Graphic Statics and Symmetry

Bernd Schulze*

Department of Mathematics and Statistics
Lancaster University, Lancaster LA1 4YF, UK
b.schulze@lancaster.ac.uk

Cameron Millar

Skidmore, Owings & Merrill
The Broadgate Tower, 20 Primrose St, London, EC2A 2EW, UK

Abstract

Reciprocal diagrams are a geometric construction dating back to Maxwell and Cremona in which a self-stressed plane framework with a planar graph is paired with another self-stressed reciprocal framework on the dual graph. Either one of the reciprocal frameworks is the form diagram of a self-stressable structure and the other is the force diagram of the corresponding axial forces. This geometric technique offers insights into the self-stresses and infinitesimal motions (mechanisms) of both frameworks in the reciprocal pair. For a symmetric framework with a fully-symmetric self-stress, we obtain an equisymmetric reciprocal pair of plane frameworks, as well as the associated symmetric discrete dual Airy stress function polyhedra. In this paper we exploit symmetry to refine the Maxwell-Cremona correspondence by considering the decomposition of the self-stress and motion spaces into invariant subspaces corresponding to the irreducible representations of the symmetry group. As such, the familiar $s = m^* + 1$ relationship for the number of self-stresses of a framework, s, and the number of mechanisms of the reciprocal, m^*, is reworked into a symmetry adapted version which provides greater insights into the properties of the reciprocal framework pair. We also show how the quotient graph of a symmetric framework and its reciprocal can be used to efficiently detect infinitesimal motions, self-stresses and polyhedral liftings of different symmetry types. This allows for symmetry-adapted simplified structural analyses of symmetric structures.

Keywords: graphic statics, reciprocal diagram, symmetry, equilibrium stress, discrete Airy stress function polyhedron

---

1. Introduction

Graphic statics is a geometry based structural design method, which has a deep history dating back to seminal work by Maxwell, Cremona and Rankine, and has appeared in various areas of Discrete and Computational Geometry (Schulze and Whiteley , 2018a) (see also Baker (2023)). These methods have recently received much attention from researchers in Engineering and Architecture for their remarkable control in design, form finding and optimization of structural solutions (see Hartz et al. (2017) e.g.). A classical example is the Maxwell-Cremona correspondence which, for a plane framework, establishes an equivalence between self-stresses, dual reciprocal diagrams, and polyhedral liftings. See Maxwell (1864, 1870); Whiteley (1982); Crapo and Whiteley (1993, 1994a); Schulze and Whiteley (2018a); Baker (2023), for example, for details.

This paper is aimed at the applied mathematician, but it is hoped that engineers also find the paper useful. To make the paper accessible to non-mathematicians, we develop the theory step by step (e.g., when analysing structures with rotational symmetry, we first consider the simplest case of half-turn symmetry before discussing the general case of n-fold rotational symmetry) and we accompany the theory with examples throughout. Engineers and mathematicians have studied graphic statics and the rigidity of structures separately for some time and developed different terminologies and principles. A recent book (Connelly and Guest , 2022) joins these two schools of thought and provides a 'translation' for engineers to access the results from mathematical rigidity theory. Another source that might be valuable for the practicing engineer might be the recent paper Millar et al. (2021), which provides a basic introduction to the symmetry approach to the analysis and design of self-stressed structures, and is specifically aimed at engineers with no background knowledge in rigidity theory or group theory; in particular, it contains a glossary of key terms.

Self-stresses in planar frameworks and corresponding polyhedral liftings are very useful within engineering, such as in the design of gridshells. Often, a designer starts with a graph, or 'topology', and wants to find a mesh, which approximates a smooth curved shell, with the same topology and planar faces (since curved glass is expensive, planar faces are desirable). Helpfully, by definition, each polyhedral lifting arising from a self-stress has planar faces and shares the same initial topology.

In the mathematical theory of geometric rigidity, there has recently been a surge of interest in the rigidity analysis of *symmetric* frameworks (see Connelly and Guest (2022); Schulze and Whiteley (2018b) e.g. for a summary of recent developments). A fundamental result in this theory is that the rigidity matrix (also known as the equilibrium matrix to engineers) of a symmetric framework can be transformed into a block-diagonalised form using methods from group representation theory (Kangwai and Guest , 2000; Owen and Power , 2010; Schulze , 2010a). Based on this block-decomposition of the rigidity matrix, one can break down the infinitesimal or static rigidity analysis of a symmetric framework into independent subproblems, one for each irreducible representation of the symmetry group of the framework (Kangwai et al. , 1999; Fowler and Guest , 2000; Owen and Power , 2010; Schulze , 2010a; Schulze et al. , 2022). In the present paper, we use this approach to refine the Maxwell-Cremona correspondence for symmetric plane frameworks into a set of symmetry-adapted correspondences, one for each irreducible representation of the symmetry group.

The starting point of the present paper is that the original framework has a non-trivial symmetry group and has a fully-symmetric self-stress so that the reciprocal framework (see Section 2.2 for a formal definition) has the same symmetry group as the original framework (i.e., the form and force diagram share the same symmetry). This is a natural assumption, as it is often helpful for engineering structures to have a fully-symmetric state of self-stress (see Millar et al. (2021) for a discussion on this). It is desirable for a gridshell to be 'self-tied'; that is, just like a bicycle wheel, all the thrust from the roof is tied back with a tension ring. This relates to a state of self-stress in the plane view; the horizontal component of the thrust in the interior members is equilibriated by the perimeter tension ring. This means that the thrust is resolved within the structure and not taken by the supporting structure. This is critical for some roofs which rest on historic walls, such as the Great Court roof of the British Museum (Williams , 2001). Another example is tension nets (or more generally, tension structures) which obtain most of their stiffness from prestressing. The domes of David Geiger and other stadium structures (like the new Tottenham Hotspur stadium engineered by Schlaich Bergermann Partner (sbp)) are essentially tensegrities. These structures are often symmetric for construction reasons and use a fully-symmetric state of self-stress to stabilise and stiffen the structure. These are just some of the areas where symmetric structures and symmetric states of self-stress are powerful within engineering.

We note that even if a self-stress is not fully-symmetric, but exhibits the symmetry of a non-trivial irreducible representation of the group, the corresponding reciprocal framework will retain non-trivial symmetry (namely the symmetry corresponding to the kernel of the irreducible representation). For example, if a framework exhibits dihedral symmetry, but the self-stress of interest has only mirror symmetry, then the reciprocal figure will share the mirror symmetry only. In this case, the methods of this paper can still be applied to the reciprocal pair corresponding to this self-stress and the smaller symmetry group, as discussed in Section 6.

Symmetry is ubiquitous in engineering structures as it allows for aesthetically pleasing and cost-efficient designs. For example, bespoke and unique glass panels and nodes within gridshells can increase costs significantly so it is desirable to have some level of modularity with repetitive components in the structure. Including a high degree of symmetry into the design is one way to achieve this, and hence many gridshell structures exhibit symmetry, as shown in the examples in Figure 1.

Symmetric buildings also have useful structural engineering properties.

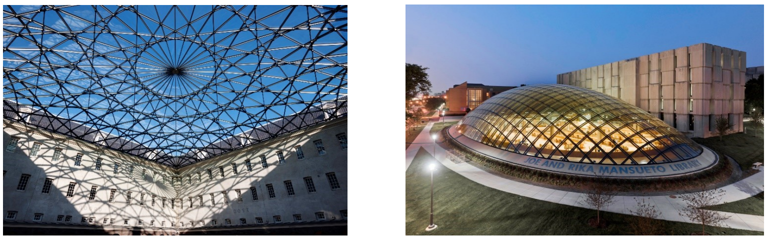

Figure 1: Examples of symmetric gridshell structures: The Dutch Maritime Museum (Ney and Partners , 2022) in Amsterdam and the Mansueto Library (Architizer , 2022) in Chicago.

Structures tend to be loaded by dead loads (such as self-weight), imposed loads (such as wind pressure) and self-stresses. Dead loads and self-stressing forces tend to be 'fully-symmetric' meaning that the forces within the symmetric structure are also symmetric. Therefore, this relates to a symmetric force diagram and polyhedral lifting (or Airy stress function). This is useful as knowledge of the symmetry can simplify the design problem. Furthermore, unbalanced live loads can often be deconstructed into 'fully-symmetric' and 'anti-symmetric' loads (see McRobie at el. (2022) e.g.) which can be more readily considered.

The paper is organised as follows. In Section 2 we first introduce the key concepts of graphic statics, such as infinitesimal motions and self-stresses of bar-joint frameworks, parallel drawings, reciprocal diagrams and polyhderal liftings. We also describe the decomposition of the motion and stress space of a symmetric framework into subspaces corresponding to the irreducible representations of the symmetry group. Based on this decomposition, we then establish symmetry-adapted correspondences between infinitesimal motions, self-stresses, parallel drawings and polyhedral liftings for the symmetry groups in the plane. We begin with a discussion of frameworks with reflection symmetry (Section 3) and half-turn symmetry (Section 4) and then describe how the theory extends to other rotational groups (Section 5), as well as dihedral groups (Section 6). Throughout the paper, we illustrate our results with examples. Finally, in Section 7 we show how symmetric infinitesimal motions and self-stresses can easily be detected by Maxwell-type rigidity counts on the quotient graph of a symmetric graph, called orbit counts.

2. Preliminaries on graphic statics and symmetry

2.1. Frameworks, rigidity, and parallel drawings

A bar-joint framework in $\mathbb{R}^2$ consists of a set of bars of fixed lengths that are connected at their ends by pin joints that allow arbitrary rotations in the plane. Mathematically, this can be modelled by a *bar-joint framework*, or simply *framework* in $\mathbb{R}^2$. In the case where all bar lengths are strictly positive, this is a pair (G, p) of a finite simple graph $G = (V, E)$ (whose edges E and vertices V correspond to the rigid bars and flexible joints, respectively) and a map $p : V \to \mathbb{R}^2$ that assigns positions to the joints in the plane, with distinct positions for the end points of each bar. More generally, we want to allow joints that are connected by a bar to have identical positions, in which case the corresponding bar length is zero. To accommodate this, we define a *(generalised) framework* as a triple (G, p, q), where $p : V \to \mathbb{R}^2$ and $q : E \to \mathbb{R}^2 \setminus \{0\}$ are maps with the property that for all $ij \in E$ there exists a scalar $\lambda_{ij} \in \mathbb{R}$ (which is possibly zero) such that $p(i) - p(j) = \lambda_{ij} q(ij)$ (Tay , 1993). Note that in the case where a bar has zero length, there is still a direction vector associated with the bar, and hence the bar still constitutes a constraint. Thus, rearranging the configuration of a framework so that adjacent vertices are assigned the same point does not change the number of point coordinates or constraints of the structure. From a practical perspective, one can consider the example in Figure 6(a). One might be investigating the impact of the distance between points p_2 and p_3; in this case the length can be as short as zero, but the vector is always aligned with the y direction. This is an important part of the mathematical definition often overlooked by engineers. If $p(i) \neq p(j)$ then we may choose $\lambda_{ij} = 1$.

For simplicity, most of the discussion of this paper focuses on frameworks with non-zero bar lengths (with the exception of the example in Section 6.2), but all the results in this paper immediately extend to generalised frameworks, with the vector $q(ij)$ playing the role of the vector $p(i) - p(j)$ for a zero-length bar.

The rigidity and flexibility analysis of frameworks is a well developed theory with a long and rich history, which has many practical applications (see Connelly and Guest (2022); Schulze and Whiteley (2018a) for example, for a summary of results). We briefly introduce the key notions from the linear theory of infinitesimal (or equivalently static) rigidity of frameworks.

An *infinitesimal motion* of a framework (G, p) in $\mathbb{R}^2$ is a function $u : V \to \mathbb{R}^2$ such that

$$(p_i - p_j) \cdot (u_i - u_j) = 0 \quad \text{for all } ij \in E, \tag{2.1}$$

where $p_i = p(i)$, $u_i = u(i)$ for each i and the $\cdot$ symbol denotes the standard inner product on $\mathbb{R}^2$. Geometrically, this condition for $ij \in E$ says that the velocity vectors u_i and u_j preserve the length of the bar joining p_i and p_j at first order (see Figure 2(a)). To the engineer, this is a mechanism such that nodes move an infinitesimal distance but members do not change length. If the framework is a generalised framework (G, p, q), then the definition of an infinitesimal motion is as above, but for each bar ij of length zero, we have the condition

$$q(ij) \cdot (u_i - u_j) = 0. \tag{2.2}$$

An infinitesimal motion u of (G, p, q) is a *trivial infinitesimal motion* if there exists a skew-symmetric matrix S and a vector t such that $u_i = Sp_i + t$ for all $i \in V$, i.e., if u corresponds to a rigid body motion in the plane. (G, p, q) is *infinitesimally rigid* if every infinitesimal motion of (G, p, q) is trivial, and *infinitesimally flexible* otherwise. The matrix corresponding to the linear system in (2.1) and (2.2), with the u_i being the unknowns, is the *rigidity matrix*, denoted $R(p, q)$ (or simply $R(p)$ if there are no zero-length bars), and it is well known that (G, p, q) is infinitesimally rigid if and only if the rank of $R(p, q)$ is $2|V| - 3$, provided that the points p_i affinely span all of $\mathbb{R}^2$.

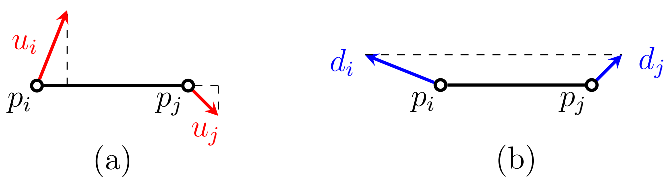

Figure 2: *A (trivial) infinitesimal motion of a single bar (a) and its corresponding parallel displacement (b) obtained by turning the velocity vectors in (a) by* 90 *degrees.*

A *self-stress* of a framework (G, p) is a function $\omega : E \to \mathbb{R}$ such that for each vertex i of G the following vector equation holds:

$$\sum_{j:ij \in E} \omega(ij)(p_i - p_j) = 0.$$

In structural engineering, $\omega(ij)(p_i - p_j)$ is the axial force in the bar ij and the stress-coefficient $\omega(ij)$ is called the force-density (scalar force divided by the bar length) of the bar ij. The summation above says that the tensions and compressions in the bars balance at each node i, and hence a self-stress is also known as an *equilibrium stress*. For the engineer, a self-stress is often considered as a set of axial forces within a framework which are in equilibrium in the absence of external loads. Note that $\omega \in \mathbb{R}^{|E|}$ is a self-stress of (G, p) if and only if $\omega^T R(p) = 0$ (i.e. it lies in the left null space of $R(p)$). Analogously, for a generalised framework (G, p, q), $\omega \in \mathbb{R}^{|E|}$ is a self-stress of (G, p, q) if and only if $\omega^T R(p, q) = 0$. A framework that is infinitesimally rigid and has no non-trivial (i.e., non-zero) self-stress is called *isostatic*.

A *parallel displacement* of a framework (G, p) in $\mathbb{R}^2$ is a function $d : V \to \mathbb{R}^2$ such that

$$(p_i - p_j)^\perp \cdot (d_i - d_j) = 0 \quad \text{for all } ij \in E, \tag{2.3}$$

where $x^\perp$ denotes the vector obtained from x by rotating it by 90 degrees (in a counterclockwise direction). A solution, d, of this linear system is called a parallel displacement of (G, p), because, geometrically, the condition for $ij \in E$ says that the *displacement vectors* d_i and d_j preserve the direction of the bar joining p_i and p_j at first order. In other words, if we change the position of p_i and p_j to $p'_i = p_i + d_i$ and $p'_j = p_j + d_j$, respectively, then the bar connecting p'_i and p'_j is parallel to the bar connecting p_i and p_j. (See Figure 2(b)). The framework (G, p') is called a *parallel redrawing* of (G, p) since corresponding bars are parallel to each other (Schulze and Whiteley , 2018a; Whiteley , 1996). Note that all the above definitions can again immediately be extended to generalised frameworks.

It is well known that $u : V \to \mathbb{R}^2$ is an infinitesimal motion of (G, p) if and only if $d : V \to \mathbb{R}^2$ defined by $d_i = u_i^\perp$ is a parallel displacement of (G, p). See Figures 2(a) and (b). So a parallel drawing and an infinitesimal motion or mechanism are directly related to each other. Moreover, the trivial infinitesimal motions of a framework (G, p) correspond to *trivial parallel displacements* of (G, p), which are always present for any framework (G, p): an infinitesimal translation by t corresponds to a translational parallel displacement by $t^\perp$ (i.e., a translation in the perpendicular direction), and an infinitesimal rotation about the origin corresponds to a dilational parallel displacement towards the origin. Because of this correspondence, which has its roots in drafting techniques from the 19^{th} century, all the combinatorial and geometric results for infinitesimal rigidity immediately transfer to parallel drawings in $\mathbb{R}^2$ (Schulze and Whiteley , 2018a; Whiteley , 1996).

2.2. Reciprocal diagrams and polyhedral liftings

In this section, we shall introduce reciprocal diagrams for frameworks whose bars all have strictly positive length. However, as mentioned above, it is immediate to extend this discussion to generalised frameworks that may also have bars of length zero.

Suppose a framework (G, p) on a planar graph G has a self-stress ω. Then, if we we cycle around a joint p_i, placing the vectors $\omega(ij)(p_i - p_j)$ end to end, we obtain a closed polygon. The length of each of these vectors is the force in the bar. This polygon is equivalent to the common 'closed force polygon' in engineering which states that the sum of forces at a node must be equal to zero. These polygons for the joints of (G, p) can be fitted together to form a framework on the dual graph G^*, called a *reciprocal diagram* of (G, p), whose edges are parallel to the corresponding edges of (G, p). See Figure 3. Furthermore, each node corresponds to a unique closed polygon within the reciprocal diagram.

In structural engineering, the original framework is usually called a *form diagram* and the reciprocal diagram is called the *force diagram*, because each bar in the form diagram has a corresponding bar in the force diagram whose length is the force in the bar. The form diagram describes the structural geometry whilst the force diagram describes the forces within the structure. The relationship between the form and force diagram is the same as the relationship between the force and form diagram (it is a two-way relationship). Therefore, the force diagram could be the structure and the form diagram would describe the forces within that structure. As such, it is common to manipulate both diagrams simultaneously so that the engineer has control of both the structural form and the forces within it. This is powerful in design as the designer can modify one diagram and see the corresponding impact on the other.

A self-stress of the 2D form diagram corresponds to a vertical lifting of the form diagram to a 3-dimensional polyhedral surface, known as the *Airy stress function* (Airy , 1862; Maxwell , 1864, 1870). Engineers may be familiar with Airy stress functions in the continuum mechanics setting where the stresses in the solid are given by the second derivatives of the stress function. A discrete version also exists but is only commonly considered in the field of graphic statics (Mitchell et al. , 2016). Here, the force in each

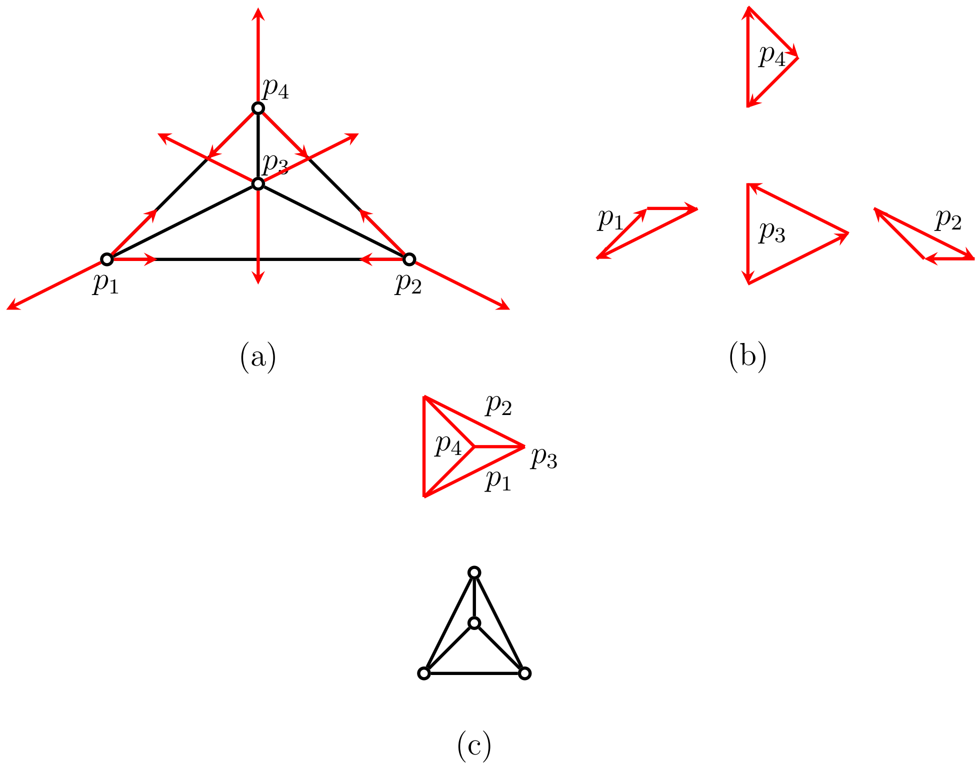

Figure 3: *(a) A plane framework* (G, p) *with a self-stress. (b) At each vertex, the equilibrium of forces of the self-stress yields a closed polygon of forces. (c) These polygons can be assembled into a drawing of the dual graph (top); if all polygons are rotated by 90 degrees, we obtain the (orthogonal) reciprocal framework of* (G, p) *(bottom).*

bar is given by the change of the normals of the two faces that are adjacent to the corresponding edge in the discrete Airy stress function which lifts the framework in the horizontal plane vertically to 3-space. See Maxwell (1864, 1870); Borcea and Streinu (2015) for details. The reciprocal diagram can only be constructed if the stress function has planar faces (Maxwell noted that the form diagram must be a projection of a plane-faced polyhedron for it to possess a state of self-stress). Such techniques can then be applied to gridshells; it is desirable that they have planar faces so by considering the gridshell as an Airy stress function, it is known that it has planar faces if a reciprocal diagram can be constructed (this was done in Adriaenssens et al. (2012) for the Dutch Maritime Museum shown in Figure 1, for example).

To make a connection to 3-dimensional polyhedral liftings of pairs of reciprocal diagrams, Maxwell rotated the reciprocal diagram by 90 degrees. He showed that (G, p) is then the vertical projection of a polyhedron if and only if the reciprocal diagram (G^*, q) is the vertical projection of the polar dual of this polyhedron (Maxwell , 1864; Crapo and Whiteley , 1993, 1994a; Schulze and Whiteley , 2018a). Moreover, (G^*, q) has the property that the coordinates of the point q_i which is dual to the face F_i of (G, p) is the gradient of the plane given by F_i (Konstantatou , 2018).

This motivates the following definition. For a framework (G, p) in $\mathbb{R}^2$ with a self-stress ω, the corresponding *(orthogonal) reciprocal framework* or simply *reciprocal framework* of (G, p) is the framework (G^*, q) in $\mathbb{R}^2$, where G^* is the dual graph of G, and every edge ij of (G^*, q) is orthogonal to the corresponding edge in (G, p) and has length $||\omega(ij)(p_i - p_j)||$; i.e. the length of the corresponding line in the reciprocal diagram is the force in the bar. Note

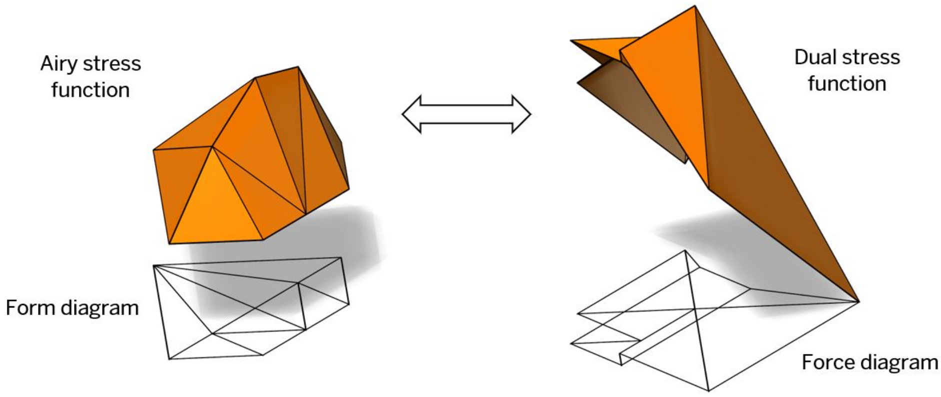

Figure 4: A form diagram and its reciprocal, together with their Airy stress functions (figure adapted from Mitchell et al. (2016)).

that (G^*, q) is unique up to dilation and translations. This is often called the Maxwell construction as opposed to the Cremona construction where lines remain parallel in the force diagram.

Recall that an infinitesimal motion of a framework is a nodal motion which causes no member extensions in the first order. This may be referred to as a 'mechanism' in mathematical and engineering literature. Let m and m^* be the dimensions of the spaces of non-trivial infinitesimal motions of a framework and its reciprocal, respectively. Similarly, let s and s^* be the dimensions of the spaces of self-stresses of a framework and its reciprocal, respectively. Simply, the framework has m mechanisms and s states of self-stress and similarly the reciprocal has m^* mechanisms and s^* states of self-stress. Then, for $s, s^* \geq 1$, we have the key relationships: $s = m^* + 1$ and $s^* = m + 1$ so that $m + s = m^* + s^*$ (Crapo and Whiteley , 1994a; McRobie et al. , 2015). In this paper, we will obtain symmetry-adapted versions of these relationships.

2.3. Symmetric frameworks

Let Γ be a finite group and let $\tau : \Gamma \to O(\mathbb{R}^2)$ be a homomorphism, where $O(\mathbb{R}^2)$ denotes the orthogonal group of $\mathbb{R}^2$. In other words, for each $\gamma \in \Gamma$, $\tau(\gamma)$ is an isometry of $\mathbb{R}^2$ (i.e., a rotation, reflection or combinations thereof). We refer to $\tau(\Gamma)$ as a *symmetry group* and call its elements $\tau(\gamma)$, $\gamma \in \Gamma$, *symmetry operations*. We use a version of the standard Schoenflies notation for symmetry groups and operations in the plane (Altmann and Herzig , 1994).

The relevant symmetry operations are the identity, denoted by id, rotations by $\frac{2\pi}{n}$, $n \in \mathbb{N}$, about the origin, denoted by C_n, and reflections in lines through the origin, denoted by σ.

The symmetry groups that can be created from these operations are the infinite sets $\mathcal{C}_n$ and $\mathcal{C}_{nv}$ for all $n \in \mathbb{N}$. $\mathcal{C}_n$ is the cyclic group generated by C_n, and $\mathcal{C}_{nv}$ is the dihedral group generated by a pair $\{C_n, \sigma\}$. The reflection group $\mathcal{C}_{1v}$ is usually denoted by $\mathcal{C}_s$. It is recommended that readers unfamiliar with this refer to Millar et al. (2021) for a further description of this.

A graph $G = (V, E)$ is called Γ-*symmetric* (with respect to ϕ) if there exists a homomorphism (i.e. group action) $\phi : \Gamma \to \mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ denotes the automorphism group of G. For simplicity, we usually denote $\phi(\gamma)(i)$ as γi for any $\gamma \in \Gamma$ and $i \in V$. Note that each automorphism $\phi(\gamma)$ of G induces a permutation of the edges of G, and we again simply write γe for $\phi(\gamma)e$ for any $\gamma \in \Gamma$ and $e \in E$.

For a Γ-symmetric graph G, a framework (G, p) in $\mathbb{R}^2$ is called $\tau(\Gamma)$-

symmetric if

$$\tau(\gamma)p_i = p_{\gamma i} \quad \text{for all } i \in V \text{ and all } \gamma \in \Gamma.$$

See Figures 5(a) and (b) for examples of $\mathcal{C}_s$-symmetric frameworks, where $\mathcal{C}_s = \tau(\mathbb{Z}_2)$ for $\mathbb{Z}_2 = \{0, 1\}$.

The definition of a $\tau(\Gamma)$-symmetric generalised framework (G, p, q) is as above, with the added condition that if a bar $e = ij$ has length zero (i.e., $p_i = p_j$), then $\tau(\gamma)(q_e) = -q_e$ if $\gamma i = j$ and $\gamma j = i$, and $\tau(\gamma)(q_e) = q_{\gamma e}$ otherwise.

A *representation* of a group Γ is a homomorphism from Γ to the general linear group of some (real or complex) vector space. The *dimension of the representation* is the dimension of that vector space. Every group has a set of irreducible representations, which can be found in standard tables (see Altmann and Herzig (1994) for example). We denote the irreducible representations (over the complex numbers) of a group by $\rho_0, \ldots, \rho_r$, where ρ_0 always denotes the trivial (or fully-symmetric) representation which assigns 1 to each element of the group.

In this paper, we will focus on Abelian groups Γ, i.e. groups whose group operations are commutative. These are the groups that only have one-dimensional irreducible representations over the complex numbers. In other words, $\rho_t(\gamma)$ is a (possibly complex) scalar for any $t \in \{0, \ldots, r\}$ and $\gamma \in \Gamma$. In this case, the number of elements in Γ equals $r + 1$, the number of irreducible representations. (The groups corresponding to $\mathcal{C}_{nv}$, $n \geq 3$, are not Abelian and will be considered in Section 6.3.) So suppose $\rho_0, \ldots, \rho_r$

are all one-dimensional. Then, for a $\tau(\Gamma)$-symmetric framework (G, p) and $t \in \{0, \ldots, r\}$, an assignment $x : V \to \mathbb{C}^2$ of (velocity or displacement) vectors, with one vector $x_i = x(i)$ to each joint p_i of (G, p), is called ρ_t-*symmetric* if

$$\tau(\gamma)x_i = \rho_t(\gamma)x_{\gamma i} \quad \text{for all } \gamma \in \Gamma \text{ and all } i \in V. \tag{2.4}$$

Similarly, an assignment $\omega : E \to \mathbb{R}$ of scalars, with one scalar $\omega_e = \omega(e)$ to each edge e, is called ρ_t-*symmetric* if

$$\omega_e = \rho_t(\gamma)\omega_{\gamma e} \quad \text{for all } \gamma \in \Gamma \text{ and all } e \in E. \tag{2.5}$$

In particular, an assignment of velocity or displacement vectors to the vertices of a $\tau(\Gamma)$-symmetric framework (G, p) is called *fully-symmetric* if it is ρ_0-symmetric. Such a fully-symmetric vector assignment has the property that the vectors remain unchanged under all symmetry operations of $\tau(\Gamma)$ since $\rho_0(\gamma) = 1$ for all $\gamma \in \Gamma$. See Figure 5(a) and (d) for an example of a fully-symmetric infinitesimal motion and a fully-symmetric parallel displacement, respectively.

Similarly, an assignment of scalars to the edges of (G, p) (say the set of force-densities in the framework) is called *fully-symmetric* if it is ρ_0-symmetric. Such a fully-symmetric scalar assignment has the property that all edges in the same edge orbit under the group action are given the same scalar.

Example 2.1. *Consider the frameworks with $\mathcal{C}_s$ symmetry in Figures 5(a)* *and (b). The reflection group $\mathcal{C}_s$ has two irreducible representations. One is the fully-symmetric representation ρ_0 and the other one is the anti-symmetric representation ρ_1 that assigns 1 to the identity operation and -1 to the reflection. The framework in (a) has a fully-symmetric infinitesimal motion, whereas the one in (b) has a ρ_1-symmetric or anti-symmetric infinitesimal motion, since the velocity vectors are all reversed by the reflection (recall Equation (2.4)). Turning the velocity vectors in (a) and (b) by 90 degrees in counterclockwise direction gives parallel displacement vectors resulting in parallel drawings of the frameworks in (a) and (b). The displacement (and hence the resulting parallel drawing) in (c) is anti-symmetric, whereas the one in (d) is fully-symmetric.*

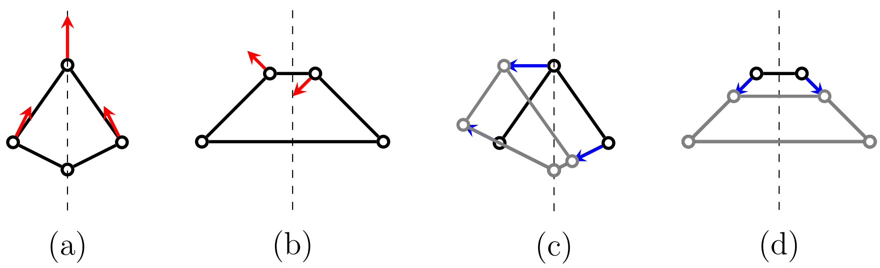

Figure 5: (a),(b): Two frameworks in $\mathbb{R}^2$ with the same underlying graph G and reflection symmetry $\mathcal{C}_s$ (but with different homomorphisms $\phi : V \to \mathrm{Aut}(G)$). The framework in (a) has a fully-symmetric infinitesimal motion and the one in (b) has an anti-symmetric infinitesimal motion. The corresponding parallel displacement vectors for the motions in (a) and (b) are anti-symmetric (c) and fully-symmetric (d), respectively.

Let (G, p) be a $\tau(\Gamma)$-symmetric framework and let $\rho_0, \ldots, \rho_r$ be the irreducible representations of $\tau(\Gamma)$. Then the space of non-trivial infinitesimal motions of (G, p), M, can be written as the direct sum $M = M_0 \oplus \cdots \oplus M_r$, where for each $t = 0, \ldots, r$, M_t is the space of ρ_t-symmetric non-trivial infinitesimal motions of (G, p). Similarly, the space of trivial infinitesimal motions of (G, p), T, can be written as $T = T_0 \oplus \cdots \oplus T_r$, where T_t is the space of ρ_t-symmetric trivial infinitesimal motions of (G, p) (Schulze , 2010a). We denote the dimension of the space M_t as m_t, so that $m = \sum_{t=0}^{r} m_t$.

Analogously, the space of non-trivial parallel displacements of (G, p), D, can be written as $D = D_0 \oplus \cdots \oplus D_r$, where for each $t = 0, \ldots, r$, D_t is the space of ρ_t-symmetric non-trivial parallel displacements of (G, p). Further, the space of trivial parallel displacements of (G, p), C, can be written as $C = C_0 \oplus \cdots \oplus C_r$, where for each $t = 0, \ldots, r$, C_t is the space of ρ_t-symmetric trivial parallel displacements of (G, p). A parallel drawing of (G, p) resulting from a ρ_t-symmetric parallel displacement is also called ρ_t-symmetric.

Finally, the space of self-stresses of (G, p), S, can also be written as $S = S_0 \oplus \cdots \oplus S_r$, where for each $t = 0, \ldots, r$, S_t is the space of ρ_t-symmetric self-stresses of (G, p) (Schulze , 2010a). We denote the dimension of the space S_t as s_t, so that $s = \sum_{t=0}^{r} s_t$. This means that all fully symmetric states of self-stress lie in S_0 and all states of self-stress with symmetry t lie in S_t. The same applies to non-trivial infinitesimal motions or parallel displacements.

3. Mirror symmetry

3.1. Refined Maxwell-Cremona correspondence for mirror symmetry

For frameworks with reflection symmetry, turning vectors by 90 degrees takes fully-symmetric parallel drawings to anti-symmetric infinitesimal motions and vice versa.

Theorem 3.1. *Let (G, p) be a plane framework with reflection symmetry group $\mathcal{C}_s$. Then*

- (G,p) *has a non-trivial fully-symmetric infinitesimal motion if and only if it has a non-trivial anti-symmetric parallel drawing.*

- (G,p) *has a non-trivial anti-symmetric infinitesimal motion if and only if it has a non-trivial fully-symmetric parallel drawing.*

Proof. Let $\mathcal{C}_s = \tau(\mathbb{Z}_2)$ for $\mathbb{Z}_2 = \{0,1\}$. Suppose that the mirror line of the reflection $\sigma \in \mathcal{C}_s$ is the y-axis and that the image of a vertex i of G under the action induced by the reflection is the vertex i'. In other words, $\phi(1)i = i'$. Then, by Equation (2.4), an infinitesimal motion $u : V \to \mathbb{R}^2$ is fully-symmetric if

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} u_i = u_{i'} \quad \text{for all } i \in V.$$

If we turn each u_i by 90 degrees in counterclockwise direction, then the velocity vector $u_i = (x_i, y_i)^T$ becomes the displacement vector $d_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u_i = (-y_i, x_i)^T$ for each i, and hence we have

$$u_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} d_i$$

for each i. Thus, we have

$$d_{i'} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u_{i'} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} u_i = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} d_i = -\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} d_i \quad \text{for all } i \in V,$$

which says that the parallel drawing corresponding to the displacement d :

$V \to \mathbb{R}^2$ is anti-symmetric. Similarly, we see that u is anti-symmetric if and only if d is fully-symmetric.

Finally, note that if we consider the space of trivial infinitesimal motions (where the mirror line is again assumed to be the y-axis), then we have the following correspondences:

- A fully-symmetric infinitesimal translation (along the y-axis) corresponds to an anti-symmetric trivial parallel drawing (translated drawing);
- An anti-symmetric infinitesimal translation (along the x-axis) corresponds to a fully-symmetric trivial parallel drawing (translated drawing);
- An anti-symmetric infinitesimal rotation corresponds to a fully-symmetric dilation.

This gives the result. □

Recall that m and m^* denote the dimensions of the spaces of non-trivial infinitesimal motions of a framework and its reciprocal, respectively. Similarly, s and s^* denote the dimensions of the spaces of self-stresses of a framework and its reciprocal, respectively. If the framework has reflection symmetry, then the motion space M and stress space S decompose as $M = M_0 \oplus M_1$ and $S = S_0 \oplus S_1$, where M_0 and S_0 are the spaces of *fully-symmetric* non-trivial infinitesimal motions and self-stresses, respectively, and M_1 and S_1 are the spaces of *anti-symmetric* non-trivial infinitesimal motions and self-stresses, respectively (recall Section 2 for the definitions). This means symmetric states of self-stress lie in S_0 and antisymmetric states of self-stress lie in S_1 (the same logic applies to the space of infinitesimal motions). The dimension of M_t and S_t are denoted by m_t and s_t, respectively. Similarly, if the reciprocal framework has reflection symmetry, then the motion space M^* and stress space S^* decompose as $M^* = M_0^* \oplus M_1^*$ and $S^* = S_0^* \oplus S_1^*$ and the dimensions of the spaces M_t^* and S_t^* are denoted by m_t^* and s_t^*.

It is known from Steinitz's theorem that the graphs of three-dimensional convex polyhedra are exactly the vertex 3-connected planar graphs. (See Grünbaum (2003), for example.) In the following we will make the assumption that the graphs under consideration are such *polyhedral graphs*. This is a standard assumption in graphic statics, as one is usually interested in polyhedral liftings of the graphs into 3-space.

Corollary 3.2. *Let G be a polyhedral graph and let (G, p) be a plane framework with reflection symmetry, σ. If (G, p) has a fully-symmetric non-trivial self-stress, then the corresponding reciprocal framework (G^*, q) also has reflection symmetry and a fully-symmetric non-trivial self-stress. In addition, we have:*

- $s_0 = m_1^* + 1$ *and* $s_0^* = m_1 + 1$*;*
- $s_1 = m_0^*$ *and* $s_1^* = m_0$*.*

Proof. Let (G, p) be a $\mathcal{C}_s$-symmetric framework with a fully-symmetric self-stress ω. Recall from Section 2.2 that the reciprocal framework of (G, p) corresponding to ω, (G^*, q), is obtained by forming a closed polygon for each vertex of (G, p) in such a way that each edge ij of the polygon is perpendicular to the original edge of (G, p) and has length $||\omega(ij)(p_i - p_j)||$. These polygons are then assembled edge to edge to obtain (G^*, q). Recall also Figure 3. Since ω is fully-symmetric, the polygon corresponding to a vertex that lies on the mirror line has the same reflection symmetry as (G, p). Moreover, the polygons corresponding to vertices of (G, p) that are images of each other under the reflection are also images of each other under the reflection. Thus, by construction, (G^*, q) is also $\mathcal{C}_s$-symmetric.

Note that, by the theory of reciprocal frameworks, (G, p) has a non-trivial self-stress if and only if it has a reciprocal framework (see (Crapo and Whiteley , 1993, Theorem 3.2), for example). So since (G, p) is the reciprocal framework of (G^*, q), it follows that (G^*, q) has a non-trivial self-stress, and by the symmetry of (G, p) and the argument from above used in reverse, this self-stress is also fully-symmetric.

It remains to show that any additional independent fully-symmetric self-stress of (G^*, q) contributing to s_0^* corresponds to a non-trivial anti-symmetric infinitesimal motion of (G, p) contributing to m_1. (Analogously it then follows that any additional independent fully-symmetric self-stress of (G, p) contributing to s_0 corresponds to a non-trivial anti-symmetric infinitesimal motion of (G^*, q) contributing to m_1^*.) If (G^*, q) has another fully-symmetric non-trivial self-stress, then this corresponds to a non-trivial fully-symmetric parallel drawing of (G, p) (again by the construction of reciprocals). By Theorem 3.1, this in turn corresponds to a non-trivial anti-symmetric infinitesimal motion of (G, p). Thus, we have $s_0^* = m_1 + 1$ (and analogously, $s_0 = m_1^* + 1$). This means that the number of symmetric states of self-stress in the reciprocal figure is equal to the number of anti-symmetric infinitesimal motions of the original framework, plus one.

Similarly, Theorem 3.1 also gives the other two equations. □

Combining the equations in Corollary 3.2, we obtain $s_0+m_0 = s_1^*+m_1^*+1$ and $s_1 + m_1 = s_0^* + m_0^* - 1$.

As observed by Maxwell in 1864, a plane framework on a polyhedral graph has a self-stress if and only if it is the vertical projection of a plane-faced polyhedron in 3-space (Maxwell , 1864, 1870). The force in each bar is given by the change in slope over the corresponding edge in the polyhedron (positive weights correspond to convex dihedral angles and negative weights to concave dihedral angles). See e.g. Maxwell (1870); Borcea and Streinu (2015). Therefore, a fully-symmetric self-stress corresponds to a polyhedral lifting (or discrete Airy stress function) that has the same symmetry group (in 3-space) as the original plane framework. Moreover, in the case of reflection symmetry, an anti-symmetric self-stress corresponds to a polyhderal lifting that no longer has reflection symmetry, but is anti-symmetric in the sense that the reflection exchanges convex and concave dihedral angles.

By Corollary 3.2, it follows that any anti-symmetric non-trivial infinitesimal motion (or fully-symmetric non-trivial parallel drawing) of the reciprocal framework corresponds to a mirror-symmetric polyhedral lifting of the original framework. Similarly, any fully-symmetric non-trivial infinitesimal motion (or anti-symmetric non-trivial parallel drawing) of the reciprocal framework corresponds to an anti-symmetric polyhedral lifting of the original framework.

We will see in Section 7 how fully-symmetric and anti-symmetric infinitesimal motions can be found very efficiently via Maxwell-type counts on the corresponding quotient graphs.

3.2. *Example*

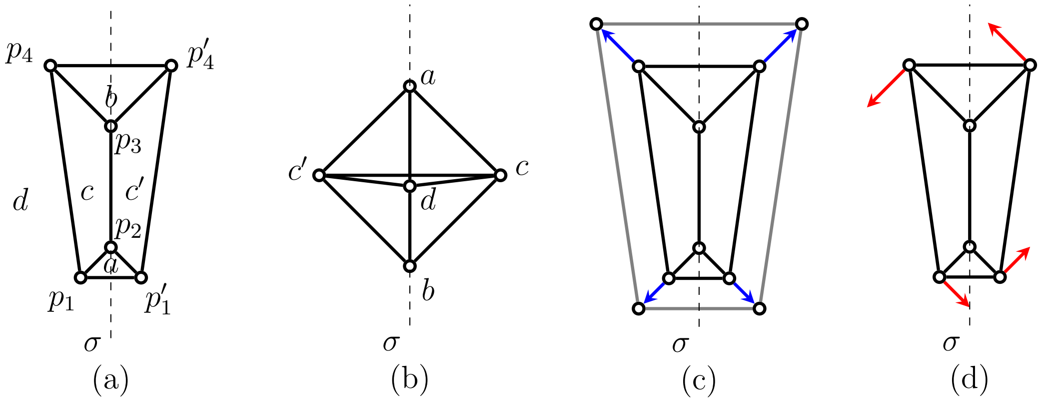

Figure 6: The self-stressed $\mathcal{C}_s$-symmetric plane framework with a fully-symmetric self-stress in (a) has the reciprocal framework (b). This reciprocal framework in (b) has two fully-symmetric self-stresses, i.e. $s_0^* = 2$. Since we have $s_0^* = m_1 + 1$ by Corollary 3.2 the additional self-stress shows up as a fully-symmetric parallel drawing (c) and a corresponding anti-symmetric infinitesimal motion ($m_1 = 1$) in the original framework (d).

The underlying graph of the framework in Figure 6(a) is the planar graph corresponding to a triangular prism in 3-space. While basic plane rigidity results show that such a graph $G = (V, E)$ with $|E| = 9$, $|V| = 6$, and $|E| = 2|V| - 3$ is isostatic (i.e., infinitesimally rigid with no self-stress or $s = m = 0$) for a generic configuration, with mirror symmetry the resulting count of vertex and edge orbits (see Schulze and Whiteley (2011) and Examples 7.2 and 7.5 in Section 7) or an analysis via the symmetry-extended Maxwell rule (Fowler and Guest , 2000; Schulze , 2010a), predicts a fully-symmetric self-stress and an anti-symmetric infinitesimal motion ($s = m = 1$). Note that this is a Desargues configuration.

Drawn with mirror symmetry as in Figure 6(a), where the reflection is denoted by σ, this framework has a reciprocal framework shown in (b) that also has reflection symmetry, as guaranteed by Corollary 3.2. Note that the reciprocal, with count $|E| = 9$, $|V| = 5$, and $|E| = 9 > 7 = 2|V| - 3$, has a 2-dimensional space of self-stresses, which are both fully-symmetric. By the discussion above, this larger space of fully-symmetric self-stresses of the reciprocal framework guarantees that there is a non-trivial fully-symmetric parallel drawing of the original framework (Figure 6(c)). This fully-symmetric parallel drawing corresponds to a non-trivial infinitesimal motion of the original framework that is anti-symmetric (Figure 6(d)).

Overall, we have $s_0 = m_1 = 1$, $s_1 = m_0 = 0$, $s_0^* = 2$, $s_1^* = 0$ and $m_0^* = m_1^* = 0$ for this example.

Note that since $s_0 = 1$ and $s_0^* = 2$, the original framework has one polyhedral lifting with reflection symmetry, whereas the reciprocal framework has two such liftings.

4. Half-turn symmetry

4.1. Refined Maxwell-Cremona correspondence for half-turn symmetry

Since the group $\mathcal{C}_2$ has the same underlying abstract group $\mathbb{Z}_2$ as $\mathcal{C}_s$, it has also only two irreducible representations, namely the fully-symmetric representation ρ_0 and the anti-symmetric representation ρ_1 which assigns 1 to the identity operation and -1 to the half-turn. For frameworks with half-turn symmetry in the plane, turning vectors by 90 degrees preserves the symmetry type of the corresponding infinitesimal motions and parallel drawings.

Theorem 4.1. *Let (G, p) be a plane framework with half-turn symmetry group $\mathcal{C}_2$. Then*

- (G, p) *has a non-trivial fully-symmetric infinitesimal motion if and only if it has a non-trivial fully-symmetric parallel drawing.*
- (G, p) *has a non-trivial anti-symmetric infinitesimal motion if and only if it has a non-trivial anti-symmetric parallel drawing.*

Proof. Let $\mathcal{C}_2 = \tau(\mathbb{Z}_2)$ for $\mathbb{Z}_2 = \{0, 1\}$. Let the image of a vertex i of G under the action induced by the half-turn be denoted by i'. In other words, $\phi(1)i = i'$. By Equation (2.4), an infinitesimal motion $u : V \to \mathbb{R}^2$ is fully-symmetric if

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} u_i = -u_i = u_{i'} \quad \text{for all } i \in V.$$

If we turn each u_i by 90 degrees in counterclockwise direction, then the velocity vector $u_i = (x_i, y_i)^T$ becomes the displacement vector $d_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u_i = (-y_i, x_i)^T$ for each i, and hence we have

$$u_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} d_i$$

for each i. Thus, we have

$$d_{i'} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u_{i'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} u_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} d_i = -d_i \quad \text{for all } i \in V,$$

which says that the parallel drawing corresponding to the displacement $d : V \to \mathbb{R}^2$ is also fully-symmetric. Similarly, it is easy to verify that u is anti-symmetric if and only if d is also.

Finally, note that if we consider the space of trivial infinitesimal motions, then we have the following correspondences:

- A fully-symmetric infinitesimal rotation corresponds to a fully-symmetric dilation;
- An anti-symmetric infinitesimal translation corresponds to an anti-symmetric parallel drawing (translated drawing).

This gives the result. □

From Theorem 4.1 we obtain:

Corollary 4.2. *Let G be a polyhedral graph and let (G, p) be a plane framework with half-turn symmetry. If (G, p) has a fully-symmetric non-trivial self-stress, then the corresponding reciprocal framework (G^*, q) also has half-turn symmetry and a fully-symmetric non-trivial self-stress. In addition, we have:*

- $s_0 = m_0^* + 1$ *and* $s_0^* = m_0 + 1$*;*
- $s_1 = m_1^*$ *and* $s_1^* = m_1$*.*

Proof. Let (G, p) be a $\mathcal{C}_2$-symmetric framework with a fully-symmetric self-stress ω. As in the proof of Corollary 3.2, we consider the construction of the reciprocal using polygons of forces for each vertex. Since ω is fully-symmetric, the polygon of forces corresponding to a vertex that lies on the centre of rotation (the origin) must have half-turn symmetry. The polygons corresponding to vertices of (G, p) that are images of each other under the half-turn are also images of each other under the half-turn. Thus, by construction, (G^*, q) is also $\mathcal{C}_2$-symmetric.

As shown in (Crapo and Whiteley , 1993, Theorem 3.2), (G, p) has a non-trivial self-stress if and only if it has a reciprocal framework. So since (G, p) is the reciprocal framework of (G^*, q), it follows that (G^*, q) has a non-trivial self-stress, and by the symmetry of (G, p) and the argument from above used in reverse, this self-stress is also fully-symmetric.

It remains to show that any additional independent fully-symmetric self-stress of (G^*, q) contributing to s_0^* corresponds to a non-trivial fully-symmetric infinitesimal motion of (G, p) contributing to m_0. (Analogously it then follows that any additional independent fully-symmetric self-stress of (G, p) contributing to s_0 corresponds to a non-trivial fully-symmetric infinitesimal motion of (G^*, q) contributing to m_0^*.) If (G^*, q) has another fully-symmetric non-trivial self-stress, then this corresponds to a non-trivial fully-symmetric parallel drawing of (G, p) (again by the construction of reciprocals). By Theorem 4.1, this in turn corresponds to a non-trivial fully-symmetric infinitesimal motion of (G, p). Thus, we have $s_0^* = m_0 + 1$ (and analogously, $s_0 = m_0^* + 1$). This means that the number of fully-symmetric states of self stress in the reciprocal figure is equal to the number of fully-symmetric infinitesimal motions of the original framework, plus one.

Similarly, Theorem 4.1 also gives the other two equations. □

Combining the equations in Corollary 4.2, we obtain $s_0 + m_0 = s_1^* + m_1^*$ and $s_1 + m_1 = s_0^* + m_0^*$.

As in the reflection case, a fully-symmetric self-stress in a plane framework with half-turn symmetry on a polyhedral graph corresponds to a polyhedral lifting that also has half-turn symmetry. Moreover, an anti-symmetric self-stress corresponds to a polyhedral lifting that is anti-symmetric, in the sense that the half-turn exchanges convex and concave dihedral angles.

By Corollary 4.2, it follows that any fully-symmetric non-trivial infinitesimal motion (or fully-symmetric non-trivial parallel drawing) of the reciprocal framework corresponds to a half-turn-symmetric polyhedral lifting of the original framework. Similarly, any anti-symmetric non-trivial infinitesimal motion (or anti-symmetric non-trivial parallel drawing) of the reciprocal framework corresponds to an anti-symmetric polyhedral lifting of the original framework.

For an efficient method for finding fully- and anti-symmetric infinitesimal motions, we again refer the reader to Section 7.

4.2. Example

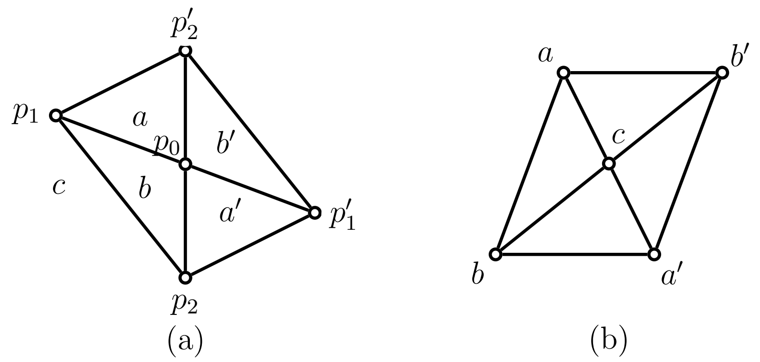

Figure 7: The self-stressed $\mathcal{C}_2$-symmetric plane framework with a fully-symmetric self-stress in (a) has the reciprocal framework (b). The reciprocal also has $\mathcal{C}_2$ symmetry and a fully-symmetric self-stress.

The underlying graph $G = (V, E)$ of the framework in Figure 7(a) is the planar graph corresponding to a quadrilateral pyramid in 3-space. Since $|E| = 8 > 7 = 2|V| - 3$, basic plane rigidity results show that the framework in Figure 7(a) must have a self-stress ($s = 1$). Note that this framework has half-turn symmetry, and a symmetry analysis such as the one described in Section 7 (see Theorem 7.1) shows that the framework has a fully-symmetric self-stress (since $|\overline{E}| = 4 > 3 = 2|V'| - 1$ in this case, with the notation from Theorem 7.1).

By the discussion above, the reciprocal framework corresponding to this self-stress also has half-turn symmetry and a fully-symmetric self-stress. See Figure 7(b). Since neither framework of the reciprocal pair has an infinitesimal motion, we have $s_0 = s_0^* = 1$ and $s_1 = s_1^* = 0$, by Corollary 4.2. The polyhedral lifting corresponding to the fully-symmetric self-stress for either framework retains the half-turn symmetry.

5. Rotational symmetry $\mathcal{C}_n$, $n \geq 3$, in the plane

5.1. Refined Maxwell-Cremona correspondence for rotational symmetry

Let $\mathbb{Z}_n = \{0, \ldots, n-1\}$ and for each $\gamma \in \mathbb{Z}_n$, let $\tau(\gamma)$ be the matrix representing the rotation about the origin by $\gamma 2\pi/n$ in the counterclockwise direction, i.e. $\tau(\gamma) = \begin{bmatrix} \cos\frac{\gamma 2\pi}{n} & -\sin\frac{\gamma 2\pi}{n} \\ \sin\frac{\gamma 2\pi}{n} & \cos\frac{\gamma 2\pi}{n} \end{bmatrix}$. This gives the symmetry group $\mathcal{C}_n = \tau(\mathbb{Z}_n)$.

When we work with complex numbers, the group $\mathcal{C}_n$ has n irreducible 1-dimensional representations whose characters are denoted by ρ_t for $t = 0, \ldots, n-1$. The representation ρ_t is defined by $\rho_t(\gamma) = \epsilon^{t\gamma}$, where ϵ denotes the complex root of unity $e^{\frac{2\pi\sqrt{-1}}{n}}$.

Recall from Section 2.3 that for a $\mathcal{C}_n$-symmetric framework (G, p), an assignment $x : V \to \mathbb{C}^2$ satisfying $\tau(\gamma)x_i = \epsilon^{t\gamma}x_{\gamma i}$ for all $\gamma \in \mathbb{Z}_n$ and $i \in V$ is called *ρ_t-symmetric.*

Theorem 5.1. *Let (G, p) be a plane framework with symmetry group $\mathcal{C}_n$, $n \geq 3$. Then for $t \in \{1, \dots, n-1\}$, (G, p) has a non-trivial ρ_t-symmetric infinitesimal motion if and only if it has a non-trivial ρ_t-symmetric parallel drawing.*

Proof. Let $t \in \{0, 1, \dots, n-1\}$ and suppose the infinitesimal motion u is ρ_t-symmetric. As before, we have $d_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u_i$ for all $i \in V$. For all $\gamma \in \mathbb{Z}_n$ and $i \in V$ we have

$$\begin{aligned} d_{\gamma i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u_{\gamma i} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \epsilon^{-t\gamma}\tau(\gamma)u_i \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \epsilon^{-t\gamma}\tau(\gamma) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} d_i \\ &= \epsilon^{-t\gamma}\tau(\gamma)d_i, \end{aligned}$$

where the last equality holds because $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tau(\gamma) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \tau(\gamma)$. Thus, d is also ρ_t-symmetric as claimed.

Finally, note that if we consider the space of trivial infinitesimal motions, then we have the following correspondences:

- A fully-symmetric infinitesimal rotation corresponds to a fully-symmetric dilation;

- The space of infinitesimal translations is spanned by a ρ_1- and a ρ_{n-1}-symmetric translation. These translations assign $(1, \sqrt{-1})^T$ and $(1, -\sqrt{-1})^T$ to each joint, respectively (Schulze and Tanigawa , 2015). A ρ_t-symmetric infinitesimal translation corresponds to a ρ_t-symmetric parallel drawing (translated drawing) for each t.

This gives the result. □

From Theorem 5.1 we obtain:

Corollary 5.2. *Let G be a polyhedral graph and let (G, p) be a plane framework with $\mathcal{C}_n$ symmetry for $n \geq 3$. If (G, p) has a fully-symmetric non-trivial self-stress, then the corresponding reciprocal framework (G^*, q) also has $\mathcal{C}_n$ symmetry and a fully-symmetric non-trivial self-stress. In addition, we have:*

- $s_0 = m_0^* + 1$ *and* $s_0^* = m_0 + 1$*;*
- $s_t = m_t^*$ *and* $s_t^* = m_t$ *for each* $t \in \{1, \dots, n-1\}$*.*

We omit the proof as it is analogous to the proofs of Corollaries 3.2 and 4.2.

Note that when we work with real numbers, rather than complex numbers, then for each $t \in \{1, \dots, n-1\}$, the pair of representations ρ_t and ρ_{n-t} of $\mathcal{C}_n$ combine to a 2-dimensional irreducible representation (see Altmann and Herzig (1994), for example). Since velocity or parallel displacement vectors in practical applications do not have complex entries, it is natural to consider these 2-dimensional real irreducible representations by pairing up ρ_t and ρ_{n-t} and adding up the corresponding counts in Corollary 5.2 for each

t. However, the complexification of the vector spaces reveals the even more refined symmetry-adapted counts given above.

5.2. Example

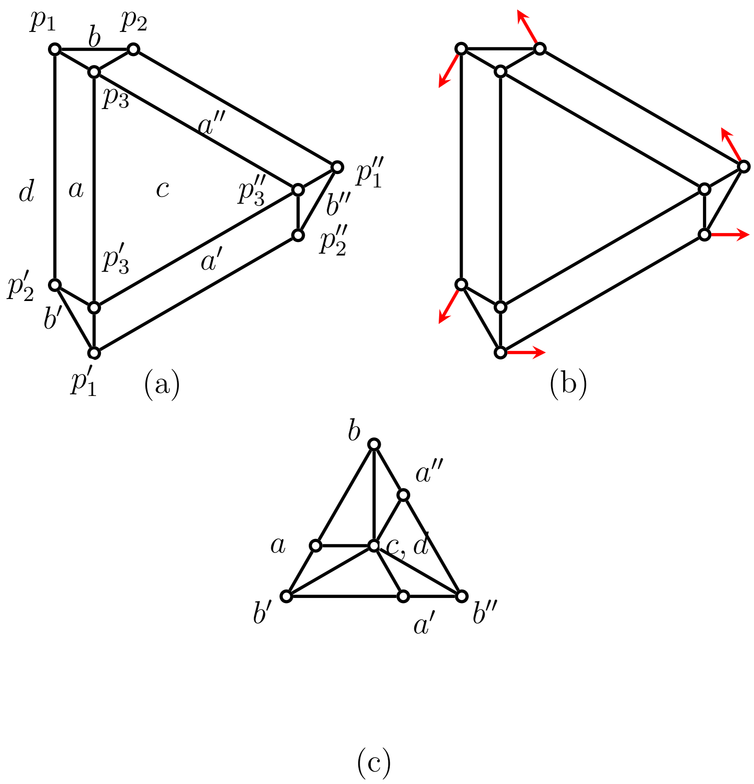

Figure 8: (a) A $\mathcal{C}_3$-symmetric framework with a fully-symmetric self-stress (and a fully-symmetric infinitesimal motion shown in (b)). The reciprocal framework shown in (c) also has $\mathcal{C}_3$ symmetry. It has two (non-adjacent) coincident vertices at the origin (namely the ones corresponding to the faces c and d in (a)) as well as overlapping edges, so that not all vertices and edges are shown in the figure.

Consider the underlying graph $G = (V, E)$ of the framework (G, p) in Figure 8(a). It has $|E| = 15$ and $|V| = 9$, and hence it satisfies the isostatic count $|E| = 2|V| - 3$. For generic configurations, the framework is in fact isostatic ($s = m = 0$). However, if realised with $\mathcal{C}_3$ symmetry as in Figure 8(a), the framework (G, p) has a fully-symmetric self-stress and a fully-symmetric infinitesimal motion, so that $s_0 = m_0 = 1$ and $s_t = m_t = 0$ for $t = 1, 2$.

Note that the $\mathcal{C}_3$ symmetry is not enough to destroy isostaticity; in fact, *almost all* realisations of G as a plane framework with $\mathcal{C}_3$ symmetry remain isostatic. Thus, even the symmetry-extended Maxwell rule (Fowler and Guest , 2000; Schulze , 2010a) or the results in Section 7 applied to the symmetry group $\mathcal{C}_3$ do not predict any infinitesimal motion or self-stress of (G, p). The infinitesimal motion and self-stress of (G, p) appear because the set of four points p_1, p_3, p_2', p_3' (and symmetrically, the sets p_1', p_3', p_2'', p_3'' and p_1'', p_3'', p_2, p_3) forms a parallelogram, so that the triangle $p_1p_2p_3$ and its two symmetric copies can each rotate in a symmetric fashion (Schulze and Whiteley , 2011).

We may construct the reciprocal framework (G^*, q) of (G, p) corresponding to the fully-symmetric self-stress; see Figure 8(c). By Corollary 5.2, it also has $\mathcal{C}_3$ symmetry and it has two fully-symmetric self-stresses: $s_0^* = 2$. Moreover, we may conclude from Corollary 5.2 that $m_0^* = 0$ and $s_t^* = m_t^* = 0$ for $t = 1, 2$. The additional fully-symmetric self-stress in (G^*, q) appears, because it corresponds to a fully-symmetric parallel drawing of the original framework (G, p), which in turn corresponds to a fully-symmetric infinitesimal motion of (G, p), by Theorem 5.1.

Conversely, using the orbit counts in Section 7, we can detect that the framework (G^*, q) has two fully-symmetric self-stresses (since $|\overline{E}| = 5$, with $\overline{E} = \{ab, ab', ac, ad, bd\}$ and $2|V'| - 1 = 3$ in the notation of Theorem 7.1). Using Corollary 5.2, this tells us that the framework (G, p) has a fully-symmetric infinitesimal motion, since $m_0 = s_0^* - 1$. Similarly, since $m_0^* = 0$, we see that

(G, p) has exactly one fully-symmetric state of self-stress.

Note that since $s_0 = 1$ and $s_0^* = 2$, the original framework has one polyhedral lifting with $\mathcal{C}_3$ symmetry, whereas the reciprocal framework has two such liftings.

6. Dihedral symmetry in the plane

We now discuss the dihedral groups which are commonly found in engineering structures. We begin with the dihedral group $\mathcal{C}_{2v}$ of order 4, which is special among the dihedral groups as it is the only dihedral group that only has one-dimensional irreducible representations over the complex numbers.

6.1. Refined Maxwell-Cremona correspondence for $\mathcal{C}_{2v}$

The characters of the four irreducible representations of $\mathcal{C}_{2v}$ are shown in Table 1. The reflections σ_x and σ_y are the reflections in the x-axis and y-axis, respectively. For readers unfamiliar with this notation, please refer to Millar et al. (2021) for a full description.

$\mathcal{C}_{2v}$	id	C_2	σ_x	σ_y
ρ_0	1	1	1	1
ρ_1	1	1	-1	-1
ρ_2	1	-1	1	-1
ρ_3	1	-1	-1	1

Table 1: The irreducible representations of $\mathcal{C}_{2v}$.

Theorem 6.1. *Let (G, p) be a plane framework with symmetry group $\mathcal{C}_{2v}$. Then*

- (G, p) *has a non-trivial fully-symmetric infinitesimal motion if and only if it has a non-trivial* ρ_1*-symmetric parallel drawing and vice versa.*
- (G, p) *has a non-trivial* ρ_2*-symmetric infinitesimal motion if and only if it has a non-trivial* ρ_3*-symmetric parallel drawing and vice versa.*

Proof. This is an immediate consequence of Theorems 3.1 and 4.1. □

Corollary 6.2. *Let* G *be a polyhedral graph and let* (G, p) *be a plane framework with* $\mathcal{C}_{2v}$ *symmetry. If* (G, p) *has a fully-symmetric non-trivial self-stress, then the corresponding reciprocal framework* (G^*, q) *also has* $\mathcal{C}_{2v}$ *symmetry and a fully-symmetric non-trivial self-stress. In addition, we have:*

- $s_0 = m_1^* + 1$ *and* $s_0^* = m_1 + 1$*;*
- $s_1 = m_0^*$ *and* $s_1^* = m_0$*;*
- $s_2 = m_3^*$ *and* $s_2^* = m_3$*;*
- $s_3 = m_2^*$ *and* $s_3^* = m_2$*.*

The proof of Corollary 6.2 is again analogous to the one of Corollaries 3.2 and 4.2, so we omit the details. If (G, p) has a fully-symmetric self-stress, then, by construction, (G^*, q) also has $\mathcal{C}_{2v}$ symmetry and a fully-symmetric self-stress. If (G^*, q) has an additional independent fully-symmetric non-trivial self-stress, then this corresponds to a non-trivial fully-symmetric parallel drawing of (G, p). By Theorem 6.1, this in turn corresponds to a non-trivial ρ_1-symmetric infinitesimal motion of (G, p). Thus, we have $s_0^* = m_1 + 1$ (and analogously, $s_0 = m_1^* + 1$).

Similarly, if (G^*, q) has an additional independent ρ_1-, ρ_2-, or ρ_3-symmetric non-trivial self-stress, then this corresponds to a non-trivial ρ_1-, ρ_2-, or ρ_3-symmetric parallel drawing of (G, p), respectively. By Theorem 6.1, these in turn correspond to non-trivial ρ_0-, ρ_3- and ρ_2-symmetric infinitesimal motions of (G, p), respectively.

6.2. Example

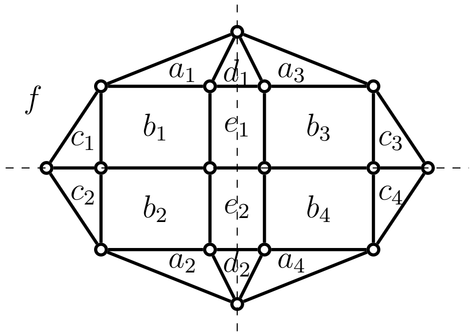

Figure 9: A framework with $\mathcal{C}_{2v}$ symmetry which has 2 fully-symmetric self-stresses, a ρ_2-symmetric self-stress and a ρ_1-symmetric infinitesimal motion.

We illustrate Corollary 6.2 by applying it to the framework shown in Figure 9.

The underlying graph $G = (V, E)$ of the framework (G, p) in Figure 9 has $|V| = 16$ and $|E| = 31$, and hence we have $|E| = 2|V| - 1$. Thus, (G, p) must have at least two independent self-stresses ($s \geq 2$). A more detailed symmetry analysis using the group $\mathcal{C}_{2v}$ reveals that (G, p) has two fully-symmetric self-stresses and another ρ_2-symmetric self-stress, as well as a ρ_1-symmetric infinitesimal motion. This can be seen by applying the symmetry-extended Maxwell rule (Fowler and Guest , 2000; Millar et al. , 2021; Schulze et al. , 2022; Schulze , 2010a), but it can also be verified more directly using the orbit counts described in Section 7 (see Example 7.8 for a detailed discussion of this example). So for (G,p) we have $s_0 = 2$, $s_2 = 1$ and $s_1 = s_3 = 0$, as well as $m_1 = 1$ and $m_0 = m_2 = m_3 = 0$.

For each of the three self-stresses, we may construct the corresponding reciprocal diagram. Since two of the self-stresses are fully-symmetric, two of the reciprocal frameworks again have $\mathcal{C}_{2v}$ symmetry, by Corollary 6.2 (see Figure 10(a) and (b)). The third self-stress is ρ_2-symmetric and hence (by definition of ρ_2) it is fully-symmetric with respect to the reflection in the horizontal mirror (but anti-symmetric with respect to the vertical mirror and the half-turn). Thus, the corresponding reciprocal framework only has $\mathcal{C}_s$ symmetry, where the reflection in $\mathcal{C}_s$ is in the horizontal mirror (see Figure 10(c)). Note that all three reciprocal frameworks are generalised frameworks, as they contain bars of length zero.

A state of self-stress of the original framework can be any linear combination of the three states of self-stress. The corresponding reciprocal diagram is a linear combination of the three individual reciprocal diagrams; the nodal coordinates are combined linearly and the framework bars are drawn between them. All bars remain perpendicular to the original bars of the framework. Therefore, the reciprocal is not unique (if there is only one state of self-stress then the reciprocal is unique up to translation and scaling). Furthermore, if we restrict to the two fully-symmetric states of self-stress, then any linear combination results in a reciprocal diagram which is also fully-symmetric. The same applies generally for ρ_t-symmetric self-stresses.

If we consider a fully-symmetric self-stress of (G,p), then, by Corollary 6.2, for the corresponding reciprocal framework (Figure 10(a) and (b)) we may conclude that $s_0^* = 2$ and $s_1^* = s_2^* = s_3^* = 0$, as well as $m_1^* = m_3^* = 1$

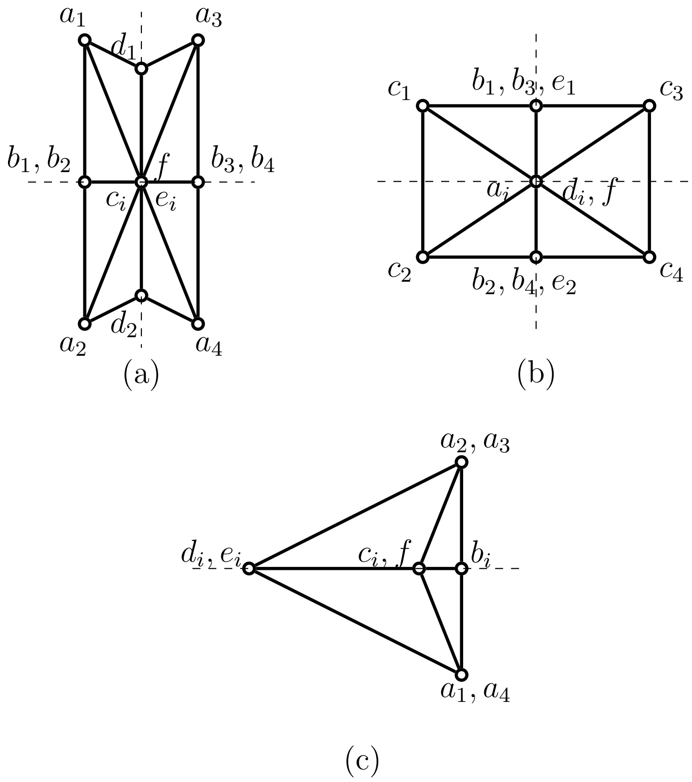

Figure 10: The reciprocal (generalised) frameworks of the example in Figure 9 corresponding to the two fully-symmetric self-stresses (a), (b) and the ρ_2-symmetric self-stress (c). Note that each reciprocal framework has some coincident vertices and overlapping edges, as well as edges of length zero, so not all vertices and edges are shown in the figure. The labelled vertices correspond to the faces with the same labels in the original framework in Figure 9.

and $m_0^* = m_2^* = 0$. In other words, the reciprocal framework has two fully-symmetric self-stresses as well as a ρ_1- and ρ_3-symmetric infinitesimal motion. The additional fully-symmetric self-stress in the reciprocal framework appears, because it corresponds to a fully-symmetric parallel drawing of the original framework (G, p), which in turn corresponds to a ρ_1-symmetric infinitesimal motion of (G, p), by Theorem 6.1.

Conversely, using the orbit counts in Section 7, we can detect that the reciprocal framework has two fully-symmetric self-stresses and an infinitesimal motion of symmetry type ρ_1 and ρ_3 (see Example 7.9 for details), from which we can deduce the rigidity properties of the original framework (G, p) using Corollary 6.2.

As mentioned above, the third self-stress of (G, p) is not fully-symmetric but ρ_2-symmetric. The kernel of ρ_2 (i.e. the subgroup of $\mathcal{C}_{2v}$ consisting of all elements that are mapped to 1 under ρ_2) is the reflection group $\mathcal{C}_s$. In this case, we can still use our methods to analyse the reciprocal pair using the $\mathcal{C}_s$ symmetry (see Figure 10(c)). For the $\mathcal{C}_s$-symmetric reciprocal framework of (G, p) corresponding to the ρ_2-symmetric self-stress of (G, p), the symmetry analysis carried out in Example 7.9 shows that it has two-fully-symmetric self-stresses and two anti-symmetric infinitesimal motions. Thus, we can use Corollary 3.2 to detect that (G, p) has three fully-symmetric self-stresses, and hence three mirror-symmetric polyhedral liftings, and one anti-symmetric infinitesimal motion ($s_0^* = 3$, $m_1^* = 1$, $s_1^* = m_0^* = 0$).

6.3. *The groups* $\mathcal{C}_{nv}$*,* $n \geq 3$

The analysis above immediately extends to dihedral groups of higher order. A key difference to the groups discussed so far is that the dihedral groups $\mathcal{C}_{nv}$ with $n \geq 3$ also have 2-dimensional irreducible representations over the complex numbers.

For example, the group $\mathcal{C}_{3v}$, which is the symmetry group of an equilateral triangle in the plane and consists of the identity, two counter-clockwise rotations about the origin by 120 and 240 degrees, and three reflections, has three irreducible representations: the 1-dimensional fully-symmetric representation ρ_0 that assigns 1 to each group element, the 1-dimensional representation ρ_1 that assigns 1 to the identity and the two rotations, and -1 to the three reflections, and the 2-dimensional representation ρ_2 that assigns to each of the six isometries in $\mathcal{C}_{3v}$ the corresponding 2×2 orthogonal matrix (with respect to a fixed basis of $\mathbb{R}^2$).

As for the symmetry groups discussed above, given a $\tau(\Gamma)$-symmetric framework (G, p), where $\tau(\Gamma) = \mathcal{C}_{nv}$ for some $n \geq 3$, the spaces M and D of non-trivial infinitesimal motions and parallel displacements of (G, p) can be decomposed as $M = M_0 \oplus \cdots \oplus M_{r'}$ and $D = D_0 \oplus \cdots \oplus D_{r'}$, where $r' + 1$ is the number of conjugacy classes of $\tau(\Gamma)$ and M_t and D_t are the spaces of ρ_t-symmetric non-trivial infinitesimal motions and parallel displacements of (G, p), respectively. The same is true for the spaces of trivial infinitesimal motions and parallel displacements. For the 1-dimensional representations ρ_t, such ρ_t-symmetric vector assignments have been defined in Section 2.3. To extend this definition to 2-dimensional representations, we need some further terminology from group representation theory.

For a group representation $\Phi : \Gamma \to GL(\mathbb{C}^n)$, a subspace $U \subseteq \mathbb{C}^n$ is called Φ-*invariant* if $\Phi(\gamma)(U) \subseteq U$ for all $\gamma \in \Gamma$. For a $\tau(\Gamma)$-symmetric framework (G, p), we let $P_V : \Gamma \to GL(\mathbb{C}^{|V|})$ be the representation that sends each element γ of Γ to the permutation matrix $P_V(\gamma) = [\delta_{i,\gamma(i')}]_{i,i'}$ that describes how $\tau(\gamma)$ permutes the vertices of (G, p). (Here δ denotes the Kronecker delta.) The representation $P_V \otimes \tau : \Gamma \to GL(\mathbb{C}^{2|V|})$ is the representation that assigns to each group element γ of Γ the Kronecker product of the permutation matrix $P_V(\gamma)$ and the orthogonal matrix $\tau(\gamma)$. The spaces M_t and D_t are then the $(P_V \otimes \tau)$-invariant subspaces corresponding to ρ_t. For the 1-dimensional representations ρ_t this definition simplifies as described in Section 2.3. We refer the reader to Kangwai and Guest (2000); Schulze (2010a) for further details.

Similarly, the space S of self-stresses of (G, p) can be decomposed as $S = S_0 \oplus \cdots \oplus S_{r'}$, where S_t is the P_E-invariant subspace corresponding to ρ_t, for $t = 0, \ldots, r'$, and $P_E : \Gamma \to GL(\mathbb{C}^{|E|})$ is the representation that sends each element γ of Γ to the permutation matrix $P_E(\gamma)$ that describes how $\tau(\gamma)$ permutes the edges of (G, p).

It is a routine calculation to verify the following result for the symmetry group $\mathcal{C}_{3v}$.

Theorem 6.3. *Let (G, p) be a plane framework with symmetry group $\mathcal{C}_{3v}$. Then*

- *(G, p) has a non-trivial fully-symmetric infinitesimal motion if and only if it has a non-trivial ρ_1-symmetric parallel drawing and vice versa.*
- *(G, p) has a non-trivial ρ_2-symmetric infinitesimal motion if and only*

if it has a non-trivial ρ_2-symmetric parallel drawing.

Thus, we obtain the following result.

Corollary 6.4. *Let G be a polyhedral graph and let (G, p) be a plane framework with $\mathcal{C}_{3v}$ symmetry. If (G, p) has a fully-symmetric non-trivial self-stress, then the corresponding reciprocal framework (G^*, q) also has $\mathcal{C}_{3v}$ symmetry and a fully-symmetric non-trivial self-stress. In addition, we have:*

- $s_0 = m_1^* + 1$ *and* $s_0^* = m_1 + 1$;
- $s_1 = m_0^*$ *and* $s_1^* = m_0$;
- $s_2 = m_2^*$ *and* $s_2^* = m_2$.

The corresponding results for dihedral groups $\mathcal{C}_{nv}$ with $n \geq 4$ can be obtained analogously in a straightforward fashion.

7. Orbit counts and simplified construction of reciprocal

As we have seen, it is often useful to be able to detect ρ_t-symmetric infinitesimal motions or self-stresses in symmetric frameworks. An efficient and powerful tool to do this is the symmetry-extended Maxwell rule, which was first established by Fowler and Guest in 2000 and is described in Fowler and Guest (2000); Schulze (2010a); Schulze et al. (2022). This rule is based on group representation theory and considers all symmetry types corresponding to the irreducible representations of the symmetry group simultaneously. Alternatively, one may focus on a particular irreducible representation ρ_t and employ a simpler and more direct analysis of whether there exist ρ_t-symmetric infinitesimal motions or self-stresses. In this section, we describe how this can be done using the concept of a quotient graph (or orbit graph), starting with the fully-symmetric case ($t = 0$). We again focus on the case where all bar lengths are strictly positive. However, Example 7.9 shows how these counts can also be applied to generalised frameworks.

7.1. Fully-symmetric orbit counts

If a framework has no non-trivial ρ_0-symmetric (or fully-symmetric) infinitesimal motions (but possibly other types of non-trivial infinitesimal motions), then the framework is said to be *forced symmetric infinitesimally rigid.* Further, if the framework is forced symmetric infinitesimally rigid and has no non-trivial fully-symmetric self-stresses, then it is called *forced symmetric isostatic* (Jordán et al. , 2016; Schulze and Whiteley , 2018b).

Necessary conditions for a $\tau(\Gamma)$-symmeric framework (G, p) to be forced symmetric isostatic in the plane have been established in Schulze and Whiteley (2011); Jordán et al. (2016). These conditions are Maxwell-type counts for the *quotient graph* $\overline{G} = G/\Gamma$ of G, which is defined as follows.

Let G be a Γ-symmetric graph (with respect to ϕ). Then the vertex set of the quotient graph $\overline{G}$ of G is the set of vertex orbits of G under the action of Γ. In other words, each vertex v of $\overline{G}$ represents the set of vertices $\{\phi(\gamma)(v)|\, \gamma \in \Gamma\}$ of G (the vertex orbit of v). Similarly, the edge set of $\overline{G}$ is the set of edge orbits of G under the action of Γ, with the edge e of $\overline{G}$ representing its edge orbit $\{\phi(\gamma)(e)|\, \gamma \in \Gamma\}$.

Note that $\overline{G}$ is a multi-graph which may contain parallel edges and loops. In particular, an edge orbit may be represented by a loop in $\overline{G}$, since an edge in G may connect a vertex with another vertex in the same vertex orbit. See

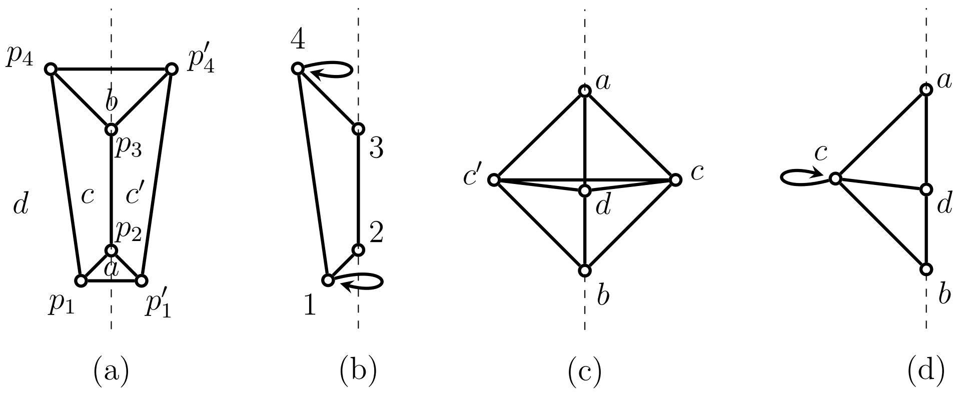

Figure 11: A plane framework (G, p) with $\mathcal{C}_s$ symmetry and a fully symmetric self-stress (a). The quotient graph of G is shown in (b). Since p_4 and p_4' are images of each other under the reflection, the corresponding vertices form one vertex orbit, which is represented by the vertex labelled 4 in the quotient graph. The same is true for p_1 and p_1'. Each of the vertices 2 and 3 forms a vertex orbit of size 1 (since p_2 and p_3 are fixed by the reflection). The underlying graph of the $\mathcal{C}_s$-symmetric reciprocal framework in (c) has the quotient graph (d).

Figure 11(a) and (b) for an example of a $\mathcal{C}_s$-symmetric framework and its corresponding quotient graph.

For a Γ-symmetric graph (with respect to ϕ), we say that a vertex i is *unshifted* by an element γ in Γ if $\phi(\gamma)(i) = i$. For a $\tau(\Gamma)$-symmetric framework (G, p) (with no coincident joints) a vertex i of G is unshifted by a reflection if and only if the joint p_i lies on the mirror line of the reflection. (This is the case for the joints p_2 and p_3 in Figure 11(a).) Similarly, a vertex of G is unshifted by a rotation if and only if the vertex lies on the centre of rotation (i.e. the origin). The set of vertices of the quotient graph $\overline{G}$ that correspond to vertices that are unshifted by a reflection σ or a rotation C_n, $n \geq 2$, are denoted by V_σ and V_n, respectively. The set of 'free' vertices of $\overline{G}$ that correspond to orbits of vertices that are not unshifted is denoted by V'. In Figure 11(b) we have $|V'| = \{1, 4\}$ and $|V_\sigma| = \{2, 3\}$ and in Figure 11(d) we have $|V'| = \{c\}$ and $|V_\sigma| = \{a, b, d\}$, for example.

The theorem below summarizes necessary counts for symmetric frameworks to be forced symmetric isostatic. For the case when the group action is free on the vertex set (i.e. no vertices are unshifted by non-trivial symmetry operations), these counts can be found in Jordán et al. (2016). Here we extend these counts to allow for vertices that are unshifted by non-trivial symmetry operations.

In the following, for sets A and B of vertices, we write $A \setminus B$ for the set of vertices that lie in A but not in B.

Theorem 7.1. *Let (G, p) be a $\tau(\Gamma)$-symmetric forced symmetric isostatic framework in the plane and let $\overline{G} = (\overline{V}, \overline{E})$ be the quotient graph of G. Then the following hold.*

- *If $\tau(\Gamma) = \mathcal{C}_s$ then $|\overline{E}| = 2|V'| + |V_\sigma| - 1$.*
- *If $\tau(\Gamma) = \mathcal{C}_n$, $n \geq 2$, then $|\overline{E}| = 2|V'| - 1$.*
- *If $\tau(\Gamma) = \mathcal{C}_{2nv}$, $n \geq 1$ then $|\overline{E}| = 2|V'| + |V_\sigma \setminus V_2| + |V'_\sigma \setminus V_2|$, where σ and σ' are two reflections lying in different conjugacy classes of $\mathcal{C}_{2nv}$.*
- *If $\tau(\Gamma) = \mathcal{C}_{(2n+1)v}$, $n \geq 1$, then $|\overline{E}| = 2|V'| + |V_\sigma \setminus V_{2n+1}|$, where σ is any reflection of $\mathcal{C}_{(2n+1)v}$.*

Intuitively, the term $2|V'|$ reflects the fact that each representative of a vertex orbit has two degrees of freedom in the plane. The velocity vectors of all other vertices in the same vertex orbit are uniquely determined by the velocity vector of the representative vertex since we restrict our attention to fully-symmetric velocity assignments. Similarly, the term $|V_\sigma|$ arises from the fact that each vertex that is unshifted by a reflection has only one degree of freedom, as the vertex has to remain on the mirror line of the reflection (so the velocity vector has to lie along the mirror line). In the forced symmetric rigidity setting, a vertex that is unshifted by a rotation must remain at the origin and hence has no degree of freedom. Finally, the dimension of the space of fully-symmetric trivial infinitesimal motions is 1 for $\mathcal{C}_s$ (the translation along the mirror) and for $\mathcal{C}_n$ (rotation about the origin), but 0 for the dihedral groups $\mathcal{C}_{nv}$.

Theorem 7.1 can be proved using the definition of the orbit rigidity matrix given in Schulze and Whiteley (2011) and a straightforward adaptation of the proof given in Jordán et al. (2016) for the case when the group action is free on the vertex set. We refer the reader to La Porta (2024); La Porta and Schulze (2023) for details.

Example 7.2. *The quotient graph in Figure 11(b) has* $|\overline{E}| = 6$*,* $|V'| = 2$ *and* $|V_\sigma| = 2$*. Thus,* $|\overline{E}| - 2|V'| - |V_\sigma| + 1 = 1$*, showing that the framework in Figure 11(a) has a fully-symmetric self-stress.*

Similarly, the quotient graph in Figure 11(d) has $|\overline{E}| = 6$*,* $|V'| = 1$ *and* $|V_\sigma| = 3$*. Thus,* $|\overline{E}| - 2|V'| - |V_\sigma| + 1 = 2$*, showing that the reciprocal framework in Figure 11(c) has two fully-symmetric self-stresses.*

Remark 7.3. There are further necessary conditions for forced symmetric isostaticity which are given in terms of sparsity counts of the *group-labelled* quotient graph (also known as a *quotient gain graph*) of the underlying graph of the framework (Schulze and Whiteley , 2011; Jordán et al. , 2016; Schulze and Tanigawa , 2015). A symmetric framework is called *symmetry-generic* if it is realised as generic as possible with the given symmetry constraints. For the groups $\mathcal{C}_s$, $\mathcal{C}_n$ and $\mathcal{C}_{(2n+1)v}$, $n \in \mathbb{N}$, it was shown in Jordán et al. (2016) that the counts in Theorem 7.1 together with the sparsity counts on the corresponding quotient gain graphs are also sufficient for symmetry-generic forced symmetric isostaticity in the plane (in the case when the group action is free on the vertex set). See also Bernstein (2022). This is not the case for the groups $\mathcal{C}_{2nv}$, $n \in \mathbb{N}$, with Bottema's mechanism (Bottema , 1960) being a classical counterexample.

7.2. Anti-symmetric orbit counts

Analogous to the forced symmetric (or ρ_0-symmetric) rigidity analysis one can carry out a ρ_t-symmetric rigidity analysis for each t. We demonstrate this for the Abelian symmetry groups in the plane, which only have one-dimensional irreducible representations over the complex numbers, i.e. for the groups $\mathcal{C}_s$, $\mathcal{C}_n$, $n \geq 2$, and $\mathcal{C}_{2v}$. (For the non-Abelian groups, this type of orbit counting becomes more difficult as it requires a modified definition of a 'quotient graph'.)

If a framework has no non-trivial ρ_t-symmetric infinitesimal motions (but possibly other types of non-trivial infinitesimal motions), then the framework is said to be *ρ_t-symmetric infinitesimally rigid*. Further, if the framework is ρ_t-symmetric infinitesimally rigid and has no non-trivial ρ_t-symmetric self-stresses, then it is called *ρ_t-symmetric isostatic*.

We first consider the reflection and half-turn group which both only have the two irreducible representations ρ_0 and ρ_1. The following theorem is a straightforward extension of the results obtained in Schulze and Tanigawa (2015) for the case when the group action is free on the vertex set. We again refer the reader to La Porta (2024); La Porta and Schulze (2023) for details.

For a quotient graph $\overline{G} = (\overline{V}, \overline{E})$, we denote $\overline{G}_\ell = (\overline{V}, \overline{E}_\ell)$ to be the multigraph obtained from $\overline{G}$ by removing all loops that correspond to edges in G joining a vertex with its image under a reflection of half-turn symmetry, and all edges joining vertices in V_σ for a reflection σ.

Theorem 7.4. *Let (G, p) be a ρ_1-symmetric isostatic framework with symmetry group $\mathcal{C}_s$ or $\mathcal{C}_2$ in the plane, and let $\overline{G} = (\overline{V}, \overline{E})$ be the quotient graph of G. Then the following hold for $\overline{G}_\ell$.*

- *If $\tau(\Gamma) = \mathcal{C}_s$ then $|\overline{E}_\ell| = 2|V'| + |V_\sigma| - 2$.*
- *If $\tau(\Gamma) = \mathcal{C}_2$ then $|\overline{E}_\ell| = 2|V'| + 2|V_2| - 2$.*

The reason for removing the loops and the edges joining vertices in V_σ from $\overline{G}$ in the counts above is that these edges do not constitute a constraint when we restrict to anti-symmetric velocity assignments (see Figure 12 for an illustration and Schulze and Tanigawa (2015); Schulze et al. (2022); La Porta and Schulze (2023), for example, for details.) The term $|V_\sigma|$ arises from the fact that each vertex that is unshifted by a reflection has only one degree of freedom, as the corresponding velocity vector in a ρ_1-symmetric infinitesimal motion has to lie perpendicular to the mirror line. Also, the term -2 reflects the fact that there is a 2-dimensional space of trivial ρ_1-symmetric infinitesimal motions for both $\mathcal{C}_s$ and $\mathcal{C}_2$.

Example 7.5. *The graph $\overline{G}_\ell$ corresponding to the quotient graph in Figure 11(b) has $|\overline{E}_\ell| = 3$. Moreover, it has $|V'| = 2$ and $|V_\sigma| = 2$. Thus, $|\overline{E}_\ell| - 2|V'| - |V_\sigma| + 2 = -1$, showing that the framework in Figure 11(a) has a ρ_1-symmetric (or anti-symmetric) infinitesimal motion.*

The graph corresponding to the quotient graph in Figure 11(d) has $|\overline{E}_\ell| = 3$, $|V'| = 1$ and $|V_\sigma| = 3$. Thus, $|\overline{E}_\ell| - 2|V'| - |V_\sigma| + 2 = 0$. So the framework in Figure 11(c) counts to be anti-symmetric isostatic.

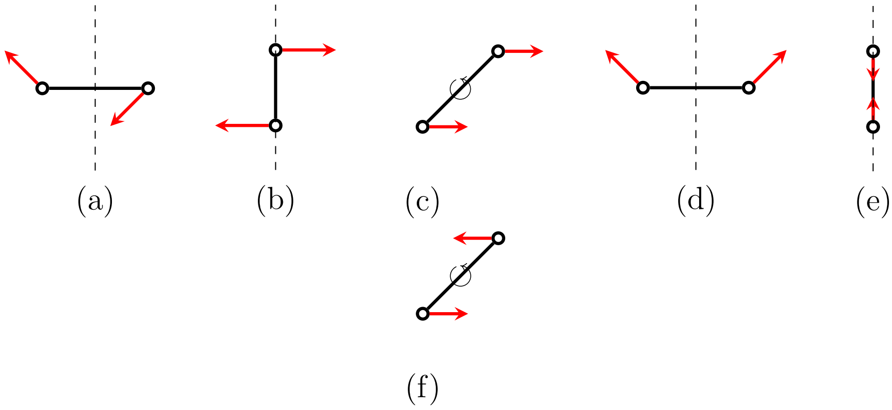

Figure 12: (a), (b): Anti-symmetric velocity vectors applied to a bar which joins vertices that are images of each other under a reflection (and hence corresponds to a loop in the quotient graph) (a) or are both unshifted by a reflection (b). By definition, *any* anti-symmetric velocity assignment yields an infinitesimal motion of such bars; hence these bars do not impose any constraint when restricting to anti-symmetric velocity assignments. Similarly, *any* anti-symmetric velocity assignment on a bar that joins a vertex and its image under a half-turn symmetry (and hence corresponds to a loop in the quotient graph) does not impose any constraint when restricting to anti-symmetric velocity assignments (c). In contrast, fully-symmetric velocity assignments can stretch (d) or compress these bars (e), (f).

The rotation group $\mathcal{C}_n$, $n \geq 3$, has the irreducible representations $\rho_0, \ldots, \rho_{n-1}$ (recall Section 5). It was shown in Schulze and Tanigawa (2015) that the space of infinitesimal translations can be written as the direct sum of a one-dimensional space of ρ_1-symmetric translations and a one-dimensional space of ρ_{n-1}-symmetric translations. (The space of infinitesimal rotations is ρ_0-symmetric.)

Also, if n is even and ρ_t maps the half-turn in $\mathcal{C}_n$ to -1, then any edge that joins a vertex with its image under the half-turn symmetry does not constitute a constraint when we restrict to ρ_t-symmetric velocity assignments (recall Figure 12(c)). Thus, for $t \in \{1, \dots, n-1\}$, we define $\overline{E}_{\ell_t}$ as follows. $\overline{E}_{\ell_t}$ is the whole edge set $\overline{E}$ of the quotient graph $\overline{G}$ of G in the cases when n is even and ρ_t maps the half-turn to 1, or when n is odd. If n is even and ρ_t maps the half-turn to -1, then $\overline{E}_{\ell_t}$ is obtained from $\overline{E}$ by removing all loops corresponding to edges of G that join a vertex with its image under the half-turn symmetry. This gives the following result.

Theorem 7.6. *Let (G, p) be a ρ_t-symmetric isostatic framework with symmetry group $\mathcal{C}_n$ in the plane, where $t \in \{1, \dots, n-1\}$ and let $\overline{G} = (\overline{V}, \overline{E})$ be the quotient graph of G. Then the following hold.*

- *If $t = 1, n-1$ then $|\overline{E}_{\ell_t}| = 2|V'| + |V_n| - 1$.*
- *If $t \in \{2, \dots, n-2\}$ then $|\overline{E}_{\ell_t}| = 2|V'|$.*

Finally, for the dihedral group $\mathcal{C}_{2v}$ we have the four irreducible representations $\rho_0, \rho_1, \rho_2, \rho_3$ (recall Section 6.1). As usual, let the reflections in the x- and y-axis be called σ_x and σ_y, respectively. It is well known (Altmann and Herzig , 1994) that for $\mathcal{C}_{2v}$, the one-dimensional space of infinitesimal rotations is ρ_1-symmetric, and the 2-dimensional space of infinitesimal translations decomposes into a one-dimensional space of ρ_2-symmetric translations and a one-dimensional space of ρ_3-symmetric translations.

We let $\overline{E}_{\ell_1}$ be the edge set that is obtained from the edge set $\overline{E}$ of the quotient graph $\overline{G}$ of G by removing all loops that correspond to edges in G joining a vertex with its image under σ_x or under σ_y, and all edges joining vertices in V_{σ_x} or V_{σ_y}. Similarly, we let $\overline{E}_{\ell_2}$ be the edge set that is obtained from $\overline{E}$ by removing all loops that correspond to edges in G joining a vertex with its image under σ_y or under the half-turn C_2, and all edges joining vertices in V_{σ_y}. Finally, we let $\overline{E}_{\ell_3}$ be the edge set that is obtained from $\overline{E}$ by removing all loops that correspond to edges in G joining a vertex with its image under σ_x or under C_2, and all edges joining vertices in V_{σ_x}.

Theorem 7.7. *Let (G, p) be a ρ_t-symmetric isostatic framework with symmetry group $\mathcal{C}_{2v}$ in the plane, where $t \in \{1, 2, 3\}$ and let $\overline{G} = (\overline{V}, \overline{E})$ be the quotient graph of G. Then the following hold.*

- *If $t = 1$ then $|\overline{E}_{\ell_1}| = 2|V'| + |V_{\sigma_x} \setminus V_2| + |V_{\sigma_y} \setminus V_2| - 1$.*
- *If $t = 2$ then $|\overline{E}_{\ell_2}| = 2|V'| + |V_2| + |V_{\sigma_x} \setminus V_2| + |V_{\sigma_y} \setminus V_2| - 1$.*
- *If $t = 3$ then $|\overline{E}_{\ell_3}| = 2|V'| + |V_2| + |V_{\sigma_x} \setminus V_2| + |V_{\sigma_y} \setminus V_2| - 1$.*

A proof can be found in La Porta (2024). We will illustrate these counts by applying them to the framework with $\mathcal{C}_{2v}$ symmetry shown in Figures 9 and 13(a).

Since for any symmetry group that only has one-dimensional irreducible representations (over the complex numbers), the information for detecting infinitesimal motions and self-stresses of various symmetry types are encoded in the quotient graphs, we may carry out the entire analysis of the graphic statics of symmetric frameworks via these quotient graphs. In particular, instead of analysing a pair of symmetric reciprocal frameworks, we may simply analyse the corresponding pair of quotient reciprocals (like the pair shown in Figure 11(b) and (d), for example).

We conclude this section with an analysis of the framework shown in Figure 13(a) and its reciprocals shown in Figure 10 via the orbit counts on the corresponding quotient graphs.

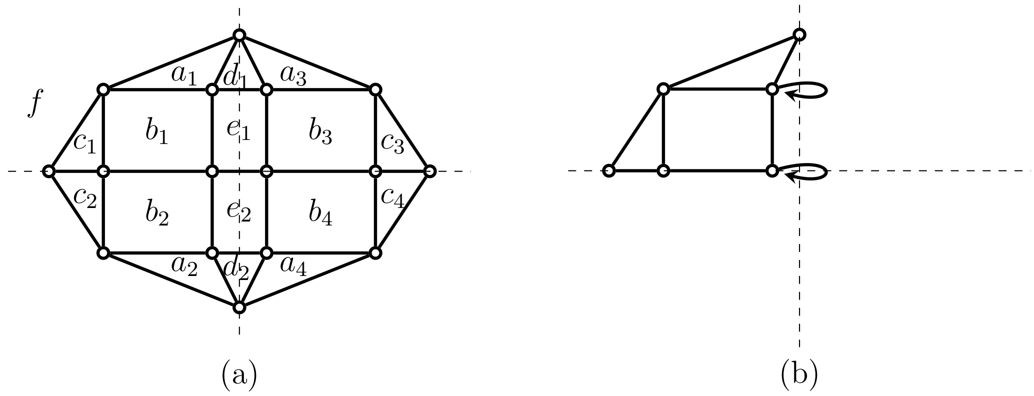

Figure 13: The framework (G, p) with $\mathcal{C}_{2v}$ symmetry from Figure 9 and its quotient graph.

Example 7.8. *We analyse the framework in Figure 13(b). Let us first consider the counts for forced symmetric infinitesimal rigidity given in Theorem 7.1. The quotient graph in Figure 13(b) has* $|\overline{E}| = 10$, $|V'| = 2$, $|V_2| = 0$, $|V_{\sigma_x}| = 3$ *and* $|V_{\sigma_y}| = 1$. *Since* $V_2 = \emptyset$, *we have* $|V_\sigma \setminus V_2| = |V_\sigma|$ *for each reflection* σ. *Thus,* $|\overline{E}| - 2|V'| - |V_{\sigma_x}| - |V_{\sigma_y}| = 2$, *showing that the framework* (G, p) *in Figure 13(a) has two fully-symmetric self-stresses.*

Let us now consider the anti-symmetric orbit counts for $t = 1, 2$ *and* 3 *given in Theorem 7.7.*

We first consider $t = 1$. *Since the quotient graph of* G *has two loops corresponding to edges joining images of vertices under* σ_y *and two edges joining vertices in* V_{σ_x}, *we have* $|\overline{E}_{\ell_1}| = 10 - 4 = 6$. *So the count for* ρ_1 *is* $|\overline{E}_{\ell_1}| - 2|V'| - |V_{\sigma_x}| - |V_{\sigma_y}| + 1 = 6 - 4 - 3 - 1 + 1 = -1$, *showing that* (G, p) *has a* ρ_1*-symmetric infinitesimal motion.*

For $t = 2$, *we have* $|\overline{E}_{\ell_2}| - 2|V'| - |V_{\sigma_x}| - |V_{\sigma_y}| + 1 = 8 - 4 - 3 - 1 + 1 = 1$, *indicating that* (G, p) *has a* ρ_2*-symmetric self-stress.*

Finally, for $t = 3$, we have $|\overline{E}_{\ell_3}| - 2|V'| - |V_{\sigma_x}| - |V_{\sigma_y}| + 1 = 7 - 4 - 3 - 1 + 1 = 0$. So we have an isostatic count for ρ_3-symmetric infinitesimal rigidity.

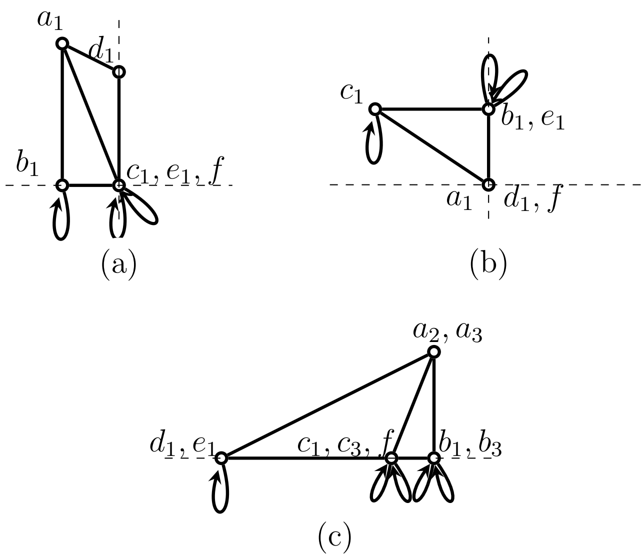

Figure 14: The quotient graphs of the reciprocal frameworks shown in Figure 10.

Example 7.9. *Consider the generalised reciprocal framework with $\mathcal{C}_{2v}$ symmetry shown in Figure 10(a). Its quotient graph is shown in Figure 14(a). The vertex orbits are represented by the six vertices $a_1, b_1, c_1, d_1, e_1, f$ and the edge orbits are represented by the 10 edges a_1d_1, a_1f, a_1b_1, b_1b_1, b_1e_1, b_1c_1, c_1c_1, c_1f, d_1e_1, e_1e_1, where b_1b_1, c_1c_1 and e_1e_1 are loops. (Since there are coincident vertices and overlapping edges, as well as edges of length zero, not all vertices and edges are shown in the figure.)*

By considering the underlying graph of the reciprocal framework in Figure 10(a), which is dual to the underlying graph of the framework in Figure 13(a), we see that $V' = \{a_1, b_1, c_1\}$, $V_2 = \{f\}$, $V_{\sigma_x} \setminus V_2 = \emptyset$ and $V_{\sigma_y} \setminus V_2 = \{d_1, e_1\}$. So for the ρ_0-symmetric count we obtain $|\overline{E}| - 2|V'| -$

$|V_{\sigma_x} \setminus V_2| - |V_{\sigma_y} \setminus V_2| = 10 - 6 - 0 - 2 = 2$, indicating that the framework has two ρ_0-symmetric self-stresses.

For ρ_1 we obtain $|\overline{E}_{\ell_1}| - 2|V'| - |V_{\sigma_x} \setminus V_2| - |V_{\sigma_y} \setminus V_2| - 1 = 6 - 6 - 0 - 2 + 1 = -1$, since $\overline{E}_{\ell_1}$ is obtained from $\overline{E}$ by removing the three loops and the edge d_1e_1. This shows that the framework has a ρ_1-symmetric infinitesimal motion.

Since $\overline{E}_{\ell_2} = \overline{E} \setminus \{e_1e_1, d_1e_1\}$ and $V_2 = \{f\}$, the ρ_2 count is isostatic: $|\overline{E}_{\ell_2}| - 2|V'| - |V_2| - |V_{\sigma_x} \setminus V_2| - |V_{\sigma_y} \setminus V_2| + 1 = 8 - 6 - 1 - 0 - 2 + 1 = 0$.

Finally, since $\overline{E}_{\ell_3} = \overline{E} \setminus \{e_1e_1, c_1c_1, b_1b_1\}$, the ρ_3 count is $|\overline{E}_{\ell_3}| - 2|V'| - |V_2| - |V_{\sigma_x} \setminus V_2| - |V_{\sigma_y} \setminus V_2| + 1 = 7 - 6 - 1 - 0 - 2 + 1 = -1$ indicating that the framework has a ρ_3-symmetric infinitesimal motion.

The symmetric counts for the quotient graph in Figure 14(b) are exactly the same as for the quotient graph in Figure 14(a).

The reciprocal framework shown in Figure 10(c) only has $\mathcal{C}_s$ symmetry and its quotient graph is shown in Figure 14(c). So here we apply the counts from Theorems 7.1 and 7.4. For ρ_0 we obtain $|\overline{E}| - 2|V'| - |V_\sigma| + 1 = 18 - 16 - 1 + 1 = 2$ since $V_\sigma = \{f\}$. So we detect two fully-symmetric self-stresses. For ρ_1 we obtain $|\overline{E}_\ell| - 2|V'| - |V_\sigma| + 2 = 13 - 16 - 1 + 2 = -2$, indicating that the framework has two anti-symmetric infinitesimal motions.

8. Conclusions, extensions, and further work

We have shown that for a symmetric framework with a fully-symmetric self-stress the graphic statics analysis of the equi-symmetric reciprocal pair of frameworks can be refined using the decomposition of the spaces of infinitesimal motions and self-stresses into invariant subspaces corresponding to the irreducible representations of the symmetry group. This refined symmetry-adapted analysis provides additional insights that cannot be obtained from the corresponding non-symmetric analysis, and it can be carried out very efficiently via Maxwell-type counts on the quotient graphs of the symmetric frameworks. Even if the original framework has a self-stress that is not fully-symmetric but only symmetric with respect to a non-trivial irreducible representation of the symmetry group, our methods can be applied to the corresponding reciprocal pair with the smaller symmetry given by the kernel of this representation.

It is typical for a paper like this to extend their concepts to higher dimensions. This is not trivial in this case as reciprocity in higher dimensions is more complex; for example, in $\mathbb{R}^3$ points are dual to volumes/cells, as is discussed in Konstantatou (2018). In fact, no existing papers discuss the $s^* = m + 1$ relationship in higher dimensional space although this is important in $\mathbb{R}^3$. As this is a significant area where basic and symmetry adapted counts can be obtained, this is left to future work.

This paper limits itself to frameworks as plane projections of spherical polyhedra, as is common within graphic statics. The study of frameworks corresponding to toroidal polyhedra is more complex and little explored (Crapo and Whiteley , 1994b; Erickson and Lin , 2021; Fowler and Guest , 2002) and a full investigation, including symmetry adapted reciprocal counts is also left to future work.

Finally, we note that if for a given framework there exists a continuous motion of the reciprocal framework, then this motion corresponds to a continuous path of parallel drawings of the reciprocal, which in turn yields a continuous path of polyhedral liftings of the original framework. This suggests some potential applications to kinematic architecture and transformable designs (see e.g. the work of Hoberman Associates).

One known result from the rigidity theory of symmetric frameworks is that for 'symmetry-generic' frameworks (i.e., for frameworks whose vertices are positioned as generic as possible with the given symmetry constraints), a fully-symmetric infinitesimal motion always extends to a continuous (finite) symmetry-preserving motion (Schulze , 2010b; Kangwai and Guest , 1999). This is true for any symmetry group in any dimension. Thus, if we detect a fully-symmetric non-trivial infinitesimal motion in a reciprocal framework, then, provided the reciprocal framework is sufficiently generic, there is a continuous motion of the reciprocal which preserves the symmetry, which then yields a continuous path of (fully- or anti-symmetric, depending on the symmetry group) polyhedral liftings of the original framework. To apply this result, we would need to find a reciprocal framework that is sufficiently generic with respect to some symmetry group and has both a fully-symmetric self-stress (by construction) and a fully-symmetric infinitesimal motion. Such examples are easy to construct by violating Maxwell-type (orbit) counts on subgraphs, so that the self-stress and the infinitesimal motion are localised to separate parts of the framework, but it remains an open problem to find non-trivial examples, where the continuous symmetry-preserving motion of the reciprocal leads to a continuous path of proper polyhedral liftings of the whole framework.

However, there are also other types of examples one may consider. The framework in Figure 8(a), for example, is non-generic with $\mathcal{C}_3$-symmetry (as discussed in Section 5.2) and it has a fully-symmetric self-stress as well as a fully-symmetric infinitesimal motion that extends to a continuous symmetry-preserving motion, as shown in Schulze and Whiteley (2011). It would be interesting to find similar examples where both frameworks of the reciprocal pair are planar with no coincident points.

Acknowledgements

We thank Bill Baker, Allan McRobie and Walter Whiteley for inspiring discussions on reciprocal diagrams.

References

Adriaenssens, S., Ney, L., Bodarwe, E and Williams, C., *Finding the Form of an Irregular Meshed Steel and Glass Shell Based on Construction Constraints*, Journal of Architectural Engineering 18 (3), 106–213, 2012.

Airy, G.B., *On the strains in the interior of beams*, Philos. Trans. R. Soc. Lond. 153, 49-79, 1862.

Altmann, S.L. and Herzig, P. *Point-Group Theory Tables*, Clarendon Press, Oxford, 1994.

Architizer, *Joe and Rika Mansueto Library*, Accessed 27/09/2022, architizer.com/blog/projects/joe-and-rika-mansueto-library.

Bernstein, D.I., *Generic Symmetry-Forced Infinitesimal Rigidity: Translations and Rotations*, SIAM Journal on Applied Algebra and Geometry, 6(2), 190–215, 2022.

Baker, W.F., *The Geometry of Equilibrium: James Clerk Maxwell and his Continuing Contributions to Engineering Mechanics*, Cambridge University Press, to appear, 2023.

Borcea, C. and Streinu, I., *Liftings and stresses for planar periodic frameworks*, Discrete Comput. Geom., 53(4):747–782, 2015.

O. Bottema, *Die Bahnkurven eines merkwürdigen Zwölfstabgetriebes*, Österr. Ing.-Arch. 14, 218–222, 1960.

Connelly, R. and Guest, S.D., *Frameworks, Tensegrities, and Symmetry*, Cambridge University Press, 2022.

Crapo, H., and Whiteley, W., *Plane stresses and projected polyhedra I: the basic pattern*, Structural Topology 20, 55–68, 1993.

Crapo, H., and Whiteley, W., *Spaces of stresses, projections, and parallel drawings for spherical polyhedra*, Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry 35, 259-281, 1994.

Crapo, H., and Whiteley, W., *Plane dependencies, parallel redrawings, and reciprocals for toroidal polyhedra*, unpublished manuscript, 1994.

Erickson, J., and Lin, P., *A Toroidal Maxwell–Cremona–Delaunay Correspondence*, Journal of Computational Geometry 12(2), 55–85, 2021.

Fowler, P.W. and Guest, S.D., *A symmetry extension of Maxwell's rule for rigidity of frames*, International Journal of Solids and Structures 37, 1793–1804, 2000.

Fowler, P.W. and Guest, S.D., *Symmetry and states of self-stress in triangulated toroidal frames*, International Journal of Solids and Structures 39(17), 4385–4393, 2002.

Grünbaum, B., Convex Polytopes, Graduate Texts in Mathematics, vol. 221 (2nd ed.), Springer-Verlag, 2003.

Hartz, C., Mazurek, A., Miki, M., Zegard, T., Mitchell, T. and Baker, W. F, *The application of 2D and 3D graphic statics in design*, In Proceedings of IASS Annual Symposia (Vol. 2017, No. 12, pp. 1–10). International Association for Shell and Spatial Structures (IASS).

Jordán, T., Kaszanitzky, V., and Tanigawa, S., *Gain-sparsity and symmetry-forced rigidity in the plane*, Discrete and Computational Geometry 55, 314–372, 2016.

Kangwai, R.D. and Guest, S.D., *Detection of finite mechanisms in symmetric structures*, International Journal of Solids and Structures 36, 5507–5527, 1999.

Kangwai, R.D. and Guest, S.D., *Symmetry-adapted equilibrium matrices*, International Journal of Solids and Structures 37, 1525–1548, 2000.

Kangwai, R.D., Guest, S.D., and Pellegrino, S., *An introduction to the analysis of symmetric structures*, Computers and Structures 71, 671–688, 1999.

Konstantatou, M., D'Acunto, P. and McRobie, A., *Polarities in structural analysis and design: n-dimensional graphic statics and structural transfor-*

mations, International Journal of Solids and Structures 152-153, 272–293, 2018.

La Porta, A. *Infinitesimal rigidity of symmetric frameworks in the plane*, PhD thesis, Lancaster University, 2024.

La Porta, A. and Schulze, B., *Combinatorial characterisations for symmetric infinitesimal rigidity in the plane*, preprint, 2023.

Maxwell, J.C., *On Reciprocal Figures and Diagrams of Forces*, Phil. Mag. Series 4, 27, 250-261, 1864.

Maxwell, J.C., *On reciprocal figures, frameworks and diagrams of forces*, Transactions of the Royal Society Edinburgh 26, 1–40, 1870.

McRobie, A., Baker, W., Mitchell, T. and Konstantatou, M., *Mechanisms and states of self-stress of planar trusses using graphic statics, Part III: Applications and extensions*, Proceedings of the IASS, 2015.

McRobie, A., Millar, C., Zastavni, D. and Baker, W., *Graphical Stability Analysis of Maillart's Roof at Chiasso*, Structural Engineering International, DOI: 10.1080/10168664.2021.2018958, 2022.

Millar, C., Schulze, B. , Mazurek, A., and Baker, W.F., *Symmetry and the design of self-stressed structures*, submitted, 2021.

Ney & Partners, *Glass Roof Dutch Maritime Museum*, Accessed 27/09/2022, ney.partners/project/glass-roof-dutch-maritime-museum.

Mitchell, T., Baker, W., McRobie, A. and Mazurek, A., *Mechanisms and states of self-stress of planar trusses using graphic statics, part I: The* *fundamental theorem of linear algebra and the Airy stress function*, International Journal of Space Structures 31 (2-4), 85–101, 2016.

Owen, J.C. and Power, S.C., *Frameworks, Symmetry and Rigidity*, International Journal of Computational Geometry & Applications 20(6),723–750, 2010.

Schulze, B. , Millar, C., Mazurek, A., and Baker, W.F., *States of self-stress in symmetric frameworks and applications*, International Journal of Solids and Structures 234-235, 2022.

Schulze, B., *Block-diagonalized rigidity matrices of symmetric frameworks and applications*, Beitrage zur Algebra und Geometrie/Contributions to Algebra and Geometry 51, No. 2, 427–466, 2010.

Schulze, B., *Symmetry as a sufficient condition for a finite flex*, SIAM Journal on Discrete Mathematics 24, No. 4, 1291–1312, 2010.

Schulze, B. and Tanigawa, S., *Infinitesimal rigidity of symmetric bar-joint frameworks*, SIAM Journal on Discrete Mathematics 29(3), 1259–1286, 2015.

Schulze, B. and Whiteley, W., *The orbit rigidity matrix of a symmetric framework*, Discrete and Computational Geometry 46(3), 561–598, 2011.

Schulze, B. and Whiteley, W., *Rigidity and scene analysis*, Handbook of Discrete and Computational Geometry, C.D. Toth, J. O'Rourke, J.E. Goodman, editors, Third Edition, Chapman & Hall CRC, 2018.

Schulze, B. and Whiteley, W., *Rigidity of symmetric frameworks*, Handbook of Discrete and Computational Geometry, C.D. Toth, J. O'Rourke, J.E. Goodman, editors, Third Edition, Chapman & Hall CRC, 2018.

Tay, T.-S., *A new proof of Laman's theorem*, Graphs and Combinatorics, 9:365–370, 1993.

Whiteley, W., *Motions and stresses of projected polyhedra* Topologie structurale / Structural Topology 7:13–38, 1982.

Whiteley, W., *Matroids from Discrete Applied Geometry*, In J. Bonin, J. Oxley, and B. Servatius, editors, Matroid Theory, vol. 197 of Contemp. Math., 171–311, AMS, Providence, 1996.

Williams, C.J.K., *The analytic and numerical definition of the geometry of the British Museum Great Court Roof*, Mathematics & design, 434–440, 2001.