms which depend on the solutions of the lower-order transport equations —i.e. $(\phi^{(p')}, \boldsymbol{\alpha}^{(p')})$ for p' such that $0 \leq p' \leq p-1$. *This observation allows implementing a recursive scheme to compute the solutions to the transport equations to any arbitrary order —modulo computational complexities.*

6.5 Initial conditions

In this section, we discuss the construction of initial data for the Maxwell-scalar field system in the Lorenz gauge. Accordingly, throughout it is assumed that

$$\nabla^a A_a = 0.$$

6.5.1 General remarks

The wave equations (6.13a)-(6.13b) suggest that a natural prescription of initial data for the Maxwell-scalar field system is

$$\phi_\star, \qquad \mathcal{D}\phi_\star, \qquad A_{AA'\star}, \qquad \mathcal{D}A_{AA'\star}.$$

Notice that the components of $A_{AA'}$ and $\mathcal{D}A_{AA'\star}$ cannot be prescribed freely. Moreover, there is some gauge freedom that can be used to set certain components to zero —see below.

Remark 46. The above is not necessarily the most physical way of prescribing initial conditions. A more physical choice is to prescribe

$$\phi_\star, \qquad \mathcal{D}\phi_\star, \qquad \eta_{AB\star}, \qquad \mu_{AB\star},$$

where η_{AB} and μ_{AB} denote, respectively, the spinorial counterparts of the electric and magnetic parts of the Faraday tensor with respect to the normal to the initial hypersurface $\mathcal{S}_\star$. In order to fix the asymptotic behaviour one requires finiteness of the energy

$$\mathcal{E}_\star \equiv \frac{1}{2}\int_{\mathbb{R}^3} \Big(|\mathfrak{D}\phi_\star|^2 + |\boldsymbol{\eta}_\star|^2 + |\boldsymbol{\mu}_\star|^2 \Big) \mathrm{d}\mu. \tag{6.18}$$

In addition, the electric and magnetic parts are subject to the Gauss constraints implied by the equation

$$\mathcal{D}^{AB}\phi_{AB} = \tfrac{1}{2}j.$$

6.5.2 Data on time symmetric hypersurfaces

In the following, we assume, for simplicity, that the initial hypersurface $\mathcal{S}_\star$ is the time-symmetric hypersurface with t = constant in the Minkowski spacetime. Accordingly the extrinsic curvature vanishes in this hypersurface —thus, we have that in the initial hypersurface $\mathcal{D}_{AB} = D_{AB}$ —that is, the Sen connection coincides with the Levi-Civita connection of the intrinsic metric to $\mathcal{S}_\star$. Notice, however, that it is not assumed that the acceleration vanishes on the initial hypersurface —this is for consistency with the conformal Gaussian gauge used to write the evolution equations.

Following the discussion in [68] we make use of the residual gauge freedom in the Lorenz gauge to set the initial value of the time components of A_a and $\mathcal{D}A_a$ to zero initially. In terms of the space-spinor split of A_a, this is equivalent to requiring

$$\alpha_\star = 0, \qquad \mathcal{D}\alpha_\star = 0.$$

It follows then from the Lorenz gauge condition that

$$D^{AB}\alpha_{AB} = \tfrac{1}{2}\chi^{AB}\alpha_{AB}. \tag{6.19}$$

This equation has to be treated as a constraint on the spatial part α_{AB}. Observe how this last equation involves the acceleration.

The definition of the Maxwell spinor in terms of $A_{AA'}$ yields the condition

$$\phi_{AB} = -\tfrac{1}{2}\dot{\alpha}_{AB} - D_{(A}{}^{Q}\alpha_{B)Q}, \qquad \text{on} \quad \mathcal{S}_\star,$$

where for brevity we have written $\dot{\alpha}_{AB} \equiv \mathcal{D}\alpha_{AB}$. Substituting this relation into the Gauss constraint

$$D^{AB}\phi_{AB} = \tfrac{1}{2}j$$

one concludes that

$$D^{AB}\dot{\alpha}_{AB} = -j. \tag{6.20}$$

Solving the constraints for α_{AB} and $\dot{\alpha}_{AB}$

In order to solve the constraint equations (6.19) and (6.20) one can make use of the Ansatz

$$\alpha_{AB} = D_{AB}\sigma + D_{(A}{}^{Q}\sigma_{B)Q}, \tag{6.21a}$$

$$\dot{\alpha}_{AB} = D_{AB}\pi + D_{(A}{}^{Q}\pi_{B)Q}, \tag{6.21b}$$

with σ, π scalars and σ_{AB}, π_{AB} symmetric, real valence 2 spinors —the latter is essentially a spinorial version of the Helmholtz decomposition. The substitution of the Ansatz (6.21a)-(6.21b) in the constraints (6.19) and (6.20) leads to elliptic equations for the scalars σ, π. The spinors σ_{AB} and π_{AB} are free data.

Time symmetric initial conditions

Time-symmetric initial data conditions for the Maxwell-scalar system, i.e. initial conditions giving rise to solutions which are time reflection symmetric with respect to the hypersurface $\mathcal{S}_\star$, are set by requiring

$$\dot{\phi} = 0, \qquad \dot{\alpha}_{AB} = 0, \qquad \text{on} \quad \mathcal{S}_\star.$$

It follows from the above that $j = 0$. Thus, the only constraint equation left to solve is equation (6.19) which can then be solved using the Ansatz (6.21a). A further consequence of $\dot{\alpha}_{AB} = 0$ is that

$$\phi_{AB} = -D_{(A}{}^{Q}\alpha_{B)Q}, \qquad \hat{\phi}_{AB} = -D_{(A}{}^{Q}\alpha_{B)Q}.$$

Now, defining the *electric* and *magnetic parts* of $\phi_\star$ with respect to the Hermitian spinor $\tau^{AA'}$ by

$$\eta_{AB} \equiv \tfrac{1}{2}\mathrm{i}(\phi_{AB} + \hat{\phi}_{AB}), \qquad \mu_{AB} \equiv \tfrac{1}{2}(\phi_{AB} - \hat{\phi}_{AB}),$$

one readily finds that

$$\mu_{AB} = 0, \qquad \text{on} \quad \mathcal{S}_\star.$$

Remark 47. This is the spinorial version of the well-known result stating that the magnetic part of time-symmetric data for the Maxwell field is vanishing.

Asymptotic conditions

The asymptotic behaviour of the initial data can be fixed in a natural way from the requirement of the finiteness of the energy on the (physical) initial hypersurface.

Scalar field. The finiteness of the energy as defined by equation (6.18) requires

$$\tilde{\mathfrak{D}}_a \tilde{\phi}_\star = o(\tilde{r}^{-2}), \qquad (\tilde{\eta}_{\boldsymbol{AB}})_\star = o(\tilde{r}^{-2}), \qquad (\tilde{\mu}_{\boldsymbol{AB}})_\star = o(\tilde{r}^{-2}).$$

These conditions are satisfied if

$$\tilde{\phi}_\star = o(\tilde{r}^{-1}), \qquad (\tilde{\alpha}_{\boldsymbol{AB}})_\star = o(\tilde{r}^{-1}).$$

In particular, one can consider an initial scalar field with leading behaviour given by

$$\tilde{\phi}_\star = \mathring{\varphi}\tilde{r}^{-1} + \cdots, \qquad \text{with} \quad \mathring{\varphi} \quad \text{a constant.}$$

In order to obtain the *conformal* version of the above condition recall that $\Theta = \rho(1-\tau^2)$ and, moreover, $\tilde{r} = 1/\rho$ and $\varpi = \Theta_\star = \tilde{r}^{-1} = \rho$. Thus,

$$\phi_\star = \varpi^{-1}\tilde{\phi}_\star = \rho^{-1}\tilde{\phi}_\star = \mathring{\varphi} + \cdots.$$

For simplicity, one can assume that $\phi_\star$ is analytic in a neighbourhood of spatial infinity. Results analogous to the ones derived here can, at the expense of complicated and lengthy recursion arguments, be obtained also for weaker differentiability assumptions.

Electric field. To analyse the asymptotic behaviour of the electric part of the Maxwell field from the conformal point of view, it is observed that

$$\tau^a = \varpi^{-1}\tilde{\tau}^a,$$

where $\tilde{\tau}^a$ and τ^a are, respectively, the physical and unphysical normals to the initial hypersurface $\mathcal{S}_\star$. Accordingly, for a Coulomb-type field $\tilde{E}_i = O(1/\tilde{r}^2)$ it follows that

$$E_i = \varpi^{-1}\tilde{E}_i = O(\rho).$$

In terms of the components with respect to the frame one has

$$E_{\boldsymbol{i}} = O(1), \qquad \eta_{\boldsymbol{AB}} = O(1).$$

Gauge potential. For the gauge potential one has that $\tilde{A}_a = A_a$. Moreover, for a Coulomb-type field one has $\tilde{A}_a = O(1/\tilde{r})$ so that in terms of the spatial components with respect to the frame:

$$A_{\boldsymbol{i}} = O(1), \qquad \alpha_{\boldsymbol{AB}} = O(1).$$

An Ansatz for the initial data

In order to give a more concrete Ansatz for the construction of the initial data for the Maxwell-scalar field system, in the following it will be assumed that *the freely specifiable data $\phi_\star$ and $\sigma_{\boldsymbol{AB}}$ are analytic in a neighbourhood of spatial infinity.* It follows then from the ellipticity of the equation for the σ that this scalar will also be analytic in a neighbourhood of i^0. Consistent with the above let

$$\phi_\star = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{m=-1}^{l} \frac{1}{n!} \varphi_{n;l,m}(Y_{lm})\rho^n, \tag{6.22a}$$

$$\sigma_i = \sum_{n=|1-i|}^{\infty} \sum_{l=|1-i|}^{n} \sum_{m=-l}^{l} \frac{1}{n!} \sigma_{i,n;l,m}({}_{1-s}Y_{lm})\rho^n, \tag{6.22b}$$

where σ_i for $i = 0, 1, 2$ denote the three (complex) independent components of $\sigma_{\boldsymbol{AB}}$ and ${}_sY_{lm}$ denote the spin-weighted spherical harmonics —see e.g. [61, 70]. Moreover, $\varphi_{n;l,m}$ and $\sigma_{i,n;l,m}$ are constants. Finally, in accordance to the above, we look for a scalar of the form

$$\sigma = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{m=-1}^{l} \frac{1}{n!} \sigma_{n;l,m}(Y_{lm})\rho^n, \tag{6.23}$$

with $\sigma_{n;l,m}$ constants.

6.6 Solution jets

In this section, we start our systematic study of the solutions to the Maxwell-scalar field in a neighbourhood of spatial infinity. In order to gain some insight into the nature of the solution we first analyse the *decoupled case* in which the charge constant $\mathfrak{q}$ is set to zero. In this case the Maxwell field and the scalar field decouple from each other and the resulting evolution equations are linear. We then contrast the behaviour of this decoupled case with that of the case where $\mathfrak{q} \neq 0$.

We recall that the system to be solved can be written as

$$\blacksquare\phi = s, \tag{6.24a}$$

$$\blacksquare\alpha_+ - 2\dot{\alpha}_+ = j_+, \tag{6.24b}$$

$$\blacksquare\alpha_- + 2\dot{\alpha}_- = j_-, \tag{6.24c}$$

$$\blacksquare\alpha_0 + \alpha_0 = j_0, \tag{6.24d}$$

$$\blacksquare\alpha_2 + \alpha_2 = j_2, \tag{6.24e}$$

where the source terms s, $j_\pm$, j_0 and j_2 are polynomial expressions (at most of third order) in the unknowns.

Remark 48. A key property of the unknowns in the system (6.24a)-(6.24e) is that they all possess a well defined spin-weight —see e.g. [61, 70, 81]. More precisely, one has that:

$$\begin{aligned} &\phi,\ \alpha_\pm \qquad \text{have spin-weight } 0,\\ &\alpha_0 \qquad \text{has spin-weight } 1,\\ &\alpha_2 \qquad \text{has spin-weight } -1. \end{aligned}$$

The above spin-weights determine the type of expansions of the coefficients in terms of spin-weighted spherical harmonics.

In order to ease the discussion we make the following simplifying assumptions:

Assumption 1. The discussion will be restricted, in first instance, to the time symmetric case. Accordingly, it is assumed that

$$\dot{\phi}_\star = 0, \qquad \dot{\alpha}_\star = 0, \qquad \dot{\alpha}_{AB\star} = 0.$$

In addition, assume initial conditions for which

$$\alpha_{\pm\star}|_{\mathcal{I}} = 0, \qquad \alpha_{0\star}|_{\mathcal{I}} = 0, \qquad \alpha_{2\star}|_{\mathcal{I}} = 0$$

and, in general,

$$\phi_\star|_{\mathcal{I}} \neq 0.$$

Observe that the class of initial data to be considered has a vector potential which, on the initial hypersurface, vanishes to leading order at spatial infinity. Crucially, the scalar field does not vanish at the leading order. *The main conclusions of our analysis can be extended, at the expense of lengthier computations, to a more general non-time symmetric setting.*

In the following, for conciseness, we write the conditions in Assumption 1 as:

$$\alpha^{(0)}_{\pm\star} = \alpha^{(0)}_{0\star} = \alpha^{(0)}_{2\star} = 0, \tag{6.25a}$$

$$\dot{\alpha}^{(0)}_{\pm\star} = \dot{\alpha}^{(0)}_{0\star} = \dot{\alpha}^{(0)}_{2\star} = 0, \tag{6.25b}$$

$$\phi^{(0)}_{\star} = \varphi_{\star}, \qquad \dot{\phi}^{(0)}_{\star} = 0, \tag{6.25c}$$

with $\varphi_\star \in \mathbb{C}$ a constant.

As we have seen before, see Section 6.4, the cylinder at spatial infinity is a total characteristic of our evolution equations. We can use this property to construct, in a recursive manner, the *jets of order p at $\mathcal{I}$*, $J^p[\phi, \boldsymbol{\alpha}]$, $p \geq 0$ of the solutions to the evolution equations. Recall that the jet is defined as

$$J^p[\phi, \boldsymbol{\alpha}] \equiv \{\partial^p_\rho \phi|_{\rho=0},\ \partial^p_\rho \boldsymbol{\alpha}|_{\rho=0}\}.$$

Knowledge of the jet $J^p[\phi, \boldsymbol{\alpha}]$ provides very precise information about the regularity of the solutions to the evolution equations in a neighbourhood of spatial infinity and its relation to the structure and properties of the initial data.

6.6.1 The decoupled case

Setting the charge parameter $\mathfrak{q} = 0$, equations (6.24a)-(6.24e) readily reduce to the linear system of equations

$$\blacksquare\phi = 0$$

$$\blacksquare\alpha_+ = 0,$$

$$\blacksquare\alpha_- = 0,$$

$$\blacksquare\alpha_0 + 2\dot{\alpha}_0 = 0,$$
$$\blacksquare\alpha_2 - 2\dot{\alpha}_2 = 0.$$

Defining

$$\phi^{(p)} \equiv \partial_\rho^p \phi|_{\rho=0}, \qquad \alpha_\pm^{(p)} \equiv \partial_\rho^p \alpha_\pm|_{\rho=0}, \qquad \alpha_0^{(p)} \equiv \partial_\rho^p \alpha_0|_{\rho=0}, \qquad \alpha_2^{(p)} \equiv \partial_\rho^p \alpha_2|_{\rho=0}, \qquad p \geq 0,$$

a calculation that shows that the solutions corresponding to the p-order elements of $J^p[\phi, \boldsymbol{\alpha}]$ satisfy the intrinsic equations

$$\blacktriangle\phi^{(p)} + 2p\tau\dot{\phi}^{(p)} = 0,$$
$$\blacktriangle\alpha_+^{(p)} + 2p\tau\dot{\alpha}_+^{(p)} = 0,$$
$$\blacktriangle\alpha_-^{(p)} + 2p\tau\dot{\alpha}_-^{(p)} = 0,$$
$$\blacktriangle\alpha_0^{(p)} + 2(p\tau + 1)\dot{\alpha}_0^{(p)} = 0,$$
$$\blacktriangle\alpha_2^{(p)} + 2(p\tau - 1)\dot{\alpha}_2^{(p)} = 0.$$

Accordingly, in the following, we study the following three model equations:

$$\blacktriangle\zeta + 2p\tau\dot{\zeta} = 0, \tag{6.26a}$$
$$\blacktriangle\zeta + 2(p\tau + 1)\dot{\zeta} = 0, \tag{6.26b}$$
$$\blacktriangle\zeta + 2(p\tau - 1)\dot{\zeta} = 0. \tag{6.26c}$$

Remark 49. In the subsequent analysis it is assumed that:

(a) the scalar ζ in equation (6.26a) has spin-weight 0 and admits an expansion of form

$$\zeta = \sum_{p=0}^{\infty} \sum_{l=0}^{p} \sum_{m=-1}^{l} \frac{1}{p!} \zeta_{p;l,m}(Y_{lm})\rho^p;$$

(b) in equation (6.26b) the scalar ζ has spin-weight 1 and admits an expansion of the form

$$\zeta = \sum_{p=1}^{\infty} \sum_{l=1}^{p} \sum_{m=-l}^{l} \frac{1}{p!} \zeta_{0,p;l,m}({}_1Y_{lm})\rho^p;$$

(c) in equation (6.26c) the scalar ζ has spin-weight -1 and admits an expansion of the form

$$\zeta = \sum_{p=1}^{\infty}\sum_{l=1}^{p}\sum_{m=-l}^{l}\frac{1}{p!}\zeta_{2,p;l,m}({}_{-1}Y_{lm})\rho^p.$$

The above expansions are consistent with the discussion regarding the freely specifiable initial data in Section 6.5.2 and equations (6.22a)-(6.22b) and (6.23), in particular. Observe that at order ρ^p the highest allowed spherical harmonic corresponds to $\ell = p$.

Substituting the Ansätze in Remark 49 into the model equations (6.26a)-(6.26c) one obtains, respectively, the ordinary differential equations

$$(1-\tau^2)\ddot{\zeta}_{p;\ell,m} + 2\tau(p-1)\dot{\zeta}_{p;\ell,m} + (p+\ell)(\ell-p+1)\zeta_{p;\ell,m} = 0, \tag{6.27a}$$

$$(1-\tau^2)\ddot{\zeta}_{0,p;\ell,m} + 2\Big((p-1)\tau+1\Big)\dot{\zeta}_{0,p;\ell,m} + (p+\ell)(\ell-p+1)\zeta_{0,p;\ell,m} = 0, \tag{6.27b}$$

$$(1-\tau^2)\ddot{\zeta}_{2,p;\ell,m} + 2\Big((p-1)\tau-1\Big)\dot{\zeta}_{2,p;\ell,m} + (p+\ell)(\ell-p+1)\zeta_{2,p;\ell,m} = 0. \tag{6.27c}$$

Equations (6.27a)-(6.27c) are examples of *Jacobi ordinary differential equations*. A discussion of the theory of these equations can be found in the monograph [71]. The subsequent analysis is strongly influenced by this reference. More details can be found in Appendix B.2.

Remark 50. It can be readily verified that if $\zeta_{0,p;p,m}(\tau)$ is a solution to equation (6.27b) then $\zeta_{0,p;p,m}(-\tau)$ solves (6.27c). Thus, it is only necessary to study two model equations.

In the decoupled case, the key insight is that the behaviour of the solutions to equations (6.27a)-(6.27c) depends on the value of the parameter ℓ. For $0 \leq \ell \leq p-1$, $p \geq 1$ the nature of the solutions is summarised in the following:

Lemma 10. *The solutions to the system* (6.27a), (6.27b) *and* (6.27c) *can be written as*

$$\zeta_{p;\ell,m}(\tau) = A_{p;\ell,m}\left(\frac{1-\tau}{2}\right)^{p} P_\ell^{(p,-p)}(\tau) + B_{p;\ell,m}\left(\frac{1+\tau}{2}\right)^{p} P_\ell^{(-p,p)}(\tau),$$

$$\zeta_{0,p;\ell,m}(\tau) = C_{p;\ell,m}\left(\frac{1-\tau}{2}\right)^{(p+1)} P_\ell^{(1+p,1-p)}(\tau) + D_{p;\ell,m}\left(\frac{1+\tau}{2}\right)^{(p-1)} P_\ell^{(-1-p,p-1)}(\tau),$$

$$\zeta_{2,p;\ell,m}(\tau) = E_{p;\ell,m}\left(\frac{1-\tau}{2}\right)^{(p+1)} P_\ell^{(1+p,1-p)}(-\tau) + F_{p;\ell,m}\left(\frac{1+\tau}{2}\right)^{(p-1)} P_\ell^{(-1-p,p-1)}(-\tau),$$

with $P_n^{(\alpha,\beta)}(\tau)$ *Jacobi polynomials of order* n *and where*

$$A_{p;\ell,m}, \quad B_{p;\ell,m}, \quad C_{p;\ell,m}, \quad D_{p;\ell,m}, \quad E_{p;\ell,m}, \quad F_{p;\ell,m} \in \mathbb{C}$$

denote some constants which can be expressed in terms of the initial conditions.

However, for equation (6.27a) in the case $\ell = p$ we have the following proposition:

Lemma 11. *For* $l = p$ *the solution to the equation* (6.27a) *can be written as*

$$\zeta_{p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^p \left(\frac{1+\tau}{2}\right)^p \left(C_{1,p;\ell,m} + C_{2,p;\ell,m}\int_0^\tau \frac{\mathrm{d}s}{(1-s^2)^{p+1}}\right),$$

where $C_{1,p;\ell,m}$, $C_{2,p;\ell,m}$ *are integration constants.*

Remark 51. Observe that the general solution given in Lemma 11 has logarithmic singularities unless the constant $C_{2,p;\ell,m}$ vanishes. Letting $\zeta_{\star p;p,m} \equiv \zeta_{p;p,m}(0)$ and $\dot{\zeta}_{\star p;p,m} \equiv \dot{\zeta}_{p;p,m}(0)$ one readily finds that

$$C_{1,p;\ell,m} = 2^{2p}\zeta_{\star p;p,m}.$$

Similarly, one has that

$$C_{2,p;\ell,m} = 2^{2p}\dot{\zeta}_{\star p;p,m}.$$

Thus, there is no logarithmic divergence if and only if $\dot{\zeta}_{\star p;p,m} = 0$ —that is, *when the initial data for* ζ *is time symmetric.* In particular

$$\zeta_{0;0,0} = C_1 + C_2\Big(\log(1-\tau) - \log(1+\tau)\Big).$$

In this case, one has that the logarithmic divergences are avoided if $\dot{\zeta}_{0;0,0}(0) = 0$.

Similarly, one obtains an analogous result for equations (6.27b) and (6.27c):

Lemma 12. *For* $\ell = p$ *the solution to equations* (6.27b) *and* (6.27c) *can be written as*

$$\zeta_{0,p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^{(p+1)} \left(\frac{1+\tau}{2}\right)^{(p-1)} \left(C_{3,p;\ell,m} + C_{4,p;\ell,m}\int_0^\tau \frac{\mathrm{d}s}{(1-s)^{p+2}(1+s)^p}\right),$$

$$\zeta_{2,p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^{(p-1)} \left(\frac{1+\tau}{2}\right)^{(p+1)} \left(C_{5,p;\ell,m} + C_{6,p;\ell,m}\int_0^\tau \frac{\mathrm{d}s}{(1+s)^{p+2}(1-s)^p}\right),$$

where $C_{3,p;\ell,m}$, $C_{4,p;\ell,m}$, $C_{5,p;\ell,m}$, $C_{6,p;\ell,m}$ *are integration constants.*

It follows from the above that the solutions for $\zeta_{0,p;p,m}(\tau)$ and $\zeta_{2,p;p,m}(\tau)$ have logarithmic singularities unless the constants $C_{4,p;\ell,m}$ and $C_{6,p;\ell,m}$ vanish. Now, if we let $\zeta_{0,\star p;p,m} \equiv \zeta_{0,p;p,m}(0)$ and $\dot{\zeta}_{0,\star p;p,m} \equiv \dot{\zeta}_{0,p;p,m}(0)$, it follows that

$$\zeta_{0,\star p;p,m} = \left(\frac{1}{2}\right)^{(p+1)}\left(\frac{1}{2}\right)^{(p-1)} a_\star = \left(\frac{1}{2}\right)^{2p} a_\star.$$

On the other hand, we have that

$$\begin{aligned}\dot{\zeta}_{0,p;p,m}(\tau) = &-\frac{1}{2}(p+1)\left(\frac{1-\tau}{2}\right)^{p}\left(\frac{1+\tau}{2}\right)^{(p-1)}\left(a_\star + \dot{a}_\star\int_0^\tau \frac{\mathrm{d}s}{(1-s)^{p+2}(1+s)^p}\right)\\ &+\frac{1}{2}(p-1)\left(\frac{1-\tau}{2}\right)^{(p+1)}\left(\frac{1+\tau}{2}\right)^{p}\left(a_\star + \dot{a}_\star\int_0^\tau \frac{\mathrm{d}s}{(1-s)^{p+2}(1+s)^p}\right)\\ &+\left(\frac{1-\tau}{2}\right)^{(p+1)}\left(\frac{1+\tau}{2}\right)^{(p-1)}\frac{\dot{a}_\star}{(1-\tau)^{p+2}(1+\tau)^p}.\end{aligned}$$

Thus, it follows that

$$\dot{\zeta}_{0\star,p;p,m} = -\frac{1}{2^{(2p-1)}}a_\star + \frac{1}{2^{2p}}\dot{a}_\star.$$

Hence in this case the condition $\dot{\zeta}_{0\star,p;p,m} = 0$ does not eliminate the logarithms in the solution. However, recalling that $\zeta_{0\star,p;p,m} = \left(\frac{1}{2}\right)^{2p} a_\star$, it follows from the previous equation that

$$\dot{a}_\star = 2^{2p}\dot{\zeta}_{0\star,p;p,m} + 2^{(2p+1)}\zeta_{0\star,p;p,m}.$$

Consequently, in order to have solutions without logarithmic divergences one needs $\dot{a}_\star = 0$ or, equivalently,

$$\dot{\zeta}_{0\star,p;p,m} = -2\zeta_{0\star,p;p,m}.$$

Remark 52. The polynomial solutions to equation (6.27b)in the case $\ell = p$ are, thus, of the form

$$\zeta_{0,p;p,m} = \zeta_{0\star,p;p,m}(1-\tau)^{(p+1)}(1+\tau)^{(p-1)}.$$

Now, since $a_s(\tau) \equiv a(-\tau)$ is a solution for the equation for $\zeta_{2,p;p,m}$ we have that to avoid logarithms in the solutions to equation (6.27c) one needs the condition

$$\dot{\zeta}_{2\star,p;p,m} = 2\zeta_{2\star,p;p,m}.$$

In this case, the polynomial solution is given by

$$\zeta_{2,p;p,m}(\tau) = \zeta_{2\star,p;p,m}(1+\tau)^{(p+1)}(1-\tau)^{(p-1)}.$$

Making use of the above results, the properties of the solutions to the transport equations implied by the decoupled (i.e. linear) Maxwell-scalar system at the cylinder at spatial infinity $\mathcal{I}$ can be succinctly summarised in the following proposition:

Proposition 12. *Given the jet $J^p[\phi, \boldsymbol{\alpha}]$ for $\mathfrak{q} = 0$ one has that:*

(i) the elements of the jet have polynomial dependence in τ for the harmonic sectors with $0 \leq \ell \leq p-1$ and, thus, they extend analytically through $\tau = \pm 1$;

(ii) generically, for $\ell = p$, the solutions have logarithmic singularities at $\tau = \pm 1$. These logarithmic divergences can be precluded by fine-tuning of the initial data.

Remark 53. The key insight from the analysis of the decoupled system is that for a given order p, the elements in $J^p[\phi, \boldsymbol{\alpha}]$ only exhibit singular behaviour at the critical sets $\mathcal{I}^{\pm}$ where spatial infinity touches null infinity for the harmonics with the highest admissible ℓ. All other sectors with $\ell < p$ are completely regular for generic initial conditions.

6.6.2 The coupled case

In this section, we provide an analysis of the behaviour of the elements of the jet $J^p[\phi, \boldsymbol{\alpha}]$ in the case $\mathfrak{q} \neq 0$ with particular emphasis on their regularity at the critical sets $\mathcal{I}^{\pm}$. In order to keep the presentation concise we focus on the differences with the decoupled case —see Remark 53.

The $p = 0$ order transport equations

We start our analysis of the full non-linear system by looking at the solutions corresponding to the jet $J^0[\phi, \boldsymbol{\alpha}]$ —that is, the order $p = 0$. By evaluating the system (6.24a)-(6.24e) one finds that

$$\blacktriangle \phi^{(0)} = s^{(0)}, \tag{6.28a}$$

$$\blacktriangle \alpha_+^{(0)} - 2\dot{\alpha}_+^{(0)} = j_+^{(0)}, \tag{6.28b}$$

$$\blacktriangle\alpha_-^{(0)} + 2\dot{\alpha}_-^{(0)} = j_-^{(0)}, \tag{6.28c}$$

$$\blacktriangle\alpha_0^{(0)} + \alpha_0^{(0)} = j_0^{(0)}, \tag{6.28d}$$

$$\blacktriangle\alpha_2^{(0)} + \alpha_2^{(0)} = j_2^{(0)}. \tag{6.28e}$$

This order is non-generic as under Assumption 1, it can be readily verified that the above transport equations decouple and it is possible to write down the solution explicitly. More precisely, one has that:

Lemma 13. *The unique solution to the* 0*-th order system* (6.28a)-(6.28e) *with initial conditions* (6.25a)-(6.25c) *is given by*

$$\phi^{(0)} = \varphi_\star, \qquad \alpha_\pm^{(0)} = 0, \qquad \alpha_0^{(0)} = 0, \qquad \alpha_2^{(0)} = 0.$$

Remark 54. As it will be seen, the 0-th order jet $J^0[\phi, \boldsymbol{\alpha}]$ given by the above lemma allows to start a recursive scheme to compute the higher order jets $J^p[\phi, \boldsymbol{\alpha}]$ with $p \geq 1$.

The $p \geq 1$ transport equations

In order to analyse the properties of the jet of order p, $J^p[\phi, \boldsymbol{\alpha}]$ for given $p = n$, we assume that we have knowledge of the jets

$$J^0[\phi, \boldsymbol{\alpha}], \quad J^1[\phi, \boldsymbol{\alpha}], \ldots, J^{n-1}[\phi, \boldsymbol{\alpha}].$$

Under this assumption and taking into account Lemma 13 one finds that the elements of $J^p[\phi, \boldsymbol{\alpha}]$ satisfy the equations —cfr. the general discussion in Subsection 6.4.1:

$$\blacktriangle\phi^{(n)} + 2n\tau\dot{\phi}^{(n)} = s^{(n)}, \tag{6.29a}$$

$$\blacktriangle\alpha_+^{(n)} + 2(n\tau - 1)\dot{\alpha}_+^{(n)} = 2\mathfrak{q}^2|\varphi_\star|^2\alpha_+^{(n)} + \tilde{j}_+^{(n)}, \tag{6.29b}$$

$$\blacktriangle\alpha_-^{(n)} + 2(n\tau + 1)\dot{\alpha}_-^{(n)} = 2\mathfrak{q}^2|\varphi_\star|^2\alpha_-^{(n)} + \tilde{j}_-^{(n)}, \tag{6.29c}$$

$$\blacktriangle\alpha_0^{(n)} + 2n\tau\dot{\alpha}_0^{(n)} + \alpha_0^{(n)} = 2\mathfrak{q}^2|\varphi_\star|^2\alpha_0^{(n)} + \tilde{j}_0^{(n)}, \tag{6.29d}$$

$$\blacktriangle\alpha_2^{(n)} + 2n\tau\dot{\alpha}_2^{(n)} + \alpha_2^{(n)} = 2\mathfrak{q}^2|\varphi_\star|^2\alpha_2^{(n)} + \tilde{j}_2^{(n)}, \tag{6.29e}$$

where $s^{(n)}$, $\tilde{j}_\pm^{(n)}$, $\tilde{j}_0^{(n)}$ and $\tilde{j}_2^{(n)}$ depend, solely, on the elements of $J^p[\phi, \boldsymbol{\alpha}]$, $0 \leq p \leq n-1$.

Remark 55. The key new feature in the above equations is the presence in (6.29b)-(6.29e) of the terms involving the constant $2\mathfrak{q}^2|\varphi_\star|^2$ in the right-hand side of the equations. These terms arise from the cubic nature of the coupling in the source terms in the Maxwell-scalar field system. This feature does not arise in systems with quadratic coupling like the conformal Einstein-field equations or the Maxwell-Dirac system. In particular, observe that one is led to consider model homogeneous equations of the form

$$\blacktriangle\zeta + 2(n\tau - 1)\dot{\zeta} - \varkappa\zeta = 0, \tag{6.30a}$$

$$\blacktriangle\zeta + 2n\tau\dot{\zeta} + (1-\varkappa)\zeta = 0 \tag{6.30b}$$

with $\varkappa \equiv 2\mathfrak{q}^2|\varphi_\star|^2$. As will be seen in the sequel, the solutions of these equations for generic choice of $\varkappa$ is radically different to that of the case $\varkappa = 0$ —i.e. $\mathfrak{q} = 0$.

Now, assuming that the various fields have an asymptotic expansion as in Remark 49 one is led to consider a hierarchy of ordinary differential equations of the form

$$(1-\tau^2)\ddot{\phi}_{n;\ell,m} + 2(n-1)\tau\dot{\phi}_{n;\ell,m} + ((\ell-n+1)(n+\ell))\phi_{n;\ell,m} = s_{n;\ell,m}, \tag{6.31a}$$

$$(1-\tau^2)\ddot{\alpha}_{+,n;\ell,m} + 2(-1+(n-1)\tau)\dot{\alpha}_{+,n;\ell,m} + (\ell(\ell+1) - n(n-1) - \varkappa)\alpha_{+,n;\ell,m} = \tilde{\jmath}_{+,n;\ell,m}, \tag{6.31b}$$

$$(1-\tau^2)\ddot{\alpha}_{-,n;\ell,m} + 2(1+(n-1)\tau)\dot{\alpha}_{-,n;\ell,m} + (\ell(\ell+1) - n(n-1) - \varkappa)\alpha_{-,n;l,m} = \tilde{\jmath}_{-,n;\ell,m}, \tag{6.31c}$$

$$(1-\tau^2)\ddot{\alpha}_{0,n;\ell,m} + 2(n-1)\tau\dot{\alpha}_{0,n;\ell,m} + ((\ell-n+1)(n+\ell) - \varkappa)\alpha_{0,n;\ell,m} = \tilde{\jmath}_{0,n;\ell,m}, \tag{6.31d}$$

$$(1-\tau^2)\ddot{\alpha}_{2,n;\ell,m} + 2(n-1)\tau\dot{\alpha}_{2,n;\ell,m} + ((\ell-n+1)(n+\ell) - \varkappa)\alpha_{2,n;\ell,m} = \tilde{\jmath}_{2,n;\ell,m}, \tag{6.31e}$$

for $0 \leq \ell \leq n$, $-\ell \leq m \leq \ell$ and with the *source terms*

$$s_{n;\ell,m}, \quad \tilde{\jmath}_{+,n;\ell,m}, \quad \tilde{\jmath}_{-,n;\ell,m}, \quad \tilde{\jmath}_{0,n;\ell,m}, \quad \tilde{\jmath}_{2,n;\ell,m},$$

known as a result of the spherical harmonics decomposition of the lower order jets $J^p[\phi, \boldsymbol{\alpha}]$ for $0 \leq p \leq n-1$. The homogeneous version of equations (6.31b)-(6.31e) does not fit the general scheme of solutions discussed in Subsection 6.6.1 for the decoupled system. In fact, one has the following general result from [71] which we quote for completeness.

Lemma 14. *The Jacobi ordinary differential equation*

$$(1-\tau^2)\ddot{a} + \Big(\beta - \alpha - (\alpha+\beta+2)\tau\Big)\dot{a} + \gamma a = 0$$

has polynomial solutions if and only if γ is rational.

So, the question is whether it is possible to characterise the solutions in an easy manner? For this, we resort to Frobenius's method to study the properties of the equations in terms of asymptotic expansions at the values $\tau = \pm 1$ —see [72], Chapter 4. The homogeneous version of the equations (6.31b)-(6.31e) can be described in terms of the model equation

$$(1-\tau^2)\ddot{\zeta} + 2(\varsigma + (n-1)\tau)\dot{\zeta} + (\ell(\ell+1) - n(n-1) - \varkappa)\zeta = 0 \qquad (6.32)$$

where

$$\varsigma = \begin{cases} -1 & \text{for} \quad \alpha_+ \\ 1 & \text{for} \quad \alpha_- \\ 0 & \text{for} \quad \phi,\ \alpha_0,\ \alpha_2 \end{cases}$$

—recall also that $\varkappa = 2\mathfrak{q}^2\varphi_\star^2$. Following Frobenius's method, we look for power series solutions of the form

$$\zeta = (1-\tau)^r \sum_{k=0}^{\infty} D_k (1-\tau)^k, \qquad D_0 \neq 0. \qquad (6.33)$$

Substitution of the Ansatz (6.33) into the model equation (6.32) leads to the *indicial equation*

$$2r(r-1) - 2\varsigma r - 2(n-1)r = 0.$$

The solutions to the indicial equation for the various values of ς are given in Table 1.

Once the solutions to the indicial equation are known, Ansatz (6.33) leads to a recurrence relation for the coefficients D_k in the series. The details of this computation are given in Appendix B.3. The key observation for the subsequent discussion is that for a given value of ς, the root $r_1 = 0$ of the indicial polynomial does not lead to a valid series solution as the recursion relation breaks down at some order. In order to obtain a second,

ς	r_1	r_2
-1	0	$n-1$
0	0	n
1	0	$n+1$

Table 1: Roots of the indicial equation.

linearly independent solution to equation (6.32) one needs to consider a more general type of Ansatz. Again, following the discussion in [72] we look for a second solution of the form

$$\zeta = \sum_{k=0}^{\infty} G_k(1-\tau)^k + (1-\tau)^{r_2}\log(1-\tau)\sum_{k=0}^{\infty} M_k(1-\tau)^k, \qquad G_0 \neq 0, \quad M_0 \neq 0. \quad (6.34)$$

A detailed inspection of the recurrence relations implied by the Ansatz (6.34) shows that all the coefficients M_k for $k = 1, 2, \ldots$ and G_k for $k = 0, 1, 2, \ldots$ can be expressed in terms of the coefficient M_0 —again, see Appendix B.3 for further details.

Remark 56. The previous analysis has been restricted, for concreteness, to the behaviour of the solutions to the homogeneous model equation (6.32) near $\tau = 1$. A similar analysis can be carried out *mutatis mutandi* to obtain the behaviour of the solutions near $\tau = -1$.

Remark 57. Observe that the logarithmic singularity of the solutions given by (6.34) is modulated by a term of the form $(1-\tau)^{r_2}$. Accordingly, within the radius of convergence of the series, the whole solution is of class C^{r_2-1} at $\tau = \pm 1$.

Remark 58. It is of some interest that the solutions to the model equation (6.32) can be written in closed form in terms of hypergeometric functions. This representation, however, makes it harder to examine the regularity properties of the solutions at the critical values $\tau = \pm 1$.

The discussion in the previous paragraphs can be summarised in the following:

Proposition 13. *The general solution to the model equation* (6.32)

$$(1-\tau^2)\ddot{\zeta} + 2(\varsigma + (n-1)\tau)\dot{\zeta} + (\ell(\ell+1) - n(n-1) - \varkappa)\zeta = 0, \qquad \varkappa \neq 0$$

with $\varsigma = -1, 0, 1,\ 0 \leq \ell \leq n,\ n = 1, 2, \ldots,$ *consists of:*

(i) one solution which is analytic for $\tau \in [-1, 1]$*;*

(ii) one solution which is analytic for $\tau \in (-1, 1)$ *and has logarithmic singularities at* $\tau = \pm 1$*. At these singular points the solution is of class* $C^{r_2 - 1}$*.*

Remark 59. The key observation from the previous analysis is the fact that the solutions to the homogeneous equations in the coupled case have one solution with logarithmic singularities for every $0 \leq \ell \leq n$ and $-\ell \leq m \leq \ell$. This is in contrast to the decoupled case where only the solutions corresponding to the spherical harmonics with $\ell = n$ had logarithmic divergences.

The solution to the inhomogeneous equations

Having analysed the behaviour of the solutions to the homogeneous part of the transport equations we proceed now to briefly discuss the behaviour to the full inhomogeneous equations (6.31a)-(6.31e). For this we rely on the method of variation of parameters as discussed in Appendix B.4.

In the following let ζ denote any of the unknowns $(\phi_{n;\ell,m},\ \alpha_{+,n;\ell,m},\ \alpha_{-,n;\ell,m},\ \alpha_{0,n;\ell,m},\ \alpha_{2,n;\ell,m})$ in the transport equations (6.31a)-(6.31e). These equations are described through the model equation

$$(1-\tau^2)\ddot{\zeta} + 2(\varsigma + (n-1)\tau)\dot{\zeta} + (n(1-n) + \ell(\ell+1) - \varkappa)\zeta = f(\tau), \qquad \varsigma = -1,\ 0,\ 1, \tag{6.35}$$

where f denotes the corresponding source terms $(s_{n;\ell,m},\ \tilde{\jmath}_{+,n;\ell,m},\ \tilde{\jmath}_{-,n;\ell,m},\ \tilde{\jmath}_{0,n;\ell,m},\ \tilde{\jmath}_{2,n;\ell,m})$. Moreover, let ζ_1 and ζ_2 denote two linearly independent solutions to the homogeneous problem. The method of variation of parameters gives the general solution to (6.35) in the form

$$\zeta(\tau) = A_1(\tau)\zeta_1(\tau) + A_2(\tau)\zeta_2(\tau), \tag{6.36}$$

where

$$A_1(\tau) = A_{1\star} - \int_0^\tau \frac{\zeta_2(s) f(s)}{W_\star (1-s^2)^n} \left(\frac{1+s}{1-s}\right)^{2\varsigma} \mathrm{d}s, \tag{6.37a}$$

$$A_2(\tau) = A_{2\star} + \int_0^\tau \frac{\zeta_1(s)f(s)}{W_\star(1-s^2)^n}\left(\frac{1+s}{1-s}\right)^{2\varsigma} \mathrm{d}s, \tag{6.37b}$$

with $A_{1\star}$ and $A_{2\star}$ constants fixed by the initial data. The details of the derivation of these expressions can be found in Appendix B.4. For ease of presentation, the discussion in this subsection is focused on the the behaviour of the solutions at $\tau = 1$. A similar discussion can be made, *mutatis mutandi*, for the behaviour at $\tau = -1$.

Consistent with Proposition (13), we distinguish two cases for the solutions of (6.36) as follows:

(i) the two solutions to the homogeneous equation are smooth at $\tau = 1$;

(ii) one of the solutions to the homogeneous problem is smooth at $\tau = 1$ while the other has a logarithmic singularity.

In the following for simplicity of the presentation it is assumed that the source term f is regular at $\tau = 1$ —i.e. it does not contain singularities of either logarithmic type or poles.

Case (i). We observe that the integrands in equations (6.37a) and (6.37b) contain a pole of order $n + 2\varsigma$ at $\tau = 1$. The decomposition in partial fractions will, for generic source $f(s)$, contain a term of the form

$$\frac{1}{1-s}$$

which, when integrated gives rise to a logarithmic term $\ln(1-\tau)$. This type of logarithmic singularity can be precluded if the zeros of the expressions

$$\zeta_1(s)f(s)(1+\tau)^{2\varsigma}, \qquad \zeta_2(s)f(s)(1+\tau)^{2\varsigma}$$

have a very fine-tuned structure. The latter can be, in principle, reexpressed in terms of conditions on the initial —this task, however, goes beyond the scope of this analysis. Thus, the generic conclusion is that even if the solutions ζ_1 and ζ_2 to the homogeneous problem do not contain logarithmic singularities at $\tau = 1$, the actual solutions to the transport equations at a given order will have this type of singularities unless the initial conditions are fine-tuned. The regularity (or more precisely, lack thereof) of these singularities is controlled by the factors of $(1-\tau)$ appearing in the functions ζ_1 and ζ_2. It can also be readily verified that the structure of these factors in ζ_1 and ζ_2 is such that the final solution as given by formula (6.36) has no poles at $\tau = 1$ —that is to say, the only possible singularities are of logarithmic type.

Case (ii). *In the following we assume that ζ_2 is the solution to the model homogeneous equation containing the logarithmic term.* A quick inspection of equation (6.37b) the shows that this term will give rise, generically, to logarithmic singularities similar to those in Case (i). The situation is, however, different for expression (6.37a) for which the denominator already contains a $\ln(1-\tau)$ term. The decomposition in terms of partial fractions gives rise to a term of the form

$$\frac{\ln(1-\tau)}{(1-\tau)},$$

which, after being integrated, gives rise to a singular term of the form

$$\ln^2(1-\tau).$$

This is the most singular term arising from the integration of the partial fractions decomposition of the integrand in (6.37a). As in Case (i), the coefficients in the partial fractions decomposition can, in principle, be expressed in terms of initial data —thus, this singular term could be removed by fine-tuning. The remaining terms in the expansion give rise, at worst, to singular terms containing $\ln(1-\tau)$ and some power of $1-\tau$. As in Case (i), it can be verified that the solution arising from formula (6.36) does not contain poles at $1-\tau$ —that is, again, all singular behaviour is of logarithmic type.

Remark 60. More generally, in view of the recursive nature of the of the transport equation in which the source terms at order n are given explicitly in terms of lower-order jets, the source terms will contain logarithmic terms involving powers of $\ln(1-\tau)$. Because of the structural properties of the variation of parameters formula will then give rise to higher-order logarithmic terms. The discussion in the previous paragraphs thus shows that even in the optimal case where the source is completely regular, logarithmic terms will arise.

6.6.3 Summary

The discussion in this section can be summarised in the following:

Theorem 5. *For generic initial data for the Maxwell-scalar field the jet $J^p[\phi, \boldsymbol{\alpha}]$, $p \geq 1$ contains logarithmic divergences at the $\tau = \pm 1$ —i.e. at the critical sets $\mathcal{I}^{\pm}$ where null infinity meets spatial infinity— for* all *spherical harmonic sectors. The logarithmic divergences are of the form*

$$(1 \pm \tau)^{\mu_1} \ln^{\mu_2}(1 \pm \tau)$$

for some non-negative integers μ_1, μ_2.

Remark 61. The situation described in Theorem 5 is to be contrasted with the situation in the decoupled case in which for the solution jet at order p, for generic initial data, there always exist spherical sectors without logarithmic singularities —see Proposition 12. Moreover, due to the absence of source terms the logarithmic singularities are of the form

$$(1 \pm \tau)^{\mu_3} \ln(1 \pm \tau)$$

for some non-negative integer μ_3. It is in this sense that the non-linear coupling of the Maxwell and scalar fields gives rise to a more singular behaviour at the conformal boundary and, consequently, a more complicated type of asymptotics.

6.7 Peeling properties of the Maxwell-scalar system

In this section, we translate the results on the regularity of the solutions of the Einstein-Maxwell at the conformal boundary obtained in Section 6.6 into statements about the asymptotic decay of the fields in the physical spacetime. The most important consequence of regularity (smoothness) at the conformal boundary of a field is the so-called *peeling* —i.e a hierarchical decay of the various components of, say, the Maxwell field along the generators of outgoing light cones. As the asymptotic expansions of Section 6.6 generically imply a non-smooth behaviour at the conformal boundary, one expects a *modified peeling behaviour.*

6.7.1 The Newman-Penrose gauge

The discussion of peeling properties fields is usually done in terms of a gauge which is adapted to null infinity —the so-called *Newman-Penrose (NP) gauge*. The relation between the NP-gauge and the F-gauge used to compute the expansions in Section 6.6 has been studied in detail, for the Minkowski spacetime, in [40]. In this subsection we briefly discuss the associated transformation formulae.

In the following, the discussion will be restricted to the case of $\mathscr{I}^+$. Analogous conditions can be formulated, *mutatis mutandi*, for $\mathscr{I}^-$. The NP gauge is adapted to the geometry of null infinity. Let $\{\boldsymbol{e}'_{\boldsymbol{AA'}}\}$ denote a frame satisfying $\boldsymbol{\eta}(\boldsymbol{e}'_{\boldsymbol{AA'}}, \boldsymbol{e}'_{\boldsymbol{BB'}}) = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}$ in a neighbourhood $\mathcal{U}$ of $\mathscr{I}^+$. The frame is said to be in the NP-gauge if it satisfies the conditions:

(i) the vector $\boldsymbol{e}'_{\boldsymbol{11'}}$ is tangent to $\mathscr{I}^+$ and is such that

$$\nabla_{\boldsymbol{11'}}\boldsymbol{e}'_{\boldsymbol{11}} = 0.$$

(ii) There exits a smooth function u (*retarded time*) on $\mathcal{U}$ that satisfies $\boldsymbol{e}'_{\boldsymbol{11'}}(u) = 1$ at $\mathscr{I}^+$.

(iii) The vector $\boldsymbol{e}'_{\boldsymbol{00'}}$ is required to satisfy

$$\boldsymbol{e}'_{\boldsymbol{00'}} = \boldsymbol{\eta}(\mathbf{d}u, \cdot).$$

(iv) Let

$$\mathcal{N}_{u_\bullet} \equiv \{p \in \mathcal{U} \mid u(p) = u_\bullet\},$$

where $u_\bullet$ is constant. Then the frame $\boldsymbol{e}'_{\boldsymbol{AA'}}$, tangent to $\mathcal{N}_{u_\bullet} \cup \mathscr{I}^+$, satisfies

$$\nabla_{\boldsymbol{00'}}\boldsymbol{e}'_{\boldsymbol{AA'}} = 0 \ \text{ on } \mathcal{N}_{u_\bullet}.$$

In [40], the relation between the NP-gauge frame $\{\boldsymbol{e}'_{\boldsymbol{AA'}}\}$ and the F-gauge frame $\{\boldsymbol{e}_{\boldsymbol{AA'}}\}$ for the Minkowski spacetime, as defined in Section 6.3.3, was explicitly computed. This computation assumes the conformal factor

$$\Theta = \rho(1-\tau^2),$$

and its key outcomes are summarised in the following:

Proposition 14. *The NP-gauge frame at $\mathscr{I}^+$ and F-gauge frame in the Minkowski spacetime are related via*

$$\boldsymbol{e}'_{\boldsymbol{AA'}} = \Lambda^{\boldsymbol{B}}{}_{\boldsymbol{A}}\bar{\Lambda}^{\boldsymbol{B'}}{}_{\boldsymbol{A'}}\boldsymbol{e}_{\boldsymbol{BB'}}, \tag{6.38}$$

with

$$\Lambda^{\boldsymbol{0}}{}_{\boldsymbol{1}} = \frac{2e^{i\omega}}{\sqrt{\rho}(1+\tau)}, \quad \Lambda^{\boldsymbol{1}}{}_{\boldsymbol{0}} = \frac{e^{-i\omega}\sqrt{\rho}(1+\tau)}{2}, \quad \Lambda^{\boldsymbol{1}}{}_{\boldsymbol{1}} = \Lambda^{\boldsymbol{0}}{}_{\boldsymbol{0}} = 0, \tag{6.39}$$

where ω is an arbitrary real number that encodes the spin rotation of the frames on $\mathbb{S}^2$. For the NP-gauge frame at $\mathscr{I}^-$, the roles of the vectors $\boldsymbol{e}'_{\boldsymbol{00'}}$ and $\boldsymbol{e}'_{\boldsymbol{11'}}$ are interchanged, and NP-gauge frame is related to the F-gauge by equation (6.38) *with $\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{B}}$ given by*

$$\Lambda^{\boldsymbol{0}}{}_{\boldsymbol{1}} = \frac{e^{-i\omega}\sqrt{\rho}(1-\tau)}{2}, \quad \Lambda^{\boldsymbol{1}}{}_{\boldsymbol{0}} = \frac{2e^{i\omega}}{\sqrt{\rho}(1-\tau)}, \quad \Lambda^{\boldsymbol{1}}{}_{\boldsymbol{1}} = \Lambda^{\boldsymbol{0}}{}_{\boldsymbol{0}} = 0. \tag{6.40}$$

6.7.2 The scalar field

We start our discussion of the peeling properties by looking at the scalar field. In order to carry out this computation we make the following assumption:

Assumption 2. *On $\mathcal{M}$, the scalar field ϕ satisfies the asymptotic expansion*

$$\phi = \sum_{p=0}^{N} \frac{1}{p!}\phi^{(p)}\rho^p + o_1(\rho^N)$$

for some sufficiently large N and where $\phi^{(p)}$ are contained in the solution jet $J^{(p)}[\phi, \boldsymbol{\alpha}]$ as discussed in Section 6.6. The remainder $o_1(\rho^N)$ is assumed to be, at least, of class C^1.

Remark 62. Making use of a generalisation of the estimates near $\mathcal{I}$ introduced in [30] for the massless spin-2 filed it is, in principle, possible to relate, in a rigorous manner, Taylor-like expansions like the one in Assumption 2 arising from the jets computed in Section 6.6 and actual solutions to the Maxwell-scalar field. The main challenge in the present case compared to the analysis in [30] is the non-linearity of the system of equations. The discussion of this problem, which would allow to reduce Assumption 2 to more basic hypothesis falls, however, outside the scope of the present thesis.

Consistent with Assumption 2 and following the discussion of Section 6.6, generically, the scalar field has, near $\mathcal{I}$ the form

$$\phi = \varphi_\star + \mathrm{O}(\rho(1-\tau)\ln(1-\tau)).$$

Now, recall that the physical scalar field $\tilde{\phi}$ is related to the unphysical one via $\tilde{\phi} = \Theta\phi$ with $\Theta = \rho(1-\tau^2) \approx \rho(1-\tau)$ near $\tau = 1$ —i.e $\mathscr{I}^+$. Accordingly, one has that

$$\tilde{\phi} = \rho(1-\tau)\varphi_\star + \mathrm{O}(\rho^2(1-\tau)^2\ln(1-\tau)).$$

Finally, expressing the latter in terms of the physical radial Bondi coordinate $\tilde{r} \approx 1-\tau$ one concludes that

$$\tilde{\phi} = \frac{\varphi_\star}{\tilde{r}} + O\left(\frac{\ln\tilde{r}}{\tilde{r}^2}\right).$$

Thus, to leading order, the physical scalar field satisfies the classic peeling behaviour. Polyhomogeneous (i.e. logarithmic contributions) are subleading.

6.7.3 The Maxwell field

In analogy to the discussion of the scalar field, we make the following assumption on the components of the Maxwell spinor —cfr. Assumption 2:

Assumption 3. *On $\mathcal{M}$, the components of the Maxwell spinor ϕ_{AB} satisfy the asymptotic expansion*

$$\phi_i = \sum_{p=0}^{N} \frac{1}{p!}\phi_i^{(p)}\rho^p + o_1(\rho^N), \qquad i = 0,\ 1,\ 2,$$

for some sufficiently large N and where $\phi_i^{(p)}$ the coefficients contained in the jet $J^p[\boldsymbol{\phi}]$ of the Maxwell field which can be computed from the solution jet $J^{(p)}[\phi, \boldsymbol{\alpha}]$ as discussed in Section 6.6. The reminder $o_1(\rho^N)$ is assumed to be, at least, of class C^1.

A careful inspection of the solutions to the Maxwell-scalar field equations at order $p = 1$ following the discussion in Section 6.6 shows that, for generic data, close to null infinity, $\mathscr{I}^+$, one has that

$$\phi_0 = O\Big((1-\tau)^2\ln(1-\tau)\Big),$$

$$\phi_1 = O\Big((1-\tau)\ln(1-\tau)\Big),$$
$$\phi_2 = O\Big(\ln(1-\tau)\Big).$$

The above expressions are given in the F-gauge. To analyse the peeling properties of solutions with this behaviour we transform into the NP gauge making use of Proposition 14. More precisely, the physical components of the Maxwell spinor in the NP gauge $\tilde{\phi}_0$, $\tilde{\phi}_1$, $\tilde{\phi}_2$, are given by:

$$\tilde{\phi}_0 = \Theta\Lambda^P{}_0\Lambda^Q{}_0\phi_{PQ},$$
$$\tilde{\phi}_1 = \Theta\Lambda^P{}_1\Lambda^Q{}_0\phi_{PQ},$$
$$\tilde{\phi}_2 = \Theta\Lambda^P{}_1\Lambda^Q{}_1\phi_{PQ}.$$

Observing that, to leading order, the physical Bondi radial coordinate satisfies $\tilde{r} \approx 1-\tau$, one concludes that

$$\tilde{\phi}_0 = O\left(\frac{\ln\tilde{r}}{\tilde{r}^3}\right), \qquad \tilde{\phi}_1 = O\left(\frac{\ln\tilde{r}}{\tilde{r}^2}\right), \qquad \tilde{\phi}_2 = O\left(\frac{\ln\tilde{r}}{\tilde{r}}\right).$$

The key point to notice in the above expressions is the presence of a logarithm in the leading term of the radiation field —ϕ_2 in the conventions used in this article. This is a specific property of the Maxwell-scalar field system — for a decoupled Maxwell field on flat spacetime, the behaviour of this particular component is always

$$\phi_2 = O\left(\frac{1}{\tilde{r}}\right)$$

—see e.g. [73].

Chapter 7

Conclusions and future perspectives

In this thesis, the conformal Einstein field equations have been discussed along with several applications. These equations, their origin and motivation have been presented in Chapter 2. The main strength of using the conformal Einstein field equations as a tool for analysing the global properties of solutions to the Einstein field equations consists of their behaviour under conformal transformations. This property allows one to study the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ through the analysis of its conformal extension $(\mathcal{M}, \boldsymbol{g})$. Furthermore, in the context of the initial value problem, it allows to reduce, in certain cases, global problems into local ones, e.g. the proof of the semiglobal non-linear stability of the Minkowski spacetime and the global non-linear stability of the de Sitter spacetime — see [23, 24].

As discussed in Chapter 2, there are two versions of the conformal Einstein field equations: the *standard conformal Einstein field equations* and a more general version represented by the *extended conformal Einstein field equations*. The former requires gauge fixing by means of *gauge source functions*, whereas for the latter this is done by exploiting the notion of *conformal Gaussian systems of coordinates*. In both cases, one obtains a first-order system of symmetric hyperbolic evolution equations. Conversely, in the classical discussion of the Cauchy problem in General Relativity due to Fourès-Bruhat [18], the hyperbolic reduction of the Einstein field equations using the gauge source functions associated with the choice of *harmonic coordinates*, reduces the Einstein field equations to a second-order hyperbolic system for the metric components.

In Chapter 3, the first application of the conformal Einstein field equations is discussed. This chapter contains a discussion on the non-linear stability of de Sitter-like spacetimes. More precisely, it is discussed how the extended conformal Einstein field equations and a gauge based on the properties of the *conformal geodesics* can be used to study the non-linear stability of this class of spacetime. This analysis identifies a class of spacetimes for which it is possible to prove non-linear stability and the existence of a regular conformal representation. These special properties are not shared by generic Cosmological solutions. Thus, it is important to identify the situations in which this is the case.

The use of conformal methods in General Relativity poses an important question: *can the conformal Einstein field equations be used to analyse the stability of black hole spacetimes as well as asymptotically simple spacetimes?* Observations have shown that the Cosmological constant in our Universe is positive, thus spacetimes that describe isolated systems within a de Sitter universe are physically relevant for this type of analysis. In this regard, the strategy described in Chapter 3 is used in Chapter 4 to discuss the non-linear stability of the Cosmological region of the Schwarzschild-de Sitter spacetime. This analysis is the first step in a programme to study the non-linear stability of this region of the Schwarzschild-de Sitter spacetime. Here we show that it is possible to construct solutions to the vacuum Einstein field equations in this region, containing a portion of the asymptotic region, which are, in a precise sense, non-linear perturbations of the exact Schwarzschild-de Sitter spacetime. Crucially, although the spacetimes constructed have an infinite extent to the future, they exclude the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$. These points correspond to the regions of the spacetime where the Cosmological horizon and the conformal boundary seem to *meet*. From the analysis of the asymptotic initial value problem in [39] it is known that the asymptotic points in the conformal boundary, from which the horizons emanate, contain singularities of the conformal structure. Thus, they cannot be dealt with by the approach used in the present work which relies on the Cauchy stability of the initial value problem for symmetric hyperbolic systems. The next step in our programme is to reformulate the existence and stability results discussed in Chapter 4 in terms of a characteristic initial value problem with data prescribed on the Cosmological horizon. However, it is necessary to prescribe the characteristic data away from the asymptotic points to avoid the singularities of the conformal structure. Alternatively, one could consider data sets which become exactly Schwarzschild-de Sitter near the asymptotic points. Given the comparative simplicity of the characteristic constraint equations, proving the existence of such data sets is not as challenging as in the case of the standard (i.e. spacelike) constraints. In what respects the evolution problem, it is expected that a generalisation of the methods used in [45] should allow us to evolve characteristics to reach a suitable hypersurface of constant coordinate r.

On the other hand, it is conjectured that the singular behaviour at the asymptotic points can be studied by methods similar to those used in the analysis of spatial infinity — see [28]. The latter consists of the introduction of a new representation of spatial infinity known as *the cylinder at spatial infinity*. In this representation, spatial infinity is not represented as a point but as a set with the topology of a cylinder. This construction allows us to formulate a regular finite initial value problem for the conformal Einstein field equations. This framework is used in Chapter 6 to analyse the effects of the interaction of a Maxwell field and a scalar field at the critical sets $\mathcal{I}^+$ and $\mathcal{I}^-$ where null infinity $\mathscr{I}$ meets spatial infinity i^0.

The Maxwell-scalar field system offers useful insights to study the linearised gravitational field and as a model for the Bianchi equations satisfied by the components of the Weyl tensor. More precisely, this provides a possible model problem for the Einstein field equations, as it can be used to understand the effects of the non-linear interactions on the regularity of solutions at the conformal boundary. The study of the non-linear interaction between the Maxwell and scalar fields shows a more singular behaviour than what can be expected by studying the behaviour of the fields when non-interacting. The cubic coupling in the Maxwell-scalar field equations generically makes the solutions more singular than what would be expected from the mere analysis of the linear analogue. This situation stands in stark contrast to that of systems with quadratic coupling like that of the Einstein field equations for which the solutions to the homogeneous transport equations in both the linear and full non-linear case share the same type of logarithmic divergences. In this sense, the Maxwell-scalar field is not a good toy model to analyse the effects of non-linear interactions in a neighbourhood of spatial infinity. Other models which potentially overcome this shortcoming are the Dirac-Maxwell system and the Yang-Mills system for which the coupling is quadratic.

Finally, we observe that for generic initial data which have finite energy and are analytic around $\mathcal{I}$ the solution to the transport equations on $\mathcal{I}$ have logarithmic singularities at the critical sets $\mathcal{I}^+$ and $\mathcal{I}^-$. The propagation of the singularities at $\mathcal{I}^\pm$ along the conformal boundary destroys the smoothness of the Faraday tensor and the scalar field tensor at $\mathcal{I}^\pm$ so that, in contrast to a decoupled context, there is no *peeling* behaviour.

Appendix A

Details of Kato's theorem for symmetric hyperbolic systems and a note on future geodesic completeness

A.1 On Kato's existence and stability result for symmetric hyperbolic systems

In this appendix, we make some remarks concerning the hypothesis in Kato's existence, uniqueness and stability result for symmetric hyperbolic equations in [49]. The results in this reference and, in particular the main Theorem II, are very general and presented in an abstract manner. This abstract presentation hinders the direct applicability of the theory. The purpose of this Appendix is to provide a guide to the use of this theorem and to verify that the main evolution system satisfies the hypothesis of the result.

Kato's theory is concerned with symmetric hyperbolic systems in which the unknown $\mathbf{u}$ is regarded as a $\mathcal{P}$-valued function over $\mathbb{R}^m$ where $\mathcal{P}$ is a Hilbert space. The Hilbert space can be real or complex and, in fact, infinite-dimensional. In the present analysis, we are interested in the case where $\mathcal{P}$ is finite-dimensional —say, of dimension N. In this case, the symmetric hyperbolic system becomes a *standard* partial differential equation. For concreteness we set here $\mathcal{P} = \mathbb{R}^N$ and $m = 3$. The following discussion of Kato's theorem will be made with this particular choice in mind.

Kato's theorem is concerned with (N-dimensional) symmetric hyperbolic quasi-linear systems of the form

$$\mathbf{A}^0(t, \underline{x}, \mathbf{u})\partial_t \mathbf{u} + \mathbf{A}^\alpha(t, \underline{x}, \mathbf{u})\partial_\alpha \mathbf{u} = \mathbf{F}(t, \underline{x}, \mathbf{u}). \tag{A.1}$$

for $0 \leq t \leq T$, $\underline{x} \in \mathbb{R}^3$, $\alpha = 1, 2, 3$, and initial conditions

$$\mathbf{u}(0, x) = \mathbf{u}_\star(\underline{x}). \tag{A.2}$$

In Kato's theory, it is convenient to regard the coefficients $\mathbf{A}^0(t, \underline{x}, \mathbf{u})$ and $\mathbf{A}^\alpha(t, \underline{x}, \mathbf{u})$ as non-linear operators depending on t sending $\mathbb{R}^N$-valued functions (i.e. the vector $\mathbf{u}$) over $\mathbb{R}^3$ into $(N \times N)$-matrix valued functions on $\mathbb{R}^3$ —in Kato's terminology these are the elements of $\mathcal{B}(\mathcal{P})$, the space of bounded linear operators over $\mathcal{P}$. Similarly, $\mathbf{F}(t, \underline{x}, \mathbf{u})$ is regarded as a non-linear operator depending on t sending $\mathbb{R}^N$-valued functions on $\mathbb{R}^3$ into $\mathbb{R}^N$-valued functions on $\mathbb{R}^3$.

Consider now $H^s(\mathbb{R}^3, \mathbb{R}^N)$, the space of $(\mathbb{R}^N)$-vector valued functions over $\mathbb{R}^3$ such that their entries have finite Sobolev norm of order s. Let $\mathcal{D}$ be a bounded open subset of $H^s(\mathbb{R}^3, \mathbb{R}^N)$. Writing

$$\mathbf{A}^\mu(t, \underline{x}, \mathbf{u}) = \Big(a^\mu_{ij}(t, \underline{x}, \mathbf{u})\Big), \qquad \mathbf{F}(t, \underline{x}, \mathbf{u}) = \Big(f_i(t, \underline{x}, \mathbf{u})\Big), \qquad i,\, j = 1, \ldots N, \qquad \mu = 0, \ldots, 3,$$

one has that for fixed t and $\mathbf{u} \in \mathcal{D}$

$$a^\mu_{ij}(t, \underline{x}, \mathbf{u}) : \mathbb{R}^m \to \mathbb{R},$$
$$f_i(t, \underline{x}, \mathbf{u}) : \mathbb{R}^m \to \mathbb{R}.$$

Key in Kato's analysis are the *uniformly local Sobolev spaces* H^s_{ul}. Let $C^\infty_0(\mathbb{R}^3, \mathbb{R})$ denote the sets of smooth functions of compact support from $\mathbb{R}^3$ to $\mathbb{R}$. Given any non-zero $\phi \in C^\infty_0(\mathbb{R}^3, \mathbb{R})$ not identically zero, then $\mathbf{u} \in H^s_{ul}$ if and only if

$$\sup_{\underline{x} \in \mathbb{R}^3} \parallel \phi_x \mathbf{u} \parallel_s < \infty, \qquad \phi_x(\underline{y}) \equiv \phi(\underline{y} - \underline{x}).$$

Remark 63. In other words, the vector-valued function $\mathbf{u}$ is in H^s_{ul} if its Sobolev norm of order s over any compact set over $\mathbb{R}^3$ is finite and remains finite as one considers larger and larger compact sets on $\mathbb{R}^3$.

Remark 64. The spaces H^s_{ul} satisfy nice embedding properties analogous to those of H^s —see Lemma 2.7 in [49].

In the following, it will be assumed that for fixed t and $\mathbf{u} \in \mathcal{D}$, the coefficients $a^\mu_{ij}(t, \underline{x}, \mathbf{u}(\underline{x}))$ are functions from $\mathcal{D}$ to $H^s_{ul}(\mathbb{R}^3, \mathbb{R})$. For $f_i(t, \underline{x}, \mathbf{u}(\underline{x}))$ one has the more relaxed condition of being a function from $\mathcal{D}$ to $H^s(\mathbb{R}^3, \mathbb{R})$. In Kato's more abstract terminology this is equivalent to requiring that $\mathbf{A}^\mu$ is a function from $\mathcal{D}$ to $H^s_{ul}(\mathbb{R}^3, \mathcal{B}(\mathcal{P}))$ and $\mathbf{F}$ from $\mathcal{D}$ to $H^s(\mathbb{R}^3, \mathcal{P})$.

One has the following reformulation of Theorem II in [49]:

Theorem 6. *Let s be a positive integer such that $s > 3/2 + 1 = 5/2$. Let $\mathbf{A}^\mu(t, \underline{x}, \mathbf{v}(\underline{x}))$, $\mathbf{F}(t, \underline{x}, \mathbf{v}(\underline{x}))$ and $\mathbf{v} \in \mathcal{D}$ as above with $0 \leq t \leq T$. Assume that the following conditions hold:*

(i) The components $a^\mu_{ij}(t, \underline{x}, \mathbf{v}(\underline{x}))$ (respectively, $f_i(t, \underline{x}, \mathbf{v}(\underline{x}))$) are bounded in the H^s_{ul}-norm (respectively H^s-norm) for $\mathbf{v} \in \mathcal{D}$, uniformly in t.

(ii) For each t, the map $\mathbf{v}(\underline{x}) \mapsto \mathbf{A}^\alpha(t, \underline{x}, \mathbf{v}(\underline{x}))$ is uniformly Lipschitz continuous on $\mathcal{D}$ from the H^0-norm to the H^0_{ul}-norm, uniformly in t. Similarly, the map $\mathbf{v}(\underline{x}) \mapsto \mathbf{F}(t, \underline{x}, \mathbf{v}(\underline{x}))$ is Lipschitz continuous from the H^0-norm to the H^0-norm, again uniformly in t.

(iii) The map $\mathbf{v}(\underline{x}) \mapsto \mathbf{A}^0(t, \underline{x}, \mathbf{v}(\underline{x}))$ is Lipschitz continuous on $\mathcal{D}$ from the H^{s-1}-norm to the H^{s-1}_{ul}-norm, uniformly in t.

(iv) The maps $t \mapsto \mathbf{A}^\alpha(t, \underline{x}, \mathbf{v}(\underline{x}))$ are continuous in the H^0_{ul}-norm for each $\mathbf{v} \in \mathcal{D}$. Similarly, the map $t \mapsto \mathbf{F}(t, \underline{x}, \mathbf{v}(\underline{x}))$ is continuous in the H^0-norm for each $\mathbf{v} \in \mathcal{D}$.

(v) The map $t \mapsto \mathbf{A}^0(t, \underline{x}, \mathbf{v}(\underline{x}))$ is Lipschitz-continuous on $[0, T]$ in the H^{s-1}_{ul}-norm, uniformly for $\mathbf{v} \in \mathcal{D}$.

(vi) *For each* $\mathbf{v} \in \mathcal{D}$ *the matrix-valued functions* $\mathbf{A}^\mu(t, \underline{x}, \mathbf{v}(\underline{x}))$ *are symmetric for each* $(t, \underline{x}) \in [0, T] \times \mathbb{R}^m$.

(vii) *The matrix* $\mathbf{A}^0(t, \underline{x}, \mathbf{v}(\underline{x}))$ *is positive definite with eigenvalues larger than, say,* 1 *for each* $(t, \underline{x})$ *and each* $\mathbf{v} \in \mathcal{D}$.

(viii) $\mathbf{u}_\star \in \mathcal{D}$.

Then there is a unique solution $\mathbf{u}$ *to* (A.1)-(A.2) *defined on* $[0, T']$ *where* $0 < T' \leq T$ *such that*

$$\mathbf{u} \in C[0, T'; \mathcal{D}] \cup C^1[0, T'; H^{s-1}(\mathbb{R}^3, \mathbb{R}^N)],$$

where T' *can be chosen common to all initial conditions* $\mathbf{u}_\star$ *in a suitably small condition of a given point in* $\mathcal{D}$.

In practice, the conditions of the above theorem are hard to verify. Kato provides sufficient conditions ensuring that conditions in the above theorem are satisfied (Theorem IV in [49]:

Theorem 7. *Suppose that* $s > 3/2 + 1 = 5/2$. *Let* Ω *be the subset of* $\mathbb{R}^3 \times \mathbb{R}^N$ *consisting of pairs* $(\underline{x}, \underline{v})$ *such that*

$$|v - v_\star(x)| < \omega, \qquad \underline{x} \in \mathbb{R}^3$$

where $\omega > 0$ *and* $v_\star \in H^s(\mathbb{R}^3, \mathbb{R}^N) \subset C^1(\mathbb{R}^3, \mathbb{R}^N)$ *are fixed. Let, as before,*

$$\mathbf{A}^\mu : [0, T] \times \Omega \longrightarrow \mathcal{B}(\mathbb{R}^N),$$
$$\mathbf{F} : [0, T] \times \Omega \longrightarrow \mathbb{R}^N,$$

where $\mathcal{B}(\mathbb{R}^N)$ *denotes the set of* $(N \times N)$*-matrix valued functions over* $\mathbb{R}^3$ *with the properties*

(a) $\mathbf{A}^\alpha \in C[0, T; C^s_b(\Omega, \mathcal{B}(\mathbb{R}^N))]$,

(b) $\mathbf{A}^0 \in \mathrm{Lip}[0, T; C^{s-1}_b(\Omega, \mathcal{B}(\mathbb{R}^N))]$,

(c) $\mathbf{F} \in C[0, T; C^{s+1}_b(\Omega, \mathbb{R}^N)]$,

(d) $\mathbf{F}_\star \in L^\infty[0,T;H^s(\mathbb{R}^3,\mathbb{R}^N)] \cap C[0,T;H^0(\mathbb{R}^3,\mathbb{R}^N)]$,

where $\mathbf{F}_\star(t,\underline{x}) \equiv \mathbf{F}(t,\underline{x},v_\star(\underline{x}))$. *Then conditions* (i)-(v) *in Theorem 6 are satisfied by* $\mathbf{A}^\mu$, $\mathbf{F}$ *provided that* $\mathcal{D}$ *is chosen as a ball in* $H^s(\mathbb{R}^3,\mathbb{R}^N)$ *with* $v_\star$ *as centre and a sufficiently small radius* $R_\star$. *In addition,* (ix) *is satisfied if* (a) *is assumed to hold with* s *replaced by* $s+1$.

Remark 65. The sets $C^r_b(\Omega,\mathcal{B}(\mathbb{R}^N))$ and $C^r_b(\Omega,\mathbb{R}^N)$ denote the spaces of functions having derivatives up to the r-th order which are continuous and bounded in the supremum norm.

Remark 66. If the $\mathbf{A}^\mu$ are polynomials in $\underline{p}$ it actually suffices that the coefficients only be in $C[0,T;H^s_{ul}]$ and also in $C^1[0,T;H^{s-1}_{ul}]$ for $\mathbf{A}^0$.

A.2 Future geodesic completeness of the background solution

The geodesic completeness of the metric (3.2) can be shown using the theory developed in [10] —in particular, Corollary 3.3 in this reference applies to the present situation.

More precisely, the theory in [10] applies to spacetimes $(\mathcal{M},\boldsymbol{g})$ such that $\mathcal{M} = [t_\bullet,\infty)\times\mathcal{S}$ where $t_\bullet > 0$ and $\mathcal{S}$ is a smooth 3-dimensional manifold. The metric $\boldsymbol{g}$ has the $3+1$ split

$$\boldsymbol{g} = -\alpha^2\boldsymbol{\omega}^0\otimes\boldsymbol{\omega}^0 + h_{\boldsymbol{ij}}\boldsymbol{\omega}^{\boldsymbol{i}}\otimes\boldsymbol{\omega}^{\boldsymbol{j}},$$

with

$$\boldsymbol{\omega}^0 = \mathbf{d}t, \qquad \boldsymbol{\omega}^{\boldsymbol{i}} = \mathbf{d}x^{\boldsymbol{i}} + \beta^{\boldsymbol{i}}\mathbf{d}t.$$

There exist numbers $0<\alpha_-$, α_+ such that

$$0<\alpha_- \leq \alpha \leq \alpha_+.$$

The metric $\boldsymbol{h} \equiv h_{\boldsymbol{ij}}\mathbf{d}x^{\boldsymbol{i}}\otimes\mathbf{d}x^{\boldsymbol{j}}$ is a *geodesically complete Riemannian metric* on $\mathcal{S}_t \equiv \{t\}\times\mathcal{S}$ such that there exists a constant $C_1>0$ such that

$$C_1 h_{\boldsymbol{ij}}(t_\bullet)v^{\boldsymbol{i}}v^{\boldsymbol{j}} \leq h_{\boldsymbol{ij}}(t)v^{\boldsymbol{i}}v^{\boldsymbol{j}}$$

for all vectors on $T\mathcal{S}$ and $t \in [t_\bullet, \infty)$. Furthermore, there exists another constant C_2 such that

$$\beta_i \beta^i \leq C_2, \qquad t \in [t_\bullet, \infty).$$

In the following let K_{ij} denote the extrinsic curvature of the hypersurfaces $\mathcal{S}_t$, $K_{\{ij\}}$ is tracefree part and K its trace.

With the above conditions, the metric $\boldsymbol{g}$ is future geodesically complete if the following two conditions hold:

(i) $D_i \alpha D^i \alpha$ is bounded by a function of t which is integrable on $[t_\bullet, \infty)$;

(ii) $K < 0$ and $K_{ij}K^{ij}$ is integrable on $[t_\bullet, \infty)$.

The metric (3.2) can be readily seen to satisfy the above conditions. In particular, as $\alpha = 1$, the norm of the spatial gradient of the lapse vanishes and, accordingly, it is integrable —this verifies condition (i) above. Moreover, the extrinsic curvature of the hypersurfaces of constant t is given by

$$K_{ij} = -\sinh t \cosh t \mathring{\gamma}_{ij},$$

so that it is pure trace. Moreover, one has that

$$K = -3 \coth t < 0, \qquad t \in [t_\bullet, \infty).$$

As $K_{\{ij\}} = 0$ in this case one has that (ii) is also satisfied. It follows then that the background metric $\mathring{\boldsymbol{g}}$ is future geodesically complete.

Appendix B

Details of the analysis of the Maxwell-scalar field system

B.1 The trace-free Ricci spinor

The use of commutators to obtain the wave equations satisfied by the components of the gauge potential leads to terms involving the spinorial counterpart of the tracefree Ricci spinor —see Equation (6.13b). The symmetries of the tracefree Ricci spinor (5.19) imply the decomposition

$$\begin{aligned}\Phi_{AB'CD'} = {} & \tfrac{1}{6}\Phi_{AC}\epsilon_{B'D'} + \tfrac{1}{3}\Phi_{CB}\tau_{AD'}\tau^{B}{}_{B'} + \tfrac{1}{6}\Phi_{CB}\tau_{AB'}\tau^{B}{}_{D'} \\ & + \tfrac{1}{6}\Phi_{AB}\tau^{B}{}_{D'}\tau_{CB'} + \tfrac{1}{3}\Phi_{AB}\tau^{B}{}_{B'}\tau_{CD'} + \Phi_{ABCD}\tau^{B}{}_{B'}\tau^{D}{}_{D'} \\ & + \tfrac{1}{3}\Phi h_{ACBD}\tau^{B}{}_{B'}\tau^{D}{}_{D'},\end{aligned}$$

where

$$\Phi_{AB} = \Phi_{(AB)}, \qquad \Phi_{ABCD} = \Phi_{(ABCD)}.$$

A direct computation of the components of the Schouten tensor of the Weyl connection associated to the covector $f_{AA'}$ in the conformal representation of the Minkowski spacetime given in the F-gauge shows that all its components vanish. Observing that

$$\hat{L}_{ba} = L_{ab} + f_a f_b - \frac{1}{2} f_c f^c g_{ab} - \nabla_b f_a,$$

it follows then that

$$\Phi_{ABA'B'} = f_{AA'}f_{BB'} - \frac{1}{2}f_{CC'}f^{CC'}\epsilon_{AB}\epsilon_{A'B'} - \nabla_{BB'}f_{AA'}.$$

In the present case, one has that

$$f_{AA'} = -x_{AB}\tau^{B}{}_{A'},$$

consistent with the fact that $f_{AA'}\tau^{AA'} = 0$. Combining the above observations, one can conclude that

$$\Phi = -1, \qquad \Phi_{AB} = 0, \qquad \Phi_{ABCD} = x_{(AB}x_{CD)}.$$

B.2 Properties of the solutions to the Jacobi ordinary differential equation

In the following, it will be convenient to define

$$D_{(n,\alpha,\beta)}a \equiv (1-\tau^2)\ddot{a} + \Big(\beta - \alpha - (\alpha+\beta+2)\tau\Big)\dot{a} + n(n+\alpha+\beta+1)a, \tag{B.1}$$

so that the general *Jacobi equation* can be written as

$$D_{(n,\alpha,\beta)}a = 0. \tag{B.2}$$

A class of solutions to (B.2) is given by the *Jacobi polynomial* of degree n with integer parameters (α, β) given by

$$P_n^{(\alpha,\beta)}(\tau) \equiv \sum_{s=0}^{n} \binom{n+\alpha}{s}\binom{n+\beta}{n-s}\left(\frac{\tau-1}{2}\right)^{n-s}\left(\frac{\tau+1}{2}\right)^{s}.$$

It follows from the above that

$$P_0^{(\alpha,\beta)}(\tau) = 1,$$

and that

$$P_n^{(\alpha,\beta)}(-\tau) = (-1)^n P_n^{(\beta,\alpha)}(\tau).$$

Solutions to (B.2) satisfy the identities

$$D_{(n,\alpha,\beta)}\left(\left(\frac{1-\tau}{2}\right)^{-\alpha}a(\tau)\right)=\left(\frac{1-\tau}{2}\right)^{-\alpha}D_{(n+\alpha,-\alpha,\beta)}a(\tau), \tag{B.3a}$$

$$D_{(n,\alpha,\beta)}\left(\left(\frac{1+\tau}{2}\right)^{-\beta}a(\tau)\right)=\left(\frac{1+\tau}{2}\right)^{-\beta}D_{(n+\beta,\alpha,-\beta)}a(\tau), \tag{B.3b}$$

$$D_{(n,\alpha,\beta)}\left(\left(\frac{1-\tau}{2}\right)^{-\alpha}\left(\frac{1+\tau}{2}\right)^{-\beta}a(\tau)\right)=\left(\frac{1-\tau}{2}\right)^{-\alpha}\left(\frac{1+\tau}{2}\right)^{-\beta}D_{(n+\alpha+\beta,-\alpha,-\beta)}a(\tau), \tag{B.3c}$$

which hold for $|\tau|<1$, arbitrary C^2-functions $a(\tau)$ and arbitrary values of the parameters α, β, n.

An alternative definition of the Jacobi polynomials, convenient for verifying when the functions vanish identically, is given by

$$P_n^{(\alpha,\beta)}(\tau)=\frac{1}{n!}\sum_{k=0}^{n}c_k\left(\frac{\tau-1}{2}\right)^k$$

with

$$\begin{aligned}
c_0 &\equiv (\alpha+1)(\alpha+2)\cdots(\alpha+n),\\
&\vdots\\
c_k &\equiv \frac{n!}{k!(n-k)!}(\alpha+k+1)(\alpha+k+2)\cdots(\alpha+n)\times(n+1+\alpha+\beta)(n+2+\alpha+\beta)\cdots\\
&\quad\cdots(n+k+\alpha+\beta),\\
&\vdots\\
c_n &\equiv (n+1+\alpha+\beta)(n+2+\alpha+\beta)\cdots(2n+\alpha+\beta).
\end{aligned}$$

Thus, for example, for $\alpha=\beta=-p$ and $n=p+\ell$ one finds that the string of products in the above coefficients start at a negative integer value and end up at a positive one indicating that one of the factors vanishes. Accordingly, the whole coefficient must vanish.

B.3 Details of the computation of the series solutions

The purpose of this appendix is to discuss some of the details in the computation of the series of solutions presented in Proposition 13. The approach followed here is a variation of the classical Frobenius method —see e.g. [72], Chapter 4.

B.3.1 The first solution

Following the main text, we consider the model equation

$$(1-\tau^2)\ddot{\zeta} + 2(\varsigma + (n-1)\tau)\dot{\zeta} + (\ell(\ell+1) - n(n-1) - \varkappa)\zeta = 0, \tag{B.4}$$

and we look for solutions satisfying the Ansatz

$$\zeta = (1-\tau)^r \sum_{k=0}^{\infty} D_k (1-\tau)^k, \qquad D_0 \neq 0. \tag{B.5}$$

Differentiation of this power series gives

$$\dot{\zeta} = (-1)\sum_{k=0}^{\infty}(k+r)D_k(1-\tau)^{k+r-1},$$
$$\ddot{\zeta} = \sum_{k=0}^{\infty}(k+r)(k+r-1)D_k(1-\tau)^{k+r-2}.$$

Hence, observing that

$$2(n-1)\tau = 2(n-1)(1-(1-\tau))$$

and by replacing the derivatives into the model equation (B.4) one obtains the indicial polynomial

$$(2r(r-1) - 2\varsigma r - 2(n-1)r) = 0. \tag{B.6}$$

Accordingly, one has that for $\varsigma = -1$ one has the roots $r_1 = 0$ and $r_2 = n-1$; for $\varsigma = 1$ the roots are $r_1 = 0$ and $r_2 = n+1$; whereas for $\varsigma = 0$ one has the roots $r_1 = 0$ and $r_2 = n$. Some further lengthy manipulations lead to the following recurrence relations for the coefficients in the Ansatz (B.5):

(i) for $\varsigma = 1$ and $r_1 = 0$

$$D_{k+1} = -\frac{k(k-3) + 2n + (n(1-n) + \ell(\ell+1) - \varkappa)}{2k(k-n-1)} D_k,$$

while if $r_2 = n+1$ one has

$$D_{k+1} = -\frac{(k+n+1)(k+n) + (n(1-n) + \ell(\ell+1) - \varkappa) - 2(k+1)}{2(k+n+1)(k+n-1)} D_k;$$

(ii) for $\varsigma = -1$ and $r_1 = 0$ one has

$$D_{k+1} = -\frac{k(k-3) + 2n + (n(1-n) + \ell(\ell+1) - \varkappa)}{2k(k-n+1)} D_k,$$

while if $r_2 = n-1$ one has

$$D_{k+1} = -\frac{(k+n-1)(k+n-4) + 2n + (n(1-n) + l(l+1) - x)}{2k(k+n-1)} D_k;$$

(iii) finally, if $\varkappa = 0$ and $r_1 = 0$ one has that

$$D_{k+1} = -\frac{k(k-3) + 2n + (n(1-n) + \ell(\ell+1) - \varkappa)}{2k(k-n+1)} D_k,$$

while if $r_2 = n$ one has

$$D_{k+1} = -\frac{(k+n)(k+n-3) + 2n + (n(1-n) + \ell(\ell+1) - \varkappa)}{2k(k+n)} D_k.$$

Two key observations can be drawn from the previous expressions:

(i) all the recurrence relations associated to non-zero roots of the indicial polynomial are well defined for $k \geq 0$. Accordingly, these lead to an infinite Taylor series for the solutions. These series can be resumed as hypergeometric functions. These solutions are regular and, in fact, analytic at $\tau = 1$. Analogous series solutions can be obtained for $\tau = -1$.

(ii) All the recurrence relations associated to zero roots of the indicial polynomial become singular for a certain non-zero value of k. Accordingly, these recurrence relations do not lead to well-defined series solutions.

In summary, the procedure described in this section only provides one independent solution to the second-order model ordinary differential equation (B.4).

B.3.2 The second solution

Motivated by the *method of reduction of order* we look for a second solution to the model equation (B.4) of the form

$$\zeta = \sum_{k=0}^{\infty} G_k (1-\tau)^k + \ln(1-\tau) \sum_{k=0}^{\infty} M_k (1-\tau)^{k+r}, \qquad G_0 \neq 0, \qquad M_0 \neq 0. \tag{B.7}$$

The substitution of the Ansatz (B.7) into equation (B.4) leads again to the indicial polynomial (B.6). Moreover, the coefficients M_k can be shown to satisfy, for the various choices of the parameter ς, the same recurrence relations as in the previous section. Accordingly, *in the following we only consider the non-zero roots of the indicial polynomial* and the series

$$\sum_{k=0}^{\infty} M_k(1-\tau)^{k+r_2}$$

is a formal solution to the model equation (B.4). The rest of the analysis is split in cases corresponding to the possible values of ς.

The case $\varsigma = 0$. In this case, the root of the indicial polynomial is $r = n$. For $k \leq n-2$ one has the recurrence relation

$$G_{k+1} = \frac{k(k-1) - 2(n-1)k - (n(1-n) + \ell(\ell+1))}{2k(k+1) - 2(n-1)(k+1)} G_k.$$

For $k = n-1$ one has

$$G_{n-1} = \frac{-2n}{-(n-1)(n-2) + 2n(n-1) - 2(n-1) + (n(1-n) + \ell(\ell+1))} M_0.$$

For $0 \geq k \geq n$ one has

$$\begin{aligned} G_{k+1} =& \frac{k(k-1) - 2nk + 2k - (n(1-n) + \ell(\ell+1))}{2k(k+1) - 2(n-1)(k+1)} G_k + \frac{-2k - 2(k+1) + 2(n-1)}{2k(k+1) - 2(n-1)(k+1)} M_{k-n+1} \\ &+ \frac{(k-1) + k - 2n + 2}{2k(k+1) - 2(n-1)(k+1)} M_{k-n}. \end{aligned}$$

In conclusion, in the case $\varsigma = 0$ one obtains a second linearly independent solution which contains a logarithmic singularity at $\tau = 1$. This solution has only one undetermined constant —namely M_0.

The case $\varsigma = 1$. In this case, the non-zero root of the indicial polynomial is given by $r_2 = n+1$. For $k \leq n-1$ we have the recurrence relation

$$G_{k+1} = \frac{k(k-1) - 2(n-1)k - (n(1-n) + \ell(\ell+1) - \varkappa)}{2k(k+1) - 2(k+1) - 2(n-1)(k+1)} G_k.$$

For $k = n$ we have the recurrence relation

$$G_n = -\frac{(n+2)}{-n(n-1) + 2n(n-1) + (n(1-n) + \ell(\ell+1) - \varkappa)} M_0.$$

For $k \geq n+1$ we have the recurrence relation

$$
\begin{aligned}
G_{k+1} = & \frac{k(k-1) - 2(n-1)k - (n(1-n) + \ell(\ell+1) - \varkappa)}{2k(k+1) - 2(k+1) - 2(n-1)(k+1)} G_k \\
& + \frac{-2k - 2(k+1) + k + 2 + 2(n-1)}{2k(k+1) - 2(k+1) - 2(n-1)(k+1)} M_{k-n} \\
& + \frac{k - 2n + 2}{2k(k+1) - 2(k+1) - 2(n-1)(k+1)} M_{k-n-1}.
\end{aligned}
$$

Again, this solution has a logarithmic singularity at $\tau = 1$ and the free constant is M_0.

The case $\varsigma = -1$. In this case, the non-zero root of the indicial polynomial is given by $r_2 = n - 1$. If $k \leq n-3$, we have the recurrence relation

$$
G_{k+1} = \frac{k(k-1) - 2(n-1)k - (n(1-n) + \ell(\ell+1) - \varkappa)}{2(k+1)(k-n+2)} G_k.
$$

If $k = n-2$ we have

$$
G_{n-2} = -\frac{2(n-1)}{(n-2)(n+1) + (n(1-n) + \ell(\ell+1) - x)} M_0.
$$

If $k \geq n-1$ the recurrence relation is the following

$$
\begin{aligned}
G_{k+1} = & \frac{k(k-1) - 2(n-1)k - (n(1-n) + \ell(\ell+1) - \varkappa)}{2(k+1)(k-n+2)} G_k - \frac{4(k+1) - 2(n-1)}{2(k+1)(k-n+2)} M_{k-n+2} \\
& + \frac{2k - 2n + 1}{2(k+1)(k-n+2)} M_{k-n+1}.
\end{aligned}
$$

All the above expressions lead to well-defined formal series solutions to the model equation (B.4) containing a logarithmic singularity at $\tau = 1$. The regularity of the solutions is regulated by the value of the corresponding root to the indicial polynomial. For example, for $\varsigma = 0$, the logarithmic part of the solution contains the factor

$$
\ln(1-\tau)(1-\tau)^n.
$$

Accordingly, the first n derivatives of the solution are finite at $\tau = 1$.

The analysis sketched in this appendix is summarised in Proposition 13 in the main text.

B.4 Solving the inhomogeneous transport equations

In this section, we discuss a general procedure to compute the solutions to the inhomogeneous equation

$$(1-\tau^2)\ddot{\zeta} + 2(\varsigma + (n-1)\tau)\dot{\zeta} + (n(1-n) + \ell(\ell+1) - \varkappa)\zeta = f(\tau), \qquad \varsigma = -1,\, 0,\, 1.$$

In the following, for convenience, we write the latter in the form

$$\ddot{\zeta} + \frac{2(\varsigma + (n-1)\tau)}{(1-\tau^2)}\dot{\zeta} + \frac{(n(1-n) + \ell(\ell+1) - \varkappa)}{(1-\tau^2)}\zeta = \tilde{f}(\tau), \qquad \tilde{f}(\tau) \equiv \frac{f(\tau)}{1-\tau^2}. \tag{B.8}$$

Let, in the following ζ_1 and ζ_2 denote solutions to the homogeneous problem

$$\ddot{\zeta} + \frac{2(\varsigma + (n-1)\tau)}{(1-\tau^2)}\dot{\zeta} + \frac{(n(1-n) + \ell(\ell+1) - \varkappa)}{(1-\tau^2)}\zeta = 0.$$

We follow the method of variation of the parameters and look for solutions of the form

$$\zeta(\tau) = A_1(\tau)\zeta_1(\tau) + A_2(\tau)\zeta_2(\tau)$$

subject to the restriction

$$\dot{A}_1\zeta_1 + \dot{A}_2\zeta_2 = 0.$$

A calculation readily yields

$$\begin{aligned}\dot{\zeta} &= A_1\dot{\zeta}_1 + A_2\dot{\zeta}_2,\\ \ddot{\zeta} &= A_1\ddot{\zeta}_1 + A_2\ddot{\zeta}_2 + \dot{A}_1\dot{\zeta}_1 + \dot{A}_2\dot{\zeta}_2,\end{aligned}$$

so that by replacing these relations into (B.8) one has that

$$\dot{A}_1\dot{\zeta}_1 + \dot{A}_2\dot{\zeta}_2 = \tilde{f}.$$

Accordingly, one obtains the algebraic system

$$\begin{aligned}\zeta_1\dot{A}_1 + \zeta_2\dot{A}_2 &= 0,\\ \dot{\zeta}_1\dot{A}_1 + \dot{\zeta}_2\dot{A}_2 &= \tilde{f}.\end{aligned}$$

For convenience, we rewrite this in matricial form as

$$\begin{pmatrix} \zeta_1 & \zeta_2 \\ \dot{\zeta}_1 & \dot{\zeta}_2 \end{pmatrix} \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{f} \end{pmatrix}. \tag{B.9}$$

The latter can be recast as

$$\begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \end{pmatrix} = \frac{1}{(1-\tau^2)W(\tau)} \begin{pmatrix} \dot{\zeta}_2 & -\zeta_2 \\ -\dot{\zeta}_1 & \zeta_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix},$$

where

$$W(\tau) \equiv \begin{vmatrix} \zeta_1 & \zeta_2 \\ \dot{\zeta}_1 & \dot{\zeta}_2 \end{vmatrix} = \zeta_1\dot{\zeta}_2 - \zeta_2\dot{\zeta}_1, \tag{B.10}$$

denotes the *Wronskian* of the system (B.9). It readily follows then that

$$\begin{aligned} \dot{A}_1(\tau) &= -\frac{\zeta_2(\tau)f(\tau)}{(1-\tau^2)W(\tau)}, \\ \dot{A}_2(\tau) &= \frac{\zeta_1(\tau)f(\tau)}{(1-\tau^2)W(\tau)}. \end{aligned}$$

Integrating, we conclude that

$$A_1(\tau) = A_{1\star} - \int_0^\tau \frac{\zeta_2(s)f(s)}{(1-s^2)W(s)}\mathrm{d}s, \tag{B.11a}$$

$$A_2(\tau) = A_{2\star} + \int_0^\tau \frac{\zeta_1(s)f(s)}{(1-s^2)W(s)}\mathrm{d}s, \tag{B.11b}$$

with $A_{1\star}$ and $A_{2\star}$ constants.

The Wronskian

Differentiating the definition of the Wronskian $W(\tau)$, Equation (B.10), and using Equation (B.8) one readily finds that

$$\dot{W}(\tau) = \alpha(\tau)W(\tau), \qquad \alpha(\tau) \equiv -\frac{2(\varsigma + (n-1)\tau)}{(1-\tau^2)}.$$

The solution to this ordinary differential equation is given by

$$W(\tau) = e^{A(\tau)}, \qquad \dot{A}(\tau) = \alpha(\tau).$$

It follows then that

$$W(\tau) = W_\star \left(\frac{1-\tau}{1+\tau}\right)^{2\varsigma} (1-\tau^2)^{n-1}, \qquad W_\star \text{ a constant.}$$

Substituting the latter expression in (B.11a)-(B.11b) one obtains the explicit expressions:

$$A_1(\tau) = A_{1\star} - \int_0^\tau \frac{a_2(s)f(s)}{W_\star(1-s^2)^n}\left(\frac{1+s}{1-s}\right)^{2\varsigma} \mathrm{d}s,$$
$$A_2(\tau) = A_{2\star} + \int_0^\tau \frac{a_1(s)f(s)}{W_\star(1-s^2)^n}\left(\frac{1+s}{1-s}\right)^{2\varsigma} \mathrm{d}s.$$

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