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A Discrete Characterization of the Solvability of Equilibrium Problems and Its Application to Game Theory

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Abstract: We state a characterization of the existence of equilibrium in terms of certain finite subsets under compactness and transfer upper semicontinuity conditions. In order to derive some consequences on game theory—Nash equilibrium and minimax inequalities—we introduce a weak convexity concept.

Keywords: equilibrium problems; game theory; minimax inequalities

MSC: 91B50; 49J35

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1. Introduction

This paper deals with the study of scalar equilibria under certain weak topological and convex assumptions. To be more precise, let $\alpha \in \mathbb{R}$, X and Y be nonempty sets, $f : X \times Y \longrightarrow \mathbb{R}$, and let us consider the so-called *equilibrium problem*:

$$\text{there exists } x_0 \in X : \ \alpha \leq \inf_{y \in Y} f(x_0, y), \tag{1}$$

or, in a more general way, this *weak equilibrium inequality*:

$$\alpha \leq \sup_{x \in X} \inf_{y \in Y} f(x, y). \tag{2}$$

Although easy examples show that (1) is stronger than (2), when X is compact and f is upper semicontinuous on X, they are equivalent problems. Indeed, our main result—Theorem 1—establishes, among other things, the equivalence of (1) and (2) under less restrictive conditions. The study of equilibrium problems can be traced back to the K. Fan minimax inequality [1], although the nomenclature is adopted from L.D. Muu and W. Oettli in [2]. Most results guaranteeing the existence of equilibrium for a scalar function assume topological hypotheses on one variable, and in addition, either convexity or concavity conditions on the other or concavity–convexity assumptions on both variables [3–10].

This kind of problem comprises the study of the celebrated *Nash equilibrium* [11–14] and the existence of saddle points or, more generally, the validity of the minimax inequality [15–21], to name only a few.

In Section 2, we state our main result, the aforementioned Theorem 1, where we provide not only the equivalence between the equilibrium problem (1) and the weak equilibrium inequality (2) under suitable conditions, but also a condition in terms of finite subsets that characterizes the existence of a solution for (1). Although it is a result of a topological nature, in order to derive applicable results we introduce in Definition 1 a convexity concept that is necessary for the existence of equilibrium. In Section 3, we obtain some consequences on game theory (Nash equilibrium and minimax inequalities), extending some known results in [22]. We finish with some conclusions.

2. A Discrete Characterization of Equilibrium

In this section, we assume certain topological hypotheses: X is a nonempty and compact topological space and f satisfies a not very restrictive continuity condition, the so-called *α-transfer upper semicontinuity* on X, more general than upper semicontinuity. Under them, we prove that the equilibrium problem (1) admits a solution when the weak equilibrium inequality (2) is valid and state a characterization of its solvability in terms of some finite subsets of Y, which leads to a quite general result on the existence of equilibrium.

Let us recall ([23], Definition 8) that if $\alpha \in \mathbb{R}$, X is a nonempty topological space, $x_0 \in X$ and Y is a nonempty set, then a function $f : X \times Y \longrightarrow \mathbb{R}$ is *α-transfer upper semicontinuous* in x_0 provided that

$$\left.\begin{array}{c}(x_0, y_0) \in X \times Y \\ f(x_0, y_0) < \alpha\end{array}\right\} \Rightarrow \text{ there exist } y_1 \in Y \text{ and a neighborhood } U \text{ of } x_0 : \ x \in U \ \Rightarrow \ f(x, y_1) < \alpha.$$

In addition, f is said to be *α-transfer upper semicontinuous* on X when it is at each $x_0 \in X$. A function is α-transfer upper semicontinuous on X as soon as it is upper semicontinuous on X, although the converse is not true:

Example 1. *Let $0 < \alpha < 1$, $0 < x_1 < x_2 < 1$ and let $f : [0,1] \times \{0,1\} \longrightarrow \mathbb{R}$ be the function given for any $0 \le x \le 1$ by*

$$f(x,0) = \begin{cases} 0, & \text{if } 0 \le x \le x_1 \\ 1, & \text{otherwise} \end{cases}$$

and

$$f(x,1) = \begin{cases} 0, & \text{if } 0 \le x < x_2 \\ 1, & \text{otherwise} \end{cases}.$$

Then f is α-transfer upper semicontinuous on $[0,1]$, since for $(x_0, y_0) \in [0,1] \times \{0,1\}$ with $f(x_0, y_0) < \alpha$ we take $y_1 := 1$ and $U := [0, x_2)$ to arrive at

$$x \in U \ \Rightarrow \ f(x, y_1) < \alpha.$$

However,

$$\{x \in [0,1] : \ \alpha \le f(x,0)\} = (x_1, 1]$$

is not closed, hence f is not upper semicontinuous on $[0,1]$.

It is a well-known fact (see [23], Remark 7) that f is α-transfer upper semicontinuous on X if, and only if,

$$\bigcap_{y \in Y} \{x \in X : \ \alpha \le f(x,y)\} = \bigcap_{y \in Y} \mathrm{cl}(\{x \in X : \ \alpha \le f(x,y)\}), \tag{3}$$

where "cl" stands for "closure".

The next result is a first version of the discrete characterization of the solvability of the equilibrium problem (1).

Lemma 1. *Suppose that X is a nonempty and compact topological space, Y is a nonempty set, $\alpha \in \mathbb{R}$ and $f : X \times Y \longrightarrow \mathbb{R}$ is α-transfer upper semicontinuous on X. Then*

$$\text{there exists } x_0 \in X : \ \alpha \le \inf_{y \in Y} f(x_0, y)$$

if, and only if, there exists a finite subset Y_1 of Y such that

$$\left.\begin{array}{c} Y_1 \subset Y_0 \subset Y \\ \emptyset \ne Y_0 \text{ finite} \end{array}\right\} \Rightarrow \bigcap_{y \in Y_0} \mathrm{cl}(\{x \in X : \alpha \le f(x,y)\}) \ne \emptyset. \tag{4}$$

Proof. The existence of $x_0 \in X$ satisfying

$$\alpha \leq \inf_{y \in Y} f(x_0, y)$$

implies the other condition with $Y_1 = \emptyset$, since for each nonempty and finite subset Y_0 of Y we have that

$$\begin{aligned} \alpha &\leq \inf_{y \in Y} f(x_0, y) \\ &\leq \min_{y \in Y_0} f(x_0, y), \end{aligned}$$

so

$$\bigcap_{y \in Y_0} \{x \in X : \alpha \leq f(x, y)\} \neq \emptyset$$

and then (4) holds.

Conversely, let Y_1 be a finite subset of Y in such a way that

$$\left. \begin{array}{c} Y_1 \subset Y_0 \subset Y \\ \emptyset \neq Y_0 \text{ finite} \end{array} \right\} \Rightarrow \bigcap_{y \in Y_0} \mathrm{cl}(\{x \in X : \alpha \leq f(x, y)\}) \neq \emptyset.$$

The compactness of X implies that

$$\bigcap_{y \in Y} \mathrm{cl}(\{x \in X : \alpha \leq f(x, y)\}) \neq \emptyset$$

and the α-transfer upper semicontinuity of f on X and (3) that

$$\bigcap_{y \in Y} \{x \in X : \alpha \leq f(x, y)\} \neq \emptyset,$$

i.e., for some $x_0 \in X$,

$$\alpha \leq \inf_{y \in Y} f(x_0, y).$$

□

The next result provides us with a discrete weak equilibrium inequality implying the condition $\bigcap_{y \in Y_0} \mathrm{cl}(\{x \in X : \alpha \leq f(x, y)\}) \neq \emptyset$:

Lemma 2. *If X is a nonempty and compact topological space, Y_0 is a nonempty and finite set, $f : X \times Y_0 \longrightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$ satisfy*

$$\alpha \leq \sup_{x \in X} \min_{y \in Y_0} f(x, y),$$

and

$$\text{there exists } \delta > 0 : \ \beta \in [\alpha - \delta, \alpha] \ \Rightarrow \ f \text{ is } \beta\text{-transfer upper semicontinuous on } X, \tag{5}$$

then

$$\bigcap_{y \in Y_0} \mathrm{cl}(\{x \in X : \alpha \leq f(x, y)\}) \neq \emptyset.$$

Proof. Let us proceed by contradiction, so, let us assume that

$$\bigcap_{y \in Y_0} \mathrm{cl}(\{x \in X : \alpha \leq f(x, y)\}) = \emptyset,$$

i.e.,

$$x_0 \in X \;\Rightarrow\; \text{there exists } y_{x_0} \in Y_0 : x_0 \notin \text{cl}(\{x \in X : \alpha \leq f(x, y_{x_0})\}),$$

in particular, there exists $0 < \delta_{x_0} < \delta$, with

$$f(x_0, y_0) < \alpha - \delta_{x_0}.$$

Therefore, according to the $\alpha - \delta_{x_0}$ transfer upper semicontinuity of f on X, there exists a $y_{x_0} \in Y_0$ and a neighborhood U_{x_0} of x_0 such that

$$x \in U_{x_0} \;\Rightarrow\; f(x, y_{x_0}) < \alpha - \delta_{x_0}.$$

Then

$$X = \bigcup_{x \in X} U_x$$

and, by compactness, there exist $x_1, \ldots, x_n \in X$ with

$$X = \bigcup_{i=1}^{n} U_{x_i}.$$

Given $x \in X$, let $i \in \{1, \ldots, n\}$ such that $x \in U_{x_i}$, so

$$f(x, y_{x_i}) < \alpha - \delta_{x_i}$$

and thus, if we set $\delta := \min\{\delta_{x_1}, \ldots, \delta_{x_n}\} > 0$, then

$$\min_{y \in Y_0} f(x, y) < \alpha - \delta$$

and the arbitrariness of $x \in X$ yields

$$\sup_{x \in X} \min_{y \in Y_0} f(x, y) \leq \alpha - \delta,$$

in particular,

$$\sup_{x \in X} \min_{y \in Y_0} f(x, y) < \alpha,$$

which contradicts the hypothesis. □

A first consequence of the previous lemmas is the equivalence of the solvability of the equilibrium problem (1) and its weak inequality (2) with the topological conditions under consideration:

Corollary 1. *Assume that X is a nonempty and compact topological space, Y is a nonempty set, $\alpha \in \mathbb{R}$ and that $f : X \times Y \longrightarrow \mathbb{R}$ satisfies condition (5). If in addition*

$$\alpha \leq \sup_{x \in X} \inf_{y \in Y} f(x, y),$$

then

$$\text{there exists } x_0 \in X : \;\alpha \leq \inf_{y \in Y} f(x_0, y).$$

Proof. Given a nonempty and finite subset Y_0 of Y, the weak equilibrium inequality yields

$$\begin{aligned} \alpha &\leq \sup_{x \in X} \inf_{y \in Y} f(x, y) \\ &\leq \sup_{x \in X} \min_{y \in Y_0} f(x, y), \end{aligned}$$

and therefore, it follows from Lemma 2 that

$$\bigcap_{y \in Y_0} \operatorname{cl}(\{x \in X : \alpha \leq f(x,y)\}) \neq \emptyset.$$

Now Lemma 1 applies with $Y_1 = \emptyset$, and we are done. □

We are in a position to establish our main result:

Theorem 1. *Let X be a nonempty and compact topological space, Y be a nonempty set, $\alpha \in \mathbb{R}$ and $f : X \times Y \longrightarrow \mathbb{R}$ be a function satisfying condition (5). Then, the following are equivalent:*

(i) *The weak equilibrium inequality (2) holds, that is,*

$$\alpha \leq \sup_{x \in X} \inf_{y \in Y} f(x,y).$$

(ii) *f admits an equilibrium (1), i.e.,*

$$\text{there exists } x_0 \in X : \alpha \leq \inf_{y \in Y} f(x_0,y).$$

(iii) *There exists a finite subset Y_1 of Y such that*

$$\left.\begin{array}{c} Y_1 \subset Y_0 \subset Y \\ \emptyset \neq Y_0 \text{ finite} \end{array}\right\} \Rightarrow \bigcap_{y \in Y_0} \{x \in X : \alpha \leq f(x,y)\} \neq \emptyset,$$

or, in other words,

$$\left.\begin{array}{c} Y_1 \subset Y_0 \subset Y \\ \emptyset \neq Y_0 \text{ finite} \end{array}\right\} \Rightarrow \text{there exists } x_0 \in X : \ \alpha \leq \min_{y \in Y_0} f(x,y).$$

(iv) *For some finite subset Y_1 of Y there holds*

$$\left.\begin{array}{c} Y_1 \subset Y_0 \subset Y \\ \emptyset \neq Y_0 \text{ finite} \end{array}\right\} \Rightarrow \alpha \leq \sup_{x \in X} \min_{y \in Y_0} f(x,y).$$

(v) *There exists a finite subset Y_1 of Y such that*

$$\left.\begin{array}{c} Y_1 \subset Y_0 \subset Y \\ \emptyset \neq Y_0 \text{ finite} \end{array}\right\} \Rightarrow \bigcap_{y \in Y_0} \operatorname{cl}(\{x \in X : \alpha \leq f(x,y)\}) \neq \emptyset.$$

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is Corollary 1, the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear, (iv) $\Rightarrow$ (v) is Lemma 2 and (v) $\Rightarrow$ (ii) is Lemma 1. □

It is worth mentioning that the equivalence (ii) $\Leftrightarrow$ (iii) with $Y_1 = \emptyset$ was stated in ([13], Theorem 3.1), but the fact that Y_1 can be nonempty is a useful extension of such a result, as we will show in Example 3. Let us also point out that the equivalence (i) $\Leftrightarrow$ (iv) is an extension of ([24], Lemma 2.8).

In view of assertions (iii) and (v) in Theorem 1, one could expect that, under the compactness of X and the condition (5), for a nonempty and finite subset Y_0 of Y there holds

$$\bigcap_{y \in Y_0} \{x \in X : \alpha \leq f(x,y)\} = \bigcap_{y \in Y_0} \operatorname{cl}(\{x \in X : \alpha \leq f(x,y)\}).$$

However, that is not the case:

Example 2. *Let $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be the function defined as*

$$(x,y) \mapsto \begin{cases} 1, & \text{if } y < x \text{ or } (x,y) = (1,1) \\ 0, & \text{otherwise} \end{cases},$$

and let $0 < \alpha < 1$. Given $0 < \delta < \alpha$, $\beta \in [\alpha - \delta, \alpha]$ and $y \in [0,1)$,

$$\{x \in [0,1] : \ \beta \le f(x,y)\} = (y,1],$$

and

$$\{x \in [0,1] : \ \beta \le f(x,1)\} = \{1\},$$

so, for any $y \in [0,1]$

$$\mathrm{cl}(\{x \in [0,1] : \ \beta \le f(x,y)\}) = [y,1].$$

In particular, f is β-transfer upper semicontinuous on $[0,1]$, because

$$\begin{aligned} \bigcap_{y \in Y} \{x \in X : \beta \le f(x,y)\} &= \{1\} \\ &= \bigcap_{y \in Y} \mathrm{cl}(\{x \in X : \beta \le f(x,y)\}), \end{aligned}$$

but for any nonempty and finite subset Y_0 of $[0,1]$ not containing $\{1\}$, let us say $Y_0 = \{y_1, \ldots, y_m\}$ with $0 \le y_1 < \cdots < y_m < 1$, we have that

$$\bigcap_{y \in Y_0} \{x \in [0,1] : \ \alpha \le f(x,y)\} = (y_m, 1],$$

while

$$\bigcap_{y \in Y_0} \mathrm{cl}(\{x \in [0,1] : \ \alpha \le f(x,y)\}) = [y_m, 1].$$

A useful way to handle Theorem 1 is given below. We first introduce an equilibrium concept of convexity. As usual, given $m \ge 1$, Δ_m stands for the *unit simplex* of $\mathbb{R}^m$:

$$\Delta_m := \left\{ \mathbf{t} \in \mathbb{R}^m : \ 0 \le t_1, \ldots, t_m \text{ and } \sum_{j=1}^{m} t_j = 1 \right\}.$$

Definition 1. *Given $\alpha \in [-\infty, +\infty]$ and X and Y nonempty sets, a function $f : X \times Y \longrightarrow \mathbb{R}$ is said to be α-*convex *on Y provided that*

$$\left. \begin{array}{l} m \ge 1, \ \mathbf{t} \in \Delta_m \\ y_1, \ldots, y_m \in Y \end{array} \right\} \Rightarrow \ \alpha \le \sup_{x \in X} \sum_{j=1}^{m} t_j f(x, y_j).$$

*And dually, if $\omega \in [-\infty, +\infty]$, then f is ω-*concave *on X when*

$$\left. \begin{array}{l} n \ge 1, \ \mathbf{s} \in \Delta_n \\ x_1, \ldots, x_n \in X \end{array} \right\} \Rightarrow \ \inf_{y \in Y} \sum_{i=1}^{n} s_i f(x_i, y) \le \omega.$$

When $\alpha = \inf_{y \in Y} \sup_{x \in X} f(x,y)$ (respectively, $\omega = \sup_{x \in X} \inf_{y \in Y} f(x,y)$), this notion of convexity (respectively, concavity) coincides with that of *infsup-convexity* on Y (respectively, *supinf-concavity* on X), a generalization of convexlikeness (or Fan's convexity) on Y ([15] p. 42) which was considered for the first time in ([16], Corollary 3.1) and arose naturally when dealing with equilibrium and minimax problems (see, for instance, [16,20,25]). Moreover, when $X = Y$ and $\alpha = \inf_{x \in X} f(x,x)$, α-convexity on Y is nothing more than the so-called *inf-*

diagonal convexity on the second variable ([26], Definition 2.1), which extends, for instance, the concept of *diagonal convexity* when X is a nonempty subset of a vector space ([27] Definition 2.5).

Let us notice that α-convexity is a necessary condition in order that problem (1) admits a solution, and even that (2) holds. Indeed, if (2) is valid, then for any $m \geq 1$, $\mathbf{t} \in \Delta_m$ and $y_1, \ldots, y_m \in Y$,

$$\begin{aligned}\alpha &\leq \sup_{x\in X} \inf_{y\in Y} f(x,y) \\ &\leq \sup_{x\in X} \sum_{j=1}^{m} t_j f(x,y_j).\end{aligned}$$

Although easy examples show that the converse is not true, under some additional hypotheses we can state this equilibrium result:

Corollary 2. *Let X be a nonempty and compact topological space, Y be a nonempty set, $f : X \times Y \longrightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$. Let us also assume that f satifies condition (5) and is α-convex on Y, and that there exists a finite subset Y_1 of Y such that*

$$\left.\begin{array}{c} Y_1 \subset Y_0 \subset Y \\ \emptyset \neq Y_0 \text{ finite} \end{array}\right\} \Rightarrow f|_{X\times Y_0} \text{ is supinf-concave on } X.$$

Then, the equilibrium problem admits a solution, i.e., there exists $x_0 \in X$ such that

$$\alpha \leq \inf_{y\in Y} f(x_0, y).$$

Proof. Let $Y_0 = \{y_1, \ldots, y_m\}$ be a nonempty and finite subset of Y and containing Y_1. The supinf-concavity of $f|_{X\times Y_0}$ on X guarantees, thanks to [25] Theorem 2.6, the existence of $\mathbf{t} \in \Delta_m$ such that

$$\sup_{x\in X} \min_{j=1,\ldots,m} f(x,y_j) = \sup_{x\in X} \sum_{j=1}^{m} t_j f(x,y_j),$$

which, together with the α-convexity of f on Y, implies

$$\alpha \leq \sup_{x\in X} \min_{j=1,\ldots,m} f(x,y_j).$$

Finally, the existence of an equilibrium x_0 for f follows from the equivalence (ii) $\Leftrightarrow$ (iv) of Theorem 1. □

The following example proves that the finite set Y_1 in the corollary above is not necessarily empty. So, by the way, we show that checking condition (iv) in Theorem 1 for a nonempty set Y_1, allows us to apply it for more general equilibria:

Example 3. *Let function $f : \{0,1\} \times [0,1] \longrightarrow \mathbb{R}$ given by*

$$f(0,y) := \begin{cases} 1, & \text{if } 0 \leq y < 0.5 \\ 0, & \text{if } 0.5 \leq y \leq 1 \end{cases},$$

$$f(1,y) := \begin{cases} 0, & \text{if } 0 \leq y \leq 0.5 \\ 1, & \text{if } 0.5 < y \leq 1 \end{cases},$$

which is continuous on the compact set $\{0,1\}$ (discrete topology) and admits the equilibrium

$$0 \leq \inf_{y\in[0,1]} f(0,y),$$

therefore, it is 0-convex. Despite the fact that for some nonempty subset Y_0 of $[0,1]$, the restriction of f to $\{0,1\} \times Y_0$ is not supinf-concave on $\{0,1\}$ ($Y_0 := \{0,1\}$), there exists a finite subset Y_1 of

$[0,1]$ *in such a way that for any nonempty and finite subset of* $[0,1]$ *with* $Y_1 \subset [0,1]$ *we have that* $f|_{\{0,1\}\times Y_0}$ *is supinf-concave on* $\{0,1\}$*: it suffices to take* $Y_1 = \{0.5\}$.

The convexity conditions considered in the above corollary are different from the concept of α-transfer quasiconvexity considered in [13,28]: if X is a nonempty and convex subset of a linear space and Y is a nonempty set, $f : X \times Y \to \mathbb{R}$ is *α-transfer quasiconvex* on Y if given $m \geq 1$ and $y_1, \ldots, y_m$ in Y, there exists $x_1, \ldots, x_m$ in X such that

$$\left.\begin{array}{c} 1 \leq k \leq m,\ \mathbf{t} \in \Delta_k \\ \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\} \end{array}\right\} \Rightarrow \alpha \leq \max_{j=1,\ldots,k} f\Big(\sum_{j=1}^{k} t_{i_j} x_{i_j}, y_{i_j}\Big).$$

The notion of α-transfer quasiconvexity requires that the set X be convex, while this strong condition is not necessary for the α-convexity. On the other hand, the following example shows a function which is α-transfer quasiconvex and not α-convex for some α.

Example 4. *Let* $X = [-1, \pi/2]$, $Y = \{0,1\}$ *and let* $f : [-1, \pi/2] \times \{0,1\} \longrightarrow \mathbb{R}$ *be given for any* $-1 \leq x \leq \pi/2$ *by*

$$f(x,0) := \sin(x-1),$$

$$f(x,1) := -\sin x.$$

The function is 0-transfer quasiconvex:

- *If* $m = 1$ *and* $y_1 = 0$, *we choose* $x_1 \in [1, \pi/2]$ *and then* $0 \leq f(x_1, 0)$.
- *If* $m = 1$ *and* $y_1 = 1$, *we choose* $x_1 \in [-1, 0]$ *and then* $0 \leq f(x_1, 1)$.
- *If* $m = 2$, $y_1 = 0$ *and* $y_2 = 1$, *we choose* $x_1, x_2 \in (1, \pi/2]$ *and then for any* $t \in [0,1]$ *we have*

$$0 \leq \max_{i=1,2} f((1-t)x_1 + tx_2, y_i).$$

However, f is not 0-convex on $\{0,1\}$, *since for* $m = 2$ *and* $\mathbf{t} = (1/2, 1/2)$ *we have that*

$$x \in [-1, \pi/2] \Rightarrow \frac{1}{2} f(x,0) + \frac{1}{2} f(x,1) < -\sin 1 < 0.$$

Let us conclude this section by stressing that Corollary 2 improves the topological assumptions in the two-function equilibrium result ([22] Theorem 2.7) when the involved functions are the same.

3. Application to Game Theory: Nash Equilibrium and Minimax Inequalities

Now we focus on deriving some consequences to game theory, and more specifically, to the existence of Nash equlibria and to establishing the validity of some minimax inequalities, both from the equilibrium results in Section 2.

We first deal with the existence of certain Nash equilibria. To this end, let us consider the following noncooperative game in the normal form

$$G = (X_i, u_i)_{i \in I},$$

where $I = \{1, \ldots, n\}$ is the finite set of players, X_i the strategy space of the player i which is a nonempty subset of a topological space E_i, and $u_i : X \to \mathbb{R}$ is the payoff function of player i, where $X := \prod_{i \in I} X_i$.

When the player i chooses a strategy $x_i \in X_i$, the situation of the game is described by the vector $x = (x_1, \ldots, x_n) \in X$. For each player $i \in I$ denote by $-i := \{j \in I$ such that $j \neq i\}$ the set of all players other than player i. Also denote by $X_{-i} = \prod_{j \neq i} X_j$ the Cartesian product of the sets of strategies of players $-i$ and $x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Note that $x = (x_1, x_{-1}) = (x_2, x_{-2}) = \cdots = (x_n, x_{-n})$ (after rearranging the components if necessary).

With this notation, a strategy profile $x^* \in X$ is a *pure strategy Nash equilibrium of a game* G if

$$u_i(y_i, x^*_{-i}) \leq u_i(x^*) \quad \forall i \in I,\ \forall y_i \in X_i. \tag{6}$$

We introduce the following topological concept:

Definition 2. *Given $\beta \geq 0$, a noncooperative game G is said to be β*-diagonally transfer continuous *if*

$$\left.\begin{array}{c} x, y \in X \\ \Phi(x,x) < \Phi(x,y) \end{array}\right\} \Rightarrow \begin{array}{c} \text{there exist } y_1 \in X \text{ and a neighborhood } U \text{ of } x: \\ z \in U \;\Rightarrow\; \Phi(z,z) + \beta < \Phi(z, y_1), \end{array}$$

where $\Phi : X \times X \to \mathbb{R}$ is the aggregator function *defined at each $(x,y) \in X \times X$ by*

$$\Phi(x,y) := \sum_{i=1}^{n} u_i(x_1, \ldots, y_i, \ldots, x_n) = \sum_{i=1}^{n} u_i(y_i, x_{-i}). \tag{7}$$

We point out that when $\beta = 0$ we recover the notion of *diagonally transfer continuity* in ([29], Definition 1).

In this result, we characterize the existence of Nash equilibrium for certain noncooperative games:

Theorem 2. *Let $G = (X_i, u_i)_{i \in I}$ be a noncooperative game such that given $i \in I$, X_i is a nonempty and compact topological space, and $f : X \times X \to \mathbb{R}$ defined by*

$$f(x,y) := \Phi(x,x) - \Phi(x,y) = \sum_{i=1}^{n} (u_i(x) - u_i(y_i, x_{-i})), \qquad (x, y \in X). \tag{8}$$

If for some $\delta > 0$, G is β-diagonally transfer continuous for all $\beta \in [0, \delta]$, then the following assertions are equivalent:

(i) *The weak equilibrium inequality (2)*

$$0 \leq \sup_{x \in X} \inf_{y \in X} f(x,y)$$

is valid.

(ii) *The game G has a Nash equilibrium.*

(iii) *For some finite subset X_1 of X there holds*

$$\left.\begin{array}{c} X_1 \subset X_0 \subset X \\ \emptyset \neq X_0 \text{ finite} \end{array}\right\} \Rightarrow \text{there exists } x^* \in X:\ 0 \leq \min_{y \in X_0} f(x^*, y).$$

(iv) *There exists a finite subset X_1 of X such that*

$$\left.\begin{array}{c} X_1 \subset X_0 \subset X \\ \emptyset \neq X_0 \text{ finite} \end{array}\right\} \Rightarrow 0 \leq \sup_{x \in X} \min_{y \in X_0} f(x,y).$$

(v) *We can find a finite subset X_1 of X satisfying*

$$\left.\begin{array}{c} X_1 \subset X_0 \subset X \\ \emptyset \neq X_0 \text{ finite} \end{array}\right\} \Rightarrow \bigcap_{y \in X_0} \operatorname{cl}(\{x \in X : 0 \leq f(x,y)\}) \neq \emptyset.$$

Proof. Note that G being β diagonally transfer continuous is equivalent to the fact that f is β-transfer upper semicontinuous on the first variable; and the game G has a Nash equilibrium if, and only if, the function f admits an equilibrium (EP). The proof is a straighforward consequence of Theorem 1. □

In view of Corollary 2, we deduce:

Corollary 3. *Given $\delta > 0$, let us assume that $G = (X_i, u_i)_{i \in I}$ is a β-diagonally transfer continuous noncooperative game for all $\beta \in [0, \delta]$, and such that the strategy sets X_i are nonempty and compact topological spaces. If f defined by (8) is 0-convex on the second variable, and there exists a finite subset X_1 of X such that*

$$\left.\begin{array}{r} X_1 \subset X_0 \subset Y \\ \emptyset \neq X_0 \text{ finite} \end{array}\right\} \Rightarrow f|_{X \times X_0} \text{ is supinf-concave on } X,$$

then, the game G has a Nash equilibrium.

To conclude this section we state some minimax results. Let us first recall that by a *minimax inequality* one means a result guaranteeing that for a function $f : X \times Y \longrightarrow \mathbb{R}$ there holds

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} f(x, y),$$

and therefore, since the reverse inequality is always valid,

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

It is worth quoting not only the pioneer work of J. von Neumann [30], but also those of K. Fan [15] and S. Sion [19], where several concepts of weak convexity were considered, namely, the aforementioned *convexlikeness* and the well-known *quasi convexity*. For a survey on minimax inequalities, we refer interested readers to [31,32].

As a consequence of Corollary 2 we state the following minimax theorem that extends ([24] Theorem 2.1), and even the two-function minimax theorem ([25] Corollary 3.11) when the two functions coincide:

Corollary 4. *Suppose that X is a nonempty and compact topological space, Y is a nonempty set and $f : X \times Y \longrightarrow \mathbb{R}$ is a function such that $\alpha := \inf_{y \in Y} \sup_{x \in X} f(x, y) \in \mathbb{R}$. If in addition, f satisfies condition (5) and for some finite subset Y_1 of Y we have that*

$$\left.\begin{array}{r} Y_1 \subset Y_0 \subset Y \\ \emptyset \neq Y_0 \text{ finite} \end{array}\right\} \Rightarrow f|_{X \times Y_0} \text{ is supinf-concave on } X,$$

then, f satisfies the minimax inequality if, and only if, it is infsup-convex on Y.

Proof. Apply Corollary 2. □

Let us mention another kind of inequality, the one that originated the study of equilibrium problems. Although it is not strictly of the minimax type, a *Fan minimax inequality* for a function $f : X \times X \longrightarrow \mathbb{R}$ is a result stating, under adequate hypotheses, the validity of the inequality

$$\inf_{x \in X} f(x, x) \leq \sup_{x \in X} \inf_{y \in X} f(x, y).$$

The celebrated result of K. Fan ([1] Theorem 1) is different from the following one, which in turn extends ([26] Theorem 3.1):

Corollary 5. *Let X be a nonempty and compact topological space and $f : X \times X \longrightarrow \mathbb{R}$ a function such that $\alpha := \inf_{x \in X} f(x, x) \in \mathbb{R}$. If f satisfies condition (5) and there exists a finite subset X_1 of X such that*

$$\left.\begin{array}{r} X_1 \subset X_0 \subset X \\ \emptyset \neq X_0 \text{ finite} \end{array}\right\} \Rightarrow f|_{X \times X_0} \text{ is supinf-concave on } X,$$

then,

$$\inf_{x \in X} f(x,x) \leq \sup_{x \in X} \inf_{y \in X} f(x,y),$$

if, and only if, f is inf-diagonally convex on its second variable.

Proof. It follows from Corollary 2. □

4. Conclusions

In this paper, we have established a result that makes equivalent the existence of equilibrium for a function $f : X \times Y \longrightarrow \mathbb{R}$ and the validity of the corresponding weak equilibrium inequality: Theorem 1. The topological condition on f that guarantees the equivalence is the condition (5). Theorem 1 provides not only the equivalence between the equilibrium problem (1) and the weak equilibrium inequality (2) but also a condition, (4), in terms of finite subsets that characterize the existence of a solution for (1). In particular, we obtain a discrete characterisation of the solvability of the equilibrium problem. When we also include a not very restrictive convexity condition in Definition 1, we obtain the existence of a solution for more general equilibrium problems than others previously established in [22]. As an application of all this, we obtain both, two results on the existence of Nash equilibria, one topological, Theorem 2 the other also convex, Corollary 3 and some minimax inequalities, Corollaries 4 and 5.

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