Stochastic optimal control for problems arising in data science

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Introduction

The goal of this thesis is to summarise the work we have done during the course of the PhD on the *Follow the Leader* (FTL) problem. The problem firstly arose in [35] in the context of search of a space probe by radar: since only a single antenna was available, the searcher could only look at one region of the sky at a time. The celestial sphere was idealized as being discrete, and the antenna looked at exactly one of n regions at a time. It is assumed that only the position is uncertain, and the time to move the antenna from one region to the next is ignored.

An a priori distribution on the n regions is given by orbit predictions. The data processes are the integrals of white Gaussian receiver noise, of known spectral density $\frac{\sigma^2}{2}$, of mean 0 for incorrect regions, and of known mean $\mu > 0$ corresponding to the radar signal for the correct region. The problem was solved by applying a certain two-stage searching strategy, which was deemed as suboptimal by the author. The author then introduced the FTL problem in [36]: consider N, $N \in \mathbf{N}$, independent standard Brownian motions $W_1, \dots, W_N$ over a probability space $(\Omega, \mathcal{F}, P)$ and define N independent stochastic processes $X_1, \dots, X_N$ such that there exist $i^* \in \{1, \dots, N\}$ for which

$$dX_{i^*}(t) = dW_{i^*}(t) + \mu dt,$$

$\mu \in \mathbf{R}$, $\mu \neq 0$, $t \geq 0$, that is, there exist a unique $i^* \in \{1, \dots, N\}$ such that $(X_{i^*}(t))_{t \geq 0}$ is a Brownian motion with drift μ, whereas the other $N-1$ processes are standard Brownian motions. The goal of the authors is to identify which Brownian motion has drift $\mu \neq 0$ in the minimum average searching time and by controlling the probability of a wrong terminal decision. This class of problems has been studied extensively in decision theory (see for example [28]), but in [36] the authors introduce a major constraint which makes the problem more interesting: at any time $t \geq 0$ only one process X_i, $i \in \{1, \dots, N\}$, can be observed. This assumption comes naturally from the setting in [35]. The authors consider a *prior distribution* $\pi = (\pi_1, \dots, \pi_N)$ and a posterior distribution $\Pi(t) = (\Pi_1(t), \dots, \Pi_N(t))$, such that $\Pi_i(t) = P(i = i^* | \mathcal{F}_t)$ for all $i \in \{1, \dots, N\}$, $t \geq 0$, $(\mathcal{F}_t)_{t \geq 0}$ the observation filtration, that is, the filtration containing the information collected by observing the processes until time t. In particular we have $\Pi(0) = \pi$, the prior distribution. Define a tolerance error $\varepsilon \in (0, \frac{1}{2})$ and set an arbitrary

threshold $1-\varepsilon$ and a $\mathcal{F}$- stopping time

$$\tau = \inf\{t \geq 0 | \exists i \in \{1, \ldots, N\} \text{ s.t } \Pi_i(t) = 1-\varepsilon\}.$$

The authors reformulate the FTL problem as finding the policy that minimizes the average searching time $\mathbf{E}_\pi(\tau)$, given that the probability of wrong terminal error does not exceed $1-\varepsilon$. In [36] the authors claim that the optimal strategy is the so-called *Follow the Leader* strategy, which also gives the name to the problem: at each time $t \geq 0$ always observe the process $X_i(t)$ associated with the largest posterior probability $\Pi_i(t)$, which we will call the *leader* from now on. To do so, the authors restrict their focus to a specific class of strategies called δ-*perturbed strategies*. A δ-perturbed strategy keeps observing one process X_i until its associated posterior probability Π_i drops below the starting probability π_i by $\frac{\delta}{N}$. In that case it starts to observe the leader. The authors show the optimality of the FTL strategy within this class of strategies and claim that such a result can be extended to the class of all strategies that observe only one process at each time $t \geq 0$.

The author of [42] approached a very similar problem in the same year: consider N independent Brownian motions defined on a common probability space $(\Omega, \mathcal{F}, P)$. One process X_{i^*} has a non-zero drift μ, whereas the others $N-1$ are standard Brownian motions. Considering just the strategies that observe one process at each time $t \geq 0$, compute the optimal searching strategy, that is, the strategy that minimises the average searching time to find the process with the drift. The only two major differences between [42] and [36] are that in [42] the author assumes that the stochastic processes all start from the same position, that is, if we denote the processes as $X = (X_1, \ldots, X_N)$, we have $X(0) = (x, \ldots, x)$. Using the notation introduced in [36], this corresponds to setting the prior distribution π as uniform. Therefore, we can regard the results in [42] as a special case of [36]. It is unclear if the author was aware of the results contained in [36] and since [42] was initially published in Russian, it is possible that the western scientific community was not aware of the results obtained in [42]. The author computes the likelihood process T of X. If we are observing the i-th process X_i in the time interval $(t, t+s)$, $t, s \geq 0$, we have

$$\begin{aligned} T_i(t+s) &= T_i(t) \exp\left\{X_i(t+s) - X_i(t) - \frac{s}{2}\right\}, \\ T_j(t+s) &= T_j(t), \end{aligned}$$

for $j \in \{1, \ldots, N\}$, $j \neq i$. The fact that the prior distribution is uniform implies that $T_k(0) = \frac{1}{N}$ for all $k \in \{1, \ldots, N\}$. The author defines the risk function to be minimized as

$$r(T_1, \ldots, T_N | d) = \sum_{k=1}^{N} \Pi_k \left(\alpha_k(T_1, \ldots, T_N | d) + cN_k(T_1, \ldots, T_N | d)\right),$$

where Π_k is the posterior probability process of the k-th process, d is a chosen strategy, N_k is the mean scanning time under the strategy d and α_k is the probability of acceptance of the hypothesis "the k-th process has a non-zero drift", as in this version of the problem the probability of acceptance is not fixed a priori. The author then consider the *Hamilton-Jacobi-Bellman* (HJB) equation ([33] is a good introduction to the topic) of the problem as

$$\inf_i \left\{ \frac{1}{2} \frac{\partial^2 r}{\partial T_i^2} T_i^2 + \frac{T_i^2}{\sum_k T_k} \frac{\partial r}{\partial T_i} \right\} = -c.$$

As far as we know, this is the first time in which the FTL problem has been studied as *dynamic programming problem* ([33] and [39] are great introductions to the topic). The problem with the results in [42] comes when the author tries to prove that the FTL strategy, that is, the strategy that consists in observing the process $X_i(t)$ with likelihood $T_i(t) = \max_k T_k(t)$ for all $t \geq 0$, is the optimal policy. As pointed out in [17], the proof lacks clarity in several passages and neither we nor the authors of [17] were able to understand it.

The FTL problem was considered again in 1971 in [25], where the authors pointed out several flaws in the reasoning of [36]: in particular, the authors show that the class of δ- perturbed strategy is not tight. Therefore, the notion of weak limit used in [36] to prove the optimality of the FTL strategy is not appropriate. The same authors considered the problem once more in [24]: the approach of the authors is to consider a new set of strategies, denoted by $D_{\delta\gamma}$. A strategy $I_{\delta\gamma} \in D_{\delta\gamma}$ is such that

$$\begin{aligned}
&\alpha_m \geq \delta, \quad \beta_m \geq \gamma, \\
&T_{m+1} = T(\Pi_1(T_m), \ldots, \Pi_N(T_m), \alpha_m, \beta_m) + T_m, \\
&I_{\delta\gamma}(t) = i_m, \quad T_m \leq t \leq T_{m+1}, \\
&\Pi_i(T_m) + \alpha_{m+1} \leq 1 - \varepsilon, \\
&\Pi_i(T_m) - \beta_{m+1} > 0,
\end{aligned}$$

where $m \in \mathbb{N}$, $T_0 = 0$, $\varepsilon, \alpha_m, \beta_m$ are arbitrarily chosen, $i_m \in \mathbb{N}$. Here the function $T(\Pi_1, \ldots, \Pi_N, \alpha, \beta, i_m)$ is the first time at which Π_i crosses either $\pi_i + \alpha$ or $\pi_i - \beta$. Note that the controls in $D_{\delta\gamma}$ correspond to the strategies which search the same target i as long as the posterior probability $\Pi_i(t)$ does not change by more than a predetermined quantity. The authors prove that the FTL policy is optimal in $D_{\delta\gamma}$. This proof, although limited to $\delta\gamma$ strategies only, seemed to satisfy the mathematical community for many years and the interest focused on variations of the FTL problem: in Section 4 of [13] the author shows that a cyclic application of the sequential probability ratio test (SPRT) (see [34] for an introduction) has the same expected searching time of the FTL strategy. In Section 5 the author shows that such a policy can be also applied for *change point detection problems*,

that is, for problems where the drift μ appears at a random time τ, rather than being present in one of the processes $X_1, \ldots, X_N$ from the beginning.

Another variation of the FTL problem was considered in [27], where the authors studied the case where the number of processes is infinite and it is not known, a priori, how many of them have a non-zero drift $\mu \neq 0$. Moreover, the authors consider processes in discrete time, that is, time-series data. All these assumptions make it possible for the authors to obtain an elegant and relatively simple solution: they prove that the cumulative sum (CUSUM) test (see [7] for an introduction) is optimal in this context. In [8] the authors address the same problem in continuous time: it is important to point out that, although similar to the FTL problem, the version with infinitely many processes allows the author to make use of methods not available in the finite case. Indeed, the results in [27] and [8] rely on the fact that, once we switch from process X_i to process X_{i+1}, $i \in \mathbf{N}$, we will never observe X_i again, as there are still infinitely many processes to be observed. Clearly, such a reasoning cannot be applied in the formulation proposed in [36]. However, as pointed out in [27] and [8], this solution is still good enough, for practical purposes, when the number of processes is very large, as it is the case in many applications. Another interesting variation of the FTL problem is considered in [31], where the authors propose the following problem: suppose the adversary picks at random a real number $y_t \in [0, 1]$, for $t = 1, \ldots, T$, and keeps it secret. We guess that number by picking another number $x_t \in [0, 1]$. We then pay the squared difference $(x_t - y_t)^2$. The author then generalizes the problem to different loss functions and shows, using machine learning techniques, that the so called *Follow the Regularized Leader* (FTRL) algorithm minimizes the loss function of the problem. The FTRL algorithm consists in choosing at each time t the minimizer of the sum of the past losses plus a time-varying regularization.

A similar problem is considered in [23], where the authors proposed the *Follow the Perturbed Leader* approach, which consists in slightly perturbing the loss function at each time t and then choose the strategy that minimizes the expected loss. The authors show that this strategy is nearly as good as the optimal strategy and more efficient from a computational point of view.

As far as the classical FTL problem is concerned the breakthrough came in 2018 with the publication of [17]. The authors propose a counterexample to the optimality of the FTL policy. The setting is the same as in [36], [25] and [24]. The authors introduce an alternative strategy, simply named *Strategy B*: let us assume that, at $t = 0$, the processes are sorted decreasingly, so that $x_1 \geq x_2 \geq \cdots \geq x_N$. In [17] the authors show that the posterior probability associated to X_i, $i \in \mathbf{N}$, can be written as

$$\Pi_i(t) = \frac{e^{\mu X_i(t)}}{\sum_{j=1}^{N} e^{\mu X_j(t)}}.$$

Strategy B observes X_2 until it reaches either x_1 or a specific level a such that

$$\Pi_1(t) = \frac{e^{\mu X_1(t)}}{\sum_{j\neq 2} e^{\mu X_j(t)} + e^{\mu a}} = 1 - \varepsilon,$$

where a is the unique value of X_2 for which $\Pi_1(t) = 1 - \varepsilon$. If X_2 reaches a first, the search is over, if X_2 reaches x_1 first, we continue with the FTL strategy. The authors show that in several cases Strategy B outperforms the FTL strategy, achieving a smaller expected searching time than the FTL policy. The complete table of counterexamples can be found in Section 5 of [17]. The authors also point out issues in the proof of the optimality of the FTL strategy given in [24]: to prove Lemma 3.4 and Theorem 3.5 the authors assume that the posterior distribution of the stopping time τ, that is, the first time at which $\Pi_1(t)$ reaches either $\pi_1 + \alpha$ or $\pi_1 - \beta$, is independent of the search rule, which is clearly incorrect. Refer to Section 6.1 of [17] for a complete explanation.

Such a result leaves several open questions: if the FTL policy is not optimal, which is the optimal strategy? What about the case where the prior distribution is uniform, that is, the case considered in [42]? The counterexamples given in [17] are insufficient to disprove the claim that the FTL strategy is optimal for a uniform prior distribution, so the question remains open. A slight variation of the problem has been studied recently in [16], where the authors considered the associated optimal stopping problem (see [32] for a comprehensive overview of optimal stopping): they consider a 3-dimensional stochastic process for which 2 coordinates are standard Brownian motions and the remaining coordinate is a Brownian motion with drift $\mu \neq 0$, but we do not know which coordinate has the drift. The goal is to find such a coordinate as soon as possible and with minimal probabilities of the wrong terminal decisions. The central difference between the FTL problem and the work in [16] is that in the latter the observer has no control over which coordinate to observe next. In fact, this approach can also be interpreted as the case where we can observe all processes at the same time. Eventually, we note that deep learning techniques have been applied to stochastic optimal control problems recently: in particular in [21] the authors used a simple neural network to approximate the value function of a market making problem, obtaining good results. It is possible that such methods can be applied to the FTL problem successfully.

The goal of this thesis is to study the FTL problem and to understand which is the optimal policy. The work will be structured as follows:

1. in Chapter 1, we will study the setting of the FTL problem and derive a rigorous formulation of the problem as a dynamic programming problem. We will show that the *value function* of the problem can be characterized as the viscosity solution (see [33] and [39] for a solid introduction) of a certain HJB equation and proceed to study it;

2. in Chapter 2, we will focus on the case studied by [42], that is, the case with uniform prior distribution. We will discuss the results of the paper, the issues with such results and show the results we achieved in that particular setting;

3. in Chapter 3 we will discuss the case with general prior distribution and show numerical results that support the results obtained in [17].

Chapter 1

The Follow The Leader problem

In this chapter we will first recall the classical setting of the FTL problem as introduced in [36]. We will provide a modern formulation to the problem: to do so, we will take advantage of the notation introduced in [17]. Consider a complete filtered probability space $\Omega = (E, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, where $E = \{1, \dots, N\} \times \mathcal{C}([0,\infty)^N)$ and $\mathbb{P}$ is a probability measure on Ω such that

$$\mathbf{P}\left(i = j^*, (X_1, \dots, X_N) \in \prod_{k=1}^{N} A_k\right) = \prod_{k=1, k\neq i}^{N} \mathbb{W}_0(A_k)\mathbb{W}_\mu(A_i)\pi_i,$$

where $\pi_i \in [0,1]$, $A_i \in \mathcal{B}(\mathcal{C}([0,\infty)))$, for $i \in \{1, \dots, N\}$ and

$$\mathbb{W}_\mu(A) = \tilde{\mathbf{P}}\left((\tilde{W}_t + \mu t)_{t\geq 0} \in A\right),$$

$\mu \in \mathbb{R}$, $\tilde{W}$ is $\tilde{\mathbf{P}}$- Brownian motion and $A \in \mathcal{B}(\mathcal{C}([0,\infty)^N))$. Hence, under $\mathbf{P}$ we choose an index j^* with probability π_{j^*} and $X_1, \dots, X_N$ are iid Brownian motions conditional on j^*, with X_{j^*} a Brownian motion with drift μ and the other X_i, $i \neq j^*$, are Brownian motions with drift 0. Then, we choose $\mathcal{F}_t$ to be the natural filtration generated by $j^*, W_1, \dots, W_N$ augmented in the usual way. Consider now an $\mathcal{F}_t$-progressively measurable process J_t taking values in $\{1, \dots, N\}$. Define the process Y_t by

$$dY_t = dX_{J_t}(t) \tag{1.1}$$
$$Y_0 = 0, \tag{1.2}$$

and the natural filtration $\mathcal{F}^Y$ generated by Y, augmented in the usual manner. Hence J is an admissible control if J is $\mathcal{F}_t^Y$-progressively measurable. Note that

such a control is well-defined: consider controls that are constant up to $\mathcal{F}^Y$-stopping times τ_i, and at τ_i choose a new $\mathcal{F}_{\tau_i}$-measurable value for J_t, $t \in [\tau_i, \tau_{i+1})$. If $\tau_i \to \infty$ as $i \to \infty$ $\mathbb{P}$-a.s., the control is admissible. We will always consider the version of $(W_1(t), \dots, W_N(t))_{t\geq 0}$ with continuous paths. Let us construct N processes $(X_1(t), \dots, X_N(t))_{t\geq 0}$ such that:

$$dX_i(t) = dW_i(t) + \mu \mathbb{1}\{i = j^*\}dt,$$

for $t \geq 0$, $\mu > 0$ and where $\mathbb{1}\{A\}$ is the indicator function of the set $A \in \mathcal{B}(\Omega)$, where $\mathcal{B}(\Omega)$ are the Borel sets of Ω.
The starting points $X(0) = (X_1(0), \dots, X_N(0)) = (x_1, \dots, x_N)$ are known real numbers, $X(0) \in \mathbb{R}^N$. Our goal is to identify j^* in the shortest possible time and by minimizing the probability of wrong terminal decisions. To do so we have one key restriction: at each time t we can observe only one particle, that is, only one $X_i(t)$, $i \in I$. The key feature of the problem is that if we observe X_i in the time period $(t, t+s)$, $t, s \geq 0$, only the value of X_i changes during $(t, t+s)$, whereas all the other processes X_j, $j \in I \setminus \{i\}$, remain at the initial level, that is, $X_j(t+s) = X_j(t)$. Consider the function $J : \mathbb{R}^+ \to I$ and denote by $J(t)$ the particle under observation at time t. The *observation process* $Y = (Y(t))_{t\geq 0}$ is the unique strong solution to the stochastic differential equation (SDE) (refer to [30] for a comprehensive treatment of the topic)

$$dY(t) = dX_{J(t)}(t), \tag{1.3}$$

for $t \geq 0$. Note that the process $J(t)$ is such that:

1. $J(t) \in I$ for all $t \geq 0$;

2. $(J(t))_{t\geq 0}$ is progressively measurable, that is, for all $t \geq 0$ the map $[0, t] \times \Omega \to \mathbf{R}$ defined by $(s, \omega) \mapsto J_s(\omega)$ is $\mathcal{B}([0, t]) \otimes \mathcal{Y}_t$-measurable. In particular, this implies that $(J(t))_{t\geq 0}$ is $(\mathcal{Y}_t)_{t\geq 0}$-adapted (see [1] for a proof of this), where $(\mathcal{Y}_t)_{t\geq 0}$ is the natural filtration of the observation process $Y = (Y(t))_{t\geq 0}$, that is

$$\mathcal{Y}_t = \sigma\left(\sigma(Y_s, 0 \leq s \leq t) \cup \mathcal{N}\right),$$

where $\mathcal{N}$ is the the collection of all null sets of the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Therefore, $(\mathcal{Y}_t)_{t\geq 0}$ is the augmented natural filtration of the process Y.

Remark 1.0.1. We consider $(\mathcal{Y}_t)_{t\geq 0}$ to be the augmented natural filtration as this assumption is needed to show that the posterior probability process Π (which we are going to define shortly) is progressively measurable (see [1] for a detailed explanation).

Note that 1 and 2 together imply that $(W_{J(t)})_{t\geq 0}$ is a well-defined Brownian motion.

Remark 1.0.2. The process $(J(t))_{t\geq 0}$ represents the control the observer has over the system. The fact that $(J(t))_{t\geq 0}$ is $\mathcal{Y}_t$-measurable implies that every decision we make is based only on the data we have observed up to time t. This makes the FTL problem a continuous- time *sequential* stochastic optimal control problem.

Remark 1.0.3. In the discussion above, we formulate the control problem in terms of an adapted control over a probability space. However because the process Y which generates the observed filtration is itself controlled, it is non-trivial to setup the control problem in a fully rigorous manner, for example, to determine the Dynamic Programming Principle (DPP) for the control problem. One approach would be to take a formal delayed approach, for example, to consider controls which are constant on deterministic time intervals, and which only switch value according to the information available at these time points. However even in this setup, it would seem difficult to formulate and prove a satisfactory DPP. We note also that while there is a significant literature on controlled filtering problems, of which this is closely related, many results in this setting are not immediately applicable, since this literature (e.g. [18], [20], [3], [2], [10], [15], [14]) tends to focus on problems where there is a controlled process X, which is not fully observed, and the control of X must be adapted to a filtration Y, where Y is typically given in terms of dynamics such as $dY_t = h(X_t)\,dt + dW_t$, where W is a Brownian noise term which may be independent of, or correlated to the noise generating the dynamics of the unseen process X. A notable exception is [2], where the function h may also depend on the control (although not in a framework that would include the problem we study).

Another approach to rigorously formulating a DPP for this problem would be via a weak control approach. In this case, rather than working on a fixed filtered probability space, and optimising over adapted controls (which is complicated by the fact that the control will determine the filtration that the process can then be adapted to), we would choose to optimise over the space of probability measures on the appropriate path space, where the construction of the control and the adapted process are produced simultaneously, that is, one optimises over classes of probability measures which support a controlled process Y, a control process α, which is adapted to the filtration generated by Y, and such that the dynamics of Y satisfy the required constraints. In this context, the difficulty is proving a sort of concatenation property, as required for the DPP. However powerful tools are available, and the paper [43] provides a good framework for establishing these types of properties.

In this thesis, we do not pursue these approaches. Instead, our approach will be similar to the 'separation' problem described in e.g. [2]: that is, we consider the process enhanced by the current posterior belief about the likelihoods of different outcomes. In this setting, it is common, e.g. as in [2], to consider the

unnormalized likelihoods, although through normalisation, we expect to get a similar characterisation, but with one fewer degree of freedom.

Remark 1.0.4. The process Y defined in (1.3) is a well-defined *controlled diffusion process*.

Define now the *prior* distribution $(\pi_1, \dots, \pi_N)$ as the probability that each particle is the correct one (that is the one with drift μ) a priori, so that

$$\pi_i = \mathbf{P}\left(i = j^*\right), \tag{1.4}$$

for $i \in I$.
We now want to compute the *likelihood* process $(L(t))_{t\geq 0} = (L_1(t), \dots, L_N(t))_{t\geq 0}$. The likelihood process L can be seen as the unnormalized version of the posterior probability process Π. Fix $J(t) = i$, for $i \in I$: a straightforward application of Girsanov's formula (see [30]) shows that the posterior likelihood that the i-th particle has drift μ, given $\mathcal{Y}(t)$, with respect to the Wiener measure, is

$$L_i(t) = \pi_i \exp\left(\mu \int_0^t \mathbf{1}\{J(s) = i\} dY(s) - \frac{1}{2}\mu^2 \int_0^t \mathbf{1}\{J(s) = i\} ds\right). \tag{1.5}$$

Note that the first integral on the right hand side of (1.5) is well-defined, as J is progressively measurable and Y is continuous (see Chapter 3 of [1]). We can now compute the *posterior probability* of the i-th particle having drift μ, given everything we observed till time t, that is, given $\mathcal{Y}(t)$, as

$$\Pi_i(t) = \frac{L_i(t)}{\sum_{k=1}^N L_k(t)}. \tag{1.6}$$

The existence of the process Π_i such that $\Pi_i(t) = \mathbf{P}\left(\hat{J} = i \middle| \mathcal{Y}(t)\right)$, for $t \geq 0$ and $i \in I$ is non-trivial. Let us recall Theorem 2.1 of [1], as this is particularly important in the construction of the posterior probability process Π.

Theorem 1.0.5 ([1])**.** *Let S be a complete separable metric space and $\mathcal{S}$ be the associated Borel σ-algebra. Then there exists a $\mathbf{P}(S)$-valued $\mathcal{Y}(t)$-adapted process $\Pi = \{\Pi(t), t \geq 0\}$ such that for any $f \in B(S)$*

$$\Pi_i(t)(f) = \mathbf{E}[f(X_i(t))|\mathcal{Y}(t)] \quad \mathbf{P} - a.s.$$

Moreover, if Y satisfies the evolution equation

$$Y_t = Y_0 + \int_0^t h(X_s) ds + W_t, \tag{1.7}$$

where W is a standard Brownian motion and $h : S \to \mathbf{R}$ is such that

$$\mathbf{E}\left[\int_0^t ||h(X_s)||^2 ds\right] < \infty,$$

for all $t \geq 0$, then Π has a $\mathcal{Y}(t)$-adapted progressively measurable modification. Furthermore, if X is càdlàg then Π can be chosen to have càdlàg paths.

Remark 1.0.6. The vast majority of Theorem 1.0.5 can be readily applied to our case. The only part that requires a more careful study is whether the observation process Y satisfies (1.7). In [1], and more in general in the *stochastic filtering* literature, the observation process is always assumed to satisfy (1.7). In practice, this boils down to assume that there exist a function h, satisfying the conditions of Theorem 1.0.5, for which Y satisfies (1.7). The modern formulation to the FTL problem was developed in [17], where the authors assume that the process Y satisfies (1.7). We will therefore assume that Y satisfies (1.7) from now on.

Let us denote by $\varepsilon \in (0,1)$ the terminal tolerance error, that is, the maximum probability of terminal error that we are ready to accept. We define the $\mathcal{Y}$-stopping time

$$\tau = \inf\{t \geq 0 | \max_j \Pi_j(t) = 1 - \varepsilon\}, \tag{1.8}$$

and we declare that the i-th particle, for some $i \in I$, is the particle with drift μ if at time τ, $\max_j \Pi_j(\tau) = \Pi_i(\tau)$. By Theorem 1.0.5 we can choose a version of Π with càdlàg paths almost surely and, since L_j, $j \in I$, has continuous paths almost surely, also Π has continuous paths almost surely.

Remark 1.0.7. By definition the posterior probability process $\Pi(t) = (\Pi_1(t), \dots, \Pi_N(t))$ is a probability measure on Ω for each $t \geq 0$. Hence for $i \in I$ we have

$$\Pi_i(t) = 1 - \sum_{k \neq i} \Pi_k(t) \tag{1.9}$$

for all $t \geq 0$. We will see that this property is very useful as it implies a "dimensionality reduction": rather than keeping track of N processes, as we would do by considering the original processes $(X_1, \dots, X_N)$, we can consider only $N-1$ processes, as the posterior probability of the N-th process is given by (1.9). However (1.9) also implies that the processes $(\Pi_1, \dots, \Pi_N)$ are dependent whereas the original processes $(X_1, \dots, X_N)$ and the likelihood processes $(L_1, \dots, L_N)$ are independent given the control J. As a consequence of the dependence of the processes $(\Pi_1, \dots, \Pi_N)$ we can rule out the case $N = 2$ from the set of non-trivial cases. The value of the FTL problem for $N = 2$ can be easily computed as, by equation (1.9), we have $\Pi_1(t) = 1 - \Pi_2(t)$ for all $t \geq 0$. This implies that at any time $t \geq 0$ it does not matter which process we are observing, as the value of the other one can be immediately deduced by (1.9). Hence for $N = 2$, all strategies that observe only one process at any time $t \geq 0$ are equivalent and there is not a unique optimal searching strategy. From now on we will consider $N \geq 3$.

Let us now write the FTL problem as a dynamic programming problem. Dynamic programming was introduced in the 1950s by Richard Bellman and has since become one of the main and most important approaches in stochastic optimal control. We will use this approach to give a formal set up to the FTL

problem and we will mostly use the notation introduced in [33] and [39]. We can write the value function of the FTL problem as

$$\begin{aligned} V(l_1,\dots,l_N) &= \inf_{J\in\mathbf{J}} \mathbf{E}[\tau | L_1(0)=l_1,\dots,L_N(0)=l_N, \text{strategy}=J] \\ &:= \inf_{J\in\mathbf{J}} \mathbf{E}_{J,l}[\tau], \end{aligned} \tag{1.10}$$

where $\mathbf{J}$ is the set of admissible searching strategies, V is the value function of the problem, $l_1,\dots,l_N$ are the starting points of the processes $L_1,\dots,L_N$ and τ is the stopping time in (1.8), rewritten in terms of likelihood processes using (1.6). We can characterize the set of admissible searching strategies $\mathbf{J}$ as the set of strategies $(J(t))_{t\geq 0}$.

Remark 1.0.8. Note that the value function of the FTL problem $V(l_1,\dots,l_N)$ does not depend on the starting time t_0 but only on the starting points $l_1,\dots,l_N$. Indeed, we can assume $t_0 = 0$ without losing generality. We can therefore formulate the Follow The Leader problem as an *infinite horizon* dynamic programming problem (see [33] for a detailed explanation of the topic).

1.1 The Generalised FTL problem

A key feature of the FTL problem is that we can only observe one process at each time $t \geq 0$. However, we can obtain very useful insights about the problem by introducing a slight variation of the original problem where we can observe multiple processes at once. To do so we consider the generalized observation process $Y(t)$,

$$Y(t) = \int_0^t \sum_{k=1}^N J_k(s) dX_k(s) \tag{1.11}$$

where $J_k(s) \in [0,1]$ for $k \in I$, $s \leq t$ and, for example,

$$\sum_{k=1}^N J_k^2(t) = 1, \tag{1.12}$$

for $t \geq 0$.

Remark 1.1.1. Model (1.11) allows us to observe "fractions" of particles at each time t, that is, we can dedicate a "portion" of our observation to several particles at once. Note that the original model, used in [17], [36], [25], [24] and [42], considers one-dimensional controls such that $J(t) \in I$, for all $t \geq 0$. It is immediate to see that such a control is equivalent to the following N-dimensional control:

1. $J_k(s) \in \{0,1\}$, $k \in I$;
2. $\sum_{k=1}^N J_k(s) = 1$.

Therefore the model we have just introduced can be seen as a generalization of the original model. Furthermore, constraint (1.12) is a further relaxation of the original model and we impose it merely for computational convenience, as it will become clear in the following.

As in the original formulation we assume that the N Brownian motions $W_1, \dots, W_N$ driving the processes $X_1, \dots, X_N$ are independent. Given the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, let us define the σ-finite probability measures:

- $\mathbf{P}^0$: probability measure under which all particles are standard Brownian motions. Under this measure no process has a non-zero drift μ and our search would never stop. In such a case the random variable $\hat{J}$ takes value 0;
- $\mathbf{P}^i$: probability measure under which particle i, for $i \in I$, is the Brownian motion with positive drift μ. Under this measure X_i is our target.

Under the measure $\mathbf{P}^0$ the observation process (1.11) can be written as

$$dY(t) = \sum_{k=1}^{N} J_k(t) dW_k^0(t) \tag{1.13}$$

where W_k^0 are $\mathbf{P}^0$- Brownian motions for all $k \in I$. Under the measure $\mathbf{P}^i$, $i \in I$, the observation process (1.11) can be written as

$$dY(t) = \sum_{k=1}^{N} J_k(s) dW_k^i(s) + \mu J_i(t) dt,$$

where W_k^i are $\mathbf{P}^i$- Brownian motions.

Remark 1.1.2. Note that the measures $\mathbf{P}^0$, $\mathbf{P}^i$, $i \in I$, are absolutely continuous w.r.t the probability measure $\mathbf{P}$ for $T > 0$ finite. Moreover, $\mathbf{P}^i$, $i \in I$, is absolutely continuous with respect to $\mathbf{P}^0$ for $T > 0$ finite.

Consider now the probability measure on E, $\mathbf{P}^{0,i}$ under which all X_i, $i \in I$, are standard Brownian motion and we are observing the i-th particle, that is $i = j^*$. Note that

$$\mathbf{P}^0 = \sum_{i=0}^{N} \pi_i \mathbf{P}^{0,i}. \tag{1.14}$$

Therefore we define the *likelihood* of the i-th particle X_i having drift $\mu \neq 0$ as

$$\hat{L}_i(t) = \frac{d\mathbf{P}^i}{d\mathbf{P}^{0,i}}(t). \tag{1.15}$$

Remark 1.1.3. Note that the likelihood $\hat{L}_i$, $i \in I$, is the unnormalized posterior probability that the i-th particle has drift $\mu > 0$. Indeed, note that

$$\begin{aligned}\Pi_i(t) &= \mathbf{P}(i = j^*|\mathcal{Y}_t) = \mathbf{E}[\mathbb{1}\{i = j^*\}|\mathcal{Y}_t] \\ &= \mathbf{E}_0\left[\mathbb{1}\{i = j^*\}\frac{d\mathbf{P}}{d\mathbf{P}^0}\middle|\mathcal{Y}_t\right] \\ &= \frac{\mathbf{E}_0[\hat{L}_i(t)|\mathcal{Y}_t]}{\mathbf{E}_0[\frac{d\mathbf{P}}{d\mathbf{P}^0}|\mathcal{Y}_t]},\end{aligned} \tag{1.16}$$

where the last equality is obtained by Bayes' rule. Note that the numerator $\mathbf{E}_0[\hat{L}_i(t)|\mathcal{Y}_t]$ is the projection of the likelihood $\hat{L}_i$ onto the observation filtration $\mathcal{Y}$. Let us denote such a process by L_i, that is,

$$L_i(t) = \frac{d\mathbf{P}^i}{d\mathbf{P}^{0,i}}(t)\Big|_{\mathcal{Y}_t}. \tag{1.17}$$

We can then write (1.16) as

$$\Pi_i(t) = \mathbf{P}(i = j^*|\mathcal{Y}_t) = \frac{L_i(t)}{\sum_j \pi_j L_j(t)}$$

Remark 1.1.4. The likelihood $\hat{L}_i$, $i \in I$, is a well-defined function. However, there is no guarantee that $\hat{L}_i(t)$ is $\mathcal{Y}_t$-adapted, which is an essential property that the likelihoods of the model have to satisfy. We overcome this issue by considering the restriction of the likelihood process to the observation filtration L_i.

By definition of Radon-Nikodym derivative (we refer to [1]) the restricted likelihood L_i can be equivalently written as

$$\begin{aligned}L_i(t) =& \mathbf{E}_0[\hat{L}_i(t)|\mathcal{Y}_t] = \\ =& \mathbf{E}_0\left[\exp\left\{\mu W_i(t) - \frac{1}{2}\mu^2 t\right\}\middle|\mathcal{Y}_t\right] = \\ =& \exp\left\{-\frac{1}{2}\mu^2 t\right\}\mathbf{E}_0\left[\exp\left\{\mu W_i(t)\right\}\middle|\mathcal{Y}_t\right],\end{aligned} \tag{1.18}$$

where $\mathbf{E}_0$ denotes the expectation under $\mathbf{P}^0$. To compute the conditional expectation $\mathbf{E}_0\left[\exp\left\{\mu W_j(t)\right\}\middle|\mathcal{Y}_t\right]$ we consider a generic process

$$dZ(t) = \sum_k \alpha_k(t) dW_k(t),$$

where $(\alpha(t))_{t\geq 0}$ is a $\mathcal{Y}_t$-adapted process. Our goal is to decompose $Z(t)$ in two orthogonal components, one that is $\mathcal{Y}_t$-measurable and a second one that is $\mathcal{Y}_t$-orthogonal, that is

$$dZ(t) = \gamma(t)dY(t) + \gamma^{\perp}(t)dY^{\perp}(t).$$

Denoting by $Y^{\perp,n}$ the n-th component of $Y^\perp \in \mathbf{R}^{N-1}$, we have $d\langle Y(t), Y^{\perp,n}(t)\rangle = 0$ for $n \in \{1,\dots,N-1\}$, $\gamma(t) \in \mathbf{R}$ and $\gamma^\perp(t) \in \mathbf{R}^{N-1}$, for $t \geq 0$. Consider an orthonormal basis $(J^\perp(t))_{t\geq 0}$ for $\mathbf{R}^{N-1}$. Then we have

$$dZ(t) = \gamma(t)\sum_k J_k(t)dW_k(t) + \sum_k \gamma_k^\perp(t) J_k^{\perp,n}(t)dW_k(t)$$

so that

$$d\langle Y(t), Y^{\perp,n}(t)\rangle = \langle J(t), J^{\perp,n}(t)\rangle dt = 0,$$

for all $n \in \{1,\dots,N-1\}$. Our goal is to find $(J^\perp(t))_{t\geq 0}$ such that

$$\alpha(t) = \gamma(t)J(t) + \gamma^\perp(t)J^\perp(t)$$

and $\langle J(t), J^{\perp,n}(t)\rangle = 0$, $n \in \{1,\dots,N-1\}$. If we take $\gamma^\perp(t) = \mathbb{1}_{N-1}$, where $\mathbb{1}_{N-1}$ is the identity matrix of dimension $N-1$, we see that

$$\langle \alpha(t) - \gamma(t)J(t), J(t)\rangle = 0,$$

hence we obtain

$$\gamma(t) = \frac{\langle \alpha(t), J(t)\rangle}{\langle J(t), J(t)\rangle}.$$

By choosing $\alpha_k = e_k$, $k \in I$, where e_k is the k-th vector of the canonical basis of $\mathbb{R}^N$, we get

$$\gamma(t) = \frac{J_k(t)}{\sum_i J_i^2(t)} = J_k(t),$$

where we used (1.12) to obtain the second equality. We can now write

$$dW_k(t) = J_k(t)\sum_{k'} J_{k'}(t)dW_{k'}(t) + \sum_k J_{k'}^{\perp,n}(t)dW_{k'}(t),$$

where $J_{k'}^{\perp,n}(t) = e_k - J_k(t)J_{k'}(t)$, $k' \in I$ and $n \in \{1,\dots,N-1\}$. Hence we have

$$dW_k(t) = J_k(t)\sum_{k'} J_{k'}(t)dW_{k'}(t) + \sum_{k'}(\delta_{k'=k} - J_k(t)J_{k'}(t))dW_{k'}(t). \qquad (1.19)$$

By (1.19) we can write (1.18) as

$$\begin{aligned}
L_j(t) &= \exp\left\{-\frac{1}{2}\mu^2 t\right\} \times \\
&\times \mathbf{E}_0\left[\exp\left\{\mu\left(\int_0^t J_j(s)dY(s) + \int_0^t \sum_k (\delta_{k=j} - J_k(t)J_j(t))dW_k(t)\right)\right\}\Big|\mathcal{Y}_t\right] \\
&= \exp\left\{-\frac{1}{2}\mu^2 t\right\}\exp\left\{\mu\int_0^t J_j(s)dY(s)\right\}\times \\
&\times \mathbf{E}_0\left[\int_0^t \sum_k (\delta_{k=j} - J_k(t)J_j(t))dW_k(t)\Big|\mathcal{Y}_t\right] \\
&= \exp\left\{-\frac{1}{2}\mu^2 t\right\}\exp\left\{\mu\int_0^t J_j(s)dY(s)\right\}\times \\
&\times \mathbf{E}_0\left[\exp\left\{\mu\int_0^t \sum_k \beta_k(s)dW_k(s)\right\}\Big|\mathcal{Y}_t\right],
\end{aligned}$$

where

$$\beta_k(t) = \begin{cases} -J_j(t)J_k(t) & k \neq j \\ -J_j(t)^2 + 1 & k = j \end{cases}.$$

Conditioning $\mathbf{E}_0\left[\exp\left\{\mu\int_0^t \sum_k \beta_k(s)dW_k(s)\right\}\Big|\mathcal{Y}_t\right]$ on $J(t)$, we see that

$$\mathbf{E}_0\left[\exp\left\{\mu\int_0^t \sum_k \beta_k(s)dW_k(s)\right\}\Big|\mathcal{Y}_t\right] = \exp\left\{\frac{1}{2}\mu^2\int_0^t (1 - J_j^2(s))ds\right\}.$$

Therefore we can write (1.18) as

$$L_j(t) = \exp\left\{-\frac{1}{2}\mu^2\int_0^t J_j^2(s)ds + \mu\int_0^t J_j(s)dY(s)\right\}. \tag{1.20}$$

By Ito's formula and using (1.12) we conclude that

$$dL_j(t) = \mu J_j(t)L_j(t)dY(t). \tag{1.21}$$

Remark 1.1.5. Note that the likelihood L_j, $j \in I$, is a martingale under $\mathbf{P}^0$. The standard way to show this is to check Novikov's condition (see [30]), but this can be quite difficult to verify. Alternatively, we can exploit Lemma 3.9 of [1], which shows that L_j is a $\mathbf{P}^0$-martingale under slightly weaker conditions.

Equation (1.21) describes the likelihood of the j-th particle L_j, $j \in I$, under the measure $\mathbf{P}^0$. However under $\mathbf{P}^0$ all particles are standard Brownian motions, which removes any information gain from the problem. To properly model the

FTL problem we need to define a new measure that we call *posterior probability measure* $\mathbf{P}_\pi$, defined as

$$\mathbf{P}_\pi = \sum_{k=1}^{N} \pi_k \mathbf{P}^k, \tag{1.22}$$

where $\pi = (\pi_1, \ldots, \pi_N)$ is the prior distribution defined in (1.4). Let us now define the process Z_π as

$$Z_\pi(t) = \frac{d\mathbf{P}_\pi}{d\mathbf{P}^0}(t)\Big|_{\mathcal{Y}(t)}, \tag{1.23}$$

the Radon-Nikodym derivative of $\mathbf{P}_\pi$ with respect to $\mathbf{P}^0$ restricted to the observation filtration $\mathcal{Y}(t)$. Since $\mathbf{P}^i$ is absolutely continuous with respect to $\mathbf{P}^0$, $i \in I$, also $\mathbf{P}_\pi$ is absolutely continuous with respect to $\mathbf{P}^0$ and (1.23) is well-defined. By definition of $\mathbf{P}_\pi$ we see that

$$Z_\pi(t) = \sum_{k=1}^{N} \pi_k \frac{d\mathbf{P}^k}{d\mathbf{P}^0}(t) = \sum_{k=1}^{N} \pi_k L_k(t). \tag{1.24}$$

To derive an equation describing the behavior of the likelihood processes L_i, $i \in I$, under the posterior measure $\mathbf{P}_\pi$, we first need to study the processes $(W_i(t))_{t\geq 0}$, $i \in I$, driving the particles: we know that W is a N-dimensional Brownian motion under $\mathbf{P}^0$, but we do not know anything about its behavior under $\mathbf{P}_\pi$. To do so, let us define the *posterior probability process* of the i-th particle Π_i, $i \in I$, as

$$\Pi_i(t) = \frac{\pi_i L_i(t)}{\sum_{k=1}^{N} \pi_k L_k(t)}, \tag{1.25}$$

and note that if the prior distribution is uniform, that is, if $\pi = \pi_1 = \pi_2 = \cdots = \pi_N$, equation (1.25) simplifies to (1.6).

Remark 1.1.6. This approach is well understood in the filtering literature. In this literature there is often a distinction between the normalized filtered distribution of the unknown state variable, that is $\Pi(t) = (\Pi_i(t), \ldots, \Pi_N(t))$, whose dynamics are described by the Kushner-Stratonovich equation, and the unnormalized filtered distribution $L(t) = (L_1(t), \ldots, L_N(t))$, whose dynamics are given by the Zakai equation (see for example [1]).

Lemma 1.1.7. Denote by W^i_π, $i \in I$, the stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, specified by the following SDE:

$$dW^i_\pi(t) = dW_i(t) - \Pi_i(t)dt, \quad t \geq 0.$$

Then W^i_π, $i \in I$, is a $\mathbf{P}_\pi$-Brownian motion.

Proof. We prove this by Lévy's characterization of Brownian motion (see for example [38]). To do so we need to verify 3 conditions:

1. $W_\pi^i(0) = 0$ $\mathbf{P}_\pi$-a.s., $i \in I$;

2. $W_\pi^i(t)$ is a $\mathbf{P}_\pi$-local martingale (see [38]), $i \in I$.

3. the quadratic variation of W_π^i is such that $[W_\pi^i]_t = t$, for all $t > 0$.

In this particular case we can show that W_π^i, $i \in I$, is a $\mathbf{P}_\pi$-martingale, which in turn implies property 2. It is readily verifiable that $W_\pi^i(0) = 0$ $\mathbf{P}_\pi$-a.s., for all $i \in I$. To show that property 2 holds true we need to prove that, for $s \leq t$, $s, t \geq 0$,

$$\mathbf{E}_\pi \left[(W_i(t) - W_i(s)) - \mu \int_s^t \Pi_i(r) dr \Big| \mathcal{Y}(s) \right] = 0.$$

Since L_i, for $i \in I$, is a $\mathbf{P}^0$-martingale, it can be readily proved that $\Pi_i(t)$ is a $\mathbf{P}_\pi$-martingale, as Π_i is defined as in (1.25). Hence

$$\begin{aligned}
&\mathbf{E}_\pi \left[(W_i(t) - W_i(s)) - \mu \int_s^t \Pi_i(r) dr \Big| \mathcal{Y}(s) \right] = \\
&= \sum_{k=1}^N \pi_k \mathbf{E}_k \left[(W_i(t) - W_i(s)) - \mu \int_s^t \Pi_i(r) dr \Big| \mathcal{Y}(s) \right] = \\
&= \mu \pi_i(t-s) - \mu \sum_{k=1}^N \pi_k \mathbf{E}_k \left[\int_s^t \Pi_i(r) dr \Big| \mathcal{Y}(s) \right] = \\
&= \mu \pi_i(t-s) - \mu \sum_{k=1}^N \pi_k \mathbf{E}_0 \left[L_k(t) \int_s^t \Pi_i(r) dr \Big| \mathcal{Y}(s) \right] = \\
&= \mu \pi_i(t-s) - \mu \sum_{k=1}^N \pi_k \mathbf{E}_0 \left[L_k(t) \int_s^t \Pi_i(r) dr \Big| \mathcal{Y}(s) \right] = \\
&= \mu \pi_i(t-s) - \mu \mathbf{E}_\pi \left[\int_s^t \Pi_i(r) dr \Big| \mathcal{Y}(s) \right] = \\
&= \mu \pi_i(t-s) - \mu \pi_i(t-s) = 0.
\end{aligned}$$

Therefore W_π^i is a $\mathbf{P}_\pi$-martingale, $i \in I$. To show property 3 recall that the quadratic variation of an Ito's process $X(t) = X(0) + \int_0^t \sigma(s) dW(s) + \int_0^t \mu(s) ds$ can be computed as $[X]_t = \int_0^t \sigma^2(s) ds$. Hence, we can immediately see that $[W_\pi^i]_t = t$, for all $t \geq 0$.
This proves that W_π^i is a $\mathbf{P}_\pi$-standard Brownian motion. □

By Lemma 1.1.7 we can rewrite the observation process (1.13) as

$$dY(t) = \sum_{k=1}^N J_k(t) dW_k(t) = \sum_{k=1}^N J_k(t) \{ dW_\pi^k(t) + \mu \Pi_k(t) dt \}, \tag{1.26}$$

which, given (1.21), leads to

$$dL_i^\pi(t) = \mu J_i(t) L_i^\pi(t) dY(t) = \\ = \mu J_i(t) L_i^\pi(t) \sum_{k=1}^{N} J_k(t) \{dW_\pi^k(t) + \mu \Pi_k(t) dt\}. \tag{1.27}$$

Remark 1.1.8. Clearly L_i^π is *not* a P_π-martingale for any $i \in I$. However, Π_i is a P_π-martingale for all $i \in I$, as we have shown in Lemma 1.1.7 (in particular, Π_i is a P_π- standard Brownian motion, $i \in I$).

Remark 1.1.9. Let $i^* \in I$ denote the observed particle at time $t \geq 0$. Then we have $J_{i^*}(t) = 1$ and $J_i(t) = 0$ for all $i \neq i^*$, $i \in I$. Therefore the likelihood process at time $t \geq 0$ looks like

$$dL_{i^*}^\pi(t) = \mu L_{i^*}^\pi(t) dY(t) = \\ = \mu L_{i^*}^\pi(t) \sum_{k=1}^{N} \{dW_\pi^k(t) + \mu \Pi_k(t) dt\}, \\ dL_i^\pi(t) = 0, \quad i \neq i^*.$$

We can rewrite the value of the FTL problem (1.10) as

$$V(\pi_1, \ldots, \pi_N) = \min_{J \in \mathbf{J}} \mathbf{E}_\pi[\tau_J | \Pi_1(0) = \pi_1, \ldots, \Pi_N(0) = \pi_N], \tag{1.28}$$

where τ_J is defined as in (1.8) and $\mathbf{E}_\pi$ denotes the expectation under the posterior measure P^π.

Remark 1.1.10. The value functions (1.10) and (1.28) are equivalent. By switching our attention from (1.10) to (1.28) we are passing from a problem on the original process X, with values in $\mathbb{R}^N$, to a different problem on the posterior probability process Π. As $\Pi(t)$ is a probability measure on Ω for all $t \geq 0$, Π takes values on $\mathcal{P}(\Omega)$, the set of probability measures on Ω. This makes the FTL problem a *measure-valued* stochastic optimal control problem. This kind of optimization problems are often used in mathematical finance and have many interesting applications (see for example [11]).

The posterior probability of the i-th particle at time $t \geq 0$, $\Pi_i(t)$, as defined in (1.25) differs from the likelihood of the i-th particle at time $t \geq 0$, $L_i^\pi(t)$ only by a multiplicative factor. Therefore we expect the optimal strategy and the value function in the space of likelihoods to be the same as in the space of posterior probabilities. This implies that we can either focus on (1.28) or on

$$V(l_1, \ldots, l_N) = \min_{J \in \mathbf{J}} \mathbf{E}_\pi[\tau_{J^L} | L_1^\pi(0) = l_1, \ldots, L_N^\pi(0) = l_N], \tag{1.29}$$

where

$$\tau_{J^L} = \inf\left\{ t \geq 0 \middle| \max_j L_j^\pi(t) = \frac{1-\varepsilon}{\varepsilon} \sum_{k \neq j} \frac{\pi_k}{\pi_j} L_k^\pi(t) \right\}. \tag{1.30}$$

This is equivalent to say that

$$V(l_1, \ldots, l_N) = V(\pi_1, \ldots, \pi_N). \tag{1.31}$$

Remark 1.1.11. More generally we expect $V(\alpha l_1, \ldots, \alpha l_N) = V(l_1, \ldots, l_N)\ \forall \alpha > 0$. This assumption is pretty natural as it states that multiplying all likelihoods by a constant does not affect the average searching time of a given strategy $J \in \mathbf{J}$. Indeed, by multiplying all particles by a common constant $\alpha > 0$ we also change the stopping time τ_{J^L} in equation (1.30) by the same quantity, to get

$$\tau_{J^L} = \inf\left\{ t \geq 0 \middle| \max_j \left(\alpha L_j^\pi(t)\right) = \frac{1-\varepsilon}{\varepsilon} \sum_{k \neq j} \frac{\pi_k}{\pi_j} \left(\alpha L_k^\pi(t)\right) \right\}.$$

Remark 1.1.12. The version of the FTL problem with value (1.28) has nice theoretical properties and we will make use of it when needed, but we will consider, for the most part, the FTL problem in the space of likelihoods, that is, the value function (1.29), as the independence of the likelihood processes $L_1^\pi, \ldots, L_N^\pi$ makes computations and numerical simulations much easier.

As pointed out in Remark 1.1.11, we expect the value function of the FTL problem to be invariant to multiplication of constants $\alpha > 0$. In the following theorem we are going to show that the value of the FTL problem is also invariant to permutation of the prior distribution π.

Theorem 1.1.13. *Consider a prior distribution $\Pi(0) = \pi = (\pi_1, \ldots, \pi_N)$ and let us sort it in a decreasing way $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_N)$, so that $\hat{\pi}_1 \geq \cdots \geq \hat{\pi}_N$. Denote by V the value function of the FTL problem as defined in* (1.28)*. Then we have*

$$V(\hat{\pi}_1, \ldots, \hat{\pi}_N) = V(\pi_1, \ldots, \pi_N).$$

Proof. Consider J_π as the set of admissible strategies given a prior distribution π. Consider a permutation $P : I \to I$ and define J_{π^p} as the set of admissible strategies given the prior distribution $\pi^p = (\pi_{P(1)}, \ldots, \pi_{P(N)})$. Consider a control $J' \in J_\pi$. Let us fix $\omega \in \Omega$ and apply J' so to get likelihoods $L^\pi(t, \omega) = (L_1^\pi(t, \omega), \ldots, L_N^\pi(t, \omega))$. By Remark 1.1.11, we can equivalently consider the problem in the space of likelihoods. Recall that $L_i^\pi(t)$ is the unique strong solution (given $L_i^\pi(0) = l_i$) of (1.27), $t \geq 0$ and $i \in I$, and that the likelihoods are driven by a collection of independent P_π-Brownian motions $W_1^\pi(t), \ldots, W_N^\pi(t)$. Consider a new filtered probability space $(\Omega, \mathcal{G}, \mathbf{G} = (\mathcal{G}_t)_{t \geq 0}, \mathbf{P})$ on which are defined N

independent Brownian motions $W^\pi_{P(1)}(t), \ldots, W^\pi_{P(N)}(t)$. On this probability space we can define a new collection of likelihood processes $L^\pi_{P(1)}, \ldots, L^\pi_{P(N)}$ as

$$dL^\pi_{P(i)}(t) = \mu J_{P(i)}(t) L^\pi_{P(i)}(t) dY(t) =$$
$$\mu J_{P(i)}(t) L^\pi_{P(i)}(t) \sum_{k=1}^{N} J_{P(k)}(t) \{dW^\pi_{P(k)}(t) + \mu \Pi_{P(k)}(t) dt\}. \tag{1.32}$$

Note that $(L^\pi_{P(1)}, \ldots, L^\pi_{P(N)})$ has prior distribution l^p and that $L^\pi_{P(i)}$ is the unique strong solution of (1.32) on $(\Omega, \mathcal{G}, \mathbf{G} = (\mathcal{G}_t)_{t\geq 0}, \mathbf{P})$. Define a strategy $J'' \in J_{\pi^p}$ such that $J''(t) = (J'_{P(1)}(t), \ldots, J'_{P(N)}(t))$ for all $t \geq 0$. By the bijectivity of P, J'' is unique. Let us apply J'' on $(L^\pi_{P(1)}, \ldots, L^\pi_{P(N)})$ and consider the same path $\omega \in \Omega$. Then $(L^\pi_{P(1)}(t,\omega), \ldots, L^\pi_{P(N)}(t,\omega)) = (L^\pi_1(t,\omega), \ldots, L^\pi_N(t,\omega))$. Moreover J'' inherits the progressive measurability and therefore the $\mathcal{Y}_t$-adaptivity of J'. □

One of the most important feature of the dynamic programming approach is to associate to each strategy $J \in \mathbf{J}'$ a value function V_J of the form

$$V_J(l_1, \ldots, l_N) = \mathbf{E}_\pi[\tau_{J^L} | L^\pi_1(0) = l_1, \ldots, L^\pi_N(0) = l_N, \text{ strategy} = J], \tag{1.33}$$

and a corresponding Hamilton-Jacobi-Bellman (HJB) equation (see [33] and [39] for an extensive introduction) which can be regarded as the "infinitesimal" version of the dynamic programming principle (DPP). The associated HJB equation is a second-order linear ordinary differential equation (ODE) of the form

$$H_i(f) = \frac{\partial f}{\partial l_i} \mu^2 \frac{l_i^2}{\sum_k l_i} + \frac{1}{2} \frac{\partial^2 f}{\partial l_i^2} \mu^2 l_i^2 + 1, \tag{1.34}$$

where $f \in C^2(\mathbf{R}^N)$, $f : \mathbf{R}^N \to \mathbf{R}$. In particular, we expect the value function V_J of the policy $J \in \mathbf{J}$ to solve equation (1.34), that is,

$$\frac{\partial V_J}{\partial l_i} \frac{\mu^2 l_i^2}{\sum_k l_i} + \frac{1}{2} \frac{\partial^2 V_J}{\partial l_i^2} \mu^2 l_i^2 + 1 = 0, \tag{1.35}$$

with associated boundary conditions. By equations (1.33) and (1.34), we can rewrite equation (1.29) as

$$V(l_1, \ldots, l_N) = \min_{J \in \mathbf{J}} V_J(l_1, \ldots, l_N),$$

and we can define the *optimal strategy* $J^* \in \mathbf{J}$ as the policy such that

$$V_{J^*}(l_1, \ldots, l_N) = V(l_1, \ldots, l_N) = \min_{J \in \mathbf{J}} V_J(l_1, \ldots, l_N). \tag{1.36}$$

We would expect the value function of the FTL problem V to solve the following ODE:

$$\min_k H_k(V)(l) = 0, \tag{1.37}$$

with associated boundary conditions. Unfortunately, there are no guarantees that $V \in C^2(\mathbf{R}^N)$. In fact, this is in general not true even with value functions of relatively simple stochastic control problems. To overcome this issue we need to resort to the notion of *viscosity solution*: viscosity solutions were introduced in the 1990s in [12] and provide a method to find solutions to second-order partial differential equations (PDE) when a classical strong solution is not well-defined. In particular, the viscosity solutions approach provides powerful means to study in great generality stochastic optimal control problems and gives a rigorous formulation of the HJB equation for functions that are only assumed to be locally bounded. By combining these results with comparison principles for viscosity solutions, we can characterize the value function as the unique viscosity solution of the associated HJB equation. [33] and [39] have great chapters about this topic. Given the central role of viscosity solutions in the following, we recall the definition of viscosity solution for a second-order PDE.

1.2 The value as viscosity solution of the HJB equation

Definition 1.2.1 (Definition 4.2.1, [33])**.** Consider an open subset $\mathcal{O}$ of $\mathbb{R}^N$, $N \in \mathbb{N}$, and w a locally bounded function $w : \mathcal{O} \to \mathbb{R}$. Consider a second-order PDE of the form

$$F(x, w(x), Dw(x), D^2w(x)) = 0, \quad x \in \mathcal{O}, \tag{1.38}$$

where F is a continuous function on $\mathcal{O} \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}_n$ taking values in $\mathbf{R}$, where $\mathcal{S}_n$ is the space of symmetric $n \times n$ matrices. We assume that F satisfies the ellipticity condition, that is, for all $x \in \mathcal{O}$, $r \in \mathbf{R}$, $p \in \mathbf{R}^n$, M,$\hat{M} \in \mathcal{S}_n$,

$$M \leq \hat{M} \implies F(x, r, p, M) \geq F(x, r, p, \hat{M}),$$

where $M \leq \hat{M} \iff M - \hat{M} \in \mathcal{S}_n^+$, and $\mathcal{S}_n^+$ is the set of non-negative definite matrices in $\mathcal{S}_n$. Denote by w^*, w_* the smallest (resp. largest) upper-semicontinuous function above (resp. lower-semicontinuous function below) w on $\mathcal{O}$. Then:

1. w is a viscosity subsolution of (1.38) on $\mathcal{O}$ if

$$F(\bar{x}, w^*(\bar{x}), D\phi(\bar{x}), D^2\phi(\bar{x})) \leq 0 \quad x \in \mathcal{O}, \tag{1.39}$$

   for $\bar{x} \in \mathcal{O}$ and for all $\phi \in C^2(\mathcal{O})$, such that $\bar{x}$ is a maximum point of $w^* - \phi$;

2. w is a viscosity supersolution of (1.38) on $\mathcal{O}$ if

$$F(\bar{x}, w_*(\bar{x}), D\phi(\bar{x}), D^2\phi(\bar{x})) \geq 0 \quad x \in \mathcal{O}, \tag{1.40}$$

   for $\bar{x} \in \mathcal{O}$ and for all $\phi \in C^2(\mathcal{O})$, such that $\bar{x}$ is a minimum point of $w_* - \phi$;

3. We say that w is a viscosity solution of (1.38) on $\mathcal{O}$, if it is both a subsolution and a supersolution of (1.38).

Remark 1.2.2. We can interpret the notion of viscosity solution as follows: we evaluate the PDE (1.38) on the "closest" semicontinuous function to w (either w^* or w_*) but at the same time we evaluate the gradient and the second derivatives on a smooth function $\phi \in C^2(\mathcal{O})$. The idea underneath it is to ignore the kinks of w and substitute the gradient and second derivatives of w on those points with gradient and second derivatives of a smooth function $\phi \in C^2(\mathcal{O})$.

Remark 1.2.3. A deep treatment of the theory of viscosity solutions is beyond the scope of this thesis. What interest us the most is to characterize the value function of an infinite horizon stochastic optimal control problem as the unique viscosity solution of the corresponding *variational* HJB equation. To do so, let us recall one more theoretical result.

Consider a general controlled diffusion process X on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, with values on $\mathbf{R}^n$ described by the following SDE:

$$dX(s) = b(X(s), \alpha(s))ds + \sigma(X(s), \alpha(s))dW(s), \tag{1.41}$$

where W is a d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ and the control process $\alpha = (\alpha(s))$ is $\mathbf{F}$-progressively measurable and valued in $A \subset \mathbf{R}^m$. The coefficients b and σ satisfy the usual regularity conditions that guarantee that (1.41) has a unique strong solution (see Chapter 1 of [33]).The Hamiltonian associated to a stochastic optimal control problem for the controlled diffusion (1.41) has the general form

$$H(x, p, M) = \sup_{\alpha \in A} \left\{ b(x, \alpha)p + \frac{1}{2} tr(\sigma(x, \alpha)\sigma'(x, \alpha)M) + f(x, \alpha) \right\}. \tag{1.42}$$

Assumption 1.2.4. Let us assume that H is continuous on $\text{int}(\text{dom}(H))$, that the function f in (1.42) has quadratic growth and that $f(., \alpha)$ is continuous in x for all $\alpha \in A$. Assume that there exist a continuous function $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}_n \to \mathbb{R}$ such that

$$(x, p, M) \in dom(H) \iff G(x, p, M) \geq 0.$$

Then the following theorem holds true:

Theorem 1.2.5 (Theorem 4.3.1, [33])**.** *Suppose the value function v is locally bounded and that assumptions 1.2.4 hold true. Then for all $\beta > 0$, where β is a discount factor, v is a viscosity solution of the HJB variational inequality:*

$$\min\{\beta v(x) - H(x, Dv(x), D^2v(x)), G(x, Dv(x), D^2v(x))\} = 0 \tag{1.43}$$

for $x \in \mathbf{R}^n$.

We cannot apply Theorem 1.2.5 directly to our case, as the result requires a discount factor $\beta > 0$. To apply Theorem 1.2.5 to our case we will approach the problem as a discounted problem with bounded horizon and discount rate $\beta \searrow 0$.

Theorem 1.2.6. *Given a prior distribution $l = (l_1, \dots, l_N)$ for the likelihood process $L^\pi = (L_1^\pi, \dots, L_N^\pi)$ defined by* (1.25)*, the value function of the Follow The Leader problem V, that is the unique solution of the minimization problem* (1.28)*, is the unique viscosity solution of the equation*

$$\min_k H_k(V)(l) = 0, \tag{1.44}$$

with boundary condition

$$V(\partial^\varepsilon D) = 0, \tag{1.45}$$

where

$$D = \left\{ l \in \mathbf{R}^N \middle| l_k \geq 0 \ \forall k \right\} \tag{1.46}$$

and $\partial^\varepsilon D = \{l \in D | l_{k^*} = \max_k l_k = \frac{1-\varepsilon}{\varepsilon} \sum_{k \neq k^*} l_k\}$.

Proof. Let us introduce the notation $\boldsymbol{L}^u = (L^{1,u}, \dots, L^{N,u})$, where $L^{i,u}$ is the likelihood of the i-th box, $i \in \{1, \dots, N\}$, under the admissible control u. For readability we will write $L^{i,u}$ in place of $L^\pi_{i,u}$ for the remainder of the proof.

In the FTL case the dynamics of the likelihood L^i are given (in law) by (see equation (1.27) with $k = 1$):

$$dL_t^{i,u} = \mathbb{1}_{\{u_t = i\}} \frac{(L_t^{i,u})^2}{\sum_k L_t^{k,u}} \mu^2, dt + \mathbb{1}_{\{u_t = i\}} \mu L_t^{i,u} dW_t.$$

Note that, as pointed out in Remark 1.1.11, the dynamics of the likelihood process are invariant to scaling. That is, scaling the likelihood process L by a constant α will scale through the future likelihood in a common manner. If our objectives in the control problem are invariant to scaling, then the value will be invariant to scaling too.

In the case of the posterior distribution, we can compute the posterior probability of the j-th box in terms of the likelihoods, by $\Pi^j = L^j / \sum_k L^k$. When we observe the i-th box, we then see:

$$d\Pi_t^{j,u} = \mu \Pi_t^{j,u} \left(\mathbb{1}_{\{j = u_t\}} - \sum_k \mathbb{1}_{\{k = u_t\}} \Pi_t^{k,u} \right) dW_t. \tag{1.47}$$

Observe also that, since $\sum_k \Pi_t^{k,u} = 1$, we have that $d \sum_k \Pi_t^{k,u} = 0$.

Note that by Theorem 1.1.13 and Remark 1.1.11, the original value function $v(\ell_1, \dots, \ell_N)$ satisfies a reordering condition $v(\ell_1, \dots, \ell_N) = v(\ell_{\sigma(1)}, \dots, \ell_{\sigma(N)})$

where σ is an arbitrary permutation of $\{1, \ldots, N\}$, and also a rescaling condition, that is

$$v(\ell_1, \ldots, \ell_N) = v(\alpha\ell_1, \ldots, \alpha\ell_N)$$

where $\alpha > 0$. Consequently, we can define the value for the problem given in terms of π in a rescaled fashion, namely:

$$\hat{v}(\pi^1, \ldots, \pi^N) := v(\pi^1, \ldots, \pi^N)$$

where we only interpret $\hat{v}$ on the set of probability vectors, $\pi^i \in [0, 1]$ and $\sum_k \pi^k = 1$.

We first aim to prove:

1. The function $\hat{v}$ is bounded on $\mathcal{P}(\{1, \ldots, N\})$.
2. The function $\hat{v}$ is continuous on $\mathcal{P}(\{1, \ldots, N\})$.

To see the first, we argue as follows. Fix a stopping probability ε, we want to run until $\tau_\varepsilon := \inf\{t \geq 0 : \pi^j > 1 - \varepsilon\}$, some $j \in \{1, \ldots, N\}\}$, and we want to show that there exists $K > 0$ such that

$$\tau_\varepsilon \leq K$$

for some observation strategy. Fix $\eta \in (0, 1/N)$.

Fix a prior distribution π. Since $\sum_k \pi^i = 1$, there exists an index j such that $\pi^j \geq \frac{1}{N}$. We observe this index j until either $\Pi^j = 1 - \varepsilon$, or $\Pi^j = \eta$. We observe that the expected time to exit for the process given we start at $\pi^j = \pi_0$ can be computed as $w(\cdot)$, where $w(\cdot)$ is the solution to the second order ODE:

$$w(\eta) = 0 = w(1 - \varepsilon), \quad \mu^2\pi^2(1 - \pi)^2 w''(\pi) = -2. \tag{1.48}$$

To see this is a straightforward application of Itô's Lemma.

Note that the resulting function w is bounded, from which we conclude that there is a maximal average time w^* that the process will take to reach either $1-\varepsilon$ or η from any given starting point in $[\eta, \varepsilon]$. Moreover, if we write H_y for the first hitting time of y, since Π is a martingale, it also follows that $\mathbb{P}(H_{1-\varepsilon} < H_\eta | \pi_0 = \pi) = \frac{\pi - \eta}{1 - \varepsilon - \eta}$. In particular, if $\pi \geq \frac{1}{N}$, then $\mathbb{P}(H_{1-\varepsilon} < H_\eta | \pi_0 = \pi) > \delta$ for some $\delta > 0$.

Finally, observe that the following strategy guarantees on average a searching time of at most w^*/δ: initially choose any index which has $\pi^i \geq \frac{1}{N}$. It is clear that at least one such index must exists. Observe this index until either the probability of this index exceeds $1 - \varepsilon$, or it drops below η. On average, this will take time at most w^*. With probability at least δ, the index will stop with probability at least $1-\varepsilon$ and we have identified the valid index. Otherwise we can choose another index, again choosing an index with initial probability at least

$1/N$, and repeat. The number of repeated trials is bounded above in law by a geometric distribution with parameter δ, and therefore we need an average of at most δ^{-1} such trials. Conditional on the initial starting point, each trial has average length at most w^*, giving the upper bound.

We now turn to the proof of continuity. First observe that using the equivalence of formulation as v or $\hat{v}$, it is sufficient to prove continuity of v, rather than continuity of $\hat{v}$ in the interior of $\mathcal{P}$. It is straightforward to see, using similar ideas to above, that for any $(\ell_1, \ldots, \ell_N) \in \mathbb{R}^N$, there exists $\delta > 0$ such that an appropriate choice of strategy can reach any point in the ball of radius δ in small time, with high probability: following a similar calculation to that in (1.48), it is straghtforward to see that one can move from $(\ell_1, \ldots, \ell_N)$ to $(\ell_1', \ell_2, \ldots, \ell_N)$ in small (average) time, with high probability of success. Repeating, continuity follows.

To show continuity in general, that is, continuity including the boundary where $\pi^i = 0$ for some $i \in \{1, \ldots, N\}$ is harder, and is not standard, see for example [9]. First observe that if π is sufficiently close to the stopping region, then we can find a strategy that stops within small time on average. That is, given $\xi > 0$, there exists $\delta > 0$ such that for any π such that $\pi^j > 1 - \varepsilon - \delta$ for some $j \in \{1, \ldots, N\}$, then there exists a strategy which will stop in average time less than ξ. This follows immediately from running the argument above.

Now consider $\hat{v}(\pi)$. We show that for any $\xi > 0$, we can find a ball around π such that $\hat{v}(\pi') \leq \hat{v}(\pi) + \xi$, for all π' in the ball.

Since the diffusion coefficient in (1.47) is Lipschitz in π, standard results on diffusions imply that for a given control u that has expected exit time smaller than $\hat{v}(\pi)+\xi/4$, and $T > 0$, we can find a ball around π such that $\mathbb{P}(\sup_{s \leq T} ||\pi_s - \pi_s'|| > \delta') < \xi/(4w^*)$, where $(\pi_s')_{s \leq T}$ is the controlled process following the same strategy as $(\pi_s)_{s \leq T}$, but starting at π'. Fix T large enough that $(\pi_s)_{s \leq T}$ reaches the stopping set

$$P := \{\pi : \pi^i \geq 1 - \varepsilon \text{ for some } i\}$$

with probability at least $1 - \xi/(4w^*)$ before time T (such a time T exists since $\hat{v}(\pi)$ is finite). Then we run the $(\pi_s')_{s \leq T}$ process until we stop π, or T. If we stop π, then by the argument above, π' is sufficiently close to the stopping region, and we can choose our original ball such that the expected time to reach the exit region from this position is less than $\xi/4$. If we reach time T, then we can still construct a strategy to reach P with average time at most w^*. The conclusion follows.

Once we have continuity of the value function on the whole domain, it is easy to verify that it is the unique value function: we can, for example, approximate from below by a well-behaved approximation (see for example [19]), and take limits, using Dini's Theorem to deduce uniform convergence. Specifically, consider an increasing sequence of connected, compact, subsets K^N of $\mathbb{R}^N$ with smooth boundaries, increasing in $\mathbb{R}^N$. Consider the problem of minimising the expected

time to exit from $int(K^N) \cap P^C$ using the control strategies as above. Standard results (e.g. [9, Theorem 4.1]) show that under these additional assumptions, the corresponding value functions are continuous viscosity solutions to the corresponding HJB equation, and taking the limit as $N \to \infty$, we recover the value of the original problem.

The limit is then also a viscosity solution by stability results (see Lemma 6.2 from [19]), and we then get the desired conclusion. □

We conclude this chapter by formally defining the FTL strategy: such a policy is deceptively simple, but in fact is not trivial to define it rigorously. To see why, consider the following situation: we start with a generic prior distribution l. By Theorem 1.1.13 we can sort the prior distribution decreasingly without affecting the value of the problem, so that we have $l = (l_1, l_2, \dots, l_N)$ with $l_1 \geq l_2 \geq \cdots \geq l_N$. We choose to apply the FTL policy and to observe the first particle L_1 at time $t = 0$. There are 2 possible outcomes:

1. the process X_1 is easily identified as having the drift μ and the likelihood L_1 reaches the threshold $l^* := m_\varepsilon \sum_{k>1} l_k$ before going down to the level l_2. In this case the search is over;

2. we still do not have enough data to confidently say that X_1 has drift μ, so the likelihood L_1 goes down to the level l_2 before touching l^*.

In principle, outcome 2 could happen an infinite number of times and infinitely often. Recalling that L_1^π is characterised as the unique strong solution of equation (1.27) (with $J_1(t) = 1$ for $t \geq 0$), we see that the behaviour in outcome 2 implies that equation (1.27) has no strong solution and, therefore, that the likelihood L_1 is not well-defined. To solve this issue, we observe that we can understand the phenomenon by introducing a *local time effect* at point l. Indeed, note that equation (1.27) under the FTL strategy can be rewritten as

$$dL_i^\pi(t) = \mathbb{1}_{\{t \geq 0 | L_i^\pi(t) = \max_k L_k^\pi(t)\}} \mu L_i^\pi(t) \{dW_i^\pi(t) + \mu \Pi_i(t) dt\}, \tag{1.49}$$

for $i \in I$, $t \geq 0$. It is evident that the coefficients of (1.49) are not Lipschitz, or even continuous, so the sense in which the SDE (1.49) has a solution needs to be clarified. Equation (1.49) has strong similarities with the celebrated Tanaka SDE: in particular, if we take $N = 2$, we ignore the drift term and we define $y(t) = L_1^\pi(t) - L_2^\pi(t)$, we see that (1.49) can be written as

$$dy(t) = sign(y(t)) \mu dW^\pi(t),$$

for $t \geq 0$, which is exactly Tanaka SDE. There is no strong solution to such SDE, but there exists a weak solution, represented by taking Y to be a Brownian motion started at $y(0) = L_1^\pi(0) - L_2^\pi(0)$, which we may as well suppose is positive,

and then defining $dW(t) = sign(y(t))dy(t) = d|y(t)| - dE(t)$, where E is the local time of y at zero. Then we have

$$L_1^\pi(t) = L_1^\pi(0) + y^+(t) - \frac{1}{2}E(t)$$
$$L_2^\pi(t) = L_2^\pi(0) + y^-(t) - \frac{1}{2}E(t).$$

It is beyond the scope of this thesis to study the theory of excursions and local times, but we will see how to use local times to construct the FTL policy in the next section. Theorem 1 of [17], which we will partially recall in the next section, will be extremely useful in this regard. We refer to [38] and [40] for a in-depth treatment of the topic.

1.3 Local time effect

Following the approach proposed by [17], we define the log-likelihood of the i-th particle Z_i^π, $i \in I$, as

$$Z_i^\pi(t) := \mu^{-1} \log(L_i^\pi(t)), \quad t \geq 0, \quad \mu \neq 0. \tag{1.50}$$

Given $Z_i^\pi(0) = z_i$, a straightforward application of Ito's formula shows that Z_i^π is the unique strong solution of

$$dZ_i^\pi(t) = J_i(t) \left(dY(t) - \frac{\mu}{2} J_i(t) dt \right). \tag{1.51}$$

Note that by applying the FTL policy we obtain

$$dZ_1^\pi(t) = \left(dY(t) - \frac{\mu}{2} dt \right) = dW_1^\pi(t) + \mu \left(\Pi_1(t) - \frac{1}{2} \right) dt, \quad t \geq 0,$$

and $dZ_i^\pi(t) = 0$ for $i \in \{2, \ldots, N\}$, $t \geq 0$.

Remark 1.3.1. Note that $Z_i^\pi(t)$ is a monotone function of $L_i^\pi(t)$, $i \in I$. Therefore the value of the FTL strategy in the space of log-likelihoods is equivalent to the value in the space of likelihoods.

Clearly, the space of likelihoods L and the space of log-likelihoods Z are isomorphic, that is, any point $l = (l_1, l_2, \ldots, l_N)$ in the space of likelihoods corresponds to only one point $z = (z_1, z_2, \ldots, z_N)$ in the space of log-likelihoods and the converse holds true as well. Hence, we consider the prior distribution $z = (z_1, \ldots, z_N)$, where $z_1 \geq z_2 \geq \cdots \geq z_N$. If we apply the FTL strategy to this configuration, one of the two outcomes listed above can happen: either Z_1^π reaches z^*, where $z^* = \mu^{-1} \log(l^*)$, or it first goes down to the level of Z_2^π. In the latter case, we obtain the configuration $Z_1^\pi(t) \simeq Z_2^\pi(t) \geq Z_3^\pi(t) \geq \cdots \geq Z_N^\pi(t)$.

In this situation, is not at all clear which, between Z_1^π and Z_2^π, is the leader and applying the FTL policy becomes non trivial. To overcome this issue we consider the local reparametrisation $Z_M^\pi(t) := \max_i Z_i^\pi(t) = \max\{Z_1^\pi(t), Z_2^\pi(t)\}$, $Z_m^\pi(t) := \min_i Z_i^\pi(t) = \min\{Z_1^\pi(t), Z_2^\pi(t)\}$. Such a parametrisation is very useful as in [17] the authors show the following results:

Theorem 1.3.2 (Theorem 1, [17])**.** *For all starting value $z = (z_1, \dots, z_N)$, for all $t \geq 0$, we have*

1. $dZ_m^\pi(t) = \frac{1}{N} dW_m^\pi(t)$;
2. $Z_M^\pi(t) - Z_m^\pi(t) = W^\pi(t) - W_m^\pi(t) \stackrel{d}{=} |W^\pi(t)|$,

where W^π is a P_π-Brownian motion, W_m^π denotes the running minimum process of the Brownian motion W^π and the last equality in point 2, due to Lèvy (see [37]), is an equality in distribution.

Remark 1.3.3. Point 2 of Theorem 1.3.2 is especially interesting as it shows that at the boundary $Z_1^\pi(t) \simeq Z_2^\pi(t) \geq Z_3^\pi(t) \geq \cdots \geq Z_N^\pi(t)$ we have a local time behavior: we are uncertain about which particle is the leader and we therefore run each particle for a very short time to establish which will be the next leader.

Hence, we have

$$dZ_M^\pi(t) = dZ_m^\pi(t) + dW^\pi(t) - dW_m^\pi(t) = dW^\pi(t) - \frac{1}{2} dW_m^\pi(t),$$

and since the set $\{t \geq 0 | Z_1^\pi(t) = Z_2^\pi(t) \geq \cdots \geq Z_N^\pi(t)\}$ is negligible with respect to the Lebesgue measure we have

$$dZ_M^\pi(t) = -\frac{1}{2} dW_m^\pi(t) = -dZ_m^\pi(t). \tag{1.52}$$

By applying Ito's formula once more we obtain

$$\begin{aligned} dV(Z_M^\pi(t), Z_m^\pi(t), Z_3^\pi(t), \dots, Z_N^\pi(t)) &= \frac{\partial V}{\partial Z_M^\pi} dZ_M^\pi(t) + \frac{\partial V}{\partial Z_m^\pi} dZ_m^\pi(t) = \\ &= \frac{1}{2} \frac{\partial V}{\partial Z_m^\pi} dW_m^\pi(t) - \frac{1}{2} \frac{\partial V}{\partial Z_M^\pi} dW_m^\pi(t). \end{aligned}$$

By standard arguments (see for example [33] or [39]), V has to be a martingale on $\{t \geq 0 | Z_M^\pi(t) = Z_m^\pi(t)\}$. Therefore we must have

$$\frac{\partial V}{\partial Z_M^\pi} = \frac{\partial V}{\partial Z_m^\pi}. \tag{1.53}$$

Remark 1.3.4. This result will be particularly important in Chapter 3, when we will study the optimal strategy of the FTL problem for a generic prior distribution. In chapter 2 we will study the optimal strategy of the FTL problem when the prior distribution is uniform. In that particular case we will make use of what we obtained so far to argue a very similar result.

1.4 Summary

In this chapter we formally defined the FTL problem. We converted a stochastic optimal control problem on the sample paths space, $\mathbb{R}^N$, $N \in \mathbb{N}$, to a measure-valued stochastic control problem on the space of probability measures on Ω, $\mathcal{P}(\Omega)$. We then showed that such a problem is equivalent to a stochastic control problem in the space of likelihoods. We characterized the value of the FTL problem (1.28) as the unique viscosity solution of the HJB equation (1.44), and we used the theory of excursions and local times to formally define the FTL strategy even at the boundary $\{t \geq 0 | L_1^\pi(t) \simeq L_2^\pi(t) \geq L_3^\pi(t) \geq \cdots \geq L_N^\pi(t)\}$. In the next chapter, we are going to study the FTL problem in the particular case where the prior distribution π is uniform, that is, $\pi = \pi_1 = \cdots = \pi_N$, for $\pi \in [0, 1]$.

Chapter 2

The optimal strategy with uniform prior distribution

In this chapter we will study the optimal strategy for the FTL problem (1.28) when the prior distribution π is uniform, that is, $\Pi(0) = (\pi, \dots, \pi)$, for $\pi \in [0, 1]$. In particular, we will show that the policy that was conjectured to be optimal, that is, the FTL policy, is in fact suboptimal. To do so, we will firstly rewrite the HJB equation (1.34) for the FTL policy when the prior distribution is uniform. We will compute the value of the FTL strategy and provide visualisations and numerical simulations to understand its behavior. Eventually, we will propose a novel approach to evaluate if the FTL strategy is the optimal strategy and we will show that the results we obtain are in contrast with the claims in [42]. We will always consider the case $N = 3$ throughout the entire chapter. We will see that our arguments are not easily generalizable to higher dimensions and we will point out where this hypothesis play an important role in our arguments. At the end of the chapter we will discuss how such results could be generalized to $N > 3$ and which difficulties we need to overcome to do so.

2.1 The FTL strategy

In this chapter we will consider the case where the prior distribution π is uniform, that is, $\pi = \pi_1 = \pi_2 = \pi_3$, for $\pi \in [0, 1]$. For the most part, we will study the problem in the space of likelihoods, in order to use the independence hypothesis to simplify computations and numerical simulations. Let us denote by l the starting level of all particles and by $\mathcal{S}$ the following region:

$$\mathcal{S} := \left\{ (l_1, l, l) \in \mathbb{R}^{\mathbb{3}^+} | l_1 \in [l, l^*] \right\}, \tag{2.1}$$

where $l^* = 2lm_\varepsilon$, $m_\varepsilon = \frac{1-\varepsilon}{\varepsilon}$ and $\varepsilon \in (0, \frac{2}{3})$.

Remark 2.1.1. The upper limit of ε is $\frac{2}{3}$ for a simple reason: if the N posterior probabilities Π_i, $i \in I$, start with uniform prior distribution, we have $(\pi_1, \pi_2, \pi_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This means that if $\varepsilon > \frac{2}{3}$ we have $1 - \varepsilon < \frac{1}{3}$. This level of confidence is trivially satisfied at the start of the detection procedure, making the problem trivial. From now on, we will always consider non-trivial searching processes, that is, $\varepsilon \in (0, \frac{2}{3})$.

Here the upper boundary l^* can be written as $l^* = 2m_\varepsilon l_3$ and it is the threshold at which we declare that the particle with largest likelihood is the correct one, that is, the threshold at which we stop our search and we are at least $1 - \varepsilon$ confident that the chosen particle has drift $\mu \neq 0$. In particular, the threshold is reached at the stopping time (1.30).

Remark 2.1.2. From now on, we will refer to the set $\mathcal{S}$ as the *south boundary*, for reasons that will become apparent in the next chapter. Note that, if we start with a uniform prior distribution l, and we apply the FTL strategy, we will always work in the set $\mathcal{S}$. In other words, by applying the FTL strategy we will never leave the set $\mathcal{S}$.

In the case of uniform prior distribution it is difficult to define the FTL strategy at the boundary $\{t \geq 0 | L_1^\pi(t) \simeq L_2^\pi(t) = L_3^\pi(t)\}$: as we have seen in section 1.3, it might be difficult to identify which particle is the leader and we might have to observe one particle for a very short time before switching to another one and try again. Luckily, the scenario is very similar to the one discussed in section 1.3 and we refer to those results to formally define the FTL strategy in this particular case as well. As we have seen in Chapter 1, the value function associated to the FTL strategy is the unique solution of the ODE (1.34) with $i = 1$ and appropriate boundary conditions. Let us now establish such boundary conditions: as defined in (1.28), the value function of the FTL problem, as much as the value associated to any strategy $J \in \mathbf{J}'$, can be interpreted as the expectation of the $\mathcal{Y}_t$- stopping time (1.30) (in the space of likelihoods). Hence, we have

$$V_J(l) \geq 0$$

for all $l \in \mathbb{R}^3$ and any strategy $J \in \mathbf{J}'$. In particular, we can see that $V_J(l) = 0$ if and only if $l = l^*$, as at l^* the target confidence is attained and we stop our search. From this we can say that

$$V_{FTL}(l^*, l, l) = 0, \tag{2.2}$$

where V_{FTL} denotes the value function associated to the FTL policy. Deriving the left boundary condition is less trivial and to do so we propose the following approach: consider the space of ordered likelihoods $(L_1^\pi(t), L_2^\pi(t), L_3^\pi(t))$, where $L_1^\pi(t) \geq L_2^\pi(t) \geq L_3^\pi(t)$. Assume that the prior distribution l is such that $l = (l_3, l_3, l_3)$, so that all 3 particles start from the same level $l_3 > 0$. Consider the set

$$D_{l_3} := \{t \geq 0 | L_1^\pi(t) \geq L_2^\pi(t) \geq L_3^\pi(t) = l_3\}. \tag{2.3}$$

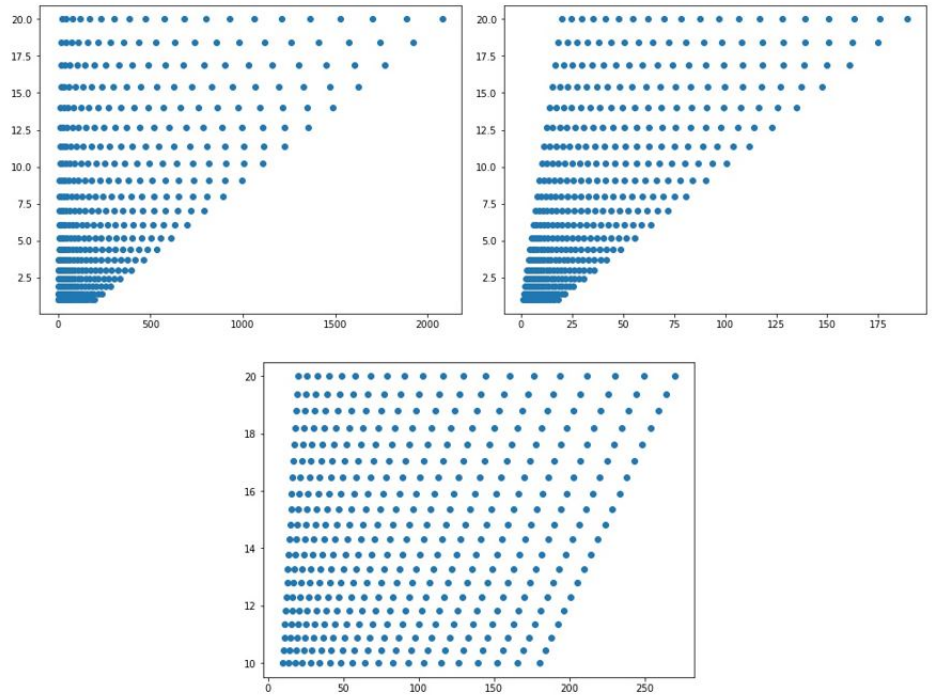

Fig. 2-1: The domain D_{l_3} for different values of the parameters ε and l_3. On the top-left we have $\varepsilon = 0.01$ and $l_3 = 1$. On the top-right we have $\varepsilon = 0.1$ and $l_3 = 1$. At the bottom we have $\varepsilon = 0.01$ and $l_3 = 10$. The dots represent the grid we used to discretise the domain D_{l_3}.

Remark 2.1.3. In equation (2.3) we are taking $L_3^\pi(t) = l_3$. This does not necessarily imply that we will never observe $L_3^\pi(t)$, as we can exploit Remark 1.1.11 to renormalize $L_3^\pi(t)$ if $L_3^\pi(t) < l_3$.

Figure 2-1 shows the shape of D_{l_3} for fixed values of l_3 and ε: on the x-axis we have the L_1^π-coordinate and on the y-axis we have the L_2^π-coordinate. The line at the bottom of the plots, corresponding to $L_2^\pi(t) = l_3$, is the set $\mathcal{S}$, hence we named it *south boundary*. On the bottom-left corner, we have the starting position $l = (l_3, l_3, l_3)$. On the left side we have the boundary $\{t \geq 0 | L_1^\pi(t) = L_2^\pi(t) \geq l_3\}$, which is a natural boundary for the FTL strategy. On the right we have the *stopping boundary* $L_1^\pi(t) = m_\varepsilon(L_2^\pi(t) + l_3) = l^*(t)$, at which our search terminates. The horizontal line at the top of the plot, that we will often call *north boundary*, is arbitrary. Note that the parameters l_3 and ε influence both the length of the south boundary $\mathcal{S}$ and the steepness of the right boundary $l^*(t)$.

Remark 2.1.4. Note that the bigger ε, the shorter the south boundary: this is coherent with the fact that ε represents the "difficulty" of the detection problem. The larger ε, the smaller the necessary confidence $1 - \varepsilon$ to finish our search.

Therefore, it makes sense that the south boundary "shrinks" as ε gets larger.

Note that having a uniform prior distribution means that we are always starting the search at the bottom-left of the domain. Starting from there, at (l_3, l_3, l_3), if we apply the FTL policy, we will always move along the south boundary $\mathcal{S}$. Therefore, the natural left boundary for the value of the FTL strategy would be the point (l_3, l_3, l_3). However, as we will show shortly, a slightly different approach is more fruitful. Let us denote by FTS the *Follow the Second* strategy, that is, the strategy that always observes the process with associated second-largest likelihood L_2^π. In the same way, we denote by FTT the strategy that always observes L_3^π. Recall now the result claimed by [42]: if we start with a uniform prior distribution, that is, if we start our search on the bottom-left corner (l_3, l_3, l_3) of D_{l_3}, the optimal strategy is to always follow the particle with maximum likelihood $L_{i^*}^\pi(t) = \max_i L_i^\pi(t)$. If the claim is correct, it implies that we will always move along the south boundary $\mathcal{S}$ and never leave it. As pointed out in the Introduction, in [17] the authors have shown that the FTL policy is not always optimal if the prior distribution is unspecified, but they did not obtain any result in the case where the prior distribution is uniform. The main goal of this chapter is to show that, even with uniform prior distribution, the FTL policy is not always optimal. To obtain such a result we ask ourselves the following question: can we find any point $l \in \mathcal{S}$ at which it is "better" to leave $\mathcal{S}$? That is, is there any point $l \in \mathcal{S}$ at which the value associated to an alternative strategy (either FTS or FTT) is smaller than the value of the FTL policy? Intuitively speaking, having a uniform prior distribution such that $L_1^\pi(0) = L_2^\pi(0) = L_3^\pi(0) = l_3$, makes it impossible to distinguish between the FTL strategy and the other competing strategies FTS and FTT. In practice, to apply the FTL policy we would need to observe a particle for a short amount of time and see whether the particle is actually the leader or if it comes back to the initial value l_3. In the latter case, we select another particle and we run the procedure once more. It is easy to see that this behavior is connected naturally to the local time behavior introduced in section 1.3. Define the line

$$\mathcal{S}_\delta := \{(l, l_3 + \delta, l_3) | \quad l \in [l_3 + \delta, m_\varepsilon(2l_3 + \delta)]\}, \tag{2.4}$$

where $\delta > 0$, $\delta << 1$. Note that $\mathcal{S}_\delta \to \mathcal{S}$ as $\delta \to 0^+$. By the scaling property of the value function (see Remark 1.1.11) we can rewrite the value of the FTL strategy V_0 at the left boundary of the set $\mathcal{S}_\delta$, that is at $(l_3 + \delta, l_3 + \delta, l_3)$ as $V_0(l_3 + \delta, l_3 + \delta, l_3) = V_0(l_3, l_3, l_3 - \delta')$, where $\delta' = \delta + O(\delta)$. By Theorem 1.1.13 we can permute the arguments and obtain

$$V_0(l_3, l_3, l_3 - \delta') = V_0(l_3 - \delta', l_3, l_3). \tag{2.5}$$

Remark 2.1.5. Note how (2.5) requires the hypothesis $N = 3$. We will discuss how this reasoning can be generalized to higher dimensions at the end of the chapter.

Let us therefore compute the value of the FTL strategy on the new domain

$$\mathcal{S}' := \{(l, l_3, l_3)| \quad l \in [l_3 - \delta', 2m_\varepsilon l_3]\}.$$

We compute the value of the FTL strategy on $\mathcal{S}'$ as the unique solution of

$$\begin{aligned} &\frac{\partial V_0}{\partial l_1}\frac{\mu^2 l_1^2}{l_1 + 2l_3} + \frac{1}{2}\frac{\partial^2 V_0}{\partial l_1^2}\mu^2 l_1^2 + 1 = 0, \\ &V_0(2m_\varepsilon l_3) = 0, \\ &V_0(l_3 - \delta) = \beta_\delta, \end{aligned} \tag{2.6}$$

where $\beta_\delta > 0$ is a constant to be determined.

Remark 2.1.6. From now on we will simply write $V_0(l)$ rather than $V_0(l, l_3, l_3)$ to simplify the notation. Note that such a simplification is appropriate, as the value function of any strategy that observes only particle i, $i \in I$, depends only on the likelihood L_i^π, as likelihoods are independent. In fact, the value will also depend on the level of other likelihoods, but these will be treated as fixed parameters, rather than variables. For example, the most appropriate notation for the value of the FTL strategy at a generic point (l_1, l_3, l_3) is $V_0(l_1; l_3, l_3)$, but we will simply write $V_0(l_1)$.

Remark 2.1.7. In principle β_δ could be any positive real number, as β_δ is a boundary condition we set arbitrarily. In the following we will show that the boundary value problem (2.6) has a unique solution on $\mathcal{S}'$ for all values of β_δ. However, in Algorithm 2.3.1, we will show how to choose a unique value for β_δ.

Remark 2.1.8. We will postpone the proof of the existence and uniqueness of the solution to the boundary value problem (2.6), given $\beta_\delta > 0$, to the next section, where we will prove a more general statement that implies the aforementioned result.

Let us now compute the unique solution of the boundary value problem (2.6). The general solution of (2.6) on $\mathcal{S}'$ is of the form

$$V_0(l) = \frac{2\log(l)(2l_3 - l)}{\mu^2(2l_3 + l)} - \frac{k_1^0}{2l_3 + l} + k_2^0, \tag{2.7}$$

where k_0^1 and k_0^2 are constants to be determined. Using the first boundary condition we get

$$k_2^0 = -\frac{2(1 - m_\varepsilon)\log(2m_\varepsilon l_3)}{\mu^2(m_\varepsilon + 1)} + \frac{k_1^0}{2l_3(m_\varepsilon + 1)}, \tag{2.8}$$

and by the second boundary condition we obtain

$$k_1^0 = \frac{2l_3(3l_3 - \delta)(m_\varepsilon + 1)}{l_3(1 - 2m_\varepsilon) - \delta}\left\{\beta_\delta - \frac{2(l_3 + \delta)\log(l_3 - \delta)}{\mu^2(3l_3 - \delta)} + \frac{2(1 - m_\varepsilon)\log(2m_\varepsilon l_3)}{\mu^2(m_\varepsilon + 1)}\right\}. \tag{2.9}$$

2.2 The δ-FTL strategy

To compute the correct value of β_δ we introduce a new strategy, that we call the δ-FTL strategy. Such a strategy is defined on

$$\mathcal{S}'_\delta := \mathcal{S}_\delta \cup \left\{(l_3+\delta, l, l_3) \in \mathbb{R}^3 | l \in [l_3, l_3+\delta]\right\} \cup \mathcal{S}'$$

and can be interpreted as follows: start at any point on $\mathcal{S}'_\delta$ and run the leader L_1^π until it hits either $l = l_3 + \delta$ or $l = m_\varepsilon(2l_3 + \delta)$. If it hits the latter first we stop, the target has been reached and we declare that particle X_1 has drift μ with confidence $1 - \varepsilon$. If it hits $l = l_3 + \delta$ first, we keep observing L_1^π and we see if it comes back to $\mathcal{S}_\delta$ or goes down to $(l_3 + \delta, l_3, l_3) \in \mathcal{S}'$. If it hits $(l_3 + \delta, l_3, l_3)$ we switch to the FTL strategy.

Remark 2.2.1. One of the main reason why we introduced the δ-FTL strategy is to understand the local time behavior at (l_3, l_3, l_3) that we already discussed in Section 1.3: note how the δ-FTL policy corresponds to observing a particle at (l_3, l_3, l_3) and wait until it becomes clear whether it is the leader.

Remark 2.2.2. The FTL and the δ-FTL strategy are inherently "coupled": the value of the problem at the left boundary of $\mathcal{S}_\delta$, $(l_3 + \delta, l_3 + \delta, l_3)$, is equal to the value of the problem at $(l_3 - \delta', l_3, l_3)$, that is,

$$V(l_3 + \delta, l_3 + \delta, l_3) = V(l_3 - \delta', l_3, l_3),$$

as we already pointed out when we defined the set $\mathcal{S}'$. Such a point lies in both $\mathcal{S}_\delta$ and $\mathcal{S}'$.

To define the value of the δ-FTL strategy V_δ properly we need to characterise it as the unique solution of a boundary value problem. The right boundary condition is similar to the one for V_0: at the stopping time τ_l, described in equation (1.30), the likelihood of the leader L_1^π is $l_\delta^* := m_\varepsilon(2l_3 + \delta)$. Hence we must have $V_\delta(l_\delta^*) = 0$. As far as the left boundary condition is concerned, we assign to it an arbitrary value, that is, $V_\delta(l_3 + \delta, l_3, l_3) = \alpha_\delta(\beta_\delta) > 0$.

Remark 2.2.3. The left boundary condition for V_δ, $\alpha_\delta(\beta_\delta)$, does depend on the left boundary condition for V_0, β_δ. Indeed, as the 2 strategies are effectively just the same strategy applied on different domains, we must have

$$V_\delta(l_3) = V_0(l_3 + \delta) = \alpha_\delta(\beta_\delta).$$

We are now going to characterize the value function of the δ-FTL strategy, V_δ, as the unique solution of the following boundary value problem:

$$\begin{aligned} &f''(l, f'(l)) = F(l, f'(l)) = -\frac{2}{\mu^2 l^2} - \frac{2}{l + 2l_3 + \delta} f'(l). \qquad (2.10)\\ &f(m_\varepsilon(2l_3 + \delta)) = 0,\\ &f(l_3 + \delta) = \alpha_\delta(\beta_\delta), \end{aligned}$$

Fig. 2-2: In orange the domain of V_δ, $\mathcal{S}'_\delta$, in blue the domain of V_0, $\mathcal{S}'$. The left boundaries of (2.6) and (2.10) are denoted by blue points. The right boundaries, at which the values V_0 and V_δ are 0, are denoted by black points.

where $\alpha_\delta(\beta_\delta) > 0$ is a constant to be determined and $f \in \mathcal{C}^2(\mathcal{S}_\delta)$. Firstly, we need to show that the boundary value problem (2.10) has a unique solution, given $\alpha_\delta(\beta_\delta)$. Note that $\alpha_\delta(\beta_\delta)$ is determined by β_δ. As we mentioned in Remark 2.1.7, this result will also prove the existence and uniqueness of the solution to the boundary value problem (2.6), given β_δ.

Proposition 2.2.4. The boundary value problem (2.10) has a unique solution on $\mathcal{S}'_\delta$, given $\beta_\delta > 0$ (hence $\alpha_\delta(\beta_\delta)$).

To prove Proposition 2.2.4 we need to prove a few intermediate results. Let us start by proving the following lemma:

Lemma 2.2.5. Let f be a solution to the boundary value problem (2.10). The second derivative of f with respect to l_1, f'', is Lipschitz in f' and continuous in l.

Proof. Consider equation (2.10) and take $l', l'' \in \mathcal{S}'$ and compute

$$|f''(l') - f''(l'')| = \left| -\frac{2}{\mu^2(l')^2} + \frac{2}{\mu^2(l'')^2} - \frac{2}{l' + 2l_3 + \delta} f'(l') + \frac{2}{l'' + 2l_3 + \delta} f'(l'') \right| \leq$$
$$\left| -\frac{2}{\mu^2(l')^2} + \frac{2}{\mu^2(l'')^2} \right| + \left| \frac{2}{l'' + 2l_3 + \delta} f'(l'') - \frac{2}{l' + 2l_3 + \delta} f'(l') \right|.$$

Let us focus on the second term, as the first one does not depend on f'. Since $l', l'' \in \mathcal{S}'$, we have $l', l'' \geq l_3 - \delta$. Hence

$$\left| \frac{2}{l'' + 2l_3 + \delta} f'(l'') - \frac{2}{l' + 2l_3 + \delta} f'(l') \right| \leq \frac{2}{3l_3 - \delta} |f'(l'') - f'(l')| =$$
$$\frac{2}{3l_3 - \delta} |f'(l') - f'(l'')|.$$

Hence $f''(l)$ is Lipschitz in $f'(l)$. Moreover $f''(l)$ is clearly continuous in l. □

Proof. (of Proposition 2.2.4)
Our goal is to apply the Picard-Lindelof theorem to show that there exists a solution of equation (2.6) in a local closed neighborhood of the initial point $l_3 - \delta'$ (see for example [22]). However, Picard-Lindelof theorem applies to initial value problems and not to boundary value problems. Therefore, before applying the theorem, we need to convert the boundary value problem (2.10) into an initial value problem. Let us rewrite the boundary value problem (2.10) as

$$\begin{aligned} &\frac{1}{2}\mu^2 l^2 f''(l) + \frac{\mu^2 l^2}{l + 2l_3 + \delta} f'(l) + 1 = 0 \\ &f(l_3 - \delta) = \beta_\delta \\ &f'(l_3 - \delta) = c, \end{aligned} \tag{2.11}$$

where $c \in \mathbb{R}$ is a constant we choose to make the initial value problem (2.11) equivalent to the boundary value problem (2.10). Let us consider the vector $V : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\underline{V}(l) = (V_1(l), V_2(l)) = (V(l), V'(l)),$$

with initial condition

$$\underline{V}(0) = (V_0, V_0').$$

We write $\underline{V}'(l) = F(\underline{V}, l)$, where $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is such that

$$F(\underline{V}, l) = (V_2, G(V_2, l)),$$

where $G : \mathbb{R} \times [l_3 - \delta', 2m_\varepsilon l_3] \to \mathbb{R}$. Therefore the second order initial value problem (2.11) can be rewritten as the following first order initial value problem:

$$\begin{aligned} &x'(l) = G(x(l), l) \\ &x(0) = x_0, \end{aligned} \tag{2.12}$$

where x is such that

$$\begin{aligned} &V_2(l) = x(l) \\ &V_1(l) = y_0 + \int_{l_3 - \delta}^{l} x(s) ds. \end{aligned} \tag{2.13}$$

By Lemma 2.2.5 we know that $G(l, x(l))$ is Lipschitz in $x(l)$ and continuous in l. To apply Picard-Lindelof we need to show that G is locally Lipschitz in l. By equation Lemma 2.2.5 the function G is continuously differentiable in l, which implies that G is locally Lipschitz in l. By Picard-Lindelof theorem we obtain the existence and uniqueness of the solution to (2.11) on a neighborhood of the initial point $l_3 - \delta \in \mathcal{S}'$. However, we would like to obtain the existence and

uniqueness of the solution to (2.11) on the entire domain $\mathcal{S}'$. The existence of the global solution is relatively easy to obtain, as the domain $\mathcal{S}'$ is a compact subset of $\mathbb{R}$: let us denote the local solution in a neighborhood of $l \in \mathcal{S}'$ by y_l. We can then "glue" the local solutions y_l together to obtain a global solution y such that $y(l) = y_l(l)$, for all $l \in \mathcal{S}'$. To prove the uniqueness of the global solution we need to show the boundedness of the gradient. By (2.12) we have

$$x(l) \leq \int_{l_3-\delta}^{l} |G(s, x(s))|ds + x_0 \leq$$
$$x_0 + l(\hat{K}l + G(l_3 - \delta, x(l_3 - \delta)))e^{Kl},$$

for all $l \in \mathcal{S}'$, where the second inequality uses the two Lipschitz conditions and Gronwall's lemma (see [22]). Here K is the x-Lipschitz constant and $\hat{K}$ is the l-Lipschitz constant. Consider now 2 solutions $x(l)$ and $\hat{x}(l)$ to the ODE (2.12) such that $|x(0) - \hat{x}(0)| < \delta$. Then

$$|x(l) - \hat{x}(l)| \leq |x(0) - \hat{x}(0)| + \int_{l_3-\delta}^{l} |G(s, x(s)) - G(s, \hat{x}(s))|ds \leq$$
$$\delta + K \int_{l_3-\delta}^{l} |x(s) - \hat{x}(s)|ds.$$

By Gronwall's inequality we obtain

$$|x(l) - \hat{x}(l)| \leq \delta \exp\left(K(l - l_3 + \delta)\right). \tag{2.14}$$

Denote the solution $x(l)$ as $x(l; x_0)$ to stress the dependence on the initial condition. Then, by (2.14) we get that the function $x_0 \mapsto x(s; x_0)$ is continuous. Since the integral operator is also continuous we have that the map $x_0 \mapsto \int_{l_3-\delta}^{2m_\varepsilon l_3} x(s; x_0)ds$ is continuous as well. Note that such an integral can be written as

$$\int_{l_3-\delta}^{2m_\varepsilon l_3} x(l)dl = y(2m_\varepsilon l_3) - y(l_3 - \delta). \tag{2.15}$$

Hence we can rewrite the initial value problem (2.11) as a boundary value problem of type (2.13) with boundary conditions given by (2.15). Therefore, the boundary value problem (2.6), equivalent to the initial value problem (2.11), can be rewritten as the boundary value problem (2.13) with boundary value condition (2.15), which has a unique solution. Hence, the boundary value problem (2.10) has a unique solution as well. □

The general solution of (2.10) has the form

$$V_\delta(l) = \frac{2\log(l)(2l_3 - l + \delta)}{\mu^2(l + 2l_3 + \delta)} + \frac{k_1^\delta}{l + 2l_3 + \delta} + k_2^\delta, \tag{2.16}$$

where k_1^δ and k_2^δ are constant to be determined. Considering the first boundary condition we get

$$k_2^\delta = \frac{2(m_\varepsilon - 1)\log(m_\varepsilon(2l_3+\delta))}{\mu^2(m_\varepsilon+1)} - \frac{k_1^\delta}{(m_\varepsilon+1)(2l_3+\delta)}, \tag{2.17}$$

and by the second boundary condition we have

$$k_1^\delta = \frac{(m_\varepsilon+1)(2l_3+\delta)(3l_3+\delta)}{l_3(2m_\varepsilon-1)+\delta m_\varepsilon}\Bigg\{\alpha_\delta(\beta_\delta) - \frac{2(l_3+\delta)\log(l_3)}{\mu^2(3l_3+\delta)} + \tag{2.18}$$

$$\frac{2(1-m_\varepsilon)\log(m_\varepsilon(2l_3+\delta))}{\mu^2(m_\varepsilon+1)}\Bigg\}. \tag{2.19}$$

Equations (2.7), (2.9), (2.8) and (2.16), (2.18), (2.17), describe the value function of the FTL and δ-FTL policy, respectively. Let us summarise the results of this section in the following theorem.

Theorem 2.2.6. *The functions $V_0 : \mathcal{S} \to \mathbb{R}$, $V_\delta : \mathcal{S} \to \mathbb{R}$ are the value functions of the FTL and $\delta - FTL$ strategy, respectively. They can be characterized as the unique solutions of boundary value problems* (2.6) *and* (2.10)*, respectively. In particular, we have that $V_0 \in \mathcal{C}^2(\mathcal{S})$ and $\mathcal{C}^2(\mathcal{S}_\delta)$.*

2.3 Numerical simulation of V_0 and V_δ

In the previous section we obtained closed- form solutions for the value functions V_0 and V_δ. However, they still depend on the arbitrary value $\beta_\delta > 0$ for the left boundary condition of V_0 in (2.6). To solve this issue we have to solve the following *fixed point* equation:

$$\alpha_\delta(\beta_\delta) = V_0(l_3+\delta;\beta_\delta). \tag{2.20}$$

To solve (2.20) we propose the following algorithm:

Algorithm 2.3.1. set $i = 0$ and fix a tolerance level $t(\delta)$. Choose an arbitrary $\beta_\delta^i > 0$. Then

1. compute $V_0(l;\beta_\delta^i)$;
2. set $\alpha_\delta^i = V_0(l_3+\delta;\beta_\delta^i)$;
3. compute $V_\delta(l;\alpha_\delta^i)$;
4. evaluate the difference $d_{\beta_\delta^i} := |\beta_\delta^i - V_\delta(l_3+\delta;\alpha_\delta^i)|$. If $d_{\beta_\delta^i} \le t(\delta)$ stop the procedure, otherwise set $\beta_\delta^{i+1} = V_\delta(l_3+\delta;\alpha_\delta^i)$, $i = i+1$ and repeat the cycle.

Let us study the behavior of the value functions V_0 and V_δ and analyse their dependence on the parameters μ and ε. Note that we are able to simulate V_0 and V_δ using the fixed point algorithm 2.3.1, setting the appropriate left boundary conditions for V_0 and V_δ, that is, the correct values for β_δ and $\alpha_\delta(\beta_\delta)$. In the followings, the tolerance function $t(\delta)$ will always be of the form $t(\delta) = \frac{\delta^2}{100}$. In Figure 2-3 we can see the behaviour of the difference $d_{\beta^i_\delta} = |\beta^i_\delta - V_\delta(l_3 + \delta; \alpha^i_\delta)|$ for different values of β^0_δ and δ, as a function of the number of iterations of the algorithm. We note that such a difference goes to 0 quickly independently on the values of β^0_δ and δ.

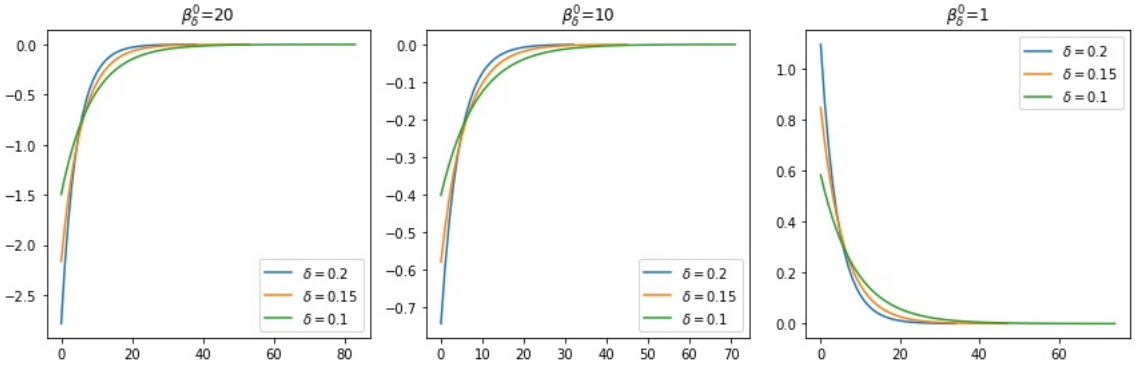

Fig. 2-3: The difference $d_{\beta^i_\delta}$ of Algorithm 2.3.1 as a function of the number of iterations, for several values of β^0_δ and δ. Note how $d_{\beta^i_\delta}$ goes to 0 independently of β^0_δ and δ.

From Figure 2-4 and Figure 2-5 we se that:

1. the detection problem becomes easier as ε gets larger: this makes sense, as a larger ε leads to a smaller tolerance level $1 - \varepsilon$. For example, in Figure 2-4 we see that for $\varepsilon = 0.3$, that is $1 - \varepsilon = 0.7$, the average searching is much smaller than the value when $\varepsilon = 0.1$, that is $1 - \varepsilon = 0.9$;

2. a similar behaviour can be observed with respect to μ, in Figure 2-5: a larger drift makes more evident more quickly which particle is the correct one, and this leads to significantly smaller values for larger values of μ. As an example, see the sharp difference between the value V_0 with $\mu = 1$ and V_0 with $\mu = 10$.

Figure 2-6 and Figure 2-7 show similar results for the value of the δ-FTL strategy V_δ. Similar interpretations to the ones provided for V_0 can be done for V_δ. In Figure 2-8 we compare the close form solution of the boundary value problem (2.6) to a Monte Carlo approximation of the value functions V_0. The Monte Carlo algorithm is structured as follows:

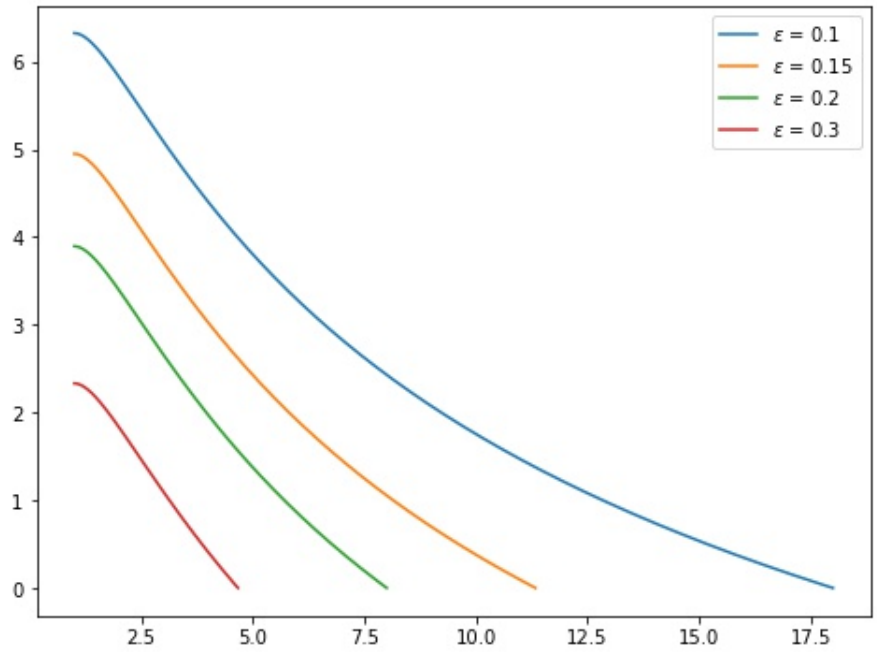

Fig. 2-4: The FTL value V_0 for different values of the tolerance parameter ε. Here $\mu = 1$, $l_3 = 1$ and $\delta = 10^{-4}$. The domain $\mathcal{S}'$ is discretised with a grid of 10^3 points. Note that the amplitude of $\mathcal{S}'$ depends on ε and δ. We can see that the smaller ε (hence the larger the tolerance $1 - \varepsilon$) the quicker the value goes to 0.

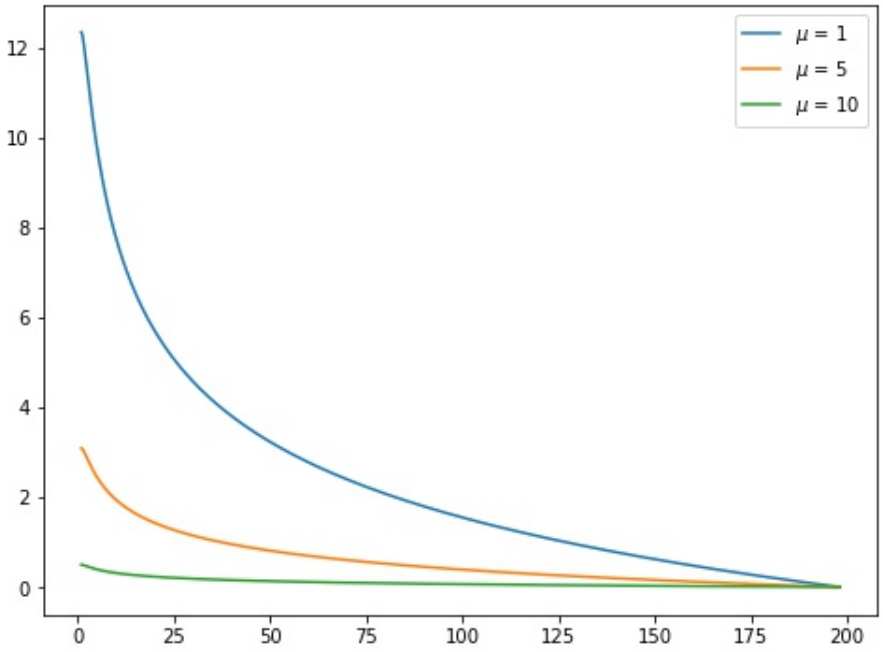

Fig. 2-5: The FTL value V_0 for different values of the drift μ. Here $\varepsilon = 0.01$, $l_3 = 1$ and $\delta = 10^{-4}$. The domain $\mathcal{S}'$ is discretised with a grid of 10^3 points. The larger the drift μ, the smaller the average searching time: indeed a very large drift becomes evident more quickly than a small one, which makes the problem easier for the detection algorithm.

Algorithm 2.3.2. 1. discretise $\mathcal{S}'$ with a grid of N points. Denote such a grid by $\mathcal{S}'_d$.

2. choose a point $\bar{l} = (l, l_3, l_3) \in \mathcal{S}'_d$ and denote by $\bar{l}_1$ the first coordinate of $\bar{l}$. Simulate the SDE of the likelihood process L_1^π (1.27) starting from $\bar{l}$, on a time interval dt up to the first exit time $\tau_{\bar{l}}$ of L_1^π from the interval $\mathcal{S}'$. Define a variable c that counts how long each simulation takes to terminate. Set $c = 0$;

3. after dt, check the value of $L_1^\pi(t)$:

   - if $l_3 - \delta < L_1^\pi(t) < l^*$, set $\bar{l}_1 = L_1^\pi(t)$ and come back to point 2. Set $c = c + 1$;
   - if $L_1^\pi(t) \leq l_3$ set $\bar{l}_1 = l_3$ and come back to point 2. Set $c = c + 1$;
   - if $L_1^\pi(t) \geq l^*$ stop the simulation;

4. repeat the procedure n times;

5. for each starting point $\bar{l} \in \mathcal{S}'_\delta$, average c over n simulations and multiply it for the time taken by each simulation dt.

Remark 2.3.3. The Monte Carlo simulation described in Algorithm 2.3.1 simulate the average time taken by a searching algorithm that applies the FTL strategy along the south boundary $\mathcal{S}'$: as we can see in Figure 2-8, this agrees with the solution of the boundary value problem (2.6).

2.4 Convergence of V_δ to V_0

As we pointed out in section 2.2, the δ-FTL strategy boils down to apply the FTL strategy on $\mathcal{S}'_\delta$. Since $\mathcal{S}'_\delta \to \mathcal{S}' \to \mathcal{S}$ as $\delta \to 0^+$, it stands to reason that V_δ should get "closer" to V_0 as $\delta \to 0^+$. In this section we will prove that this is the case. Let us start with a simple visualisation: in Figure 2-9 we see that as δ gets smaller the difference between V_δ and V_0 becomes smaller. To prove the result suggested by Figure 2-9, we note that the second-order differential operator in (2.10), $\mathcal{L}_\delta$, that is

$$\mathcal{L}_\delta(f(l)) = \frac{1}{2}\mu^2 l^2 f''(l) + \frac{\mu^2 l^2}{l + 2l_3 + \delta} f'(l) + 1, \tag{2.21}$$

tends to the second-order differential operator in (2.6), $\mathcal{L}_0$, that is

$$\mathcal{L}_0(f(l)) = \frac{1}{2}\mu^2 l^2 f''(l) + \frac{\mu^2 l^2}{l + 2l_3} f'(l) + 1, \tag{2.22}$$

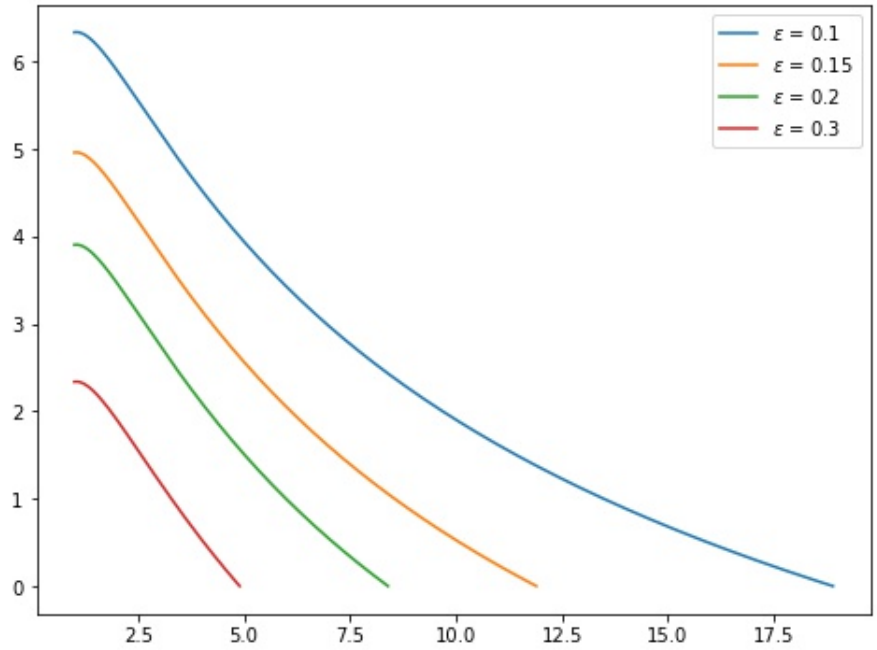

Fig. 2-6: The δ-FTL value V_δ for different values of the tolerance parameter ε. Here $\mu = 1$, $l_3 = 1$ and $\delta = 10^{-1}$. The domain $\mathcal{S}'_\delta$ is discretised with a grid of 10^3 points. Note that the amplitude of $\mathcal{S}'_\delta$ depends on ε and δ.

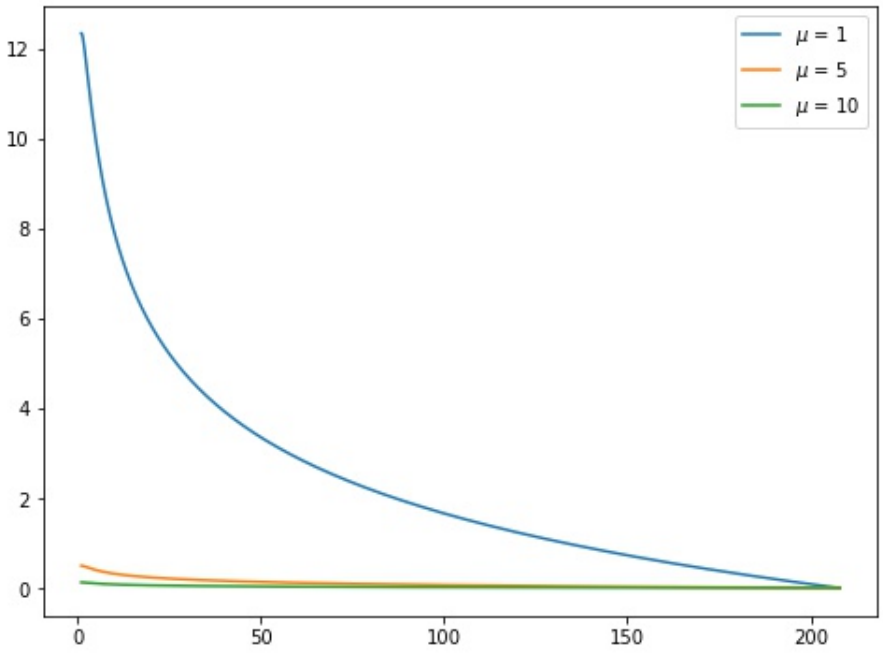

Fig. 2-7: The δ-FTL value V_δ for different values of the drift μ. Here $\varepsilon = 0.01$, $l_3 = 1$ and $\delta = 10^{-1}$. The domain $\mathcal{S}'_\delta$ is discretised with a grid of 10^3 points.

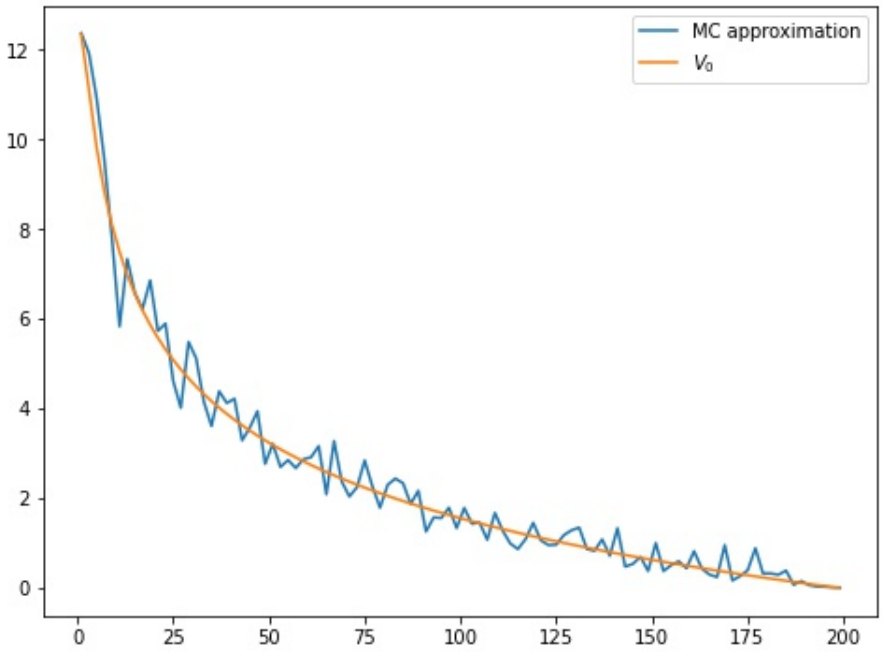

Fig. 2-8: The FTL value V_0 compared with a Monte Carlo simulation of the FTL strategy on $\mathcal{S}'$. Here $\varepsilon = 0.01$, $l_3 = 1$, $\delta = 10^{-2}$ and $\mu = 1$. The Monte Carlo simulation consists of 100 simulations for each of the 700 points of the grid over $\mathcal{S}'$. The time step is $dt = 10^{-4}$.

for all smooth functions $f \in \mathcal{C}^2(\mathcal{S}')$ as $\delta \to 0^+$. At the same time the right boundary conditions of (2.6) and (2.10) are both 0. Hence, to prove that $V_\delta \to V_0$ as $\delta \to 0^+$, we just need to show that the left boundary condition of (2.10) converges to the left boundary condition of (2.6), that is, we need to show that $\alpha_\delta(\beta_\delta) \to \beta_\delta$ as $\delta \to 0^+$.

Lemma 2.4.1. The left boundary condition of (2.10) converges to the left boundary condition of (2.6), that is,

$$\alpha_\delta(\beta_\delta) \to \beta_\delta,$$

as $\delta \to 0^+$.

Proof. Recall that

$$\alpha_\delta = V_0(l_3 + \delta) = -\frac{2(l_3 - \delta)\log(l_3 + \delta)}{\mu^2(3l_3 + \delta)} - \frac{k_1^0}{3l_3 + \delta} + k_2^0.$$

Let us make explicit the dependence of $\alpha_\delta(\beta_\delta)$ on β_δ: by (2.8) we have

$$\begin{aligned}\alpha_\delta(\beta_\delta) = &\frac{2(l_3 - \delta)\log(l_3 + \delta)}{\mu^2(3l_3 + \delta)} - \frac{2(1 - m_\varepsilon)\log(l^*)}{\mu^2(1 + m_\varepsilon)} \\ &+ \frac{\delta + l_3(1 - 2m_\varepsilon)}{2l_3(2l_3 + \delta)(1 + m_\varepsilon)} k_1^0. \end{aligned} \tag{2.23}$$

By equation (2.9) we can write the last term in (2.23)as

$$\begin{aligned}&\frac{\delta + l_3(1 - 2m_\varepsilon)}{2l_3(2l_3 + \delta)(1 + m_\varepsilon)} k_1^0 = \\ &\frac{l_3(1 - 2m_\varepsilon) + \delta}{l_3(1 - 2m_\varepsilon) - \delta} \left\{ \beta_\delta - \frac{2(l_3 + \delta)\log(l_3 - \delta)}{\mu^2(3l_3 - \delta)} + \frac{2(1 - m_\varepsilon)\log(l^*)}{\mu^2(1 + m_\varepsilon)} \right\}. \end{aligned} \tag{2.24}$$

Clearly, as $\delta \to 0^+$ (2.24) tends to

$$\beta_\delta - \frac{2\log(l_3)}{3\mu^2} + \frac{2(1 - m_\varepsilon)\log(l^*)}{\mu^2(1 + m_\varepsilon)},$$

so that equation (2.23) tends to β_δ. In particular, from (2.24) we can see that the order of convergence is $O(\delta)$. In particular, note that

$$\begin{aligned}\frac{\alpha_\delta(\beta_\delta) - \beta_\delta}{\delta} = \frac{1}{\delta} \Bigg\{ &\frac{2l_3(1 - 2m_\varepsilon)}{l_3(1 - 2m_\varepsilon) - \delta} \left(\beta_\delta + \frac{2(1 - m_\varepsilon)\log l^*}{\mu^2(1 + m_\varepsilon)} \right) + \frac{2(l_3 - \delta)\log(l_3 + \delta)}{\mu^2(3l_3 + \delta)} - \\ &\frac{2(l_3 + \delta)\log(l_3 - \delta)}{\mu^2(3l_3 - \delta)} \frac{l_3(1 - 2m_\varepsilon) + \delta}{l_3(1 - 2m_\varepsilon) - \delta} \Bigg\} \to 0,\end{aligned}$$

as $\delta \to 0^+$. □

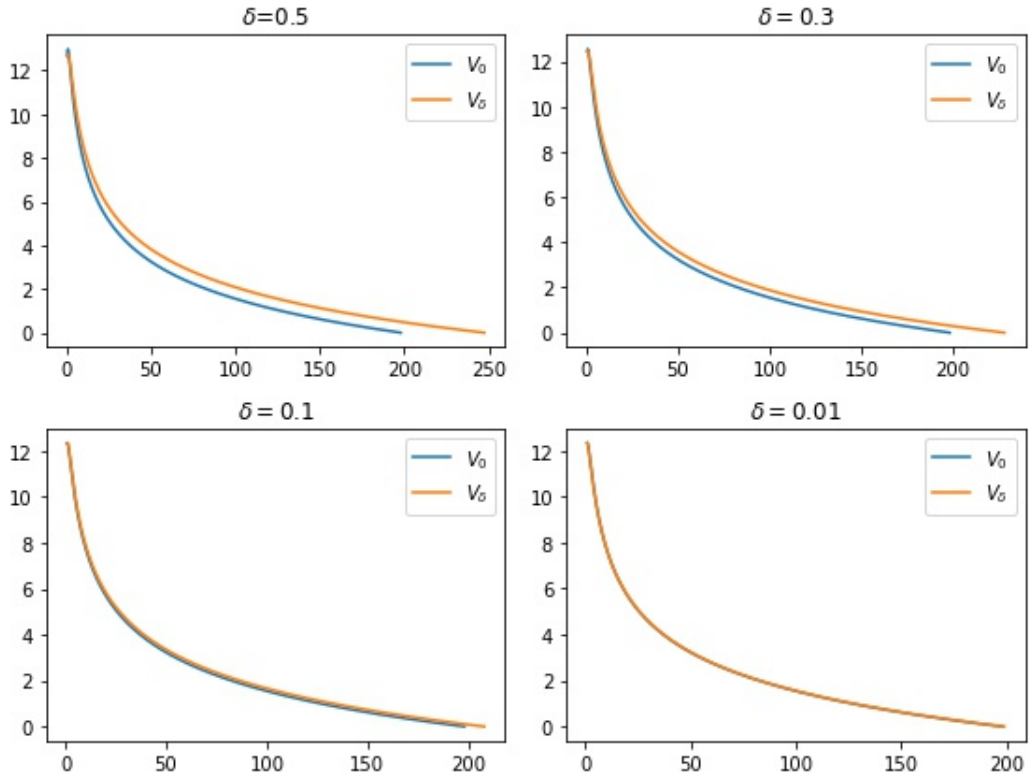

Fig. 2-9: The FTL value V_0 and the δ-FTL value V_δ on $\mathcal{S}'$ compared for fixed δ. Here $\varepsilon = 0.01$, $l_3 = 1$ and $\mu = 1$. The trend is clear: the smaller δ, the smaller the difference between V_0 and V_δ.

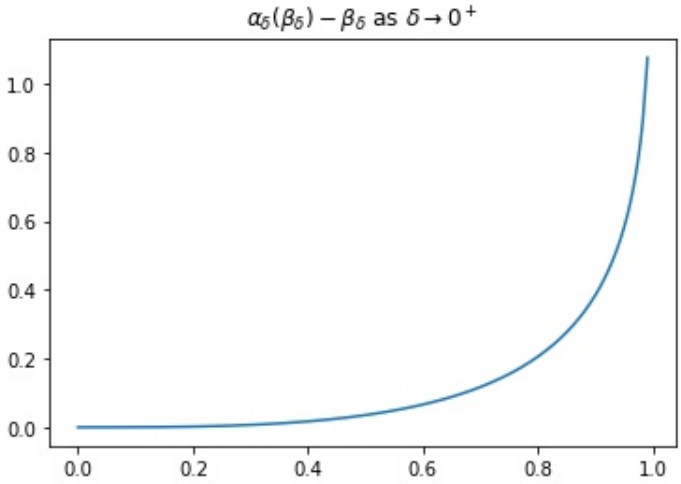

Fig. 2-10: An illustration of Lemma 2.4.1. On the x-axis the values of δ. We can see that the difference $\alpha_\delta(\beta_\delta) - \beta_\delta$ tends to 0 as $\delta \to 0^+$.

In Figure $2-10$ we can see an illustration of the result of Lemma 2.4.1. Therefore $V_\delta \to V_0$ as $\delta \to 0^+$.

Remark 2.4.2. We have proven that $V_\delta \to V_0$ as $\delta \to 0^+$, but we still do not have a way to compare V_δ and V_0. In particular, we would like to have a way to choose between the δ-FTL strategy and the FTL strategy when δ is small. To do so, let us introduce the function h.

2.5 The function h

In this section we are going to introduce the function h. As we mentioned above, our goal is to find a way to compare the FTL and δ-FTL policy, when δ is small. To do so we need to define a common domain over which to compare the value functions V_δ and V_0 for all $\delta > 0$. To do so consider the south boundary $\mathcal{S}$ introduced in (2.1) and note that $\mathcal{S} \subset \mathcal{S}'$ for all $\delta' > 0$. At the same time we have $\mathcal{S} \subset \mathcal{S}_\delta$, for all $\delta > 0$. Define the first-order differential operator $\hat{\mathcal{L}} : \mathcal{C}^1(\mathcal{S}) \to \mathbb{R}$ as

$$\hat{\mathcal{L}} f := \lim_{\delta \to 0^+} \frac{1}{\delta} \left(\mathcal{L}_\delta - \mathcal{L}_0 \right)(f), \tag{2.25}$$

for $f \in \mathcal{C}^1(\mathcal{S})$, and the linearised versions of the second-order differential operators $\mathcal{L}_\delta$ and $\mathcal{L}_0$ in (2.21) and (2.22) as

$$\mathcal{L}_\delta^{lin} f := \mathcal{L}_\delta f - 1 \tag{2.26}$$

$$\mathcal{L}_0^{lin} f := \mathcal{L}_0 f - 1, \tag{2.27}$$

for $f \in \mathcal{C}^2(\mathcal{S})$.

Theorem 2.5.1. *Consider the function $h : \mathcal{S} \to \mathbf{R}$ such that*

$$\begin{aligned} &\mathcal{L}_0^{lin} h = -\hat{\mathcal{L}} V_0 \\ &h(l_3) = -V_0'(l_3) \\ &h(l^*) = -m_\varepsilon V_0'(l^*). \end{aligned} \tag{2.28}$$

Then h is such that

$$\left|\left| \frac{V_\delta(l) - V_0(l) - \delta h(l)}{\delta} \right|\right|_\infty \to 0, \tag{2.29}$$

as $\delta \to 0^+$, for all $(l, l_3, l_3) \in \mathcal{S}$.

To prove such a result we structure the proof in three main steps:

- compute the operator $\hat{\mathcal{L}}$ explicitly;
- show that the limiting function (2.29) satisfies the boundary conditions of the boundary value problem (2.28);

- apply the strong maximum principle to show that h satisfies the limit (2.29).

Let us start by computing the operator $\hat{\mathcal{L}}$: by equations (2.25) we can see that for small $\delta > 0$ we have

$$\mathcal{L}_\delta f = \left(\mathcal{L}_0 + \delta\hat{\mathcal{L}}\right) f, \tag{2.30}$$

for $f \in \mathcal{C}^1(\mathcal{S})$. Therefore we have

$$\hat{\mathcal{L}} f(l) = -\frac{\mu^2 l^2}{(l + 2l_3 + \delta)(l + 2l_3)} f'(l), \tag{2.31}$$

for $f \in \mathcal{C}^1(\mathcal{S})$ and $\delta > 0$. This form is not satisfactory, as we need the operator $\hat{\mathcal{L}}$, as well as the function h, not to depend on δ. To remove such a dependence note that $\forall \delta > 0$ it exists a positive constant k such that

$$\left| -\frac{\mu^2 l^2}{(l + 2l_3 + \delta)(l + 2l_3)} + \frac{\mu^2 l^2}{(l + 2l_3)^2} \right| \leq \frac{\mu^2 l^2}{(l + 2l_3)^3}\delta + k\delta^2.$$

Hence we can write $\hat{\mathcal{L}}$ as

$$\hat{\mathcal{L}} f(l) = -\frac{\mu^2 l^2}{(l + 2l_3)^2} f'(l) + O(\delta^2), \tag{2.32}$$

for $f \in \mathcal{C}^2(\mathcal{S})$, so that

$$\mathcal{L}_\delta f(l) = \left(\mathcal{L}_0 + \delta\hat{\mathcal{L}}\right) f(l) = \left(\mathcal{L}_0 - \delta\frac{\mu^2 l^2}{(l + 2l_3)^2}\right) f(l),$$

for all $f \in \mathcal{C}^2(\mathcal{S})$.

Let us now show that the limiting function in (2.29) satisfies the right boundary condition of (2.28):

Lemma 2.5.2. Denote by $\hat{h} : \mathcal{S} \to \mathbf{R}$ the function such that

$$\hat{h}(l) = \lim_{\delta \to 0^+} \frac{V_\delta(l) - V_0(l)}{\delta},$$

for all $l \in \mathcal{S}$. Then $\hat{h}$ is such that

$$\hat{h}(l^*) = -m_\varepsilon V_0'(l^*). \tag{2.33}$$

Proof. By definition we have;

$$\hat{h}(l^*) = \lim_{\delta \to 0^+} \frac{V_\delta(l^*) - V_0(l^*)}{\delta} = \lim_{\delta \to 0^+} \frac{V_\delta(l^*)}{\delta},$$

as $V_0(l^*) = 0$ by (2.6). Note that the distance between the right boundary of $\mathcal{S}'_\delta$ and the right boundary of $\mathcal{S}$ is $|l^*_\delta - l^*| = \delta m_\varepsilon$. By Theorem 2.2.6 we have that V_δ is a $\mathcal{C}^2(\mathcal{S}_\delta)$ function, we can use the first-order approximation

$$V_\delta(l^*) = V_\delta(l^*_\delta) - \delta m_\varepsilon V'_\delta(l^*) + O(\delta^2).$$

By (2.10) we have

$$\hat{h}(l^*) = \lim_{\delta\to 0^+} \frac{1}{\delta}\left(V_\delta(l^*_\delta) - \delta m_\varepsilon V'_\delta(l^*)\right) = \lim_{\delta\to 0^+} -m_\varepsilon V'_\delta(l^*). \tag{2.34}$$

To get the right boundary condition in (2.28) we need to show that $\lim_{\delta\to 0^+} V'_\delta(l^*) = V'_0(l^*)$. To do so recall that the value function associated to any strategy $J \in \mathcal{J}'$, is the expected searching time under strategy J, given the initial position of the particles l. Consider a point $\bar{l}$ such that $l_3 < \bar{l} < l^*$. Consider two positive constants $\eta, \eta_\delta > 0$ such that $\frac{\eta_\delta}{l^*_\delta - \bar{l}_\delta} = \frac{\eta}{l^* - \bar{l}}$, for all $\delta > 0$. Denote by $H_l(\hat{l})$, for $l, \hat{l} \in \mathcal{S}$, the first hitting time of the likelihood L^π_1 starting from $\hat{l}$, that is $L^\pi_1(0) = \hat{l}$, of level l. For $l \in (\bar{l}, l^*)$, $l_\delta \in (\bar{l}_\delta, l^*_\delta)$, we can write the value functions at $l^* - \eta$ and $l^*_\delta - \eta_\delta$, for all $\delta > 0$, as

$$\begin{aligned} V_\delta(l^*_\delta - \eta_\delta) = {} & \mathbf{E}_\pi\left[V_\delta(\bar{l}_\delta)\mathbf{1}\left\{H_{\bar{l}_\delta}(l_\delta) < H_{l^*_\delta}(l_\delta)\right\} + V_\delta(l^*_\delta)\mathbf{1}\left\{H_{l^*_\delta}(l_\delta) < H_{\bar{l}_\delta}(l_\delta)\right\}\right] \\ & - \mathbf{E}_\pi\left[H_{l^*_\delta}(l_\delta) \wedge H_{\bar{l}_\delta}(l_\delta)\right], \end{aligned}$$

and

$$\begin{aligned} V_0(l^* - \eta) = {} & \mathbf{E}_\pi\left[V_0(\bar{l})\mathbf{1}\left\{H_{\bar{l}}(l) < H_{l^*}(l)\right\} + V_0(l^*)\mathbf{1}\left\{H_{l^*}(l) < H_{\bar{l}}(l)\right\}\right] \\ & - \mathbf{E}_\pi\left[H_{l^*}(l) \wedge H_{\bar{l}}(l)\right]. \end{aligned}$$

Since $V_0(l^*) = V_\delta(l^*_\delta) = 0$ we have

$$\begin{aligned} V_\delta(l^*_\delta - \eta_\delta) = {} & \mathbf{E}_\pi\left[V_\delta(\bar{l}_\delta)\mathbf{1}\left\{H_{\bar{l}_\delta}(l_\delta) < H_{l^*_\delta}(l_\delta)\right\}\right] \\ & - \mathbf{E}_\pi\left[H_{l^*_\delta}(l_\delta) \wedge H_{\bar{l}_\delta}(l_\delta)\right], \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} V_0(l^* - \eta) = {} & \mathbf{E}_\pi\left[V_0(\bar{l})\mathbf{1}\left\{H_{\bar{l}}(l) < H_{l^*}(l)\right\}\right] \\ & - \mathbf{E}_\pi\left[H_{l^*}(l) \wedge H_{\bar{l}}(l)\right], \end{aligned} \tag{2.36}$$

where $\mathbf{1}\{A\}$ is the indicator function of the set A. We can rewrite (2.35) and (2.36) as

$$V_\delta(l^*_\delta - \eta) = V_\delta(\bar{l}_\delta)\mathbf{P}_\pi\left(H_{\bar{l}_\delta}(l_\delta) < H_{l^*_\delta}(l_\delta)\right) - \mathbf{E}_\pi\left[H_{l^*_\delta}(l_\delta) \wedge H_{\bar{l}_\delta}(l_\delta)\right], \tag{2.37}$$

and

$$V_0(\bar{l} - \eta) = V_0(\bar{l})\mathbf{P}_\pi\left(H_{\bar{l}}(l) < H_{l^*}(l)\right) - \mathbf{E}_\pi\left[H_{l^*}(l) \wedge H_{\bar{l}}(l)\right]. \tag{2.38}$$

To solve equations (2.38) and (2.37) we have to compute probabilities of the type $\mathbf{P}_\pi(H_x < H_y)$ and expectations of the form $\mathbf{E}_\pi[H_x \wedge H_y]$, for $x, y \in \mathbb{R}$. These are easy to compute when the underlying process is a standard Brownian motion, whereas closed form expressions are either not known or difficult to compute in all other cases. The best approach is to construct an equivalent measure $\mathbf{Q}_\pi$ under which L_1^π is a $\mathbf{Q}_\pi$-standard Brownian motion.

Lemma 2.5.3. Denote by $\frac{d\mathbf{P}_\pi}{d\mathbf{Q}_\pi}(l)$, $l \in \mathcal{S}$, the *Radon-Nikodym derivative* of $\mathbf{P}_\pi$ with respect to $\mathbf{Q}_\pi$ (see for example [30]). Denote by $\mathbf{Q}_\pi$ the measure under which the likelihood process L_1^π is a standard Brownian motion. Then $\mathbf{Q}_\pi$ is absolutely continuous with respect to $\mathbf{P}_\pi$, the Radon-Nikodym derivative $\frac{d\mathbf{P}_\pi}{d\mathbf{Q}_\pi}(l)$ is well-defined on $\mathcal{S}$, and there exists a previsible process $(\gamma_s)_s$, satisfying Novikov's condition, such that

$$\frac{d\mathbf{P}_\pi}{d\mathbf{Q}_\pi}(l) = \exp\left\{-\int_0^{H_{\bar{l}}(l)\wedge H_{l^*}(l)} \gamma_s dW_s^\pi(l) - \frac{1}{2}\int_0^{H_{\bar{l}}(l)\wedge H_{l^*}(l)} \gamma_s^2 ds\right\},$$

holds true for all $l \in \mathcal{S}$. Moreover $\frac{d\mathbf{P}_\pi}{d\mathbf{Q}_\pi}(l) \to 1$ as $\bar{l} \to l^*$.

Proof. The measure $\mathbf{Q}_\pi$ is absolutely continuous with respect to $\mathbf{P}_\pi$ by standard results (see for example [38]). By the same token, the Radon-Nikodym derivative $\frac{d\mathbf{P}_\pi}{d\mathbf{Q}_\pi}$ is well-defined over $\mathcal{S}$. To show that $\frac{d\mathbf{P}_\pi}{d\mathbf{Q}_\pi}(l) \to 1$ as $\bar{l} \to l^*$ note that $H_{\bar{l}}(l) \wedge H_{l^*}(l) \to 0$ as $\bar{l} \to l^*$, for all $l \in (\bar{l}, l^*)$. In turn , this implies that $\frac{d\mathbf{P}_\pi}{d\mathbf{Q}_\pi}(l) \to 1$ as $\bar{l} \to l^*$, for all $l \in \mathcal{S}$. □

Noting that $H_{\bar{l}} \wedge H_{l^*}$ is an almost surely finite $\mathcal{Y}$-stopping time and that L_1^π is an almost surely bounded process, as $l_3 \leq L_1^\pi(l) \leq l^*$ for all $l \in \mathcal{S}$, we can apply Doob's optional stopping theorem (see for example [38]) to write

$$\mathbf{P}_\pi(H_{\bar{l}}(l) < H_{l^*}(l)) = \mathbf{E}_\pi^{\mathbf{P}}[\mathbf{1}\{H_{\bar{l}}(l) < H_{l^*}(l)\}] = \mathbf{E}_\pi^{\mathbf{Q}}\left[\frac{d\mathbf{P}_\pi}{d\mathbf{Q}_\pi}(l)\mathbf{1}\{H_{\bar{l}}(l) < H_{l^*}(l)\}\right]$$
$$\xrightarrow{\bar{l}\to l^*} \mathbf{E}_\pi^{\mathbf{Q}}[\mathbf{1}\{H_{\bar{l}}(l) < H_{l^*}(l)\}] = \frac{\eta}{l^* - \bar{l}}, \tag{2.39}$$

and

$$\mathbf{P}_\pi\left(H_{\bar{l}_\delta}(l_\delta) < H_{l_\delta^*}(l_\delta)\right) \xrightarrow{\bar{l}_\delta \to l_\delta^*} \frac{\eta_\delta}{l_\delta^* - \bar{l}_\delta}. \tag{2.40}$$

We can also compute the expectation of $H_{l^*} \wedge H_{\bar{l}}$ (and similarly for $H_{l_\delta^*} \wedge H_{\bar{l}_\delta}$) by applying Doob's optional stopping theorem on the martingale $L_1^2(l) - l$. Then we get

$$\mathbf{E}_\pi^{\mathbf{Q}}[H_{l^*}(l) \wedge H_{\bar{l}}(l)] = \frac{\eta}{l^* - \bar{l}}(\bar{l}^2 - (l^*)^2) + (l^*)^2 \tag{2.41}$$

Similarly we have,

$$\mathbf{E}_\pi^{\mathbf{Q}}\left[H_{l_\delta^*}(l_\delta) \wedge H_{\bar{l}_\delta}(l_\delta)\right] = \frac{\eta_\delta}{l_\delta^* - \bar{l}_\delta}(\bar{l}_\delta^2 - (l_\delta^*)^2) + (l_\delta^*)^2. \tag{2.42}$$

Putting (2.41) and (2.39) together we obtain

$$V_0(l^* - \eta) = \frac{\eta}{l^* - \bar{l}} \left(V_0(\bar{l}) - (\bar{l}^2 - (l^*)^2) \right) - (l^*)^2, \tag{2.43}$$

and, by (2.42) and (2.40), we have

$$V_\delta(l^*_\delta - \eta_\delta) = \frac{\eta_\delta}{l^*_\delta - \bar{l}_\delta} \left(V_\delta(\bar{l}_\delta) - (\bar{l}^2_\delta - (l^*_\delta)^2) \right) - (l^*_\delta)^2. \tag{2.44}$$

By (2.43) and (2.44) we see that

$$\lim_{\eta \to 0^+} \lim_{\delta \to 0^+} \frac{V_\delta(l^*_\delta - \eta_\delta)}{\eta_\delta} = \lim_{\eta \to 0^+} \frac{V_0(l^* - \eta)}{\eta} = V_0'(l^*), \tag{2.45}$$

as $V_\delta \to V_0$ as $\delta \to 0^+$ by Lemma (2.4.1) and $l^*_\delta \to l^*$ as $\delta \to 0^+$. Therefore h satisfies (2.33). □

We now study the left-hand side boundary.

Lemma 2.5.4. The limiting function $\hat{h} : \mathcal{S} \to \mathbf{R}$ is such that

$$\hat{h}(l_3) = -V_0'(l_3). \tag{2.46}$$

Proof. By definition $\hat{h}$ satisfies (2.29). Therefore we have

$$\begin{aligned} \hat{h}(l_3) &= \lim_{\delta \to 0^+} \frac{V_\delta(l_3) - V_0(l_3)}{\delta} \\ &= \lim_{\delta \to 0^+} \frac{V_\delta(l_3) - (V_0(l_3 - \delta) + \delta V_0'(l_3 - \delta))}{\delta} \\ &= \lim_{\delta \to 0^+} \frac{(\alpha_\delta(\beta_\delta) - \beta_\delta) - \delta V_0'(l_3 - \delta)}{\delta}, \end{aligned} \tag{2.47}$$

where we used a first order Taylor's approximation in the second equality. By applying Lemma 2.4.1 to equation (2.47) we obtain

$$\hat{h}(l_3) = -V_0'(l_3), \tag{2.48}$$

which concludes the proof. □

We are now going to show that h satisfies the limit (2.29) in the domain $int(\mathcal{S})$.

Lemma 2.5.5. Consider the function $\hat{V} : \mathcal{S} \to \mathbf{R}$ defined as $\hat{V}(l) := V_0(l) + \delta h(l)$, for $\delta > 0$. Then there exists a positive constant $k > 0$ such that

$$||V_\delta(l) - \hat{V}(l)||_\infty \leq k\delta^2,$$

for $l \in \mathcal{S}$. In particular, we have that for all $\delta > 0$, there exists $k > 0$ such that

$$||\mathcal{L}_\delta^{lin}(V_\delta(l) - \hat{V}(l))||_\infty \leq k\delta^2,$$
$$V_\delta(l_3) = \hat{V}(l_3) + k\delta^2$$
$$V_\delta(l_\delta^*) = \hat{V}(l_\delta^*) + k\delta^2.$$

where $\mathcal{L}_\delta^{lin}$ is the linearised version of the differential operator $\mathcal{L}_\delta$.

Proof. Note that $\mathcal{L}_\delta^{lin}$ is uniformly elliptic, as there exist a constant $\lambda > 0$ such that

$$\frac{1}{2}(\mu l)^2 > \lambda,$$

for all $l \in \mathcal{S}$. In particular, we may take $\lambda = \frac{1}{2}(\mu l_3)^2$. We note that $\mathcal{L}_\delta^{lin}$ has constant term $c = 0$. Define the function $u := \hat{V} - V_\delta$. Our goal is to apply the strong maximum principle for uniformly elliptic operators (see for example [22]): to do so we need to verify that $\mathcal{L}_\delta^{lin} u \geq 0$, which in turns requires to check that $\mathcal{L}_\delta^{lin}\hat{V} \geq -1$. To check such a condition note that

$$\mathcal{L}_\delta^{lin}\hat{V} = \mathcal{L}_\delta^{lin}(V_0 + \delta h) = (\mathcal{L}_0^{lin} + \delta\hat{\mathcal{L}})(V_0 + \delta h) = -1 + \delta^2\hat{\mathcal{L}}h, \tag{2.49}$$

as $V_0 \in \mathcal{C}^2(\mathcal{S})$ and $h \in \mathcal{C}^1(\mathcal{S})$, where we used (2.28) and (2.30). Hence, there exist $\xi > 0$ such that

$$\mathcal{L}_\delta^{lin} u = \mathcal{L}_\delta^{lin}(\hat{V} - V_\delta) \geq \delta^2\hat{\mathcal{L}}h \geq -\xi,$$

by (2.10) and (2.49). Our goal is to find a function $g : \mathcal{S} \to \mathbf{R}$ such that $\mathcal{L}_\delta^{lin} g \geq 1$, in such a way that the function $\hat{u} := \hat{V} - V_\delta + \xi g$ is such that

$$\mathcal{L}_\delta^{lin}\hat{u} \geq 0.$$

Let us choose $g(l) = Bl$, where $B \in \mathbb{R}$. To choose B we need to solve $\mathcal{L}_\delta^{lin} g \geq 1$: from doing so we get

$$B \geq \frac{l + 2l_3 + \delta}{\mu^2 l^2}.$$

Let us take

$$B := \frac{l^* + 2l_3 + \delta}{\mu^2 l_3^2}$$

Thus, for $g(l) = Bl$ we have

$$\mathcal{L}_\delta^{lin}\hat{u}(l) \geq 0.$$

As $\hat{u}$ is clearly not a constant function, by the strong maximum principle we have that $\sup_{l\in\mathcal{S}} \hat{u}(l)$ is attained at the boundary of the domain ∂S. In particular, we see that

$$\arg\max_{l\in\mathcal{S}} \hat{u}(l) = l_3.$$

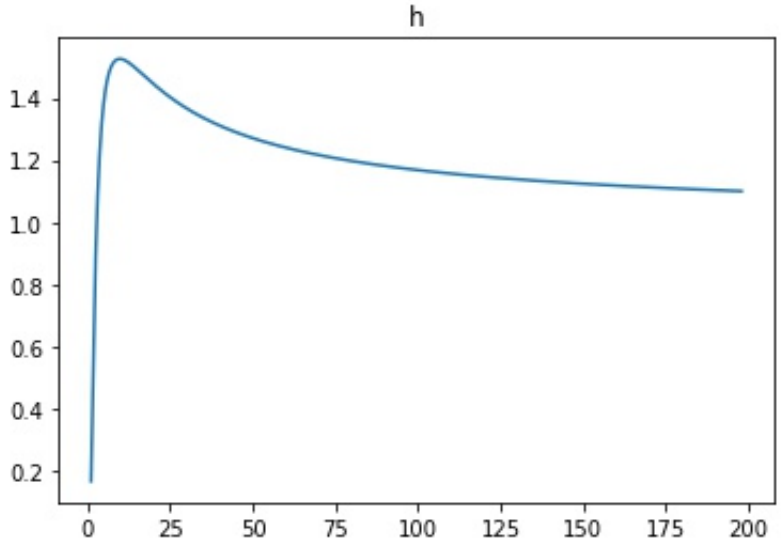

Fig. 2-11: The function h simulated using equation (2.29). Here $\varepsilon = 0.01$, $l_3 = 1$, $\mu = 1$. We consider $\delta = 10^{-5}$ to simulate the limit (2.29).

Moreover , since $g(l) \geq 0$ for all $l \in \mathcal{S}$, we have

$$||\hat{V} - V_\delta||_\infty \leq ||\hat{V} - V_\delta + \xi g||_\infty = \hat{V}(l_3) - V_\delta(l_3) + \xi g(l_3)$$
$$= -\delta V_0'(l_3) + \xi g(l_3),$$

where the last equality uses the left boundary condition of h in equation (2.48). Note that the dependence of V_0' on δ is described by the constant k_1^0 in equation (2.9): thus, it has the same order of convergence of α_δ to β_δ, which we proved to be $O(\delta)$ in Lemma 2.4.1. As g is clearly linear in δ, we may conclude that it exists $k > 0$ such that

$$||\hat{V} - V_\delta||_\infty \leq k\delta^2.$$

□

Putting Lemma 2.5.5, 2.5.4 and 2.5.2 together we see that the solution of the boundary value problem (2.28) satisfies the limit (2.29). Figure 2-11 shows a simulation of h as the limit (2.29). Figure 2-12 shows a comparison between h computed as the limit (2.29) and h obtained as solution of the boundary value problem (2.28).

2.6 A counterexample to the optimality of the FTL strategy along the south boundary

The goal of this section is to show that we can find an alternative strategy which gives a smaller expected searching time than the FTL policy for some

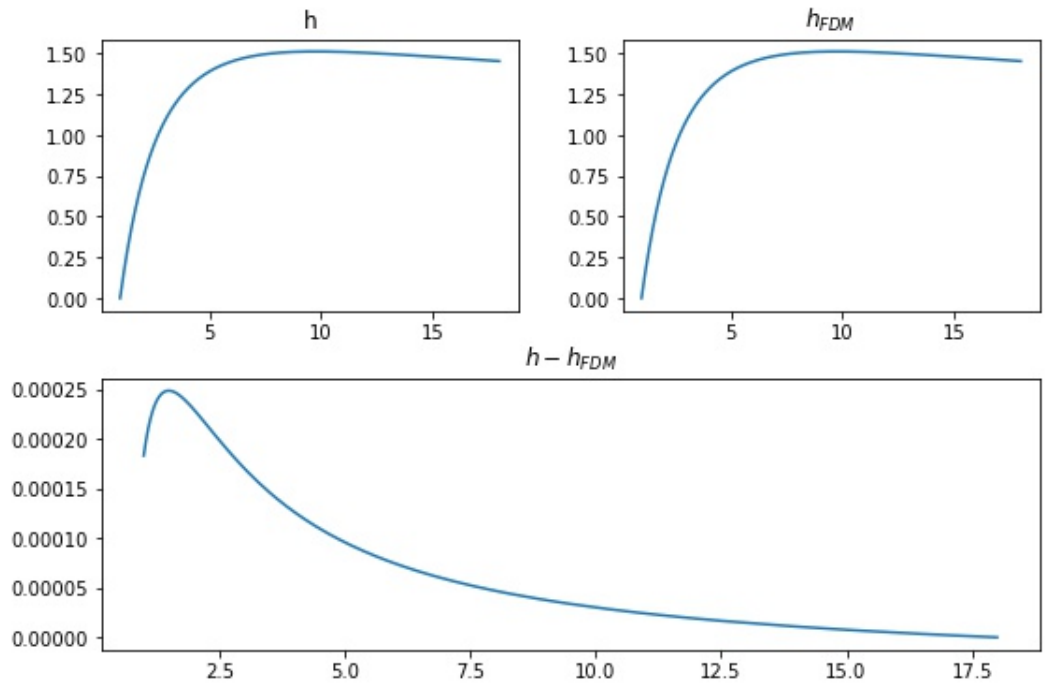

Fig. 2-12: A comparison between h simulated as the limit (2.29) (on the top-left) and h as solution of the boundary value problem (2.28) (on the top-right). Here we used a finite difference method with $\varepsilon = 0.1$, $l_3 = 1$, $\mu = 1$. We considered $\delta = 10^{-5}$ and discretised the south boundary with $N = 10^5$ points. The difference between the 2 solutions is shown in plot at the bottom of the figure.

$l \in \mathcal{S}$ and a certain set of parameters $\mu, l_3 > 0$ and $\varepsilon \in (0, \frac{2}{3})$. This is clearly in contradiction with the main claim of [42]: as we mentioned in the Introduction, the results of [42] were deemed as unclear by the authors in [17], but they did not provide a counterexample to the main claim. Let us start from a generic point $(l, l_3, l_3) \in \mathcal{S}$ and consider 2 possible strategies:

FTO: the Follow The Other (FTO) strategy prescribes to apply either the FTS or the FTT (Follow the Third) strategy, that is, to move to either $(l, l_3 - \delta^*, l_3)$ or $(l, l_3 + \delta, l_3)$. The parameter δ^* has to be chosen in such a way that $\frac{l_3^2}{l_3 - \delta^*} = l_3 + \delta$, $\delta > 0$;

FTL: move to either $(l - \eta, l_3, l_3)$ or $(l + \eta, l_3, l_3)$ where $\eta > 0$ is chosen such that

$$\mathbf{E}_{FTL}[H_{l-\eta}(l) \wedge H_{l+\eta}(l)] = \mathbf{E}_{FTO}[H_{l+\delta}(l) \wedge H_{l-\delta^*}(l)], \tag{2.50}$$

where $H_{l+\eta}(l)$ (respectively $H_{l-\eta}(l)$) is the first hitting time of $(l + \eta, l_3, l_3)$ (respectively $(l - \eta, l_3, l_3)$) starting from (l, l_3, l_3) and $H_{l+\delta}(l)$ (respectively $H_{l-\delta^*}(l)$) is the first hitting time of $(l, l_3 + \delta, l_3)$ (respectively $(l, l_3 - \delta^*, l_3)$) starting from (l, l_3, l_3).

Remark 2.6.1. Intuitively speaking, at $(l, l_3, l_3) \in \mathcal{S}$ the second and third particles, L_π^2 and L_π^3, are at the same level. Hence we cannot distinguish between the FTS and FTT strategy and the FTO policy consists in just observing either L_π^2 or L_π^3. If the particle goes "up" by δ, we are actually applying the FTS policy and we reach the point $(l, l_3 + \delta, l_3) \in \mathcal{S}_\delta$. If it goes "down" by δ^*, we we are actually applying the FTT policy and we can use the scalability property of the value function (see Remark 1.1.11) to ensure that the value at $(l, l_3 - \delta^*, l_3)$ is equal to the value of a corresponding point $(l + \hat{\delta}, l_3 + \delta, l_3) \in \mathcal{S}_\delta$. In this section we will study how to properly choose η, δ^* and $\hat{\delta}$.

Theorem 2.6.2. *Let $g : \mathcal{S} \to \mathbf{R}$ be the function defined by*

$$g(l) := h(l) + \frac{l}{2l_3} V_0'(l). \tag{2.51}$$

Then the FTL strategy is suboptimal at $(l, l_3, l_3) \in \mathcal{S}$ if $g(l) < 0$.

Proof. To set appropriate η and δ^* recall that the likelihood process L_π^1 under the FTL policy is a $\mathbf{P}_\pi$- Brownian motion with drift $\mu > 0$ and $L_\pi^1(0) = l \in [l_3, 2m_\varepsilon l_3]$. At $(l, l_3, l_3) \in \mathcal{S}$ the residual expected searching time under the FTL strategy is

$$\mathbb{E}_{FTL}^{\eta, -\eta} := \mathbf{P}_\pi(H_{l+\eta}(l) < H_{l-\eta}(l)) V_0(l+\eta) + \mathbf{P}_\pi(H_{l-\eta}(l) < H_{l+\eta}(l)) V_0(l-\eta), \tag{2.52}$$

whereas the residual expected searching time under the FTO strategy is

$$\mathbb{E}_{FTO}^{\delta, -\delta^*} := \mathbf{P}_\pi(H_{l_3+\delta}(l_3) < H_{l_3-\delta^*}(l_3)) V_\delta(l_3+\delta) + \mathbf{P}_\pi(H_{l_3-\delta^*}(l_3) < H_{l_3+\delta}(l_3)) V_\delta(l_3-\delta^*). \tag{2.53}$$

Once more we need to compute probabilities of the type $\mathbf{P}_\pi(H_x < H_y)$ for some $x, y \in \mathbf{R}$. To do so, we can apply the same methods we used in section 2.5: find an equivalent measure $\mathbf{Q}_\pi$ under which the likelihood process is a standard Brownian motion. We refer to section 2.5 for a complete explanation of such methods. We have

$$\mathbf{P}_\pi(H_{l+\eta} < H_{l-\eta}) = \mathbb{E}^{\mathbf{P}}_\pi(\mathbb{1}\{H_{l+\eta} < H_{l-\eta}\}) = \mathbb{E}^{\mathbf{Q}}_\pi\left(\frac{d\mathbf{P}_\pi}{d\mathbf{Q}_\pi}\mathbb{1}\{H_{l+\eta} < H_{l-\eta}\}\right).$$

Once more, note that as $\eta \to 0^+$, $H_{l-\eta} \wedge H_{l+\eta} \to 0^+$, so that $\frac{d\mathbf{P}_\pi}{d\mathbf{Q}_\pi} \to 1$. Therefore, for $\eta \approx 0^+$, we have

$$\mathbf{P}_\pi(H_{l+\eta}(l) < H_{l-\eta}(l)) = \mathbb{E}^{\mathbf{Q}}_\pi\left(\mathbb{1}\{H_{l+\eta}(l) < H_{l-\eta}(l)\}\right) = \mathbf{Q}_\pi(H_{l+\eta}(l) < H_{l-\eta}(l)).$$

Since L^1_π is a $\mathbf{Q}_\pi$- Brownian motion we have

$$\mathbf{P}_\pi(H_{l+\eta}(l)(< H_{l-\eta}(l)) = \mathbf{Q}_\pi(H_{l+\eta}(l) < H_{l-\eta}(l)) = \frac{l+\eta-l}{l+\eta-(l-\eta)} = \frac{\eta}{2\eta} = \frac{1}{2}, \tag{2.54}$$

which is consistent with L^1_π being a $\mathbf{Q}_\pi$- standard Brownian motion. By the same reasoning we compute $\mathbf{P}_\pi(H_{l_3+\delta}(l) < H_{l_3-\delta^*}(l))$ as

$$\mathbf{P}_\pi(H_{l_3+\delta}(l_3) < H_{l_3-\delta^*}(l_3)) = \mathbf{Q}_\pi(H_{l_3+\delta}(l_3) < H_{l_3-\delta^*}(l_3)) = \frac{\delta^*}{\delta+\delta^*}, \tag{2.55}$$

where $\frac{l_3^2}{l_3-\delta^*} = l_3+\delta$. Here the idea is that we want to find a point $(l+\hat{\delta}, l_3+\delta, l_3) \in \mathcal{S}_\delta$ such that $V(l+\hat{\delta}, l_3+\delta, l_3) = V(l, l_3-\delta^*, l_3)$, for some $\hat{\delta} > 0$. This can be done by exploiting the scalability property (see Remark 1.1.11): we have that $\hat{\delta}$ is such that $l + \hat{\delta} = l\frac{l_3}{l_3-\delta^*}$, for all $(l, l_3+\delta, l_3) \in \mathcal{S}_\delta$. Hence we can rewrite (2.55) as

$$\mathbf{Q}_\pi(H_{l_3+\delta}(l_3) < H_{l_3-\delta^*}(l_3)) = \mathbf{Q}_\pi(H_l < H_{l+\hat{\delta}}) = \frac{l_3}{2l_3+\delta}. \tag{2.56}$$

By (2.54) and (2.56) we obtain

$$\mathbf{E}_{FTL}[H_{l-\eta}(l) \wedge H_{l+\eta}(l)] = l^2 + \eta^2, \tag{2.57}$$

$$\mathbf{E}_{FTO}[H_l \wedge H_{l+\hat{\delta}}] = l^2\tfrac{l_3}{2l_3+\delta} + l^2\left(\tfrac{l_3+\delta}{l_3}\right)^2\tfrac{l_3+\delta}{2l_3+\delta}. \tag{2.58}$$

Hence we select η, δ, δ^* in such a way that

$$l^2 + \eta^2 = l^2\frac{l_3}{2l_3+\delta} + l^2\left(\frac{l_3+\delta}{l_3}\right)^2\frac{l_3+\delta}{2l_3+\delta},$$

that is,

$$\eta = \sqrt{\delta\frac{l^2(l_3+\delta)}{l_3^2}}.$$

We can now rewrite (2.52) and (2.53) as

$$\mathbf{E}_{FTL}^{\eta,-\eta} = \tfrac{1}{2}V_0\left(l + \sqrt{\delta\tfrac{l^2(l_3+\delta)}{l_3^2}}\right) + \tfrac{1}{2}V_0\left(l - \sqrt{\delta\tfrac{l^2(l_3+\delta)}{l_3^2}}\right) \tag{2.59}$$

$$\mathbf{E}_{FTO}^{\delta,-\delta^*} = \tfrac{l_3}{2l_3+\delta}V_\delta(l) + \tfrac{l_3+\delta}{2l_3+\delta}V_\delta(l+\hat{\delta}). \tag{2.60}$$

Note that, as $\delta \to 0^+$, (2.59) tends to

$$\mathbf{E}_{FTL}^{\eta,-\eta} = \frac{1}{2}V_0(l) + \frac{1}{2}V_0(l) = V_0(l).$$

As $V_\delta = V_0 + \delta h + \mathcal{O}(\delta^2)$, and taking $\delta \to 0^+$, we can rewrite (2.60) as

$$\mathbf{E}_{FTO}^{\delta,-\delta^*} = V_0(l) + \delta\left(h(l) + \frac{l}{2l_3}V_0'(l)\right). \tag{2.61}$$

Therefore the FTL strategy is suboptimal at $(l, l_3, l_3) \in \mathcal{S}$ if

$$h(l) + \frac{l}{2l_3}V_0'(l) < 0, \tag{2.62}$$

which proves our claim. □

We now have a criterion to understand at which point and for which values of the parameters l_3, μ and $\varepsilon \in (0, \frac{2}{3})$ the FTL policy returns a lower expected searching time with respect to the alternative FTO policy. Let us note that the FTL value V_0 in equation (2.7), with constants (2.9), (2.8), is clearly decreasing as

$$V_0'(l) = \frac{l(\mu^2 k_1^0 - 2l) + 8l_3(l_3 - l\log(l))}{\mu^2 l(l+2l_3)^2} \leq 0,$$

for all $(l, l_3, l_3) \in \mathcal{S}$, $l_3 > 0$ and $\mu > 0$. As we do not have an analytical solution to the boundary value problem (2.29), we have to rely on numerical methods to study the sign of g. Let us start by noting that

$$g(l_3) = h(l_3) + \frac{1}{2}V_0'(l_3) = -V_0'(l_3) + \frac{1}{2}V_0'(l_3) = -\frac{1}{2}V_0'(l_3), \tag{2.63}$$

$$g(l^*) = h(l^*) + \frac{2m_\varepsilon l_3}{2l_3}V_0'(l^*) = -m_\varepsilon V_0'(l^*) + m_\varepsilon V_0'(l^*) = 0. \tag{2.64}$$

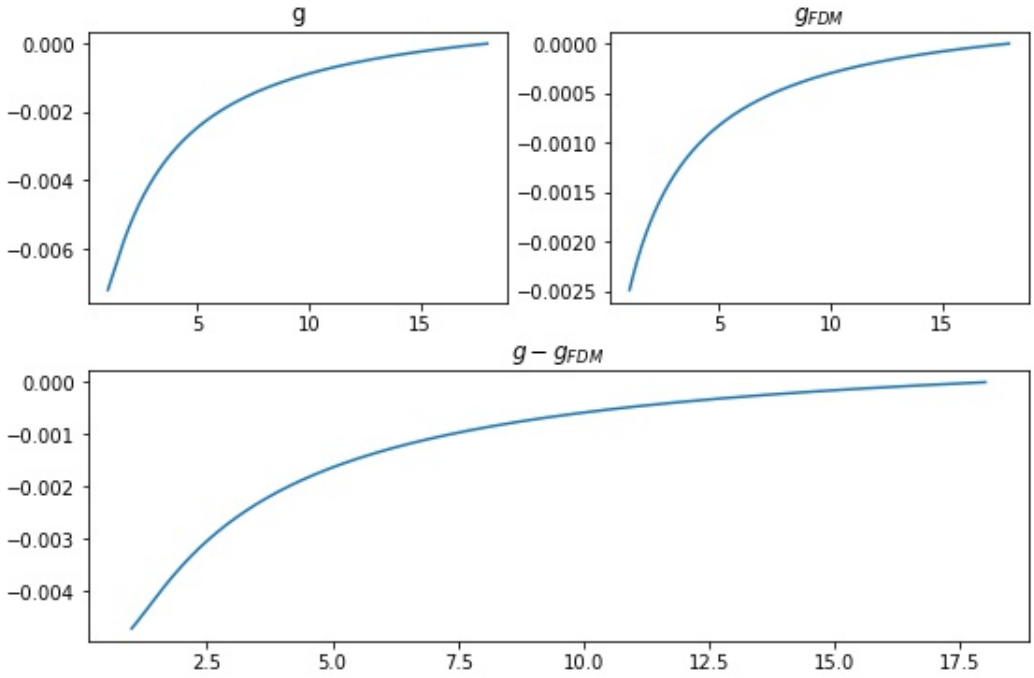

Fig. 2-13: A comparison between g simulated through the limiting process (on the top-left) and g constructed using h obtained by finite difference method (on the top-right). Here we used a finite difference method with $\varepsilon = 0.1$, $l_3 = 1$, $\mu = 1$. We considered $\delta = 10^{-5}$ and discretised the south boundary with $N = 10^5$ points. The difference between the 2 solutions is shown in plot at the bottom of the figure.

To simulate the function g numerically, we are going to use the finite difference approximation of the function h, whereas we have the closed form solution of the gradient of V_0, V_0'. We will also compare it to the "limiting" version of g, that is, the function g computed using as h the function that numerically satisfies limit (2.29). As we can see in Figure 2-13, the function g is negative for all $(l, l_3, l_3) \in \mathcal{S}$, for $l_3 = 1$, $\mu = 1$ and $\varepsilon = 0.1$. This is in direct contradiction with the claim of [42]: in such a paper the author proposes the FTL policy as the optimal strategy on the south boundary $\mathcal{S}$. This would imply that the function g, as defined in (2.51), should be positive, $g(l) > 0$, for any $(l, l_3.l_3) \in \mathcal{S}$. This is in contradiction with our numerical findings in Figure $2 - 13$. As we already mentioned, the proof of the optimality of the FTL policy on the south boundary $\mathcal{S}$ offered in [42] is considered unclear by both us and the authors of [17]: in the latter paper the authors showed that the FTL strategy is not always optimal when the prior distribution is not uniform, but did not obtain any conclusive findings in the case where the prior distribution is uniform. We can now confirm that the FTL strategy is not optimal even on the south boundary $\mathcal{S}$, at least not for all $\varepsilon \in (0, \frac{2}{3})$. Note that, as shown in Figure 2-14, there are values of ε for

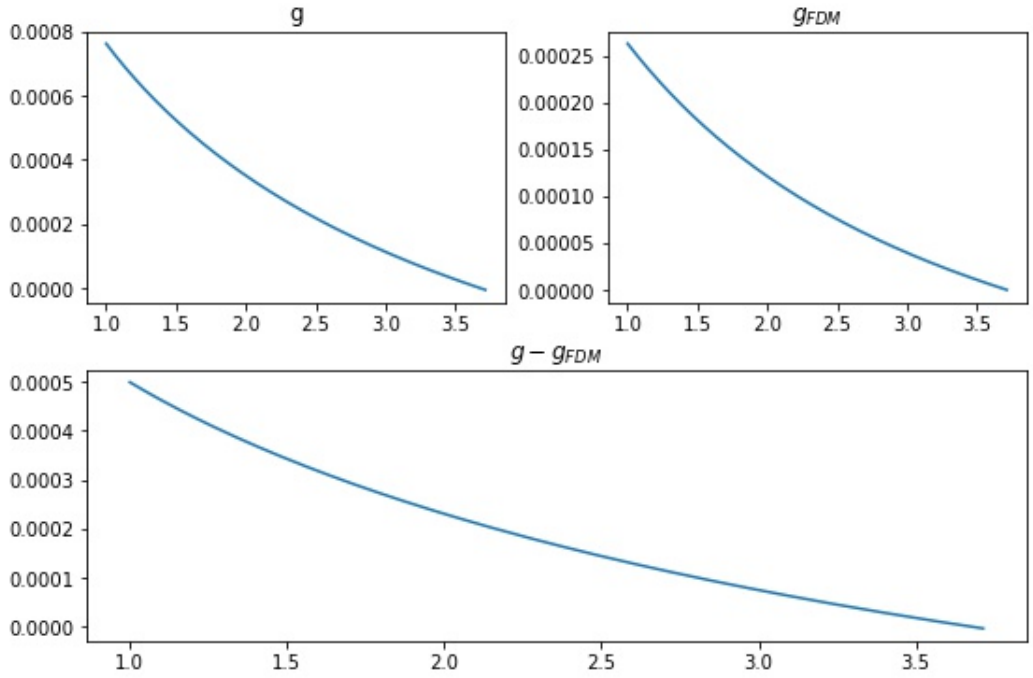

Fig. 2-14: A comparison between g simulated through the limiting process (on the top-left) and g constructed using h obtained by finite difference method (on the top-right). Here we used a finite difference method with $\varepsilon = 0.4$, $l_3 = 1$, $\mu = 1$. We considered $\delta = 10^{-5}$ and discretised the south boundary with $N = 10^5$ points. The difference between the 2 solutions is shown in plot at the bottom of the figure.

which $g(l) > 0$ for all $l \in \mathcal{S}$.

Remark 2.6.3. A comparison between Figure 2-13 and 2-14 shows that the optimality of the FTL strategy depends on the tolerance parameter ε. This finding is quite interesting and, to some extent, counter intuitive: recall that ε sets the level of confidence $1 - \varepsilon$ that we require to declare which particle has the drift $\mu \neq 0$. This makes clear that the smaller ε, the larger the confidence $1 - \varepsilon$ required, the longer the average time needed to reach such a level of confidence. However, considering the result we have just shown, we can say that the value of ε affects not only the average searching time, but the optimal strategy as well.

We still do not have a clear result to show for which values of ε the FTL strategy is optimal, but we can see in Figure $2-15$ that $g(l) > 0$ for $(l, l_3, l_3) \in \mathcal{S}$, with $l \in [0.35, 0.7]$. Note that:

1. even for $(l, l_3, l_3) \in \mathcal{S}$ and $\varepsilon \in [0.35, 0.7]$, we cannot conclude that the FTL strategy is optimal, only that it yields a lower expected searching time than the alternative FTO policy;

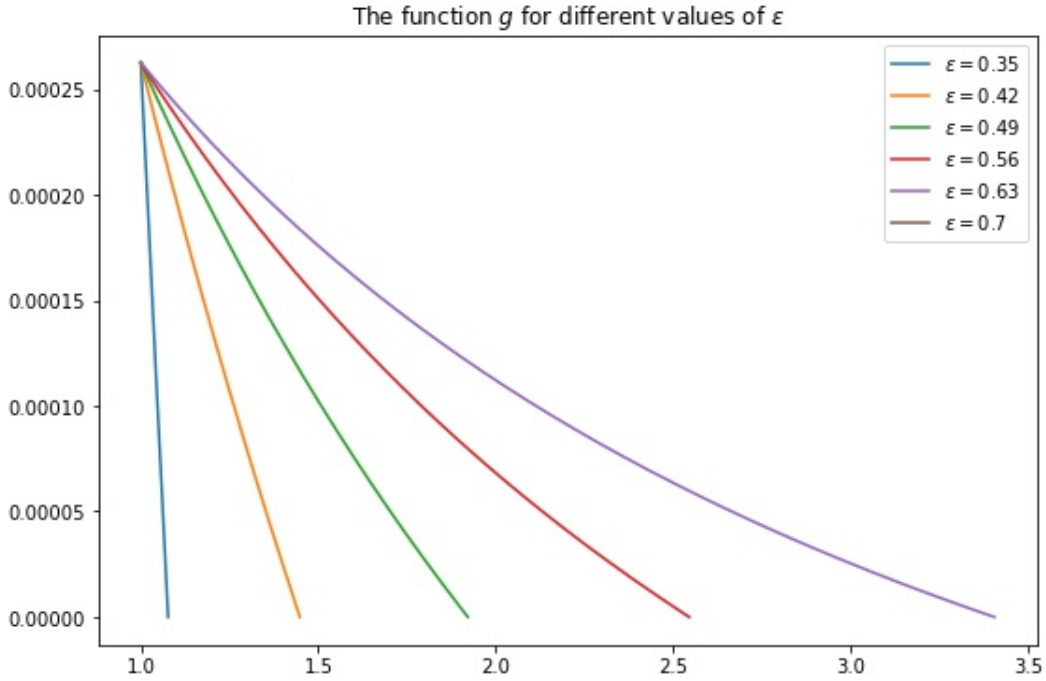

Fig. 2-15: The function g for different values of ε. Here g is approximated using the finite difference method and $l_3 = 1$, $\mu = 1$. We considered $\delta = 10^{-5}$ and discretised the south boundary with $N = 10{-}^5$ points.

2. the FTL policy seems to perform better than the FTO strategy when ε is large, that is, when the required confidence level $1 - \varepsilon$ is small.

2.7 Extension to the case $N > 3$

As we pointed out at the beginning of the chapter, the results we have discussed hold true for the case $N = 3$. In particular, as we have shown in Remark 2.1.5, the study of the boundary behavior of the value function at (l_3, l_3, l_3) is specific to the case $N = 3$. We believe that such an approach can be extended to the case $N > 3$. Suppose for instance $N = 4$: define the south boundary as

$$\mathcal{S}_4 := \{(l, l_3, l_3, l_3) | l \in [l_3, l_4^*)\},$$

where $l_4^* = 3m_\varepsilon l_3$. Suppose we start at $(l_3, l_3, l_3, l_3) \in \mathcal{S}_4$ and we apply the FTL strategy. There are two possible outcomes:

1. if $L_1^\pi(t)$ increases by $\delta > 0$ we reach the level $(l_3 + \delta, l_3, l_3, l_3)$;
2. if $L_1^\pi(t)$ decreases by $\delta' > 0$ we reach the level $(l_3 - \delta', l_3, l_3, l_3)$, where $\delta' = \delta + O(\delta)$ is such that $V(l_3 - \delta', l_3, l_3, l_3) = V(l_3 + \delta, l_3 + \delta, l_3 + \delta, l_3)$.

Therefore, we should be able to study the boundary behavior of the value function using similar arguments to the ones we exploited for $N = 3$.

2.8 Summary

In this chapter we studied the optimality of the FTL policy when the prior distribution is uniform, that is, when the processes all start from the same initial level l. We characterized the FTL strategy as the unique solution of the boundary value problem (2.6). We have also properly constructed the FTL strategy by exploiting the notion of local time. We then introduced the $\mathcal{S}_\delta$ domain and studied the δ-FTL strategy on $\mathcal{S}_\delta$. We used such a strategy to study the limit behaviour of the value function on the south boundary $\mathcal{S}$. To do so we defined the function h in equation (2.29). We then used such a function to compare the FTL strategy to an alternative strategy, defined on the south boundary $\mathcal{S}$, that we denoted as FTO, the Follow The Other strategy. Using numerical methods, which included both direct simulations of the value functions and finite difference methods to approximate the solutions to boundary value problems, we have shown that there exist values of the tolerance parameter ε for which the FTL policy is not optimal for any $(l, l_3, l_3) \in \mathcal{S}$. We can therefore reiterate the conclusion of [17]: we do not know which is the optimal strategy, but we know that, even when the prior distribution is uniform, is not always best (optimal) to follow the leader!

Chapter 3

The optimal strategy with generic prior distribution

In the previous chapter we studied the case where the prior distribution is uniform. This is equivalent to saying that all likelihoods L_i^π, $i \in I$ (respectively all the probability processes Π_i, $i \in I$) start from a same common level l (respectively π). In this chapter we remove such an assumption and study the behaviour of the value function of the FTL problem (1.10) when we do not have any information about the prior distribution. We will denote the prior distribution as $\pi = (\pi_1, \dots, \pi_N)$, $N \in \mathbb{N}$, $\pi \in D_\pi$, where

$$D_\pi = \left\{ \pi = (\pi_1, \dots, \pi_N) \middle| \pi_i \in [0,1] \text{ and } \sum_i \pi_i = 1 \right\}. \tag{3.1}$$

As we have done in the previous chapter, we will mostly work in the space of likelihoods L^π, so to exploit the independence between the processes L_i^π, $i \in I$, but we will also exploit relevant properties of the posterior probability processes Π_i, $i \in I$, when necessary. Let us start by constructing the FTL strategy for generic prior distributions.

3.1 The FTL strategy

Consider the prior distribution $l = (l_1, \dots, l_N)$, $N \in \mathbb{N}$. To define the FTL policy we sort the prior distribution $l = (l_1, \dots, l_N)$, $N \in \mathbb{N}$, in decreasing order, in such a way that $l_1 \geq l_2 \geq \cdots \geq l_N$. By Remark (1.1.13), this does not affect the value of the problem. As we have seen in section 1.3 of Chapter 1, it is not immediate to define the FTL policy when two or more particles have the same value. To see this take for example $N = 3$ and consider a generic prior distribution $l = (l_1, l_2, l_3)$, $l_1 > l_2 \geq l_3$. At time $t = 0$ the FTL strategy prescribes to observe the likelihood process $L_1^\pi(t) = \max_{i \in I} L_i^\pi(t)$ until:

1. at some time $t > 0$, $L_1^\pi(t)$ reaches the threshold $L_1^\pi(t) = m_\varepsilon(L_2^\pi(t) + L_3^\pi(t))$ before reaching the level $L_1^\pi(t) = l_2$. In this case we declare that the first process X_1 has drift $\mu \neq 0$ with probability $1 - \varepsilon$ and the search is over;

2. at some time $t > 0$, $L_1^\pi(t)$ reaches the level $L_1^\pi(t) = l_2$ before reaching the target value $L_1^\pi(t) = m_\varepsilon(L_2^\pi(t) + L_3^\pi(t))$. In this case $L_1^\pi(t) = L_2^\pi(t) \geq L_3^\pi(t)$ and it is not clear which particle is now the leader.

To resolve case 2 we can exploit the results developed in section 1.3: we map the likelihood processes L_1^π, L_2^π into the running maximum process L_M^π and the running minimum process L_m^π. See section 1.3 for a detailed analysis of the problem. In Chapter 1 we have shown that the value function of the FTL policy V_{FTL} is the unique viscosity solution to the second-order ODE (2.6). We now need to compute the boundary of the problem: the right boundary associated to the ODE (2.6) is naturally $(m_\varepsilon(l_2 + l_3), l_2, l_3)$: at this point the search is over as we can declare that the particle X_1 has drift $\mu \neq 0$ with probability $1 - \varepsilon$, $\varepsilon \in (0, \frac{N-1}{N})$. This means that the residual average time to complete the searching process under the FTL strategy at $(m_\varepsilon(l_2 + l_3), l_2, l_3)$ is 0, that is,

$$V_{FTL}(m_\varepsilon(l_2 + l_3), l_2, l_3) = 0. \tag{3.2}$$

As far as the left boundary is concerned, a natural choice is at (l_2, l_2, l_3): as we already mentioned, here the particles L_1^π and L_2^π are at the same level and we cannot easily distinguish which one is the leader. By exploiting the results obtained in section 1.3 and making use of the martingale condition, as detailed in Remark 1.3.3, we impose the following boundary condition:

$$\frac{\partial V_{FTL}}{\partial l_1}\Big|_{l_1 = l_2^+} = \frac{\partial V_{FTL}}{\partial l_2}\Big|_{l_2 = l_1^+}. \tag{3.3}$$

Remark 3.1.1. Starting at any point $l \in D$, where D is defined as in equation (1.46), we can interpret the FTL strategy as a *horizontal* move in D. To see this, recall Figure 2-1 and note that the FTL policy, starting at $l \in D$, prescribes to move horizontally between the west boundary $l_1 = l_2$ and the east boundary $l_1 = m_\varepsilon(l_2 + l_3)$, which corresponds to observing the likelihood process L_1^π until it reaches either l_2 or $m_\varepsilon(l_2 + l_3)$.

Considering the boundary conditions (3.3) and (3.2), and the second-order ODE (2.6), we claim that the value of the FTL policy V_{FTL} is a viscosity solution to the following boundary value problem

$$\begin{aligned} \frac{\partial f}{\partial l_1}\frac{(\mu l_1)^2}{l_1 + l_2 + l_3} + \frac{1}{2}\frac{\partial^2 f}{\partial l_1^2}(\mu l_1)^2 + 1 &= 0, \\ f\left(\frac{1-\varepsilon}{\varepsilon}(l_2 + l_3), l_2, l_3\right) &= 0, \\ \frac{\partial f}{\partial l_1}\Big|_{l_1 = l_2^+} &= \frac{\partial f}{\partial l_2}\Big|_{l_2 = l_1^+}, \end{aligned} \tag{3.4}$$

where $f \in \mathcal{C}(D)$.

Remark 3.1.2. The boundary value problem (3.4) needs to be interpreted in the sense of viscosity theory. In particular the boundary conditions (3.3) and (3.2) have the form described in section 7.A of [12]. We can see that the Dirichlet condition (3.2) can be obtained by taking the function $B(x, r, p)$, which in [12] represents the boundary value of the problem, as $B(x, r, p) = r - f(x)$, with $f(x) = 0$ for all $x \in \partial\Omega$, and the Neumman condition (3.3) can be obtained by taking $B(x, r, p) = \langle n(x), p\rangle - f(x)$ with $f(x) = 0$ for all $x \in \partial\Omega$, where $n(x)$ denotes the outward unit normal to $x \in \partial\Omega$.

Theorem 3.1.3. *The value function of the FTL policy V_{FTL} is the unique viscosity solution to the boundary value problem* (3.4).

Proof. Denote the left boundary by $D^l = \{(l, l, l_3) | l \geq l_3\}$ and consider a function $g : D^l \to \mathbb{R}^+$ such that $V_{FTL}(l_2, l_2, l_3) = g(l_2, l_2)$. In the following we will simply write $g(l_2)$ in place of $g(l_2, l_2)$. For each $l_2 \geq l_3$ it exists a function $h : \mathbb{R}^+ \times D^l \to \mathbb{R}$ such that by solving the ODE (3.4) on $[l_2, m_\varepsilon(l_3 + l_2)]$ with boundary conditions

$$\begin{aligned} f\left(\frac{1-\varepsilon}{\varepsilon}(l_2 + l_3), l_2, l_3\right) &= 0, \\ f(l_2, l_2, l_3) &= g(l_2), \end{aligned} \tag{3.5}$$

we have

$$V'_{+,FTL}(l_2) = h(l_2, g).$$

Consider now $\delta > 0$. Then we have

$$\begin{aligned} V_{FTL}(l_2 - \delta, l_2, l_3) &= V_{FTL}(l_2 - \delta, l_2 - \delta, l_3) + h(l_2 - \delta, g(l_2) - g(l_2)'\delta) \\ &= g(l_2) - h(l_2, g(l_2)). \end{aligned}$$

A straightforward application of Taylor's expansion gives

$$g' = \left(h + \frac{\partial h}{\partial l_2}\right)\left(\frac{\partial h}{\partial l_2} + \delta\right)^{-1}. \tag{3.6}$$

Equation (3.6) is a first-order ODE. To obtain a unique solution we augment (3.6) with the boundary condition

$$g(l_3) = V_0(l_3, l_3, l_3), \tag{3.7}$$

where V_0 denotes the value function of the FTL policy in the case of uniform prior distribution, as computed in (2.7). Denote the value function of the FTL policy with such a boundary condition by $V(l_1, l_2) := V_{FTL}(l_1; g(l_3), l_2)$. Consider now a

bounded stopping time τ and let us apply Ito-Tanaka's equation to $V(L^1_{t\wedge\tau}, L^2_{t\wedge\tau})$ to get

$$dV(l_1, l_2) = -dt + \left(\frac{\partial V_+}{\partial l_1} + \frac{\partial V_-}{\partial l_2}\right) dT,$$

where T is the local time process as defined in Remark 1.3.3. By construction we have $\frac{\partial V_+}{\partial l_1} = -\frac{\partial V_-}{\partial l_2}$, so that the local time effect goes to zero. Therefore we have a unique solution to the first order ODE (3.6) augmented with the boundary condition (3.7), which implies a unique solution to the boundary value problem (3.4). □

Remark 3.1.4. The boundary condition (3.3) is much weaker than a classical boundary condition. In view of that, we cannot hope to uniquely identify the two constants k_1 and k_2 that we will obtain from the general solution of equation (3.4).

The general solution of the second-order differential equation (3.4) can be written as

$$V_{FTL}(l_1, l_2, l_3) = \frac{2\log(l_1)(l_2 + l_3 - l_1)}{\mu^2(l_2 + l_3 + l_1)} + \frac{k_1(l_2)}{l_2 + l_3 + l_1} + k_2(l_2). \tag{3.8}$$

Remark 3.1.5. Note that the functions k_1 and k_2 are constants with respect to l_1 but depend on l_2. Condition (3.3) is not enough to write k_1 and k_2 as constants, as we have done in the case where the prior distribution is uniform. In the following we are going to characterise k_1 as the solution of a first-order ODE. Condition (3.2) will then allow us to write k_2 as a function of k_1.

By condition (3.2) we have

$$k_2 = \frac{2(m_\varepsilon - 1)\log(m_\varepsilon(l_2 + l_3))}{\mu^2(1 + m_\varepsilon)} - \frac{k_1}{(l_2 + l_3)(1 + m_\varepsilon)}. \tag{3.9}$$

To use condition (3.3) we first need to compute the gradient of V_{FTL} at (l_2, l_2, l_3). We have

$$\left.\frac{\partial V_{FTL}}{\partial l_1}\right|_{l_1 = l_2^+} = \frac{2(l_3^2 - 2l_2^2\log(l_2) + 2l_2l_3(1 - \log(l_2)))}{\mu^2 l_2(2l_2 + l_3)^2} - \frac{k_1(l_2)}{(2l_2 + l_3)^2} \tag{3.10}$$

$$\left.\frac{\partial V_{FTL}}{\partial l_2}\right|_{l_1 = l_2^+} = \frac{\partial k_2(l_2)}{\partial l_2} + \frac{1}{2l_2 + l_3}\frac{\partial k_1(l_2)}{\partial l_2} - \frac{k_1(l_2)}{(2l_2 + l_3)^2} + \frac{4l_2\log(l_2)}{\mu^2(2l_2 + l_3)^2}. \tag{3.11}$$

By equation (3.9) we have

$$\frac{\partial k_2(l_2)}{\partial l_2} = \frac{2(m_\varepsilon - 1)}{\mu^2(1 + m_\varepsilon)(l_2 + l_3)} - \frac{1}{(1 + m_\varepsilon)(l_2 + l_3)}\frac{\partial k_1(l_2)}{\partial l_2} + \frac{k_1(l_2)}{(1 + m_\varepsilon)(l_2 + l_3)^2},$$

so that (3.11) can be rewritten as

$$\frac{\partial V_{FTL}}{\partial l_2}\Big|_{l_1=l_2^+} = \frac{2(m_\varepsilon - 1)}{\mu^2(1+m_\varepsilon)(l_2+l_3)} + \frac{4l_2\log(l_2)}{\mu^2(2l_2+l_3)^2} + \frac{l_2(m_\varepsilon-1)+m_\varepsilon l_3}{(1+m_\varepsilon)(l_2+l_3)(2l_2+l_3)}\frac{\partial k_1(l_2)}{\partial l_2} + \frac{l_2^2(3-m_\varepsilon)+2l_2l_3(1-m_\varepsilon)-m_\varepsilon l_3^2}{(1+m_\varepsilon)(l_2+l_3)^2(2l_2+l_3)^2}k_1(l_2). \tag{3.12}$$

By equating equations (3.10) and (3.12) we obtain

$$\frac{l_2(m_\varepsilon-1)+m_\varepsilon l_3}{(1+m_\varepsilon)(l_2+l_3)(2l_2+l_3)}\frac{\partial k_1(l_2)}{\partial l_2} + \frac{4l_2(l_2+l_3)+l_3^2}{(1+m_\varepsilon)(l_2+l_3)^2(2l_2+l_3)^2}k_1(l_2) = \frac{2l_3(2l_2+l_3)-(4l_2+l_3)4l_2\log(l_2)}{\mu^2 l_2(2l_2+l_3)^2} + \frac{2(1-m_\varepsilon)}{\mu^2(1+m_\varepsilon)(l_2+l_3)}. \tag{3.13}$$

We rewrite equation (3.13) by introducing the integrating factor

$$M(l_2) := \exp\left\{\int_{l_3}^{l_2} \frac{4l(l+l_3)+l_3^2}{(l+l_3)(2l+l_3)(l_2(m_\varepsilon-1)+m_\varepsilon l_3)}dl\right\}.$$

Using $M(l_2)$ we can write equation (3.13) as

$$k_1(l_2) = \frac{1}{M(l_2)}\int_{l_3}^{l_2} \frac{(1+m_\varepsilon)(l+l_3)(2l+l_3)}{l(m_\varepsilon-1)+m_\varepsilon l_3} \left\{\frac{2l_3(2l+l_3)-(4l+l_3)4l\log(l)}{\mu^2 l(2l+l_3)^2} + \frac{2(1-m_\varepsilon)}{\mu^2(1+m_\varepsilon)(l+l_3)}\right\}M(l)dl + \frac{1}{M(l_2)}. \tag{3.14}$$

To solve equation (3.14) we need to add one boundary condition: here we can use the value of k_1 when the likelihood of the second particle L_2^π is equal to the likelihood of the third particle L_3^π, that is, the value of k_1 at (l_3, l_3). In particular, note that this is the value of the constant k_0^1 in the case where the prior distribution is uniform. We have

$$k_1(l_3, l_3) = 2\mu^{-2}l_3(3 - 4\log(l_3)). \tag{3.15}$$

Hence, we can compute $k_1(l_2)$ as solution of the initial value problem constituted by equation (3.14) with initial condition (3.15). Solving the integral equation (3.14) is not straightforward, which makes finding the solution to the boundary value problem (3.4) particularly hard. Hence, we will use numerical methods to solve the initial value problem (3.14) and (3.15). We will then use such approximation to compute the numerical solution to the boundary value problem (3.4).

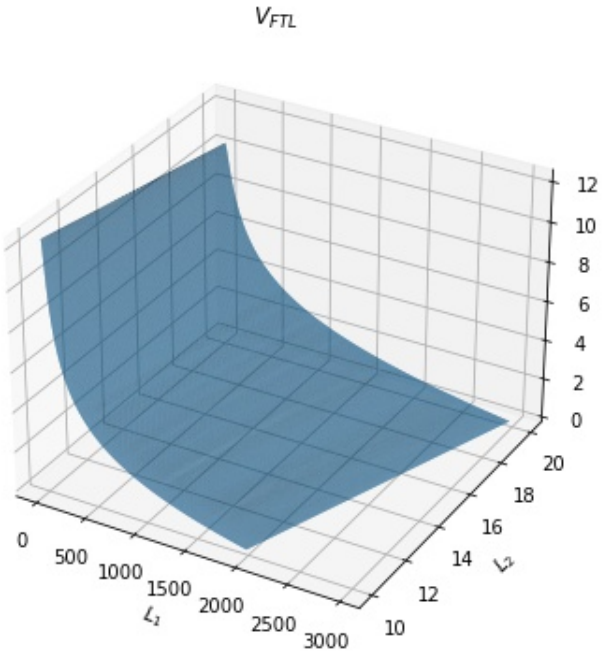

Fig. 3-1: The value of the FTL strategy V_{FTL} as a function of L_1^π and L_2^π. Here $\mu = 1$, $l_3 = 10$ and $\varepsilon = 0.01$. The domain D is discretised with 40000 points.

Figures 3-1, 3-2 and 3-3 show the value of the FTL function as a function of the likelihoods L_1^π and L_2^π. The likelihood of the third particle L_3^π is kept fixed. By comparing Figures 3-1, 3-2 and 3-3 we can understand some interesting pattern:

1. a larger drift μ makes the detection problem easier, as the signal to be found is larger. Indeed, we can see that the values attained by V_{FTL} in Figure 3-3 are much smaller than the values of V_{FTL} in figures 3-1 and 3-2;

2. the smaller the error tolerance ε, the harder the detection problem. To see this, note that the values attained by V_{FTL} in Figure 3-2 are much smaller than the values of V_{FTL} in Figure 3-1;

3. the average searching time reaches its maximum at the left boundary D^l, where the uncertainty about which particle is the leader is maximal;

4. the value function reaches 0 at the right boundary, in accordance with the boundary condition (3.2).

In Figure 3-4 we can see that the value of the FTL strategy restricted on the south boundary $\mathcal{S}$ coincides with the value we have computed in Figure 2-5 in Chapter 2 in the case of uniform prior distribution.

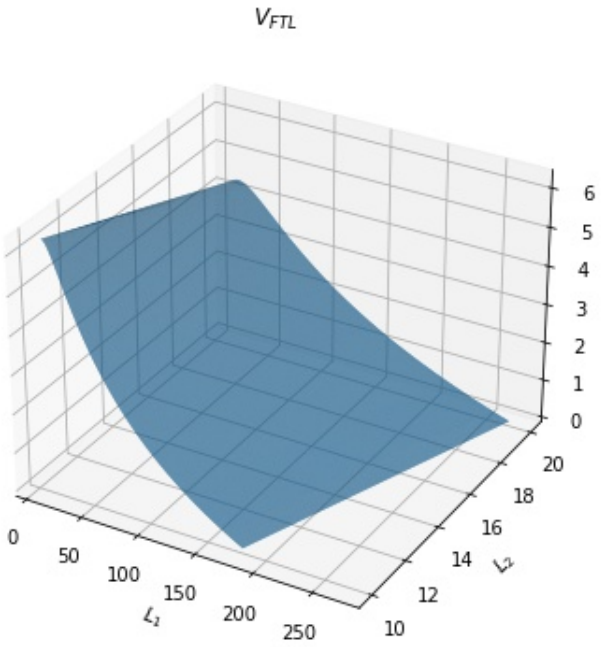

Fig. 3-2: The value of the FTL strategy V_{FTL} as a function of L_1^π and L_2^π. Here $\mu = 1$, $l_3 = 10$ and $\varepsilon = 0.1$. The domain D is discretised with 40000 points.

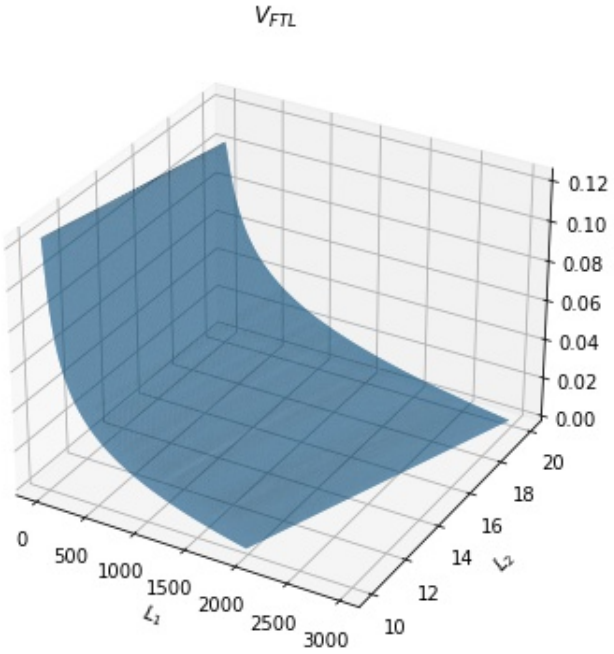

Fig. 3-3: The value of the FTL strategy V_{FTL} as a function of L_1^π and L_2^π. Here $\mu = 10$, $l_3 = 10$ and $\varepsilon = 0.01$. The domain D is discretised with 40000 points.

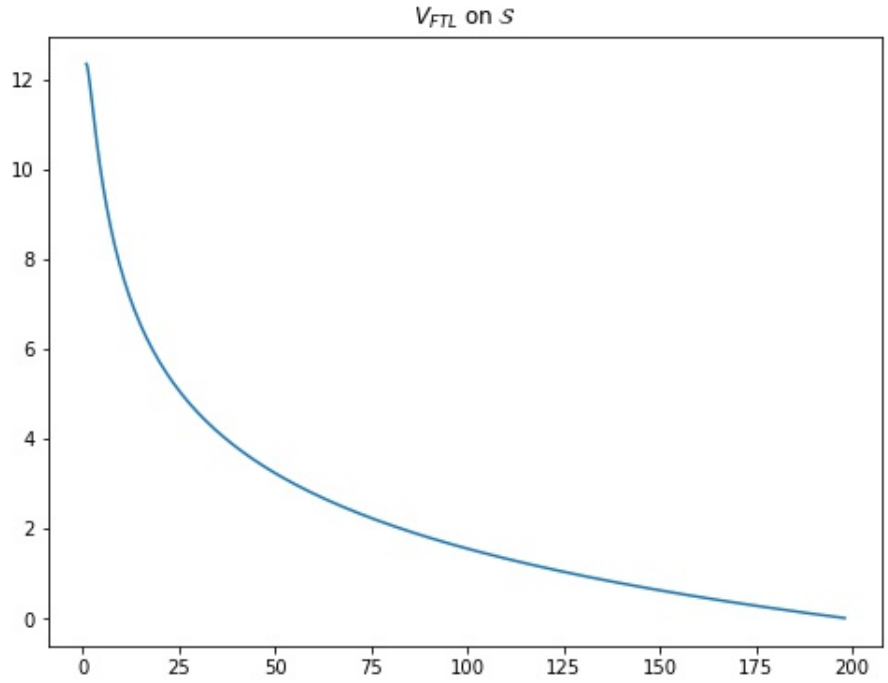

Fig. 3-4: The value of the FTL strategy V_{FTL} on the south boundary $\mathcal{S}$. Here $l_3 = 1$, $\mu = 1$ and $\varepsilon = 0.01$. The south boundary $\mathcal{S}$ has been discretised with 200 points.

3.2 A study on the optimality of the FTL strategy

In the previous section we studied the value function of the FTL strategy V_{FTL}. However, as the authors have shown in [17], the FTL strategy is not the optimal policy. In particular, the authors have shown that an alternative strategy, which they simply named Strategy B, outperforms the FTL policy for certain values of the parameters μ and ε. The complete table of counterexamples can be found in section 5 of [17]. Recall that Strategy B consists of:

- at time $t = 0$ start with a prior distribution π such that $\pi_1 > \pi_2 > \pi_3$. Observe the second largest posterior probability $\Pi_2(t)$ until it reaches either π_1 or a specific level a;
- if it reaches π_1 first, switch to the FTL strategy;
- recalling that
$$\Pi_1(t) = \frac{e^{\mu X_1(t)}}{\sum_{j=1}^{3} e^{\mu X_j(t)}},$$
and by taking a as the unique value of $\Pi_2(t)$ such that $\Pi_1(t) = 1 - \varepsilon$, if $\Pi_2(t)$ reaches a first then the search is over.

Although the insights in [17] are of great interest, the numerical examples provided are limited. Therefore our first goal is to construct a coherent and well-defined numerical scheme to evaluate the optimal strategy at any point $l \in D$. In Chapter 1, Theorem 1.2.6, we have shown that the value function of the optimal strategy is the unique viscosity solution of equation (1.44). In our particle case, where $N = 3$, Theorem 1.2.6 tells us that the value function V of the optimal strategy at $l \in D$ is the unique viscosity solution of

$$H_{J^*}(V)(l) = \min\{H_1(V)(l), H_2(V)(l), H_3(V)(l)\}, \tag{3.16}$$

with associated boundary conditions (1.45), where the operator H_i, $i \in I$, is the second-order differential operator described in equation (1.34). Here H_{J^*} denotes the Hamiltonian operator associated to the optimal strategy $J^* \in \mathcal{J}$. The minimization problem is relatively straightforward in the interior of the domain, but it requires more care at the boundaries:

- at the east boundary $l_1 = l_2$, the boundary condition (3.3) implies that we cannot distinguish between FTL and FTS strategy. One possibility is to run the third particle by $\delta > 0$, which amounts to move by δ along the east boundary, as by normalization moving up by δ along the third coordinate

amounts to move up by δ along both the first and second coordinate. We have

$$(l_2, l_2, l_3) \to (l_2, l_2, l_3 + \delta) = (l_2 + \delta, l_2 + \delta, l_3),$$

for $0 < l_3 < l_2$, where once more we exploited Remark 1.1.11. Another possible strategy is to move either the first or the second particle, which amounts to evaluate the boundary condition (3.3), so that the optimal strategy on the east boundary is such that

$$H_{J^*}(V) = \min\left\{H_3(V), \frac{\partial V}{\partial l_1}\Big|_{l_1=l_2^+} - \frac{\partial V}{\partial l_2}\Big|_{l_2=l_1^+}\right\}; \tag{3.17}$$

- a similar argument can be used on the south boundary: we can either run the leader, which will diffuse along the south boundary, or run either the second or the third particle. We can see that the optimal strategy is such that

$$H_{J^*}(V) = \min\left\{H_1(V), \frac{\partial V}{\partial l_2}\Big|_{l_1=l_3^+} - \frac{\partial V}{\partial l_3}\Big|_{l_3=l_2^+}\right\}. \tag{3.18}$$

  To see why this is the case recall that the south boundary is in fact the domain $\mathcal{S}$ considered in Chapter 2 when studying the FTL problem with uniform prior distribution. On such a boundary the second and third likelihood, L_2^π and L_3^π, are equal, $l_2 = l_3$. At $l_2 = l_3$ we cannot distinguish between L_2^π and L_3^π, and if we choose not to move L_1^π we can only observe either L_2^π or L_3^π, which amounts to the FTO strategy that we introduced in Chapter 2. Such a situation leads to the following boundary condition:

$$\frac{\partial V}{\partial l_2}\Big|_{l_2=l_3^+} = \frac{\partial V}{\partial l_3}\Big|_{l_3=l_2^+}. \tag{3.19}$$

Taking this into account, our goal is to construct a numerical scheme to solve equations (3.16), (3.17) and (3.18) iteratively. As we are going to see, the construction of such an algorithm is non trivial, as standard methods, as well as standard ways to discretise the domain D, are not suitable for this problem. Let us start by describing the iterative procedure:

Algorithm 3.2.1. 1. Denote by n the current iteration. At $n = 0$, for all $l \in D$ compute V_0 as $V_0(l) = \min\{V_L(l), V_S(l), V_T(l)\}$ where $V_L, V_S, V_T : D \to \mathbf{R}_+$ are such that

$$\begin{aligned} H_1(V_L)(l) &= 0, \\ H_2(V_S)(l) &= 0, \\ H_3(V_T)(l) &= 0. \end{aligned}$$

2. for $n > 0$, for all $l \in int(D)$, compute V_n as

$$V_n(l) = \min\{H_1 V_{n-1}(l), H_2 V_{n-1}(l), H_3 V_{n-1}(l)\}.$$

For $l \in \partial D$ we instead evaluate the appropriate boundary condition;

3. stop the procedure at step $m > 0$ such that

$$|V_m(l) - V_{m-1}(l)| < \delta,$$

for all $l \in D$ and $\delta > 0$ a fixed tolerance level.

In light of Algorithm 3.2.1, we see that a critical condition is the *monotonicity* of the numerical scheme, that is, we need to guarantee that $V_n(l) \geq V_{n-1}(l)$ for all $l \in D$ and all iterations n. This condition is essential, as we cannot allow the searching time at iteration n to be smaller than the searching time at a previous iteration. Unfortunately, the monotonicity of the scheme is not guaranteed by most standard algorithms to solve second-order ODEs. The natural approach to discretise an irregular domain such as D would be a triangular mesh, but to numerically solve a second-order ODE on a triangular mesh we would need to implement a finite element method (FEM) (see [41] for a detailed explanation of such an algorithm). However, the FEM does not satisfy the monotonicity property and, to the best of our knowledge, there are no modifications of the FEM that guarantee such a property. Therefore, we decide to use an irregular rectangular grid to discretise the domain D: for each value of l_2, we consider a one dimensional grid with M points, where M is the same for all $l_2 \geq l_3$.

Remark 3.2.2. Such a choice of mesh will result in a *slanted* grid over D: consider for example the Hamiltonian of the FTL strategy in equation (1.34) (i=1). For each $l_2 \geq l_3$, the first and last value of the corresponding grid are $l_1 = l_2$ and $l_1 = m_\varepsilon(l_2 + l_3)$ (as shown in section 2.1 of Chapter 2). This means that the boundaries of the boundary value problem vary with the parameter l_2, resulting in the aforementioned slanted grid.

Remark 3.2.3. Choosing a slanted rectangular grid does not automatically solve the issue of the monotonicity of Algorithm 3.2.1: for each $l_2 \geq l_3$ (each l_1, respectively) the value of the FTL strategy (FTS strategy, respectively) can be computed by solving equation (1.34) by finite difference method (FDM). However, the standard version of the FDM does not guarantee the monotonicity of Algorithm 3.2.1 yet. That said, we can introduce an appropriate modification of the FDM that satisfies such a property.

We now have to find a suitable numerical scheme that guarantees the monotonicity of the approximation $(V_n)_n$ of the value function V. Such a problem has been studied extensively in the last 20 years: in [26] the author used a mixture of PDE theory and probabilistic methods to compute the convergence rate of an

appropriately modified FDM. These results were then extended by [5], [4] and [6]. Following their presentation, let us consider a sufficiently smooth function $\phi : D \to \mathbf{R}$ and the second-order differential operator

$$L^\alpha \phi(l) = tr(a^\alpha(l) D^2 \phi(l)) + b^\alpha(l) D\phi(l), \tag{3.20}$$

where $tr(a)$ denotes the trace of the matrix of coefficients $a \in \mathbf{R}^{N \times N}$ and $\alpha \in \mathcal{A}$ is an admissible control. Let us write the finite difference approximation

$$L_h^\alpha \phi(l) = \sum_{\beta \in \mathcal{S}} C_h^\alpha(l, \beta)\{\phi(l + \beta h) - \phi(l)\}, \tag{3.21}$$

where $h > 0$ is the spatial step and $\mathcal{S}$ denotes a stencil over which we approximate the operator, which is a subset of $\mathbf{Z}^N \setminus \{0\}$. It can be proven (see [5]) that a sufficient assumption for the monotonicity of the scheme (3.21) is

$$C_h^\alpha(l, \beta) \geq 0, \tag{3.22}$$

for all $l \in D$ and $\beta \in \mathcal{S}$.

Remark 3.2.4. It is straightforward to see that the standard FDM approximation of a second-order differential operator of type (3.20), written in the form (3.21), does not necessarily satisfy condition (3.22). Indeed, the FDM has no constraints on the positivity of coefficients $C_h^\alpha(l, \beta)$.

One of the main contribution of [5] and [4] is to provide *Kushner's approximation* of the second order differential operator (3.20): we can write the approximation (3.21) as

$$L_h^\alpha \phi = \sum_{i=1}^{N} \left\{ \frac{a_{ii}^\alpha}{2} \Delta_{ii} + \sum_{j \neq i} \left(\frac{a_{ij}^{\alpha+}}{2} \Delta_{ij}^+ - \frac{a_{ij}^{\alpha-}}{2} \Delta_{ij}^- \right) + b_i^{\alpha+} \delta_i^+ - b_i^{\alpha-} \delta_i^- \right\} \phi, \tag{3.23}$$

where $b^+ = \max\{b, 0\}$, $b^- = (-b)^+$ and

$$\begin{aligned}
\delta_i^\pm \phi(l) &= \pm \frac{1}{h} \{\phi(l \pm e_i h) - \phi(l)\} \\
\Delta_{ii} &= \frac{1}{h^2} \{\phi(l + e_i h) - 2\phi(l) + \phi(l - e_i h)\} \\
\Delta_{ij}^+ \phi(l) &= \frac{1}{2h^2} \{2\phi(l) + \phi(l + e_i h + e_j h) + \phi(l - e_i h - e_j h)\} - \\
&\quad \frac{1}{h^2} \{\phi(l + e_i h) + \phi(l - e_i h) + \phi(l + e_j h) + \phi(l - e_j h)\} \\
\Delta_{ij}^- \phi(l) &= \frac{1}{2h^2} \{2\phi(l) + \phi(l + e_i h - e_j h) + \phi(l - e_i h + e_j h)\} - \\
&\quad \frac{1}{h^2} \{\phi(l + e_i h) + \phi(l - e_i h) + \phi(l - e_j h) + \phi(l + e_j h)\}.
\end{aligned}$$

Using such an approximation the coefficients $C_h^\alpha(l,\beta)$ in (3.21) can be written as

$$C_h(l,\pm e_i) = \frac{a_{ii}(l)}{2h^2} - \sum_{j\neq i}\frac{|a_{ij}(l)|}{4h^2} + \frac{b_{ij}^{\pm}(l)}{h} \tag{3.24}$$

$$C_h(l, e_i h \pm e_j h) = \frac{a_{ij}^{\pm}(l)}{2h^2} \quad i \neq j$$

$$C_h(l, -e_i h \pm e_j h) = \frac{a_{ij}^{\mp}(l)}{2h^2} \quad i \neq j,$$

where $(e_k)_k$ is the canonical base of $\mathbf{R}^N$. In [5] and [4] the authors show that the approximation (3.21) of (3.20) is of positive type, that is, satisfies (3.22), if and only if the matrix a is diagonally dominant. Having such a simple criteria to verify the positivity of the scheme is extremely useful, especially since the operators we are interested in, that is the Hamiltonians (1.34) (for $i = 1, 2, 3$), are uni-dimensional operators, which makes checking the positivity condition straightforward. Indeed, it is enough to note that the second-order coefficient of the operator (1.34) is positive for all $l \in D$, $\frac{1}{2}\mu^2 l^2 > 0$.

One of the limitations of the results in [5], [4] and [6] is that they do not take into account the boundary conditions, but consider only the ODE itself over an unbounded domain. That is, the Kushner approximation (3.23) is constructed over an open subset $A \subset \mathbb{R}$. Unfortunately, this is not enough in our case and we have to understand how to deal with the boundary conditions. To do so, we first have to establish an important difference between the boundaries of D (refer to Figure 2-1 for a graphical representation):

1. the "west" boundary $l_1 = l_2$, with boundary condition (3.3), and the "east" boundary, $l_1 = m_\varepsilon(l_2 + l_3)$, with boundary condition (3.2), are natural boundaries of D, that is, they are part of the definition of the FTL problem;

2. the same is true for the "south" boundary $l_2 = l_3$;

3. the "north" boundary, that is, the upper boundary in Figure 2-1, is not an natural boundary. Instead, it is an artificial boundary needed to guarantee that the modified FDM (3.23) terminates in a finite time.

Because of such reasons, setting an appropriate boundary condition for the north boundary is not trivial, as in doing so we need to take care not to modify the value of the FTL problem. To do so we impose the following consistency condition:

$$\lim_{N\to\infty} V\left(N\frac{l_1}{l_2}, N, l_3\right) = V_2(l_1, l_2), \tag{3.25}$$

where $V_2(l_1, l_2)$ is the value function of the FTL problem in the 2-dimensional case.

Remark 3.2.5. The idea underneath (3.25) is that if the second particle L_2^π is much bigger than the third particle L_3^π, $L_2^\pi >> L_3^\pi$, then the third particle should have a negligible effect on the value of the problem. Note that, since we are studying the FTL problem on D, $L_1^\pi > L_2^\pi$, so that on the north boundary we have $L_1^\pi > L_2^\pi >> L_3^\pi$. In the limit, we expect the value function of the FTL problem on the north boundary to behave as the value function in the case $N = 2$.

Remark 3.2.6. As far as the numerical scheme is concerned, we are going to implement the consistency condition (3.25) in the following way: denote by l_2^M the maximum value of the coordinate l_2 in the finite case, that is, the value of L_2^π on the north boundary. Then we choose l_2^M as the smallest value of l_2 such that

$$|V(l_1, l_2, l_3) - V_2(l_1, l_2)| < \xi, \tag{3.26}$$

where $\xi > 0$ is a numerical tolerance.

Remark 3.2.7. The value function of the 2-dimensional FTL problem $V_2(l_1, l_2)$ can be easily computed as in the 2-dimensional case, all searching strategies are equivalent and we can just compute the value of the FTL policy as the value of the problem. To see why that is the case, note that for $N = 2$ we must have $\Pi_1(t) = 1 - \Pi_2(t)$ for all $t \geq 0$, as Π is a probability measure on D. This means that we can infer information about both particles no matter which one we decide to run, making FTL and FTS equivalent. To compute such value, note that in the 2-dimensional case the value function of the FTL policy is the unique solution (see [29] for a detailed analysis) of the following boundary value problem:

$$\begin{aligned} \frac{\partial f}{\partial l_1}\frac{(\mu l_1)^2}{l_1 + l_2} + \frac{1}{2}\frac{\partial^2 f}{\partial l_1^2}(\mu l_1)^2 + 1 &= 0, \\ f\left(\frac{1-\varepsilon}{\varepsilon} l_2, l_2\right) &= 0, \\ \frac{\partial f}{\partial l_1}\Big|_{l_1 = l_2^+} &= 0, \end{aligned} \tag{3.27}$$

for $f \in \mathcal{C}^2([l_2, m_\varepsilon l_2])$, where $m_\varepsilon = \frac{1-\varepsilon}{\varepsilon}$. By standard calculations we see that the value function of the FTL policy V_{FTL}, that is, the unique solution of (3.27) on $[l_2, m_\varepsilon l_2]$, can be written as

$$V_{FTL}(l_1, l_2) = \frac{2(l_2 - l_1)\log(l_1)}{\mu^2(l_1 + l_2)} + \frac{k_1}{l_1 + l_2} + k_2,$$

with

$$k_2 = -\frac{2(1 - m_\varepsilon)\log(m_\varepsilon l_2)}{\mu^2(1 + m_\varepsilon)} - \frac{k_1}{l_2(1 + m_\varepsilon)},$$

and

$$k_1 = \frac{1}{1 - 2\varepsilon}\mu^{-2}\log(l_2).$$

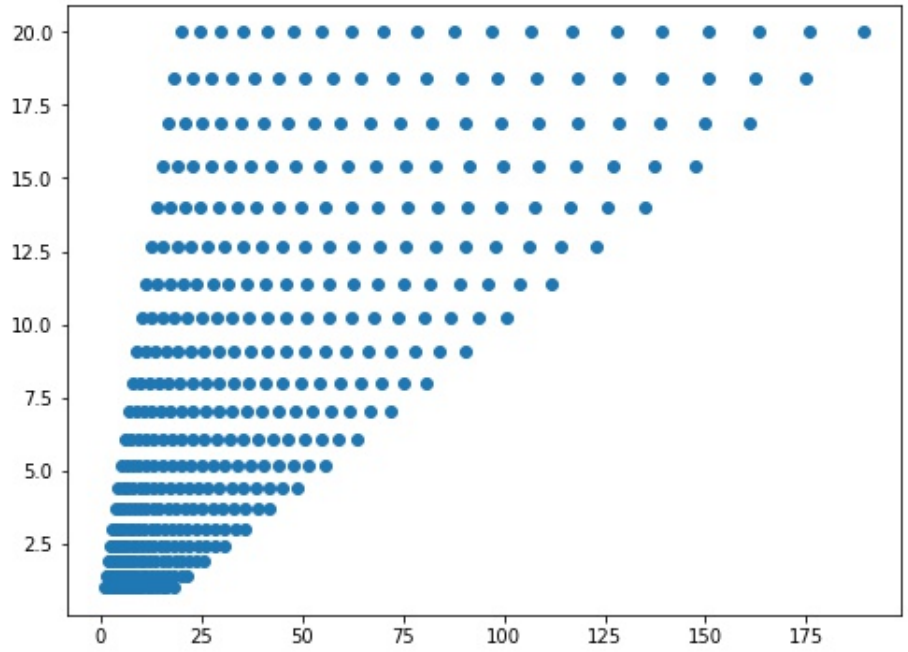

Fig. 3-5: The domain D discretised by the slanted grid introduced in Remark 3.2.2. In this example we have $\varepsilon = 0.1$ and $l_3 = 1$.

We have now established how to implement the boundary conditions, therefore we can implement Algorithm 3.2.1 taking care of using the appropriate finite differences to ensure the positivity of the coefficients (see equation (3.22)) and therefore the monotonicity of the schema.

From Figure 3-5 we see that the slanted grid introduced in Remark 3.2.2 is not uniform in the l_2 direction. In particular, we choose to have more points near the west and south boundary, as the possible local times effects detailed in Remark 1.3.3 require more numerical precision. Another consequence of the slanted grid is that computing the finite differences needed to approximate the first and second-order derivatives in the Hamiltonian operators (1.34) is not trivial: to compute the finite differences we need to identify the grid points immediately to the left/right and up/down with respect to the current grid point. Indeed, to approximate the derivatives in the internal of the domain $int(D)$ we need to compute the value function as a weighted average of two points immediately to the north-east and north-west with respect to the current grid point. We need to be more careful at the boundaries. Depending on the position, there are various possibilities. In some cases, immediately to the north/south will be a boundary value, and we should either compute this value (on the west boundary, via interpolating the nearest boundary points), or on the east boundary by using the boundary condition. It is clear that at the boundaries we may need to use one-sided differences to compute the first derivatives. In Figure 3-6 we evaluate the

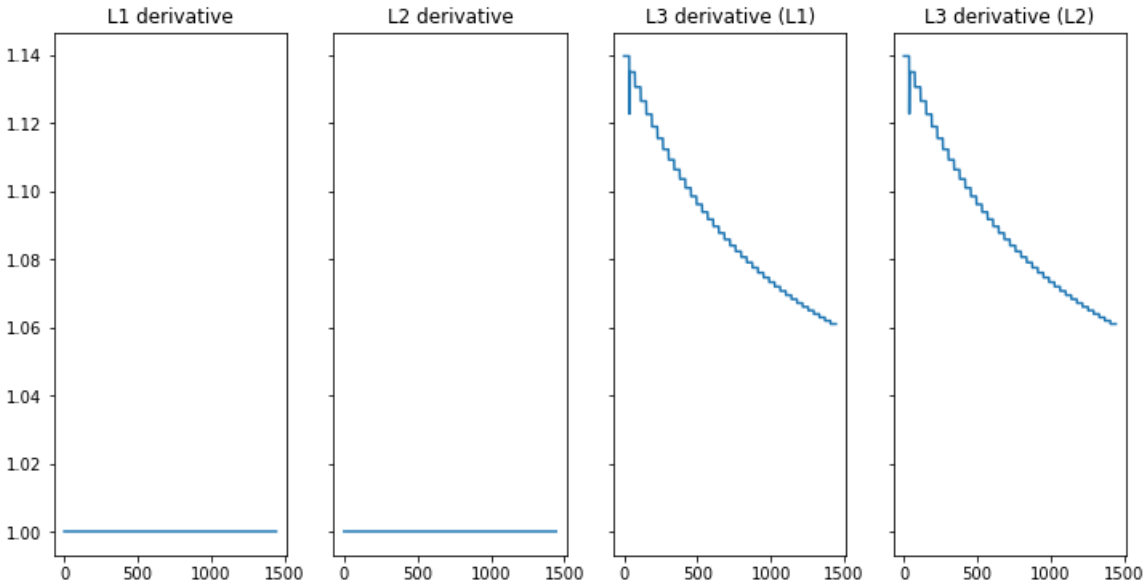

Fig. 3-6: The matrix multiplication between the approximation of the derivatives and the corresponding coordinates. From the left $\frac{\partial}{\partial l_1} \times l_1$, $\frac{\partial}{\partial l_2} \times l_2$, $-\frac{\partial}{\partial l_3} \times l_1$ and $-\frac{\partial}{\partial l_3} \times l_2$. On the x-axis we have the number of grid points.

goodness of the approximation of the first derivatives: the plots show the matrix multiplication between the approximations of $\frac{\partial}{\partial l_1}$ and l_1 (on the left), between $\frac{\partial}{\partial l_2}$ and l_2 and between $-\frac{\partial}{\partial l_3}$ and either l_1 or l_2. Ideally the results should converge to 1 as the grid gets finer, which is the case for l_1 and l_2. The approximation is not as good for the derivative in l_3, but that is to be expected: by normalization we have $\frac{\partial}{\partial l_3} \times l_1 = -1$ (respectively $\frac{\partial}{\partial l_3} \times l_2 = -1$), but this relationship is not linear, so the approximation does not behave as well.

We will now set up differential operators for FTL, FTS and FTT which are chosen to ensure that the "signs" of the coefficients are always positive away from the point at which we evaluate, and negative at the centre. This ensures that in Algorithm 3.2.1 the value function will increase with iterations, and will converge to the solution of the (discretised) HJB equation, and for sufficiently fine grids, also to the true solution. Key to being able to do this are two observations:

1. for second order derivatives, the usual approximation satisfies this property, and this is not disturbed when we have to approximate on the "slanted" grid, provided we approximate by taking positive averages at nearby points;

2. for first order derivatives, we need to be more careful. Specifically, if the first order derivative of a function u appears with a positive sign, we can approximate by the right-sided derivative, $u_x = (u(x+\eta) - u(x))/\eta$, while if it has a negative sign, we use the left-sided approximation, $u_x = (u(x) -$

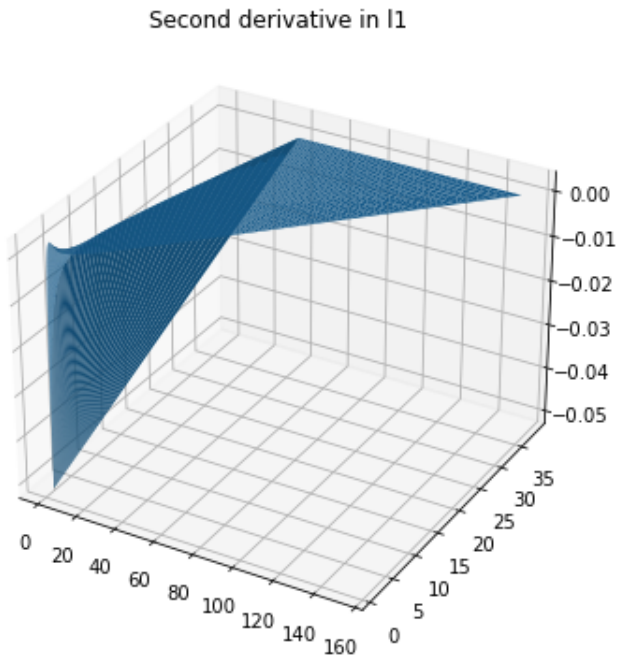

Fig. 3-7: The approximation of the second derivative in l_1 of the value of the FTL strategy on a 40x40 grid.

$u(x-\eta))/\eta$, where $\eta > 0$. In practice, because the sign is positive for FTL and FTS, we can always get away with using the right-sided derivative. This is more complicated in the FTT case, because the "right" derivative in l_3 becomes in fact a "southwest" derivative in l_1 or l_2. As a result, and because the sign difference in the l_3 case is the same as the other cases, we want to use the "right derivative" in the l_3 case, which corresponds to computing in the south west direction.

Figures 3-7 and 3-8 show the approximation of the second-order derivatives in l_1 and l_2 of the value of the FTL strategy V_{FTL} using the method described above. By running Algorithm 3.2.1 we obtain an approximation of the value function of the FTL policy, as shown in Figure 3-9. In Figure 3-10 we compare it to the value obtained by numerically solving the boundary value problem (3.4). As we can see the difference between the two solutions is relatively small across the whole domain but at the south west corner.

In Figures 3-11 and 3-12 we can see the iterative approximations of the FTS and FTT value functions. One way to check the goodness of such approximations is to recall that V_{FTL} is the unique viscosity solution of the HJB equation (3.4). In Figure 3-13 we can see the result of $\hat{H}_{FTL}\hat{V}_{FTL}$, where $\hat{H}_{FTL}$ is the iterative approximation of the Hamiltonian operator H_{FTL} and $\hat{V}_{FTL}$ is the iterative approximation of the FTL value. We expect the result to tend to 0 as the grid gets

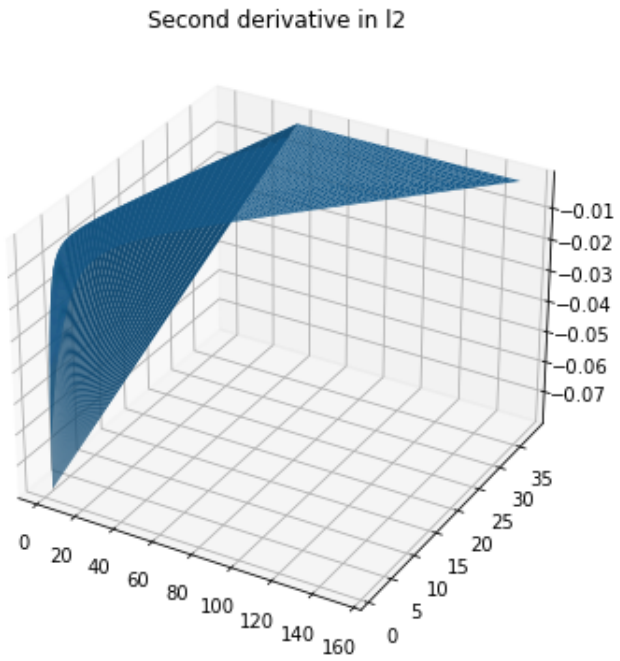

Fig. 3-8: The approximation of the second derivative in l_2 of the value of the FTL strategy on a 40x40 grid.

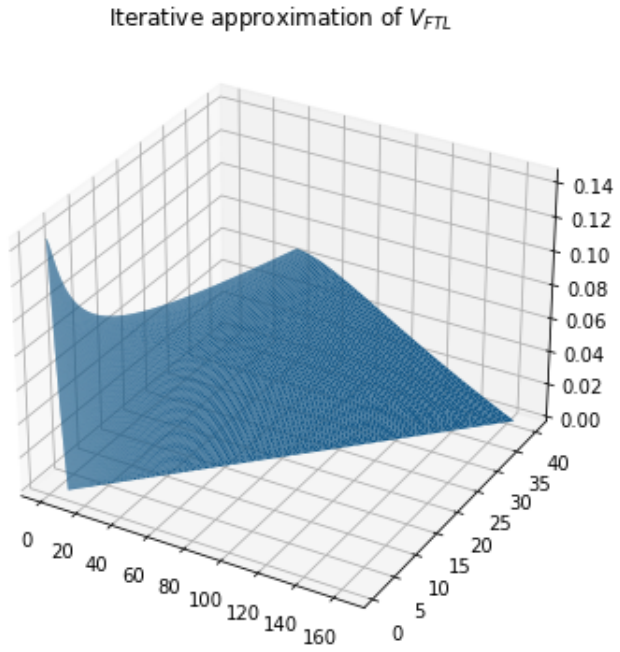

Fig. 3-9: The iterative approximation of the value function of the FTL policy obtained by implementing Algorithm 3.2.1. Here $\mu = 5$, $\varepsilon = 0.2$ and $l_3 = 1$. We used a 70x70 grid and run the algorithm for $N = 10000$ iterations.

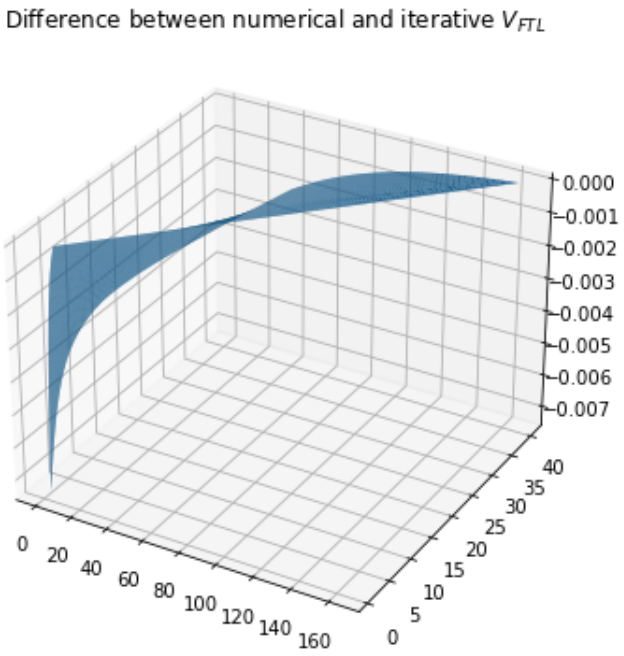

Fig. 3-10: The difference between the iterative approximation of V_{FTL} and the numerical solution of the boundary value problem (3.4). Here we used a 70x70 grid and run the algorithm for $N = 10000$ iterations.

finer. In Figures 3-14 and 3-15 we do the same for the FTS and FTT policies. Once more, we can see that the approximation is not as good close to the south west corner. The approximation is also better for FTL then for FTS and FTT.

One last sanity check on Algorithm 3.2.1 can be obtained by looking at the convergence of the value function through iterations: in Figure 3-16 we can see the values of the 3 strategies as a function of the number of iterations. Note that:

- the value of the strategies increases with the number of iterations. This confirms that Algorithm 3.2.1 is indeed monotonic, as shown in [5] and [4];
- the rate of convergence starts to slow down around $N = 20000$ iterations;
- the rate of convergence at the south west corner behaves similarly to the rate of convergence inside the domain D. We decided to study the rate of convergence at the bottom left separately, as in that region the approximation of the value function via iterative method does not behave as well.

Let us now study the result of Algorithm 3.2.1: we will use the approximation of the value function to detect which strategy is optimal at each point $l \in D$. Our main interest is to study the "switching" boundary, that is, the region of D where the optimal strategy changes. Our goal is to study the geometry of the

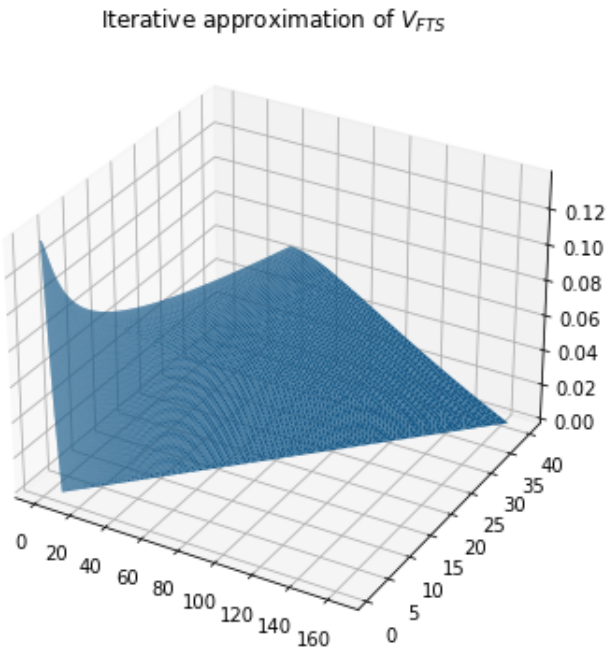

Fig. 3-11: The iterative approximation of the value function of the FTS policy obtained by implementing Algorithm 3.2.1. Here $\mu = 5$, $\varepsilon = 0.2$ and $l_3 = 1$. We used a 70x70 grid and run the algorithm for $N = 10000$ iterations.

switching boundary, so to understand in which subset of D the FTL strategy is optimal and where is better to run either the second or the third particle rather than the leader. We will study the switching boundary for different values of the parameters ε and μ: recall that these parameters represent the tolerance error and the signal or drift of the target process, respectively. Together they provide information about the difficulty of the detection problem, as a larger μ (respectively a larger ε) make the detection problem "easier" with respect to the same problem with smaller parameters. Considering the rate of convergence illustrated in Figure 3-16, from now on we will run Algorithm 3.2.1 with at least $N = 20000$ iterations. In Figure 3-17 we can see the switching boundary with $\varepsilon = 0.1$ and $\mu = 1$. In Figure 3-18 we can see the same simulation but where the size of the grid points is proportional to the difference between the two values, that is, if a grid point l is blue than is optimal to apply the FTL strategy and the size of the grid point is proportional to $V_{FTS}(l) - V_{FTL}(l)$. From this simulation we can see that:

1. the switching boundary has an irregular shape and it is not easy to interpret;
2. the FTT policy is never optimal: there is no point $l \in D$ where we are better off running the third particle rather than one of the others;

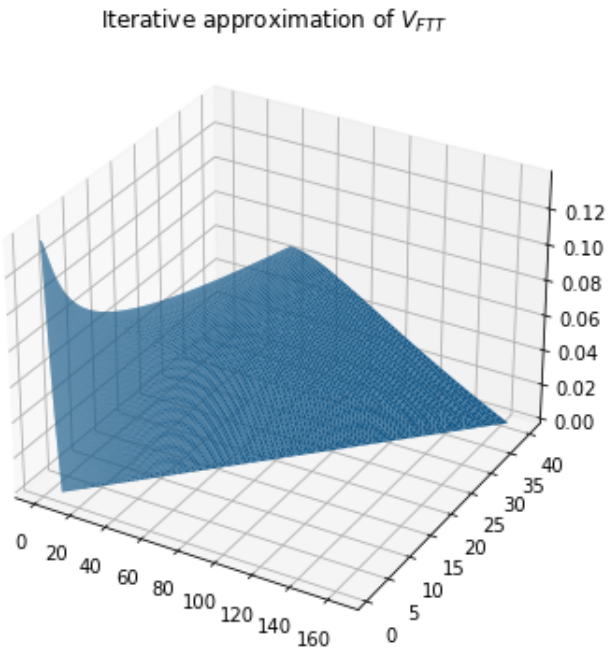

Fig. 3-12: The iterative approximation of the value function of the FTT policy obtained by implementing Algorithm 3.2.1. Here $\mu = 5$, $\varepsilon = 0.2$ and $l_3 = 1$. We used a 70x70 grid and run the algorithm for $N = 10000$ iterations.

3. the FTS strategy is optimal over a large subset of the south boundary S: this confirms our findings in Chapter 2, that is, it is not necessarily optimal to run the leader when the prior distribution is uniform.

In Figures 3-19 and 3-20 we can see another simulation with $\varepsilon = 0.05$: note how the region where FTS is optimal is now more compact and " squeezed" on the west boundary.

Recalling that the FTL problem can be studied both in the space of likelihoods L^π and the space of posterior probabilities Π, we can see in Figure 3-21 the same simulation in the space of posterior probabilities. Note how the shape of the switching boundary is more regular in this space: in particular, we can see that the unusual behavior close to the north boundary in the space of likelihoods is less evident here. To properly interpret Figure 3-21 note that the west boundary in the space of likelihoods corresponds to the line $\pi_1 = \pi_2$ in the space of probabilities and that the east boundary in the space of likelihoods is projected onto the boundary $\pi_1 = 1 - \varepsilon$, on the bottom-right of the domain. In Figures 3-22 and 3-23 we can see another simulation with $\varepsilon = 0.2$ and $N = 50000$. We also used a finer grid of 90x90 points. Thanks to the large number of iterations we can see a clear division between the two regions:

1. the subset of D where the FTS strategy is optimal is now larger and seems

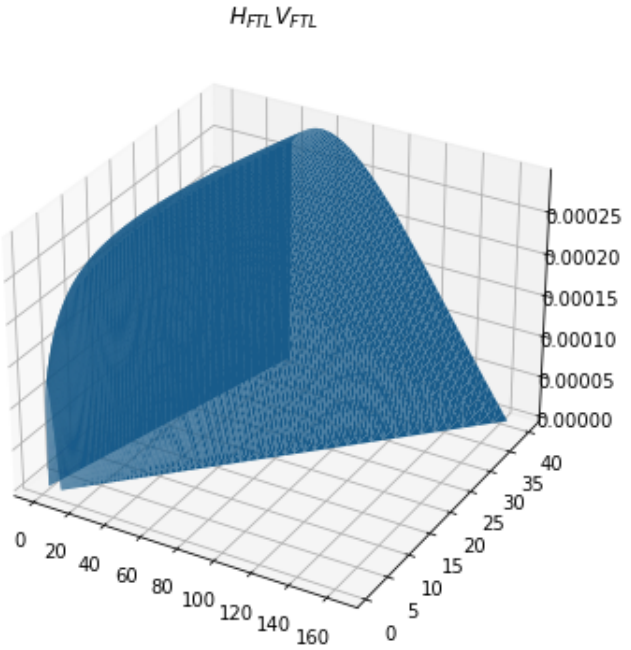

Fig. 3-13: The iterative approximation of $H_{FTL}V_{FTL}$ obtained by implementing Algorithm 3.2.1. This should tend to 0 as the grid gets finer. Here $\mu = 5$, $\varepsilon = 0.2$ and $l_3 = 1$. We used a 70x70 grid and run the algorithm for $N = 10000$ iterations.

to cover a large region of the domain. The only regions where FTL is optimal are close to the south and north boundary. This can be seen in the space of posterior probability as well;

2. the switching boundary has now a more pronounced "butterfly" shape in the space of likelihoods;

3. the region above the south boundary where FTL is optimal has now a clearly defined shape.

Considering all results obtained from the simulations so far, we can now draw some conclusions:

1. the shape of the switching boundary seems to be highly sensitive to the value of the tolerance parameter ε. In particular, the region where the FTS strategy is optimal seems to get larger as ε increases, that is, as the detection problem becomes "easier";

2. such strong sensitivity on the parameter ε could explain why the optimal strategy is so hard to compute: there does not seem to be an optimal strategy for all values of ε and μ;

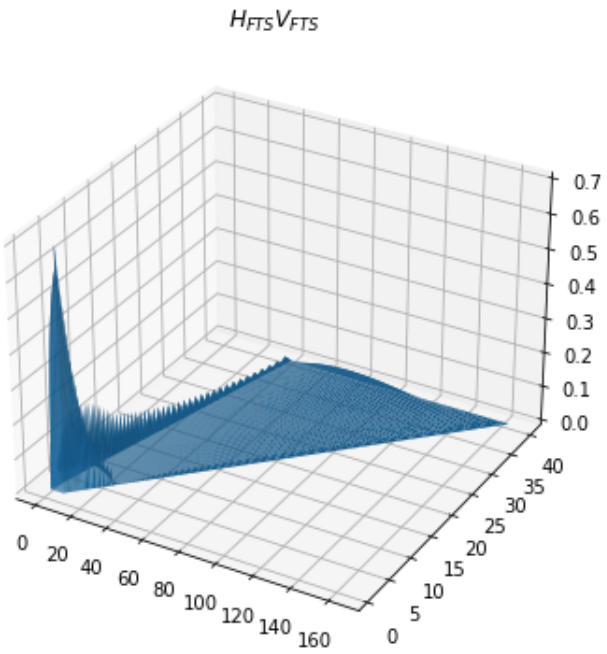

Fig. 3-14: The iterative approximation of $H_{FTS}V_{FTS}$ obtained by implementing Algorithm 3.2.1. These should tend to 0 as the grid gets finer. Here $\mu = 5$, $\varepsilon = 0.2$ and $l_3 = 1$. We used a 70x70 grid and run the algorithm for $N = 10000$ iterations.

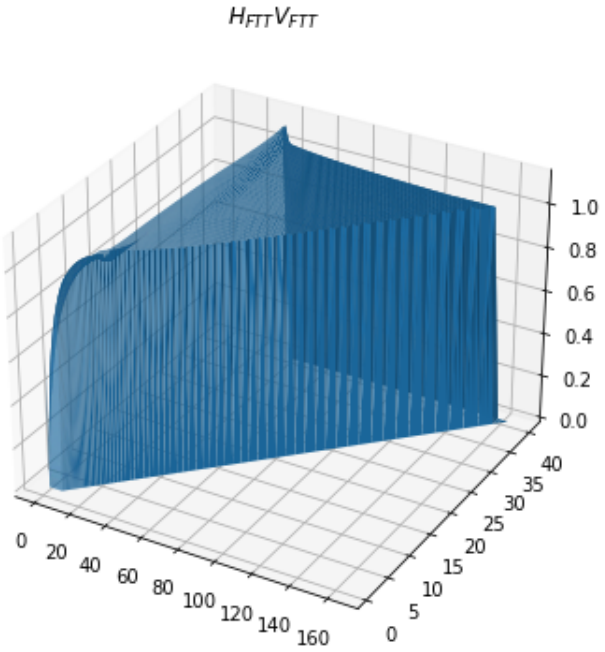

Fig. 3-15: The iterative approximation of $H_{FTT}V_{FTT}$ obtained by implementing Algorithm 3.2.1. These should tend to 0 as the grid gets finer. Here $\mu = 5$, $\varepsilon = 0.2$ and $l_3 = 1$. We used a 70x70 grid and run the algorithm for $N = 10000$ iterations.

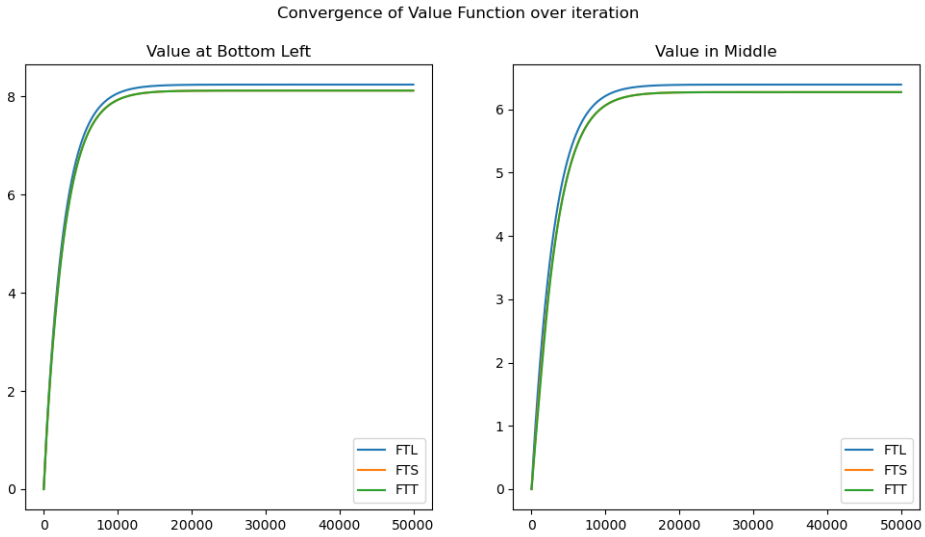

Fig. 3-16: The rate of convergence of the value functions V_{FTL}, V_{FTS} and V_{FTT} as a function of the number of iterations. Here $\mu = 1$, $\varepsilon = 0.05$ and $l_3 = 1$. We used a 70x70 grid.

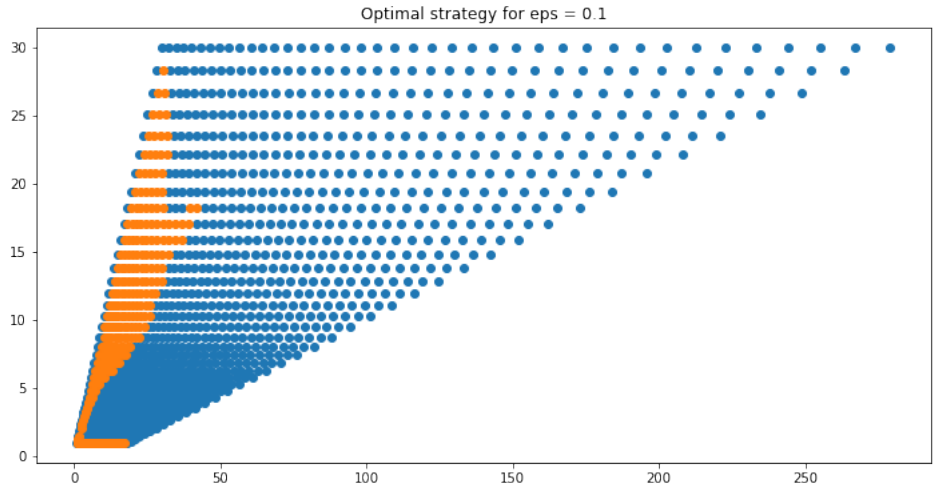

Fig. 3-17: The switching boundary between the FTL and FTS strategies. In blue the area where FTL is optimal, in orange the region where FTS is optimal. Here $\mu = 1$, $\varepsilon = 0.1$ and $l_3 = 1$. We used a 70x70 grid and run the algorithm for $N = 20000$ iterations.

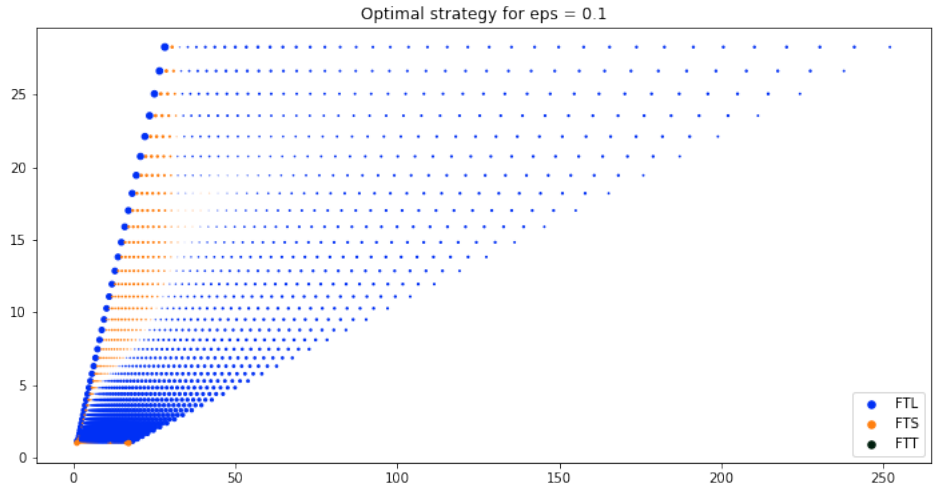

Fig. 3-18: The switching boundary between the FTL and FTS strategies. In blue the area where FTL is optimal, in orange the region where FTS is optimal. The size of the grid points is proportional to the difference between the optimal strategy and the second best strategy. Here $\mu = 1$, $\varepsilon = 0.1$ and $l_3 = 1$. We used a 70x70 grid and run the algorithm for $N = 20000$ iterations.

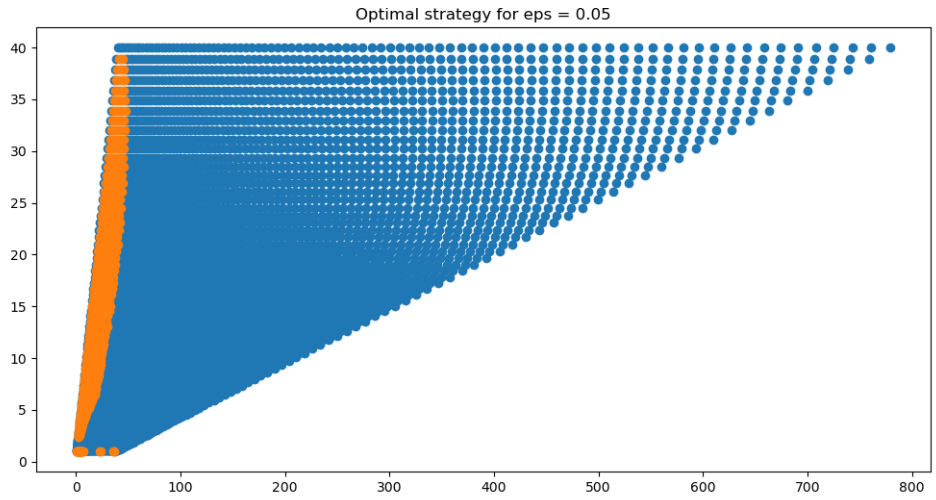

Fig. 3-19: The switching boundary between the FTL and FTS strategies. In blue the area where FTL is optimal, in orange the region where FTS is optimal. Here $\mu = 1$, $\varepsilon = 0.05$ and $l_3 = 1$. We used a 70x70 grid and run the algorithm for $N = 20000$ iterations.

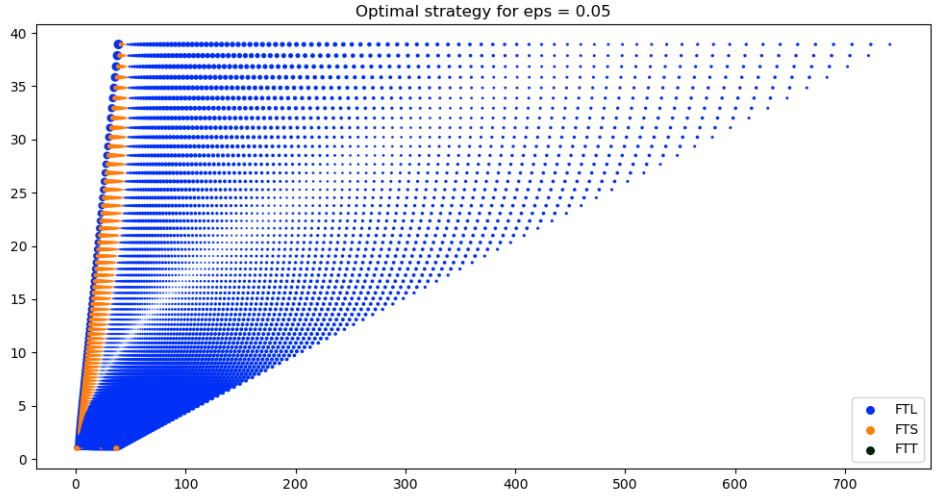

Fig. 3-20: The switching boundary between the FTL and FTS strategies. In blue the area where FTL is optimal, in orange the region where FTS is optimal. The size of the grid points is proportional to the difference between the optimal strategy and the second best strategy. Here $\mu = 1$, $\varepsilon = 0.05$ and $l_3 = 1$. We used a 70x70 grid and run the algorithm for $N = 20000$ iterations.

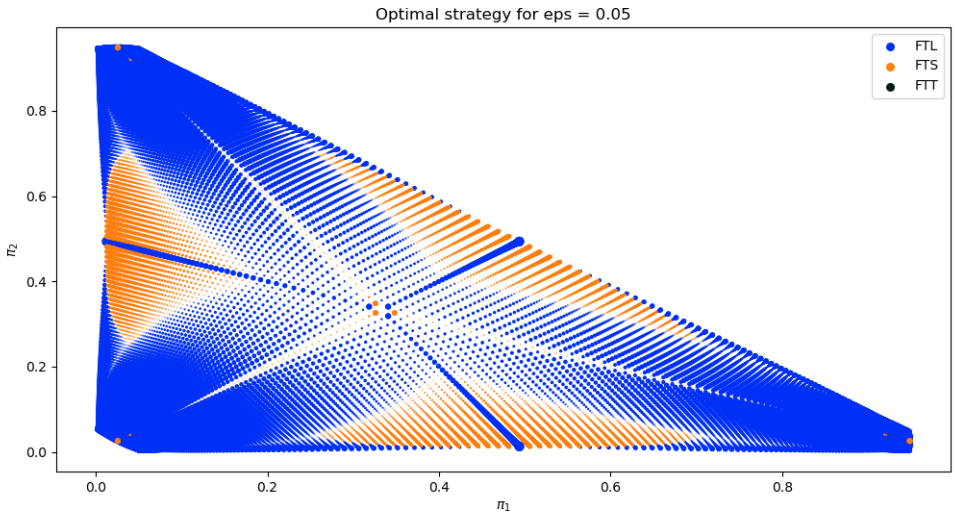

Fig. 3-21: The switching boundary between the FTL and FTS strategies in the space of posterior probabilities. In blue the area where FTL is optimal, in orange the region where FTS is optimal. The size of the grid points is proportional to the difference between the optimal strategy and the second best strategy. Here $\mu = 1$, $\varepsilon = 0.05$ and $l_3 = 1$. We used a 70x70 grid and run the algorithm for $N = 20000$ iterations.

3. the simulations confirm the results we obtained in Chapter 2: there exist $l \in S$ and parameters ε, μ, for which the FTS policy is "better" than the FTL strategy;

4. there is no $l \in D$ for which the FTT strategy is optimal. We speculate that, for any $N \in \mathbb{N}$, it is never optimal to observe the N-th particle for the N-dimensional FTL problem.

3.3 Summary

In this chapter we have studied the FTL problem in the case of non-uniform prior distribution. We first found a numerical solution to the boundary value problem (3.4), that is, we numerically computed the value function associated with the Follow The Leader strategy. We have seen that this agrees with the results in Chapter 1 about the value of the FTL policy when the prior distribution is uniform. Starting from the numerical counter examples to the optimality of the FTL strategy provided in [17], we performed an in-depth numerical analysis of the FTL problem when the prior distribution is not uniform and $N = 3$. In doing so we have:

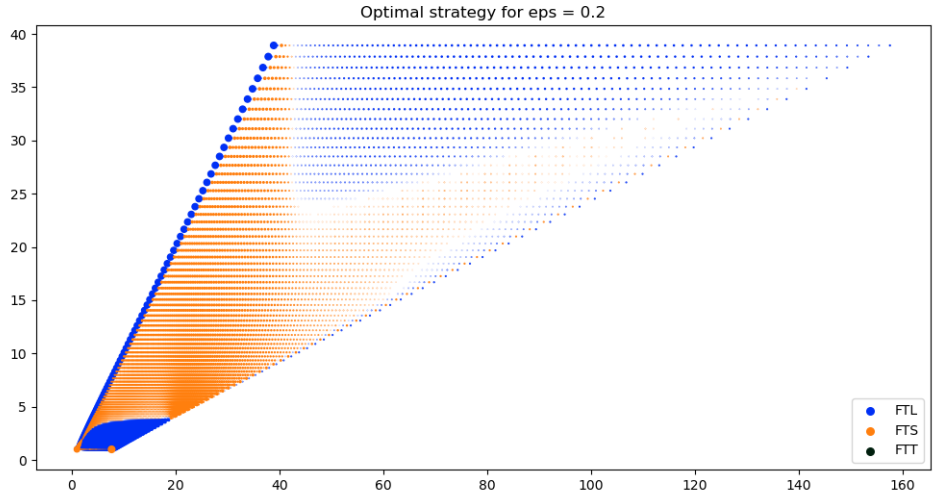

Fig. 3-22: The switching boundary between the FTL and FTS strategies. In blue the area where FTL is optimal, in orange the region where FTS is optimal. The size of the grid points is proportional to the difference between the optimal strategy and the second best strategy. Here $\mu = 1$, $\varepsilon = 0.2$ and $l_3 = 1$. We used a 90x90 grid and run the algorithm for $N = 50000$ iterations.

1. studied the boundary conditions of the FTL problem in the space of likelihoods. We also introduced a new "artificial" boundary, that we named north boundary, so to make the problem solvable by means of numerical methods;

2. we introduced the iterative Algorithm 3.2.1 to approximate the value function of the FTL problem: in essence this algorithm iteratively applies the Hamiltonian operators describing the FTL, FTS and FTT policies to the value function of the problem and evaluate which strategy "performs better" at each step;

3. we introduced an irregular grid over the space of likelihoods L^π: this was necessary to guarantee to properly approximate the first and second-order derivatives needed to compute the approximation of the Hamiltonian operators;

4. we exploited the Kushner's approximation introduced in [5], [4] and [6] to define a modified version of Algorithm 3.2.1. Such a version guarantees that the approximation of the value function is increasing as a function of the number of iterations;

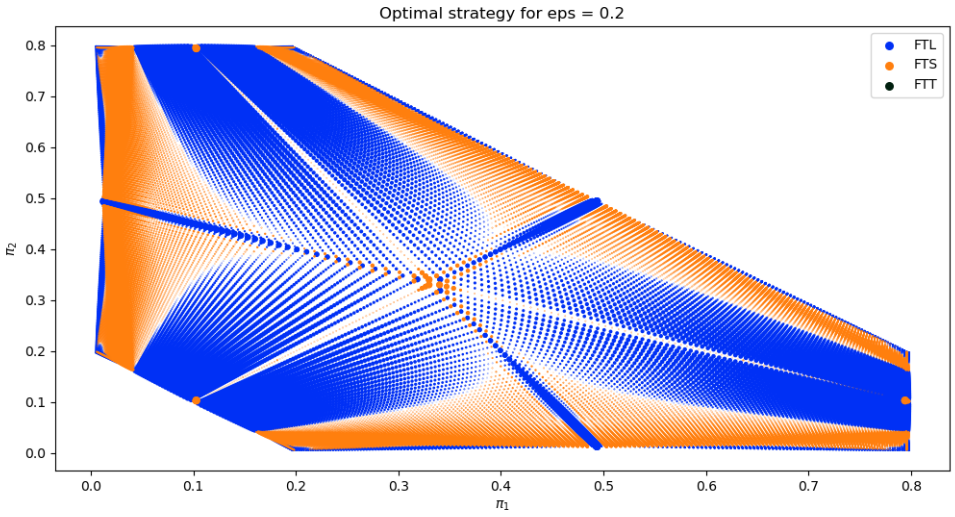

Fig. 3-23: The switching boundary between the FTL and FTS strategies in the space of posterior probabilities. In blue the area where FTL is optimal, in orange the region where FTS is optimal. The size of the grid points is proportional to the difference between the optimal strategy and the second best strategy. Here $\mu = 1$, $\varepsilon = 0.2$ and $l_3 = 1$. We used a 90x90 grid and run the algorithm for $N = 50000$ iterations.

5. we implemented the modified, monotonic version of Algorithm 3.2.1, showing that the approximation of the value of the FTL strategy is consistent with what computed by standard numerical techniques;

6. we studied the switching boundaries of the the FTL problem. Most interestingly, we have seen that the geometry of the switching boundary is highly sensitive to the tolerance parameter ε and that the FTT strategy, that is, the policy that prescribes to run the third particle, never seems to be optimal.

Considering all results we just summarised, we believe we can now augment the conclusions of the authors in [17]: we still do not know which is the optimal strategy, but we do know that (for a specific subset of the domain) it is better *not* to Follow The Leader!

Future work

The FTL problem has a long history of publications and interesting research based on it, and we believe there are still many open questions to study:

1. a first step would be to perform a more in-depth analysis of the sensitivity of the optimal strategy to the tolerance parameter ε. Such a parameter seems to have a fundamental role in determining the optimal strategy and by studying the sensitivity of the value on it, we could learn more about the value function of the FTL problem;

2. although interesting, the numerical simulations of the value function are not enough to compute the optimal strategy. A first step to do so would be to find a good interpretation for the geometry of the switching boundary, which we do not have at the moment;

3. the numerical simulations suggest that the Follow The Third policy is never optimal. An intermediate step to compute the optimal strategy would be to prove that;

4. we still do not know much about the value function of the FTL problem when we consider more than 3 particles, $N > 3$. We believe that the behavior of the value function remains the same for all N but further analyses is needed to confirm that. The main difficulty of the generic N-dimensional problem is to consider the further boundary conditions that arise as the dimensionality of the problem increases. These could pose problems both in the theoretical and numerical analysis of the value function.

5. the iterative algorithm 3.2.1 is an application of the Kushner's approximation introduced in [5], [4] and [6]. It could be of interest to improve such a method, both from the efficiency and the numerical precision point of view. We believe such an approach could be used to numerically study other similar stochastic control problems.

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