

## ENHANCED DISSIPATION AND TRANSITION THRESHOLD FOR THE POISEUILLE FLOW IN A PERIODIC STRIP\*

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**Abstract.** We consider the solution to the two-dimensional Navier–Stokes equations around the Poiseuille flow  $(y^2, 0)$  on  $\mathbb{T} \times \mathbb{R}$  with small viscosity  $\nu > 0$ . Via a hypocoercivity argument, we prove that the  $x$ -dependent modes of the solution to the linear problem undergo the enhanced dissipation effect with a rate proportional to  $\nu^{\frac{1}{2}}$ . Moreover, we study the nonlinear enhanced dissipation effect and we establish a transition threshold of  $\nu^{\frac{2}{3}+}$  for initial data in  $L^2$ . Namely, when the  $L^2$  norm of the perturbation of the Poiseuille flow is size at most  $\nu^{\frac{2}{3}+}$ , its size remains so for all times and the enhanced dissipation persists with a rate proportional to  $\nu^{\frac{1}{2}}$ .

**Key words.** Navier–Stokes equations, Poiseuille flow, transition threshold, enhanced dissipation, metastability

**MSC codes.** 35Q30, 35Q35, 76D05, 76E05

**DOI.** 10.1137/21M1444011

**1. Introduction.** We study the two-dimensional (2D) Navier–Stokes equations

$$(1) \quad \begin{cases} \partial_t U + (U \cdot \nabla)U + \nabla P - \nu \Delta U = 0, \\ \nabla \cdot U = 0, \end{cases}$$

defined on the domain  $\mathbb{T} \times \mathbb{R}$ .  $U = (U_1, U_2)$  is the velocity vector field,  $P$  is the scalar pressure, and  $\nu$  is the viscosity coefficient of the fluid, proportional to the inverse of the Reynolds number. Defining the vorticity of  $U$  as  $\Omega = \nabla^\perp \cdot U$ , where  $\nabla^\perp = (-\partial_y, \partial_x)$ , it is possible to remove the pressure term and to rewrite the above system as

$$(2) \quad \begin{cases} \partial_t \Omega + U \cdot \nabla \Omega - \nu \Delta \Omega = 0, \\ \Omega = \Delta \Psi, \\ U = \nabla^\perp \Psi. \end{cases}$$

Here  $\Psi$  is the corresponding stream function for the vector field  $U$ . It is easy to see that the so-called Poiseuille Flow

$$U_P = (y^2, 0), \quad \Omega_P = -2y$$

is a stationary solution of (2). To study the dynamic near the Poiseuille flow we consider a small perturbation of it. We set  $U = U_P + u$ , so  $\Omega = \Omega_P + \omega$ , where  $u, \omega$  are the perturbations for the velocity field and the vorticity, respectively. The vorticity formulation for  $\omega$  reads as follows:

$$(3) \quad \begin{cases} \partial_t \omega + y^2 \partial_x \omega - 2 \partial_x \psi - \nu \Delta \omega = -u \cdot \nabla \omega, \\ \omega = \Delta \psi, \\ u = \nabla^\perp \psi. \end{cases}$$

\* Received by the editors September 3, 2021; accepted for publication (in revised form) February 23, 2023; published electronically September 14, 2023.  
<https://doi.org/10.1137/21M1444011>

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The boundary conditions here are set to be periodic for the  $x$  variable, while for the  $y$  direction  $\omega$  is assumed to have sufficient decay at infinity. The stability of the Poiseuille flow can then be seen as the decay of the solution to (3). An overview of the paper is presented in the following subsections of the introduction. The first part is devoted to the analysis of the linear problem while the second part focuses on the transition threshold problem for the fully nonlinear equation.

**1.1. Linear enhanced dissipation and estimates for  $\mathcal{L}_\nu$ .** We define the linear operator

$$(4) \quad \mathcal{L}_\nu = -y^2 \partial_x + 2\partial_x \Delta^{-1} + \nu \Delta$$

associated to the linearized counterpart of (3). In the first part, we establish some decay estimates for the semigroup generated by  $\mathcal{L}_\nu$  in the usual  $L^2$  norm. Our first result is the following; here  $\mathbb{P}_\neq$  is the projection on the nonzero  $x$ -Fourier modes.

**THEOREM 1.1.** *Let  $\nu < 1$  and  $g \in L^2(\mathbb{T} \times \mathbb{R})$ . There exist constants  $C_0, c_0 > 0$  independent of  $\nu$  such that*

$$(5) \quad \|e^{\mathcal{L}_\nu t} \mathbb{P}_\neq(g)\|_{L^2} \leq C_0 e^{-c_0 \nu^{\frac{1}{2}} t} \|\mathbb{P}_\neq(g)\|_{L^2}$$

for all  $t \geq 0$ .

This result gives a quantitative estimate of the linear *enhanced dissipation* effect. Indeed, the timescale obtained for the nonzero modes of the initial datum is proportional to  $\nu^{-1/2}$ , which is much faster than the heat equation timescale, proportional to  $\nu^{-1}$ .

We refer to enhanced dissipation as the phenomenon where the mixing properties of the fluid allow one to improve the natural heat dissipation timescale  $\mathcal{O}(\nu^{-1})$  to a faster timescale  $\mathcal{O}(d(\nu)^{-1})$  that satisfies

$$\lim_{\nu \rightarrow 0} \frac{\nu}{d(\nu)} = 0.$$

This phenomenon has been widely studied in the physics literature (see, for example, [11, 16, 17]) and mathematics literature [7, 8]. In the context of the Navier–Stokes equations near shear flows, we cite results for the well known Couette flow [4, 20], with a dissipation timescale  $\mathcal{O}(\nu^{-1/3})$ , and the Kolmogorov flow [12, 21, 22], where the rate is known to be  $\mathcal{O}(\nu^{-1/2})$ . Regarding the Poiseuille flow, the first linear enhanced dissipation result was given by Coti Zelati, Elgindi, and Widmayer [9] for the unbounded 2D domain  $(x, y) \in \mathbb{T} \times \mathbb{R}$ . In their paper the linear enhanced dissipation effect is established around the Poiseuille flow. The rate obtained is proportional to  $\nu^{1/2}(1 + |\log \nu|)^{-1}$  in the weighted  $L^2$  space with norm

$$\|f\|_X^2 = \|f\|_{L^2}^2 + \|yf\|_{L^2}^2.$$

Ding and Lin [10] proved the same decay rate, without the logarithmic correction, for the Poiseuille flow in a bounded 2D channel  $\mathbb{T} \times [-1, 1]$  with Navier–Slip boundary conditions. We cite also the paper by Chen, Wei, and Zhang [6], where a  $\mathcal{O}(\nu^{-1/2})$  rate is obtained for the 3D pipe Poiseuille flow. The approach used in the last two papers is completely different from the hypocoercivity method. It relies on resolvent estimates and a Gearhart–Prüss type theorem introduced by Wei in [19].

Our first result is a sharpening of [9]; indeed we are able to remove the logarithmic correction and to get a decay rate of  $\mathcal{O}(\nu^{-1/2})$  in  $L^2(\mathbb{T} \times \mathbb{R})$ . The proof of

Theorem 1.1 relies on a hypocoercivity argument [18], similar to the one in [9]. Here, we construct an energy functional following Wei and Zhang's idea [21], namely, each term of the functional has a time dependent weight. Theorem 1.1 follows then from an iteration argument. Furthermore, thanks to the time dependent weights of the energy functional, we are able to prove additional estimates on the semigroup  $e^{\mathcal{L}_\nu t}$  generated by the linearized operator  $\mathcal{L}_\nu$ . These estimates play a crucial role in establishing the transition threshold.

**1.2. Nonlinear enhanced dissipation and transition threshold.** Our second result concerns the transition threshold for the 2D Poiseuille flow. The asymptotic stability of fluid motion between parallel plates was first analyzed by Kelvin [13], who introduced the following concept: the stability may depend on the viscosity coefficient  $\nu$  in such a way that the stability threshold decreases whenever  $\nu$  decreases. The mathematical formulation of this problem can be given as follows. Given a norm  $\|\cdot\|_X$ , find a  $\gamma = \gamma(X)$  such that

$$\begin{aligned} \|u\|_X \leq \nu^\gamma &\Rightarrow \text{stability,} \\ \|u\|_X \gg \nu^\gamma &\Rightarrow \text{instability;} \end{aligned}$$

here  $u$  is a perturbation of the flow.

The transition threshold problem for the Couette flow has been deeply studied recently [1, 2, 3, 5], in both the frameworks of Sobolev spaces and Gevrey classes. In the 2D case the transition threshold is known to be  $\gamma \leq \frac{1}{3}$  [15].

For the 2D Kolmogorov flow in the periodic box, it holds that  $\gamma \leq \frac{2}{3} + \varepsilon$  for any  $\varepsilon > 0$  (see [22]). For the Poiseuille flow, it has been proved by Coti Zelati, Elgindi, and Widmayer [9] that  $\gamma \leq \frac{3}{4} + \varepsilon$  in  $\mathbb{T} \times \mathbb{R}$  and by Ding and Lin [10] that  $\gamma \leq \frac{3}{4}$  in  $\mathbb{T} \times [-1, 1]$  with Navier-slip boundary conditions, i.e.,  $\omega(\pm 1) = \psi(\pm 1) = 0$ .

In the second part of this paper we are going to show a transition threshold for the Poiseuille Flow on  $\mathbb{T} \times \mathbb{R}$  with  $\gamma \leq \frac{2}{3} + \varepsilon$ . We are able to prove the following theorem using the linear enhanced dissipation and the estimates on the semigroup generated by the linearized operator.

**THEOREM 1.2.** *There exist constants  $\varepsilon_0 \in (0, 1)$ ,  $C_1 > 0$ ,  $c_1 > 0$  such that for all  $0 < \nu < 1$  and for every  $u_{in}, \omega_{in} \in L^2$  with*

$$\|\omega_{in}\|_{L^2} + \|\mathbb{P}_0 u_{in}\|_{L^2} \leq \varepsilon_0 (1 + |\log \nu|^{\frac{1}{2}})^{-1} \nu^{2/3},$$

*the solution  $\omega(t)$  of (3) is global in time with the bound*

$$\|\mathbb{P}_{\neq}(\omega)(t)\|_{L^2} \leq C_1 e^{-c_1 \nu^{1/2} t} \|\mathbb{P}_{\neq}(\omega_{in})\|_{L^2}.$$

The proof is based on careful estimates of the nonzero modes of the nonlinear term in (3). We remark that our theorem gives a better transition threshold for the planar Poiseuille flow in  $\mathbb{T} \times \mathbb{R}$ , bringing it from  $\nu^{\frac{3}{4}+}$  to  $\nu^{\frac{2}{3}+}$ , more precisely  $(1 + |\log \nu|^{\frac{1}{2}})^{-1} \nu^{\frac{2}{3}}$ .

*Structure of the paper.* In section 2 we define the modified energy functional (11). We prove Theorem 2.1, which describes the key behavior of the energy functional needed to deduce the enhanced dissipation for the linear problem (10). Section 3 is devoted to proving Lemma 3.1, which establishes additional estimates on the semigroup generated by the linearized operator. Section 4 concludes the paper and contains the proof of Theorem 4.2.

*Notation.* Throughout this paper we use

$$\nabla_k = \mathbb{P}_k \nabla = (ik, \partial_y)$$

for the projected gradient onto the  $\pm k$ -th  $x$ -Fourier modes and

$$\Delta_k = \mathbb{P}_k \Delta = -k^2 + \partial_y^2$$

for the projected Laplacian. Moreover, we use  $C > 0$  to indicate a constant independent of  $\nu, k$ , and  $t$  that may vary line by line. We also denote by  $\|\cdot\| = \|\cdot\|_{L^2}$  the usual  $L^2$  norm.

**2. Hypocoercivity estimates.** For an  $x$ -periodic function  $f$  we write its Fourier expansion as

$$(6) \quad f(t, x, y) = \sum_{j \in \mathbb{Z}} a_j(t, y) e^{ijx}, \quad a_j(t, y) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t, x, y) e^{-ijx} dx.$$

For  $k \in \mathbb{N}_0$  we set

$$(7) \quad f_k(t, x, y) = \sum_{|j|=k} a_j(t, y) e^{ijx},$$

so that

$$(8) \quad f(t, x, y) = \sum_{k \in \mathbb{N}_0} f_k(t, x, y)$$

can be expressed as a sum of real valued functions localized in the  $x$ -frequency on a single band  $\pm k$ ,  $k \in \mathbb{N}_0$ . We also introduce the following operators: given a function  $f$  we define

$$(9) \quad \mathbb{P}_0(f) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x, y) dx, \quad \mathbb{P}_{\neq}(f) = f - \mathbb{P}_0(f),$$

and for any  $k \in \mathbb{N}_0$ , we denote  $\mathbb{P}_k$  the projection to the sum of the  $\pm k$ th Fourier modes in  $x$ . We start by considering the linearized system associated to (3) with initial datum  $g \in L^2$ , i.e.,

$$(10) \quad \begin{cases} \partial_t \omega + y^2 \partial_x \omega - 2 \partial_x \psi - \nu \Delta \omega = 0, \\ \Delta \psi = \omega, \\ \omega|_{t=0} = g. \end{cases}$$

The solution is given by  $\omega(t) = e^{\mathcal{L}_\nu t} g$  for all  $t \geq 0$ , where  $\mathcal{L}_\nu$  is the linear operator (4). To establish decay estimates for the semigroup we proceed by defining the following energy functional, where  $\alpha, \beta, \gamma$  have to be determined:

$$(11) \quad \Phi(t) = \frac{1}{2} \|\omega\|^2 + \frac{1}{2} \alpha \nu t \|\nabla \omega\|^2 + 2\beta \nu t^2 \langle \partial_y \omega, y \partial_x \omega \rangle + \frac{1}{2} \gamma \nu t^3 \left[ \|y \partial_x \omega\|^2 + 2 \|\nabla \partial_x \psi\|^2 \right].$$

Note that by collecting factors of the form  $\nu^{1/2} t$ , the last three terms of the functional read as

$$(12) \quad \frac{1}{2} \alpha \nu^{1/2} (\nu^{1/2} t) \|\nabla \omega\|^2 + 2\beta (\nu^{1/2} t)^2 \langle \partial_y \omega, y \partial_x \omega \rangle + \frac{1}{2} \gamma \nu^{-1/2} (\nu^{1/2} t)^3 \left[ \|y \partial_x \omega\|^2 + 2 \|\nabla \partial_x \psi\|^2 \right]$$

and resemble the ones used in [9], differing from them by the time dependent weights. This modification has already been used by Wei and Zhang studying the 2D Kolmogorov flow [21] and by Li and Zhao in the context of the 2D quasi-geostrophic equation on the torus [14]. Inserting time dependent weights is the key to achieve enhanced dissipation in the  $L^2$  norm and to remove the  $1 + |\log \nu|$  correction obtained in [9].

THEOREM 2.1. Fix  $0 < \varepsilon < \frac{1}{36}$  and define  $\alpha = \varepsilon^2, \beta = \varepsilon^3, \gamma = 16\varepsilon^4$ . Then

$$(13) \quad \alpha < \frac{1}{2}, \quad 16\beta^2 \leq \alpha\gamma, \quad 4\alpha^2 < \beta < \frac{1}{16}\alpha, \quad 9\gamma < 4\beta,$$

and it follows that  $\Phi(t)$  satisfies

$$(14) \quad \Phi(t) \geq \frac{1}{2} \|\omega\|^2 + \frac{1}{4} \alpha \nu t \|\nabla \omega\|^2 + \frac{1}{4} \gamma \nu t^3 \left[ \|y \partial_x \omega\|^2 + 2 \|\nabla \partial_x \psi\|^2 \right]$$

and

$$(15) \quad \Phi'(t) \leq -\gamma \nu^2 t^3 \|\partial_x \omega\|^2$$

for every  $t \geq 0$ .

Once this statement is proved, linear enhanced dissipation follows directly.

THEOREM 2.2. Let  $\nu < 1$  and  $g \in L^2$ . There exist constants  $C_0, c_0 > 0$  independent of  $\nu$  such that

$$(16) \quad \|e^{\mathcal{L}\nu t} \mathbb{P}_k(g)\| \leq C_0 e^{-c_0 \nu^{\frac{1}{2}} |k|^{\frac{1}{2}} t} \|\mathbb{P}_k(g)\|$$

for all  $t \geq 0$ . Combining all nonzero modes together we get

$$(17) \quad \|e^{\mathcal{L}\nu t} \mathbb{P}_{\neq}(g)\| \leq C_0 e^{-c_0 \nu^{\frac{1}{2}} t} \|\mathbb{P}_{\neq}(g)\|.$$

In addition, for the nonzero modes of the velocity  $u_k$ , we have the following decay:

$$(18) \quad \|\mathbb{P}_k(u)(t)\|^2 \leq \frac{2}{\gamma \nu k^2 t^3} \|\mathbb{P}_k(g)\|^2$$

for all  $t \geq 0$ .

We briefly give the proof. To simplify the notation we will write  $f_k$  instead of  $\mathbb{P}_k(f)$ .

*Proof of Theorem 2.2.* Since the equations decouple in  $k$ , we can apply separately Theorem 2.1 to the  $\pm k$ th  $x$ -frequency and get

$$(19) \quad \Phi_k(t) \geq \frac{1}{2} \|e^{\mathcal{L}\nu t} g_k\|^2 \quad \text{and} \quad \Phi'_k(t) \leq -\gamma \nu^2 k^2 t^3 \|e^{\mathcal{L}\nu t} g_k\|^2.$$

Then,

$$\begin{aligned} \frac{1}{2} \|e^{\mathcal{L}\nu t} g_k\|^2 &\leq \Phi_k(t) \\ &= \Phi_k(0) + \int_0^t \Phi'_k(s) ds \\ &\leq \frac{1}{2} \|g_k\|^2 - \int_0^t \gamma \nu^2 k^2 s^3 \|e^{\mathcal{L}\nu s} g_k\|^2 ds \\ &\leq \frac{1}{2} \|g_k\|^2 - \frac{\gamma}{4} \nu^2 k^2 t^4 \|e^{\mathcal{L}\nu t} g_k\|^2. \end{aligned}$$

Rearranging this inequality we obtain

$$(20) \quad \|e^{\mathcal{L}\nu t} g_k\|^2 \leq \frac{1}{1 + \frac{\gamma}{2} \nu^2 k^2 t^4} \|g_k\|^2.$$

To prove the enhanced dissipation we proceed by iteration. Fix a time  $t_0$  for which

$$\frac{1}{1 + \frac{\gamma}{2}\nu^2 k^2 t_0^4} = \frac{1}{2}.$$

Now, for every time  $t > t_0$ , write it as  $t = \lfloor t_0^{-1}t \rfloor t_0 + t^*$  with  $t^* \in [0, t_0)$ . Using the fact that (10) is autonomous, we deduce from (20) that for every  $s \in [0, t_0]$

$$(21) \quad \left\| e^{\mathcal{L}_\nu(t_0+s)} g_k \right\|^2 \leq \frac{1}{1 + \frac{\gamma}{2}\nu^2 k^2 t_0^4} \left\| e^{\mathcal{L}_\nu s} g_k \right\|^2 \leq \frac{1}{2} \left\| e^{\mathcal{L}_\nu s} g_k \right\|^2.$$

Then, using the fact that  $t \mapsto \|e^{\mathcal{L}_\nu t} g_k\|$  is decreasing, by iteration we have

$$\begin{aligned} \left\| e^{\mathcal{L}_\nu t} g_k \right\|^2 &\leq \left\| e^{\mathcal{L}_\nu \lfloor t_0^{-1}t \rfloor t_0} g_k \right\|^2 \\ &\leq \left( \frac{1}{2} \right)^{\lfloor t_0^{-1}t \rfloor} \|g_k\|^2 \\ &\leq C_0 e^{-c_0 \nu^{\frac{1}{2}} |k|^{\frac{1}{2}} t} \|g_k\|^2. \end{aligned}$$

Finally, from the monotonicity of  $\Phi(t)$  we can deduce (18). Indeed,

$$(22) \quad \frac{1}{2} \gamma \nu t^3 \|\partial_x \nabla \psi\|^2 \leq \|g\|^2,$$

and recalling that

$$\|u_k\|^2 = \|\nabla_k^\perp \psi_k\|^2 = \|\nabla_k \psi_k\|^2,$$

we have the last inequality. □

In order to prove Theorem 2.1 we state some preliminary identities that will be used to compute the derivative of the functional  $\Phi$ .

PROPOSITION 2.3. *Let  $\omega$  be a solution to (10). Then the following holds:*

$$(23) \quad \frac{1}{2} \frac{d}{dt} \|\omega\|^2 = -\nu \|\nabla \omega\|^2;$$

$$(24) \quad \frac{1}{2} \frac{d}{dt} \|\nabla \omega\|^2 = -\nu \|\Delta \omega\|^2 - 2\langle y \partial_x \omega, \partial_y \omega \rangle;$$

$$(25) \quad \frac{d}{dt} \langle \partial_y \omega, y \partial_x \omega \rangle = -2 \|y \partial_x \omega\|^2 - 4 \|\partial_{xy} \psi\|^2 - 2\nu \langle \Delta \omega, y \partial_{xy} \omega \rangle;$$

$$(26) \quad \frac{1}{2} \frac{d}{dt} \left[ \|y \partial_x \omega\|^2 + 2 \|\nabla \partial_x \psi\|^2 \right] = -\nu \|\partial_x \omega\|^2 - \nu \|y \partial_x \nabla \omega\|^2.$$

The proof of these identities can be found in [9, Lemma 2.4]. We proceed with the proof of Theorem 2.1.

*Proof of Theorem 2.1.* For the sake of clarity we recall here that  $\Phi(t)$  is defined as

$$(27) \quad \Phi(t) = \frac{1}{2} \|\omega\|^2 + \frac{1}{2} \alpha \nu t \|\nabla \omega\|^2 + 2\beta \nu t^2 \langle \partial_y \omega, y \partial_x \omega \rangle + \frac{1}{2} \gamma \nu t^3 \left[ \|y \partial_x \omega\|^2 + 2 \|\nabla \partial_x \psi\|^2 \right].$$

Using the Cauchy–Schwarz inequality and the Young inequality we get

$$(28) \quad 2\beta t^2 |\langle \partial_y \omega, y \partial_x \omega \rangle| \leq 2\beta t^2 \|\partial_y \omega\| \|y \partial_x \omega\| \leq \frac{\alpha t}{4} \|\partial_y \omega\|^2 + \frac{4\beta^2 t^3}{\alpha} \|y \partial_x \omega\|^2.$$

Plugging this into  $\Phi(t)$  we get

$$(29) \quad \Phi(t) \geq \frac{1}{2} \|\omega\|^2 + \frac{1}{4} \alpha \nu t \|\nabla \omega\|^2 + \frac{1}{2} \left( \gamma - \frac{8\beta^2}{\alpha} \right) \nu t^3 \left[ \|y \partial_x \omega\|^2 + 2 \|\nabla \partial_x \psi\|^2 \right],$$

and hence the lower bound. We now proceed by computing the derivative of the functional. By Proposition 2.3 we have

$$\begin{aligned} \frac{d}{dt} \Phi(t) &= -\nu \|\nabla w\|^2 + \frac{\alpha \nu}{2} \|\nabla w\|^2 + \alpha \nu t \left( -\nu \|\Delta w\|^2 - 2 \langle y \partial_x w, \partial_y w \rangle \right) + 4\beta t \nu \langle \partial_y w, y \partial_x w \rangle \\ &\quad + 2\beta \nu t^2 \left( -2\nu \langle \Delta w, y \partial_x \partial_y w \rangle - 2 \|y \partial_x w\|^2 - 4 \|\partial_x \partial_y \psi\|^2 \right) \\ &\quad + \frac{3}{2} \gamma t^2 \nu \left( \|y \partial_x w\|^2 + 2 \|\nabla \partial_x \psi\|^2 \right) + \gamma \nu t^3 \left( -\nu \|y \partial_x \nabla w\|^2 - \nu \|\partial_x w\|^2 \right). \end{aligned}$$

After rearranging all the terms we obtain

$$(30) \quad \frac{d}{dt} \Phi(t) = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned} I_1 &= -\frac{\nu}{2} \|\nabla w\|^2 + \alpha \nu \|\nabla w\|^2 - \gamma \nu^2 t^3 \|\partial_x w\|^2, \\ I_2 &= -\alpha \nu^2 t \|\Delta w\|^2 - 4\beta \nu^2 t^2 \langle \Delta w, y \partial_{xy} w \rangle - \gamma \nu^2 t^3 \|y \partial_x \nabla w\|^2, \\ I_3 &= -\frac{\nu}{2} \|\nabla w\|^2 - 2\alpha \nu t \langle y \partial_x w, \partial_y w \rangle - \beta \nu t^2 \|y \partial_x w\|^2 - 2\beta \nu t^2 \|\partial_{xy} \psi\|^2, \\ I_4 &= -\frac{\nu \alpha}{2} \|\nabla w\|^2 + 4\beta \nu t \langle y \partial_x w, \partial_y w \rangle - \beta \nu t^2 \|y \partial_x w\|^2 - 2\beta \nu t^2 \|\partial_{xy} \psi\|^2, \\ I_5 &= -2\nu \beta t^2 \left[ \|y \partial_x w\|^2 + 2 \|\partial_{xy} \psi\|^2 \right] + \frac{3}{2} \gamma \nu t^2 \left[ \|y \partial_x w\|^2 + 2 \|\nabla \partial_x \psi\|^2 \right]. \end{aligned}$$

Recalling now the conditions (13) on the constants  $\alpha, \beta, \gamma$ , namely,

$$\alpha < \frac{1}{2}, \quad 16\beta^2 \leq \alpha\gamma, \quad 4\alpha^2 < \beta < \frac{1}{16}\alpha, \quad 9\gamma < 4\beta,$$

we use again the Cauchy–Schwarz inequality and the Young inequality to get

$$(31) \quad I_1 \leq -\gamma \nu^2 t^3 \|\partial_x w\|^2;$$

$$(32) \quad I_2 \leq -\frac{1}{2} \alpha \nu^2 t \|\Delta w\|^2 - \left( \gamma - \frac{8\beta^2}{\alpha} \right) \nu^2 t^3 \|y \nabla \partial_x w\|^2 < 0;$$

$$(33) \quad I_3 \leq -\frac{\nu}{4} \|\nabla w\|^2 - (\beta - 4\alpha^2) \nu t^2 \|y \partial_x w\|^2 - 2\beta \nu t^2 \|\partial_{xy} \psi\|^2 < 0;$$

$$(34) \quad I_4 \leq -\frac{\nu}{2} (\alpha - 16\beta) \|\nabla w\|^2 - \frac{\beta}{2} \nu t^2 \|y \partial_x w\|^2 - 2\beta \nu t^2 \|\partial_{xy} \psi\|^2 < 0.$$

Moreover, from

$$(35) \quad \langle \partial_{xy} \psi, y \partial_x w \rangle = -\frac{1}{2} \|\partial_{xy} \psi\|^2 + \frac{1}{2} \|\partial_{xx} \psi\|^2$$

we can deduce that

$$(36) \quad \|\nabla \partial_x \psi\|^2 \leq \|y \partial_x w\|^2 + 3 \|\partial_{xy} \psi\|^2,$$

and hence

$$(37) \quad I_5 \leq -\left( 2\beta - \frac{9}{2}\gamma \right) \nu t^2 \left[ \|y \partial_x w\|^2 + 2 \|\partial_{xy} \psi\|^2 \right] \leq 0.$$

Combining all together we get the upper bound on  $\Phi'(t)$ .  $\square$

**3. Additional linear estimates.** In this section we give some estimates on the semigroup generated by the linearized operator  $\mathcal{L}_\nu$ . These estimates play a crucial role in the estimates for the nonlinear term in the full perturbed system.

LEMMA 3.1. *Let  $g \in L^2$  such that  $\mathbb{P}_0(g) = 0$ . The following estimates hold for every  $T > 0$ :*

$$(38) \quad \int_0^T \|\nabla(e^{\mathcal{L}_\nu t} g)\|^2 \leq \frac{1}{2} \nu^{-1} \|g\|^2;$$

$$(39) \quad \int_0^T \|\partial_x(e^{\mathcal{L}_\nu t} g)\| \leq C \nu^{-2/3} \|g\|;$$

$$(40) \quad \int_0^T \|\nabla \Delta^{-1}(e^{\mathcal{L}_\nu t} g)\|_{L^\infty}^2 \leq C(|\log \nu| + 1) \nu^{-1/3} \|g\|^2.$$

*Proof.* Throughout the proof,  $T$  will be any positive time. The first estimate follows directly from the energy inequality for the linearized problem

$$(41) \quad \|e^{\mathcal{L}_\nu t} g\|^2 + 2\nu \int_0^t \|\nabla(e^{\mathcal{L}_\nu s} g)\|^2 \leq \|g\|^2$$

and it holds for any  $g \in L^2$ . For the second estimate we note that, for any  $k \neq 0$ ,

$$(42) \quad \begin{cases} k^2 \|(e^{\mathcal{L}_\nu t} g)_k\|^2 \leq k^2 e^{-2\nu k^2 t} \|g_k\|^2, \\ k^2 \|(e^{\mathcal{L}_\nu t} g)_k\|^2 \leq k^2 e^{-\gamma|k|^{\frac{1}{2}} \nu^{\frac{1}{2}} t} \|g_k\|^2. \end{cases}$$

Using the fact that the function  $x^n e^{-x}$  is bounded for  $x \geq 0$  for every  $n \in \mathbb{N}$ , we have

$$(43) \quad \begin{cases} k^2 \|(e^{\mathcal{L}_\nu t} g)_k\|^2 \leq \frac{C}{\nu t} \|g_k\|^2, \\ k^2 \|(e^{\mathcal{L}_\nu t} g)_k\|^2 \leq \frac{C}{\nu^2 t^4} \|g_k\|^2, \end{cases}$$

and hence

$$(44) \quad \|\partial_x(e^{\mathcal{L}_\nu t} g)\|^2 = \sum_{k \neq 0} |k|^2 \|(e^{\mathcal{L}_\nu t} g)_k\|^2 \leq C \min \left\{ \frac{1}{\nu t}, \frac{1}{\nu^2 t^4} \right\} \|g\|^2.$$

We have that

$$(45) \quad \min \left\{ \frac{1}{\sqrt{\nu t}}, \frac{1}{\nu t^2} \right\} = \begin{cases} (\nu t)^{-\frac{1}{2}} & \text{for } t \leq \nu^{-\frac{1}{3}}, \\ (\nu t^2)^{-1} & \text{for } t \geq \nu^{-\frac{1}{3}}, \end{cases}$$

and so

$$\begin{aligned} \int_0^T \|\partial_x(e^{\mathcal{L}_\nu t} g)\| &\leq C \left( \int_0^{\nu^{-\frac{1}{3}}} \frac{1}{\sqrt{\nu t}} dt + \int_{\nu^{-\frac{1}{3}}}^T \frac{1}{\nu t^2} dt \right) \|g\| \\ &\leq C \left( \nu^{-\frac{1}{2}} \nu^{-\frac{1}{6}} + \nu^{-1} \nu^{\frac{1}{3}} \right) \|g\| \\ &\leq C \nu^{-\frac{2}{3}} \|g\|. \end{aligned}$$

For the third estimate, we apply the Minkowski inequality and we reduce to

$$(46) \quad \int_0^T \|\nabla \Delta^{-1}(e^{\mathcal{L}_\nu t} g)\|_{L^\infty}^2 \leq \left( \sum_{k \neq 0} \left( \int_0^T \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}_\nu t} g)_k\|_{L^\infty}^2 dt \right)^{\frac{1}{2}} \right)^2.$$



Consider  $\|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\|_{L^\infty}^2$  for  $k \neq 0$ . Using the 1D Gagliardo–Nirenberg–Sobolev inequality we have

$$(47) \quad \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\|_{L^\infty}^2 \leq C \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\| \|(e^{\mathcal{L}\nu t} g)_k\|.$$

Moreover, interpolating the  $L^2$  norm we get

$$(48) \quad \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\|_{L^\infty}^2 \leq C \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\|^{\frac{3}{2}} \|\nabla_k(e^{\mathcal{L}\nu t} g)_k\|^{\frac{1}{2}}.$$

Combining (47) with (22) we deduce

$$(49) \quad \begin{cases} \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\|_{L^\infty}^2 \leq C(\nu k^2 t^3)^{-\frac{1}{2}} \|g_k\|^2, \\ \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\|_{L^\infty}^2 \leq |k|^{-1} \|g_k\|^2, \end{cases}$$

where the second inequality follows from  $|k| \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\| \leq \|(e^{\mathcal{L}\nu t} g)_k\| \leq \|g_k\|$ . Hence, we deduce that

$$(50) \quad \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\|_{L^\infty}^2 \leq C |k|^{-1} \min\{1, (\nu t^3)^{-\frac{1}{2}}\} \|g_k\|^2.$$

As before, using (50) we have

$$\begin{aligned} \int_0^T \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\|_{L^\infty}^2 &\leq C |k|^{-1} \left( \int_0^T \min\{1, (\nu t^3)^{-\frac{1}{2}}\} dt \right) \|g_k\|^2 \\ &\leq C |k|^{-1} \left( \nu^{-\frac{1}{3}} + \int_{\nu^{-\frac{1}{3}}}^T (\nu t^3)^{-\frac{1}{2}} dt \right) \|g_k\|^2 \\ &\leq C |k|^{-1} \left( \nu^{-\frac{1}{3}} + \nu^{-\frac{1}{2}} \nu^{\frac{1}{6}} \right) \|g_k\|^2 \\ &\leq C |k|^{-1} \nu^{-\frac{1}{3}} \|g_k\|^2. \end{aligned}$$

Using (48) and the Hölder inequality we obtain

$$\begin{aligned} \int_0^T \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\|_{L^\infty}^2 &\leq C \int_0^T \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\|^{\frac{3}{2}} \|\nabla_k(e^{\mathcal{L}\nu t} g)_k\|^{\frac{1}{2}} \\ &\leq C \left( \int_0^T \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t} g)_k\|^2 \right)^{\frac{3}{4}} \left( \int_0^T \|\nabla_k(e^{\mathcal{L}\nu t} g)_k\|^2 \right)^{\frac{1}{4}} \\ &\leq C \left( \int_0^T |k|^{-2} \min\{1, (\nu t^3)^{-1}\} \|g_k\|^2 dt \right)^{\frac{3}{4}} \nu^{-\frac{1}{4}} \|g_k\|^{\frac{1}{2}} \\ &\leq C |k|^{-\frac{3}{2}} \nu^{-\frac{1}{2}} \|g_k\|^2. \end{aligned}$$

So, from (46) and using the Cauchy–Schwarz inequality we deduce

$$\begin{aligned}
 \int_0^T \|\nabla \Delta^{-1}(e^{\mathcal{L}\nu t}g)\|_{L^\infty}^2 &\leq C \left( \sum_{k \neq 0} \left( \int_0^T \|\nabla_k \Delta_k^{-1}(e^{\mathcal{L}\nu t}g)_k\|_{L^\infty}^2 \right)^{\frac{1}{2}} \right)^2 \\
 &\leq C \left( \sum_{0 < |k| < \nu^{-\frac{1}{3}}} \left( k^{-1} \nu^{-\frac{1}{3}} \|g_k\|^2 \right)^{\frac{1}{2}} + \sum_{|k| > \nu^{-\frac{1}{3}}} \left( k^{-\frac{3}{2}} \nu^{-\frac{1}{2}} \|g_k\|^2 \right)^{\frac{1}{2}} \right)^2 \\
 &\leq C \left( \nu^{-\frac{1}{3}} \sum_{0 < |k| < \nu^{-\frac{1}{3}}} k^{-1} + \nu^{-\frac{1}{2}} \sum_{|k| > \nu^{-\frac{1}{3}}} k^{-\frac{3}{2}} \right) \sum_{k \neq 0} \|g_k\|^2 \\
 &\leq C \left( \nu^{-\frac{1}{3}} |\log \nu| + \nu^{-\frac{1}{2}} \nu^{\frac{1}{6}} \right) \|g\|^2 \\
 &\leq C \nu^{-\frac{1}{3}} (1 + |\log \nu|) \|g\|^2. \quad \square
 \end{aligned}$$

*Remark 3.2.* Regarding the third estimate, note that using only (47) leads to the series  $\sum_{k>0} \frac{1}{k}$ , which is clearly not convergent. The logarithmic correction term arises when we use (48) to overcome this problem.

**4. Nonlinear transition threshold.** We start by considering the nonlinear problem (3). We decompose  $\omega = \omega_s + \tilde{\omega}$ , as well as  $\psi = \psi_s + \tilde{\psi}$  and  $u = u_s + \tilde{u}$ , where

$$\omega_s = \mathbb{P}_0 \omega = \frac{1}{2\pi} \int_{\mathbb{T}} \omega dx$$

and

$$\tilde{\omega} = \mathbb{P}_{\neq} \omega = (1 - \mathbb{P}_0) \omega.$$

Note that the second component of  $u_s$  is zero since  $\mathbb{P}_0 u^2 = \mathbb{P}_0(\partial_x \psi) = 0$ . For this reason we will identify  $u_s$  with its first component and therefore it will be treated as a scalar. It follows that  $\omega_s$  and  $\tilde{\omega}$  satisfy

$$(51) \quad \begin{cases} \partial_t \omega_s - \nu \partial_y^2 \omega_s = -\mathbb{P}_0(\tilde{u} \cdot \nabla \tilde{\omega}), \\ \partial_t \tilde{\omega} + y^2 \partial_x \tilde{\omega} - 2 \partial_x \tilde{\psi} - \nu \Delta \tilde{\omega} = -\mathbb{P}_{\neq}(u_s \partial_x \tilde{\omega} + \tilde{u} \cdot \nabla \omega), \end{cases}$$

while  $u_s$  satisfies

$$(52) \quad \partial_t u_s + \mathbb{P}_0(\tilde{u} \cdot \nabla \tilde{u}^1) = \nu \partial_y^2 u_s.$$

We prove the following lemma, which gives some a priori estimates on the equations for  $\tilde{\omega}$  and  $u_s$ . These inequalities will be used in the bootstrap argument for the main theorem.

**LEMMA 4.1.** *There exist constants  $\bar{C}, K > 0$  such that for any  $T > 0$ , if  $t \in [0, T]$ , the following inequalities hold true:*

$$(53) \quad \|u_s(t)\|^2 \leq \|u_{s,in}\|^2 + 2\bar{C} \sup_{\tau \in [0, T]} \|u_s(\tau)\|^{1/2} \|\omega_{in}\|^{1/2} \int_0^t \|\tilde{\omega}(\tau)\|^2 d\tau.$$

Define

$$\varepsilon = \bar{C} \nu^{-2/3} \sup_{\tau \in [0, T]} \|u_s(\tau)\|^{1/2} \|\omega_{in}\|^{1/2} + \nu^{-2/3} (1 + |\log \nu|)^{1/2} \|\omega_{in}\|,$$

and then

$$(54) \quad \int_0^t \|u_s \partial_x \tilde{\omega}\| \, ds + \int_0^t \|\tilde{u} \cdot \nabla \omega\| \, ds \leq K\varepsilon \left( \|\tilde{\omega}_{in}\| + \int_0^t \|u_s \partial_x \tilde{\omega}\| \, ds + \int_0^t \|\tilde{u} \cdot \nabla \omega\| \, ds \right).$$

*Proof.* Consider first the equation for  $u_s$ , namely,

$$\partial_t u_s = \nu \partial_y^2 u_s - \mathbb{P}_0(\tilde{u} \cdot \nabla \tilde{u}^1).$$

Taking the time derivative of the  $L^2$  norm we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_s\|^2 &\leq \langle u_s, \mathbb{P}_0(\tilde{u} \cdot \nabla \tilde{u}^1) \rangle \\ &\leq \|u_s\|_{L^\infty} \|\tilde{u}\| \|\tilde{\omega}\| \\ &\leq \bar{C} \|u_s\|^{1/2} \|\omega_s\|^{1/2} \|\tilde{\omega}\|^2, \end{aligned}$$

where in the last inequality we used the 1D Gagliardo–Nirenberg inequality. Recalling that

$$(55) \quad \|\omega(t)\|^2 + 2\nu \int_0^t \|\nabla \omega(\tau)\|^2 \, d\tau = \|\omega_{in}\|^2$$

we have  $\|\omega_s(t)\| \leq \|\omega_{in}\|$ , and hence by integrating in time the above inequality we get (53).

For (54) we proceed as follows. Denoting by  $\mathcal{L}_\nu$  the linearized operator, we have

$$(56) \quad \partial_t \tilde{\omega} - \mathcal{L}_\nu \tilde{\omega} = -\mathbb{P}_\neq(u_s \partial_x \tilde{\omega} + \tilde{u} \cdot \nabla \omega),$$

and thus, applying the Duhamel formula, we obtain

$$(57) \quad \tilde{\omega}(t) = e^{t\mathcal{L}_\nu} \tilde{\omega}_{in} - \int_0^t e^{(t-s)\mathcal{L}_\nu} \mathbb{P}_\neq(u_s \partial_x \tilde{\omega} + \tilde{u} \cdot \nabla \omega) \, ds.$$

Consider the following equation:

$$(58) \quad \tilde{\omega}(t) = e^{t\mathcal{L}_\nu} \tilde{\omega}(0) + \int_0^t e^{(t-s)\mathcal{L}_\nu} f(s) \, ds.$$

Using Lemma 3.1 we obtain the following additional estimates. We consider first

$$(59) \quad \int_0^t \|\partial_x \tilde{\omega}\| \, dt \leq \int_0^t \|\partial_x e^{\tau\mathcal{L}_\nu} \tilde{\omega}(0)\| \, d\tau + \int_0^t \left\| \partial_x \int_0^\tau e^{(\tau-s)\mathcal{L}_\nu} f(s) \, ds \right\| \, d\tau.$$

The first term on the right hand side can be bounded using (39) as

$$(60) \quad \int_0^t \|\partial_x e^{\tau\mathcal{L}_\nu} \tilde{\omega}(0)\| \, d\tau \leq C\nu^{-2/3} \|\tilde{\omega}(0)\|,$$

while for the second term we use the Minkowski integral inequality and estimate (39) again as follows:

$$(61) \quad \begin{aligned} \int_0^t \left\| \int_0^\tau \partial_x e^{(\tau-s)\mathcal{L}_\nu} f(s) \, ds \right\| \, d\tau &\leq \int_0^t \int_s^t \left\| \partial_x e^{(\tau-s)\mathcal{L}_\nu} f(s) \right\| \, d\tau \, ds \\ &\leq C\nu^{-2/3} \int_0^t \|f(s)\| \, ds. \end{aligned}$$

Combining the two estimates we have

$$(62) \quad \nu^{2/3} \int_0^t \|\partial_x \tilde{\omega}\| \leq C \left( \|\tilde{\omega}(0)\| + \int_0^t \|f(s)\| \, ds \right).$$

Analogously, using estimate (40) we have

$$(63) \quad \left( \int_0^t \|\nabla \Delta^{-1} \tilde{\omega}(\tau)\|_{L^\infty}^2 \, d\tau \right)^{1/2} \leq \left( \int_0^t \|\nabla \Delta^{-1} e^{\tau \mathcal{L}_\nu} \tilde{\omega}(0)\|_{L^\infty}^2 \, d\tau \right)^{1/2} + \left( \int_0^t \left\| \nabla \Delta^{-1} \int_0^\tau e^{(\tau-s)\mathcal{L}_\nu} f(s) \, ds \right\|_{L^\infty}^2 \, d\tau \right)^{1/2} \leq C(|\log \nu| + 1)^{1/2} \nu^{-1/6} \left( \|\tilde{\omega}(0)\| + \int_0^t \|f(s)\| \, ds \right)$$

and we deduce

$$(64) \quad \nu^{2/3} \int_0^t \|\partial_x \tilde{\omega}\| \, dt + (|\log \nu| + 1)^{-1/2} \nu^{1/6} \left( \int_0^t \|\nabla \Delta^{-1} \tilde{\omega}\|_{L^\infty}^2 \, dt \right)^{1/2} \leq C \|\tilde{\omega}(0)\| + C \int_0^t \|f(s)\| \, ds.$$

By renaming  $K$  the constant  $C$  appearing after these computations, and defining

$$A(t) = \nu^{2/3} \int_0^t \|\partial_x \tilde{\omega}\| + \nu^{1/6} (1 + |\log \nu|)^{-1/2} \left( \int_0^t \|\tilde{u}\|_{L^\infty}^2 \right)^{1/2},$$

from (56) we have that

$$(65) \quad A(t) \leq K \left( \|\tilde{\omega}_{in}\| + \int_0^t \|u_s \partial_x \tilde{\omega}\| \, ds + \int_0^t \|\tilde{u} \cdot \nabla \omega\| \, ds \right).$$

For the second term on the right hand side in (65), we have by using the definition of  $A(t)$  and the hypothesis

$$(66) \quad \begin{aligned} \int_0^t \|u_s \partial_x \tilde{\omega}\| \, ds &\leq \int_0^t \|u_s\|_{L^\infty} \|\partial_x \tilde{\omega}\| \, ds \\ &\leq \bar{C} \sup_{\tau \in [0, T]} \left( \|u_s(\tau)\|^{1/2} \|\omega_s(\tau)\|^{1/2} \right) \int_0^t \|\partial_x \tilde{\omega}\| \, ds \\ &\leq \bar{C} \sup_{\tau \in [0, t]} \left( \|u_s(\tau)\|^{1/2} \right) \|\omega_{in}\|^{1/2} \nu^{-\frac{2}{3}} A(t). \end{aligned}$$

Analogously, for the third term on the right hand side in (69), we have

$$(67) \quad \begin{aligned} \int_0^t \|\tilde{u} \cdot \nabla \omega\| &\leq \int_0^t \|\tilde{u}\|_{L^\infty} \|\nabla \omega\| \\ &\leq \left( \int_0^t \|\tilde{u}\|_{L^\infty}^2 \right)^{\frac{1}{2}} \left( \int_0^t \|\nabla \omega\|^2 \right)^{\frac{1}{2}} \\ &\leq \left( \nu^{-\frac{1}{6}} (|\log \nu| + 1)^{\frac{1}{2}} A(t) \right) \left( \nu^{-\frac{1}{2}} \|\omega_{in}\| \right) \\ &\leq \nu^{-\frac{2}{3}} (|\log \nu| + 1)^{\frac{1}{2}} \|\omega_{in}\| A(t). \end{aligned}$$

Recalling that

$$\varepsilon = \bar{C}\nu^{-2/3} \sup_{\tau \in [0, T]} \|u_s(\tau)\|^{1/2} \|\omega_{in}\|^{1/2} + \nu^{-2/3}(1 + |\log \nu|)^{1/2} \|\omega_{in}\|,$$

from (65), (66), and (67), we can deduce (54).  $\square$

**THEOREM 4.2.** *There exist constants  $\varepsilon_0 \in (0, 1)$ ,  $C_1 > 0$ ,  $c_1 > 0$  such that for all  $\nu < 1$  and for every  $u_{in}, \omega_{in} \in L^2$  with*

$$\|\omega_{in}\| + \|\mathbb{P}_0 u_{in}\| \leq \varepsilon_0(1 + |\log \nu|)^{-1/2} \nu^{2/3},$$

*the solution of (3) is global in time with the bound*

$$\|\tilde{\omega}(t)\| \leq C_1 e^{-c_1 \nu^{1/2} t} \|\tilde{\omega}_{in}\|.$$

The proof follows a bootstrap argument.

*Bootstrap step.* Let  $t_0$  such that  $C_0 e^{-c_0 \nu^{1/2} t_0} = \frac{1}{6}$ , and let  $T \leq t_0$ . Assume that

$$\sup_{t \in [0, T]} e^{c_0 \nu^{1/2} t} \|\tilde{\omega}(t)\| \leq 4C_0 \|\tilde{\omega}_{in}\|$$

and

$$\sup_{t \in [0, T]} \|u_s(t)\| \leq 4\varepsilon_0 \nu^{2/3} (1 + |\log \nu|)^{-1/2}.$$

Then, this implies

$$\sup_{t \in [0, T]} e^{c_0 \nu^{1/2} t} \|\tilde{\omega}(t)\| \leq 2C_0 \|\tilde{\omega}_{in}\|$$

and

$$\sup_{t \in [0, T]} \|u_s(t)\| \leq 2\varepsilon_0 \nu^{2/3} (1 + |\log \nu|)^{-1/2}.$$

*Proof of theorem.* Assuming the bootstrap step, the theorem follows. Indeed, for  $t = t_0$  we have

$$\|\tilde{\omega}(t_0)\| \leq 2C_0 e^{-c_0 \nu^{1/2} t_0} \|\tilde{\omega}_{in}\| = \frac{1}{3} \|\tilde{\omega}_{in}\|.$$

We remark that estimates in Lemma 4.1 hold whenever the bootstrap assumptions hold true, and hence we can iterate this process, as is done for the linear case, with time intervals of length  $t_0 \sim \nu^{-1/2}$  to obtain a global solution that satisfies the decay rate

$$\|\tilde{\omega}(t)\| \leq C_1 e^{-c_1 \nu^{1/2} t} \|\tilde{\omega}_{in}\|. \quad \square$$

*Proof of the bootstrap step.* From Lemma 4.1 and the smallness assumptions on the initial data we deduce easily that the bootstrap step holds. Indeed, from (53) we have

$$\begin{aligned} \|u_s(t)\|^2 &\leq \varepsilon_0^2 (1 + |\log \nu|)^{-1} \nu^{4/3} + 2\bar{C}4\varepsilon_0 \nu^{2/3} (1 + |\log \nu|)^{-1/2} 4C_0 \|\tilde{\omega}_{in}\|^2 \int_0^t e^{-c_0 \nu^{1/2} \tau} d\tau \\ &\leq \varepsilon_0^2 (1 + |\log \nu|)^{-1} \nu^{4/3} + 2\bar{C}4\varepsilon_0 \nu^{2/3} (1 + |\log \nu|)^{-1/2} 4C_0 \|\omega_{in}\|^2 \int_0^{+\infty} e^{-c_0 \nu^{1/2} \tau} d\tau \\ &\leq \varepsilon_0^2 (1 + |\log \nu|)^{-1} \nu^{4/3} \left(1 + C\varepsilon_0 \nu^{2/3} (1 + |\log \nu|)^{-1/2} \nu^{-1/2}\right) \\ &\leq 4\varepsilon_0^2 (1 + |\log \nu|)^{-1} \nu^{4/3}, \end{aligned}$$

and hence

$$\sup_{t \in [0, T]} \|u_s(t)\| \leq 2\varepsilon_0 \nu^{2/3} (1 + |\log \nu|)^{-1/2}.$$

For  $\tilde{\omega}$  consider

$$(68) \quad \tilde{\omega}(t) = e^{t\mathcal{L}_\nu} \tilde{\omega}_{in} - \int_0^t e^{(t-s)\mathcal{L}_\nu} \mathbb{P}_\neq (u_s \partial_x \tilde{\omega} + \tilde{u} \cdot \nabla \omega) ds.$$

Taking the  $L^2$  norm of (68), using the semigroup bounds and (54), we obtain, for all  $t \in [0, T]$ ,

$$(69) \quad \begin{aligned} e^{c_0 \nu^{1/2} t} \|\tilde{\omega}(t)\| &\leq C_0 \|\tilde{\omega}_{in}\| + C_0 e^{c_0 \nu^{1/2} T} \int_0^t \|u_s \partial_x \tilde{\omega} + \tilde{u} \cdot \nabla \omega\| ds \\ &\leq C_0 \|\tilde{\omega}_{in}\| + C_0 e^{c_0 \nu^{1/2} t_0} \frac{\varepsilon K}{1 - \varepsilon K} \|\tilde{\omega}_{in}\| \\ &\leq C_0 \|\tilde{\omega}_{in}\| + 6C_0^2 \frac{\varepsilon K}{1 - \varepsilon K} \|\tilde{\omega}_{in}\|. \end{aligned}$$

By the bootstrap assumption and hypothesis, for  $\varepsilon_0$  small we have

$$(70) \quad \begin{aligned} \varepsilon &= \bar{C} \nu^{-2/3} \sup_{\tau \in [0, T]} \|u_s(\tau)\|^{1/2} \|\omega_{in}\|^{1/2} + \nu^{-2/3} (1 + |\log \nu|)^{1/2} \|\omega_{in}\| \\ &\leq \bar{C} 4\varepsilon_0 (1 + |\log \nu|)^{-1/2} + \varepsilon_0 \\ &\leq \frac{1}{12KC_0} \leq \frac{1}{2K}. \end{aligned}$$

Thus

$$6C_0^2 \frac{\varepsilon K}{1 - \varepsilon K} \leq 12C_0^2 \varepsilon K \leq C_0$$

and the bootstrap step holds true.  $\square$

**Acknowledgment.** The author wants to thank Michele Coti Zelati for his support and all the useful and stimulating discussions about this problem.

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