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DISSERTATION

Bifurcations from codimension-one D_{4m}-equivariant homoclinic cycles.

vorgelegt von

M.Sc. Maria Kellner

betreuender Hochschullehrer

Priv.-Doz. Dr. Jürgen Knobloch

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Gutachter

Prof. Dr. Carsten Trunk, Technische Universität Ilmenau

Prof. Dr. Ale Jan Homburg, University of Amsterdam

Prof. Dr. Jeroen Lamb, Imperial College London

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Abstract

The topic of this thesis is a detailed description of the dynamics near D_{4m}-symmetric relative homoclinic cycles by using Lin's method.

The homoclinic cycles have codimension-one, that is we observe the generic unfolding within a one-parameter family. They consist of several trajectories that are homoclinic to a hyperbolic equilibrium and which are all related to each other by means of the symmetry induced by a finite group. We assume real leading eigenvalues and connecting trajectories that approach the equilibrium along leading directions. The homoclinics are situated in flow-invariant subspaces.

Especially for such homoclinic cycles in differential equations with D_k-symmetry (D_k is the symmetry group of a regular k-gon in the plane) where k is a multiple of 4 some of these flow-invariant subspaces are perpendicular to each other. This implies the vanishing of the typically appearing leading order terms in some of the determination equations gained from Lin's method. In order to give a precise description of the nonwandering dynamics of such a homoclinic cycle, that is a description of the solutions that remain in the neighbourhood of the cycle both in phase and parameter space, further information about the residual terms in the determination equations are needed.

In this thesis we present a more sophisticated representation of the residual terms in the determination equations and identify two further terms of next leading exponential rates. Based on this we discuss the solvability of the resulting determination equations for homoclinic cycles in $\mathbb{R}^4$. Thereby two cases must be distinguished, depending on the size ratio of the two new terms. In one case we observe subshifts of finite type. In the other case the analysis turns out to be more difficile so we restrict the investigation to periodic solutions.

Beyond that we show how vector fields in $\mathbb{R}^4$ containing a homoclinic cycle with D_k-symmetry can be constructed. Those can be used for numerical investigations. One of these examples we consider numerically to verify some of the analytic results.

Contents

1 Introduction

Bifurcation theory is one of the big topics in the modern theory of dynamical systems. Roughly speaking it studys a sudden change of the dynamical behavior when changing the parameter in a family of dynamical systems. This can range from the change in the number of equilibria up to the transition from tame to wild (chaotic) dynamics. The roots of the bifurcation theory date back to Poincaré, [P1890], but it is still a vibrant research topic with many applications in several scientific disciplines.

Thereby the bifurcation theory of heteroclinic and homoclinic solutions play a key role in our understanding of complex (chaotic) dynamics. Heteroclinic solutions are solutions that connect in infinite time, invariant solution sets of the differential equations such as i.e. equilibrium points or periodic orbits. If the connected sets are identical, we speak of a homoclinic solution. A review of the contemporary results and literature of bifurcation theory of homoclinic and heteroclinic orbits is provided by Homburg and Sandstede [HomSan10]. An introduction to chaotic dynamics can be found in [Dev89].

The modern bifurcation theory of heteroclinic and homoclinic solutions is significantly influenced by the fundamental work of Shil'nikov from the 1960s. An outline can be found in the monographs [SSTC98, SSTC01]. Shil'nikov's approach to study homoclinic bifurcation problems is based on Poincaré first return maps. This has become the standard technique for treating this type of bifurcation problems.

At the beginning of the 1990s, Lin established a method for constructing orbits in neighbourhoods of heteroclinic chains, [Lin90]. Nowadays, this procedure is also known as Lin's method. Essentially, it is based on a Liapunov/Schmidt reduction. In the course of the reduction process information on the dynamics is lost (e.g. stability statements). However, in several cases the existence of certain orbits can be proved more easily. With respect to chaotic dynamics it is even possible to prove by means of that method the existence of an invariant set on which the dynamics is topologically conjugated to a finite subshift on a finite number of symbols, [HJKL11]. We want to remark that due to the loss of stability results the existence of a suspended (hyperbolic) horseshoe cannot be proved.

Of growing interest is the study of heteroclinic cycles or more generally, heteroclinic networks. In the simplest case, such networks consists of equilibria and orbits connecting these equilibria (heteroclinic orbits). Such networks have been identified as a "source" of non-trivial dynamics and appear, among other things, in physical problems such as convection [GuHo88, Ruc01], in population dynamics [Hof94, Hof98, MaLe75] or also in neuronal networks [AOWT07].

In general, a heteroclinic trajectory connecting hyperbolic equilibria with same saddle index (same dimension of the unstable manifold) has at least codimension-one, i.e. such a trajectory occurs robustly in one-parameter families of differential equations. If there are no dependencies between the heteroclinic trajectories, the codimension of a heteroclinic network is the sum of the codimensions of the individual heteroclinic trajectories of the network. The study of such networks is thus only meaningful in correspondingly multi-parameter families of differential equations.

The situation is different for symmetric differential equations. The symmetry can enforce flow-invariant subspaces in which heteroclinic trajectories are robust or at least of low codimension, [Kru97]. This can lead to complicated heteroclinic networks with low codimension or even codimension-zero. Codimension-zero networks in particular are robust - they persist under perturbations of the underlying differential equation.

Symmetries of differential equations respectively vector fields are described by means of group actions – a vector field has a certain symmetry or equivalently it is equivariant under the (linear) action (representation) of a group G if it commutes with the representation operators of G. In this respect G is also called the symmetry group of the vector field. The text books [Van82, GSch85, GSS88, ChoLau00, Fie07] provide general treatises on equivariant bifurcation theory. In [Fie96] among others also symmetric heteroclinic networks are considered.

Of particular interest are heteroclinic networks, which result as a group orbit of the symmetry group of a single heteroclinic trajectory. In a certain sense, this trajectory defines the network. Then the codimension of the network can coincide with the codimension of the defining trajectory. Surprisingly, such networks can generate very complex dynamics.

Robust heteroclinic networks have been increasingly investigated since the 90s, compare [AgCaLa05, AgLaRo10, HomKno10, KLPRS10, KruMel04]. A detailed review on robust heteroclinic cycles is provided by [Kru97]. Part II of that paper describes in detail the state of mathematical research, while in Part III experiments and numerical applications are discussed.

More recently bifurcation problems of non-robust symmetric heteroclinic networks came into focus. Even simple networks of this kind can generate very complex dynamics. For example this is the case for a network consisting of two homoclinics which emerge from each other by a reflectional symmetry and approach a hyperbolic equilibrium along the same direction – a so-called $\mathbb{Z}_2$-symmetric Bellows. Note that $\mathbb{Z}_2$-symmetric Bellows form at least a codimension-1 relative homoclinic cycle. They generate shift-dynamik (full shift on two symbols), [Hom93].

In [Mat99] Matthies shows that in the course of a D_3-Takens-Bogdanov bifurcation D_3- symmetric relative homoclinic cycles arise, which generate a subshift of finite type. In [Hom93] as well as in [Mat99], first return maps were used to study the dynamics of the network.

In [HJKL11] the dynamics near codimension-one homoclinic cycles are considered by applying Lin's method. Under open conditions and in a wide range of cases bifurcation scenarios were established describing how shift dynamics appear or disappear in the bifurcation. It turned out that the analysis in [HJKL11] fails for homoclinic cycles with specific symmetries. The prototype bifurcation where this analysis fails arises for homoclinic cycles in differential equations with D_k-symmetry where k is a multiple of 4.

The goal of this thesis is to extend the analysis in [HJKL11] in such a way that we obtain a more precise description of the nonwandering dynamics in the neighbourhood of a homoclinic cycle where the analysis in [HJKL11] does not provide a complete picture. Beyond that we show how vector fields in $\mathbb{R}^4$ containing a homoclinic cycle with D_k-symmetry can be constructed. Those can be used for numerical investigations. One of these examples we consider numerically to verify some of the analytic results.

In what follows we briefly discuss the reasons for the failure of the analysis given in [HJKL11] in case of D_{4m}-symmetric homoclinic cycles. We start by introducing the precise framework used there.

In [HJKL11] a one-parameter family of differential equations

$$\dot{x} = f(x, \lambda) \tag{1.1}$$

is considered, where $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, and f is sufficiently smooth. Further (1.1) is assumed to be ***equivariant (symmetric)*** under the linear action of a finite group G, cf. [GSS88]:

$$gf(x, \lambda) = f(gx, \lambda), \qquad \forall g \in G, \tag{1.2}$$

and has for $\lambda = 0$ a ***heteroclinic trajectory*** γ connecting two hyperbolic equilibria p and hp with

$h \in G$. That is γ is a solution of (1.1) at $\lambda = 0$ with

$$\lim_{t\to-\infty} \gamma(t) = p \quad \text{and} \quad \lim_{t\to\infty} \gamma(t) = hp.$$

As a consequence of the symmetry we find for any solution q of (1.1) that also gq is a solution of (1.1) for all $g \in G$. Hence, with γ being a heteroclinic trajectory connecting p and hp also $g\gamma$ is a heteroclinic trajectory connecting gp and $g(hp)$. Let Γ be the heteroclinic network generated by γ, that is

$$\Gamma = G(\overline{\gamma}). \tag{1.3}$$

In other words, Γ is the group orbit of the closure of a single heteroclinic trajectory γ. Such a heteroclinic network, where each connecting trajectory is related to a single trajectory by symmetry, is called ***relative homoclinic cycle***. It consists of the hyperbolic equilibrium p, the heteroclinic trajectory γ and all further G-images of γ and p.

The ***connectivity matrix*** $\boldsymbol{C} = (c_{ij})$ of a heteroclinic network (with heteroclinic trajectories γ_i) is a 0-1 matrix, where $c_{ij} = 1$ if the endpoint (the ω***-limit*** $\omega(\gamma_i)$) of the heteroclinic connection γ_i is equal to the starting point (the α***-limit*** $\alpha(\gamma_j)$) of the heteroclinic connection γ_j.

The main theorem of [HJKL11] uses notation for topological Markov chains which we recall in the following, cf. also [Shu86, Definition 10.1].

Definition 1.0.1. *Let*

$$\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}}$$

denote the set of double-infinite sequences, $\kappa : \mathbb{Z} \to \{1, \ldots, k\}$, $i \mapsto \kappa_i$, equipped with the product topology. Let further $A = (a_{ij})_{i,j\in\{1,\ldots,k\}}$ be a 0-1 matrix, that is $a_{i,j} \in \{0, 1\}$. By

$$\Sigma_A = \{\kappa \in \Sigma_k \mid a_{\kappa_i \kappa_{i+1}} = 1\}$$

we denote the ***topological Markov chain*** *defined by A. The* ***left shift*** *σ operating on Σ_k by*

$$\sigma : \Sigma_k \to \Sigma_k, \qquad (\sigma\kappa)_i = \kappa_{i+1}$$

leaves Σ_A invariant. The pair (Σ_A, σ) is called a ***subshift of finite type****.*

With that we can formulate the main statement in [HJKL11]. Thereby we leave out some assumptions which we address as "generic conditions". For more details concerning these assumptions we refer to Section 4.1 and [HJKL11].

Theorem 1.0.2 ([HJKL11], Theorem 1.1)**.** *Let $\dot{x} = f(x, \lambda)$ be a one parameter family of differential equations equivariant with respect to a finite group G, cf. (1.2), which has at $\lambda = 0$ a codimension-one relative homoclinic cycle Γ with hyperbolic equilibrium as defined in (1.3). Assume further some generic conditions concerning minimal intersection of tangent spaces at the stable and unstable manifolds along γ and non-orbit-flip and non-inclination flip properties. Write $\gamma_1, \ldots, \gamma_k$ for the connecting trajectories that constitute Γ.*

There is an explicit construction of $k \times k$ matrices A_- and A_+ with coefficients in $\{0, 1\}$ and the nonzero coefficients in mutually disjoint positions, so that the following holds for any generic family unfolding a relative homoclinic cycle as above.

Take cross sections $\mathcal{S}_i$ transverse to γ_i and write Π_λ for the first return map on the collection of cross *sections* $\cup_{j=1}^k \mathcal{S}_j$. *For* $\lambda > 0$ *small enough, there is an invariant set* $\mathcal{D}_\lambda \subset \cup_{j=1}^k \mathcal{S}_j$ *for* Π_λ *such that for each* $\kappa \in \Sigma_{A_+}$ *there exists a unique* $x \in \mathcal{D}_\lambda$ *with* $\Pi_\lambda^i(x) \in \mathcal{S}_{\kappa_i}$. *Moreover,* $(\mathcal{D}_\lambda, \Pi_\lambda)$ *is topologically conjugate to* (Σ_{A_+}, σ). *An analogous statement holds for* $\lambda < 0$ *with* Σ_{A_+} *replaced by* Σ_{A_-}.

This above description of the dynamics provides a complete picture of the local nonwandering dynamics near Γ *if and only if*

$$A_+ + A_- = \boldsymbol{C}, \tag{1.4}$$

where $\boldsymbol{C}$ *denotes the connectivity matrix of the relative homoclinic cycle.*

Due to the topological conjugation between $(\mathcal{D}_\lambda, \Pi_\lambda)$ and the finite subshift (Σ_{A_+}, σ) one speaks of shift dynamics. We illustrate the statement of Theorem 1.0.2 by means of a D_4 symmetric relative homoclinic cycle as it is depicted in Figure 1.1, cf. also [HJKL11, Table 1, Case 6]. Here the group orbit Γ is obtained from a single homoclinic trajectory asymptotic to a G-invariant hyperbolic equilibrium p. The connectivity matrix of this homoclinic cycle is given as $\boldsymbol{C} = \mathbf{1}$, the matrix where all entries are equal to one. Recall that D_k is the symmetry group of a regular k-gon in the plane. Hence it is generated by two elements, the reflection ζ which generates a cyclic subgroup of order two and the rotation θ_k which generates a cyclic subgroup of order k. According to the achievements in [HJKL11] we find that the matrices A_-, A_+ are given by

$$A_- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{1.5}$$

Obviously $A_+ + A_- \neq \mathbf{1}$. That means that Theorem 1.0.2 does not provide a complete description of the nonwandering dynamics in the neighbourhood of the homoclinic cycle under consideration.

Indeed for this example Theorem 1.0.2 merely confirms well-known facts: If $\lambda < 0$ the nonwandering set consists of four 1-periodic solutions, shadowing the individual connecting trajectories γ_i $(i = 1, 2, 3, 4)$. If $\lambda > 0$ the nonwandering dynamics consists of two 2-periodic orbits shadowing in each case the figure eight configuration consisting of the pairs of homoclinic trajectories being opposite. Note in this respect that those pairs are located within invariant subspaces, cf. also Section 4.1 and 5.1. Hence for each pair [HomSan10, Theorem 5.79] applies. Figure 6.2 in Section 6 illustrates the bifurcation within the invariant subspace.

The aim of this thesis is to show that under further appropriate assumptions on the vector field (1.1) the analysis in [HJKL11] can be refined to the effect that the statement of Theorem 1.0.2 remains true for instance with matrices

$$A_- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_+ = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{1.6}$$

According to this A_+ for $\lambda > 0$ the nonwandering dynamics consists of all trajectories avoiding shadowing twice the same homoclinic trajectory in a row. Further we want to remark that with the matrices given in (1.6) the corresponding dynamics is completely described. In this respect we refer to the main statement of this thesis formulated as Theorem 5.3.3 below.

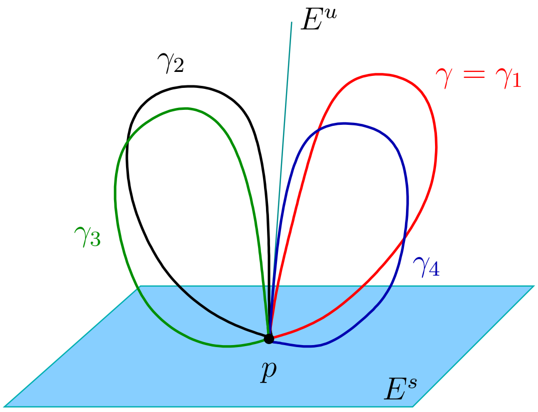


Figure 1.1: A relative homoclinic cycle which is built up as group orbit of the closure of the homoclinic trajectory γ. The underlying symmetry group is D_4. In this particular example is D_4 the isotropy group of the equilibrium p, and $\mathbb{Z}_2$ is the isotropy group of the homoclinic trajectory γ. The single homoclinic trajectories are located within invariant subspaces, and what is more the subspaces related to adjacent homoclinic trajectories are orthogonal to each other.

Before we elaborate on our goal, let us uncover the reason why the analysis in [HJKL11] provides only an incomplete description. To this end let $\gamma_1, \ldots, \gamma_k$ be, as above, the connecting trajectories that constitute the relative homoclinic cycle Γ. The matrices $A_\pm$ are constructed by showing that a related succession of connecting trajectories $\gamma_i \subset \Gamma$, or in other words an itinerary along Γ, can be shadowed by an actual trajectory of (1.1). For more detailed explanations we use the following definition.

Definition 1.0.3. *Let $\kappa \in \Sigma_k$ be fixed. A* ***heteroclinic chain*** *Γ^κ is a double infinite sequence of connecting trajectories γ_{κ_i}, $i \in \mathbb{Z}$, so that $\omega(\gamma_{\kappa_{i-1}}) = \alpha(\gamma_{\kappa_i})$.*

One of the key statements of Lin's method, [Lin90, San93, Kno04] concerns the existence of so-called ***Lin trajectories***: For a given sequence of $\boldsymbol{\omega} = (\omega_i)_{i\in\mathbb{Z}}$ there exists a unique piecewise continuous trajectory $\mathbf{X} = (X_i)_{i\in\mathbb{Z}}$, $X_i : [-\omega_i, \omega_i] \to \mathbb{R}^n$ that follows the succession of Γ^κ. The X_i are solutions of (1.1) and between the final point of each X_i and the initial point of its successor X_{i+1} there may appear a jump $\Xi_i = X_{i+1}(-\omega_{i+1}) - X_i(\omega_i) \neq 0$. Note that $\Xi_i = \Xi_i(\boldsymbol{\omega}, \lambda, \kappa)$. So actual trajectories $\mathbf{X}$ shadowing Γ^κ are related to solutions of $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = 0$, $i \in \mathbb{Z}$.

According to Sandstede [San93], the jump can be written as

$$\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \xi^\infty_{\kappa_i}(\lambda) + \xi_i(\boldsymbol{\omega}, \lambda, \kappa).$$

The first addend $\xi^\infty_{\kappa_i}$ measures the distance between the stable and unstable manifold and is thus only influenced by the system parameter λ. Under appropriate assumptions $\xi^\infty_{\kappa_i}$ can be chosen as the system parameter itself. The second one, ξ_i, becomes exponentially small with increasing ω_i.

In the case of the homoclinic cycle discussed here, we find, cf. [HJKL11, Proposition 3.7],

$$\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \lambda - e^{2\mu^s(\lambda)\omega_i} \langle \eta^s_{\kappa_{i-1}}(\lambda), \eta^-_{\kappa_i}(\lambda) \rangle + R_i(\boldsymbol{\omega}, \lambda, \kappa) = 0, \tag{1.7}$$

for all $i \in \mathbb{Z}$ where the residual terms have the form $R_i(\boldsymbol{\omega}, \lambda, \kappa) = O(e^{2\mu^s(\lambda)\omega_{i+1}\delta}) + O(e^{2\mu^s(\lambda)\omega_i\delta})$ for some $\delta > 1$. Thereby μ^s denotes the leading stable eigenvalue of $D_1 f(p, \lambda)$. These equations we will call ***determination equations***.

If for given κ all products $\langle \eta^s_{\kappa_{i-1}}, \eta^-_{\kappa_i} \rangle$ are different from zero, system (1.7) is solvable if and only if all these products have the same sign as λ. At this point we want to mention that for fixed κ the sign of the scalar product does not depend on λ. With the matrix $M = (m_{i,j})_{i,j\in\{1,\dots,k\}}$, $m_{i,j} := sgn\langle \eta_i^s, \eta_j^- \rangle$ we find

$$A_- = \frac{1}{2}(|M| - M) \quad \text{and} \quad A_+ = \frac{1}{2}(|M| + M). \tag{1.8}$$

If $\langle \eta^s_{\kappa_{i-1}}, \eta^-_{\kappa_i} \rangle = 0$ for some i a more involved analysis than the one carried out in [HJKL11] is necessary to solve equation (1.7). More precisely, in order to decide whether or not (1.7) is solvable knowledge about the leading order terms of the residual terms $R_i(\boldsymbol{\omega}, \lambda, \kappa)$ is needed.

Next we discuss under which circumstances the scalar products under consideration become zero. To this end we introduce

$$e_i^s := \lim_{t\to\infty} \frac{\gamma_i(t) - p}{\|\gamma_i(t) - p\|}, \tag{1.9}$$

which defines the direction the homoclinic trajectory γ_i is approaching the equilibrium p for positive time. Further we define

$$e_j^- := \lim_{t\to-\infty} \frac{\psi_j(t)}{\|\psi_j(t)\|} \tag{1.10}$$

where $\psi_j(t)$ is a solution of the adjoint variational equation along $\gamma_j(t)$

$$\dot{x} = -[D_1 f(\gamma_j(t), \lambda)]^T x, \quad x(0) = \psi_j$$

and ψ_j is a unit vector satisfying

$$\text{span}\{\psi_j\} = \left(T_{\gamma_j(0)} W^s(\omega(\gamma_j)) + T_{\gamma_j(0)} W^u(\alpha(\gamma_j))\right)^\perp .$$

The orthogonal complement is defined by a G-invariant scalar product $\langle \cdot, \cdot \rangle$. We want to remark that in the context of [HJKL11] the vector ψ_j is, up to scalar multiples, uniquely defined. The existence of the limit in (1.10) is ensured by the considerations in [HJKL11, Section 3], cf. also Lemma 2.4.1 below.

It turns out that

$$\eta^s_{\kappa_{i-1}}(\lambda) \in \text{span}\{e^s_{\kappa_{i-1}}\} \quad \text{and} \quad \eta^-_{\kappa_i}(\lambda) \in \text{span}\{e^-_{\kappa_i}\}$$

and what is more

$$\langle \eta^s_{\kappa_{i-1}}(\lambda), \eta^-_{\kappa_i}(\lambda) \rangle = \tilde{A}_i(\lambda, \kappa) \langle e^s_{\kappa_{i-1}}, e^-_{\kappa_i} \rangle, \quad \tilde{A}_i(\lambda, \kappa) > 0.$$

The scalar product $\langle e_i^s, e_i^- \rangle$ is due to the symmetry equal to $\langle e_1^s, e_1^- \rangle$. Of course the sign of $\langle e_1^s, e_1^- \rangle$ depends on the choice of ψ_1. We choose ψ_1 such that

$$\langle e_1^s, e_1^- \rangle < 0. \tag{1.11}$$

Now, in terms of the above introduced D_4-symmetric homoclinic cycle, cf. also Figure 1.1, we find that the homoclinic trajectories γ_i are located in invariant subspaces. Consequently the corresponding η_i^s and η_i^- are also located within these subspaces. As the D_4 is the symmetry group of the square some of these subspaces are perpendicular to each other. This implies $\langle \eta_i^s, \eta_{i+1}^- \rangle = 0$, $i = 1, 2, 3, 4$. In the end this provides the zeros off the minor diagonal in the matrix A_+ given in (1.5). In other words, by means of the results in [HJKL11] it cannot be decided whether a heteroclinic chain Γ^κ with κ of the form that for one $i \in \mathbb{Z}$ there is a $j \in \{1, \dots, 4\}$ such that $\kappa_i = j$ and $\kappa_{i+1} = j + 1$ has a shadowing actual trajectory.

As already indicated above the topic of this thesis is a more detailed analysis of D_{4m}-symmetric relative homoclinic cycles by using Lin's method. For this purpose a more sophisticated representation of the residual terms $R_i(\boldsymbol{\omega}, \lambda, \kappa)$ is required – especially for those i for which the scalar product $\langle e^s_{\kappa_{i-1}}, e^-_{\kappa_i} \rangle$ disappears. Let denote J_κ the set of all those i.

According to our findings the determination equation takes in this case the form

$$\Xi_i(\boldsymbol{\omega},\lambda,\kappa) := \lambda - e^{4\mu^s(\lambda)\omega_i}B_i(\lambda) - e^{2\mu^s(\lambda)(\omega_{i-1}+\omega_i)}C_i(\lambda,\kappa), + \check{R}_i(\boldsymbol{\omega},\lambda,\kappa) = 0, \quad i \in J_\kappa, \tag{1.12}$$

For the structure of the residual terms $\check{R}_i(\boldsymbol{\omega},\lambda,\kappa)$ we refer to Theorem 5.3.1.

If $i, i-1 \in J_\kappa$ the quantity $C_i(0,\kappa)$ at $\lambda = 0$ arises from the scalar product of $\eta^-_{\kappa_i}(0)$ on one side and a direction within $\operatorname{span}\{e^s_{\kappa_{i-2}}\}$ on the other side. The second direction results from $\eta^s_{\kappa_{i-2}}(0)$ through transportation along the homoclinic solution $\gamma_{\kappa_{i-1}}(t)$ via the adjoint variational equation $\dot{x} = -[D_1 f(\gamma_{\kappa_{i-1}}(t), 0)]^T x$ from $-\omega_{i-1}$ to ω_i. With $i, i-1 \in J_\kappa$, $\eta^-_{\kappa_i}$ and $\eta^s_{\kappa_{i-2}}$ are situated in the same one-dimensional subspace and it turns out that all $C_i(\lambda,\kappa)$ have the same absolute value and are different from zero. The sign of $C_i(\lambda,\kappa)$ depends on the topological structure of - to put it simple - the stable manifold within a tubular neighbourhood of the homoclinic trajectory γ and whether $\kappa_i = \kappa_{i-2}$ or not.

Unfortunately we are unable to give a nice geometrical interpretation for the quantity $B_i(\lambda)$. However we are able to prove some helpful properties of this quantity. First we want to note that $B(\lambda) = B_i(\lambda)$ does not depend on i. Though we could not prove that $B(\lambda) \neq 0$, but we show that the term does not vanish as a consequence of symmetry. Moreover we show in Chapter 7 numerically that the $B(0)$ related to the example constructed in Chapter 6 is different from zero. We take that as justification for a corresponding assumption on $B(0)$ in our analysis carried out in Chapter 5.

After establishing the advanced system of determination equations, we will discuss its solvability. The investigations in [HJKL11] show that if (1.7) holds for all $i \in \mathbb{Z}$, the solving transition times ω_i are for fixed λ of approximately the same size. More precisely, they satisfy the equation $\omega_i(\lambda,\kappa) = \frac{1}{2\mu^s(0)}(\ln(|\lambda|) + \ln(r_i))$, with uniformly (that is independent of λ and κ) bounded terms r_i, cf. [HJKL11, Equation 4.8]. Recall in this respect that λ is close to zero. Then $\ln(|\lambda|)$ dominates the other addend and for sufficiently small λ the transition times are approximately given by $\omega_i(\lambda,\kappa) \approx \frac{1}{2\mu^s(0)}\ln(|\lambda|)$.

Now, the $i^{\text{ th}}$ equation in the set of determination equations has the form (1.7) or (1.12) depending on whether $\langle e^s_{\kappa_{i-1}}, e^-_{\kappa_i}\rangle \neq 0$ or not. Thus the system of determination equations has become considerably more complicated. This also has an effect on the resulting dynamics, which is particularly evident in the transition times $\boldsymbol{\omega} = (\omega_i)_{i\in\mathbb{Z}}$ that solve the system for given λ and κ.

When solving the set of determination equations, three cases must be distinguished, $|B(0)| > |C_i(0,\kappa)|$, $|B(0)| = |C_i(0,\kappa)|$ and $|B(0)| < |C_i(0,\kappa)|$. In all three cases, for the solving ω_i the following equation applies

$$\omega_i(\lambda,\kappa) = \begin{cases} \frac{1}{2\mu^s(0)}(\ln(|\lambda|) + \ln(r_i(\lambda,\kappa))), & i \in \mathbb{Z}\setminus J_\kappa, \\ \frac{1}{4\mu^s(0)}(\ln(|\lambda|) + \ln(r_i^2(\lambda,\kappa))), & i \in J_\kappa. \end{cases} \tag{1.13}$$

However, the properties of the terms r_i are different in the cases.

In the first case, $|B(0)| > |C_i(0,\kappa)|$, we find, similarly to [HJKL11], that the r_i are uniformly bounded. Hence for λ sufficiently small, $\ln(|\lambda|)$ dominates the term and we find with

$$\omega_i(\lambda,\kappa) \approx \begin{cases} \frac{1}{2\mu^s(0)}\ln(|\lambda|), & i \in \mathbb{Z}\setminus J_\kappa, \\ \frac{1}{4\mu^s(0)}\ln(|\lambda|), & i \in J_\kappa. \end{cases}$$

that the transition times are essentially of the same size (for fixed λ), only differing by a factor of 2, depending on the course of the trajectory. More precisely, it takes only half as much time to run along the homoclinic trajectories γ_i to γ_j if they lie in mutually orthogonal subspaces than in the other cases.

Consequently the first case, $|B(0)| > |C_i(0,\kappa)|$, can be seen as a generalization of the results in [HJKL11]. There are matrices $A_\pm$ so that the corresponding topological Markov chains $\Sigma_{A_\pm}$ describe the nonwandering dynamics as explained in Theorem 1.0.2. These matrices are defined as in (1.8) but here by means of the matrix $M = (m_{ij})$ with

$$m_{ij} := \begin{cases} \operatorname{sgn}\langle \eta_i^s(\lambda), \eta_j^-(\lambda)\rangle, & \operatorname{sgn}\langle \eta_i^s(\lambda), \eta_j^-(\lambda)\rangle \neq 0, \\ \operatorname{sgn} B(\lambda), & \operatorname{sgn}\langle \eta_i^s(\lambda), \eta_j^-(\lambda)\rangle = 0. \end{cases} \tag{1.14}$$

What is more, $A_+ + A_- = \mathbf{1}$. That is in this respect the description of the dynamics provides a complete picture of the local nonwandering dynamics near Γ. So in this case, the nonwandering dynamics in the neighbourhood of the cycle can be described by a subshift of finite type. Regarding the D_4-symmetric homoclinic cycle, cf. Figure 1.1, this results for $B(\lambda) > 0$ in (1.6). For the precise formulation of the bifurcation result we refer to Theorem 5.3.3.

A different picture emerges in the remaining cases $|B(0)| = |C_i(0,\kappa)|$ and $|B(0)| < |C_i(0,\kappa)|$. To realize this it is enough to consider periodic trajectories. Here, for such a trajectory N denotes the length of the longest sequence of determination equations with $i \in J_\kappa$ and $\operatorname{sgn} C_i(\lambda,\kappa) \neq \operatorname{sgn} B(\lambda)$. It turns out that the periodic trajectories exist for a λ-range $(0, \hat\lambda(N))$ or $(-\hat\lambda(N), 0)$, respectively, where $\hat\lambda(N) \to 0$ as N tends to infinity.
The reason for this is that the r_i in (1.13) are no longer uniformly bounded, but can have huge differences in size depending on the course of the trajectory, that is given by κ. It can be seen that the r_i are larger the larger the value of N becomes. Thus, the second summand in (1.13) can have a non-negligible influence on the transition times ω_i. To ensure that for fixed κ $\inf(\boldsymbol{\omega})$ is still sufficiently large to satisfy our analysis, λ may have to be very small. To be more precise, the following estimate has to be satisfied

$$\lambda < \left({}^{B(0)}/_{C(0)}\right)^{2N} e^{4\mu^s(0)\inf(\boldsymbol{\omega})}.$$

Hence the existence of a shadowing trajectory depends on the size of λ. Note that the solutions according to Theorem 1.0.2 are not affected by the restricted λ-ranges, since here $J_\kappa = \emptyset$ and consequently $N = 0$. In order to decide at what sign of λ the periodic orbits do exist we can in most cases rely on the determination by means of the matrices $A_\pm$ from (1.8), with $M = (m_{ij})$ as in (1.14). The only exceptions that oppose the assignment via the matrices $A_\pm$ are the periodic orbits for which $i \in J_\kappa$ and $\operatorname{sgn} C_i(\lambda,\kappa) \neq \operatorname{sgn} B(\lambda)$ holds for all i. For $|B(0)| = |C_i(0,\kappa)|$ these orbits do not exist, whereas for $|B(0)| < |C_i(0,\kappa)|$ they exist exactly for the opposite sign of λ than in the case $|B(0)| > |C_i(0,\kappa)|$.
Finally, it should be emphasised that in the cases $|B(0)| = |C_i(0,\kappa)|$ and $|B(0)| < |C_i(0,\kappa)|$ the local nonwandering dynamics cannot be described by a subshift. For the precise formulation of these cases see Theorem 5.3.4.

This thesis is organized as follows. In the subsequent Chapter 2 we introduce the subject of exponential dichotomy that imposes a fundamental part of Lin's method. Briefly speaking exponential dichotomies allows to define (time-depending) stable subspaces of linear differential equations $\dot x = A(t)x$ via the image of a certain projection $P(\cdot)$. Using this concept allows to extract leading terms and their rates of convergences from certain solutions of differential equations. We have compiled the information presented from various literature, for example [Cop78, Kla06, San93, Kno99], to derive more accurate estimates of convergence rates, especially for projections $P(\cdot)$ associated to exponential dichotomies themselves. Our findings will be used in Chapter 4 to determine terms of leading exponential rates of the residuals $R_i(\boldsymbol{\omega}, \lambda, \kappa)$ in (1.7). Further in Section 2.6 we clarify the meaning of the expression 'Codimension-one' for homoclinic trajectories. In doing so we simply recall statements from [HomSan10].

To keep this thesis self-contained we provide in Chapter 3 a detailed derivation of Lin's method. In particular we proof the existence of Lin trajectories and derive a suitable representation of the jumps $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ and present finally some basic estimates related to it. We are aware that most of that material can be found in the existing literature, cf. for example [Lin90, San93, Kno04]. However for the further analysis of the residual terms $R_i(\boldsymbol{\omega}, \lambda, \kappa)$ in Chapter 4 it is essential to have this background at our disposal. Further, due to the introduction of an additional projection in (3.38) and the in Lemmata 2.1.14 and 2.5.6 stated convergence rates of the projections the here presented estimates are more accurate than existing estimates found in the literature. Finally we consider the derivatives of the jump $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ with respect to the transition times ω_j and present corresponding estimates.

In Chapter 4 we derive representations of the residual terms $R_i(\boldsymbol{\omega}, \lambda, \kappa)$ specifically tailored to codimension-one symmetric homoclinic cycles that are generated by a single homoclinic trajectory. The Theorems 4.3.1 and 4.3.3 provide correspondingly adapted representations of the residual terms. Here we take in particular into account that invariant subspaces may be perpendicular to each other. Theorem 4.3.3 is of particular interest for further considerations of symmetric homoclinic cycles in $\mathbb{R}^4$. The basis for these theorems are the Lemmata 4.3.14, 4.3.20 and 4.3.22. In the last two lemmata the above mentioned terms B_i and C_i are defined.

Having now a more distinct representation of the jump ξ_i we discuss the solvability of the resulting system of determination equations $\big(\Xi_i(\boldsymbol{\omega}, \lambda, \kappa)\big)_{i \in \mathbb{Z}} = 0$ in Chapter 5. Here we restrict ourselves to D_{4m}-symmetric vector fields in $\mathbb{R}^4$. Section 5.2 provides the information about the signs of B and C_i. Then, Section 5.3 contains with Theorem 5.3.3 and Theorem 5.3.4 our main results that describe the dynamic behaviour in the neighbourhood of the homoclinic cycle Γ differentiated according to the size ratios of $|C_i|$ to $|B|$. In case of Theorem 5.3.4 we discuss only periodic trajectories, that is the sequence in κ is finite. The proofs of the theorems can be found in the subsequent subsections.

To verify the analytic results numerically it calls for examples of vector fields containing a symmetric homoclinic cycle. In Chapter 6 we present with Theorem 6.2.8 an explicit construction of families of D_k-symmetric polynomial vector fields in $\mathbb{R}^4$ possessing a codimension-one homoclinic cycle. Further we prove the validity of the in Section 5.1 assumed hypotheses for the constructed cycles, cf. Lemma 6.1.1 and Section 6.3. This Chapter basically is a version of [HKK14].

Finally in Section 7 we make use of the constructed D_4-symmetric vector field in Section 6, namely (6.7), and examine it numerically. First we use MATLAB to show graphically the non-vanishing of the term $B(0)$ in Section 7.1. In Section 7.2 we confirm the analytic results of Theorem 5.3.3 in case of the example vector field (6.7). To this end we use the continuation package AUTO and examine the neighbourhood of the homoclinic cycle Γ for certain periodic orbits. We especially verify the existence of the periodic orbits for $\kappa = \overline{12}$ and $\kappa = \overline{1234}$ for the same sign of λ. That is, we find the 2-periodic trajectory that shadows the pathway of the homoclinic trajectories γ_1 and γ_2, cf. Figure 1.1 as well as the 4-periodic trajectory that follows γ_1, γ_2, γ_3 and γ_4 before it closes. This verifies that for this example the relation $|B(0)| > |C_i(0, \kappa)|$ applies. Otherwise these two periodic trajectories should not exist for the same sign of λ. Also we find that these periodic trajectories $\kappa = \overline{12}$ and $\kappa = \overline{1234}$ exist for the same sign of λ as the trajectory that shadows a figure-eight configuration $\kappa = \overline{13}$, which concludes that $B(\lambda) > 0$. This also can be inferred by the fact that we failed to find the trajectory $\kappa = \overline{121}$ - that is the trajectory that follows γ_1 twice, moves on to trace γ_2 and then closes - no matter for which sign of λ.
When investigating the trajectory $\kappa = \overline{1243}$ different transition times could also be identified. Staying in the same invariant subspace took almost twice as long as moving to the perpendicular subspace.

2 Preliminaries

As we have indicated in the introduction an essential part of Lin's method consists of the exact estimation of leading terms within the determination equations $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = 0$. For this purpose we provide in this section some fundamentals that lead to assertions about rates of convergence of special solutions of ordinary differential equations.

First we introduce in Section 2.1 the concept of exponential dichotomies that is used to determine stable subspaces of linear time depending differential equations. In Section 2.2 we apply this concept on variational equations along solutions of autonomous equations, since this is the context in which exponential dichotomies appear in Lin's method. Then in Sections 2.3, 2.4 and 2.5 we use exponential dichotomies to extract leading terms and their rates of convergence

- from solutions within the stable manifold of hyperbolic equilibria (Lemma 2.3.1),
- from stable solutions of disturbed linear equations (Lemma 2.4.1),
- from transition matrices composed with projections of exponential dichotomies (Lemma 2.5.2)
- and from projections of exponential dichotomies themselves (Lemma 2.5.6).

Each of these lemmata is used when it comes to find the leading terms in the determination equations that we gain from Lin's method when applying it on codimension-one D_{4k}-symmetric homoclinic cycles. Finally in Section 2.6, we collect the assumptions on a vector field to unfold a codimension-1 homoclinic trajectory.

To begin with we recall some basic statements about linear mappings, especially concerning coherencies between a linear mapping and its transpose regarding image and kernel as well as eigenvalues and eigenspaces. Most lemmata and theorems listed here can be found in any textbook for linear algebra. For example we refer to [Gan, Fis03].

By $\mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ we denote the set of all linear mappings that map from $\mathbb{R}^n$ into $\mathbb{R}^n$. Let $A \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ be such a linear mapping. The kernel and the image of A are defined as

$$\ker A := \{x \in \mathbb{R}^n \mid Ax = 0\} \qquad \text{and} \qquad \operatorname{im} A := \{y \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^n : Ax = y\}.$$

Both the kernel and the image of A are linear subspaces of $\mathbb{R}^n$ that satisfy the dimension formula

$$\dim(\ker A) + \dim(\operatorname{im} A) = n,$$

see [Fis03, Satz 2.2.4]. Now, let $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an arbitrary scalar product and $U \subseteq \mathbb{R}^n$ be a linear subspace of $\mathbb{R}^n$. Then we denote by $U^\perp$ the orthogonal complement of U with respect to the scalar product $\langle \cdot, \cdot \rangle$:

$$U^\perp := \{x \in \mathbb{R}^n \mid \langle x, y \rangle = 0 \quad \forall y \in U\}.$$

Of course $U^\perp$ is a linear subspace of $\mathbb{R}^n$ as well, [Fis03, p.294].

By A^T we denote the transpose of A with respect to the scalar product $\langle \cdot, \cdot \rangle$, that is A^T satisfies

$$\langle Ax, y \rangle = \langle x, A^T y \rangle$$

for all $x, y \in \mathbb{R}^n$. Between the kernel and the image of a linear mapping A and its transpose A^T the following relations apply:

Lemma 2.0.1 ([Fis03], Satz 6.2.4)**.** *Let $A, A^T \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ be a linear mapping and its transpose. Then*

$$imA^T = (kerA)^\perp \qquad and \qquad kerA^T = (imA)^\perp.$$

In the following we denote by $\sigma(A)$ the spectrum, that is the set of all (including complex) eigenvalues of $A \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n) \subset \mathbb{L}(\mathbb{C}^n, \mathbb{C}^n)$,

$$\sigma(A) := \{\mu \in \mathbb{C} \mid \det(A - \mu \cdot id) = 0\}.$$

It is a well known fact that for every $A \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ there exists a non-singular matrix $T \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that $T^{-1}AT$ has Jordan normal form, see [Fis03, p.259ff]. The column vectors of T associated to the Jordan block corresponding to the eigenvalue $\mu \in \sigma(A)$ span the generalized eigenspace of A with respect to the eigenvalue μ. We denote this generalised eigenspace by $E_A(\mu)$. For a partial spectrum $\sigma_1 \subset \sigma(A)$ of A we denote the corresponding generalised eigenspace by $E_A(\sigma_1)$. Further we define the complement spectrum of A with respect to μ by

$$\sigma_\mu^c := \sigma(A) \setminus \{\mu\}$$

and denote by $E_A(\sigma_\mu^c)$ the generalized eigenspace of A with respect to the spectrum σ_μ^c. If it is evident from the context to which matrix the eigenspaces refer, we omit the subscript.

It is easy to see that A and A^T have the same spectrum, that is $\sigma(A) = \sigma(A^T)$. Hence for any $\mu \in \sigma(A)$ we also find $\sigma_\mu^c = \sigma(A^T) \setminus \{\mu\}$. As a direct consequence of Lemma 2.0.1 the following relation yields between eigenspaces of A and A^T.

Lemma 2.0.2. *Let $A, A^T \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ be a linear mapping and its transpose. Let further $\sigma(A) = \sigma_1 \cup \sigma_2$, $\sigma_1 \cap \sigma_2 = \emptyset$. Then we find*

$$E_{A^T}(\sigma_1) = E_{-A^T}(-\sigma_1) = E_A(\sigma_2)^\perp.$$

Especially for an eigenvalue $\mu \in \sigma(A)$ of A we obtain

$$E_{A^T}(\mu) = E_{-A^T}(-\mu) = E_A(\sigma_\mu^c)^\perp.$$

We conclude this section with the definition of projection.

Definition 2.0.3. *A* ***projection*** *$P \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ is a linear mapping satisfying $P^2 = P$.*

Remark 2.0.4. *Due to $P^2 = P$ we find that $(id - P)$ also is a projection satisfying $P(id - P) = (id - P)P = 0$. Therefore the following relations yield*

$$imP = ker(id - P) \qquad and \qquad im(id - P) = kerP.$$

2.1 Exponential dichotomies and trichotomies

Exponential dichotomies play a central role in the theory of Lin's method. Therefore it is necessary to be familiar with the basic definitions and theorems which we will provide in this section. A standard reference on this subject is [Cop78]. However for our further analysis we need more detailed information which extend the statements presented in [Cop78]. To be precise we have need of a more specific roughness-theorem, see Lemma 2.1.7, that can be found in [Kla06] and which is inspired by [San93] and [Kno99]. Furthermore, we look at the relations between the exponential dichotomies of a linear differential equations and its adjoint equation. And finally, we also consider the concept of exponential trichotomy, see for example [HaLi86, Bey94, Kla06].

Consider the linear differential equation in $\mathbb{R}^n$

$$\dot{x} = A(t)x, \tag{2.1}$$

where $A(\cdot)$ is a continuous matrix function on an interval $J \subset \mathbb{R}$. The corresponding transition matrix we denote by $\Phi(\cdot,\cdot)$.

Definition 2.1.1. *Equation (2.1) is said to have an* ***exponential dichotomy*** *on J, if there exist a projection $P(\cdot)$ and constants $\alpha < \beta$ and $K > 0$ such that for all $s,t \in J$*

$$\left.\begin{array}{lrcll} (i) & \Phi(t,s)P(s) & = & P(t)\Phi(t,s), & \\ (ii) & \|\Phi(t,s)P(s)\| & \leq & Ke^{\alpha(t-s)}, & t \geq s, \\ (iii) & \|\Phi(t,s)(id-P(s))\| & \leq & Ke^{-\beta(s-t)}, & s \geq t. \end{array}\right\} \tag{2.2}$$

We say that (2.1) has an exponential dichotomy with projection $P(\cdot)$ and exponential rates α and β.

Basically this definition can be found in [Cop78]. However there it was formulated in terms of fundamental matrices instead of transition matrices and it was explicitly demanded, that α and β have different signs. Indeed the condition $\alpha < 0 < \beta$ is not necessary for the concept of exponential dichotomies but it displays the classical approach in which exponential dichotomies first have been used. In i.e. [San93] the additional condition in the definition was spared.

The most interesting cases are those in which the interval J is equal to a half-line $\mathbb{R}^+$ or $\mathbb{R}^-$ or where J is equal to the whole line $\mathbb{R}$.

To explain the basic idea of exponential dichotomies we assume for a moment that (2.1) has an exponential dichotomy on $\mathbb{R}^+$ with the constants α and β satisfying the additional condition $\alpha < 0 < \beta$. Then Definition 2.1.1(ii) says that for $x \in \mathrm{im}P(s)$ the term $\Phi(t,s)x$ tends exponentially fast with the rate α to zero as t tends to infinity. On the contrary, Definition 2.1.1(iii) provides that for $y \in \mathrm{im}(id - P(t)) \setminus \{0\}$ the term $\Phi(s,t)y$ tends to infinity with an exponential rate of at least β, as $s \to \infty$. This can be seen as follows: Define $x := \Phi(s,t)y$. Then $x \in \mathrm{im}(id - P(s))$ and with Definition 2.1.1(iii) we find $\|\Phi(t,s)x\| \leq Ke^{-\beta(s-t)}\|x\|$ or equivalently $\|y\|K^{-1}e^{\beta(s-t)} \leq \|\Phi(s,t)y\|$.

Remark 2.1.2. *If equation (2.1) has an exponential dichotomy on $\mathbb{R}^+$ with projection $P^+(\cdot)$ and constants $\alpha < 0 < \beta$ then it holds*

$$\begin{array}{lcl} x \in imP^+(s) & \Leftrightarrow & \|\Phi(t,s)x\| \underset{t\to\infty}{\to} 0, \\ x \in kerP^+(s) \setminus \{0\} & \Rightarrow & \|\Phi(t,s)x\| \underset{t\to\infty}{\to} \infty. \end{array}$$

This statement can be found in [Cop78]. Due to condition (i) in Definition 2.1.1 all projections $P^+(t)$, $t > 0$, are determined by $P^+(0)$. But note that $P^+(0)$ is not uniquely given. By Definition 2.1.1(ii) and (iii) only its image is entirely settled, while there is some freedom in choosing its kernel. This also holds true, when α and β have the same sign. Indeed one can show the following Lemma that was formulated in [Cop78] for classical dichotomies.

Lemma 2.1.3 ([Kla06], Lemma A.2.3.). *Assume that (2.1) has an exponential dichotomy on $\mathbb{R}^+$ with constants α and β and associated projection $P(\cdot)$. Let $Q(0)$ be a projection on $\mathbb{R}^+$ with $imP(0) = imQ(0)$. Then the projection $Q(t) := \Phi(t,0)Q(0)\Phi(0,t)$ is also associated to the exponential dichotomy of (2.1) with constants α, β.*

Corresponding to Remark 2.1.2, if (2.1) has an exponential dichotomy on $\mathbb{R}^-$, then Definition 2.1.1(*iii*) provides that for $x \in \ker P(s)$ the solution $\Phi(t,s)x$ tends exponentially fast (with the rate $-\beta$) to zero for $t \to -\infty$.

Remark 2.1.4. *If equation (2.1) has an exponential dichotomy on $\mathbb{R}^-$ with projection $P^-(\cdot)$ and constants $\alpha < 0 < \beta$ then it holds*

$$\begin{aligned} x \in imP^-(s) \setminus \{0\} \quad &\Rightarrow \quad \|\Phi(t,s)x\| \underset{t\to-\infty}{\rightarrow} \infty, \\ x \in kerP^-(s) \quad &\Leftrightarrow \quad \|\Phi(t,s)x\| \underset{t\to-\infty}{\rightarrow} 0. \end{aligned}$$

Here the kernel of $P^-(0)$ is determined and we have freedom in choosing its image. Again we refer to [Cop78] for this statement.

Indeed the projection is uniquely determined, if (2.1) has an exponential dichotomy on the whole line $\mathbb{R}$. The image of P is then given by the subspace of initial values of solutions bounded on the positive half-line $\mathbb{R}^+$ and the kernel of P is the subspace of initial values of solutions bounded on the negative half-line $\mathbb{R}^-$.

Hence we see that the classical approach of exponential dichotomies (the approach where we chose the constants α and β to have different signs) describes a separation of the set of solutions into those that converge exponentially fast to zero and those increasing exponentially. Thus, by using exponential dichotomies one can define stable or unstable subspaces at time t of the non-autonomous equation (2.1), respectively, by means of the corresponding projections.

Definition 2.1.5. *Assume that equation (2.1) has an exponential dichotomy (2.2) on $\mathbb{R}^+$ with projection $P^+(\cdot)$ and constants $\alpha < 0 < \beta$. Then the* `stable subspace at time` *t, $E^s_{A(\cdot)}(t)$, is well defined by $E^s_{A(\cdot)}(t) := imP^+(t)$.*
If equation (2.1) has an exponential dichotomy on $\mathbb{R}^-$ with projection $P^-(\cdot)$ and constants $\alpha < 0 < \beta$ then the `unstable subspace at time` *t, $E^u_{A(\cdot)}(t)$, is given by $E^u_{A(\cdot)}(t) := kerP^-(t)$.*

In the more generalized concept of exponential dichotomies, where the assumption $\alpha < 0 < \beta$ is replaced by $\alpha < \beta$, one simply separates the set of solutions into those having an exponential upper bound and those having an exponential lower bound. This approach can be used to describe strong stable or strong unstable subspaces of the non-autonomous equation (2.1).

Now, a first example of a linear differential equation having an exponential dichotomy we get by considering the autonomous equation $\dot{x} = Ax$ with constant coefficient matrix A. Let the spectrum $\sigma(A)$ be composed of two non-empty sets σ_1 and σ_2; $\sigma(A) = \sigma_1 \cup \sigma_2$ and $\sigma_1 \cap \sigma_2 = \emptyset$. If there are constants α and β such that

$$\mathrm{Re}(\mu^1) < \alpha < \beta < \mathrm{Re}(\mu^2), \qquad \forall \mu^1 \in \sigma_1, \forall \mu^2 \in \sigma_2 \tag{2.3}$$

then $\dot{x} = Ax$ has an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$ with constants α and β. The associated projection $P(t) \equiv P$ can be chosen to be the spectral projection according to the given decomposition of the spectrum of A.

A special case we obtain, if A has no eigenvalues with zero real part, that is the spectrum $\sigma(A) = \sigma_s \cup \sigma_u$, with $\sigma_s := \{\mu \in \sigma(A)|\ \mathrm{Re}(\mu) < 0\}$ and $\sigma_u := \{\mu \in \sigma(A)|\ \mathrm{Re}(\mu) > 0\}$. In that case we call A ***hyperbolic***. We outline this in the following lemma. Its statement was proven by Coppel in [Cop78].

Lemma 2.1.6. *Let $A \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ be hyperbolic with $\sigma(A) = \sigma_s(A) \cup \sigma_u(A)$. Let further α and β be two constants satisfying*

$$Re(\mu^s) < \alpha < 0 < \beta < \ Re(\mu^u)$$

for all $\mu^s \in \sigma^s(A)$ and $\mu^u \in \sigma^u(A)$. Then the differential equation $\dot{x} = Ax$ has an exponential dichotomy (2.2) on both $\mathbb{R}^+$ and $\mathbb{R}^-$ with exponential rates α and β. The corresponding projections P^+ and P^- satisfy

$$\left.\begin{array}{lllll} imP^+ & = & E_A^s & := & \{x \in \mathbb{R}^n \mid \|e^{At}x\| \underset{t\to\infty}{\to} 0\}, \\ kerP^- & = & E_A^u & := & \{x \in \mathbb{R}^n \mid \|e^{At}x\| \underset{t\to-\infty}{\to} 0\}. \end{array}\right\} \tag{2.4}$$

Indeed one can choose a projection $P(\cdot) \equiv P$ that way that $\dot{x} = Ax$ has an exponential dichotomy on the whole line $\mathbb{R}$: just define $\mathrm{im}P := \mathrm{im}P^+$ and $\ker P := \ker P^-$. Of course $\mathrm{im}P^+ = E_A^s$ is the generalised stable eigenspace of A corresponding to the eigenvalues in $\sigma_s(A)$; $E_A^s = E_A(\sigma_s)$. Analogously $\ker P^- = E_A^u$ is the unstable eigenspace of A; $E_A^u = E_A(\sigma_u)$.

Further examples can be gained by perturbing systems having an exponential dichotomy. For one of the most important properties of exponential dichotomies is their so-called roughness, their persistence under certain perturbations. Over time many different roughness-theorems have been proven, see for example [Cop78] and [JuWig01]. In the following we give the kind of roughness-theorem we need for our analysis. Here we restrict ourselves to exponentially bounded perturbation of autonomous systems.

Lemma 2.1.7 ([Kla06], Lemma A.2.4.)**.** *Let $A \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ and for $t \in [t_0, \infty)$ let $B(t) \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$. We assume that $\dot{x} = Ax$ has an exponential dichotomy on $[t_0, \infty)$ with constants α and β and that there are positive constants K_B and δ such that*

$$\|B(t)\| \leq K_B e^{-\delta t}. \tag{2.5}$$

Then the equation

$$\dot{x} = [A + B(t)]x \tag{2.6}$$

has an exponential dichotomy on $[t_0, \infty)$ with the same constants α and β.

For the proof ideas of Sandstede (see [San93]) and Knobloch (see [Kno99]) were used instead of going along the lines of Coppel's proof, where $\mathrm{sgn}\alpha \neq \mathrm{sgn}\beta$ was explicitly used, see [Cop78, Lemma 4.1]. In order to keep this thesis self-contained we present the proof exactly as it can be found in [Kla06].

Proof. Let $\Phi(\cdot, \cdot)$ be the transition matrix of (2.6) and recall that there is a decomposition $\sigma(A) = \sigma_1 \cup \sigma_2$ of the spectrum of A such that (2.3) is true. The corresponding spectral projection we denote by P, where P projects on the generalised eigenspace of σ_1.

First projections $\hat{P}_s(t)$ on $\mathbb{R}^n$ are constructed which satisfy the second inequality (ii) in (2.2). For that purpose consider for $\eta \in \mathrm{im}P$ the fixed point equation

$$\begin{aligned} x(t,s) &= e^{A(t-s)}\eta + \int_s^t e^{A(t-\tau)}PB(\tau)x(\tau,s)d\tau - \int_t^\infty e^{A(t-\tau)}(id-P)B(\tau)x(\tau,s)d\tau \\ &=: \left(T_s(x,\eta)\right)(t,s) \end{aligned} \tag{2.7}$$

in the Banach space

$$\mathbb{S}_\alpha := \{x : [t_0,\infty)\times[t_0,\infty) \to \mathbb{R}^n \mid \sup_{t\geq s\geq t_0} e^{\alpha(s-t)}\|x(t,s)\| < \infty\}$$

with the norm $\|x\|_\alpha := \sup_{t\geq s\geq t_0} e^{\alpha(s-t)}\|x(t,s)\|$. Indeed the Banach fixed point theorem can be applied to prove that (2.7) has a unique fixed point $x_s(\eta)(\cdot,\cdot)$. For that the following estimates for some positive constant K are exploited:

$$\left.\begin{aligned} \|e^{A(t-s)}P\| &\leq Ke^{\alpha(t-s)}, && t\geq s, \\ \|e^{A(t-s)}(id-P)\| &\leq Ke^{\beta(t-s)} \leq Ke^{\alpha(t-s)}, && s\geq t, \\ \|x(\tau,s)\| &\leq \|x\|_\alpha e^{\alpha(\tau-s)}, && \tau\geq s. \end{aligned}\right\} \tag{2.8}$$

Together with (2.5) these estimates show that T_s maps $\mathbb{S}_\alpha$ into itself and that (at least for sufficiently large t_0) T_s is a contraction. More precisely it yields

$$e^{\alpha(s-t)}\|T_s(x,P\eta)(t,s)\| \leq K\|\eta\| + \frac{KK_B\|x\|_\alpha}{\delta}e^{-\delta t_0} < \infty \tag{2.9}$$

and

$$\|T_s(x,\eta) - T_s(y,\eta)\|_\alpha \leq \frac{KK_B}{\delta}e^{-\delta t_0}\|x-y\|_\alpha. \tag{2.10}$$

This conclusion is true independently on the sign of α. By construction $x_s(\eta)(\cdot,\cdot)$ solves (2.6). On the other hand, if

$$x(\cdot,\cdot), \qquad x(t,s) := \Phi(t,s)\Phi(s,t_0)\xi, \tag{2.11}$$

belongs to $\mathbb{S}_\alpha$, then it solves (2.7). With that projections $\hat{P}_s(s)(\cdot)$ can be defined by

$$\begin{aligned} \hat{P}_s(s) : \quad \mathbb{R}^n &\to \mathbb{R}^n \\ \xi &\mapsto x_s(P\xi)(s,s). \end{aligned}$$

Indeed $\hat{P}_s(s)(\cdot)$ is linear, because $x_s(\eta)(\cdot,\cdot)$ depends linearly on η, and $P\xi = Px_s(P\xi)(s,s)$, see (2.7), shows that $\hat{P}_s(s)^2(\cdot) = \hat{P}_s(s)(\cdot)$. Further, by construction we have

$$x_s(P\xi)(t,s) = \Phi(t,s)x_s(P\xi)(s,s) = \Phi(t,s)\hat{P}_s(s)\xi. \tag{2.12}$$

Hence for $x(\cdot,\cdot)$ defined in (2.11) it applies

$$x(\cdot,\cdot) \in \mathbb{S}_\alpha \quad \Leftrightarrow \quad x(s,s) = \Phi(s,t_0)\xi \in \mathrm{im}\hat{P}_s(s). \tag{2.13}$$

Finally (2.12) and the third estimate in (2.8) yield that for some positive K_s

$$\|\Phi(t,s)\hat{P}_s(s)\| \leq K_s e^{\alpha(t-s)}, \qquad t\geq s\geq t_0. \tag{2.14}$$

Next it is shown that the image of $\hat{P}_s$ are invariant under Φ, that is

$$\Phi(t,s)\mathrm{im}\hat{P}_s(s) = \mathrm{im}\hat{P}_s(t). \tag{2.15}$$

Consider $x(\cdot,\cdot)$, $x(\tau,t) := \Phi(\tau,t)\Phi(t,s)\xi$ with $\xi \in \mathrm{im}\hat{P}_s(s)$. Then, because of (2.14), $x(\cdot,\cdot)$ belongs to $\mathbb{S}_\alpha$. Now the invariance property (2.15) follows from (2.13).

With that projections $P^+(t)$ can be defined which are associated to the exponential dichotomy of (2.6). To this end consider the direct sum decompositions of $\mathbb{R}^n$

$$\mathbb{R}^n = \mathrm{im}\hat{P}_s(t) \oplus \Phi(t,t_0)(\ker\hat{P}_s(t_0)). \tag{2.16}$$

Denote by $P^+(t)$ the corresponding projections with

$$\mathrm{im}P^+(t) = \mathrm{im}\hat{P}_s(t). \tag{2.17}$$

Of course $P^+(t)$ commutes with Φ: $P^+(t)\Phi(t,s) = \Phi(t,s)P^+(s)$. It remains to verify the estimates (ii) and (iii) in (2.2). For that solutions of (2.6) are considered which start in $\ker P^+(s) = \mathrm{im}(id - P^+(s))$ as solutions of the fixed point equation

$$\begin{aligned} x(s,t) &= e^{A(s-t)}(id-P)\eta + \int_{t_0}^{s} e^{A(s-\tau)}PB(\tau)x(\tau,t)d\tau - \int_{s}^{t} e^{A(s-\tau)}(id-P)B(\tau)x(\tau,t)d\tau \\ &=: \left(T_u(x,(id-P)\eta)\right)(s,t) \end{aligned} \tag{2.18}$$

in the Banach space

$$\mathbb{S}_\beta := \{x : [t_0,\infty)\times[t_0,\infty) \to \mathbb{R}^n \mid \sup_{t\geq s\geq t_0} e^{\beta(t-s)}\|x(s,t)\| < \infty\}$$

with the norm $\|x\|_\beta := \sup\limits_{t\geq s\geq t_0} e^{\beta(t-s)}\|x(s,t)\|$. For that the following estimates are used

$$\left.\begin{aligned} \|e^{A(t-s)}P\| &\leq Ke^{\alpha(t-s)} \leq Ke^{\beta(t-s)}, & t\geq s, \\ \|e^{A(t-s)}(id-P)\| &\leq Ke^{\beta(t-s)}, & s \geq t, \\ \|x(\tau,t)\| &\leq \|x\|_\beta e^{\beta(\tau-t)}, & t\geq \tau. \end{aligned}\right\} \tag{2.19}$$

Again the Banach fixed point theorem yields that for each $\eta \in \mathrm{im}(id-P)$ equation (2.18) has a unique fixed point $x_u(\eta)(\cdot,\cdot)$, since

$$e^{\beta(t-s)}\|T_u(x,(id-P)\eta)(s,t)\| \leq K\|\eta\| + \frac{KK_B\|x\|_\beta}{\delta}e^{-\delta t_0} < \infty \tag{2.20}$$

and

$$\|T_u(x,\eta) - T_u(y,\eta)\|_\beta \leq \frac{KK_B}{\delta}e^{-\delta t_0}\|x-y\|_\beta. \tag{2.21}$$

And again

$$\begin{aligned} \hat{P}_u(t) : \quad \mathbb{R}^n &\to \mathbb{R}^n \\ \xi &\mapsto x_u((id-P)\xi)(t,t) \end{aligned}$$

are projections. From the fixed point equation (2.18) we read that $\mathrm{im}\hat{P}_u(t_0) = \mathrm{im}(id-P)$ and similarly we read from (2.7) that $\ker \hat{P}_s(t_0) = \operatorname{im}(id - P)$. This finally gives

$$\operatorname{im}\hat{P}_u(t) = \Phi(t, t_0)\ker\hat{P}_s(t_0) = \ker P^+(t).$$

The function $x_u(\eta)(\cdot, s)$ is a solution of (2.6). Therefore

$$x_u((id - P)\xi)(t, s) = \Phi(t, s)x_u((id - P)\xi)(s, s) = \Phi(t, s)\hat{P}_u(s)(\xi). \tag{2.22}$$

With that we get similar to (2.14) that for some positive K_u

$$\|\Phi(s, t)\hat{P}_u(t)\| \leq K_u e^{\beta(s-t)}, \qquad t \geq s \geq t_0. \tag{2.23}$$

Because of $\operatorname{im}P^+(t) = \operatorname{im}\hat{P}_s(t)$ and $\operatorname{im}(id - P^+(t)) = \operatorname{im}\hat{P}_u(t)$ it yields for $t \geq s \geq t_0$

$$\left.\begin{aligned} \Phi(t, s)P^+(s) &= \Phi(t, s)\hat{P}_s(t)P^+(s) \\ \Phi(s, t)(id - P^+(t)) &= \Phi(s, t)\hat{P}_u(t)(id - P^+(t)). \end{aligned}\right\} \tag{2.24}$$

If $\{\|P^+(t)\|, t \geq t_0\}$ is bounded then, due to (2.14) and (2.23), we find a positive constant C such that

$$\left.\begin{aligned} \|\Phi(t, s)P^+(s)\| &= Ce^{\alpha(t-s)}, \quad t \geq s \geq t_0, \\ \|\Phi(s, t)(id - P^+(t))\| &= Ce^{-\beta(t-s)}, \quad t \geq s \geq t_0. \end{aligned}\right\} \tag{2.25}$$

Then, see Definition 2.1.1, equation (2.6) has an exponential dichotomy as stated in the lemma.

The proof is concluded with the verification that $\{\|P^+(t)\|, t \geq t_0\}$ is indeed bounded. For $\xi \in \mathbb{R}^n$ define $\xi_s := P^+(t)\xi$ and $\xi_u := (id - P^+(t))\xi$. Since $\operatorname{im}P^+(t) = \operatorname{im}\hat{P}_s(t)$ it applies

$$P^+(t)\xi = \hat{P}_s(t)P^+(t)\xi = \hat{P}_s(t)\xi_s = x_s(P\xi_s)(t, t).$$

Then due to (2.7) it yields

$$\begin{aligned} P^+(t)\xi &= P\xi_s - \int_t^\infty e^{A(t-\tau)}(id - P)B(\tau)x_s(P\xi_s)(\tau, t)d\tau \\ &= P\xi - P\xi_u - \int_t^\infty e^{A(t-\tau)}(id - P)B(\tau)x_s(P\xi_s)(\tau, t)d\tau \\ &= P\xi - P\hat{P}_u(t)\xi_u - \int_t^\infty e^{A(t-\tau)}(id - P)B(\tau)x_s(P\xi_s)(\tau, t)d\tau \\ &= P\xi - Px_u((id - P)\xi_u)(t, t) - \int_t^\infty e^{A(t-\tau)}(id - P)B(\tau)x_s(P\xi_s)(\tau, t)d\tau. \end{aligned}$$

Exploiting now (2.18) one obtains

$$P^+(t)\xi = P\xi - \int_{t_0}^{t} e^{A(t-\tau)}PB(\tau)x_u((id - P)\xi_u)(\tau, t)d\tau - \int_t^\infty e^{A(t-\tau)}(id - P)B(\tau)x_s(P\xi_s)(\tau, t)d\tau. \tag{2.26}$$

Because of (2.12) and (2.22) it follows

$$\left.\begin{array}{rl} P^+(t)\xi = P\xi & -\int\limits_{t_0}^{t} e^{A(t-\tau)}PB(\tau)\Phi(\tau,t)\hat{P}_u(t)(id-P^+(t))\xi d\tau \\ & -\int\limits_{t}^{\infty} e^{A(t-\tau)}(id-P)B(\tau)\Phi(\tau,t)\hat{P}_s(t)P^+(t)\xi d\tau. \end{array}\right\} \tag{2.27}$$

Next the integral terms in the last equation are estimated: By means of (2.5), (2.19) and (2.23) we find

$$\|\int\limits_{t_0}^{t} e^{A(t-\tau)}PB(\tau)\Phi(\tau,t)\hat{P}_u(t)(id-P^+(t))\xi d\tau\| \leq \int\limits_{t_0}^{t} KK_BK_u e^{(\alpha-\beta)(t-\tau)}e^{-\delta\tau}(1+\|P^+(t)\|)\|\xi\|\,d\tau.$$

Because of $\alpha-\beta<0$ we get

$$\|\int\limits_{t_0}^{t} e^{A(t-\tau)}PB(\tau)\Phi(\tau,t)\hat{P}_u(t)(id-P^+(t))\xi d\tau\| \leq \left(\int\limits_{t_0}^{t} KK_BK_u e^{-\delta\tau}(1+\|P^+(t)\|)d\tau\right)\|\xi\|. \tag{2.28}$$

In the same way, but this time exploiting (2.5), (2.8) and (2.14) we get

$$\|\int\limits_{t}^{\infty} e^{A(t-\tau)}(id-P)B(\tau)\Phi(\tau,t)\hat{P}_s(t)P^+(t)\xi d\tau\| \leq \left(\int\limits_{t}^{\infty} KK_BK_s e^{-\delta\tau}d\tau\|P^+(t)\|\right)\|\xi\|. \tag{2.29}$$

Then choose t_0 that large that $KK_B\max\{K_s,K_u\}\int_{t_0}^{\infty}e^{-\delta\tau}d\tau \leq \frac{1}{2}$. Finally, combining (2.27) – (2.29) yields

$$\|P^+(t)\| \leq 2\left(\|P\| + KK_BK_u\int\limits_{t_0}^{\infty}e^{-\delta\tau}d\tau\right) \leq 2\|P\|+1.$$

□

We will need some details of the proof again in Chapter 6.3. To make it easier for us later on, we already present the needed statements here as remarks.

Remark 2.1.8. *Indeed the constants K_s and K_u can be chosen equally with*

$$K_s = K_u = \frac{\delta K}{\delta - KK_Be^{-\delta t_0}}.$$

This can be seen as follows. From (2.9) we derive for the unique fixed point $x_s(P\xi)(\cdot,\cdot)$ of (2.7) the estimate

$$\|x_s(P\xi)\|_\alpha \leq \frac{K\delta}{\delta - KK_Be^{-\delta t_0}}\|\xi\|.$$

Recall from (2.10) that t_0 was chosen large enough so that $\delta - KK_Be^{-\delta t_0}>0$. This together with (2.12) gives

$$\|\Phi(t,s)\hat{P}_s(s)\xi\| = \|x_s(P\xi)(t,s)\| \leq e^{\alpha(t-s)}\frac{K\delta}{\delta - KK_Be^{-\delta t_0}}\|\xi\|$$

and compared with (2.14) we obtain the possible constant K_s as presented above.
Analogously we find for the unique fixed point $x_u((id-P)\xi)(\cdot,\cdot)$ of (2.18) that due to (2.20) the following estimate yields

$$\|x_u((id-P)\xi)\|_\beta \leq \frac{K\delta}{\delta - KK_Be^{-\delta t_0}}\|\xi\|.$$

In combination with (2.22) *and* (2.23) *this yields* K_u*.*

Remark 2.1.9. *For* K_s *and* K_u *chosen as in Remark 2.1.8 we can choose*

$$t_0 = \frac{1}{\delta} \ln \left(\frac{(2K+1)KK_B}{\delta} \right).$$

Then the contraction arguments in (2.10) *and* (2.21) *are satisfied with the constant* $\frac{1}{2K+1} < 1$ *and we find with* $e^{-\delta t_0} = \frac{\delta}{(2K+1)KK_B}$ *that*

$$KK_B \max\{K_s, K_u\} \int_{t_0}^{\infty} e^{-\delta\tau} d\tau = \frac{K^2 K_B \delta}{\delta - KK_B e^{-\delta t_0}} \frac{e^{-\delta t_0}}{\delta} = \frac{K}{1 - \frac{1}{2K+1}} \frac{1}{(2K+1)} = \frac{1}{2}.$$

Remark 2.1.10. *Due to* (2.14), (2.23) *and* (2.24) *we obtain that the constant* C *of the exponential dichotomy in* (2.25) *can be estimated with* $C = \sup_{t \geq t_0} \|P^+(t)\| \max\{K_s, K_u\}$*. Applying the norm of* P^+*, Remark 2.1.8 and Remark 2.1.9 this yields*

$$C = (2\|P\| + 1) \frac{\delta K}{\delta - KK_B e^{-\delta t_0}} = \frac{1}{2}(2\|P\| + 1)(2K + 1).$$

We continue the list of remarks with more general statements.

Remark 2.1.11. *If* (2.1) *has an exponential dichotomy on the interval* $[t_0, \infty)$*,* $t_0 > 0$ *then* (2.1) *also has an exponential dichotomy on* $\mathbb{R}^+$ *with the same projection and the same exponential rates* α *and* β*, see [Cop78]. Hence the exponential dichotomy of* (2.6) *on* $[t_0, \infty)$ *can be extended on* $\mathbb{R}^+$*.*

Remark 2.1.12 ([Kla06], Remark A.2.5.)**.** *Let* $\sigma(A) = \sigma^1 \cup \sigma^2$ *be the decomposition of the spectrum of* A *as introduced at the beginning of the proof. Then* $\dot{x} = Ax$ *and therefore also* $\dot{x} = (A + B(t))x$ *has an exponential dichotomy with constants* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *if*

$$Re(\mu^1) < \tilde{\alpha} < \tilde{\beta} < Re(\mu^2), \quad \forall \mu^1 \in \sigma^1, \forall \mu^2 \in \sigma^2.$$

Moreover the corresponding x_s *and* x_u *do not depend on the choice of* $\tilde{\alpha}$ *and* $\tilde{\beta}$*.*

Remark 2.1.13. *Due to the construction of the projection* $P^+(\cdot)$ *we find with* $kerP^+(t) = im\hat{P}_u(t)$ *and* $im\hat{P}_u(t_0) = kerP$ *that*

$$kerP^+(t_0) = kerP.$$

Hence the dimensions of the images of the projections P *and* $P^+(\cdot)$ *of the exponential dichotomies of* $\dot{x} = Ax$ *and* $\dot{x} = (A + B(t))x$ *remain equal.*

Of course a similar lemma to Lemma 2.1.7 holds for exponential dichotomies on $(-\infty, t_0]$. We simply need to claim that $\|B(t)\| \leq K_B e^{\delta t}$ for some $\delta > 0$.

Next we want to state that for any projection $P^+(\cdot)$ and P associated to the exponential dichotomy on $\mathbb{R}^+$ of (2.6) and $\dot{x} = Ax$, respectively, the norm $\|P^+(t) - P\|$ tends exponentially fast to zero for $t \to \infty$. We do this in view of the later estimation of the jump terms gained from Lin's method.

Lemma 2.1.14 ([Kla06], Lemma A.2.6.)**.** *Let the assumptions of Lemma 2.1.7 hold true. Let further* P *be the spectral projection of* A *associated to the exponential dichotomy of* $\dot{x} = Ax$ *on* $[t_0, \infty)$*, c.f.* (2.3)*, and let* $P^+(\cdot)$ *be the projection of the exponential dichotomy of* (2.6) *on* $[t_0, \infty)$ *in accordance with Lemma 2.1.7. Then there are positive constants* ϑ *and* $\mathcal{K}$ *such that* $\|P^+(t) - P\| \leq \mathcal{K} e^{-\vartheta t}$*. In particular* *it can be shown that any* $\vartheta > 0$ *with*

$$\alpha - \beta + \vartheta < 0, \qquad Re(\mu) < \alpha - \vartheta, \ \forall \mu \in \sigma_1, \qquad \vartheta + \delta < 0 \tag{2.30}$$

is suitable.

Remark 2.1.15. *Let* $\tilde{\mu}_1 \in \sigma_1$ *and* $\tilde{\mu}_2 \in \sigma_2$ *denote leading eigenvalues of* σ_1 *and* σ_2*, respectively, in the sense of the inequalities*

$$Re(\mu_1) \leq \ Re(\tilde{\mu}_1), \qquad Re(\tilde{\mu}_2) \leq \ Re(\mu_2), \quad \forall \mu_1 \in \sigma_1, \mu_2 \in \sigma_2.$$

The first and the second condition in (2.30) *can be combined to one restriction on* ϑ*. Indeed within the first inequality* $\vartheta < \beta - \alpha$ *we can make* ϑ *as large as possible if we choose* β *as close to* $Re(\tilde{\mu}_2)$ *and* α *as close to* $Re(\tilde{\mu}_1)$ *as possible, see* (2.3)*. The second restriction* $\vartheta < \alpha - \ Re(\tilde{\mu}_1)$ *tells that* α *should at best be chosen far from* $Re(\tilde{\mu}_1)$ *to increase* ϑ*.*

To optimize ϑ *within these two restrictions we therefore choose* $\alpha = (\beta + \ Re(\tilde{\mu}_1))/2$ *and obtain from both the first and the second inequality in* (2.30)

$$\vartheta < \frac{\beta - \ Re(\tilde{\mu}_1)}{2} = \beta - \alpha = \alpha - \ Re(\tilde{\mu}_1).$$

Now, ϑ *increases with* β *hence we choose* β *as close to* $Re(\tilde{\mu}_2)$ *as possible.*

In the following we want to inspect the correlation between (2.1) and its adjoint equation

$$\dot{x} = -A(t)^T x, \tag{2.31}$$

with regard to exponential dichotomy. The transition matrix of (2.31) we denote by $\Psi(\cdot,\cdot)$. Note, that $\Psi(t,s) = \Phi(s,t)^T$, for all $s,t \in J$, where $\Phi(\cdot,\cdot)$ denotes the transition matrix of (2.1). This simply follows from differentiating the condition $\Phi(t,s)\Phi(s,t) = id$ with respect to t. Therefore solutions of the linear equation (2.1) and solutions of the adjoint equation (2.31) to initial values that stay orthogonal to each other remain orthogonal for all time:

Lemma 2.1.16. *Let* $u(\cdot)$ *and* $v(\cdot)$ *be solutions of* (2.1) *and* (2.31)*, respectively, with* $u(0) = u_0$ *and* $v(0) = v_0$*. If* $\langle u_0, v_0 \rangle = 0$ *then* $\langle u(t), v(t) \rangle = 0$ *for all* $t \in \mathbb{R}$*.*

Proof. With $u(t) = \Phi(t,0)u_0$ and $v(t) = \Psi(t,0)v_0$ we find

$$\langle u(t), v(t) \rangle = \langle \Phi(t,0)u_0, \Psi(t,0)v_0 \rangle = \left\langle \Phi(t,0)u_0, \Phi(0,t)^T v_0 \right\rangle = \langle \Phi(0,t)\Phi(t,0)u_0, v_0 \rangle = \langle u_0, v_0 \rangle = 0.$$

□

Naturally with (2.1) having an exponential dichotomy also the adjoint system (2.31) has an exponential dichotomy, cf. i.e [Pal84]. The following lemma shows the relation of the corresponding projections.

Lemma 2.1.17. *Assume that* (2.1) *has an exponential dichotomy* (2.2) *on* $\mathbb{R}^+$ *with constants* α, β *and projection* $P^+(\cdot)$*. Then we find that the adjoint equation* (2.31) *also has an exponential dichotomy on* $\mathbb{R}^+$*with constants* $-\beta, -\alpha$*. For the associated projection* Q^+ *holds the relation*

$$im Q^+(s) = im(id - P^+(s))^T = (im P^+(s))^\perp.$$

Proof. We still denote by $\Phi(\cdot,\cdot)$ and $\Psi(\cdot,\cdot)$ the transition matrices of (2.1) and (2.31), respectively. Recall that $\Psi(t,s) = \Phi(s,t)^T$. Now, let (2.1) have an exponential dichotomy (2.2) on $\mathbb{R}^+$ with projection $P^+(\cdot)$. Then we find that the projection $P^+(\cdot)^T$ commutes with $\Psi(\cdot,\cdot)$, since for all $s,t \in \mathbb{R}^+$

$$\Psi(s,t)P^+(t)^T = \Phi(t,s)^T P^+(t)^T = [P^+(t)\Phi(t,s)]^T = [\Phi(t,s)P^+(s)]^T = P^+(s)^T\Phi(t,s)^T = P^+(s)^T\Psi(s,t).$$

In finite dimensional spaces every matrix norm is equivalent, that is for any two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ we find two constants $c_1, c_2 > 0$ such that for all $A \in \mathbb{L}(\mathbb{R}^n,\mathbb{R}^n)$: $c_1\|A\|_a \le \|A\|_b \le c_2\|A\|_a$. Let in addition $\|\cdot\|_T$ denote a self-adjoint norm, that is $\|A^T\|_T = \|A\|_T$ for all $A \in \mathbb{L}(\mathbb{R}^n,\mathbb{R}^n)$. Then there exist constants c_1 and c_2 such that for all $A \in \mathbb{L}(\mathbb{R}^n,\mathbb{R}^n)$

$$c_1\|A^T\| \le \|A^T\|_T = \|A\|_T \le c_2\|A\|.$$

Therefore we find with $\Psi(s,t)P^+(t)^T = [\Phi(t,s)P^+(s)]^T$ that

$$\begin{array}{lllll} \|\Psi(s,t)P^+(t)^T\| & \le & \frac{c_2}{c_1}\,\|\Phi(t,s)P^+(s)\| & \le & \frac{c_2}{c_1}Ke^{\alpha(t-s)}, \quad t \ge s \ge 0, \\ \|\Psi(s,t)(id-P^+(t))^T\| & \le & \frac{c_2}{c_1}\,\|\Phi(t,s)(id-P^+(s))\| & \le & \frac{c_2}{c_1}Ke^{-\beta(s-t)}, \quad s \ge t \ge 0. \end{array}$$

Hence (2.31) also has an exponential dichotomy on $\mathbb{R}^+$ with constants $-\beta, -\alpha$ and for example, projection $(id - P^+(\cdot))^T$. Therefore any projection $Q^+(\cdot)$ associated with the exponential dichotomy of (2.31) on $\mathbb{R}^+$ needs to satisfy $\text{im}Q^+(s) = \text{im}(id - P^+(s))^T = \{x \in \mathbb{R}^n \mid \|\Psi(t,s)x\| \underset{t\to\infty}{\to} 0\}$. The identity $\text{im}(id - P^+(s))^T = (\text{im}P^+(s))^\perp$ follows from Lemma 2.0.1 and Remark 2.0.4. □

An analogous lemma is true for exponential dichotomy on $\mathbb{R}^-$.

Remark 2.1.18. *Assume that equation (2.1) has an exponential dichotomy on $\mathbb{R}^+$ with corresponding projections $P^+(\cdot)$ and the stable subspace $E^s_{A(\cdot)}(t) = imP^+(t)$. Then we get the stable subspace $E^s_{-A(\cdot)^T}(t)$ of the adjoint equation (2.31) from*

$$E^s_{-A(\cdot)^T}(t) = kerP^+(t)^T = [imP^+(t)]^\perp = [E^s_{A(\cdot)}(t)]^\perp.$$

Analogously, if equation (2.1) has an exponential dichotomy on $\mathbb{R}^-$ with projections $P^-(\cdot)$ and the unstable subspace $E^u_{A(\cdot)}(t) = kerP^-(t)$ we get the unstable subspace $E^u_{-A(\cdot)^T}(t)$ of the adjoint equation (2.31) from

$$E^u_{-A(\cdot)^T}(t) = imP^-(t)^T = [kerP^-(t)]^\perp = [E^u_{A(\cdot)}(t)]^\perp.$$

In the special case of $A(t) \equiv A \in \mathbb{L}(\mathbb{R}^n,\mathbb{R}^n)$ the assertion of Remark 2.1.18 is confirmed by Lemma 2.0.2.

Now, the concept of exponential dichotomy can be extended by introducing two instead of just one dividing cut. This leads us to the concept of exponential trichotomy, cf. [HaLi86].

Definition 2.1.19. *Equation (2.1) is said to have an* **exponential trichotomy** *on the interval J, if there exist projections $P(t)$, $Q(t)$ and $R(t)$ with $P(t) + Q(t) + R(t) = id$, $\forall t \in J$, and constants $\alpha < \gamma - c < \gamma + c < \beta$, $c > 0$ and $K > 0$ such that for all $s,t \in J$*

$$\Phi(t,s)P(s) = P(t)\Phi(t,s), \quad \Phi(t,s)Q(s) = Q(t)\Phi(t,s), \quad \Phi(t,s)R(s) = R(t)\Phi(t,s)$$

and

$$\begin{aligned} \|\Phi(t,s)P(s)\| &\le Ke^{\alpha(t-s)}, & \|\Phi(t,s)Q(s)\| &\le Ke^{(\gamma+c)(t-s)}, & t &\ge s,\\ \|\Phi(t,s)Q(s)\| &\le Ke^{(\gamma-c)(t-s)}, & \|\Phi(t,s)R(s)\| &\le Ke^{\beta(t-s)}, & s &\ge t. \end{aligned}$$

If we choose $\gamma = 0$ Definition 2.1.19 conforms with the classical definition of exponential trichotomy as it can be found in [Bey94]. Assume equation (2.1) has an exponential trichotomy on $\mathbb{R}^+$ with $\gamma = 0$. Then again, cf. our considerations on exponential dichotomies that lead to Remark 2.1.2, we find that solutions starting in $\mathrm{im}P(s)$ decay with an exponential rate of at least α, as $t \to \infty$, whereas solutions starting in $\mathrm{im}R(t)$ increase with an exponential rate of at least β as $s \to \infty$. Finally solutions starting in $\mathrm{im}Q(s)$ do not decay faster than e^{-ct} and simultaneously do not increase faster than e^{ct}. In this sense the images of $P(t), Q(t)$ and $R(t)$ can be seen as stable, centre and unstable subspaces at time t corresponding to equation (2.1). Note, that in this case only the images of $P(t)$ and $P(t) + Q(t)$ are uniquely determined, that is only the stable and the centre-stable subspaces are fixed.

Let $A \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ and the spectrum $\sigma(A) = \sigma_s \cup \sigma_c \cup \sigma_u$ with $\sigma_c := \{\mu \in \sigma(A) |\ \mathrm{Re}(\mu) = 0\}$. If $\sigma_s, \sigma_c, \sigma_u \neq \emptyset$, then the autonomous system $\dot{x} = Ax$ has an exponential trichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$ with $\gamma = 0$.
However, setting $\gamma = \mathrm{Re}(\mu)$ for any $\mu \in \sigma(A)$ allows a cut at an arbitrary point of the spectrum. This way one can for example separate the leading stable subspace from the strong stable subspace. Let $\sigma(A) = \sigma_{ss} \cup \{\mu^s\} \cup \sigma_u$ with $\mathrm{Re}(\mu^s) < 0$ and $\mathrm{Re}(\mu) < \mathrm{Re}(\mu^s)$ for all $\mu \in \sigma_{ss}$. Then the system $\dot{x} = Ax$ has an exponential trichotomy on $\mathbb{R}^+$ with $\gamma = \mathrm{Re}(\mu^s)$. Again only the images of $P(t)$ and $P(t) + Q(t)$ are settled.

As in the case of exponential dichotomies there are also certain roughness-theorems for exponential trichotomies. Again we confine ourselves to exponentially bounded perturbations of autonomous systems. The following lemma can already be found in [Kla06, Lemma A.2.9] for the case $\gamma = 0$.

Lemma 2.1.20. *Let $A \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ and for $t \in [t_0, \infty)$ let $B(t) \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$. We assume that $\dot{x} = Ax$ has an exponential trichotomy on $[t_0, \infty)$ with constants $\alpha < \gamma - c < \gamma + c < \beta$, $c > 0$, and that there are positive constants K_B and δ such that*

$$\|B(t)\| \le K_B e^{-\delta t}.$$

Then equation (2.6) has an exponential trichotomy on $[t_0, \infty)$ with the same constants $\alpha < \gamma - c < \gamma + c < \beta$.

Proof. The proof parallels the proof in [Kla06]. The idea is to define the projections P, Q and R by means of projections of exponential dichotomies of (2.6). According to Lemma 2.1.7 equation (2.6) has two exponential dichotomies on $[t_0, \infty)$: one with constants $\alpha < \gamma - c$ and corresponding projection $P_s(t)$ and another one with constants $\gamma + c < \beta$ and projection $P_{cs}(t)$. Since $\alpha < \gamma + c$ we find $\mathrm{im}P_s(t) \subset \mathrm{im}P_{cs}(t)$. Further, thanks to Lemma 2.1.3, we may choose the kernels of these projections such that $\ker P_{cs}(0) \subset \ker P_s(0)$ and hence $\ker P_{cs}(t) \subset \ker P_s(t)$ for all t. This yields that $P_s(t)P_{cs}(t) = P_{cs}(t)P_s(t) = P_s(t)$ and therefore $Q(t) := P_{cs}(t) - P_s(t)$ is a projection. This shows that equation (2.6) has an exponential trichotomy with projections $P(t) := P_s(t)$, $Q(t)$ and $R(t) := id - P_{cs}(t)$ and constants $\alpha < \gamma - c < \gamma + c < \beta$. □

2.2 Variational equations

In this section we apply Lemma 2.1.7 and 2.1.20 to prove that certain variational equations have exponential dichotomies or trichotomies, respectively. Further we present geometrical interpretations of the corresponding stable and unstable subspaces. To this end we start with the autonomous differential equation in $\mathbb{R}^n$

$$\dot{x} = f(x), \tag{2.32}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth. Denote by $\{\varphi^t(\cdot)\}$ the corresponding flow of the system (2.32) and let p be a ***hyperbolic equilibrium*** of (2.32), that is $Df(p)$ is a hyperbolic matrix.

Remark 2.2.1. *Due to Lemma 2.1.6 the linear autonomous differential equation $\dot{x} = Df(p)x$ has an exponential dichotomy (2.2) on $\mathbb{R}$ with constants α^s and α^u, where*

$$Re(\mu^s) < \alpha^s < 0 < \alpha^u < \ Re(\mu^u)$$

for all $\mu^s \in \sigma_s$ and $\mu^u \in \sigma_u$; $\sigma(Df(p)) = \sigma_s \cup \sigma_u$. The corresponding projection P is the spectral projection with respect to the generalised eigenspaces of σ_s and σ_u, that is $imP = E_{Df(p)}(\sigma_s)$.

The ***stable*** and ***unstable manifold*** to the equilibrium p are defined by

$$\left.\begin{aligned} W^s(p) &:= \{x \in \mathbb{R}^n \mid \|\varphi^t(x) - p\| \underset{t\to\infty}{\to} 0\}, \\ W^u(p) &:= \{x \in \mathbb{R}^n \mid \|\varphi^t(x) - p\| \underset{t\to-\infty}{\to} 0\}. \end{aligned}\right\} \tag{2.33}$$

With $n_s := \dim W^s(p)$ and $n_u := \dim W^u(p)$ we find that $n_s + n_u = n$. Solutions within the stable manifold converge exponentially fast towards the equilibrium for positive time. Analogously solutions in the unstable manifold converge exponentially fast towards the equilibrium for negative time. We express this in the following Lemma which is based on [Rob99, Chapter 5, Theorem 10.1].

Lemma 2.2.2. *Consider the differential equation (2.32) with the corresponding flow $\{\varphi^t(\cdot)\}$ and the hyperbolic equilibrium p.*

(i) Let $x_0 \in W^s(p)$ then

$$\|\varphi^t(x_0) - p\| \le e^{\alpha^s t}\|x_0\|, \qquad \forall t > 0$$

(ii) Let $x_0 \in W^u(p)$ then

$$\|\varphi^t(x_0) - p\| \le e^{\alpha^u t}\|x_0\|, \qquad \forall t < 0$$

The exponential rates α^s and α^u are bounds of the corresponding stable and unstable spectrum, respectively, as presented in Remark 2.2.1.

Recall that the generalised stable eigenspace $E_{Df(p)}(\sigma_s)$ is tangent to the stable manifold $W^s(p)$ in the equilibrium point p. Analogously $E_{Df(p)}(\sigma_u)$ is tangent to $W^u(p)$ in p. Regarding the theory of stable and unstable manifolds we refer to [Shu86, HiPuSh77, Rob99] where the assertions made above can be found.

The ***linear variational equation*** of (2.32) along a solution $\varphi^t(x_0)$ of (2.32) through any point $x_0 \in \mathbb{R}^n$ reads as follows

$$\dot{x} = Df(\varphi^t(x_0))x. \tag{2.34}$$

The ***adjoint variational equation*** corresponding to (2.34) reads

$$\dot{x} = -[Df(\varphi^t(x_0))]^T x. \tag{2.35}$$

Alike the case of the linear equation (2.1) and its adjoint equation (2.31) we again denote the transition matrices of (2.34) and (2.35) by $\Phi(t,\tau)$ and $\Psi(t,\tau) := \Phi(\tau,t)^T$, respectively.

Thanks to the roughness theorem, cf. Lemma 2.1.7, the exponential dichotomy of $\dot{x} = Df(p)x$ passes on to variational equation (2.34) along solutions of (2.32) that start within the stable or unstable manifold, respectively.

Lemma 2.2.3. *Consider the variational equation* (2.34) *with the corresponding transition matrix* $\Phi(t,s)$. *Write* $A(t) := Df(\varphi^t(x_0))$.

(i) Let $x_0 \in W^s(p)$. *Then the variational equation* (2.34) *has an exponential dichotomy* (2.2) *on* $\mathbb{R}^+$ *with constants* $\alpha = \alpha^s$ *and* $\beta = \alpha^u$ *given in Remark 2.2.1. The corresponding projection* $P^+(\cdot)$ *satisfies*

$$im P^+(\tau) = \{x \in \mathbb{R}^n \mid \|\Phi(t,\tau)x\| \underset{t\to\infty}{\to} 0\} =: E^s_{A(\cdot)}(\tau)$$

with $\dim(im P^+(\tau)) = n_s$.

(ii) Let $x_0 \in W^u(p)$. *Then the variational equation* (2.34) *has an exponential dichotomy* (2.2) *on* $\mathbb{R}^-$ *with constants* $\alpha = \alpha^s$ *and* $\beta = \alpha^u$ *given in Remark 2.2.1. The corresponding projection* $P^-(\cdot)$ *satisfies*

$$ker P^-(\tau) = \{x \in \mathbb{R}^n \mid \|\Phi(t,\tau)x\| \underset{t\to-\infty}{\to} 0\} =: E^u_{A(\cdot)}(\tau).$$

with $\dim(ker P^-(\tau)) = n_u$.

This Lemma basically can be found in [Sch95, Lemma 1.3]. However, there the exponential rates α and β were not mentioned. This is due to the fact that a different version of the roughness theorem was used to prove this result, where the exponential rates do not explicitly arise from. Therefore we give the detailed proof using Lemma 2.1.7 even though it hardly differs from the proof given in [Sch95].

Proof. We rewrite (2.34) into

$$\dot{x} = Df(p)x + \left[Df(\varphi^t(x_0)) - Df(p)\right] x.$$

The differential equation $\dot{x} = Df(p)x$ has an exponential dichotomy on $\mathbb{R}^+$ with projection P, $\dim \mathrm{im} P = n_s$, and constants α^s and α^u, cf. Remark 2.2.1. Since $x_0 \in W^s(p)$ the solution $\varphi^t(x_0)$ converges exponentially fast towards p as $t \to \infty$, cf. Lemma 2.2.2. Therefore we find due to the Lipschitz continuity

$$\|Df(\varphi^t(x_0)) - Df(p)\| \le K\|\varphi^t(x_0) - p\| \le KCe^{\alpha^s t}$$

for some positive constants K and C. Hence we can apply Lemma 2.1.7 with $A = Df(p)$ and $B(t) = Df(\varphi^t(x_0)) - Df(p)$ and find that (2.34) has an exponential dichotomy with the same exponential rates α^s and α^u. Due to *(ii)* and *(iii)* in Definition 2.1.1 we find

$$\mathrm{im} P^+(\tau) = \{x \in \mathbb{R}^n \mid \|\Phi(t,\tau)x\| \underset{t\to\infty}{\to} 0\},$$

cf. also Remark 2.1.2. Finally the dimensions of the image and the kernel of the corresponding projections do not change, cf. Remark 2.1.13, so $\dim \mathrm{im} P^+(\tau) = n_s$. Analogously one can prove *(ii)*. □

In case of the adjoint variational equation (2.35) a similar lemma holds true. Again the following result can be found in [Sch95, Lemma 1.4], appart from the exponential rates α and β.

Lemma 2.2.4. *Consider the adjoint variational equation (2.35) with the corresponding transition matrix $\Psi(t,s)$. Let $A(t) := Df(\varphi^t(x_0))$.*

(i) Let $x_0 \in W^s(p)$. Then the adjoint variational equation (2.35) has an exponential dichotomy (2.2) on $\mathbb{R}^+$ with the constants $\alpha = -\alpha^u$ and $\beta = -\alpha^s$ given in Remark 2.2.1. The corresponding projection $Q^+(\cdot)$ fulfils

$$im Q^+(\tau) = \{x \in \mathbb{R}^n \mid \|\Psi(t,\tau)x\| \underset{t\to\infty}{\to} 0\} =: E^s_{-A(\cdot)^T}(\tau)$$

with $\dim(im Q^+(\tau)) = n_u$.

(ii) Let $x_0 \in W^u(p)$. Then the adjoint variational equation (2.35) has an exponential dichotomy (2.2) on $\mathbb{R}^-$ with the constants $\alpha = -\alpha^u$ and $\beta = -\alpha^s$ given in Remark 2.2.1. The corresponding projection $Q^-(\cdot)$ satisfies

$$ker Q^-(\tau) = \{x \in \mathbb{R}^n \mid \|\Psi(t,\tau)x\| \underset{t\to-\infty}{\to} 0\} =: E^u_{-A(\cdot)^T}(\tau)$$

with $\dim(ker Q^-(\tau)) = n_s$.

Proof. The proof runs along the same lines as the proof of Lemma 2.2.3. Again we rewrite (2.35) into

$$\dot{x} = -Df(p)^T x + \left[Df(p) - Df(\varphi^t(x_0))\right]^T x$$

and apply Lemma 2.1.7 with $A = -Df(p)^T$ and $B(t) = [Df(p) - Df(\varphi^t(x_0))]^T$. Since p is a hyperbolic equilibrium of (2.32) also $-Df(p)^T$ has no eigenvalues with zero real part. We just have to bear in mind that $\sigma_s(-A^T) = -\sigma_u(A)$ and $\sigma_u(-A^T) = -\sigma_s(A)$. Therefore $\dot{x} = -Df(p)^T x$ has an exponential dichotomy (2.2) with constants $\alpha = -\alpha^u$ and $\beta = -\alpha^s$, cf. Lemma 2.1.17. □

Remark 2.2.5. *Due to Lemma 2.1.17 the following relations apply between the projection $P^\pm(\cdot)$ of Lemma 2.2.3 and the projections $Q^\pm(\cdot)$ of Lemma 2.2.4:*

$$im Q^+(t) = (im P^+(t))^\perp \quad \text{and} \quad ker Q^-(t) = (ker P^-(t))^\perp.$$

In the following we take a closer look at the stable and unstable subspaces of the variational equations and find that they can be represented by the tangent spaces of the stable and unstable manifolds of (2.32) in any point x_0 of the stable or unstable manifold.

Lemma 2.2.6 ([Sch95], Lemma 1.5, Lemma 1.6)**.** *Let p be a saddle point of (2.32) and let $\Phi(t,s)$ and $\Psi(t,s)$ be the transition matrices of the variational equation (2.34) and the adjoint variational equation (2.35) along the solution $\varphi^t(x_0)$, respectively.*

(i) If $x_0 \in W^s(p)$ then

$$\begin{aligned} T_{x_0}W^s(p) &= \{x \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}^+} \|\Phi(t,0)x\| < \infty\} &= E^s_{A(\cdot)}(0), \\ (T_{x_0}W^s(p))^\perp &= \{x \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}^+} \|\Psi(t,0)x\| < \infty\} &= E^s_{-A(\cdot)^T}(0). \end{aligned}$$

(ii) If $x_0 \in W^u(p)$ then

$$\begin{aligned} T_{x_0}W^u(p) &= \{x \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}^-} \|\Phi(t,0)x\| < \infty\} &= E^u_{A(\cdot)}(0), \\ (T_{x_0}W^u(p))^\perp &= \{x \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}^-} \|\Psi(t,0)x\| < \infty\} &= E^u_{-A(\cdot)^T}(0). \end{aligned}$$

We omit the proof and simply refer to [Sch95].

In the following we adopt the concept of exponential trichotomies on the set of variational equations (2.34) and their adjoint equations (2.35) along solutions within the stable or unstable manifold of a hyperbolic equilibrium. To this end let p be a hyperbolic equilibrium of (2.32). The spectrum $\sigma(Df(p))$ shall be divided into leading stable and unstable eigenvalues $\sigma_{ls/lu}$ and strong stable and unstable spectrum, that is $\sigma(Df(p)) = \sigma_{ss} \cup \sigma_{ls} \cup \sigma_{lu} \cup \sigma_{uu}$ where

$$\sigma_{ls} := \{\mu^s \in \sigma(Df(p)) \mid \ \mathrm{Re}(\mu^s) = \max\{\mathrm{Re}(\mu), \mu \in \sigma_s\}\},$$

$$\sigma_{lu} := \{\mu^u \in \sigma(Df(p)) \mid \ \mathrm{Re}(\mu^u) = \min\{\ \mathrm{Re}(\mu), \mu \in \sigma_u\}\},$$

and $\sigma_{ss} = \sigma_s \setminus \sigma_{ls}$, $\sigma_{uu} = \sigma_u \setminus \sigma_{lu}$. Hence we obtain $\sigma_s = \sigma_{ls} \cup \sigma_{ss}$ and $\sigma_u = \sigma_{lu} \cup \sigma_{uu}$.

Analogously to the considerations in Remark 2.2.1 regarding exponential dichotomies we separate the spectrum by real constants in the following manner:

$$\mathrm{Re}\ \mu < \alpha^{ss} < \beta^s < \ \mathrm{Re}\ \mu^s < \alpha^s < 0 < \alpha^u < \ \mathrm{Re}\ \mu^u < \beta^u < \alpha^{uu} < \ \mathrm{Re}\ \hat{\mu} \tag{2.36}$$

for all $\mu \in \sigma_{ss}$, $\mu^s \in \sigma_{ls}$, $\mu^u \in \sigma_{lu}$ and $\hat{\mu} \in \sigma_{uu}$. In the following we denote by n_{ls} and n_{lu} the dimensions of the generalized eigenspaces $E_{Df(p)}(\sigma_{ls})$ and $E_{Df(p)}(\sigma_{lu})$, respectively. Further let n_{ss} be the dimension of the generalized strong stable eigenspace $E_{Df(p)}(\sigma_{ss})$, that is the sum of the algebraic multiplicity of all eigenvalues $\mu \in \sigma_{ss}$ and let analogously n_{uu} be the dimension of the generalized strong unstable eigenspace $E_{Df(p)}(\sigma_{uu})$. Then we find $n_s = n_{ls} + n_{ss}$ and $n_u = n_{lu} + n_{uu}$. Recall Definition 2.1.19 for the introduction of the projections P, Q and R and their relation to the exponential rates $\alpha, \beta, \gamma - c$ and $\gamma + c$.

Lemma 2.2.7. *Consider the linear variational equation* (2.34) *and its adjoint variational equation* (2.35) *along $\varphi^t(x_0)$ with their corresponding transition matrices $\Phi(t,\tau)$ and $\Psi(t,\tau)$. Let $A(t) := Df(\varphi^t(x_0))$.*

(i) Let $x_0 \in W^s(p)$. Then the variational equation (2.34) *has an exponential trichotomy on $\mathbb{R}^+$ with projections $P = P^+_{ss}$, $Q = P^+_{ls}$ and $R = P^+_u$ and constants $\alpha = \alpha^{ss}$, $\beta = \alpha^u$, $\gamma - c = \beta^s$ and $\gamma + c = \alpha^s$ such that*

$$\begin{aligned} im(P^+_{ss}(\tau)) &= \{x \in \mathbb{R}^n \mid \|\Phi(t,\tau)x\| \le Ke^{\alpha^{ss}(t-\tau)}\|x\|, \quad t \ge \tau \ge 0\} &=: E^{ss}_{A(\cdot)}(\tau), \\ im(P^+_{ss}(\tau) + P^+_{ls}(\tau)) &= \{x \in \mathbb{R}^n \mid \|\Phi(t,\tau)x\| \le Ke^{\alpha^{s}(t-\tau)}\|x\|, \quad t \ge \tau \ge 0\} &= E^{s}_{A(\cdot)}(\tau). \end{aligned}$$

Further we have $\dim im(P^+_{ss}) = n_{ss}$, $\dim im(P^+_{ls}) = n_{ls}$ and $\dim im(P^+_u) = n_{lu} + n_{uu}$.

(ii) Let $x_0 \in W^u(p)$. Then the variational equation (2.34) *has an exponential trichotomy on $\mathbb{R}^-$ with projections $P = P^-_s$, $Q = P^-_{lu}$ and $R = P^-_{uu}$ and constants $\alpha = \alpha^s$, $\beta = \alpha^{uu}$, $\gamma - c = \alpha^u$ and $\gamma + c = \beta^u$ such that*

$$im(P^-_{uu}(\tau)) = \{x \in \mathbb{R}^n \mid \|\Phi(t,\tau)x\| \le Ke^{\alpha^{uu}(t-\tau)}\|x\|, \quad t \le \tau \le 0\} =: E^{uu}_{A(\cdot)}(\tau),$$

$$im(P^-_{uu}(\tau)+P^-_{lu}(\tau)) \quad = \quad \{x\in\mathbb{R}^n \mid \|\Phi(t,\tau)x\|\le Ke^{\alpha^u(t-\tau)}\|x\|, \quad t\le\tau\le 0\} \quad = \quad E^u_{A(\cdot)}(\tau).$$

Further we have $\dim im(P^-_s)=n_{ls}+n_{ss}$*,* $\dim im(P^+_{lu})=n_{lu}$ *and* $\dim im(P^+_{uu})=n_{uu}$*.*

(iii) Let $x_0\in W^s(p)$*. Then the adjoint variational equation* (2.35) *has an exponential trichotomy on* $\mathbb{R}^+$ *with projections* $P=Q^+_{ss}$*,* $Q=Q^+_{ls}$ *and* $R=Q^+_u$ *and constants* $\alpha=-\alpha^{uu}$*,* $\beta=-\alpha^s$*,* $\gamma-c=-\beta^u$ *and* $\gamma+c=-\alpha^u$ *such that*

$$im(Q^+_{ss}(\tau)) = \quad \{x\in\mathbb{R}^n \mid \|\Psi(t,\tau)x\|\le Ke^{-\alpha^{uu}(t-\tau)}\|x\|, \quad t\ge\tau\ge 0\} \quad =: E^{ss}_{-A(\cdot)^T}(\tau),$$

$$im(Q^+_{ss}(\tau)+Q^+_{ls}(\tau)) = \quad \{x\in\mathbb{R}^n \mid \|\Psi(t,\tau)x\|\le Ke^{-\alpha^{u}(t-\tau)}\|x\|, \quad t\ge\tau\ge 0\} \quad = E^{s}_{-A(\cdot)^T}(\tau).$$

Further we have $\dim im(Q^+_{ss})=n_{uu}$*,* $\dim im(Q^+_{ls})=n_{lu}$ *and* $\dim im(Q^+_u)=n_{ls}+n_{ss}$*.*

(iv) Let $x_0\in W^u(p)$*. Then the adjoint variational equation* (2.35) *has an exponential trichotomy on* $\mathbb{R}^-$ *with projections* $P=Q^-_s$*,* $Q=Q^-_{lu}$ *and* $R=Q^-_{uu}$ *and constants* $\alpha=-\alpha^u$*,* $\beta=-\alpha^{ss}$*,* $\gamma-c=-\alpha^s$ *and* $\gamma+c=-\beta^s$ *such that*

$$im(Q^-_{uu}(\tau)) = \quad \{x\in\mathbb{R}^n \mid \|\Psi(t,\tau)x\|\le Ke^{-\alpha^{ss}(t-\tau)}\|x\|, \quad t\le\tau\le 0\} \quad =: E^{uu}_{-A(\cdot)^T}(\tau),$$

$$im(Q^-_{uu}(\tau)+Q^-_{lu}(\tau)) = \quad \{x\in\mathbb{R}^n \mid \|\Psi(t,\tau)x\|\le Ke^{-\alpha^{s}(t-\tau)}\|x\|, \quad t\le\tau\le 0\} \quad = E^{u}_{-A(\cdot)^T}(\tau).$$

Further we have $\dim im(Q^-_s)=n_{lu}+n_{uu}$*,* $\dim im(Q^-_{lu})=n_{ls}$ *and* $\dim im(Q^-_{uu})=n_{ss}$*.*

The proofs run along the same lines as the proofs of Lemma 2.2.3 and 2.2.4, respectively, by applying the rougness-Lemma 2.1.20.

Similar to the statements of Lemma 2.2.6 we find an explicit representation of the strong stable subspaces $E^{ss}_{A(\cdot)}(\tau)$ and $E^{uu}_{-A(\cdot)^T}(\tau)$. To this end we first introduce the geometrical objects which will be identified as the images of the projections $P^{\pm}_{ss}$ and $Q^{\pm}_{uu}$.

Remark 2.2.8. *Invariant manifold theory provides the existence of the so-called* ***extended unstable manifold*** $W^{ls,u}(p)$ *of* p *which is a locally invariant manifold whose tangent space at* p *is the sum of the generalized unstable eigenspace* $E_{Df(p)}(\sigma_u)$ *and the leading stable eigenspace* $E_{Df(p)}(\sigma_{ls})$*. Likewise the* ***extended stable manifold*** $W^{s,lu}(p)$ *is a locally invariant manifold having the tangent space in* p *consisting of the generalised stable eigenspace* $E_{Df(p)}(\sigma_s)$ *plus the leading unstable eigenspace* $E_{Df(p)}(\sigma_{lu})$*. The construction of these manifolds can take place via the graph transform method and it provides that any* $x\in W^{s,lu}(p)$ *does not run away from* p *faster than* $e^{\beta^u t}$*, that is*

$$x\in W^{s,lu}(p) \qquad \Rightarrow \qquad \|\varphi^t(x)-p\|\le Ke^{\beta^u t}\|x-p\|, \quad t>0. \tag{2.37}$$

Analogously we find

$$x\in W^{ls,u}(p) \qquad \Rightarrow \qquad \|\varphi^t(x)-p\|\le Ke^{\beta^s t}\|x-p\|, \quad t<0.$$

See (2.36) *for the introduction of* β^u *and* β^s*. Note that the local extended (un)stable manifolds are not uniquely defined but they possess unique tangent spaces along the (un)stable manifold. The first general proofs of the existence of these manifolds were given by Kelley [Kel67] and by Hirsch, Pugh and Shub [HiPuSh77].*

Remark 2.2.9. *Assume the leading stable eigenvalue of a hyperbolic equilibrium* p *to be real and semisimple. Then, within the stable manifold one find an invariant* ***strong stable foliation*** F^{ss}*. The leaves* *of F^{ss}, the so-called* **`strong stable fibres`**, *$\mathcal{F}^{ss}$, have the same dimension as $E_{Df(p)}(\sigma_{ss})$ and they are local submanifolds of the stable manifold. The foliation is locally invariant. To be precise, the single fibres are transported into each other via the flow of the differential equation:*

$$\varphi^t(\mathcal{F}^{ss}(x_0)) \subseteq \mathcal{F}^{ss}(\varphi^t(x_0)), \quad t \geq 0.$$

Further we find for all $x \in \mathcal{F}^{ss}(x_0)$ that

$$\|\varphi^t(x) - \varphi^t(x_0)\| < Ke^{\alpha^{ss}t}\|x - x_0\|. \tag{2.38}$$

See Figure 2.1 for illustration. The existence of F^{ss} was shown in [Hom96, Theorem 2.1.1] in case of real simple leading eigenvalues and it can be extended to real semisimple eigenvalues.

With this we find

Lemma 2.2.10. *Assume the leading eigenvalues μ^s and μ^u to be real and semisimple. Let $x_0 \in W^s(p)$. Then*

$$\begin{array}{llcl} (i) & E^{ss}_{A(\cdot)}(\tau) & = & T_{\varphi^\tau(x_0)}\mathcal{F}^{ss}(\varphi^\tau(x_0)), \\ (ii) & E^{ss}_{-A(\cdot)^T}(\tau) & = & \left(T_{\varphi^\tau(x_0)}W^{s,lu}(p)\right)^\perp. \end{array}$$

For $x_0 \in W^u(p)$ we have

$$\begin{array}{llcl} (iii) & E^{uu}_{A(\cdot)}(\tau) & = & T_{\varphi^\tau(x_0)}\mathcal{F}^{uu}(\varphi^\tau(x_0)), \\ (iv) & E^{uu}_{-A(\cdot)^T}(\tau) & = & \left(T_{\varphi^\tau(x_0)}W^{ls,u}(p)\right)^\perp. \end{array}$$

Proof. The proof of the lemma runs along the same lines as the proof of Lemma 2.2.6 for the stable subspaces, cf. [Sch95]. We confine ourselves with the proof of (i) and (ii). The proof of (iii) and (iv) follows analogously. Without loss of generality we assume $p = 0$.

We begin with the proof of (i). To start with we prove the assertion for $x_0 \in W^s_{loc}(p)$. Then $\varphi^\tau(x_0) \in W^s_{loc}(p)$ for all $\tau \geq 0$.

We use the fact that the strong stable fibre $\mathcal{F}^{ss}(\varphi^\tau(x_0))$ can be expressed as graph of a smooth function

$$h_\tau : E(\sigma_{ss}) \to E(\sigma_{ls}) + E(\sigma_u)$$

such that locally around p we have $\mathcal{F}^{ss}(\varphi^\tau(x_0)) = graph(h_\tau)$, cf. Figure 2.1. With $H_\tau : E(\sigma_{ss}) \to \mathbb{R}^n$, $H_\tau(\xi) = \xi + h_\tau(\xi)$ we obtain $\mathcal{F}^{ss}(\varphi^\tau(x_0)) = im(H_\tau)$. Using the flow $\{\varphi^t(\cdot)\}$ of (2.32) we obtain a parametrization of $\mathcal{F}^{ss}(\varphi^\tau(x_0))$ by

$$\mathcal{F}^{ss}(\varphi^\tau(x_0)) = \{H_\tau(\xi) \mid \xi \in E(\sigma_{ss})\} = \{\varphi^0(H_\tau(\xi)) \mid \xi \in E(\sigma_{ss})\}.$$

Of course the last relation in this equation is trivial but it serves as a basis for the definition of the function ψ introduced below.

Let ξ_0 be such that $H_\tau(\xi_0) = \varphi^\tau(x_0) \in \mathcal{F}^{ss}(\varphi^\tau(x_0))$. Then we have

$$T_{\varphi^\tau(x_0)}\mathcal{F}^{ss}(\varphi^\tau(x_0)) = \{DH_\tau(\xi_0)\eta \mid \eta \in E(\sigma_{ss})\} = \{D_\xi\big(\varphi^0(H_\tau(\xi_0))\big)\eta \mid \eta \in E(\sigma_{ss})\}.$$

We define for $\eta \in E(\sigma_{ss})$ a function ψ by

$$\psi(\cdot,\eta) : \mathbb{R}^+ \to \mathbb{R}^n; \quad \psi(t,\eta) = D_\xi\big(\varphi^t(H_\tau(\xi_0))\big)\eta, \quad t \geq 0$$

and show that it is continuous differentiable on $\mathbb{R}^+$ and that it solves the variational equation

$$\dot{x} = Df(\varphi^t(H_\tau(\xi_0)))x$$

along the solution $\varphi^t(H_\tau(\xi_0))$ of (2.32):

$$\begin{aligned}\dot{\psi}(t,\eta) &= \tfrac{d}{dt}(D_\xi\big(\varphi^t(H_\tau(\xi_0))\big)\eta) = D_\xi(\tfrac{d}{dt}\varphi^t(H_\tau(\xi_0))\eta) = D_\xi f(\varphi^t(H_\tau(\xi_0)))\eta \\ &= Df(\varphi^t(H_\tau(\xi_0)))D_\xi\big(\varphi^t(H_\tau(\xi_0))\big)\eta \\ &= Df(\varphi^t(H_\tau(\xi_0)))\psi(t,\eta).\end{aligned}$$

On that behalf we assume at least $f \in C^2(\mathbb{R}^n,\mathbb{R}^n)$.

Since ξ_0 was chosen such that $H_\tau(\xi_0) = \varphi^\tau(x_0)$, we find that ψ is a solution of the variational equation $\dot{x} = Df(\varphi^t(\varphi^\tau(x_0)))x = Df(\varphi^{t+\tau}(x_0))x$ with the corresponding transition matrix $U(t,0) = \Phi(t+\tau,\tau)$. Hence ψ satisfies the representation $\psi(t,\eta) = U(t,0)\psi(0,\eta) = \Phi(t+\tau,\tau)\psi(0,\eta)$.

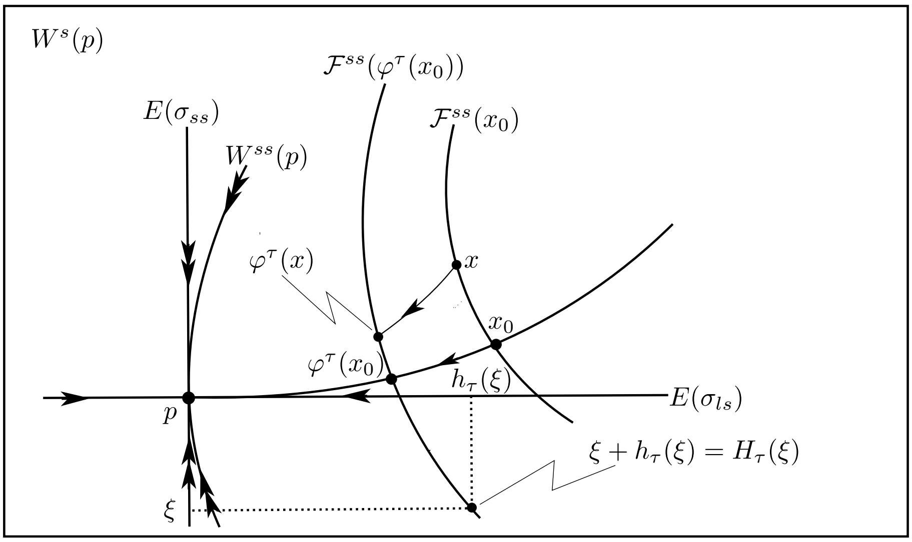


Figure 2.1: Illustration of the strong stable fibres $\mathcal{F}^{ss}(x_0)$ and $\mathcal{F}^{ss}(\varphi^\tau(x_0))$ within the stable manifold $W^s(p)$.

Due to the properties of the strong stable fibre $\mathcal{F}^{ss}(\varphi^\tau(x_0))$, cf.(2.38), the difference of two solutions of (2.32) starting in this fibre goes exponentially fast to zero with a convergence rate of $e^{\alpha^{ss}t}$. Using this property we declare the function

$$\Omega : E(\sigma_{ss}) \to C^1_{\alpha^{ss}}(\mathbb{R}^+,\mathbb{R}^n); \quad \xi \mapsto \Omega(\xi) := \varphi^{(\cdot)}(H_\tau(\xi)) - \varphi^{(\cdot)}(\varphi^\tau(x_0)).$$

Here we denote by $C^1_{\alpha^{ss}}(\mathbb{R}^+,\mathbb{R}^n)$ the space of all continuous differentiable functions $g : \mathbb{R}^+ \to \mathbb{R}^n$ satisfying $\|g(t)\| \leq Ke^{\alpha^{ss}t}$ for some constant $K > 0$ and for all $t \geq 0$.

Then we find that $\psi(\cdot,\eta) \in C^1_{\alpha^{ss}}(\mathbb{R}^+,\mathbb{R}^n)$ for all $\eta \in E(\sigma_{ss})$ since $\psi(\cdot,\eta) = D\Omega(\xi_0)\eta$ with $D\Omega(\xi_0) \in \mathbb{L}(E(\sigma_{ss}), C^1_{\alpha^{ss}}(\mathbb{R}^+,\mathbb{R}^n))$. Thus we have

$$\|\Phi(t+\tau,\tau)\psi(0,\eta)\| = \|\psi(t,\eta)\| \leq Ke^{\alpha^{ss}t} = Ke^{\alpha^{ss}((t+\tau)-\tau)}$$

and hence

$$T_{\varphi^\tau(x_0)}\mathcal{F}^{ss}(\varphi^\tau(x_0)) \subseteq \{x \in \mathbb{R}^n \mid \|\Phi(t,\tau)x\| \leq Ke^{\alpha^{ss}(t-\tau)}\|x\|, \quad t \geq \tau \geq 0\}.$$

Due to Lemma 2.2.7 we find that the right-hand side has the same dimension as the tangent space of the stable fibre $T_{\varphi^\tau(x_0)}\mathcal{F}^{ss}$. Hence we have proven

$$T_{\varphi^\tau(x_0)}\mathcal{F}^{ss}(\varphi^\tau(x_0)) = \{x \in \mathbb{R}^n \mid \|\Phi^+(t,\tau)x\| \leq Ke^{\alpha^{ss}(t-\tau)}\|x\|, \quad t \geq \tau \geq 0\}.$$

for all $x_0 \in W^s_{loc}(p)$.

Now we assume $x_0 \in W^s(p)$ and $x_0 \notin W^s_{loc}(p)$. Then there exists a $T \in \mathbb{R}^+$ such that $x_1 := \varphi^T(x_0) \in W^s_{loc}(p)$.

Let U_0 be a neighbourhood of $\varphi^\tau(x_0)$ in $W^s(p)$. Then we can choose T large enough such that $U_1 := \varphi^T(U_0)$ is a neighbourhood of $\varphi^\tau(x_1)$ in $W^s_{loc}(p)$. Since $\varphi^T(\cdot)$ is a local diffeomorphism from U_0 to U_1 we find

$$D\varphi^T(\varphi^\tau(x_0))T_{\varphi^\tau(x_0)}\mathcal{F}^{ss}(\varphi^\tau(x_0)) = T_{\varphi^\tau(x_1)}\mathcal{F}^{ss}(\varphi^\tau(x_1)).$$

The solutions $D\varphi^{(\cdot)}(\varphi^\tau(x_0))\eta$, $\eta \in \mathbb{R}^n$, of the linear variational equation $\dot{x} = Df(\varphi^{t+\tau}(x_0))x$ on $\mathbb{R}^+$ can be expressed as

$$D\varphi^t(\varphi^\tau(x_0))\eta = \Phi(t+\tau,\tau)\underbrace{D\varphi^0(\varphi^\tau(x_0))}_{id}\eta = \Phi(t+\tau,\tau)\eta.$$

From this it follows

$$\Phi(t,\tau)\eta = \Phi(t,T+\tau)\Phi(T+\tau,\tau)\eta = \Phi(t,T+\tau)D\varphi^T(\varphi^\tau(x_0))\eta.$$

Hence we find

$$\begin{aligned}
&\eta \in T_{\varphi^\tau(x_0)}\mathcal{F}^{ss}(\varphi^\tau(x_0)) \\
&\iff D\varphi^T(\varphi^\tau(x_0))\eta \in T_{\varphi^\tau(x_1)}\mathcal{F}^{ss}(\varphi^\tau(x_1)) \\
&\iff \|\Phi(t,\tau+T)D\varphi^T(\varphi^\tau(x_0))\eta\| \leq Ke^{\alpha^{ss}(t-\tau-T)}\|D\varphi^T(\varphi^\tau(x_0))\eta\|, \quad t \geq \tau+T \geq 0 \\
&\iff \|\Phi(t,\tau)\eta\| \leq Ke^{-\alpha^{ss}T}\|D\varphi^T(\varphi^\tau(x_0))\|e^{\alpha^{ss}(t-\tau)}\|\eta\|, \qquad t \geq \tau+T \geq 0 \\
&\iff \|\Phi(t,\tau)\eta\| \leq \tilde{K}e^{\alpha^{ss}(t-\tau)}\|\eta\|, \qquad t \geq \tau \geq 0
\end{aligned}$$

The last equivalence holds since the existence of an exponential dichotomy on $[T+\tau,\infty)$ implies the existence of an exponential dichotomy on $\mathbb{R}^+$ with the same projections, cf. [Cop78] and Remark 2.1.11.

The proof of (ii) is carried out in three steps. At first we prove in the same way as we have proven (i) that for any $x_0 \in W^s_{loc}(p)$

$$T_{\varphi^\tau(x_0)}W^{s,lu}_{loc}(p) \subseteq \{x \in \mathbb{R}^n \mid \|\Phi(t,\tau)x\| \leq Le^{\beta^u(t-\tau)}\|x\|,\ t \geq \tau \geq 0\}.$$

Then we extend this assertion for all $x_0 \in W^s(p)$. Finally we prove

$$\left[T_{\varphi^\tau(x_0)}W^{s,lu}(p)\right]^\perp = \{x \in \mathbb{R}^n \mid \|\Psi(t,\tau)x\| \le Le^{-\alpha^{uu}(t-\tau)}\|x\|,\ t \ge \tau \ge 0\}.$$

Let $x_0 \in W^s_{loc}(p)$. Recall that $W^{s,lu}(p)$ is not uniquely defined. However its tangent space along elements of the stable manifold is uniquely given. So we simply pick one manifold and express it as graph of a smooth function $h : E(\sigma_s \cup \sigma_{lu}) \to E(\sigma_{uu})$. With the function

$$H : E(\sigma_s \cup \sigma_{lu}) \to \mathbb{R}^n; \quad H(\xi) = \xi + h(\xi)$$

and the flow $\{\varphi^t(\cdot)\}$ of (2.32) we obtain a parametrization of the chosen manifold $W^{s,lu}_{loc}(p)$ by

$$W^{s,lu}_{loc}(p) = \{H(\xi) \mid \xi \in E(\sigma_s \cup \sigma_{lu})\} = \{\varphi^0(H(\xi)) \mid \xi \in E(\sigma_s \cup \sigma_{lu})\}.$$

Now we choose $\xi_0 \in E(\sigma_s \cup \sigma_{lu})$ such that $H(\xi_0) = \varphi^\tau(x_0) \in W^s_{loc}(p) \subset W^{s,lu}_{loc}(p)$, $\tau > 0$. Then we have

$$T_{H(\xi_0)}W^{ls,u}(p) = \{DH(\xi_0)\eta \mid \eta \in E(\sigma_s \cup \sigma_{lu})\} = \{D_\xi\varphi^0(H(\xi_0))\eta \mid \eta \in E(\sigma_s \cup \sigma_{lu})\}.$$

Analogously to the considerations in (i) we find that the function

$$\psi(\cdot,\eta) : \mathbb{R}^+ \to \mathbb{R}^n; \quad \psi(t,\eta) = D_\xi\varphi^t(H(\xi_0))\eta, \quad t \ge 0$$

is continuous differentiable on $\mathbb{R}^+$ and solves the variational equation

$$\dot{x} = Df(\varphi^t(H(\xi_0)))x = Df(\varphi^{t+\tau}(x_0))x.$$

Therefore ψ satisfies the representation $\psi(t,\eta) = U(t,0)\psi(0,\eta)$ with transition matrix $U(\cdot,\cdot)$ satisfying $U(t,0) = \Phi(t+\tau,\tau)$.
Now, due to (2.37) we can declare the function

$$\Omega : E(\sigma_s \cup \sigma_{lu}) \to C^1_{\beta^u}(\mathbb{R}^+,\mathbb{R}^n); \quad \xi \mapsto \Omega(\xi) := \varphi^{(\cdot)}(H(\xi)).$$

Then $\psi(\cdot,\eta) \in C^1_{\beta^u}(\mathbb{R}^-,\mathbb{R}^n)$ for all $\eta \in E(\sigma_s \cup \sigma_{lu})$ since $\psi(\cdot,\eta) = D\Omega(\xi_0)\eta$ with $D\Omega(\xi_0) \in \mathbb{L}(E(\sigma_s \cup \sigma_{lu}), C^1_{\beta^u}(\mathbb{R}^+,\mathbb{R}^n))$. Therefore we have

$$\|\Phi(t+\tau,\tau)\psi(0,\eta)\| = \|\psi(t,\eta)\| \le Le^{\beta^u t}.$$

and hence

$$T_{\varphi^\tau(x_0)}W^{s,lu}_{loc}(p) \subseteq \{x \in \mathbb{R}^n \mid \ \|\Phi(t,\tau)x\| \le Le^{\beta^u(t-\tau)}, \quad t \ge \tau \ge 0\}.$$

Now we assume $x_0 \in W^s(p)$, $x_0 \notin W^s_{loc}(p)$. Then again we can choose a $T \in \mathbb{R}^+$ such that $x_1 := \varphi^T(x_0) \in W^s_{loc}(p)$ and $U_1 := \varphi^T(U_0)$ is a neighbourhood of $\varphi^\tau(x_1)$ in $W^s_{loc}(p)$ for a neighbourhood U_0 of $\varphi^\tau(x_0)$. Since $\varphi^T(\cdot)$ is a local diffeomorphism from U_0 to U_1 we find

$$D\varphi^T(\varphi^\tau(x_0))T_{\varphi^\tau(x_0)}W^{s,lu}(p) = T_{\varphi^\tau(x_1)}W^{s,lu}(p).$$

The solutions $D\varphi^{(\cdot)}(\varphi^\tau(x_0))\eta$, $\eta \in \mathbb{R}^n$, of the linear variational equation $\dot{x} = Df(\varphi^{t+\tau}(x_0))x$ on $\mathbb{R}^+$ can be expressed as

$$D\varphi^t(\varphi^\tau(x_0))\eta = \Phi(t+\tau,\tau)\underbrace{D\varphi^0(\varphi^\tau(x_0))}_{id}\eta = \Phi(t+\tau,\tau)\eta.$$

From this it follows

$$\Phi(t,\tau)\eta = \Phi(t,\tau+T)\Phi(\tau+T,\tau)\eta = \Phi(t,\tau+T)D\varphi^T(\varphi^\tau(x_0))\eta.$$

Hence we find

$$\begin{aligned}
&\eta \in T_{\varphi^\tau(x_0)}W^{s,lu}(p) \\
\iff\quad & D\varphi^T(\varphi^\tau(x_0))\eta \in T_{\varphi^\tau(x_1)}W^{s,lu}(p) \\
\implies\quad & \|\Phi(t,\tau+T)D\varphi^T(\varphi^\tau(x_0))\eta\| \le Le^{\beta^u(t-\tau-T)}\|D\varphi^T(\varphi^\tau(x_0))\eta\|, && t \ge \tau+T \ge 0 \\
\iff\quad & \|\Phi(t,\tau)\eta\| \le Le^{-\beta^u T}\|D\varphi^T(\varphi^\tau(x_0))\|e^{\beta^u(t-\tau)}\|\eta\|, && t \ge \tau+T \ge 0 \\
\iff\quad & \|\Phi(t,\tau)\eta\| \le \tilde{L}e^{\beta^u(t-\tau)}\|\eta\|, && t \ge \tau \ge 0
\end{aligned}$$

We conclude with the third step. In this regard we choose $\xi \in \mathbb{R}^n$ such that $\|\Psi(t,\tau)\xi\| \le Ke^{-\alpha^{uu}(t-\tau)}\|\xi\|$ for all $t \ge \tau \ge 0$. Further let $\eta \in T_{\varphi^\tau(x_0)}W^{s,lu}(p)$. Recall from (2.36) that $0 < \beta^u < \alpha^{uu}$. Then we have on the one hand

$$\begin{aligned}
\left|\left\langle e^{-\beta^u(t-\tau)}\Phi(t,\tau)\eta, e^{\alpha^{uu}(t-\tau)}\Psi(t,\tau)\xi\right\rangle\right| &= e^{(\alpha^{uu}-\beta^u)(t-\tau)}\left|\langle\Phi(t,\tau)\eta, \Psi(t,\tau)\xi\rangle\right| \\
&= e^{(\alpha^{uu}-\beta^u)(t-\tau)}\left|\left\langle\Phi(t,\tau)\eta, \Phi(\tau,t)^T\xi\right\rangle\right| \\
&= e^{(\alpha^{uu}-\beta^u)(t-\tau)}\left|\langle\Phi(\tau,t)\Phi(t,\tau)\eta, \xi\rangle\right| \\
&= e^{(\alpha^{uu}-\beta^u)(t-\tau)}\left|\langle\eta,\xi\rangle\right| \\
&\overset{t\to\infty}{\longrightarrow} \begin{cases} \infty, & \langle\eta,\xi\rangle \ne 0 \\ 0, & \langle\eta,\xi\rangle = 0 \end{cases}
\end{aligned}$$

and on the other hand

$$\left|\left\langle e^{-\beta^u(t-\tau)}\Phi(t,\tau)\eta, e^{\alpha^{uu}(t-\tau)}\Psi(t,\tau)\xi\right\rangle\right| \le \|e^{-\beta^u(t-\tau)}\Phi(t,\tau)\eta\|\|e^{\alpha^{uu}(t-\tau)}\Psi(t,\tau)\xi\| \le \tilde{L}\|\eta\|\cdot K\|\xi\|.$$

Therefore the scalar product of η and ξ has to be zero. Thus we have shown the inclusion

$$\{x \in \mathbb{R}^n \mid \|\Psi(t,\tau)x\| \le Ke^{-\alpha^{uu}(t-\tau)}\|x\|, \quad t \ge \tau \ge 0\} \subseteq \left(T_{\varphi^\tau(x_0)}W^{s,lu}_{loc}(p)\right)^\perp.$$

Again the equality follows from the equality of the dimensions, cf. Lemma 2.2.7(iii). □

2.3 Behaviour in the stable manifold

In the following sections we will apply the concept of exponential dichotomies and trichotomies to determine leading terms and corresponding rates of convergence of special solutions of differential equations. Now this section is attended to solutions of autonomous differential equations starting in the stable manifold of a hyperbolic equilibrium. That kind of solutions converge exponentially fast towards the equilibrium point. We are interested in the exact rate of convergence and the form of the leading term.

We consider the differential equation

$$\dot{x} = f(x, \lambda).$$

First we name the required assumptions.

(A2.1). *Let $p = 0$ be an asymptotically stable equilibrium of a smooth family of vector fields $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and assume that*

(i) $\sigma(D_1 f(0, \lambda)) = \{\mu^s(\lambda)\} \cup \sigma_{ss}(\lambda)$, *where* $Re(\mu) < \alpha^{ss} < Re(\mu^s(\lambda)) < \alpha^s < 0$ *for all* $\mu \in \sigma_{ss}(\lambda)$.

(ii) for all λ the leading stable eigenvalue $\mu^s(\lambda)$ is semisimple, that is the algebraic multiplicity equals the geometric multiplicity.

(iii) we choose α^s such that $2\alpha^s < Re(\mu^s(\lambda))$ for sufficiently small λ,

(iv) we choose $\nu \in \mathbb{N}$, $\nu \geq 2$ such that for all $k \in \{0, \ldots, \nu-1\} \setminus \{1\}$ $D_1^{(k)} f(0, \lambda) = 0$ and $D_1^{(\nu)} f(0, \lambda) \neq 0$.

With $E(\mu^s(\lambda))$ and $E(\sigma_{ss}(\lambda))$ we name the generalised eigenspaces of $D_1 f(0, \lambda)$ assigned to $\mu^s(\lambda)$ and $\sigma_{ss}(\lambda)$, respectively. Then $P_s(\lambda)$ denotes the projection on the leading stable eigenspace $E(\mu^s(\lambda))$ along the strong stable eigenspace $E(\sigma_{ss}(\lambda))$.

Since $p = 0$ is a hyperbolic equilibrium of the vector field f, that is $f(0, \lambda) = 0$ and $D_1 f(0, \lambda) \neq 0$, (A2.1)(iv) holds in general for $\nu = 2$. However, ν might be greater than two, if the derivatives of the vector field f at the equilibrium point $p = 0$ vanish from the second up to the $(\nu - 1)$th order.

In addition to (A2.1) we also need the following assumption.

(A2.2). *Let (A2.1) hold true. Additionally we assume*

(v) $\sigma_{ss}(\lambda) = \{\mu^{ss}(\lambda)\} \cup \sigma_{sss}(\lambda)$, *where* $Re(\mu) < \alpha^{sss} < Re(\mu^{ss}(\lambda)) < \alpha^{ss}$ *for all* $\mu \in \sigma_{sss}(\lambda)$.

(vi) for all λ the eigenvalue $\mu^{ss}(\lambda)$ is semisimple.

(vii) we can choose α^s such that $\nu\alpha^s < Re(\mu^{ss}(\lambda))$ for sufficiently small λ.

Let $E(\mu^{ss}(\lambda))$ be the eigenspaces assigned to $\mu^{ss}(\lambda)$ and let $P_{ss}(\lambda)$ be the projection on $E(\mu^{ss}(\lambda))$ along $E(\sigma^c_{\mu^{ss}}(\lambda)) = E(\mu^s(\lambda)) \oplus E(\sigma_{sss}(\lambda))$.

Unlike Assumption (A2.1)(iii) which always can be satisfied with an appropriate choice of α^s the Assumption (A2.2)(vii) means a restriction. The spectral gap between $\mu^s(\lambda)$ and $\mu^{ss}(\lambda)$ may not be too large so that the inequality $\nu\alpha^s < \mathrm{Re}(\mu^{ss}(\lambda))$ can be satisfied.

The following lemma is an extension of [San93, Lemma 1.7] where it was formulated for simple leading eigenvalues and for $\nu = 2$. The assertion under part a) also can be found in [HJKL11, Lemma 3.3] for $\nu = 2$.

Lemma 2.3.1. *a) Let the Assumption (A2.1) be satisfied. Then there is a $d > 0$ such that for all trajectories $x(\cdot)$ of $\dot{x} = f(x, \lambda)$ with $\|x(0)\| < d$ there exists the limit*

$$\eta^s(x(0), \lambda) = \lim_{t \to \infty} e^{-D_1 f(0,\lambda)t} P_s(\lambda) x(t) \in E(\mu^s(\lambda))$$

and a constant $c_s > 0$ such that

$$\left\| x(t) - e^{D_1 f(0,\lambda)t} \eta^s(x(0), \lambda) \right\| \leq c_s e^{\max\{\alpha^{ss}, \nu\alpha^s\}t}.$$

b) If in addition (A2.2) holds, then there is a $d > 0$ such that for all trajectories $x(\cdot)$ of $\dot{x} = f(x, \lambda)$ with $\|x(0)\| < d$ there also exists the limit

$$\eta^{ss}(x(0), \lambda) = \lim_{t\to\infty} e^{-D_1 f(0,\lambda)t} P_{ss}(\lambda)x(t) \in E(\mu^{ss}(\lambda)).$$

Furthermore there is a constant $c_{ss} > 0$ such that

$$\left\| x(t) - e^{D_1 f(0,\lambda)t}\eta^s(x(0), \lambda) - e^{D_1 f(0,\lambda)t}\eta^{ss}(x(0), \lambda) \right\| \leq c_{ss} e^{\max\{\alpha^{sss}, \nu\alpha^s\}t}.$$

Proof. The vector field allows a representation $f(x, \lambda) = D_1 f(0, \lambda) + g(x, \lambda)$, where $g(0, \lambda) = D_1 g(0, \lambda) = 0$. If $\nu > 2$ then we find in addition that $D_1^{(k)} g(0, \lambda) = 0$ for all $k < \nu$. For simplicity we write $A(\lambda) := D_1 f(0, \lambda)$. Then $x(\cdot)$ is a trajectory of $\dot{x} = f(x, \lambda)$ if and only if it solves for any $s < t$ the integral equation

$$x(t) = e^{A(\lambda)(t-s)} x(s) + \int_s^t e^{A(\lambda)(t-\tau)} g(x(\tau), \lambda) d\tau. \tag{2.39}$$

To prove the existence of the limits η^s and η^{ss} we show that $e^{-A(\lambda)t} P_s(\lambda)x(t)$ and $e^{-A(\lambda)t} P_{ss}(\lambda)x(t)$ are fundamental sequences. Since $A(\lambda)E(\mu^s(\lambda)) \subset E(\mu^s(\lambda))$ and $A(\lambda)E(\sigma_{ss}(\lambda)) \subset E(\sigma_{ss}(\lambda))$ we find that P_s and $e^{A(\lambda)t}$ commute. Hence we obtain from (2.39)

$$P_s(\lambda)x(t) = e^{A(\lambda)(t-s)} P_s(\lambda)x(s) + \int_s^t e^{A(\lambda)(t-\tau)} P_s(\lambda) g(x(\tau), \lambda) d\tau, \tag{2.40}$$

and therefore

$$\left\| e^{-A(\lambda)t} P_s(\lambda)x(t) - e^{-A(\lambda)s} P_s(\lambda)x(s) \right\| \leq \int_s^t \|e^{-A(\lambda)\tau} P_s(\lambda)\| \|g(x(\tau), \lambda)\| d\tau.$$

Due to the exponential dichotomy of $\dot{x} = A(\lambda)x$ and since $\mu^s(\lambda)$ is semisimple we find $\|e^{-A(\lambda)\tau} P_s(\lambda)\| \leq K_1 e^{-\operatorname{Re}(\mu(\lambda))\tau}$. Additionally we exploit (iv) in (A2.1) which leads to

$$\|g(x, \lambda)\| \leq c_g \|x\|^\nu, \quad \forall x : \|x\| < \varepsilon \tag{2.41}$$

for some positive constant c_g and a $\nu \geq 2$. Finally $\|x(\tau)\| \leq K_2 e^{\alpha^s \tau}$ and we obtain

$$\left\| e^{-A(\lambda)t} P_s(\lambda)x(t) - e^{-A(\lambda)s} P_s(\lambda)x(s) \right\| \leq K_1 K_2^\nu c_g \int_s^t e^{(\nu\alpha^s - \operatorname{Re}(\mu^s(\lambda)))\tau} d\tau,$$

with $\nu\alpha^s - \operatorname{Re}(\mu^s(\lambda)) < 2\alpha^s - \operatorname{Re}(\mu^s(\lambda)) < 0$ due to (A2.1)(iii). Hence, for all $\varepsilon > 0$ there is a t_0 such that for all $t > s > t_0$ the right-hand side of the last inequality is smaller than ε. This makes $e^{-A(\lambda)t} P_s(\lambda)x(t)$ a fundamental sequence.

Analogously to the projection P_s we also find that P_{ss} and $e^{A(\lambda)t}$ commute and again for any t, s with $t > s$ there is a constant $C > 0$ such that for $x(\cdot)$ with $\|x(0)\| < d$

$$\left\| e^{-A(\lambda)t} P_{ss}(\lambda)x(t) - e^{-A(\lambda)s} P_{ss}(\lambda)x(s) \right\| \leq \int_s^t \|e^{-A(\lambda)\tau} P_{ss}(\lambda)\| \|g(x(\tau), \lambda)\| d\tau \leq C \int_s^t e^{(\nu\alpha^s - \operatorname{Re}(\mu^{ss}(\lambda)))\tau} d\tau.$$

Also here we exploit that $\mu^{ss}(\lambda)$ is semisimple. Due to (A2.2)(vii) we find $\nu\alpha^s - \mathrm{Re}(\mu^{ss}(\lambda)) < 0$ and this shows that $e^{-A(\lambda)t}P_{ss}(\lambda)x(t)$ is a fundamental sequence as well.

Next we turn towards the estimates. For that we first decompose (2.39) by means of the projection P_s into

$$\begin{aligned} P_s(\lambda)x(t) &= e^{A(\lambda)(t-s)}P_s(\lambda)x(s) + \int_s^t e^{A(\lambda)(t-\tau)}P_s(\lambda)g(x(\tau),\lambda)d\tau, \\ (id - P_s(\lambda))x(t) &= e^{A(\lambda)(t-s)}(id - P_s(\lambda))x(s) + \int_s^t e^{A(\lambda)(t-\tau)}(id - P_s(\lambda))g(x(\tau),\lambda)d\tau. \end{aligned}$$

In the first equation the limit $s \to \infty$ does exist and we get

$$\begin{aligned} \|x(t) - e^{A(\lambda)t}\eta^s(x(0),\lambda)\| \leq\ & \|e^{A(\lambda)(t-s)}(id - P_s(\lambda))x(s)\| + \|\int_s^t e^{A(\lambda)(t-\tau)}(id - P_s(\lambda))g(x(\tau),\lambda)d\tau\| \\ & + \|\int_t^\infty e^{A(\lambda)(t-\tau)}P_s(\lambda)g(x(\tau),\lambda)d\tau\|. \end{aligned}$$

Without loss of generality we set $s = 0$. Now, the single terms on the right hand side can be estimated as follows:

$$\begin{aligned} \|e^{A(\lambda)t}(id - P_s(\lambda))x(0)\| &\leq K_1 e^{\alpha^{ss}t}\|x(0)\| &&= O(e^{\alpha^{ss}t}), \\ \|\int_0^t e^{A(\lambda)(t-\tau)}(id - P_s(\lambda))g(x(\tau),\lambda)d\tau\| &\leq K_1K_2^\nu c_g \int_0^t e^{\alpha^{ss}(t-\tau)}e^{\nu\alpha^s\tau}d\tau &&= O(e^{\nu\alpha^s t} + e^{\alpha^{ss}t}), \\ \|\int_t^\infty e^{A(\lambda)(t-\tau)}P_s(\lambda)g(x(\tau),\lambda)d\tau\| &\leq K_1K_2^\nu c_g \int_t^\infty e^{\,\mathrm{Re}(\mu^s(\lambda))(t-\tau)}e^{\nu\alpha^s\tau}d\tau &&= O(e^{\nu\alpha^s t}). \end{aligned}$$

This proves the estimate in a).

To obtain the estimate in b) we need to refine the decomposition of (2.39) by also using the projection P_{ss}. To this end recall that between the two projections the relations $P_{ss}P_s = 0$, $(id - P_{ss})P_s = P_s$ and $P_{ss}(id - P_s) = P_{ss}$ hold. Then we find

$$\begin{aligned} P_s(\lambda)x(t) &= e^{A(\lambda)(t-s)}P_s(\lambda)x(s) + \int_s^t e^{A(\lambda)(t-\tau)}P_s(\lambda)g(x(\tau),\lambda)d\tau \\ P_{ss}(\lambda)x(t) &= e^{A(\lambda)(t-s)}P_{ss}(\lambda)x(s) + \int_s^t e^{A(\lambda)(t-\tau)}P_{ss}(\lambda)g(x(\tau),\lambda)d\tau \\ (id - P_{ss}(\lambda))(id - P_s(\lambda))x(t) &= e^{A(\lambda)(t-s)}(id - P_s(\lambda))(id - P_{ss}(\lambda))x(s) \\ &\quad + \int_s^t e^{A(\lambda)(t-\tau)}(id - P_{ss}(\lambda))(id - P_s(\lambda))g(x(\tau),\lambda)d\tau. \end{aligned}$$

Again in the first and second equation the limits $s \to \infty$ do exist and we get

$$\begin{aligned} & \|x(t) - e^{A(\lambda)t}\eta^s(x(0),\lambda) - e^{A(\lambda)t}\eta^{ss}(x(0),\lambda)\| \\ & \leq \|e^{A(\lambda)t}(id - P_{ss}(\lambda))(id - P_s(\lambda))x(0)\| + \|\int_0^t e^{A(\lambda)(t-\tau)}(id - P_{ss}(\lambda))(id - P_s(\lambda))g(x(\tau),\lambda)d\tau\| \\ & \quad + \|\int_t^\infty e^{A(\lambda)(t-\tau)}P_{ss}(\lambda)g(x(\tau),\lambda)d\tau\| + \|\int_t^\infty e^{A(\lambda)(t-\tau)}P_s(\lambda)g(x(\tau),\lambda)d\tau\| \end{aligned}$$

by setting $s = 0$ in the third equation. We already estimated the last term with $O(e^{\nu\alpha^s t})$. The remaining terms on the right hand side can be estimated as follows:

$$\begin{aligned}
\|e^{A(\lambda)t}(id-P_{ss}(\lambda))(id-P_s(\lambda))x(0)\| &\leq K_1 e^{\alpha^{sss}t}\|x(0)\| \\
&\leq O(e^{\alpha^{sss}t}), \\
\|\int_0^t e^{A(\lambda)(t-\tau)}(id-P_{ss}(\lambda))(id-P_s(\lambda))g(x(\tau),\lambda)d\tau\| &\leq K_1K_2^{\nu}c_g\int_0^t e^{\alpha^{sss}(t-\tau)}e^{\nu\alpha^s\tau}d\tau \\
&\leq O(e^{\nu\alpha^s t}+e^{\alpha^{sss}t}), \\
\|\int_t^\infty e^{A(\lambda)(t-\tau)}P_{ss}(\lambda)g(x(\tau),\lambda)d\tau\| &\leq K_1K_2^{\nu}c_g\int_t^\infty e^{\mathrm{Re}(\mu^{ss}(\lambda))(t-\tau)}e^{\nu\alpha^s\tau}d\tau \\
&\leq O(e^{\nu\alpha^s t}),
\end{aligned}$$

This concludes the proof. □

Remark 2.3.2. *If the leading eigenvalues $\mu^s(\lambda)$ and $\mu^{ss}(\lambda)$ are real, we have*

$$\begin{aligned}
e^{D_1f(0,\lambda)t}\eta^s(x(0),\lambda) &= e^{\mu^s(\lambda)t}\eta^s(x(0),\lambda), \\
e^{D_1f(0,\lambda)t}\eta^{ss}(x(0),\lambda) &= e^{\mu^{ss}(\lambda)t}\eta^{ss}(x(0),\lambda).
\end{aligned}$$

Remark 2.3.3. *If $\sigma_{ss}(\lambda)=\emptyset$, that is $D_1f(0,\lambda)$ has no strong stable eigenvalues the constant α^{ss} can be choosen arbitrarily small and we simply find the estimate:*

$$\left\|x(t)-e^{D_1f(0,\lambda)t}\eta^s(x(0),\lambda)\right\|\leq c_s e^{\nu\alpha^s t}.$$

Lemma 2.3.1 does not say anything about η^s possibly becoming zero. In the following we want to go into this matter. To this end we need more information on the stable manifold $W^s(p)$ of an equilibrium p.

For $\sigma_{ss}\neq\emptyset$ one can prove the existence of a submanifold of $W^s(p)$ called the ***strong stable manifold*** $W^{ss}(p)$. This manifold distinguish itself in a stronger rate of convergence of the rest of $W^s(p)$, cf. [HiPuSh77, Shu86],

$$W^{ss}(p):=\{x\in\mathbb{R}^n \mid \|\varphi^t(x_0)-p\|\leq e^{\alpha^{ss}t}\|x_0\|\}\subset W^s(p). \tag{2.42}$$

Locally around p, that is in a neighbourhood $U(p,\varepsilon)$, $\varepsilon>0$ sufficiently small, one speaks of the ***local stable manifold***

$$W^s_{loc}(p):=W^s(p)\cap U(p,\varepsilon).$$

Analogously one defines the ***local strong stable manifold***

$$W^{ss}_{loc}(p):=W^{ss}(p)\cap U(p,\varepsilon).$$

Via a coordinate transformation one can embed the local manifolds into the generalized eigenspaces of the corresponding eigenvalues, that is $W^s_{loc}(p)\subset E_{Df(p)}(\sigma_s)$ and $W^{ss}_{loc}(p)\subset E_{Df(p)}(\sigma_{ss})$, see i.e. [Rob99, Shu86] and Section 4.2.

Corollary 2.3.4. *Assume (A2.1) and let $\eta^s(x(0),\lambda)$ be given as in Lemma 2.3.1. Then $\eta^s(x(0),\lambda)\neq 0$ if and only if $x(0)\notin W^{ss}(0)$.*

Proof. Recall that $\eta^s(x(0),\lambda)=\lim_{t\to\infty}e^{-A(\lambda)t}P_s(\lambda)x(t)$. With $x(0)\in W^{ss}(0)$ also $x(t)\in W^{ss}(0)$ for all $t>0$. Especially for t sufficiently large we find $x(t)\in W^{ss}_{loc}(0)$ which can be embedded into the strong stable eigenspace $E(\sigma_{ss}(\lambda))$. Therefore, for t sufficiently large, $P_s(\lambda)x(t) = 0$. Hence $x(0) \in W^{ss}(0)$ implies $\eta^s(x(0), \lambda) = 0$.

Now, let $x(0) \notin W^{ss}(0)$. Then we find for $s > 0$ sufficiently large that $x(s) \notin W^{ss}_{loc}(0) \subset E(\sigma_{ss}(\lambda))$ and hence $P_s(\lambda)x(s) \neq 0$. With (2.40) we find for $t > s$

$$e^{-A(\lambda)(t-s)}P_s(\lambda)x(t) = P_s(\lambda)x(s) + \int_s^t e^{-A(\lambda)(\tau-s)}P_s(\lambda)g(x(\tau), \lambda)d\tau. \tag{2.43}$$

In the following we will estimate the integral term. To this end we use the estimate for g, cf. (2.41), $\|e^{-A(\lambda)(\tau-s)}P_s(\lambda)\| \leq e^{-\mu^s(\lambda)(\tau-s)}\|P_s(\lambda)\|$ and $\|x(\tau)\| \leq e^{\alpha^s(\tau-s)}\|x(s)\|$, see Lemma 2.2.2. Thus we gain with $2\alpha^s - \mu^s(\lambda) < 0$

$$\left\| \int_s^t e^{-A(\lambda)(\tau-s)}P_s(\lambda)g(x(\tau), \lambda)d\tau \right\| \leq \int_s^t e^{(2\alpha^s - \mu^s(\lambda))(\tau-s)}\|P_s(\lambda)\|c_g\|x(s)\|^2 d\tau \leq \frac{c_g\|P_s(\lambda)\|\|x(s)\|^2}{|2\alpha^s - \mu^s(\lambda)|}.$$

For $d = \|x(0)\|$ sufficiently small, that is for $\|x(s)\|$ sufficiently small, $s > 0$, we then find that

$$\left\| \int_s^t e^{-A(\lambda)(\tau-s)}P_s(\lambda)g(x(\tau), \lambda)d\tau \right\| \leq \frac{c_g\|P_s(\lambda)\|\|x(s)\|^2}{|2\alpha^s - \mu^s(\lambda)|} < \|P_s(\lambda)x(s)\|.$$

Hence we conclude with (2.43) and the reverse triangle inequality

$$\|e^{-A(\lambda)(t-s)}P_s(\lambda)x(t)\| \geq \left| \|P_s(\lambda)x(s)\| - \| \int_s^t e^{-A(\lambda)(\tau-s)}P_s(\lambda)g(x(\tau), \lambda)d\tau\| \right|$$

that for d sufficiently small there is a $\hat{d} > 0$ independent of t such that

$$\|e^{-A(\lambda)(t-s)}P_s(\lambda)x(t)\| \geq \hat{d}.$$

Therefore $\eta^s(x(0), \lambda) = \lim_{t\to\infty} e^{-A(\lambda)t}P_s(\lambda)x(t) \neq 0$. □

Remark 2.3.5. *We assumed the equilibrium $p = 0$ to be asymptotically stable. If p is a hyperbolic equilibrium then Lemma 2.3.1 describes the behaviour of solutions in the stable manifold. One obtains a similar lemma for solutions in the unstable manifold by reversing time.*

2.4 Perturbed linear equations

The following lemma states a similar assertion to Lemma 2.3.1 for non-autonomous perturbed linear differential equations

$$\dot{x} = [A(\lambda) + B(t, \lambda)]x \tag{2.44}$$

with $A(\lambda), B(t, \lambda) \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$ for all $\lambda \in \mathbb{R}$, $t \in \mathbb{R}$. But unlike Lemma 2.3.1 we do not consider trajectories given on the whole semiaxis $\mathbb{R}^+$ but only on $[s, \infty)$ for some $s \geq 0$. Part a) of this lemma can be found in [HJKL11, Lemma 3.4], for $s = 0$. First we state the required preconditions.

(A2.3). *Consider a smooth family of linear nonautonomous differential equation* (2.44) *and assume that*

(i) $\sigma(A(\lambda)) = \{\mu^s(\lambda)\} \cup \sigma_u(\lambda) \cup \sigma_{ss}(\lambda)$, *where* $Re(\mu) < \alpha^{ss} < Re(\mu^s(\lambda)) < \alpha^s < 0 < Re(\hat{\mu})$ *for all*

$\mu \in \sigma_{ss}(\lambda)$ and for all $\hat{\mu} \in \sigma_u(\lambda)$.

(ii) the leading eigenvalue $\mu^s(\lambda)$ is for all λ semisimple.

(iii) there is a $\delta < 0$ such that $\|B(t,\lambda)\| < K_B e^{\delta t}$ and $\alpha^s + \delta <$ Re$(\mu^s(\lambda))$.

Let further $E(\mu^s(\lambda))$ and $E(\sigma^c_{\mu^s}(\lambda))$ be the generalised eigenspaces of $A(\lambda)$ assigned to $\mu^s(\lambda)$ and $\sigma^c_{\mu^s}(\lambda) = \sigma_{ss}(\lambda) \cup \sigma_u(\lambda)$, respectively, and let $P_s(\lambda)$ be the projection on $E(\mu^s(\lambda))$ along $E(\sigma^c_{\mu^s}(\lambda))$.

(A2.4). *Additionally to (A2.3) we assume that*

(iv) $\sigma_{ss}(\lambda) = \{\mu^{ss}(\lambda)\} \cup \sigma_{sss}(\lambda)$ where Re$(\mu) < \alpha^{sss} <$ Re$(\mu^{ss}(\lambda)) < \alpha^{ss}$ for all $\mu \in \sigma_{sss}(\lambda)$.

(v) the eigenvalue $\mu^{ss}(\lambda)$ is for all λ semisimple.

(vi) we can choose δ and α^s such that $\alpha^s + \delta <$ Re$(\mu^{ss}(\lambda))$ for sufficiently small λ.

Further $E(\mu^{ss}(\lambda))$ denotes the eigenspaces of $A(\lambda)$ assigned to $\mu^{ss}(\lambda)$ and $P_{ss}(\lambda)$ is the projection on $E(\mu^{ss}(\lambda))$ along $E(\sigma^c_{\mu^{ss}}(\lambda))$.

Analogously to the considerations in the forgoing section we find that Assumption (A2.3)(iii) can always be satisfied by an appropriate choice of α^s whereas Assumption (A2.4)(vi) means a restriction. Again the spectral gap between $\mu^s(\lambda)$ and $\mu^{ss}(\lambda)$ may not be too large so that the inequality $\alpha^s + \delta <$ Re$(\mu^{ss}(\lambda))$ can be satisfied.

Lemma 2.4.1. *a) Let the asumption (A2.3) be satisfied. Then there is a $d > 0$ such that for all trajectories $x(\cdot)$ of (2.44) with $\|x(0)\| < d$ and for all $s \in [0,t]$ with $x(s) = \xi$ and $\lim_{t\to\infty} x(t;s,\xi) = 0$ there exists the limit*

$$\eta^+(\xi,\lambda) = \lim_{t\to\infty} e^{-A(\lambda)(t-s)} P_s(\lambda) x(t;s,\xi) \in E(\mu^s(\lambda))$$

and a constant $c > 0$ such that

$$\left\| x(t;s,\xi) - e^{A(\lambda)(t-s)}\eta^+(\xi,\lambda) \right\| \leq c e^{\max\{\alpha^{ss}(t-s),(\alpha^s(t-s)+\delta t)\}}.$$

b) If in addition (A2.4) holds true, then there is a $d > 0$ such that for all trajectories $x(\cdot)$ of (2.44) with $\|x(0)\| < d$ and $\lim_{t\to\infty} x(t;s,\xi) = 0$ there also exists the limit

$$\eta^{++}(\xi,\lambda) = \lim_{t\to\infty} e^{-A(\lambda)(t-s)} P_{ss}(\lambda) x(t;s,\xi) \in E(\mu^{ss}(\lambda)).$$

Furthermore there is a constant $c > 0$ such that

$$\left\| x(t) - e^{A(\lambda)(t-s)}\eta^+(x(0),\lambda) - e^{A(\lambda)(t-s)}\eta^{++}(x(0),\lambda) \right\| \leq c e^{\max\{\alpha^{sss}(t-s),(\alpha^s(t-s)+\delta t)\}}.$$

Proof. $x(\cdot;s,\xi)$ is a solution of (2.44) with $x(s;s,\xi) = \xi$ if and only if it solves the integral equation

$$x(t;s,\xi) = e^{A(\lambda)(t-u)} x(u;s,\xi) + \int_u^t e^{A(\lambda)(t-\tau)} B(\tau,\lambda) x(\tau;s,\xi) d\tau$$

for any $u \in \mathbb{R}$. Now, let $P(\lambda)$ be the spectral projection of the exponential dichotomy of $\dot{x} = A(\lambda)x$ (on the whole line $\mathbb{R}$) with im$P(\lambda) = E(\sigma_s(\lambda))$ and ker$P(\lambda) = E(\sigma_u(\lambda))$. Then we find that $P_s(\lambda)P(\lambda) = P_s(\lambda)$

and we can decompose the solution $x(\cdot; s, \xi)$ by means of the projection $P(\lambda)$ into

$$\left.\begin{aligned} P(\lambda)x(t;s,\xi) &= e^{A(\lambda)(t-u)}P(\lambda)x(u;s,\xi) + \int_u^t e^{A(\lambda)(t-\tau)}P(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau, \\ (id - P(\lambda))x(t;s,\xi) &= -\int_t^\infty e^{A(\lambda)(t-\tau)}(id - P(\lambda))B(\tau,\lambda)x(\tau;s,\xi)d\tau. \end{aligned}\right\} \tag{2.45}$$

Here we get the second equation by letting u tend to infinity. This can be done since both $x(u; s, \xi)$ and $e^{-A(\lambda)u}(id - P(\lambda))$ tend to zero as u goes to infinity.

Analogously to the proof of Lemma 2.3.1 we show the existence of $\eta^+(\xi, \lambda)$ by verifying that

$$e^{-A(\lambda)(t-s)}P_s(\lambda)x(t; s, \xi)$$

is a fundamental sequence. From (2.45) we get with $P_s(\lambda)x(t; s, \xi) = P_s(\lambda)P(\lambda)x(t; s, \xi)$

$$\left\| e^{-A(\lambda)(t-s)}P_s(\lambda)x(t;s,\xi) - e^{-A(\lambda)(u-s)}P_s(\lambda)x(u;s,\xi) \right\| = \left\| \int_u^t e^{A(\lambda)(s-\tau)}P_s(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau \right\|$$

Since $\|B(\tau, \lambda)\| \le K_B e^{\delta\tau}$ we find, due to the roughness theorem 2.1.7, that equation (2.44) has an exponential dichotomy on $\mathbb{R}^+$ with exponential rate α^s and some constant $K_1 > 0$ such that $\|x(\tau; s, \xi)\| \le K_1 e^{\alpha^s(\tau-s)}$. This is because $x(\tau; s, \xi)$ tends to zero as $\tau \to \infty$ and is therefore situated within the stable subspace of (2.44). Finally we use $\|e^{A(\lambda)(s-\tau)}P_s(\lambda)\| \le K_2 e^{\mathrm{Re}(\mu^s(\lambda))(s-\tau)}$, since $\mu^s(\lambda)$ is semisimple. This leads to

$$\left\| \int_u^t e^{A(\lambda)(s-\tau)}P_s(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau \right\| \le K_1K_2K_B e^{(\ \mathrm{Re}(\mu^s(\lambda))-\alpha^s)s} \int_u^t e^{(\alpha^s+\delta-\ \mathrm{Re}(\mu^s(\lambda)))\tau}d\tau$$

with $e^{(\ \mathrm{Re}(\mu^s(\lambda))-\alpha^s)s} < 1$ and $\alpha^s+\delta-\ \mathrm{Re}(\mu^s(\lambda)) < 0$. Therefore, for all $\varepsilon > 0$, we find a t_0 such that for $t > u > t_0$ the right hand side of the previous equation is smaller than ε. That makes $e^{-A(\lambda)(t-s)}P_s(\lambda)x(t; s, \xi)$ a fundamental sequence.

Since $P_{ss}(\lambda)x(t; s, \xi) = P_{ss}(\lambda)P(\lambda)x(t; s, \xi)$ we obtain analogously

$$\begin{aligned} \left\| e^{-A(\lambda)(t-s)}P_{ss}(\lambda)x(t;s,\xi) - e^{-A(\lambda)(u-s)}P_{ss}(\lambda)x(u;s,\xi) \right\| &= \left\| \int_u^t e^{A(\lambda)(s-\tau)}P_{ss}(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau \right\| \\ &\le K_1K_2K_B e^{(\ \mathrm{Re}(\mu^{ss}(\lambda))-\alpha^s)s} \int_u^t e^{(\alpha^s+\delta-\ \mathrm{Re}(\mu^{ss}(\lambda)))\tau}d\tau \end{aligned}$$

with $e^{(\ \mathrm{Re}(\mu^{ss}(\lambda))-\alpha^s)s} < 1$ and $\alpha^s + \delta - \ \mathrm{Re}(\mu^{ss}(\lambda)) < 0$, due to the restrictive condition (A2.4) (vi). That concludes the proof that $e^{-A(\lambda)(t-s)}P_{ss}(\lambda)x(t; s, \xi)$ is a fundamental sequence, since for all $\varepsilon > 0$ there exists a t_0 such that for $t > u > t_0$ the right hand side of the last equation is smaller than ε.

It remains to prove the estimates. In accordance with (2.45) we decompose the solution $x(t; s, \xi)$ by means of the projection $P_s(\lambda)$ and obtain

$$\left.\begin{aligned} P_s(\lambda)x(t;s,\xi) &= e^{A(\lambda)(t-u)}P_s(\lambda)x(u;s,\xi) + \int\limits_u^t e^{A(\lambda)(t-\tau)}P_s(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau, \\ (id-P_s(\lambda))x(t;s,\xi) &= e^{A(\lambda)(t-u)}(id-P_s(\lambda))P(\lambda)x(u;s,\xi) \\ &\quad + \int\limits_u^t e^{A(\lambda)(t-\tau)}(id-P_s(\lambda))P(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau \\ &\quad - \int\limits_t^\infty e^{A(\lambda)(t-\tau)}(id-P(\lambda))B(\tau,\lambda)x(\tau;s,\xi)d\tau. \end{aligned}\right\} \tag{2.46}$$

In the first equation the limit $u \to \infty$ does exist and we get

$$P_s(\lambda)x(t;s,\xi) = e^{A(\lambda)(t-s)}\eta^+(\xi,\lambda) - \int\limits_t^\infty e^{A(\lambda)(t-\tau)}P_s(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau.$$

Therefore we find

$$\begin{aligned} &\left\|x(t;s,\xi) - e^{A(\lambda)(t-s)}\eta^+(\xi,\lambda)\right\| \\ &\leq \left\|e^{A(\lambda)(t-u)}(id-P_s(\lambda))P(\lambda)x(u;s,\xi)\right\| + \left\|\int\limits_u^t e^{A(\lambda)(t-\tau)}(id-P_s(\lambda))P(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau\right\| \\ &\quad + \left\|\int\limits_t^\infty e^{A(\lambda)(t-\tau)}(id-P(\lambda))B(\tau,\lambda)x(\tau;s,\xi)d\tau\right\| + \left\|\int\limits_t^\infty e^{A(\lambda)(t-\tau)}P_s(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau\right\|. \end{aligned}$$

Without loss of generality we set $u = s$. Then the single terms on the right hand side of this inequality can be estimated as follows

$$\begin{aligned} \left\|e^{A(\lambda)(t-s)}(id-P_s(\lambda))P(\lambda)x(s;s,\xi)\right\| &\leq K_2 e^{\alpha^{ss}(t-s)}\|\xi\| &&= O(e^{\alpha^{ss}(t-s)}), \\ \left\|\int\limits_s^t e^{A(\lambda)(t-\tau)}(id-P_s(\lambda))P(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau\right\| &\leq K_1K_2K_B \int\limits_s^t e^{\alpha^{ss}(t-\tau)}e^{\delta\tau}e^{\alpha^s(\tau-s)}d\tau &&= O(e^{\alpha^s(t-s)}e^{\delta t} \\ & && \quad + e^{\alpha^{ss}(t-s)}), \\ \left\|\int\limits_t^\infty e^{A(\lambda)(t-\tau)}(id-P(\lambda))B(\tau,\lambda)x(\tau;s,\xi)d\tau\right\| &\leq K_1K_2K_B \int\limits_t^\infty e^{\alpha^u(t-\tau)}e^{\delta\tau}e^{\alpha^s(\tau-s)}d\tau &&= O(e^{\alpha^s(t-s)}e^{\delta t}), \\ \left\|\int\limits_t^\infty e^{A(\lambda)(t-\tau)}P_s(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau\right\| &\leq K_1K_2K_B \int\limits_t^\infty e^{\alpha^s(t-\tau)}e^{\delta\tau}e^{\alpha^s(\tau-s)}d\tau &&= O(e^{\alpha^s(t-s)}e^{\delta t}). \end{aligned}$$

This gives the estimate in a).

To obtain the estimate in b) we additionally decompose (2.46) by means of the projection $P_{ss}(\lambda)$. To this end recall that $P_{ss}(\lambda)P_s(\lambda) = 0$, $(id - P_{ss}(\lambda))P_s(\lambda) = P_s(\lambda)$ and $P_{ss}(\lambda)(id - P_s(\lambda)) = P_{ss}(\lambda)$. Therefore we end up with

$$\begin{aligned} P_s(\lambda)x(t;s,\xi) &= e^{A(\lambda)(t-u)}P_s(\lambda)x(u;s,\xi) + \int\limits_u^t e^{A(\lambda)(t-\tau)}P_s(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau \\ P_{ss}(\lambda)x(t;s,\xi) &= e^{A(\lambda)(t-u)}P_{ss}(\lambda)x(u;s,\xi) + \int\limits_u^t e^{A(\lambda)(t-\tau)}P_{ss}(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau \end{aligned}$$

$$\begin{aligned}(id-P_{ss}(\lambda))(id-P_s(\lambda))x(t;s,\xi) &= e^{A(\lambda)(t-u)}(id-P_{ss}(\lambda))(id-P_s(\lambda))P(\lambda)x(u;s,\xi)\\ &\quad+\int_u^t e^{A(\lambda)(t-\tau)}(id-P_{ss}(\lambda))(id-P_s(\lambda))P(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau\\ &\quad-\int_t^\infty e^{A(\lambda)(t-\tau)}(id-P(\lambda))B(\tau,\lambda)x(\tau;s,\xi)d\tau.\end{aligned}$$

Since in the first and second equation the limits $u\to\infty$ do exist we get

$$\begin{aligned}&\|x(t)-e^{A(\lambda)(t-s)}\eta^+(\xi,\lambda)-e^{A(\lambda)(t-s)}\eta^{++}(\xi,\lambda)\|\\ \le\ &\|e^{A(\lambda)(t-u)}(id-P_{ss}(\lambda))(id-P_s(\lambda))P(\lambda)x(u;s,\xi)\|+\|\int_t^\infty e^{A(\lambda)(t-\tau)}(id-P(\lambda))B(\tau,\lambda)x(\tau;s,\xi)d\tau\|\\ &+\|\int_u^t e^{A(\lambda)(t-\tau)}(id-P_{ss}(\lambda))(id-P_s(\lambda))P(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau\|\\ &+\|\int_t^\infty e^{A(\lambda)(t-\tau)}P_s(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau\|+\|\int_t^\infty e^{A(\lambda)(t-\tau)}P_{ss}(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau\|\end{aligned}$$

We already estimated the second and the fourth term with $O(e^{\alpha^s(t-s)+\delta t})$ each. By setting $u=s$ the remaining terms can be estimated as follows:

$$\begin{aligned}\|e^{A(\lambda)(t-s)}(id-P_{ss}(\lambda))(id-P_s(\lambda))P(\lambda)x(s;s,\xi)\| &\le K_2e^{\alpha^{sss}(t-s)}\|\xi\|\\ &=O(e^{\alpha^{sss}(t-s)}),\\ \|\int_s^t e^{A(\lambda)(t-\tau)}(id-P_{ss}(\lambda))(id-P_s(\lambda))P(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau\| &\le K_1K_2K_B\int_0^t e^{\alpha^{sss}(t-\tau)}e^{\delta\tau}e^{\alpha^s(\tau-s)}d\tau\\ &=O(e^{\alpha^s(t-s)+\delta t}+e^{\alpha^{sss}t}),\\ \|\int_t^\infty e^{A(\lambda)(t-\tau)}P_{ss}(\lambda)B(\tau,\lambda)x(\tau;s,\xi)d\tau\| &\le K_1K_2K_B\int_t^\infty e^{\mu^{ss}(\lambda)(t-\tau)}e^{\delta\tau}e^{\alpha^s(\tau-s)}d\tau\\ &=O(e^{\alpha^s(t-s)+\delta t}).\end{aligned}$$

This concludes the proof. □

Remark 2.4.2. *If the leading eigenvalues $\mu^s(\lambda)$ and $\mu^{ss}(\lambda)$ of $A(\lambda)$ are real, we have*

$$\begin{aligned}e^{A(\lambda)t}\eta^+(x(0),\lambda) &= e^{\mu^s(\lambda)t}\eta^+(x(0),\lambda),\\ e^{A(\lambda)t}\eta^{++}(x(0),\lambda) &= e^{\mu^{ss}(\lambda)t}\eta^{++}(x(0),\lambda).\end{aligned}$$

Remark 2.4.3. *Analogously to Remark 2.3.3 we can choose α^{ss} arbitrarily small, if $A(\lambda)$ has no strong stable eigenvalues and we obtain the estimate:*

$$\left\|x(t;s,\xi)-e^{A(\lambda)(t-s)}\eta^+(\xi,\lambda)\right\|\le ce^{\alpha^s(t-s)+\delta t}.$$

Analogously to the considerations concerning η^s in the forgoing section we now determine when η^+ can become zero.

Corollary 2.4.4. *Assume (A2.3) and let $\eta^+(\xi,\lambda)$ be given as in Lemma 2.4.1. Then $\eta^+(\xi,\lambda)\neq 0$ if and only if $\xi\notin imP^+_{ss}(s)$, where $P^+_{ss}(\cdot)$ denotes the projection of the exponential dichotomy of (2.44) that projects onto the strong stable subspace $E^{ss}_{A+B(\cdot)}(s)$, also cf. Lemma 2.2.7.*

Proof. Recall that $\eta^+(\xi,\lambda)=\lim_{t\to\infty}e^{-A(\lambda)(t-s)}P_s(\lambda)x(t;s,\xi)$. With $\xi\in \mathrm{im}P^+_{ss}(s)$ we find $x(t;s,\xi)\in$

$\mathrm{im}P_{ss}^+(t)$ for all $t > s$. For t sufficiently large we find $\mathrm{im}P_{ss}^+(t) = E_A(\sigma_{ss}(\lambda))$ and therefore $P_s(\lambda)x(t; s, \xi) = 0$. Hence $\xi \in \mathrm{im}P_{ss}^+(s)$ implies $\eta^+(\xi, \lambda) = 0$.

Now, let $\xi \notin \mathrm{im}P_{ss}^+(s)$. Then for s sufficiently large we have $P_s(\lambda)\xi \neq 0$ and with (2.45) we find

$$e^{-A(\lambda)(t-s)}P_s(\lambda)x(t) = P_s(\lambda)\xi + \int_s^t e^{-A(\lambda)(\tau-s)}P_s(\lambda)B(\tau, \lambda)x(\tau)d\tau. \tag{2.47}$$

In the following we will estimate the integral term. To this end we use $\|B(\tau, \lambda)\| < K_B e^{\delta\tau}$, $\|x(\tau)\| \leq e^{\alpha^s(\tau-s)}\|\xi\|$ and $\|e^{-A(\lambda)(\tau-s)}P_s(\lambda)\| \leq e^{-\mu^s(\lambda)(\tau-s)}\|P_s(\lambda)\|$. Thus we gain with $\alpha^s + \delta - \mu^s(\lambda) < 0$

$$\begin{aligned}
\left\|\int_s^t e^{-A(\lambda)(\tau-s)}P_s(\lambda)B(\tau, \lambda)x(\tau)d\tau\right\| &\leq e^{(\mu^s(\lambda)-\alpha^s)s}\int_s^t e^{(\alpha^s+\delta-\mu^s(\lambda))\tau}\|P_s(\lambda)\|K_B\|\xi\|\, d\tau \\
&\leq \frac{K_B\|P_s(\lambda)\|\|\xi\|}{|\alpha^s + \delta - \mu^s(\lambda)|}\left(e^{\delta s} - e^{(\alpha^s+\delta-\mu^s(\lambda))t}e^{(\mu^s(\lambda)-\alpha^s)s}\right) \\
&\leq e^{\delta s}\frac{K_B\|P_s(\lambda)\|\|\xi\|}{|\alpha^s + \delta - \mu^s(\lambda)|}.
\end{aligned}$$

For $s < t$ sufficiently large we then find that

$$\left\|\int_s^t e^{-A(\lambda)(\tau-s)}P_s(\lambda)B(\tau, \lambda)x(\tau)d\tau\right\| \leq e^{\delta s}\frac{K_B\|P_s(\lambda)\|\|\xi\|}{|\alpha^s + \delta - \mu^s(\lambda)|} < \|P_s(\lambda)\xi\|$$

and with (2.47) follows from the reverse triangle inequality

$$\|e^{-A(\lambda)(t-s)}P_s(\lambda)x(t)\| \geq \left|\|P_s(\lambda)\xi\| - \|\int_s^t e^{-A(\lambda)(\tau-s)}P_s(\lambda)B(\tau, \lambda)x(\tau)d\tau\|\right| > 0.$$

Therefore for s sufficiently large there is a $\hat{d} > 0$ independent of t such that

$$\|e^{-A(\lambda)(t-s)}P_s(\lambda)x(t)\| \geq \hat{d}.$$

Hence $\eta^+(\xi, \lambda) = \lim_{t\to\infty} e^{-A(\lambda)(t-s)}P_s(\lambda)x(t; s, \xi) \neq 0$. □

In what follows we give a similar lemma to Lemma 2.4.1 for linear non-autonomous systems (2.44) where $A(\lambda)$ is non-hyperbolic. Then $x = 0$ might not be the only equilibrium of $\dot{x} = A(\lambda)x$. Here we are interested in the rate of convergence of solutions $x(\cdot)$ of (2.44) that tend to an element $p \in \ker A(\lambda)$ as $t \to \infty$. Again we start with stating the requirements.

(A2.5). *Consider a smooth family of linear non-autonomous differential equation* (2.44) *and assume that*

(i) $\sigma(A(\lambda)) = \sigma_s(\lambda) \cup \sigma_c(\lambda) \cup \sigma_u(\lambda)$, *where* $Re(\mu) < \alpha^s < -\alpha^c < 0 < \alpha^c < \alpha^u < Re(\hat{\mu})$ *for all* $\mu \in \sigma_s(\lambda)$ *and for all* $\hat{\mu} \in \sigma_u(\lambda)$.

(ii) *there is a* $\delta < 0$ *such that* $\|B(t, \lambda)\| < K_B e^{\delta t}$,

(iii) *there is a* $p \in \ker A(\lambda)$ *and*

(iv) $x(\cdot)$ *is a solution of* (2.44) *that converges exponentially to* p *as* $t \to \infty$, *that is, there exist constants* $\vartheta > 0$ *and* $M > 0$ *such that* $\|x(t) - p\| \leq Me^{-\vartheta t}$, $t > 0$.

(A2.6). *Additionally to (A2.5) we assume that*

(v) $\sigma_s(\lambda) = \{\mu^s(\lambda)\} \cup \sigma_{ss}(\lambda)$ *where* $Re(\mu) < \alpha^{ss} < Re(\mu^s(\lambda)) < \alpha^s$ *for all* $\mu \in \sigma_{ss}(\lambda)$.

(vi) the eigenvalue $\mu^s(\lambda)$ *is for all* λ *semisimple.*

(vii) the constant δ *satisfies* $\delta < Re(\mu^s(\lambda))$ *for sufficiently small* λ.

Further $E(\mu^s(\lambda))$ *denotes the eigenspaces of* $A(\lambda)$ *assigned to* $\mu^s(\lambda)$ *and* $Q_s(\lambda)$ *is the projection on* $E(\mu^s(\lambda))$ *along* $E(\sigma^c_{\mu^s}(\lambda))$.

Obviously condition (vii), $\delta < \mathrm{Re}(\mu^s(\lambda))$, is again a restriction.

Lemma 2.4.5. *a) Let the Assumption (A2.5) be satisfied. Then there exists a constant* $C > 0$ *such that*

$$\|x(t) - p\| \leq Ce^{\max\{\alpha^s,\delta\}t}.$$

b) If in addition Assumption (A2.6) holds true then there exists the limit

$$\eta(x(0),\lambda) = \lim_{t\to\infty} e^{-A(\lambda)t} Q_s(\lambda)[x(t) - p] \quad \in E(\mu^s(\lambda)).$$

Furthermore there is a constant $c > 0$ *such that*

$$\left\| x(t) - p - e^{A(\lambda)t}\eta(x(0),\lambda) \right\| \leq ce^{\max\{\alpha^{ss},\delta\}t}.$$

Proof. We define $y(t) := x(t) - p$. Then y is a solution of the inhomogeneous differential equation

$$\dot{x} = [A(\lambda) + B(t,\lambda)]x + B(t,\lambda)p \tag{2.48}$$

that converges towards zero as $t \to \infty$. The variation of constants formula yields

$$y(t) = e^{A(\lambda)(t-s)}y(s) + \int_s^t e^{A(\lambda)(t-\tau)}B(\tau,\lambda)[y(\tau) + p]d\tau.$$

Now, let $P_s(\lambda), P_c(\lambda)$ and $P_u(\lambda)$ be projections of the exponential trichotomy of $\dot{x} = A(\lambda)x$ with $\mathrm{im}P_s(\lambda) = E(\sigma_s(\lambda))$, $\mathrm{im}P_c(\lambda) = E(\sigma_c(\lambda))$ and $\mathrm{im}P_u(\lambda) = E(\sigma_u(\lambda))$. Then we can decompose the solution $y(t) = P_s(\lambda)y(t) + P_c(\lambda)y(t) + P_u(\lambda)y(t)$ and obtain

$$\begin{aligned}
P_s(\lambda)y(t) &= e^{A(\lambda)(t-s)}P_s(\lambda)y(s) + \int_s^t e^{A(\lambda)(t-\tau)}P_s(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau \\
P_c(\lambda)y(t) &= e^{A(\lambda)(t-s)}P_c(\lambda)y(s) + \int_s^t e^{A(\lambda)(t-\tau)}P_c(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau \\
P_u(\lambda)y(t) &= e^{A(\lambda)(t-s)}P_u(\lambda)y(s) + \int_s^t e^{A(\lambda)(t-\tau)}P_u(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau
\end{aligned}$$

Looking closely at the second and third equation we see that the terms on the right-hand side converge for $s \to \infty$. This can be seen as follows. Due to the estimates of the exponential trichotomy, cf. Definition 2.1.19, and $\|y(t)\| \leq Me^{-\vartheta t}$, $\vartheta > 0$, we obtain

$$\|e^{A(\lambda)(t-s)}P_c(\lambda)y(s)\| \leq KMe^{-\alpha^c(t-s)}e^{-\vartheta s} \leq KMe^{-\alpha^c t}e^{(-\vartheta+\alpha^c)s}, \quad s \geq t,$$

$$\|e^{A(\lambda)(t-s)}P_u(\lambda)y(s)\| \leq KMe^{\alpha^u(t-s)}e^{-\vartheta s} \leq KMe^{\alpha^u t}e^{(-\vartheta-\alpha^u)s}, \quad s \geq t.$$

Further, since $\|y(t)+p\| < K_p$ is bounded and $\|B(t,\lambda)\| \leq K_B e^{\delta t}$, $\delta < 0$ we see with $C = KK_BK_p$

$$\|\int_s^t e^{A(\lambda)(t-\tau)}P_c(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau\| \leq C\int_s^t e^{-\alpha^c(t-\tau)}e^{\delta\tau}d\tau \leq Ce^{-\alpha^c t}\int_s^t e^{(\delta+\alpha^c)\tau}d\tau, \quad s\geq t,$$
$$\|\int_s^t e^{A(\lambda)(t-\tau)}P_u(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau\| \leq C\int_s^t e^{\alpha^u(t-\tau)}e^{\delta\tau}d\tau \leq Ce^{\alpha^u t}\int_s^t e^{(\delta-\alpha^u)\tau}d\tau, \quad s\geq t.$$

Since $\alpha^u > 0$ we have $\delta - \alpha^u < 0$ and $-\vartheta - \alpha^u < 0$. With an appropriate choice of $\alpha^c > 0$ close to zero we also have $-\vartheta + \alpha^c < 0$ and $\delta + \alpha^c < 0$. Hence we find that both $e^{A(\lambda)(t-s)}P_c(\lambda)y(s)$ and $e^{A(\lambda)(t-s)}P_u(\lambda)y(s)$ converge to zero as $s\to\infty$. Additionally also the integral terms $\int_s^t e^{A(\lambda)(t-\tau)}P_{c/u}(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau$ converge for $s\to\infty$. This finally yields by letting s tend to infinity:

$$\left.\begin{array}{rcl} P_s(\lambda)y(t) &=& e^{A(\lambda)(t-s)}P_s(\lambda)y(s) + \int_s^t e^{A(\lambda)(t-\tau)}P_s(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau \\ P_c(\lambda)y(t) &=& -\int_t^\infty e^{A(\lambda)(t-\tau)}P_c(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau \\ P_u(\lambda)y(t) &=& -\int_t^\infty e^{A(\lambda)(t-\tau)}P_u(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau \end{array}\right\} \tag{2.49}$$

So to prove a) it remains to estimate the four terms on the right-hand side. In case of the first term we set $s = 0$.

$$\begin{array}{rll} \|e^{A(\lambda)t}P_s(\lambda)y(0)\| & \leq Ke^{\alpha^s t}\|y(0)\| & = O(e^{\alpha^s t}), \\ \|\int_0^t e^{A(\lambda)(t-\tau)}P_s(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau\| & \leq KK_pK_B\int_0^t e^{\alpha^s(t-\tau)}e^{\delta\tau}d\tau & = O(e^{\alpha^s t}+e^{\delta t}), \\ \|\int_t^\infty e^{A(\lambda)(t-\tau)}P_c(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau\| & \leq KK_pK_B\int_t^\infty e^{-\alpha^c(t-\tau)}e^{\delta\tau}d\tau & = O(e^{\delta t}), \\ \|\int_t^\infty e^{A(\lambda)(t-\tau)}P_u(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau\| & \leq KK_pK_B\int_t^\infty e^{\alpha^u(t-\tau)}e^{\delta\tau}d\tau & = O(e^{\delta t}). \end{array}$$

In order to show b) we again prove the existence of $\eta(x(0),\lambda)$ by verifying that $e^{-A(\lambda)t}Q_s(\lambda)y(t)$ is a fundamental sequence. From (2.49) we obtain with $Q_s(\lambda)y(t) = Q_s(\lambda)P_s(\lambda)y(t)$ that

$$\left\|e^{-A(\lambda)t}Q_s(\lambda)y(t) - e^{-A(\lambda)s}Q_s(\lambda)y(s)\right\| \leq \int_s^t \|e^{-A(\lambda)\tau}Q_s(\lambda)\|\|B(\tau,\lambda)\|\|y(\tau)+p\|d\tau \leq C\int_s^t e^{(\delta-\mu^s(\lambda))\tau}d\tau.$$

This combined with (A2.6)(vii) shows that $\lim_{t\to\infty} e^{-A(\lambda)t}Q_s(\lambda)y(t)$ indeed exists.

Next we turn towards the estimates. To this end we decompose $y(t)$ by means of the projection $Q_s(\lambda)$ and obtain from (2.49)

$$\begin{array}{rcl} Q_s(\lambda)y(t) &=& e^{A(\lambda)(t-s)}Q_s(\lambda)y(s) + \int_s^t e^{A(\lambda)(t-\tau)}Q_s(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau \\ (id-Q_s(\lambda))y(t) &=& e^{A(\lambda)(t-s)}(id-Q_s(\lambda))P_s(\lambda)y(s) \\ && +\int_s^t e^{A(\lambda)(t-\tau)}(id-Q_s(\lambda))P_s(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau \\ && -\int_t^\infty e^{A(\lambda)(t-\tau)}(id-Q_s(\lambda))[P_c(\lambda)+P_u(\lambda)]B(\tau,\lambda)[y(\tau)+p]d\tau \end{array}$$

In the first equation the limit $s \to \infty$ does exist and with $s = 0$ in the second equation we get

$$\begin{aligned}\|y(t) - e^{A(\lambda)t}\eta(x(0),\lambda)\| \quad \leq \quad & \|e^{A(\lambda)t}(id - Q_s(\lambda))P_s(\lambda)y(0)\| \\ & + \|\int_t^\infty e^{A(\lambda)(t-\tau)}Q_s(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau\| \\ & + \|\int_0^t e^{A(\lambda)(t-\tau)}(id - Q_s(\lambda))P_s(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau\| \\ & + \|\int_t^\infty e^{A(\lambda)(t-\tau)}(id - Q_s(\lambda))[P_c(\lambda)+P_u(\lambda)]B(\tau,\lambda)[y(\tau)+p]d\tau\|\end{aligned}$$

The single terms on the right hand side of the inequality can be estimated as follows:

$$\begin{aligned}\|e^{A(\lambda)t}(id - Q_s(\lambda))P_s(\lambda)y(0)\| &\leq Ke^{\alpha^{ss}t}\|y(0)\| &= O(e^{\alpha^{ss}t}), \\ \|\int_t^\infty e^{A(\lambda)(t-\tau)}Q_s(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau\| &\leq KK_pK_B\int_t^\infty e^{\mu^s(\lambda)(t-\tau)}e^{\delta\tau}d\tau &= O(e^{\delta t}), \\ \|\int_0^t e^{A(\lambda)(t-\tau)}(id - Q_s(\lambda))P_s(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau\| &\leq KK_pK_B\int_0^t e^{\alpha^{ss}(t-\tau)}e^{\delta\tau}d\tau &= O(e^{\alpha^{ss}t}+e^{\delta t}), \\ \|\int_t^\infty e^{A(\lambda)(t-\tau)}(id - Q_s(\lambda))P_c(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau\| &\leq KK_pK_B\int_t^\infty e^{-\alpha^{c}(t-\tau)}e^{\delta\tau}d\tau &= O(e^{\delta t}), \\ \|\int_t^\infty e^{A(\lambda)(t-\tau)}(id - Q_s(\lambda))P_u(\lambda)B(\tau,\lambda)[y(\tau)+p]d\tau\| &\leq KK_pK_B\int_t^\infty e^{\alpha^{u}(t-\tau)}e^{\delta\tau}d\tau &= O(e^{\delta t}).\end{aligned}$$

This concludes the proof. □

Remark 2.4.6. *If we assume that all eigenvalues of σ_c are semisimple we find that $e^{A(\lambda)(t-s)}P_c(\lambda)$ is bounded. In that case we do not need to assume, that $x(t)$ converges* exponentially *to p as $t \to \infty$. It suffices to simply assume convergence. To be precise, Assumption (A2.5)(iv) can be replaced by*

(iv) 1) all eigenvalues of $\sigma_c(\lambda)$ are semisimple,

(iv) 2) $x(\cdot)$ is a solution of (2.44) converging to p as $t \to \infty$.

In order to support the understanding of the forgoing lemma we give the following example.

Example 2.4.7. *Consider the linear equation*

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & -a \end{bmatrix} x + e^{-t}x$$

with $a \in \mathbb{R}^+$. Then we find the two decoupled one-dimensional equations $\dot{x}_1 = e^{-t}x_1$ and $\dot{x}_2 = (-a + e^{-t})x_2$ which can be solved via separation of variables. This leads to

$$x_1(t) = c_1 e^{-e^{-t}} \qquad \text{and} \qquad x_2(t) = c_2 e^{-at}e^{-e^{-t}},$$

$c_1, c_2 \in \mathbb{R}$. Now we find $\lim\limits_{t\to\infty}(x_1(t), x_2(t))^T = (c_1, 0)^T \in kerA$. The rate of convergence of x_2 is given by $-a$. The rule of l'Hospital shows, that the rate of convergence of x_1 is given by $\delta = -1$:

$$\lim_{t\to\infty}\frac{1 - e^{-e^{-t}}}{e^{-t}} = \lim_{t\to\infty}\frac{-e^{-t}e^{-e^{-t}}}{-e^{-t}} = \lim_{t\to\infty} e^{-e^{-t}} = 1.$$

Now, if $x(\cdot)$ is a solution that converges to $(0,0)^T$ then $c_1 = 0$ and $x(\cdot)$ has the rate of convergence $-a$. In *case that* $x(\cdot)$ *is converging to any other element in* $kerA$ *it has the rate of convergence of* $\max\{-1,-a\}$ *which might be worse, if* $-a < -1$.

2.5 Application on projections

According to (2.2) we find for projections corresponding to exponential dichotomies on $\mathbb{R}^+$ with rate $\alpha < 0$ that $\lim_{t\to\infty} \Phi(t,s)P^+(s) = 0$. That is the term $\Phi(\cdot,s)P^+(s)$ which is solution of the matrix differential equation $\dot{X} = A(t)X$ tends towards the equilibrium point $X = 0 \in \mathbb{R}^{n\times n}$. So, just as we have investigated the question of leading exponential rates during the forgoing section we now determine the leading convergence rates of $\Phi(\cdot,s)P^+(s)$.

Moreover, we find that the projections $P^+(\cdot)$ corresponding to the exponential dichotomy of a perturbed linear equation $\dot{x} = [A+B(t)]x$ on $\mathbb{R}^+$ with rate $\alpha < 0$ converge towards the spectral projection P of $\dot{x} = Ax$, as t tends to infinity, cf. Lemma 2.1.14. So it also seems possible to provide exact rates of convergence and leading terms of the projections $P^+(t)$ themselves.

2.5.1 Behaviour of $\Phi(\lambda)(t,s)P^{\pm}(\lambda,s)$ and $\Phi(\lambda)(s,t)(id - P^{\pm}(\lambda,t))$

To begin with, consider the differential equation (2.44) and let $\Phi(\lambda)(\cdot,\cdot)$ denote the corresponding transition matrix. We start with an assertion based on Lemma 2.4.1 that provides a representation of the compositions of the transition matrix $\Phi(\lambda)(\cdot,\cdot)$ and the projection of the exponential dichotomy. To this end we itemize the following assumptions:

(A2.7). *Consider a smooth family of linear nonautonomous differential equation* (2.44) *and assume that*

(i) $\sigma(A(\lambda)) = \sigma_s(\lambda) \cup \sigma_u(\lambda)$, *with* $Re(\mu) < \alpha^s < 0 < \alpha^u < Re(\hat{\mu})$ *for all* $\mu \in \sigma_s(\lambda)$ *and for all* $\hat{\mu} \in \sigma_u(\lambda)$,

(ii) there is a $\delta < 0$ *such that* $\|B(t,\lambda)\| < K_B e^{\delta t}$.

(A2.8). *Let (A2.7) hold true. Additionally assume* $\sigma_s(\lambda) = \{\mu^s(\lambda)\} \cup \sigma_{ss}(\lambda)$, *where* $\mu^s(\lambda)$ *is for all* λ *semisimple and* $Re(\mu) < \alpha^{ss} < Re(\mu^s(\lambda)) < \alpha^s$ *for all* $\mu \in \sigma_{ss}(\lambda)$. *Further let* α^s *be chosen such that* $\alpha^s + \delta < \mu^s(\lambda)$.

(A2.9). *Let (A2.7) hold true. Additionally assume* $\sigma_u(\lambda) = \{\mu^u(\lambda)\} \cup \sigma_{uu}(\lambda)$, *where* $\mu^u(\lambda)$ *is for all* λ *semisimple and* $\alpha^u < Re(\mu^u(\lambda)) < \alpha^{uu} < Re(\mu)$ *for all* $\mu \in \sigma_{uu}(\lambda)$. *Further let* α^u *be chosen such that* $-\alpha^u + \delta < -\mu^u(\lambda)$.

Remark 2.5.1. *With Assumption (A2.7) to hold true equation* (2.44) *has, due to Lemma 2.1.7, an exponential dichotomy* (2.2) *on* $\mathbb{R}^+$ *with projection* $P^+(\lambda,\cdot)$ *and exponential rates* α^s *and* α^u.
Moreover, with (A2.8) and $\sigma_{ss}(\lambda) \neq \emptyset$, *equation* (2.44) *even has an exponential trichotomy on* $\mathbb{R}^+$ *with projections* $P^+_{ss}(\lambda,\cdot)$, $P^+_s(\lambda,\cdot)$ *and* $P^+_{u,uu}(\lambda,\cdot)$, *cf. Lemma 2.2.7(i), where* $P^+_{ss}(t)$ *defines the strong stable subspace at time* t *and* $imP^+(t) = im(P^+_s(t) + P^+_{ss}(t))$, $imP^+_{u,uu} = im(id - P^+)$.
If (A2.9) is satisfied with $\sigma_{uu}(\lambda) \neq \emptyset$ *then the adjoint equation of* (2.44),

$$\dot{x} = -[A(\lambda) + B(t,\lambda)]^T x, \tag{2.50}$$

has an exponential trichotomy on $\mathbb{R}^+$, *cf. Lemma 2.2.7(iii), with projections* $Q^+_{ss}(\lambda,\cdot)$, $Q^+_s(\lambda,\cdot)$ *and* $Q^+_{u,uu}(\lambda,\cdot)$.

Lemma 2.5.2. *Assume (A2.7) and let $P^+(\lambda,\cdot)$ denote the projection of the exponential dichotomy of (2.44) on $\mathbb{R}^+$, as declared in Remark 2.5.1.*

a) Let (A2.8) hold true. Then there exists a linear time-depending operator $S^+(\lambda,s):\mathbb{R}^n\to\mathbb{R}^n$ with

$$im S^+(\lambda,s)=E_A(\mu^s(\lambda)),\qquad ker S^+(\lambda,s)=ker P^+(\lambda,s)\oplus im P^+_{ss}(\lambda,s),$$

such that

$$\Phi(\lambda)(t,s)P^+(\lambda,s)=e^{A(\lambda)(t-s)}S^+(\lambda,s)+O(e^{\alpha^{ss}(t-s)}+e^{\alpha^s(t-s)}e^{\delta t}),\quad t\geq s\geq 0.$$

b) If (A2.9) holds true, then there exists a linear time-depending operators $R^+(\lambda,s):\mathbb{R}^n\to\mathbb{R}^n$ with

$$ker R^+(\lambda,s)=E_A(\sigma^c_{\mu^u}(\lambda)),\quad im R^+(\lambda,s)=ker P^+(\lambda,s)\cap[im Q^+_{ss}(\lambda,s)]^\perp,$$

such that

$$\Phi(\lambda)(s,t)(id-P^+(\lambda,t))=e^{-A(\lambda)^T(t-s)}R^+(\lambda,s)+O(e^{-\alpha^{uu}(t-s)}+e^{-\alpha^u(t-s)}e^{\delta t}),\quad t\geq s\geq 0.$$

Proof. We start with the proof of a), which is a direct consequence of Lemma 2.4.1.

With (A2.7) and (A2.8) the Assumption (A2.3) is satisfied. The mapping $\Phi(\lambda)(\cdot,s)P^+(\lambda,s)\xi$, $\xi\in\mathbb{R}^n$, satisfies the differential equation (2.44) and due to the exponential dichotomy (2.2) we find that $\lim_{t\to\infty}\Phi(\lambda)(t,s)P^+(\lambda,s)\xi=0$ for all $\xi\in\mathbb{R}^n$. Thus we can apply Lemma 2.4.1 which leads to the pointwise definition of S^+

$$S^+(\lambda,s)\xi:=\lim_{t\to\infty}e^{-A(\lambda)(t-s)}P_s(\lambda)\Phi(\lambda)(t,s)P^+(\lambda,s)\xi=\eta^+(P^+(\lambda,s)\xi,\lambda),$$

and to the estimate

$$\left\|\Phi(\lambda)(t,s)P^+(\lambda,s)\xi-e^{A(\lambda)(t-s)}S^+(\lambda,s)\xi\right\|\leq c(e^{\alpha^{ss}(t-s)}+e^{\alpha^s(t-s)}e^{\delta t}),$$

for any $c>0$. Here $P_s(\lambda)$ denotes the spectral projection onto the leading stable eigenspace $E_A(\mu^s(\lambda))$ of $A(\lambda)$ along $E_A(\sigma^c_{\mu^s}(\lambda))$. Hence we easily see that $\mathrm{im}S^+(\lambda,s)\subseteq E_A(\mu^s(\lambda))$ and $\ker S^+(\lambda,s)\supseteq\ker P^+(\lambda,s)$.

With Corollary 2.4.4 we see that ξ is an element of the kernel of $S^+(\lambda,s)$ if and only if ξ lies in the kernel of $P^+(\lambda,s)$ or within the image of $P^+_{ss}(\lambda,s)$. This implies that the dimension of $\ker S^+(\lambda,s)$ is equal to $n-n_s$ and hence the dimension formular yields $\mathrm{im}S^+(\lambda,s)=E_A(\mu^s(\lambda))$.

The proof of b) follows along the same line. Only, instead of $\Phi(\lambda)(s,\cdot)(id-P^+(\lambda,\cdot))$ we consider its transpose $\Phi(\lambda)(s,\cdot)^T(id-P^+(\lambda,s))^T=\Psi(\lambda)(\cdot,s)(id-P^+(\lambda,s))^T$. Then $\Phi(\lambda)(s,\cdot)^T(id-P^+(\lambda,s))^T\xi$, $\xi\in\mathbb{R}^n$, is a solution of the adjoint differential equation (2.50) that converge due to the exponential dichotomy to zero.

Again applying Lemma 2.4.1 provides for any $\xi\in\mathbb{R}^n$ the pointwise definition of

$$R^+(\lambda,s)^T\xi:=\lim_{t\to\infty}e^{A(\lambda)^T(t-s)}Q_s(\lambda)\Phi(\lambda)(s,t)^T(id-P^+(\lambda,s))^T\xi=\eta^+((id-P^+(\lambda,s))^T\xi,\lambda)$$

and the estimate

$$\left\|\Phi(\lambda)(s,t)^T(id-P^+(\lambda,s))^T\xi-e^{-A(\lambda)^T(t-s)}R^+(\lambda,s)^T\xi\right\|\leq c(e^{-\alpha^{uu}(t-s)}+e^{-\alpha^u(t-s)}e^{\delta t}).$$

Here $Q_s(\lambda)$ denotes the projection on the leading stable eigenspace $E_{-A^T}(-\mu^u(\lambda))$ of the matrix $-A^T$ along $E_{-A^T}(\sigma^c_{-\mu^u}(\lambda))$. Therefore we find $\mathrm{im}R^+(\lambda,s)^T \subseteq E_{-A^T}(-\mu^u(\lambda))$ and $\ker R^+(\lambda,s)^T \supseteq \mathrm{im}P^+(\lambda,s)^T$. Again we find, due to Corollary 2.4.4, that ξ lies in the kernel of $R^+(\lambda,s)^T$ if and only if ξ lies in $\mathrm{im}P^+(\lambda,s)^T$ or it lies within the image of the projection $Q^+_{ss}(\lambda,s)$, which is the projection of the exponential trichotomy of (2.50) that projects onto the strong leading subspace. Using the dimension formula then implies $\mathrm{im}R^+(\lambda,s)^T = E_{-A^T}(-\mu^u(\lambda))$. Finally, applying Lemmata 2.0.1 and 2.0.2 we gain

$$\ker R^+(\lambda,s) = [\mathrm{im}R^+(\lambda,s)^T]^\perp = [E_{-A^T}(-\mu^u(\lambda))]^\perp = E_A(\sigma^c_{\mu^u(\lambda)})$$

and

$$\mathrm{im}R^+(\lambda,s) = \left[\ker R^+(\lambda,s)^T\right]^\perp = \left[\mathrm{im}P^+(\lambda,s)^T \oplus \mathrm{im}Q^+_{ss}(\lambda,s)\right]^\perp = \ker P^+(\lambda,s) \cap \ker Q^+_{ss}(\lambda,s)^T.$$

□

Remark 2.5.3. *If in addition*

(i) $\sigma(A(\lambda))$ has no strong stable eigenvalues, we find $kerS^+(\lambda,s) = kerP^+(\lambda,s)$ and

$$\Phi(\lambda)(t,s)P^+(\lambda,s) = e^{A(\lambda)(t-s)}S^+(\lambda,s) + O(e^{\alpha^s(t-s)}e^{\delta t}), \quad t \geq s \geq 0.$$

(ii) $\sigma(A(\lambda))$ has no strong unstable eigenvalues, we find $imR^+(\lambda,s) = kerP^+(\lambda,s)$ and

$$\Phi(\lambda)(s,t)(id - P^+(\lambda,t)) = e^{-A(\lambda)^T(t-s)}R^+(\lambda,s) + O(e^{-\alpha^u(t-s)}e^{\delta t}), \quad t \geq s \geq 0.$$

(iii) the leading stable eigenvalue $\mu^s(\lambda)$ is real, we obtain

$$e^{A(\lambda)(t-s)}S^+(\lambda,s) = e^{\mu^s(\lambda)(t-s)}S^+(\lambda,s).$$

(iv) the leading unstable eigenvalue $\mu^u(\lambda)$ is real, we obtain

$$e^{-A(\lambda)^T(t-s)}R^+(\lambda,s) = e^{-\mu^u(\lambda)(t-s)}R^+(\lambda,s).$$

Remark 2.5.4. *An analogous lemma holds for projections of exponential dichotomies of (2.44) on $\mathbb{R}^-$. Namely there exist linear time-depending operators $S^-(\lambda,s)$ and $R^-(\lambda,s) : \mathbb{R}^n \to \mathbb{R}^n$ with*

$$\begin{aligned} imS^-(s) &= E_A(\mu^u(\lambda)), & kerS^-(s) &= imP^-(s) \oplus imP^-_{uu}(\lambda,s), \\ kerR^-(s) &= E_A(\sigma^c_{\mu^s}(\lambda)), & imR^-(s) &= imP^-(s) \cap [imQ^-_{uu}(\lambda,s)]^\perp, \end{aligned}$$

such that

$$\begin{aligned} \Phi(\lambda)(t,s)(id - P^-(\lambda,s)) &= e^{A(\lambda)(t-s)}S^-(\lambda,s) + O(\ e^{\alpha^{uu}(t-s)} + \ e^{\alpha^u(t-s)}e^{\delta t}) \quad t \leq s \leq 0, \\ \Phi(\lambda)(s,t)P^-(\lambda,t) &= e^{-A(\lambda)^T(t-s)}R^-(\lambda,s) + O(e^{-\alpha^{ss}(t-s)} + \ e^{-\alpha^s(t-s)}e^{\delta t}) \quad t \leq s \leq 0. \end{aligned}$$

Here δ denotes a positive constant such that $\|B(t,\lambda)\| \leq K_B e^{\delta t}$ and $\alpha^s - \delta < \mu^s(\lambda)$ and $-\alpha^u - \delta < -\mu^u(\lambda)$ uniformly in λ. The projections $P^-_{uu}(\lambda,s)$ and $Q^-_{uu}(\lambda,s)$ are projections of the exponential trichotomy on

$\mathbb{R}^-$, *cf. Lemma 2.2.7(ii) and (iv). If the leading eigenvalues* $\mu^s(\lambda)$ *and* $\mu^u(\lambda)$ *are real, we obtain*

$$\begin{aligned} e^{A(\lambda)(t-s)}S^-(\lambda,s) &= e^{\mu^u(\lambda)(t-s)}S^-(\lambda,s), \\ e^{-A(\lambda)^T(t-s)}R^-(\lambda,s) &= e^{-\mu^s(\lambda)(t-s)}R^-(\lambda,s). \end{aligned}$$

2.5.2 Behaviour of $P^{\pm}(\lambda,t)$

Now we consider the single projection P^+ without being paired with the corresponding transition matrix. To this end, we consider the differential equation (2.44) and assume Assumption (A2.7). Then, cf. Remark 2.5.1, (2.44) has an exponential dichotomy on $\mathbb{R}^+$ with corresponding projection P^+.

At first we look at the differential equation this projection is a solution of. Since P^+ is a projection of the exponential dichotomy of (2.44) we find, cf. (2.2)(i), $P^+(\lambda,t) = \Phi(\lambda)(t,\tau)P^+(\lambda,\tau)\Phi(\lambda)(\tau,t)$. Differentiating this equation with respect to t yields

$$\begin{aligned} \dot{P}^+(\lambda,t) &= \dot{\Phi}(\lambda)(t,\tau)P^+(\lambda,\tau)\Phi(\lambda)(\tau,t) + \Phi(\lambda)(t,\tau)P^+(\lambda,\tau)\dot{\Phi}(\lambda)(\tau,t) \\ &= [A(\lambda)+B(t,\lambda)]\Phi(\lambda)(t,\tau)P^+(\lambda,\tau)\Phi(\lambda)(\tau,t) \\ &\quad +\Phi(\lambda)(t,\tau)P^+(\lambda,\tau)\Phi(\lambda)(\tau,t)[-A(\lambda)-B(t,\lambda)] \\ &= [A(\lambda)+B(t,\lambda)]P^+(\lambda,t) - P^+(\lambda,t)[A(\lambda)+B(t,\lambda)]. \end{aligned}$$

Therefore $P^+(\lambda,\cdot)$ satisfies the matrix differential equation

$$\dot{X} = [A(\lambda)+B(t,\lambda)]X - X[A(\lambda)+B(t,\lambda)]. \tag{2.51}$$

This matrix differential equation can be read as linear differential equation. To this end we introduce for any $A \in \mathbb{R}^{n\times n}$ the linear operator $ad(A)$, cf. [Ros02],

$$\begin{aligned} ad(A): \quad \mathbb{R}^{n\times n} &\to \mathbb{R}^{n\times n} \\ X &\mapsto ad(A)X := AX - XA. \end{aligned}$$

Hence equation (2.51) then reads

$$\dot{X} = [ad(A(\lambda)) + ad(B(t,\lambda))]X \tag{2.52}$$

which has a similar structure to equation (2.44).

Further, we have $\lim_{t\to\infty} P^+(\lambda,t) = P(\lambda)$, cf. Lemma 2.1.14, where $P(\lambda)$ denotes the spectral projection of $A(\lambda)$ with respect to the stable and unstable spectrum, that is $\mathrm{im}P(\lambda) = E_A(\sigma_s(\lambda))$ and $\ker P(\lambda) = E_A(\sigma_u(\lambda))$. This projection commutes with $A(\lambda)$ and hence $ad(A(\lambda))P(\lambda) = 0$, i.e. $P(\lambda)$ is a non-trivial element of $\ker ad(A(\lambda))$.

In fact the following lemma holds for the eigenvalues of $ad(A)$:

Lemma 2.5.5 ([Ros02], §1.2 Lemma 8)**.** *If* $A \in \mathbb{R}^{n\times n}$ *has* n *eigenvalues* $\{\mu_j \mid j = 1,\dots,n\}$, *then* $ad(A)$ *has* n^2 *eigenvalues* $\{\mu_j - \mu_k \mid j,k = 1,2,\dots,n\}$.

So we find that 0 is an eigenvalue of $ad(A(\lambda))$ and for $n \geq 2$ the spectrum of $ad(A(\lambda))$ always decomposes into stable, centre and unstable spectrum:

$$\sigma(ad(A)) = \sigma_s(ad(A)) \cup \sigma_c(ad(A)) \cup \sigma_u(ad(A)).$$

Thus Assumption (A2.5)(i) applies with constants α^s_{ad}, α^c_{ad} and α^u_{ad} where

$$\text{Re}(\mu) < \alpha^s_{ad} < -\alpha^c_{ad} < 0 < \alpha^c_{ad} < \alpha^u_{ad} < \text{Re}(\hat{\mu})$$

for all $\mu \in \sigma_s(ad(A))$ and $\hat{\mu} \in \sigma_u(ad(A))$.

With $P(\lambda) \in \ker ad(A(\lambda))$ and $\|P^+(\lambda,t) - P(\lambda)\| < \mathcal{K}e^{-\vartheta t}$, cf. Lemma 2.1.14, also (A2.5)(iii) and (vi) are valid. Finally with

$$\begin{aligned} \|ad(B(t,\lambda))\| = \sup_{\|X\|=1} \|ad(B(t,\lambda))X\| &= \sup_{\|X\|=1} \|B(t,\lambda)X - XB(t,\lambda)\| \\ &\leq \sup_{\|X\|=1} 2\|B(t,\lambda)\|\|X\| \leq 2\,\|B(t,\lambda)\| \leq 2K_B e^{\delta t}, \end{aligned}$$

$\delta < 0$, also (A2.5)(ii) is satisfied. Hence one can apply Lemma 2.4.5 a), resulting in the following Lemma.

Lemma 2.5.6. *Consider the linear differential equation* (2.44) *and assume (A2.7). Let $P^+(\lambda,\cdot)$ and $P(\lambda)$ be the projections corresponding to the exponential dichotomy on $\mathbb{R}^+$ of* (2.44) *and $\dot{x} = A(\lambda)x$, respectively. Then there is a constants $C > 0$ such that*

$$\left\|P^+(\lambda,t) - P(\lambda)\right\| < Ce^{\max\{\alpha^s_{ad},\delta\}t}.$$

Remark 2.5.7. *Due to Lemma 2.1.14 we already know that $\|P^+(\lambda,t) - P(\lambda)\| < \mathcal{K}e^{-\vartheta t}$ with $\vartheta > 0$ restricted by certain inequalities. To be precise the inequalities related to our case read, due to Remark 2.1.15 with $\beta = \alpha^u$ and $Re\tilde{\mu}_1 < \alpha^s$ for all $\tilde{\mu}_1 \in \sigma_s(A(\lambda))$,*

$$\vartheta < (\alpha^u - \alpha^s)/2 \quad \textit{and} \quad \vartheta < -\delta.$$

Hence Lemma 2.5.6 is an improvement towards Lemma 2.1.14 if δ and α^s_{ad} are smaller than $(\alpha^s - \alpha^u)/2$.

The constant α^s_{ad} is bounded below by the real parts of the stable spectrum of $ad(A(\lambda))$

$$\sigma_s(ad(A(\lambda))) = \{\mu_j - \mu_k \mid \mu_j, \mu_k \in \sigma(A(\lambda)),\quad Re(\mu_j) < Re(\mu_k)\}.$$

So the size of α^s_{ad} is determined by the size of the smallest spectral gap of $A(\lambda)$, that is the size of the smallest non-zero distance between the real parts of the eigenvalues of $A(\lambda)$:

$$\alpha^s_{ad} > -\min\{|\ Re(\mu_j - \mu_k)| \mid \mu_j, \mu_k \in \sigma(A(\lambda)),\quad Re(\mu_j) \neq Re(\mu_k)\}. \tag{2.53}$$

Thus, if $-|\ Re(\mu_j - \mu_k)| < (\alpha^s - \alpha^u)/2$ for all $\mu_j, \mu_k \in \sigma(A)$, $Re(\mu_j) \neq Re(\mu_k)$, then Lemma 2.5.6 is an improvement compared to Lemma 2.1.14.

Suppose for example that the spectrum of $A(\lambda)$ only consists of two eigenvalues $\mu^s(\lambda)$ and $\mu^u(\lambda)$, $Re(\mu^s) < \alpha^s < 0 < \alpha^u < Re(\mu^u)$. Then $\sigma_s(ad(A(\lambda))) = \{\mu^s - \mu^u\}$ and α^s_{ad} can be chosen as

$$\alpha^s_{ad} := \alpha^s - \alpha^u > Re(\mu^s) - Re(\mu^u),$$

which indeed is an improvement compared to Lemma 2.1.14.

2.6 Codimension-one homoclinic trajectories

This thesis deals with the nonwandering dynamics near a homoclinic network of codimension-one. The specific homoclinic network we introduce in Section 4. However, beforehand we declare the terminus ***codimension-one*** in context of a single homoclinic trajectory. Shortly speaking this means that we observe the generic unfolding, i.e. the splitting up, of this homoclinic trajectory within a one-dimensional parameter space. Hence we need to make sure that the homoclinic trajectory indeed splits up in the characteristic way and that there are no further effects that might influence the dynamics near the homoclinic trajectory atypically. Since this description is quite vague we now go into more details. As a reference for the following explanations, see for example [Kuz04, HomSan10].

Consider the differential equation $\dot{x} = f(x, \lambda)$, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and assume that at $\lambda = \lambda_0$ we find a hyperbolic equilibrium p which is connected to itself via a homoclinic trajectory γ. Then γ is situated within the intersection of the stable manifold $W^s(p)$ and the unstable manifold $W^u(p)$ of the equilibrium p.

Definition 2.6.1. *Let X be a finite dimensional smooth manifold. An intersection of two submanifolds $U, V \subseteq X$ is called* ***transversal*** *in $x \in U \cap V$ if the tangent spaces of the submanifolds U and V in x complement each other to the tangent space of X. We write $U \pitchfork_x V$.*

The intersection of $W^s(p)$ and $W^u(p)$ along a homoclinic trajectory γ cannot be transversal with regard to the whole phase space $\mathbb{R}^n$. In the generic case one finds

$$T_{\gamma(t)}W^s(p) \cap T_{\gamma(t)}W^u(p) = \text{ span}\{\dot{\gamma}(t)\} \tag{2.54}$$

which implies $\dim(T_{\gamma(t)}W^s(p) + T_{\gamma(t)}W^u(p)) = n - 1$.

Definition 2.6.2. *A homoclinic trajectory γ is called* ***non-degenerate****, if (2.54) holds true.*

Then in general one can expect the homoclinic trajectory to exist permanently at an isolated parameter value $\lambda = \lambda_0$ within a family of differential equations $\dot{x} = f(x, \lambda)$ having a one-dimensional parameter space. That is for $\lambda \neq \lambda_0$ the homoclinic trajectory γ does not exist any more. This is the generic situation and can be explained by the transversal intersection of the manifolds

$$\begin{aligned} \mathcal{W}^s &:= \bigcup_\lambda W^s(p(\lambda)) \times \{\lambda\} \quad \text{and} \\ \mathcal{W}^u &:= \bigcup_\lambda W^u(p(\lambda)) \times \{\lambda\}, \end{aligned} \tag{2.55}$$

of the extended differential equation

$$\begin{aligned} \dot{x} &= f(x, \lambda) \\ \dot{\lambda} &= 0 \end{aligned}$$

within the product space $\mathbb{R}^n \times \mathbb{R}$ of phase and parameter space. Here $p(\lambda)$ denotes the family of saddle points for λ close to λ_0 with $p(\lambda_0) = p$. This situation is depicted in Figure 2.2 in case of a homoclinic trajectory in $\mathbb{R}^2 \times \mathbb{R}$. Note that the transversality of the intersection of $\mathcal{W}^s$ and $\mathcal{W}^u$ is sufficient for the splitting of the homoclinic trajectory but not necessary. Indeed it is necessary for the splitting of the homoclinic trajectory with non-zero speed:

Definition 2.6.3. *Let γ be a homoclinic trajectory existing at an isolated parameter value $\lambda = \lambda_0 \in \mathbb{R}$ within a family of differential equations $\dot{x} = f(x, \lambda)$. Further, let γ satisfy (2.54). We say that γ* ***splits with non-zero speed****, if $d'(\lambda)|_{\lambda_0} \neq 0$ where d denotes a scalar split function $d : \mathbb{R} \to \mathbb{R}$, $\lambda \mapsto d(\lambda)$ that* *measures the distance between the stable and unstable manifolds* $W^s(p(\lambda))$ *and* $W^u(p(\lambda))$ *near* γ *for* λ *close to* λ_0.

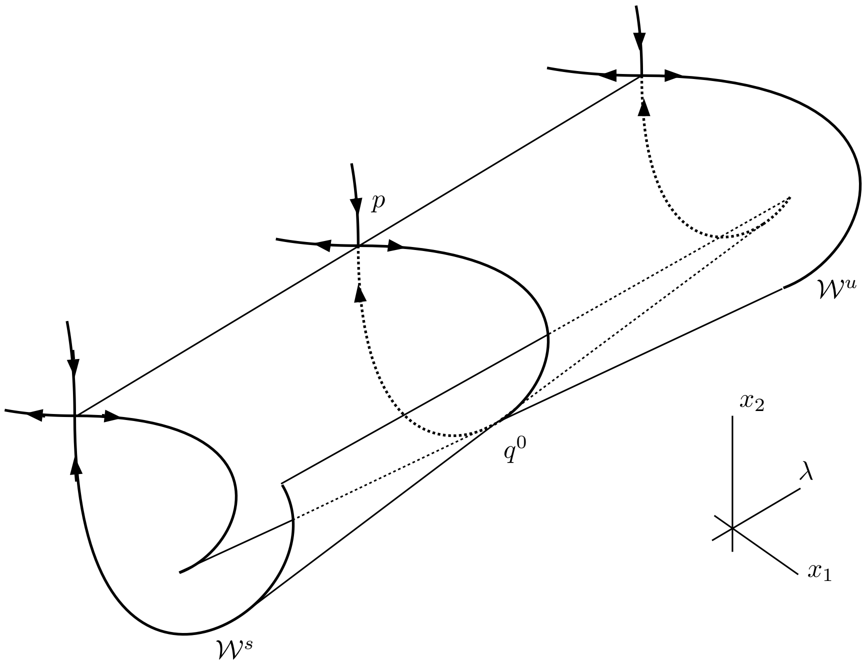


Figure 2.2: Transverse intersection of the invariant manifolds $\mathcal{W}^s$ and $\mathcal{W}^u$ in case of a homoclinic trajectory.

In accordance to these considerations the first demand on a generic homoclinic trajectory reads as follows:

(H2.1). *The homoclinic trajectory* γ *is non-degenerate and splits up with non-zero speed in* λ*, that is the stable and unstable manifolds unfold generically with respect to the parameter* λ.

In the following we exclude further interesting scenarios which complicate the behaviour of the homoclinic trajectory. At first we leave out the orbit flip situation, where a connecting trajectory approaches the equilibrium along non-leading directions.

(H2.2). *The connecting trajectory* γ *is in a non-orbit flip scenario, i.e.* γ *approaches the equilibrium along leading directions:*

$$\gamma \not\subset W^{ss}_{\lambda=0}(p) \quad \textit{and} \quad \gamma \not\subset W^{uu}_{\lambda=0}(p).$$

Generically one can expect a connecting trajectory to approach the equilibrium along leading directions. Hence in general the orbit flip is a bifurcation of at least codimension two; one parameter controls the breaking up of the trajectory, the other one describes the flipping of the direction the trajectory approaches the equilibrium from non-leading to leading. In [San93] orbit-flip bifurcations were considered.

In view of the next hypothesis recall the definition of extended stable and unstable manifolds $W^{s,lu}(p)$ and $W^{ls,u}(p)$ of a hyperbolic equilibrium we outlined in Remark 2.2.8. Further assume that the leading stable and unstable eigenvalues are real and simple.
Following the unstable manifold along a homoclinic trajectory one normally expects this manifold to tend towards the strong unstable directions $E_{D_1 f(p,0)}(\sigma_{uu})$. Likewise the stable manifold generically tends towards the strong stable directions $E_{D_1 f(p,0)}(\sigma_{ss})$ when followed along the homoclinic trajectory backwards in time. In case of the stable manifold the situation is depicted in Figure 2.3. This behaviour of the global stable manifold is a consequence of the strong λ-Lemma, see [Den89]. In short, this lemma

states that a disk D which intersects the extended unstable manifold $W^{ls,u}(p)$ transversally, becomes C^k exponentially close to $W^{ss}_{loc}(p)$ under the flow of the system. So we see that in the generic situation the stable manifold $W^s(p)$ and the extended unstable manifold $W^{ls,u}(p)$ intersect transversally in any point along γ within the local unstable manifold. In the so-called inclination flip situation this property is violated, which we want to prohibit. This leads to the hypothesis:

(H2.3). *The leading eigenvalues $\mu^s(\lambda)$ and $\mu^u(\lambda)$ of $D_1 f(p,\lambda)$ are real and simple and the connecting trajectory γ satisfies a non-inclination flip condition. That is for all $x_u \in W^u_{loc}(p) \cap \gamma$ and all $x_s \in W^s_{loc}(p) \cap \gamma$ applies*

$$W^s(p) \pitchfork_{x_u} W^{ls,u}(p) \quad \text{and} \quad W^u(p) \pitchfork_{x_s} W^{s,lu}(p)$$

Recall that the tangent space of $W^{ls,u}(p)$ along the unstable manifold is uniquely defined. So is the tangent space of $W^{s,lu}(p)$ along the stable manifold.

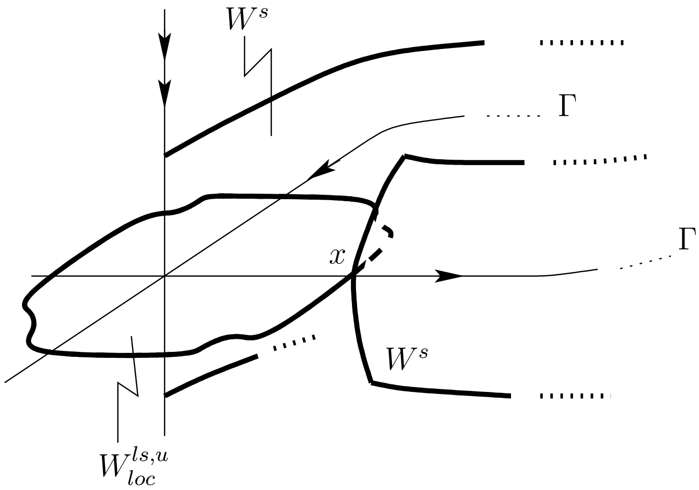


Figure 2.3: The relative position of $W^s(p)$ and $W^{ls,u}$ in the generic case.

In the following we want to state a consequence of Hypothesis (H2.3). To this end we recall some notations from previous sections. Let $A(t) = D_1 f(\gamma(t), 0)$. By $E^s_{-A(\cdot)^T}(t)$ and $E^u_{-A(\cdot)^T}(t)$ we denote the (time-depending) stable subspaces at t of the adjoint variational equation $\dot{x} = -[A(t)]^T x$ for $t \in \mathbb{R}^+$ and $t \in \mathbb{R}^-$, respectively. These subspaces are defined by the image of corresponding projections of the exponential dichotomy, cf. Lemma 2.2.6. Analogously we can define the strong stable subspaces $E^{ss}_{-A(\cdot)^T}(t)$ and $E^{uu}_{-A(\cdot)^T}(t)$ of $\dot{x} = -[A(t)]^T x$, cf. Lemma 2.2.7.

Let further $Z = (T_{\gamma(0)} W^s(p) + T_{\gamma(0)} W^u(p))^\perp$ and

$$\psi(t) := \Psi(t,0)\psi, \qquad \psi \in Z, \quad \psi \neq 0, \tag{2.56}$$

where $\Psi(t,0)$ denotes the transition matrix of the adjoint variational equation $\dot{x} = -[D_1 f(\gamma(t),0)]^T x$. Due to (2.54) we find that Z is one-dimensional and by construction we have, cf. Lemma 2.2.6, $Z \subset E^s_{-A(\cdot)^T}(0) \cap E^u_{-A(\cdot)^T}(0)$. Additionally the following lemma applies.

Lemma 2.6.4 ([Kno04], Lemma 2.3.4, Lemma 2.3.5)**.** *Let the leading eigenvalues μ^s and μ^u be real and simple.*

(i) Assume for all $x_u \in \gamma \cap W^u_{loc}(p)$ *that* $\dim(T_{x_u}W^s(p) + T_{x_u}W^{ls,u}(p)) = n$.
Then $Z \cap E^{uu}_{-A(\cdot)^T}(0) = \{0\}$.

(ii) Assume for all $x_u \in \gamma \cap W^u_{loc}(p)$ *that* $\dim(T_{x_u}W^s(p) + T_{x_u}W^{ls,u}(p)) = n - 1$.
Then $\psi(t) \in E^{uu}_{-A(\cdot)^T}(t)$, $t \in \mathbb{R}^-$.

Proof. Both $T_{\gamma(t)}W^s(p)$ and $T_{\gamma(t)}W^u(p)$ are invariant under the flow of the transition matrix $\Phi(\cdot,\cdot)$ of the variational equation along $\gamma(\cdot)$. At first, recall that the transition matrices $\Phi(\cdot,\cdot)$ and $\Psi(\cdot,\cdot)$ of the variational equation and its adjoint mutually preserve orthogonality, cf. Lemma 2.1.16. Further recall from Lemma 2.2.10 that $E^{uu}_{-A(\cdot)^T}(\tau) = \left(T_{\gamma(\tau)}W^{ls,u}(p)\right)^{\perp}$. From this it follows that for any $z \in E^{uu}_{-A(\cdot)^T}(0)$ we find

$$\lim_{t\to-\infty} \frac{\Psi(t,0)z}{\|\Psi(t,0)z\|} \in \left(T_pW^{ls,u}(p)\right)^{\perp}.$$

Now, we start with the proof of (i).
Let $D_{\gamma(t)}$ be a disk within the stable manifold such that for sufficiently large $|t|$ we find $T_{\gamma(t)}D_{\gamma(t)} \oplus T_{\gamma(t)}W^{ls,u}_{loc}(p) = \mathbb{R}^n$. Consequently we find with $\psi \in Z$ that $\psi(t)/\|\psi(t)\| \perp \left(T_{\gamma(t)}D_{\gamma(t)} \oplus T_{\gamma(t)}W^u(p)\right)$ for all $t \in \mathbb{R}^-$. Then the strong λ-lemma, cf. [Den89], and the continuity of the scalar product provide

$$\lim_{t\to-\infty} \frac{\psi(t)}{\|\psi(t)\|} \in (T_pW^{ss}(p) \oplus T_pW^u(p))^{\perp}.$$

Hence $\psi \notin E^{uu}_{-A(\cdot)^T}(0)$, that is $Z \cap E^{uu}_{-A(\cdot)^T}(0) = \{0\}$.
Next we verify (ii).
In this case we find with $\psi \in Z$ that $\lim_{t\to-\infty} \psi(t)/\|\psi(t)\| \in (E(\mu^s(0)) \oplus T_pW^u(p))^{\perp}$. Hence $\psi(t) \in E^{uu}_{-A(\cdot)^T}(t)$. □

In case of the leading eigenvalues being real and simple Lemma 2.6.4 shows that Hypothesis (H2.3) is equivalent to the condition

$$Z \not\subset E^{ss}_{-A(\cdot)^T}(0) \quad \text{and} \quad Z \not\subset E^{uu}_{-A(\cdot)^T}(0). \tag{2.57}$$

While (H2.3) presents the geometrical constellation of the non-inclination flip situation, (2.57) provides a necessary information for estimating the jump $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ as we will outline in Section 4.3.

In case that the leading stable and unstable eigenvalues of the equilibrium p are real and simple and the homoclinic trajectory γ is non-degenerate and satisfies (H2.2) and (H2.3), one can distinguish twisted and non-twisted homoclinic trajectories, see [Den89, CDF90]. These are two essentially different geometrical situations, cf. Figure 2.4. Even though in the further course of this thesis the leading eigenvalues will be semisimple rather than simple due to the symmetry, such that the concept of twisted and non-twisted homoclinic solutions is not applicable, we want to mention it here. In [Juk06], among others, bifurcations of twisted and non-twisted homoclinic solutions in symmetric vector fields are investigated.

In the following we transcribe the definition of twisted and non-twisted from [CDF90]. To this end we define in accordance with (1.9)

$$e^s := \lim_{t\to\infty} (\gamma(t) - p)/\|\gamma(t) - p\| \qquad \text{and} \qquad e^u := \lim_{t\to-\infty} (\gamma(t) - p)/\|\gamma(t) - p\|. \tag{2.58}$$

Definition 2.6.5. *Let γ be a non-degenerate homoclinic trajectory asymptotic to a saddle p where the leading stable and unstable eigenvalues are real and simple. Further assume Hypotheses (H2.2) and (H2.3).*

*(i) We call γ **twisted**, if e^u and e^s point to opposite sides of $T_{\gamma(\omega)}W^s(p)+T_{\gamma(\omega)}W^u(p)$ and $T_{\gamma(-\omega)}W^s(p)+T_{\gamma(-\omega)}W^u(p)$, respectively, for $\omega > 0$ sufficiently large.*

*(ii) If e^u, e^s point to the same side of $T_{\gamma(\omega)}W^s(p) + T_{\gamma(\omega)}W^u(p)$ and $T_{\gamma(-\omega)}W^s(p) + T_{\gamma(-\omega)}W^u(p)$, respectively, $\omega > 0$ sufficiently large, then we call γ **non-twisted**.*

Regarding this definition, note that $T_{\gamma(\omega)}W^s(p) + T_{\gamma(\omega)}W^u(p)$ and $T_{\gamma(-\omega)}W^s(p) + T_{\gamma(-\omega)}W^u(p)$ denote two different hyperplanes in $\mathbb{R}^n$. However, the hyperplanes are related to each other via the transport by means of the transition matrix of the variational equation along γ. The statement that the directions e^u and e^s point to the same or opposite sides of the hyperplanes therefore has to be read in this context. Due to the definition of $\psi(t)$, cf. (2.56), we find that $\psi(t) \in \left(T_{\gamma(t)}W^s(p) + T_{\gamma(t)}W^u(p)\right)^{\perp}$ for all $t \in \mathbb{R}$. Thus, the statement of the following lemma is readily apparent.

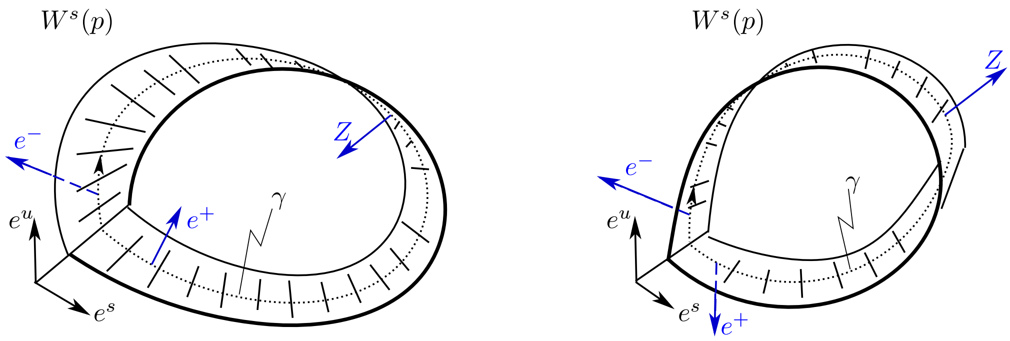


Figure 2.4: Twisted and non-twisted homoclinic trajectories in $\mathbb{R}^3$. The blue arrows display the transportation of the Z-direction along the homoclinic trajectory γ for positive and negative time via the transition matrix Ψ of the adjoint variational equation $\dot{x} = -[D_1 f(\gamma(t), 0)]^T x$.

Lemma 2.6.6 ([Kno04], Lemma 2.3.7)**.** *Let γ be a non-degenerate homoclinic trajectory asymptotic to a saddle p where the leading stable and unstable eigenvalue are real and simple. Further assume Hypotheses (H2.2) and (H2.3). Then the homoclinic trajectory γ is twisted or non-twisted, respectively, if*

$$\mathrm{sgn} \lim_{t\to-\infty} \langle e^s, \psi(t)\rangle = -\mathrm{sgn} \lim_{t\to\infty} \langle e^u, \psi(t)\rangle \quad \text{or} \quad \mathrm{sgn} \lim_{t\to-\infty} \langle e^s, \psi(t)\rangle = \mathrm{sgn} \lim_{t\to\infty} \langle e^u, \psi(t)\rangle.$$

In accordance with (1.10) we define

$$e^+ := \lim_{t\to\infty} \psi(t)/\|\psi(t)\| \qquad \text{and} \qquad e^- := \lim_{t\to-\infty} \psi(t)/\|\psi(t)\|. \tag{2.59}$$

With this the twist condition reads:

Corollary 2.6.7. *Let γ be a non-degenerate homoclinic trajectory asymptotic to a saddle p where the leading stable and unstable eigenvalue are real and simple. Further assume Hypotheses (H2.2) and (H2.3). Define*

$$\mathcal{O} := \mathrm{sgn}(\langle e^s, e^-\rangle\langle e^u, e^+\rangle).$$

Then the homoclinic trajectory γ is twisted or non-twisted, respectively, if

$$\mathcal{O} = -1 \quad \text{or} \quad \mathcal{O} = 1.$$

Remark 2.6.8. *In [HomSan10] the quantity $\mathcal{O}$ is called orientation index and it is used to describe the orientability of the homoclinic centre manifold $W^c_{\mathrm{hom}}(\lambda)$, which is 2-dimensional under the conditions* *mentioned in Corollary 2.6.7. If $\mathcal{O} = -1$ the homoclinic centre manifold has the topological structure of a Möbius band, if $\mathcal{O} = 1$ it has the structure of an annulus.*

Looking at the fibre bundle of tangent directions of the stable manifold, that are complementary to $\dot{\gamma}$, along the homoclinic trajectory

$$\mathcal{F}(W^s_\gamma) := \bigcup_{t\in\mathbb{R}} \left(T_{\gamma(t)}W^s(p) \cap [\ span\{\dot{\gamma}(t)\}]^\perp\right)$$

we see that according to Figure 2.4 in $\mathbb{R}^3$ with 2-dimensional stable manifold also $\mathcal{F}(W^s_\gamma)$ has the topological structure of a Möbius band or an annulus, if $\mathcal{O} = -1$ or $\mathcal{O} = 1$, respectively. However, in higher dimensions with semisimple leading eigenvalues the correlation between the topological structure of $\mathcal{F}(W^s_\gamma)$ and the value of $\mathcal{O}$ is not any more given, cf. Remark 5.2.3.

For the sake of completeness we conclude this section with the demand that the leading eigenvalues are not resonant.

(H2.4). *The leading stable eigenvalue $\mu^s(\lambda)$ and the leading unstable eigenvalue $\mu^u(\lambda)$ of $D_1 f(p,\lambda)$ satisfy*

$$|\ Re(\mu^s(\lambda))| \neq\ Re(\mu^u(\lambda)).$$

Consequently the vector field f is neither hamiltonian nor reversible, which would have implied the homoclinic trajectory to be robust, that is of codimension-zero.

Remark 2.6.9. *A homoclinic trajectory satisfying Hypotheses (H2.1) - (H2.4) is of codimension-one.*

3 Lin's method

Lin's method is beside first return maps one of the basic tools to analyse the dynamics near connecting trajectories. It is named after Xiao-Bao Lin who first introduced this method in 1990, [Lin90]. Lin establishes a method to construct trajectories that stay in a surrounding area of a heteroclinic network for all time. Later modifications can be found in [San93] and [Kno04].

In the following we give a brief description of Lin's method, before we go into more details in Section 3.2. The proof of Lin's method can be found in [Lin90], [San93] and [Kno04]. However, to understand the modifications we are about to accomplish in this thesis it is essential to present the whole proof of Lin's method, since it explains certain geometric terms which we will use later. The only formal change we have made here is the introduction of another projector F_{κ_i} in (3.38), which does not matter for the proof of Lin's method, but does matter for the more precise representation of the residuals.
In order to not perturb the train of thoughts too much we postpone the proofs of all assertions from Section 3.2 to Section 3.3.

We orientate ourselves by the notations and explanations made in [Kno04]. Note, that in [Kno04] Lin's method was considered in the context of a homoclinic trajectory asymptotic to a hyperbolic equilibrium that is allowed to be degenerate. The considerations we undertake in this section are made in the context of a heteroclinic chain, where each heteroclinic trajectory satisfies a minimal intersection condition.

Now, the basic idea of Lin's method can be described as follows. Consider a family of ordinary differential equations

$$\dot{x} = f(x, \lambda), \tag{3.1}$$

that possesses at $\lambda = \lambda_0$ a heteroclinic network Γ consisting of a finite number of equilibrium points p and finitely many connecting (heteroclinic) trajectories γ.

Any solution of (3.1) that stays for all time in the surrounding area of a heteroclinic network Γ is determined by the biinfinite sequence of heteroclinic trajectories it follows. Such a sequence of trajectories we call heteroclinic chain and denote it by

$$\Gamma^\kappa := \bigcup_{i\in\mathbb{Z}} (\{p_{\kappa_i}\} \cup \gamma_{\kappa_i}) \subseteq \Gamma,$$

see Definition 1.0.3, with $\kappa \in \Sigma_{\boldsymbol{C}}$. Here $\Sigma_{\boldsymbol{C}}$ denotes the topological Markov chain defined by the connectivity matrix $\boldsymbol{C} = (c_{ij})$ of the heteroclinic network Γ, see Definition 1.0.1. Thereby each trajectory γ_{κ_i} connects p_{κ_i} to $p_{\kappa_{i+1}}$ in forward time. Thus γ_{κ_i} lies in the intersection of the unstable manifold $W^u(p_{\kappa_i})$ of p_{κ_i} and the stable manifold $W^s(p_{\kappa_{i+1}})$ of $p_{\kappa_{i+1}}$. It can be possible that some or even all of the p_{κ_i} and γ_{κ_i} coincide. For example, when considering the case $p_{\kappa_i} = p$, for all $i \in \mathbb{Z}$, the heteroclinic chain reduces to a homoclinic chain, consisting of homoclinic loops γ_{κ_i} all asymptotic to p in forward and backward time. If in addition also $\gamma_{\kappa_i} = \gamma$ for all $i \in \mathbb{Z}$ the heteroclinic chain simply consists of a homoclinic trajectory γ asymptotic to p.

For a given sequence $\kappa \in \Sigma_{\boldsymbol{C}}$ and a sequence $\boldsymbol{\omega} := (\omega_i)_{i\in\mathbb{Z}}$ of sufficiently large transition times $\omega_i > 0$ Lin proved the existence of the so called ***Lin trajectory***, which we define in the following.

To this end let $\mathcal{S}_{\kappa_i}$ be a hyperplane transversal to $\gamma_{\kappa_i} \subset \Gamma^\kappa$ at some point q_{κ_i} 'in the middle' of γ_{κ_i}, that is far enough away from both p_{κ_i} and $p_{\kappa_{i+1}}$. Let further Z_{κ_i} be a subspace that is complementary to the sum of the tangent spaces at q_{κ_i} of the stable manifold $W^s(p_{\kappa_{i+1}})$ and the unstable manifold $W^u(p_{\kappa_i})$ such that $q_{\kappa_i} + Z_{\kappa_i} \subset \mathcal{S}_{\kappa_i}$.

Definition 3.0.1. *A* **Lin trajectory** *is a piecewise continuous trajectory* $\mathbf{X} = (X_i)_{i\in\mathbb{Z}}$ *satisfying the following two properties:*

- *Each* X_i *is an actual trajectory of the vector field, starting at a point on* $\mathcal{S}_{\kappa_{i-1}}$*, staying close to* $\gamma_{\kappa_{i-1}}$ *until it reaches a neighbourhood of* p_{κ_i}*, and then continuing close to* γ_{κ_i} *until it finally reaches* $\mathcal{S}_{\kappa_i}$ *at exactly the time* $2\omega_i$*.*
- *The 'jump'* Ξ_i *given by the difference between the initial point of* X_{i+1} *and the final point of* X_i *is situated within the subspace* Z_{κ_i}*.*

A visualisation of the described situation is depicted in Figure 3.2. Since the vector field f depends on a parameter λ, both the trajectory $\mathbf{X}$ and the corresponding jump $\boldsymbol{\Xi} := (\Xi_i)_{i\in\mathbb{Z}}$ depend on $\boldsymbol{\omega}$, λ and the sequence κ:

$$\mathbf{X} = \mathbf{X}(\boldsymbol{\omega}, \lambda, \kappa), \qquad \boldsymbol{\Xi} = \boldsymbol{\Xi}(\boldsymbol{\omega}, \lambda, \kappa).$$

In order to obtain an actual trajectory of the vector field which stays for all time near the heteroclinic chain Γ^κ one has to set all the jumps equal to zero, leading to the system of ***determination equations***

$$\boldsymbol{\Xi}(\boldsymbol{\omega}, \lambda, \kappa) = 0. \tag{3.2}$$

That way Lin reduced the problem of detecting trajectories staying close to Γ to that of discussing the solvability of equation (3.2) depending on κ and λ.

A major part of Lin's method is to gain good expressions and approximations of the single jumps Ξ_i. Due to B. Sandstede [San93] we know that indeed each jump Ξ_i can be split into two parts

$$\Xi_i = \xi^\infty_{\kappa_i}(\lambda) + \xi_i(\boldsymbol{\omega}, \lambda, \kappa), \tag{3.3}$$

where the first one $\xi^\infty_{\kappa_i}$ only depends on λ, since it simply measures the distance between the stable manifold $W^s(p_{\kappa_{i+1}})$ and the unstable manifold $W^u(p_{\kappa_i})$. The second part is known to be exponentially small, as $\inf(\boldsymbol{\omega})$ tends to infinity.

For discussing the solvability of equation (3.2) it is further necessary to know explicit expressions for the terms of leading exponential rates of ξ_i. However, this task will be dealt with in Chapter 4. In this chapter we focus on proving the existence of Lin trajectories and providing first estimates of the jump and its derivatives with respect to the transition times ω_j. To this end we begin in Section 3.1 with declaring the setting within we consider Lin's method. Afterwards we firstly sketch the idea of proving the existence and uniqueness of Lin trajectories in Section 3.2, before we go into more detail in Section 3.3. In Section 3.4 we consider the two parts of the jump (3.3) and present first estimates of $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$. We conclude this chapter with Section 3.5 by investigating the derivative of $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ with respect to $\boldsymbol{\omega}$ and show that the derivatives of $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ with respect to ω_j satisfy the same estimates as $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ itself.

3.1 Setting

We consider a family of autonomous differential equations (3.1) where we assume:

(H3.1).

(i) The vector field $f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ *is smooth, i.e.* $f \in C^{l+3}(\mathbb{R}^n \times \mathbb{R}^d, \mathbb{R}^n)$*,* $l \geq 3$*.*

(ii) *For* $\lambda = 0$ *equation* (3.1) *has a heteroclinic network* Γ *consisting of finitely many hyperbolic equilibria and a finite number of connecting trajectories* $\gamma := \{\gamma(t) | t \in \mathbb{R}\}$.

(iii) *Each orbit* γ *satisfies the minimal intersection condition* $\dim(T_{\gamma(0)}W^u(\alpha(\gamma)) \cap T_{\gamma(0)}W^s(\omega(\gamma))) = 1$,

(iv) *and we assume additionally* $\dim T_{\gamma(0)}W^u(\alpha(\gamma)) + \dim T_{\gamma(0)}W^s(\omega(\gamma)) = n$.

In case of a homoclinic network, that is Γ has only one hyperbolic equilibrium, condition (iv) is automatically fulfilled, while condition (iii) requires that the homoclinic trajectories γ are non-degenerate. As we have mentioned above, in the considerations in [Kno04] condition (iii) was spared.

We introduce the following direct sum decomposition of $\mathbb{R}^n$ that we shall use throughout this thesis. To this end let Γ^κ be a double infinite sequence of connecting trajectories for a given sequence $\kappa \in \Sigma_{\boldsymbol{C}}$ as defined in Definition 1.0.3, with $p_{\kappa_i} = \alpha(\gamma_{\kappa_i})$ for all $i \in \mathbb{Z}$. Recall that $\boldsymbol{C}$ denotes the connectivity matrix of Γ. For each heteroclinic trajectory $\gamma_{\kappa_i} \subset \Gamma^\kappa$ we find due to Hypothesis (H3.1)(iii), the one-dimensional subspace

$$U_{\kappa_i} := T_{\gamma_{\kappa_i}(0)}W^s(p_{\kappa_{i+1}}) \cap T_{\gamma_{\kappa_i}(0)}W^u(p_{\kappa_i}) \tag{3.4}$$

which is equal to the vector field direction $\operatorname{span}\{f(\gamma_{\kappa_i}(0), 0)\}$ along $\gamma_{\kappa_i}(0)$. Further we define complements $W^\pm_{\kappa_i}$ of U_{κ_i} within the tangent spaces of the stable and unstable manifolds, that is $W^+_{\kappa_i} \oplus U_{\kappa_i} = T_{\gamma_{\kappa_i}(0)}W^s(p_{\kappa_{i+1}})$ and $W^-_{\kappa_i} \oplus U_{\kappa_i} = T_{\gamma_{\kappa_i}(0)}W^u(p_{\kappa_i})$. To be precise we define for an arbitrary scalar product $\langle \cdot, \cdot \rangle$

$$\left.\begin{array}{rcl} W^+_{\kappa_i} & := & T_{\gamma_{\kappa_i}(0)}W^s(p_{\kappa_{i+1}}) \cap U^\perp_{\kappa_i}, \\ W^-_{\kappa_i} & := & T_{\gamma_{\kappa_i}(0)}W^u(p_{\kappa_i}) \cap U^\perp_{\kappa_i}, \\ Z_{\kappa_i} & := & \Big(T_{\gamma_{\kappa_i}(0)}W^s(p_{\kappa_{i+1}}) + T_{\gamma_i(0)}W^u(p_{\kappa_i})\Big)^\perp. \end{array}\right\} \tag{3.5}$$

Altogether this leads to the direct sum decomposition of $\mathbb{R}^n$

$$\mathbb{R}^n = U_{\kappa_i} \oplus W^+_{\kappa_i} \oplus W^-_{\kappa_i} \oplus Z_{\kappa_i}, \tag{3.6}$$

where the subspaces are via construction pairwise orthogonal to each other except for $W^+_{\kappa_i}$ and $W^-_{\kappa_i}$. Due to Hypothesis (H3.1)(iii) and (iv) Z_{κ_i} is one-dimensional. Finally we define the cross-section $\mathcal{S}_{\kappa_i}$ of the heteroclinic trajectory γ_{κ_i} by

$$\mathcal{S}_{\kappa_i} = \gamma_{\kappa_i}(0) + (Z_{\kappa_i} \oplus W^+_{\kappa_i} \oplus W^-_{\kappa_i}). \tag{3.7}$$

Now, for each heteroclinic solution $\gamma_{\kappa_i}(\cdot)$ we find solutions $\gamma^\pm_{\kappa_i}(\lambda)(\cdot)$ of (3.1) defined on $\mathbb{R}^\pm$, respectively that fulfil the following properties:

(P3.1).

(i) *the trajectories* $\gamma^\pm_{\kappa_i}(\lambda)(\cdot)$ *are close to* γ_{κ_i},

(ii) $\lim_{t\to\infty} \gamma^+_{\kappa_i}(\lambda)(t) = p_{\kappa_{i+1}}(\lambda)$ *and* $\lim_{t\to-\infty} \gamma^-_{\kappa_i}(\lambda)(t) = p_{\kappa_i}(\lambda)$,

(iii) $\gamma^\pm_{\kappa_i}(\lambda)(0) \in \mathcal{S}_{\kappa_i}$,

(iv) $\xi^\infty_{\kappa_i} := \gamma^+_{\kappa_i}(\lambda)(0) - \gamma^-_{\kappa_i}(\lambda)(0) \in Z_{\kappa_i}$.

Figure 3.1 gives a visualisation of this scenario. Indeed the following lemma applies:

Lemma 3.1.1. *For all* λ *close to* 0 *there is a unique pair* $(\gamma^+_{\kappa_i}(\lambda), \gamma^-_{\kappa_i}(\lambda))$ *of solutions of* (3.1) *satisfying the properties (P3.1)* (i)*-*(iv)*. The mappings* $\gamma^\pm_{\kappa_i}(\cdot) : \mathbb{R}^d \to C_b(\mathbb{R}^\pm, \mathbb{R}^n)$ *are smooth.*

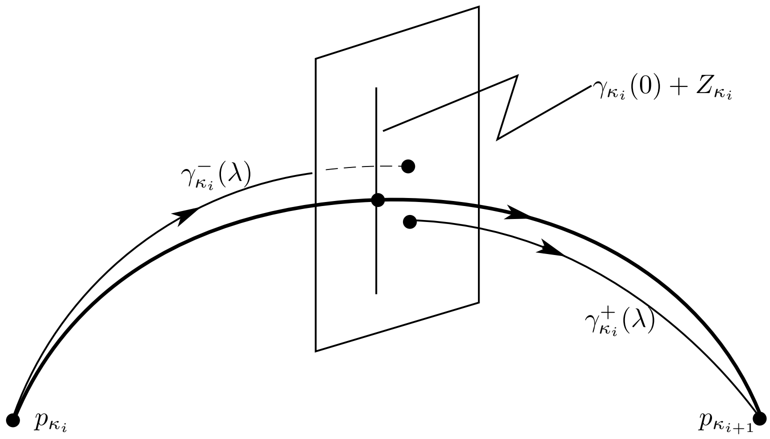


Figure 3.1: Visualisation of Lemma 3.1.1

As already mentioned this result is due to Sandstede. Hence an adequately formulated lemma can be found in [San93, Lemma 3.3] in context of a homoclinic trajectory. In [HomSan10, Lemma 2.4] it was formulated for a heteroclinic trajectory, in [Kno04, Lemma 2.1.2] in the context of a degenerated homoclinic trajectory.

Having the solutions $\gamma^\pm_{\kappa_i}(\lambda)$ of (3.1) we can declare the variational equations along $\gamma^\pm_{\kappa_i}$

$$\dot{x} = D_1 f(\gamma^\pm_{\kappa_i}(\lambda)(t), \lambda)x, \tag{3.8}$$

and their adjoint variational equations

$$\dot{x} = -\big[D_1 f(\gamma^\pm_{\kappa_i}(\lambda)(t), \lambda)\big]^T x. \tag{3.9}$$

Let $\Phi^\pm_{\kappa_i}(\lambda)(\cdot,\cdot)$ denote the transition matrices of (3.8).

Now, in order to facilitate our further analysis we assume the following Hypotheses.

(H3.2). *For sufficiently small λ and all $i \in \mathbb{Z}$ let $p_{\kappa_i}(\lambda) \equiv p_{\kappa_i}$.*

(H3.3). *For sufficiently small λ and all $i \in \mathbb{Z}$:*

$$W^s_{loc,\lambda}(p_{\kappa_i}) \subseteq T_{p_{\kappa_i}} W^s_{\lambda=0}(p_{\kappa_i}), \qquad W^u_{loc,\lambda}(p_{\kappa_i}) \subseteq T_{p_{\kappa_i}} W^u_{\lambda=0}(p_{\kappa_i}).$$

Additionally to embedding the local stable and unstable manifolds into the generalised eigenspaces of $D_1 f(p_{\kappa_i}, \lambda)$ as stipulated in (H3.3) we wish to flatten the stable and unstable manifolds simultaneously along the orbits of the particular solutions $\gamma^\pm_{\kappa_i}(\lambda)(t)$. To this end we define

$$\mathcal{S}^{\kappa_i}_{\lambda,t} := \begin{cases} \gamma^+_{\kappa_i}(\lambda)(t) + \Phi^+_{\kappa_i}(\lambda)(t,0)(W^+_{\kappa_i} \oplus W^-_{\kappa_i} \oplus Z_{\kappa_i}), & t \geq 0, \\ \gamma^-_{\kappa_i}(\lambda)(t) + \Phi^-_{\kappa_i}(\lambda,)(t,0)(W^+_{\kappa_i} \oplus W^-_{\kappa_i} \oplus Z_{\kappa_i}), & t \leq 0. \end{cases}$$

Due to Lemma 3.1.1 we have $\gamma^+_{\kappa_i}(\lambda)(0), \gamma^-_{\kappa_i}(\lambda)(0) \in \mathcal{S}_{\kappa_i} = \gamma_{\kappa_i}(0) + (W^+_{\kappa_i} \oplus W^-_{\kappa_i} \oplus Z_{\kappa_i})$. Hence the definition above is admissible and we find $\mathcal{S}^{\kappa_i}_{\lambda,0} = \mathcal{S}_{\kappa_i}$. Eventually denote by $W^s_{\mathcal{S}^{\kappa_i}_{\lambda,t}}(p_{\kappa_{i+1}})$ and $W^u_{\mathcal{S}^{\kappa_i}_{\lambda,t}}(p_{\kappa_i})$ the traces of the stable and unstable manifolds, respectively, in $\mathcal{S}^{\kappa_i}_{\lambda,t}$:

$$W^s_{\mathcal{S}^{\kappa_i}_{\lambda,t}}(p_{i+1}) = W^s(p_{\kappa_{i+1}}) \cap \mathcal{S}^{\kappa_i}_{\lambda,t}, \quad W^u_{\mathcal{S}^{\kappa_i}_{\lambda,t}}(p_{\kappa_i}) = W^u(p_{\kappa_i}) \cap \mathcal{S}^{\kappa_i}_{\lambda,t}.$$

(H3.4). *There is an $\varepsilon > 0$ such that for sufficiently small λ and for all $i \in \mathbb{Z}$*

$$
\begin{aligned}
(i) \quad & W^s_{\mathcal{S}^{\kappa_i}_{\lambda,0}}(p_{\kappa_{i+1}}) \cap B(\gamma_{\kappa_i}(0), \varepsilon) \subseteq \gamma^+_{\kappa_i}(\lambda)(0) + W^+_{\kappa_i},\\
(ii) \quad & W^u_{\mathcal{S}^{\kappa_i}_{\lambda,0}}(p_{\kappa_i}) \cap B(\gamma_{\kappa_i}(0), \varepsilon) \subseteq \gamma^-_{\kappa_i}(\lambda)(0) + W^-_{\kappa_i}.
\end{aligned}
$$

(H3.5). *There is an $\varepsilon > 0$ such that for sufficiently small λ and for all $i \in \mathbb{Z}$*

$$
\begin{aligned}
(i) \quad & W^s_{\mathcal{S}^{\kappa_i}_{\lambda,t}}(p_{\kappa_{i+1}}) \cap B(\gamma^+_{\kappa_i}(\lambda)(t), \varepsilon) \subseteq \gamma^+_{\kappa_i}(\lambda)(t) + \Phi_{\kappa_i}(t,0) W^+_{\kappa_i}, \quad t \geq 0,\\
(ii) \quad & W^u_{\mathcal{S}^{\kappa_i}_{\lambda,t}}(p_{\kappa_i}) \cap B(\gamma^-_{\kappa_i}(\lambda)(t), \varepsilon) \subseteq \gamma^-_{\kappa_i}(\lambda)(t) + \Phi_{\kappa_i}(t,0) W^-_{\kappa_i}, \quad t \leq 0.
\end{aligned}
$$

Here $\Phi_{\kappa_i}(\cdot,\cdot)$ denotes the transition matrix of the linear variational equation along the heteroclinic solution γ_{κ_i}:

$$
\dot{x} = D_1 f(\gamma_{\kappa_i}(t), 0)x. \tag{3.10}
$$

Indeed (H3.5) can be seen as a continuation of (H3.4) along $\gamma^\pm_{\kappa_i}(\lambda)(t)$.

Remark 3.1.2. *As a direct consequence of Hypotheses (H3.4) and (H3.5) we find that*

$$
\begin{aligned}
T_{\gamma^+_{\kappa_i}(\lambda)(0)} W^s_{\mathcal{S}^{\kappa_i}_{\lambda,0}}(p_{\kappa_{i+1}}) &= W^+_{\kappa_i},\\
T_{\gamma^+_{\kappa_i}(\lambda)(t)} W^s_{\mathcal{S}^{\kappa_i}_{\lambda,t}}(p_{\kappa_{i+1}}) &= \Phi_{\kappa_i}(t,0) W^+_{\kappa_i} \qquad t \geq 0.
\end{aligned}
$$

We also find that $T_{\gamma^+_{\kappa_i}(\lambda)(t)} W^s_{\mathcal{S}^{\kappa_i}_{\lambda,t}}(p_{\kappa_{i+1}})$ can be obtained from $T_{\gamma^+_{\kappa_i}(\lambda)(0)} W^s_{\mathcal{S}^{\kappa_i}_{\lambda,0}}(p_{\kappa_{i+1}})$ through transportation via the transition matrix $\Phi^+_{\kappa_i}(\lambda)(t,0)$:

$$
T_{\gamma^+_{\kappa_i}(\lambda)(t)} W^s_{\mathcal{S}^{\kappa_i}_{\lambda,t}}(p_{\kappa_{i+1}}) = \Phi^+_{\kappa_i}(\lambda)(t,0) T_{\gamma^+_{\kappa_i}(\lambda)(0)} W^s_{\mathcal{S}^{\kappa_i}_{\lambda,0}}(p_{\kappa_{i+1}}).
$$

Therefore we conclude that for any vector field f satisfying Hypotheses (H3.1) and (H3.5) we find for λ sufficiently small

$$
\begin{aligned}
\Phi_{\kappa_i}(t,0) W^+_{\kappa_i} &= \Phi^+_{\kappa_i}(\lambda)(t,0) W^+_{\kappa_i}, \quad t \geq 0,\\
\Phi_{\kappa_i}(t,0) W^-_{\kappa_i} &= \Phi^-_{\kappa_i}(\lambda)(t,0) W^-_{\kappa_i}, \quad t \leq 0.
\end{aligned}
$$

Indeed Hypotheses (H3.2) - (H3.5) do not imply a restriction to the vector fields under consideration since there always exist certain transformation to obtain the claimed results independent of the sequence $\kappa \in \Sigma_{\boldsymbol{C}}$, see [Kno04]. However, we want to point out that we lose some degree of differentiability of the vector field. To be precise, after the vector field transformation we find $f \in C^l(\mathbb{R}^n \times \mathbb{R}^d, \mathbb{R}^n)$. We will see to the justification of these hypotheses in Section 4.2, where we do this in the context of homoclinic networks in symmetric vector fields.

In order to estimate the jump $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ we denote by $\mu^s_i(\lambda)$ and $\mu^u_i(\lambda)$ the leading stable and unstable eigenvalues of $D_1 f(p_i, \lambda)$ and we furthermore assume

(H3.6). *For sufficiently small λ and all $i \in \mathbb{Z}$ let $|\ Re(\mu^s_i(\lambda))| < \ Re(\mu^u_i(\lambda))$.*

That is the stable eigenvalues $\mu^s_i(\lambda)$ lie closer to the imaginary axis than the unstable eigenvalues $\mu^u_i(\lambda)$. In view of the upcoming symmetry, a uniform requirement of the relations of the real parts of the leading eigenvalues for all $i \in \mathbb{Z}$ makes sense. The choice of which of the two is closer to the imaginary axis is thereby done without loss of generality. The estimates presented in Section 3.4 can be transcribed for the case $\mathrm{Re}(\mu^u_i(\lambda)) < |\ \mathrm{Re}(\mu^s_i(\lambda))|$.

We conclude this section with a final assumption which we will call occasionally but not exclusively.

(H3.7). *For all $i \in \mathbb{Z}$ the spectrum of $D_1 f(p_i, \lambda)$ has neither strong stable nor strong unstable eigenvalues.*

3.2 Outline of Lin's method

The first step of Lin's method is already completed by solving the problem of finding solutions $\gamma^{\pm}_{\kappa_i}(\lambda)(\cdot)$ of (3.1) defined on $\mathbb{R}^{\pm}$ that fulfil the Property (P3.1). This part was based on the idea of Sandstede, cf. [San93]. Next we describe the second step of Lin's method which introduces the Lin trajectories $\mathbf{X} := (X_i)_{i\in\mathbb{Z}}$.

We start with an introduction of notations that we will use throughout this thesis.

Definition 3.2.1.

(1) Let U be a normed space. By l^{∞}_U we denote the space of all bounded sequences $\boldsymbol{x} := (x_i)_{i\in\mathbb{Z}}$, $x_i \in U$. l^{∞}_U is equipped with the supremum norm.

(2) $\boldsymbol{\omega} := (\omega_i)_{i\in\mathbb{Z}}$, $\omega_i \in \mathbb{R}^+$.

(3) By $V_{\boldsymbol{\omega}}$ we denote the space of all sequences $\boldsymbol{v} := (v_i^+, v_i^-)_{i\in\mathbb{Z}}$ where $v_i^+ \in C([0,\omega_{i+1}],\mathbb{R}^n)$ and $v_i^- \in C([-\omega_i, 0],\mathbb{R}^n)$. $V_{\boldsymbol{\omega}}$ is equipped with the norm $\|\boldsymbol{v}\|_{V_{\boldsymbol{\omega}}} := \max\{\sup_{i\in\mathbb{Z}} \|v_i^+\|, \sup_{i\in\mathbb{Z}} \|v_i^-\|\}$.

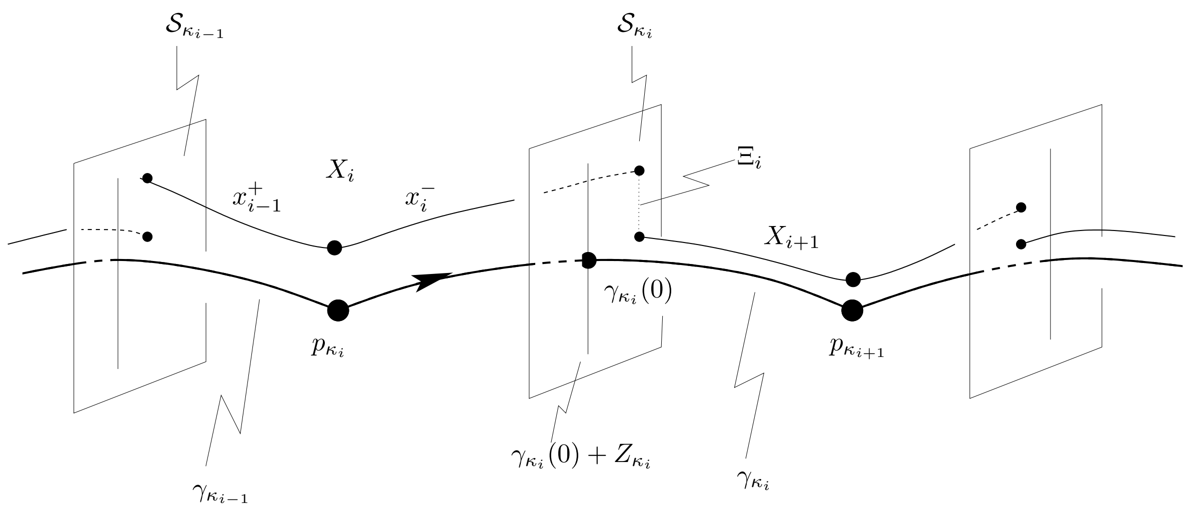


Figure 3.2: Idea of Lin's method

The following theorem ensures the existence of the above described Lin trajectories.

Theorem 3.2.2. *Consider a vector field $f(\cdot,\lambda)$ and assume (H3.1)-(H3.5). Then there are constants c, Ω such that for each $(\boldsymbol{\omega},\lambda,\kappa)$ with $|\lambda| < c$, $\omega_i > \Omega$ and $\kappa \in \Sigma_{\boldsymbol{C}}$ there are unique sequences of solutions $x_i^{\pm}(\boldsymbol{\omega},\lambda,\kappa)(\cdot)$ of (3.1) satisfying:*

(i) $x_i^+(\boldsymbol{\omega},\lambda,\kappa)(\cdot) : [0,\omega_{i+1}] \to \mathbb{R}^n$, $\quad x_i^-(\boldsymbol{\omega},\lambda,\kappa)(\cdot) : [-\omega_i, 0] \to \mathbb{R}^n$;

(ii) $x_i^+(\boldsymbol{\omega},\lambda,\kappa)(\omega_{i+1}) = x_{i+1}^-(\boldsymbol{\omega},\lambda,\kappa)(-\omega_{i+1})$;

(iii) $x_i^\pm(\boldsymbol{\omega},\lambda,\kappa)(0) \in \mathcal{S}_{\kappa_i}$ *are close to* $\gamma_{\kappa_i}^\pm(\lambda)(0)$*;*

(iv) $x_i^+(\boldsymbol{\omega},\lambda,\kappa)(0) - x_i^-(\boldsymbol{\omega},\lambda,\kappa)(0) \in Z_{\kappa_i}$

(v) $x_i^\pm(\boldsymbol{\omega},\lambda,\kappa)(0) - \gamma_{\kappa_i}^\pm(\lambda)(0) \in W_{\kappa_i}^+ \oplus W_{\kappa_i}^- \oplus Z_{\kappa_i}$.

Theorem 3.2.2 is a special case of [Kno04, Theorem 2.1.4], where degenerate homoclinic trajectories are considered. In [VanFie92, Theorem 6] an adequate theorem was formulated for a single homoclinic trajectory. In the following we merely outline the proof and refer to Chapter 3.3 for detailed explanations.

The objects $x_i^\pm$ will be characterised as perturbations of $\gamma_{\kappa_i}^\pm$ from Lemma 3.1.1:

$$x_i^\pm(\boldsymbol{\omega},\lambda,\kappa)(\cdot) = \gamma_{\kappa_i}^\pm(\lambda)(\cdot) + v_i^\pm(\boldsymbol{\omega},\lambda,\kappa)(\cdot). \tag{3.11}$$

So the task to find the quantities $x_i^\pm$ is rephrased into the task of finding appropriate $v_i^\pm$. To this end let $\boldsymbol{\omega} := (\omega_i)_{i\in\mathbb{Z}}$ be a sequence of sufficiently large transition times ω_i. In order to prove the theorem we look for $v_i^\pm$ as solutions of the following boundary value problem

$$\dot{v}_i^\pm(t) = D_1 f(\gamma_{\kappa_i}^\pm(\lambda)(t),\lambda) v_i^\pm(t) + h_{\kappa_i}^\pm(t, v_i^\pm(t),\lambda) \tag{3.12}$$

$$\left.\begin{aligned} & v_i^+(\omega_{i+1}) - v_{i+1}^-(-\omega_{i+1}) = \gamma_{\kappa_{i+1}}^-(\lambda)(-\omega_{i+1}) - \gamma_{\kappa_i}^+(\lambda)(\omega_{i+1}) =: d_{i+1}(\omega_{i+1},\lambda),\\ & v_i^+(0), v_i^-(0) \in W_{\kappa_i}^+ \oplus W_{\kappa_i}^- \oplus Z_{\kappa_i}, \qquad \text{close to zero,}\\ & v_i^+(0) - v_i^-(0) \in Z_{\kappa_i}, \end{aligned}\right\} \tag{3.13}$$

where $h_{\kappa_i}^\pm$ is defined as

$$h_{\kappa_i}^\pm(t,v,\lambda) := f(\gamma_{\kappa_i}^\pm(\lambda)(t) + v,\lambda) - f(\gamma_{\kappa_i}^\pm(\lambda)(t),\lambda) - D_1 f(\gamma_{\kappa_i}^\pm(\lambda)(t),\lambda)v. \tag{3.14}$$

Due to (P3.1) a solution $(v_i^+, v_i^-)_{i\in\mathbb{Z}} \in V_{\boldsymbol{\omega}}$ of the boundary value problem ((3.12),(3.13)) provides a unique sequence of solutions $x_i^\pm(\boldsymbol{\omega},\lambda,\kappa)(\cdot)$ of (3.1) that fulfils Theorem 3.2.2.

The boundary value problem ((3.12),(3.13)) will be solved in the following way. First we consider the inhomogeneous equation

$$\dot{v}_i^\pm(t) = D_1 f(\gamma_{\kappa_i}^\pm(\lambda)(t),\lambda) v_i^\pm(t) + g_i^\pm(t), \tag{3.15}$$

where $\mathbf{g} := (g_i^+, g_i^-)_{i\in\mathbb{Z}} \in V_{\boldsymbol{\omega}}$. Here we prove, that for all $\mathbf{g}$ we find a unique solution $\hat{\mathbf{v}} = \hat{\mathbf{v}}(\dots,\mathbf{g})$ of the boundary value problem ((3.15),(3.13)). Then substituting $\mathbf{g}$ by the non-linearity h of equation (3.12) gives a fixed point equation whose solution satisfies the boundary value problem ((3.12),(3.13)).

Let us now start with more detailed explanations. Recall that $\Phi_{\kappa_i}^\pm(\lambda)(\cdot,\cdot)$ denotes the transition matrix of the homogeneous part of equation (3.15) which is given by equation (3.8). This variational equation has exponential dichotomies on $\mathbb{R}^\pm$, cf. Section 2.1. We denote the projections associated to equation (3.8) by $P_{\kappa_i}^\pm$ such that there are positive constants K and $\alpha_{\kappa_i}^+, \beta_{\kappa_i}^+, \alpha_{\kappa_i}^-$ and $\beta_{\kappa_i}^-$ with

$$\left.\begin{aligned} \|\Phi_{\kappa_i}^+(\lambda)(t,s)(id - P_{\kappa_i}^+(\lambda,s))\| &\leq Ke^{-\alpha_{\kappa_i}^+(t-s)}, & t \geq s \geq 0,\\ \|\Phi_{\kappa_i}^+(\lambda)(t,s)P_{\kappa_i}^+(\lambda,s)\| &\leq Ke^{-\beta_{\kappa_i}^+(s-t)}, & s \geq t \geq 0,\\ \|\Phi_{\kappa_i}^-(\lambda)(t,s)P_{\kappa_i}^-(\lambda,s)\| &\leq Ke^{-\alpha_{\kappa_i}^-(t-s)}, & 0 \geq t \geq s,\\ \|\Phi_{\kappa_i}^-(\lambda)(t,s)(id - P_{\kappa_i}^-(\lambda,s))\| &\leq Ke^{-\beta_{\kappa_i}^-(s-t)}, & 0 \geq s \geq t. \end{aligned}\right\} \tag{3.16}$$

Remark 3.2.3. *In contrast to the Definition 2.1.1 of the exponential dichotomy in Section 2.1 the role of the projections* $(id-P^+_{\kappa_i})$ *and* $P^+_{\kappa_i}$ *associated with the exponential dichotomy on* $\mathbb{R}^+$ *is vice versa. That is to say here the kernel instead of the image of* $P^+_{\kappa_i}$ *is determined to be the stable subspace at time* t *of* (3.8) *along* $\gamma^+_{\kappa_i}$*. We have done this for convenience since also the kernel of* $P^-_{\kappa_i}$ *is settled to be the unstable subspace at time* t *of* (3.8) *along* $\gamma^-_{\kappa_i}$ *That way our notation correspond to that in [HJKL11].*

Recall that we have some freedom in choosing the image of $P^\pm_{\kappa_i}$. Here we set

$$\left.\begin{array}{rclrcl} \ker P^+_{\kappa_i}(\lambda,0) &=& T_{\gamma^+_{\kappa_i}(\lambda)(0)}W^s(p_{\kappa_{i+1}}), & \operatorname{im}P^+_{\kappa_i}(\lambda,0) &=& W^-_{\kappa_i}\oplus Z_{\kappa_i}, \\ \ker P^-_{\kappa_i}(\lambda,0) &=& T_{\gamma^-_{\kappa_i}(\lambda)(0)}W^u(p_{\kappa_i}), & \operatorname{im}P^-_{\kappa_i}(\lambda,0) &=& W^+_{\kappa_i}\oplus Z_{\kappa_i}, \end{array}\right\} \tag{3.17}$$

and with the condition

$$P^\pm_{\kappa_i}(\lambda,t)\Phi^\pm_{\kappa_i}(\lambda)(t,s) = \Phi^\pm_{\kappa_i}(\lambda)(t,s)P^\pm_{\kappa_i}(\lambda,s)$$

the projections are now uniquely defined. As for the constants $\alpha^+_{\kappa_i}, \beta^+_{\kappa_i}, \alpha^-_{\kappa_i}$ and $\beta^-_{\kappa_i}$ the considerations in Section 2.2 show that they are closely related to the leading eigenvalues of the linearisation of the vector field $D_1f(p_{\kappa_i},\lambda)$ at the equilibrium points p_{κ_i}. So let $\mu^s_{\kappa_i}(\lambda)$ and $\mu^u_{\kappa_i}(\lambda)$ denote a leading stable and unstable eigenvalue of $D_1f(p_{\kappa_i},\lambda)$, respectively. Then the constants $\alpha^+_{\kappa_i}, \beta^+_{\kappa_i}, \alpha^-_{\kappa_i}$ and $\beta^-_{\kappa_i}$ can be chosen (for all λ sufficiently small) within the limits of the following inequalities, cf. Lemma 2.2.3

$$\left.\begin{array}{rcccccl} \operatorname{Re}(\mu^s_{\kappa_{i+1}}(\lambda)) &<& -\alpha^+_{\kappa_i} &<& \beta^+_{\kappa_i} &<& \operatorname{Re}(\mu^u_{\kappa_{i+1}}(\lambda)), \\ \operatorname{Re}(\mu^s_{\kappa_i}(\lambda)) &<& -\alpha^-_{\kappa_i} &<& \beta^-_{\kappa_i} &<& \operatorname{Re}(\mu^u_{\kappa_i}(\lambda)). \end{array}\right\} \tag{3.18}$$

The following Lemma is the first step towards the proof of Theorem 3.2.2. An analogous lemma can be found in [Kno04, Lemma 2.1.5] again in the context of a degenerate homoclinic trajectory.

Lemma 3.2.4. *Assume Hypotheses (H3.1)-(H3.5). Let* $\Omega \in \mathbb{R}^+$ *be sufficiently large and further let* $\boldsymbol{\omega}$ *be a sequence with* $\omega_i > \Omega$, $i \in \mathbb{Z}$*. Then there is a constant* c *such that for each* $\kappa \in \Sigma_{\boldsymbol{C}}$, $\boldsymbol{g} \in V_{\boldsymbol{\omega}}$, $\boldsymbol{a} \in l^\infty_{\mathbb{R}^n}$ *and* λ *with* $|\lambda| < c$ *the system* (3.15) *has exactly one solution* $\boldsymbol{v}_{\boldsymbol{\omega}} \in V_{\boldsymbol{\omega}}$ *satisfying*

(i) $P^+_{\kappa_{i-1}}(\lambda,\omega_i)(v^+_{i-1}(\omega_i)-a_i) = 0, \quad P^-_{\kappa_i}(\lambda,-\omega_i)(v^-_i(-\omega_i)-a_i) = 0$;

(ii) $v^+_i(0), v^-_i(0) \in W^+_{\kappa_i}\oplus W^-_{\kappa_i}\oplus Z_{\kappa_i}$;

(iii) $v^+_i(0) - v^-_i(0) \in Z_{\kappa_i}$.

$\boldsymbol{v}_{\boldsymbol{\omega}}$ *and hence* $v^\pm_i$ *depend on* $(\lambda,\kappa,\boldsymbol{g},\boldsymbol{a})$ *and the mapping*

$$\begin{array}{rccc} \boldsymbol{v}_{\boldsymbol{\omega}}: & \mathbb{R}^d\times\Sigma_{\boldsymbol{C}}\times V_{\boldsymbol{\omega}}\times l^\infty_{\mathbb{R}^n} & \to & V_{\boldsymbol{\omega}} \\ & (\lambda,\kappa,\boldsymbol{g},\boldsymbol{a}) & \mapsto & \boldsymbol{v}_{\boldsymbol{\omega}}(\lambda,\kappa,\boldsymbol{g},\boldsymbol{a}) \end{array}$$

is smooth in $(\lambda,\boldsymbol{g},\boldsymbol{a})$ *and moreover linear in* $(\boldsymbol{g},\boldsymbol{a})$*. Further there is a constant* C *such that*

$$\|\boldsymbol{v}_{\boldsymbol{\omega}}\|_{V_{\boldsymbol{\omega}}} \le C(\|\boldsymbol{a}\|_{l^\infty_{\mathbb{R}^n}} + \|\boldsymbol{g}\|_{V_{\boldsymbol{\omega}}}). \tag{3.19}$$

For λ sufficiently small and ω_i sufficiently large the images of the projections $P^+_{\kappa_{i-1}}$ and $P^-_{\kappa_i}$ form a direct sum decomposition of $\mathbb{R}^n$:

$$\mathbb{R}^n = \operatorname{im}P^+_{\kappa_{i-1}}(\lambda,\omega_i)\oplus\operatorname{im}P^-_{\kappa_i}(\lambda,-\omega_i), \tag{3.20}$$

see [VanFie92, Lemma 7] and Lemma 3.3.2 below. Therefore we find for each $d_i \in \mathbb{R}^n$ an a_i such that $P^+_{\kappa_{i-1}} a_i - P^-_{\kappa_i} a_i = d_i$. In this context Lemma 3.2.4 can be seen as an "approximation" of the boundary value problem ((3.15),(3.13)), because Lemma 3.2.4(i) says that $P^+_{\kappa_{i-1}} v^+_{i-1} - P^-_{\kappa_i} v^-_i = d_i$ instead of $v^+_{i-1} - v^-_i = d_i$ as demanded in (3.13). The values $\|(id - P^+_{\kappa_{i-1}})v^+_{i-1}\|$ and $\|(id - P^-_{\kappa_i})v^-_i\|$ decrease exponentially fast as ω_i increases. In Figure 3.3 the full nonlinear situation is depicted. So the quantity $\boldsymbol{a}$ is only a device to handle the coupling condition between v^+_{i-1} and v^-_i near the equilibrium. Indeed we prove, cf. [Kno04, Lemma 2.1.6], that for given $\boldsymbol{\omega}$ and $\boldsymbol{d} := (d_i)_{i\in\mathbb{Z}} \in l^\infty_{\mathbb{R}^n}$ there is an $\boldsymbol{a}$ according to Lemma 3.2.4 satisfying $v^+_{i-1}(\lambda, \kappa, \mathbf{g}, \boldsymbol{a})(\omega_i) - v^-_i(\lambda, \kappa, \mathbf{g}, \boldsymbol{a})(-\omega_i) = d_i$.

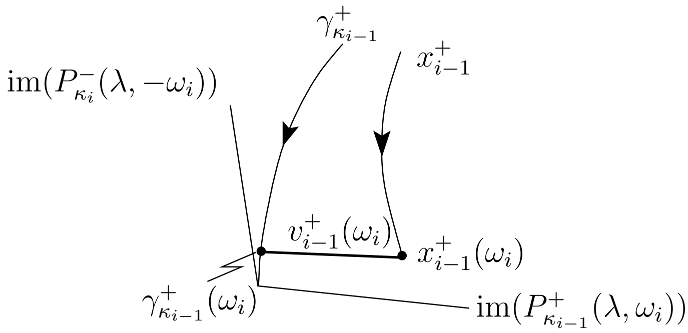


Figure 3.3: $v^+_{i-1}(\omega_i)$ in the nonlinear problem

Lemma 3.2.5. *Assume Hypotheses (H3.1)-(H3.5). Let $\Omega \in \mathbb{R}^+$ be sufficiently large and further let $\boldsymbol{\omega}$ be a sequence with $\omega_i > \Omega$, $i \in \mathbb{Z}$. Then there is a constant c such that for each $\kappa \in \Sigma_{\boldsymbol{C}}$, $\boldsymbol{g} \in V_{\boldsymbol{\omega}}$, $\boldsymbol{d} \in l^\infty_{\mathbb{R}^n}$ and λ with $|\lambda| < c$ there is exactly one $\hat{\boldsymbol{v}}_{\boldsymbol{\omega}} \in V_{\boldsymbol{\omega}}$ solving (3.15) and satisfying the boundary conditions*

(i) $\hat{v}^+_{i-1}(\omega_i) - \hat{v}^-_i(-\omega_i) = d_i$;

(ii) $\hat{v}^+_i(0), \hat{v}^-_i(0) \in W^+_{\kappa_i} \oplus W^-_{\kappa_i} \oplus Z_{\kappa_i}$;

(iii) $\hat{v}^+_i(0) - \hat{v}^-_i(0) \in Z_{\kappa_i}$.

The mapping

$$\begin{array}{rccc} \hat{\boldsymbol{v}}_{\boldsymbol{\omega}}: & \mathbb{R}^d \times \Sigma_{\boldsymbol{C}} \times V_{\boldsymbol{\omega}} \times l^\infty_{\mathbb{R}^n} & \to & V_{\boldsymbol{\omega}} \\ & (\lambda, \kappa, \boldsymbol{g}, \boldsymbol{d}) & \mapsto & \hat{\boldsymbol{v}}_{\boldsymbol{\omega}}(\lambda, \kappa, \boldsymbol{g}, \boldsymbol{d}) \end{array}$$

is smooth in $(\lambda, \boldsymbol{g}, \boldsymbol{d})$ and depend linearly on $(\boldsymbol{g}, \boldsymbol{d})$ and there exists a constant $\hat{C}$ such that

$$\|\hat{\boldsymbol{v}}_{\boldsymbol{\omega}}\|_{V_{\boldsymbol{\omega}}} \le \hat{C}(\|\boldsymbol{g}\|_{V_{\boldsymbol{\omega}}} + \|\boldsymbol{d}\|_{l^\infty_{\mathbb{R}^n}}). \tag{3.21}$$

Now, by replacing the function $\mathbf{g}$ in $\hat{\boldsymbol{v}}$ by

$$\begin{array}{rccc} \boldsymbol{\mathcal{H}}: & V_{\boldsymbol{\omega}} \times \mathbb{R}^d \times \Sigma_{\boldsymbol{C}} & \to & V_{\boldsymbol{\omega}} \\ & (\boldsymbol{v}, \lambda, \kappa) & \mapsto & (h^+_{\kappa_i}(\cdot, v^+_i(\cdot), \lambda), h^-_{\kappa_i}(\cdot, v^-_i(\cdot), \lambda))_{i\in\mathbb{Z}} \end{array}$$

we are back to the non-linear boundary value problem ((3.12),(3.13)). So analogously to [Kno04, Lemma 2.1.7] we state:

Lemma 3.2.6. *Assume Hypotheses (H3.1)-(H3.5). There are constants c, Ω such that for fixed $\boldsymbol{\omega}$ with $\omega_i > \Omega$ the following holds true: For each $\lambda \in \mathbb{R}^d$ with $|\lambda| < c$ the fixed point problem*

$$\boldsymbol{v} = \hat{\boldsymbol{v}}_{\boldsymbol{\omega}}(\lambda, \kappa, \boldsymbol{\mathcal{H}}(\boldsymbol{v}, \lambda, \kappa), \boldsymbol{d}(\boldsymbol{\omega}, \lambda, \kappa)) =: \mathcal{F}_{\boldsymbol{\omega}}(\boldsymbol{v}, \lambda, \kappa) \tag{3.22}$$

has a unique solution $\bar{\boldsymbol{v}}_{\boldsymbol{\omega}}$ *in a sufficiently small neighbourhood of* $0 \in V_{\boldsymbol{\omega}}$*. The mapping* $\lambda \mapsto \bar{\boldsymbol{v}}_{\boldsymbol{\omega}}(\lambda, \kappa)$ *is smooth.*

We want to point out that all the considerations have been done in the space $V_{\boldsymbol{\omega}}$, that is for fixed $\boldsymbol{\omega}$. Therefore it is not yet clear how $\bar{\boldsymbol{v}}(\boldsymbol{\omega}, \lambda, \kappa) = (\bar{v}_i^+(\boldsymbol{\omega}, \lambda, \kappa), \bar{v}_i^-(\boldsymbol{\omega}, \lambda, \kappa))_{i\in\mathbb{Z}} := \bar{\boldsymbol{v}}_{\boldsymbol{\omega}}(\lambda, \kappa)$ depends on $\boldsymbol{\omega}$. However, for the analysis of the determination equations only $(\bar{v}_i^+(\boldsymbol{\omega}, \lambda, \kappa)(0), \bar{v}_i^-(\boldsymbol{\omega}, \lambda, \kappa)(0))_{i\in\mathbb{Z}}$ is of importance, see (3.57) and Section 3.4. Thus we define

$$\bar{\boldsymbol{v}}^{\pm}(0) : B_\Omega \times \mathbb{R}^d \times \Sigma_{\boldsymbol{C}} \to l^\infty_{\mathbb{R}^n}, \qquad (\boldsymbol{\omega}, \lambda, \kappa) \mapsto (\bar{v}_i^{\pm}(\boldsymbol{\omega}, \lambda, \kappa)(0))_{i\in\mathbb{Z}}, \tag{3.23}$$

where $B_\Omega := (\Omega, \infty)^{\mathbb{Z}}$.

Lemma 3.2.7 ([Kno04] Lemma 2.1.8)**.** *The mappings defined in* (3.23) *depend smoothly on* $\boldsymbol{\omega}$*.*

The proof of this Lemma can be transcribed from [VanFie92, Lemma 12], [San93, Bemerkung 3.17] or [Kno04, Lemma 2.1.8]. Although in [VanFie92] only periodic solutions are searched and therefore $\bar{\boldsymbol{v}}$ is only considered as a mapping $(\Omega, \infty) \times \mathbb{R} \to \mathbb{R}$ it gives the basic idea of proving the smoothness of $\bar{\boldsymbol{v}}$ with respect to the l^∞ variable $\boldsymbol{\omega}$: performing a time rescaling.

To this end let $\boldsymbol{\omega} = (\omega_i)_{i\in\mathbb{Z}}$ be a fixed sequence. Further let $0 < \delta < 1$ and let $\boldsymbol{\beta} = (\beta_i)_{i\in\mathbb{Z}} \in l^\infty_{\mathbb{R}}$ with $|\beta_i| < \delta$. Then we replace $\dot{x} = f(x, \lambda)$ by $\dot{x} = (1 + \beta_i) f(x, \lambda)$ in the i^{th} circulation. This correspond to a time rescaling $t \mapsto (1 + \beta_i)t$ in each case. Note that $\dot{x} = f(x, \lambda)$ and $\dot{x} = (1 + \beta_i) f(x, \lambda)$ have the same orbits, but they will be passed through in different time.

The further proceeding in the proof of the smoothness with respect to $\boldsymbol{\omega} \in l^\infty_{\mathbb{R}}$ is sketched in [Kno04]. With $(\mathbf{1} + \boldsymbol{\beta})\boldsymbol{\omega} := ((1 + \beta_i)\omega_i)_{i\in\mathbb{Z}}$ one can define $\check{\bar{\boldsymbol{v}}}_{\boldsymbol{\omega}}(\boldsymbol{\beta}, \lambda, \kappa) = \check{\bar{\boldsymbol{v}}}(\boldsymbol{\beta}, \boldsymbol{\omega}, \lambda, \kappa) = (\check{\bar{v}}_i^+, \check{\bar{v}}_i^-)_{i\in\mathbb{Z}}$ by

$$\left.\begin{aligned} \check{\bar{v}}_i^+(\boldsymbol{\beta}, \boldsymbol{\omega}, \lambda, \kappa)(t) &:= \bar{v}_i^+((\mathbf{1} + \boldsymbol{\beta})\boldsymbol{\omega}, \lambda, \kappa)((1 + \beta_{i+1})t), \\ \check{\bar{v}}_i^-(\boldsymbol{\beta}, \boldsymbol{\omega}, \lambda, \kappa)(t) &:= \bar{v}_i^-((\mathbf{1} + \boldsymbol{\beta})\boldsymbol{\omega}, \lambda, \kappa)((1 + \beta_i)t). \end{aligned}\right\} \tag{3.24}$$

Note that the sequence $(\bar{v}_i^+((\mathbf{1} + \boldsymbol{\beta})\boldsymbol{\omega}, \lambda, \kappa)(\cdot), \bar{v}_i^-((\mathbf{1} + \boldsymbol{\beta})\boldsymbol{\omega}, \lambda, \kappa)(\cdot))_{i\in\mathbb{Z}}$ is the unique solution of $\boldsymbol{v} = \mathcal{F}_{(\mathbf{1}+\boldsymbol{\beta})\boldsymbol{\omega}}(\boldsymbol{v}, \lambda, \kappa)$, see (3.22).

Then one shows that $\check{\bar{\boldsymbol{v}}}_{\boldsymbol{\omega}}$ is the unique solution of

$$\boldsymbol{v} = \check{\mathcal{F}}_{\boldsymbol{\omega}}(\boldsymbol{\beta}, \boldsymbol{v}, \lambda, \kappa), \tag{3.25}$$

a fixed point equation that arises in the same way as (3.22). The mapping $\check{\mathcal{F}}_{\boldsymbol{\omega}}(\cdot, \boldsymbol{v}, \lambda, \kappa)$ is smooth and therefore $\check{\bar{\boldsymbol{v}}}_{\boldsymbol{\omega}}(\boldsymbol{\beta}, \lambda, \kappa)$ depends smoothly on $\boldsymbol{\beta} \in l^\infty_{\mathbb{R}}$. Then, via (3.24), $\bar{\boldsymbol{v}}(\boldsymbol{\omega}, \lambda, \kappa)$ depends smoothly on $\boldsymbol{\omega} \in l^\infty_{\mathbb{R}}$.

In order to derive the addressed fixed point equation (3.25) and to make clear it solutions depend smoothly on $\boldsymbol{\beta}$ one proves similar statements to Lemmata 3.2.4, 3.2.5 and 3.2.6 in case of the rescaled equations. Thereby one make use of [Kno04, Lemma 3.3.3] that states that a function $F : l^\infty \to l^\infty$, $x \mapsto (f^i(x))_{i\in\mathbb{Z}}$ is differentiable in x_0 if

- f^i is differentiable for all $i \in \mathbb{Z}$,
- there is a $K > 0$ such that $\|Df^i(x_0)\| < K$ for all $i \in \mathbb{Z}$
- $Df^i(\cdot)$ are continuous in x_0, uniformly in i.

Indeed we find for all quantities appearing in the lemmata that the i^{th} component of it never depend on the entire sequence $\boldsymbol{\beta}$ but only on β_{i-1} and β_i. So the differentiability of these i^{th} components with respect to $\boldsymbol{\beta}$ reduces to the partial differentiability with respect to β_{i-1} and β_i.

3.3 Existence and Uniqueness of Lin trajectories - the proofs

This section is dedicated to the proof of Theorem 3.2.2. To this end we trace the path we have described in Section 3.2 and prove the Lemmata 3.2.4, 3.2.5 and 3.2.6. In doing so we simply redraw the steps in [Kno04]. The basic difference is that we do this in the context of a heteroclinic chain Γ^κ within a heteroclinic network as presented in Hypothesis (H3.1) instead of a degenerated homoclinic trajectory. Indeed this means a simplification of the considerations in [Kno04].

We start with two technical lemmata.

Lemma 3.3.1 ([Kno04] Lemma 3.2.1)**.** *Let $P^{\pm}_{\kappa_i}$ be the projections associated with the exponential dichotomies of the variational equation (3.8) along $\gamma^{\pm}_{\kappa_i}(\lambda)(\cdot)$ as introduced in (3.17). Then the mappings $P^{\pm}_{\kappa_i}(\cdot,t)$ depend smoothly on λ.*

The second lemma is dedicated to the direct sum decomposition of $\mathbb{R}^n$ we have mentioned in (3.20). To this end we define P_{κ_i} as the spectral projection of $D_1 f(p_{\kappa_i},\lambda)$ associated to the exponential dichotomy of $\dot{x} = D_1 f(p_{\kappa_i},\lambda)x$ on $\mathbb{R}$, that is

$$\mathrm{im} P_{\kappa_i} = T_{p_{\kappa_i}} W^s(p_{\kappa_i}), \qquad \ker P_{\kappa_i} = T_{p_{\kappa_i}} W^u(p_{\kappa_i}). \tag{3.26}$$

Due to Hypotheses (H3.2) and (H3.3) P_{κ_i} is independend of λ.

Lemma 3.3.2 ([VanFie92] Lemma 7)**.** *Assume Hypotheses (H3.1)-(H3.5). There exist constants Ω and c such that for all $|\lambda| < c$ and $\omega > \Omega$ we have*

$$\mathbb{R}^n = im P^{+}_{\kappa_{i-1}}(\lambda,\omega_i) \oplus im P^{-}_{\kappa_i}(\lambda,-\omega_i). \tag{3.27}$$

Moreover, the norm of the projection $\tilde{P}_{\kappa_i}(\lambda,\omega_i)$ defining the decomposition (3.27) is uniformly bounded, meaning that there is a constant $\tilde{M}_{\kappa_i}$ such that $\|\tilde{P}_{\kappa_i}(\lambda,\omega_i)\| \leq 4\tilde{M}_{\kappa_i}$. We stipulate

$$im \tilde{P}_{\kappa_i}(\lambda,\omega_i) = im P^{+}_{\kappa_{i-1}}(\lambda,\omega_i), \qquad ker \tilde{P}_{\kappa_i}(\lambda,\omega_i) = im P^{-}_{\kappa_i}(\lambda,\omega_i). \tag{3.28}$$

Proof. The proof follows along the lines of the proof of Lemma 7 in [VanFie92].
Let $\tilde{M}_{\kappa_i} := \max\{\|P_{\kappa_i}\|, \|id - P_{\kappa_i}\|\} \geq 1$, where P_{κ_i} denotes the spectral projection of $D_1 f(p_{\kappa_i},\lambda)$ defined in (3.26). The projections $P^{-}_{\kappa_i}$ and $P^{+}_{\kappa_{i-1}}$ converge to the projection P_{κ_i}, that is

$$\lim_{t\to-\infty} \|P^{-}_{\kappa_i}(\lambda,t) - P_{\kappa_i}\| = 0, \quad \lim_{t\to\infty} \|P^{+}_{\kappa_{i-1}}(\lambda,t) - (id - P_{\kappa_i})\| = 0. \tag{3.29}$$

Therefore we find c, Ω such that for all $\omega_i > \Omega$ and all $|\lambda| < c$ we have $\|P^{-}_{\kappa_i}(\lambda,-\omega_i) - P_{\kappa_i}\| \leq 1/(4\tilde{M}_{\kappa_i})$ and $\|P^{+}_{\kappa_{i-1}}(\lambda,\omega_i) - (id - P_{\kappa_i})\| \leq 1/(4\tilde{M}_{\kappa_i})$. For each $\omega_i > \Omega$ we define

$$S_{\kappa_i}(\lambda,\omega_i) := P^{-}_{\kappa_i}(\lambda,-\omega_i)P_{\kappa_i} + P^{+}_{\kappa_{i-1}}(\lambda,\omega_i)(id - P_{\kappa_i}). \tag{3.30}$$

This mapping is invertible since

$$S_{\kappa_i}(\lambda,\omega_i) = id - \Big[(id - P^-_{\kappa_i}(\lambda,-\omega_i))P_{\kappa_i} + (id - P^+_{\kappa_{i-1}}(\lambda,\omega_i))(id - P_{\kappa_i})\Big]$$

where

$$\|(id - P^-_{\kappa_i}(\lambda,-\omega_i))P_{\kappa_i}\| + \|(id - P^+_{\kappa_{i-1}}(\lambda,\omega_i))(id - P_{\kappa_i})\| \leq 1/2.$$

Therefore it follows that $\|S_{\kappa_i}(\lambda,\omega_i)\|, \|S_{\kappa_i}(\lambda,\omega_i)^{-1}\| \leq 2$.

Now we define

$$\tilde{P}_{\kappa_i}(\lambda,\omega) := S_{\kappa_i}(\lambda,\omega)(id - P_{\kappa_i})S_{\kappa_i}(\lambda,\omega)^{-1} \tag{3.31}$$

and show that $\tilde{P}_{\kappa_i}$ is the claimed projection. It is obvious that $\tilde{P}^2_{\kappa_i} = \tilde{P}_{\kappa_i}$ so $\tilde{P}_{\kappa_i}$ is clearly a projection. From $\tilde{P}_{\kappa_i}(\lambda,\omega_i)S_{\kappa_i}(\lambda,\omega_i) = S_{\kappa_i}(\lambda,\omega_i)(id - P_{\kappa_i}) = P^+_{\kappa_{i-1}}(\lambda,\omega_i)(id - P_{\kappa_i})$ we conclude that

$$\mathrm{im}\tilde{P}_{\kappa_i}(\lambda,\omega_i) = \mathrm{im}P^+_{\kappa_{i-1}}(\lambda,\omega_i)(id - P_{\kappa_i}) = \mathrm{im}P^+_{\kappa_{i-1}}(\lambda,\omega_i). \tag{3.32}$$

The last equality in (3.32) is due to (3.29) since $P^+_{\kappa_{i-1}}(\lambda,\omega_i)$ is injective on $\mathrm{im}(id - P_{\kappa_i})$ for $\omega_i > \Omega$ sufficiently large and both projections $P^+_{\kappa_{i-1}}(\lambda,\omega_i)$ and $(id - P_{\kappa_i})$ have the same dimension. Analogous we find from $(id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))S_{\kappa_i}(\lambda,\omega_i) = S_{\kappa_i}(\lambda,\omega_i)P_{\kappa_i} = P^-_{\kappa_i}(\lambda,-\omega_i)P_{\kappa_i}$ that

$$\mathrm{im}(id - \tilde{P}_{\kappa_i}(\lambda,\omega_i)) = \mathrm{im}P^-_{\kappa_i}(\lambda,-\omega_i)P_{\kappa_i} = \mathrm{im}P^-_{\kappa_i}(\lambda,-\omega_i).$$

Finally we gain from (3.31)

$$\|\tilde{P}_{\kappa_i}(\lambda,\omega_i)\| \leq 4\|id - P_{\kappa_i}\| \leq 4\tilde{M}_{\kappa_i} \quad \text{and} \quad \|id - \tilde{P}_{\kappa_i}(\lambda,\omega_i)\| \leq 4\|P_{\kappa_i}\| \leq 4\tilde{M}_{\kappa_i}$$

which concludes the proof. □

Remark 3.3.3. *The inverse $S_{\kappa_i}(\lambda,\omega_i)^{-1}$ is given by the Neumann-Series*

$$S^{-1}_{\kappa_i}(\lambda,\omega_i) = \sum_{k=0}^{\infty}\Big[(id - P^-_{\kappa_i}(\lambda,-\omega_i))P_{\kappa_i} + (id - P^+_{\kappa_{i-1}}(\lambda,\omega_i))(id - P_{\kappa_i})\Big]^k. \tag{3.33}$$

Before we begin to prove the Lemmata 3.2.4, 3.2.5 and 3.2.6 from the latest section we want to introduce an abridging notation we often use throughout the following considerations.

Definition 3.3.4. *For any $\boldsymbol{x} = (x_i^+, x_i^-)_{i\in\mathbb{Z}} \in V_{\boldsymbol{\omega}}$ we define*

$$\begin{aligned} x_i^{\pm,s}(t) &:= (id - P^\pm_{\kappa_i}(t))x_i^\pm(t); \\ x_i^{\pm,u}(t) &:= P^\pm_{\kappa_i}(t)x_i^\pm(t). \end{aligned}$$

3.3.1 Proof of Lemma 3.2.4

Now that all necessary prearrangements are made we can focus on the proof of Lin's method. In the first step we solve the boundary value problem ((3.15),(3.13)). But instead of $v_i^+(\omega_{i+1}) - v_{i+1}^-(-\omega_{i+1}) = d_{i+1}(\omega_{i+1},\lambda)$ we introduce $\boldsymbol{a} \in l^\infty_{\mathbb{R}^n}$ to handle this coupling condition. To this end we define for any $\boldsymbol{a}$

$$\begin{aligned} a_i^+(\lambda,\omega_i,\kappa_{i-1}) &:= P^+_{\kappa_{i-1}}(\lambda,\omega_i)a_i; \\ a_i^-(\lambda,-\omega_i,\kappa_i) &:= P^-_{\kappa_i}(\lambda,-\omega_i)a_i. \end{aligned}$$

Then condition (i) in Lemma 3.2.4 tells that

$$\left.\begin{array}{rcl} P^+_{\kappa_{i-1}}(\lambda,\omega_i)v^+_{i-1}(\lambda,\mathbf{g},\boldsymbol{a})(\omega_i) & = & a^+_i(\lambda,\omega_i,\kappa_{i-1}); \\ P^-_{\kappa_i}(\lambda,-\omega_i)v^-_i(\lambda,\mathbf{g},\boldsymbol{a})(-\omega_i) & = & a^-_i(\lambda,-\omega_i,\kappa_i). \end{array}\right\} \tag{3.34}$$

Proof of Lemma 3.2.4. We follow along the lines of the proof of Lemma 2.1.5 and Lemma 3.2.3 in [Kno04]. We start with the proof of the existence and uniqueness of $v^\pm_i$ and hence of $\boldsymbol{v_\omega}$. To solve equation (3.15) we use variation of constant formula

$$\left.\begin{array}{l} v^+_i(t) = \Phi^+_{\kappa_i}(\lambda)(t,0)v^+_i(0) + \int\limits_0^t \Phi^+_{\kappa_i}(\lambda)(t,s)g^+_i(s)ds, \\ v^-_i(t) = \Phi^-_{\kappa_i}(\lambda)(t,0)v^-_i(0) - \int\limits_t^0 \Phi^-_{\kappa_i}(\lambda)(t,s)g^-_i(s)ds. \end{array}\right\} \tag{3.35}$$

Now we put t equal to ω_{i+1} in the equation for v^+_i, while we set $t = -\omega_i$ in the equation for v^-_i. Then we multiply these equations with $\Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})$ or $\Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)$ and apply $P^+_{\kappa_i}(0)$ or $P^-_{\kappa_i}(0)$, respectively. This leads to

$$\left.\begin{array}{l} P^+_{\kappa_i}(\lambda,0)v^+_i(0) = \Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})\underbrace{P^+_{\kappa_i}(\lambda,\omega_{i+1})v^+_i(\omega_{i+1})}_{a^+_{i+1}} - \int\limits_0^{\omega_{i+1}} \Phi^+_{\kappa_i}(\lambda)(0,s)P^+_{\kappa_i}(\lambda,s)g^+_i(s)ds, \\ P^-_{\kappa_i}(\lambda,0)v^-_i(0) = \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)\underbrace{P^-_{\kappa_i}(\lambda,-\omega_i)v^-_i(-\omega_i)}_{a^-_i} + \int\limits_{-\omega_i}^0 \Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)g^-_i(s)ds. \end{array}\right\} \tag{3.36}$$

In view of the statements (ii) and (iii) in Lemma 3.2.4 we set

$$v^+_i(0) = w^+_i + w^-_i + z^+_i \quad \text{and} \quad v^-_i(0) = w^+_i + w^-_i + z^-_i$$

with $w^+_i + w^-_i \in W^+_{\kappa_i} \oplus W^-_{\kappa_i}$ and $z^+_i, z^-_i \in Z_{\kappa_i}$. Plugging this into (3.36) we find that the left-hand side of (3.36) can be considered as a linear mapping L depending on λ from $l^\infty_{((W^+\oplus W^-)\times Z\times Z)}$ into $l^\infty_{(W^-\oplus Z)\times(W^+\oplus Z)}$. Indeed these equations are decoupled over i, so let L_i be the part of L that acts on the i^{th} equation.

If we accept for a moment, that L_i and thus L is bijective, then we can solve (3.36) for $(v^+_i(0), v^-_i(0))$ depending on $(\lambda, g^+_i, g^-_i, a^+_{i+1}, a^-_i)$. Putting this into (3.35) gives our solution $\boldsymbol{v_\omega}$ satisfying (i)-(iii). We remark that the $v^\pm_i$ do not depend on the entire sequence $\boldsymbol{a}$ and κ but only on a^+_{i+1}, a^-_i and κ_i:

$$v^\pm_i = v^\pm_{\boldsymbol{\omega},i}(\lambda,\kappa_i,\mathbf{g},(a^+_{i+1},a^-_i)). \tag{3.37}$$

In [Kno04] the bijectivity of L_i is formally proven. However, for our purposes it is useful to solve (3.36) explicitly for $v^+_i(0)$ and $v^-_i(0)$, as we will see later, when estimating the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$. To this end we introduce a projection F_{κ_i} with

$$\mathrm{im}F_{\kappa_i} = U_{\kappa_i} \oplus Z_{\kappa_i}, \qquad \ker F_{\kappa_i} = W^+_{\kappa_i} \oplus W^-_{\kappa_i}. \tag{3.38}$$

Note that F_{κ_i} is independent of λ. We further make use of Hypothesis (H3.4) and the corresponding Remark 3.1.2, which states that $W^+_{\kappa_i} \subset T_{\gamma^+_{\kappa_i}(0)}W^s(p)$ and $W^-_{\kappa_i} \subset T_{\gamma^-_{\kappa_i}(0)}W^s(p)$. This together with the definition (3.17) of $P^{\pm}_{\kappa_i}$ gives

$$P^+_{\kappa_i}(\lambda,0)v^+_i(0) = w^-_i + z^+_i \quad \text{and} \quad P^-_{\kappa_i}(\lambda,0)v^-_i(0) = w^+_i + z^-_i$$

and finally

$$(id - F_{\kappa_i})P^-_{\kappa_i}(\lambda,0)v^-_i(0) = w^+_i \quad \text{and} \quad (id - F_{\kappa_i})P^+_{\kappa_i}(\lambda,0)v^+_i(0) = w^-_i.$$

Summarizing this leads to

$$\left.\begin{aligned} v^+_i(0) &= P^+_{\kappa_i}(\lambda,0)v^+_i(0) + (id - F_{\kappa_i})P^-_{\kappa_i}(\lambda,0)v^-_i(0), \\ v^-_i(0) &= P^-_{\kappa_i}(\lambda,0)v^-_i(0) + (id - F_{\kappa_i})P^+_{\kappa_i}(\lambda,0)v^+_i(0). \end{aligned}\right\} \tag{3.39}$$

Proof of the smoothness: The linearity of $\boldsymbol{v_\omega}$ on $(\mathbf{g},\boldsymbol{a})$ is obvious by the construction. Hence the differentiability of $\boldsymbol{v_\omega}$ with respect to $(\mathbf{g},\boldsymbol{a})$ follows from the estimate (3.45) below.
All L_i depend smoothly on λ. So the differentiability of $v^{\pm}_i(\ldots)(0)$ with respect to λ follows from the differentiability of L_i and $P^{\pm}_{\kappa_i}$ with respect to λ, see Lemma 3.3.1. Hence the differentiability of $v^{\pm}_i(\ldots)(t)$ follows from (3.35).

Proof of estimate (3.19): From equation (3.36) we find by exploiting exponential dichotomies (3.16) that there are constants K and $\beta^+_{\kappa_i}$, $\alpha^-_{\kappa_i}$ allowing the estimate

$$\left.\begin{aligned} \|P^+_{\kappa_i}(\lambda,t))v^+_i(t)\| &\le Ke^{-\beta^+_{\kappa_i}(\omega_{i+1}-t)}\|a^+_{i+1}\| + K\int_t^{\omega_{i+1}} e^{\beta^+_{\kappa_i}(t-s)}\|g^{+,u}_i(s)\|ds; \\ \|P^-_{\kappa_i}(\lambda,t)v^-_i(t)\| &\le Ke^{-\alpha^-_{\kappa_i}(\omega_i+t)}\|a^-_i\| + K\int_{-\omega_i}^{t} e^{-\alpha^-_{\kappa_i}(t-s)}\|g^{-,u}_i(s)\|ds. \end{aligned}\right\} \tag{3.40}$$

Now, $(id - P^+_{\kappa_i}(\lambda,\omega_{i+1}))v^+_i(\omega_{i+1})$ and $(id - P^-_{\kappa_i}(\lambda,\omega_i))v^-_i(\omega_i)$ can be written by means of the variation of constants formula (3.35) by applying the projections $(id - P^{\pm}_{\kappa_i})$

$$\left.\begin{aligned} (id - P^+_{\kappa_i}(t))v^+_i(t) &= \Phi^+_{\kappa_i}(\lambda)(t,0)(id - P^+_{\kappa_i}(0))v^+_i(0) + \int_0^t \Phi^+_{\kappa_i}(\lambda)(t,s)(id - P^+_{\kappa_i}(s))g^+_i(s)ds, \\ (id - P^-_{\kappa_i}(t))v^-_i(t) &= \Phi^-_{\kappa_i}(\lambda)(t,0)(id - P^-_{\kappa_i}(0))v^-_i(0) - \int_t^0 \Phi^-_{\kappa_i}(\lambda)(t,s)(id - P^-_{\kappa_i}(s))g^-_i(s)ds. \end{aligned}\right\} \tag{3.41}$$

Again exploiting exponential dichotomies (3.16) by additionally using the decomposition of $v^{\pm}_{\kappa_i}(0)$ given in equation (3.39) combined with the above estimate (3.40) of $P^{\pm}_{\kappa_i}v^{\pm}_{\kappa_i}$ at $t=0$ we find the estimate

$$\left.\begin{aligned} \|(id - P^+_{\kappa_i}(\lambda,t))v^+_i(t)\| &\le KLe^{-\alpha^+_{\kappa_i}t}\|P^-_{\kappa_i}(\lambda,0)v^-_i(0)\| + K\int_0^t e^{-\alpha^+_{\kappa_i}(t-s)}\|g^{+,s}_i(s)\|ds \\ &\le KLe^{-\alpha^+_{\kappa_i}t}\left[Ke^{-\alpha^-_{\kappa_i}\omega_i}\|a^-_i\| + K\int_{-\omega_i}^{0} e^{\alpha^-_{\kappa_i}s}\|g^{-,u}_i(s)\|ds\right] \\ &\quad + K\int_0^t e^{-\alpha^+_{\kappa_i}(t-s)}\|g^{+,s}_i(s)\|ds \\ \|(id - P^-_{\kappa_i}(\lambda,t))v^-_i(t)\| &\le KLe^{\beta^-_{\kappa_i}t}\|P^+_{\kappa_i}(\lambda,0)v^+_i(0)\| + K\int_t^0 e^{\beta^-_{\kappa_i}(t-s)}\|g^{-,s}_i(s)\|ds \\ &\le KLe^{\beta^-_{\kappa_i}t}\left[Ke^{-\beta^+_{\kappa_i}\omega_{i+1}}\|a^+_{i+1}\| + K\int_0^{\omega_{i+1}} e^{-\beta^+_{\kappa_i}s}\|g^{+,u}_i(s)\|ds\right] \\ &\quad + K\int_t^0 e^{\beta^-_{\kappa_i}(t-s)}\|g^{-,s}_i(s)\|ds \end{aligned}\right\} \tag{3.42}$$

We will come back to the estimates (3.40) and (3.42) when it comes to estimating the jump $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$. In order to go on with the proof of Lin's method it suffices to use the following rougher estimates

$$\left.\begin{array}{rcl} \|P^+_{\kappa_i}(\lambda)v_i^+\| & \leq & K\|a^+_{i+1}\| + M\|g_i^{+,u}\|; \\ \|P^-_{\kappa_i}(\lambda)v_i^-\| & \leq & K\|a^-_i\| + M\|g_i^{-,u}\|; \end{array}\right\} \tag{3.43}$$

which follow from (3.40) and

$$\left.\begin{array}{rcl} \|(id - P^+_{\kappa_i}(\lambda))v_i^+\| & \leq & KL\left[Ke^{-\alpha^-_{\kappa_i}\omega_i}\|a^-_i\| + M\|g_i^{-,u}\|\right] + M\|g_i^{+,s}\|; \\ \|(id - P^-_{\kappa_i}(\lambda))v_i^-\| & \leq & KL\left[Ke^{-\beta^+_{\kappa_i}\omega_{i+1}}\|a^+_{i+1}\| + M\|g_i^{+,u}\|\right] + M\|g_i^{-,s}\|; \end{array}\right\} \tag{3.44}$$

which follow from (3.42). Combining (3.43) and (3.44) finally leads to

$$\|v_i^+\|,\ \|v_i^-\| \leq \tilde{K}(\|a_i^-\| + \|a^+_{i+1}\|) + \tilde{M}(\|g_i^-\| + \|g_i^+\|). \tag{3.45}$$

Further we find from (3.40) and (3.42) the estimate

$$\|v_i^{\pm}(0)\| \leq \tilde{K}e^{\alpha\Omega}(\|a_i^-\| + \|a^+_{i+1}\|) + \tilde{M}(\|g_i^-\| + \|g_i^+\|) \tag{3.46}$$

for some $\alpha < 0$. □

3.3.2 Proof of Lemma 3.2.5

In the following we use the device $\boldsymbol{a}$ to integrate the coupling condition $v_i^+(\omega_{i+1}) - v^-_{i+1}(-\omega_{i+1}) = d_{i+1}(\omega_{i+1}, \lambda)$.

Proof of Lemma 3.2.5. We follow along the lines of the proof of Lemma 2.1.6 and Lemma 3.2.5 in [Kno04]. We show that for a given $\boldsymbol{d}$ there is an $\boldsymbol{a}$ such that $\boldsymbol{v}_{\boldsymbol{\omega}}(\lambda, \kappa, \mathbf{g}, \boldsymbol{a})$ satisfies

$$v^+_{i-1}(\lambda, \kappa, \mathbf{g}, \boldsymbol{a})(\omega_i) - v_i^-(\lambda, \kappa, \mathbf{g}, \boldsymbol{a})(-\omega_i) = d_i, \quad i \in \mathbb{Z}. \tag{3.47}$$

Therefore $\boldsymbol{a}$ is defined by a system consisting of (3.47) and (3.34). Combining these equations leads to

$$a_i^+ - a_i^- = d_i - (id - P^+_{\kappa_{i-1}}(\lambda, \omega_i))v^+_{i-1}(\ldots)(\omega_i) + (id - P^-_{\kappa_i}(\lambda, -\omega_i))v_i^-(\ldots)(-\omega_i).$$

Now, using projection $\tilde{P}_{\kappa_i}$ defined in Lemma 3.3.2 we get

$$\left.\begin{array}{l} a_i^+ = \tilde{P}_{\kappa_i}\left(d_i - (id - P^+_{\kappa_{i-1}}(\lambda, \omega_i))v^+_{i-1}(\ldots)(\omega_i) + (id - P^-_{\kappa_i}(\lambda, -\omega_i))v_i^-(\ldots)(-\omega_i)\right), \\ a_i^- = -(id - \tilde{P}_{\kappa_i})\left(d_i - (id - P^+_{\kappa_{i-1}}(\lambda, \omega_i))v^+_{i-1}(\ldots)(\omega_i) + (id - P^-_{\kappa_i}(\lambda, -\omega_i))v_i^-(\ldots)(-\omega_i)\right) \end{array}\right\} \tag{3.48}$$

which yields the fixed point equation for $\boldsymbol{a}$:

$$a_i = (2\tilde{P}_{\kappa_i} - id)\left(d_i - (id - P^+_{\kappa_{i-1}}(\lambda, \omega_i))v^+_{i-1}(\ldots, \boldsymbol{a})(\omega_i) + (id - P^-_{\kappa_i}(\lambda, -\omega_i))v_i^-(\ldots, \boldsymbol{a})(-\omega_i)\right). \tag{3.49}$$

Note that $(id - P^+_{\kappa_{i-1}})v^+_{i-1} + (id - P^-_{\kappa_i})v_i^-$ depend linearly on $(\mathbf{g}, \boldsymbol{a})$ - cf. Lemma 3.2.4. Hence the fixed point equation (3.49) can be rewritten in the form

$$\boldsymbol{a} = L_1(\lambda,\kappa)\boldsymbol{a} + L_2(\lambda,\kappa)\mathbf{g} + L_3(\lambda,\kappa)\boldsymbol{d} \tag{3.50}$$

where $L_j(\lambda,\kappa)(\cdot)$, $j = 1,2,3$ are linear operators depending on (λ,κ). The estimate (3.44) ensures, that Ω can be chosen large enough such that for sufficiently small λ the linear operator $(id - L_1)$ is invertible. Thus (3.50) can be solved for $\boldsymbol{a} = \hat{\boldsymbol{a}}_{\boldsymbol{\omega}}(\lambda,\kappa,\mathbf{g},\boldsymbol{d})$ and the mapping $\hat{\boldsymbol{a}}_{\boldsymbol{\omega}}$ depends linearly on $(\mathbf{g},\boldsymbol{d})$. Altogether we have

$$\hat{\boldsymbol{v}}_{\boldsymbol{\omega}}(\lambda,\kappa,\mathbf{g},\boldsymbol{d}) = \boldsymbol{v}_{\boldsymbol{\omega}}(\lambda,\kappa,\mathbf{g},\hat{\boldsymbol{a}}_{\boldsymbol{\omega}}(\lambda,\kappa,\mathbf{g},\boldsymbol{d})). \tag{3.51}$$

Due to (3.48) we find that $a_i^{\pm}$ do not depend on the entire sequence $\boldsymbol{d}$ and κ but only on d_i and κ_{i-1},κ_i:

$$a_i^{\pm} = \hat{a}^{\pm}_{\boldsymbol{\omega},i}(\lambda,(\kappa_{i-1},\kappa_i),\mathbf{g},d_i). \tag{3.52}$$

Proof of the smoothness: The smoothness of $\hat{\boldsymbol{v}}_{\boldsymbol{\omega}}$ with respect to $(\mathbf{g},\boldsymbol{d})$ again follows from the linear dependence on these variables and estimate (3.21), which we prove below. The differentiability of $\hat{\boldsymbol{a}}_{\boldsymbol{\omega}}$ with respect to λ is obvious and with the result of Lemma 3.2.4 it passes on to $\hat{\boldsymbol{v}}_{\boldsymbol{\omega}}$.

Proof of estimate (3.21): We have shown that $\hat{v}_i^{\pm}(\lambda,\kappa,\mathbf{g},\boldsymbol{d}) = v_i^{\pm}(\lambda,\kappa,\mathbf{g},\hat{\boldsymbol{a}}_{\boldsymbol{\omega}}(\lambda,\kappa,\mathbf{g},\boldsymbol{d}))$. Therefore estimate (3.19) provides

$$\|\hat{\boldsymbol{v}}_{\boldsymbol{\omega}}\|_{V_{\boldsymbol{\omega}}} \le \hat{C}_1(\|\hat{\boldsymbol{a}}_{\boldsymbol{\omega}}\|_{l^\infty_{\mathbb{R}^n}} + \|\mathbf{g}\|_{V_{\boldsymbol{\omega}}}).$$

Further, from equation (3.50) we gain the estimate $\|\hat{\boldsymbol{a}}_{\boldsymbol{\omega}}\|_{l^\infty_{\mathbb{R}^n}} \le \hat{C}_2(\|\boldsymbol{d}\|_{l^\infty_{\mathbb{R}^n}} + \|\mathbf{g}\|_{V_{\boldsymbol{\omega}}})$ which finally leads to the claimed estimate (3.21). □

3.3.3 Proof of Lemma 3.2.6

Eventually we can deal with the nonlinear boundary value problem ((3.12),(3.13)). To this end we first define the following Nemyzki operators

Definition 3.3.5. *Let*

$$\begin{aligned} H_i^+ &: C([0,\omega_{i+1}],\mathbb{R}^n)\times\mathbb{R}^d\times\Sigma_{\boldsymbol{C}} \to C([0,\omega_{i+1}],\mathbb{R}^n),\\ H_i^- &: C([-\omega_i,0],\mathbb{R}^n)\times\mathbb{R}^d\times\Sigma_{\boldsymbol{C}} \to C([-\omega_i,0],\mathbb{R}^n), \end{aligned}$$

be the Nemyzki operators where

$$H_i^{\pm}(v,\lambda,\kappa)(t) := h^{\pm}_{\kappa_i}(t,v(t),\lambda).$$

Further we set $\boldsymbol{\mathcal{H}} := (H_i^+,H_i^-)_{i\in\mathbb{Z}}$. *More precisely*

$$\begin{array}{rccc} \boldsymbol{\mathcal{H}}: & V_{\boldsymbol{\omega}}\times\mathbb{R}^d\times\Sigma_{\boldsymbol{C}} & \to & V_{\boldsymbol{\omega}} \\ & (\boldsymbol{v},\lambda,\kappa) & \mapsto & (H_i^+(v_i^+,\lambda,\kappa),H_i^-(v_i^-,\lambda,\kappa))_{i\in\mathbb{Z}}. \end{array}$$

Lemma 3.3.6. H_i^+ *and* H_i^- *are smooth mappings in* (v,λ). *Moreover*

$$\left(D_1 H_i^{\pm}(v,\lambda,\kappa)w\right)(t) := D_2 h^{\pm}_{\kappa_i}(t,v(t),\lambda)w(t).$$

Proof. The proof is based on the proof of Lemma 3.2.7 in [Kno04]. We will show the proof exemplarily for H_i^+. Invoking the mean-value-theorem gives

$$\begin{aligned} &\|H_i^+(v+w,\lambda,\kappa) - H_i^+(v,\lambda,\kappa) - D_2 h_{\kappa_i}^+(\cdot, v(\cdot),\lambda)w(\cdot)\|\frac{1}{\|w\|} \\ = &\sup_{t\in[0,\omega_{i+1}]} \|h_{\kappa_i}^+(t,(v+w)(t),\lambda) - h_{\kappa_i}^+(t,v(t),\lambda) - D_2 h_{\kappa_i}^+(t,v(t),\lambda)w(t)\|\frac{1}{\|w\|} \\ \leq &\sup_{t\in[0,\omega_{i+1}]} \int_0^1 \|D_2 h_{\kappa_i}^+(t,v(t)+\tau w(t),\lambda) - D_2 h_{\kappa_i}^+(t,v(t),\lambda)\| \to 0 \end{aligned}$$

as $\|w\|$ tends to zero. This proves the differentiability with respect to v. Since $D_2 h_{\kappa_i}^+$ is continuous with respect to v also $D_1 H_i^\pm$ is continuous with respect to v. The existence of higher derivatives can be proven analogously. The differentiability with respect to λ follows from the differentiability of $h_{\kappa_i}^\pm$ with respect to λ. □

The smoothness of $H_i^\pm$ passes on to $\boldsymbol{\mathcal{H}}$ because the $H_i^\pm$ are defined by means of the same $h_{\kappa_i}^\pm$ and depend only on v_i and not on the entire sequence $\boldsymbol{v}$. Hence we find $D_1\boldsymbol{\mathcal{H}} = (D_1 H_i^+, D_1 H_i^-)_{i\in\mathbb{Z}}$, where

$$D_1\boldsymbol{\mathcal{H}}(\boldsymbol{v},\lambda,\kappa)\boldsymbol{w} = (D_1 H_i^+(v_i^+(t),\lambda,\kappa)w_i^+(t), D_1 H_i^-(v_i^-(t),\lambda,\kappa)w_i^-(t))_{i\in\mathbb{Z}}.$$

Then $D_1\boldsymbol{\mathcal{H}}$ is continuous with respect to $\boldsymbol{v}$ since for all $i \in \mathbb{Z}$ $D_1 H_i^\pm$ are continuous with respect to $v_i^\pm$.

Now, we can rewrite the boundary value problem ((3.12),(3.13)) into the fixed point problem

$$\boldsymbol{v} = \hat{\boldsymbol{v}}_{\boldsymbol{\omega}}(\lambda, \boldsymbol{\mathcal{H}}(\boldsymbol{v},\lambda,\kappa), \boldsymbol{d}(\boldsymbol{\omega},\lambda,\kappa)) =: \mathcal{F}_{\boldsymbol{\omega}}(\boldsymbol{v},\lambda,\kappa). \tag{3.53}$$

The mapping $\boldsymbol{d}$ is smooth in λ. Together with the smoothness of $\hat{\boldsymbol{v}}_{\boldsymbol{\omega}}$ and $\boldsymbol{\mathcal{H}}$ it follows that $\mathcal{F}_{\boldsymbol{\omega}}$ is smooth in $(\boldsymbol{v},\lambda)$.

Proof of Lemma 3.2.6. We follow along the lines of the proof of Lemma 2.1.7 and Lemma 3.2.10 in [Kno04]. The proof will be given by means of the Banach fixed point theorem. First we show that there is a $\mathcal{F}_\omega(\cdot,\lambda,\kappa)$ invariant closed neighbourhood of $0 \in V_{\boldsymbol{\omega}}$. Due to estimate (3.21) we find as an immediate consequence that

$$\|\mathcal{F}_\omega\|_{V_{\boldsymbol{\omega}}} \leq \hat{C}(\|\boldsymbol{d}\|_{l^\infty_{\mathbb{R}^n}} + \|\boldsymbol{\mathcal{H}}\|_{V_{\boldsymbol{\omega}}}). \tag{3.54}$$

From the definition of $h_{\kappa_i}^\pm$, cf. (3.14), we see that $\boldsymbol{\mathcal{H}}(0,\lambda,\kappa) = 0$. Therefore we obtain an estimate of $\|\boldsymbol{\mathcal{H}}\|_{V_\omega}$ by using the mean value theorem:

$$\|\boldsymbol{\mathcal{H}}(\boldsymbol{v},\lambda,\kappa)\|_{V_{\boldsymbol{\omega}}} \leq \int_0^1 \|D_1\boldsymbol{\mathcal{H}}(\tau\boldsymbol{v},\lambda,\kappa)\|_{V_{\boldsymbol{\omega}}} d\tau \|\boldsymbol{v}\|_{V_{\boldsymbol{\omega}}}.$$

Looking at Lemma 3.3.6 and the property $D_2 h_{\kappa_i}^\pm(t,0,\lambda) \equiv 0$ we find that $D_1\boldsymbol{\mathcal{H}}(0,\lambda,\kappa) = 0$. Hence there is an ε such that for $\|\boldsymbol{v}\|_{V_{\boldsymbol{\omega}}}$, $\|\lambda\| < \varepsilon$ it holds

$$\|D_1\boldsymbol{\mathcal{H}}(\tau\boldsymbol{v},\lambda,\kappa)\|_{V_{\boldsymbol{\omega}}} \leq (2\hat{C})^{-1} \tag{3.55}$$

and thus

$$\|\boldsymbol{\mathcal{H}}(\boldsymbol{v},\lambda,\kappa)\|_{V_{\boldsymbol{\omega}}} \leq \frac{\varepsilon}{2\hat{C}}.$$

Now, let Ω be the constant corresponding to Lemma 3.3.2. Then there is an $\tilde{\Omega} > \Omega$ such that for all $\omega > \tilde{\Omega}$ we have $\|\gamma_{\kappa_i}(\omega) - \gamma_{\kappa_{i+1}}(-\omega)\| \leq \varepsilon/(6\hat{C})$. By construction of $\gamma_{\kappa_i}^\pm$ we know that $\gamma_{\kappa_i}^\pm(\lambda)(t) \to \gamma_{\kappa_i}(t)$ uniformly in $t \in \mathbb{R}^\pm$ as $\lambda \to 0$. Therefore we find an $\bar{\varepsilon} < \varepsilon$ such that for all $\|\lambda\| < \bar{\varepsilon}$ it holds

$\|\gamma^{\pm}_{\kappa_i}(\lambda)(\pm\omega) - \gamma_{\kappa_i}(\pm\omega)\| \le \varepsilon/(6\hat{C})$. Hence for $\|\lambda\| < \bar{\varepsilon}$ and $\boldsymbol{\omega}$ with $\omega_i > \tilde{\Omega}$ we find with (3.13) that the norm of $\boldsymbol{d}$ can be estimated with

$$\|\boldsymbol{d}\|_{l^{\infty}_{\mathbb{R}^n}} \le \frac{\varepsilon}{2\hat{C}}.$$

Altogether (3.54) says that for $\|\lambda\| < \bar{\varepsilon} =: c$ and $\boldsymbol{\omega}$ with $\omega_i > \tilde{\Omega}$ the mapping $\mathcal{F}_{\boldsymbol{\omega}}$ leaves the ball $B[0,\varepsilon] \subset V_{\boldsymbol{\omega}}$ invariant.

Further for these variables we have, cf. (3.53) in combination with (3.55) and (3.21)

$$\|D_1\mathcal{F}_{\boldsymbol{\omega}}(\boldsymbol{v},\lambda,\kappa)\|_{V_{\boldsymbol{\omega}}} \le \|D_2\hat{\boldsymbol{v}}_{\boldsymbol{\omega}}(\ldots)\|_{V_{\boldsymbol{\omega}}} \cdot \|D_1\boldsymbol{\mathcal{H}}(\ldots)\|_{V_{\boldsymbol{\omega}}} < \hat{C}\frac{1}{2\hat{C}} = \frac{1}{2}.$$

So we see by invoking the mean value theorem that $\mathcal{F}_{\boldsymbol{\omega}}$ is a contraction on $B[0,\varepsilon]$. Hence the existence and uniqueness follows by the Banach fixed point theorem. □

This concludes the proof of Theorem 3.2.2 about the existence and uniqueness of the Lin trajectories.

3.4 The jump $\Xi(\boldsymbol{\omega},\lambda,\kappa)$

Summarising, Lin's method provides for a given sequence κ and a sequence $\boldsymbol{\omega} := (\omega_i)_{i\in\mathbb{Z}}$ of sufficiently large transition times a unique Lin trajectory $(X_i(\boldsymbol{\omega},\lambda,\kappa))_{i\in\mathbb{Z}}$ with

$$X_i(t) := \begin{cases} x^+_{i-1}(t), & t \in [0,\omega_i] \\ x^-_i(t-2\omega_i), & t \in [\omega_i, 2\omega_i] \end{cases}, \tag{3.56}$$

that is a piecewise continuous solutions of (3.1) which shadows the pathway of the heteroclinic chain Γ^{κ}. Now, to gain an actual, that is a continuous solution of (3.1) each single jump

$$\Xi_i(\boldsymbol{\omega},\lambda,\kappa) := x^+_i(\boldsymbol{\omega},\lambda,\kappa)(0) - x^-_i(\boldsymbol{\omega},\lambda,\kappa)(0),$$

has to be equal to zero. Therefore, in order to obtain statements about the nonwandering dynamics in the neighbourhood of the heteroclinic network, one has to discuss the solvability of the system of determination equations $\boldsymbol{\Xi}(\boldsymbol{\omega},\lambda,\kappa) = 0$ in dependence on the sequences κ and the sign of λ.

Thanks to the idea of Sandstede, cf. [San93], we know that $\boldsymbol{\Xi}$ consists of two components: $\xi^{\infty}_{\kappa_i}$ which only depend on λ and $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$ that decreases exponentially when $\boldsymbol{\omega}$ tends to infinity. This is due to the partition (3.11) of $x^{\pm}_i$ into $\gamma^{\pm}_{\kappa_i}$ and $v^{\pm}_i$ which yields

$$\left.\begin{aligned} \Xi_i(\boldsymbol{\omega},\lambda,\kappa) &= \underbrace{\gamma^+_{\kappa_i}(\lambda)(0) - \gamma^-_{\kappa_i}(\lambda)(0)} + \underbrace{v^+_i(\boldsymbol{\omega},\lambda,\kappa)(0) - v^-_i(\boldsymbol{\omega},\lambda,\kappa)(0)} \\ &= \qquad \xi^{\infty}_{\kappa_i}(\lambda) \qquad + \qquad \xi_i(\boldsymbol{\omega},\lambda,\kappa). \end{aligned}\right\} \tag{3.57}$$

Within this section we want to take a first look at these two components, as far as it is possible in the broader context of heteroclinic chains. First we consider $\xi^{\infty}_{\kappa_i}(\lambda)$ in Section 3.4.1, before we are looking at the estimates of the jump ξ_i, $i \in \mathbb{Z}$. To this end we derive a suitable representation of $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$ in Section 3.4.2, as it can be found in [Kno04]. Afterwards we present estimates for the individual terms appearing in the representation. Explicit expressions of some of the terms we will present in Section 4 in case of G-symmetric homoclinic networks.

3.4.1 Melnikov integral - $\xi_{\kappa_i}^{\infty}(\lambda)$

Lemma 3.1.1 provides solutions $\gamma_{\kappa_i}^{\pm}(\lambda)(\cdot)$ of (3.1) close to γ_{κ_i} situated within the stable manifold $W^s(p_{i+1})$ or the unstable manifold $W^u(p_i)$, respectively. Since $\xi_{\kappa_i}^{\infty}(\lambda)$ is defined as

$$\xi_{\kappa_i}^{\infty}(\lambda) := \gamma_{\kappa_i}^{+}(\lambda)(0) - \gamma_{\kappa_i}^{-}(\lambda)(0)$$

it measures the distance of the stable and unstable manifold and is due to Lemma 3.1.1 situated within the one-dimensional subspace Z_i. In the following we show, how this information is used in solving the determination equations $\boldsymbol{\Xi}(\boldsymbol{\omega}, \lambda, \kappa) = 0$.

To this end we consider $\dot{x} = f(x, \lambda)$ having for $\lambda = \lambda_0$ a single heteroklinic trajectory γ connecting two hyperbolic equilibria p^- and p^+. In accordance with Hypothesis (H3.1) we demand

(P3.2).

(i) $\lim\limits_{t\to\infty} \gamma(t) = p^+$ *and* $\lim\limits_{t\to-\infty} \gamma(t) = p^-$,

(ii) $\dim(T_{\gamma(0)}W^u(p^-) \cap T_{\gamma(0)}W^s(p^+)) = 1$,

(iii) $\dim T_{\gamma(0)}W^u(p^-) + \dim T_{\gamma(0)}W^s(p^+) = n$.

Due to (iii) and (ii) the intersection of the unstable manifold $W^u(p^-)$ of the equilibrium p^- and the stable manifold $W^s(p^+)$ of p^+ cannot be transversal. Indeed we have

$$\dim \left(T_{\gamma(0)}W^u(p^-) + T_{\gamma(0)}W^s(p^+)\right) = n - 1$$

With $Z := (T_{\gamma(0)}W^s(p^+) + T_{\gamma(0)}W^u(p^-))^{\perp}$ we find an one-dimensional subspace that not only complement the tangent spaces of the manifolds to the whole phase-space. Additionally we have for all $\psi \in Z$ that $\Psi(t,0)\psi$ is a bounded solution of $\dot{x} = -[D_1 f(\gamma(t), \lambda_0)]^T x$ for all $t \in \mathbb{R}$, cf. Lemma 2.2.6. Here again $\Psi(\cdot,\cdot)$ denotes the transition matrix of the adjoint variational equation $\dot{x} = -[D_1 f(\gamma(t), \lambda_0)]^T x$.

Analogously to the considerations in Section 2.6 we then generically find the heteroclinic trajectory to exist at an isolated parameter value $\lambda = \lambda_0$ within a family of differential equations $\dot{x} = f(x, \lambda)$, with $\lambda \in \mathbb{R}$. This again can be explained by the transversal intersection of the manifolds $\mathcal{W}^s := \bigcup\limits_{\lambda} W^s(p^+(\lambda)) \times \{\lambda\}$ and $\mathcal{W}^u := \bigcup\limits_{\lambda} W^u(p^-(\lambda)) \times \{\lambda\}$, cf. (2.55), of the extended differential equation

$$\left.\begin{array}{rcl} \dot{x} & = & f(x, \lambda) \\ \dot{\lambda} & = & 0 \end{array}\right\} \tag{3.58}$$

within the product space $\mathbb{R}^n \times \mathbb{R}$. Here $p^+(\lambda)$ and $p^-(\lambda)$ denote the families of saddle points for λ close to λ_0 with $p^{\pm}(\lambda_0) = p^{\pm}$. Recall Figure 2.2 in case of a homoclinic trajectory in $\mathbb{R}^2 \times \mathbb{R}$.

Hence for the generic case we demand:

(P3.3). *The heteroclinic trajectory γ splits up with non-zero speed.*

As well as for homoclinic trajectories we mean by this the existence of a scalar split function $d : \mathbb{R} \to \mathbb{R}$, $\lambda \mapsto d(\lambda)$ that measures the displacement of the stable and unstable manifolds $W^s(p^+(\lambda))$ and $W^u(p^-(\lambda))$ near γ for λ close to λ_0, that satisfies $d'(\lambda)|_{\lambda_0} \neq 0$, cf. Definition 2.6.3.

Indeed an appropriate split function d can be defined by

$$d(\lambda) := \langle \psi, \gamma^+(\lambda)(0) - \gamma^-(\lambda)(0) \rangle \tag{3.59}$$

where $\psi \in Z$, $\|\psi\| = 1$ and $\gamma^\pm(\lambda)(\cdot)$ are the unique solutions of $\dot{x} = f(x, \lambda)$ which are close to γ, satisfying

- $\lim_{t \to \pm\infty} \gamma^\pm(\lambda)(t) = p^\pm(\lambda)$,
- $\gamma^\pm(\lambda)(0) \in \gamma(0) + [f(\gamma(0), 0)]^\perp$,
- $\gamma^+(\lambda)(0) - \gamma^-(\lambda)(0) \in Z$.

Note, that with this we are in the context of Lemma 3.1.1. Hence we find $\xi^\infty(\lambda) = d(\lambda)\psi$.

The first derivative of the split function (3.59) in $\lambda = \lambda_0$, $d'(\lambda)|_{\lambda_0}$, is known as the Melnikov integral $\mathcal{M}$, cf. [HomSan10, Kr11], and has the following form, cf. [GuHo83, Kuz04],

$$d'(\lambda)|_{\lambda_0} = \mathcal{M} := \int_{-\infty}^{\infty} \langle \Psi(\lambda_0)(t, 0)\psi, D_\lambda f(\gamma(t), \lambda_0) \rangle dt. \tag{3.60}$$

Then the following theorem applies.

Theorem 3.4.1 ([Kuz04], p.230)**.** *Let γ be a heteroclinic trajectory existing at an isolated parameter value $\lambda = \lambda_0 \in \mathbb{R}$ within a family of differential equations $\dot{x} = f(x, \lambda)$ that satisfies (P3.2). The manifolds $\mathcal{W}^s$ and $\mathcal{W}^u$ of the extended differential equation (3.58) intersect transversally if and only if the Melnikov integral (3.60) is different from zero.*

Indeed the assertion in [Kuz04] was only made in context of homoclinic trajectories but it also holds true for heteroclinic trajectories.

So, if we assume a heteroclinic trajectory γ to satisfy Properties (P3.2) and (P3.3), the geometrical consequence is that we find the generic case of the splitting up of the heteroclinic trajectory, due to the transversal intersection of $\mathcal{W}^s$ and $\mathcal{W}^u$. Analytically this means with $d'(\lambda)|_{\lambda_0} \neq 0$ that due to the inverse-function-theorem the scalar split function (3.59) is invertible for all λ sufficiently close to λ_0. Hence we can reparameterize the system with the parameter d and obtain for the jump:

$$\xi^\infty_{\kappa_i}(d) = \xi^\infty_{\kappa_i}(\lambda(d)) = d\psi.$$

The considerations above show that in a generic system one real parameter is needed to describe the splitting up of a single heteroclinic connection satisfying (P3.2) and (P3.3). This parameter can be chosen that way, that it measures the distance of the stable and unstable manifold. Hence in a generic system holding k different heteroclinic trajectories it takes k different real parameters. That is, in the case of the heteroclinic network Γ, declared in Hypothesis (H3.1), the dimension of the parameter space would in general be at least as large as the number of the different heteroclinic trajectories the network consists of. Thereby each term $\langle \xi_i^\infty, \psi_i \rangle$ can be interpreted as one component of the parameter $\lambda \in \mathbb{R}^d$ for each heteroclinic trajectory γ_i that splits up with non-zero speed.

3.4.2 Representation of the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$

Here we focus on the second component of the jump. First we present a suitable representation of ξ_i as it can also be found in [Kno04]. In accordance with (3.57) we have

$$\xi_i(\boldsymbol{\omega},\lambda,\kappa) = v_i^+(\boldsymbol{\omega},\lambda,\kappa)(0) - v_i^-(\boldsymbol{\omega},\lambda,\kappa)(0). \tag{3.61}$$

We decompose $v_i^+(\boldsymbol{\omega},\lambda,\kappa)(0)$ and $v_i^-(\boldsymbol{\omega},\lambda,\kappa)(0)$ by means of the projections $P^+_{\kappa_i}(\lambda,0)$ and $P^-_{\kappa_i}(\lambda,0)$, respectively, cf. (3.17). Indeed, due to Hypothesis (H3.4), this is the same as the decomposition of $v_i^+(\boldsymbol{\omega},\lambda,\kappa)(0)$ in its components of $(W^-_{\kappa_i}\oplus Z_{\kappa_i})$ and $W^+_{\kappa_i}$ and similarly the decomposition of $v_i^-(\boldsymbol{\omega},\lambda,\kappa)(0)$ in its components of $(W^+_{\kappa_i}\oplus Z_{\kappa_i})$ and $W^-_{\kappa_i}$.

Now, recall that in our case Z_{κ_i} is one-dimensional and denote by ψ_{κ_i} a normal vector with $\operatorname{span}\{\psi_{\kappa_i}\} = Z_{\kappa_i}$. Recall further from the introduction (cf. (1.11)) that we choose the directions of ψ_j, $j\in\mathbb{Z}$, such that $\langle \lim\limits_{t\to\infty}\frac{\gamma_j(t)-p_{j+1}}{\|\gamma_j(t)-p_{j+1}\|}, \lim\limits_{t\to-\infty}\frac{\psi_j(t)}{\|\psi_j(t)\|}\rangle < 0$. Then ξ_i can be written as

$$\xi_i(\boldsymbol{\omega},\lambda,\kappa) = \langle \psi_{\kappa_i}, \xi_i(\boldsymbol{\omega},\lambda,\kappa)\rangle\, \psi_{\kappa_i}. \tag{3.62}$$

Since Z_{κ_i} is orthogonal to $W^+_{\kappa_i}$ and $W^-_{\kappa_i}$ the decompositions described above yield

$$\langle \psi_{\kappa_i}, \xi_i(\boldsymbol{\omega},\lambda,\kappa)\rangle = \left\langle \psi_{\kappa_i}, P^+_{\kappa_i}(\lambda,0)v_i^+(\boldsymbol{\omega},\lambda,\kappa)(0)\right\rangle - \left\langle \psi_{\kappa_i}, P^-_{\kappa_i}(\lambda,0)v_i^-(\boldsymbol{\omega},\lambda,\kappa)(0)\right\rangle. \tag{3.63}$$

The addends on the right-hand side we denote by

$$\boldsymbol{T}^{\mathbf{1}}_{\boldsymbol{\kappa_i}} := \left\langle \psi_{\kappa_i}, P^+_{\kappa_i}(\lambda,0)v_i^+(\boldsymbol{\omega},\lambda,\kappa)(0)\right\rangle \quad\text{and}\quad \boldsymbol{T}^{\mathbf{2}}_{\boldsymbol{\kappa_i}} := -\left\langle \psi_{\kappa_i}, P^-_{\kappa_i}(\lambda,0)v_i^-(\boldsymbol{\omega},\lambda,\kappa)(0)\right\rangle. \tag{3.64}$$

Next we derive appropriate representations of $\boldsymbol{T}^{\mathbf{1}}_{\boldsymbol{\kappa_i}}$ and $\boldsymbol{T}^{\mathbf{2}}_{\boldsymbol{\kappa_i}}$. We will do this exemplarily for $\boldsymbol{T}^{\mathbf{1}}_{\boldsymbol{\kappa_i}}$. Due to (3.36) we have

$$\begin{aligned} P^+_{\kappa_i}(\lambda,0)v_i^+(\boldsymbol{\omega},\lambda,\kappa)(0) &= \Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})P^+_{\kappa_i}(\lambda,\omega_{i+1})a^+_{i+1} \\ &\quad - \int_0^{\omega_{i+1}} \Phi^+_{\kappa_i}(\lambda)(0,s)P^+_{\kappa_i}(\lambda,s)h^+_{\kappa_i}(s,v_i^+(\boldsymbol{\omega},\lambda,\kappa)(s),\lambda)ds. \end{aligned}$$

Here we substitute a^+_{i+1}, cf. (3.48) with d_i given in the first equation in (3.13), by using

$$\begin{aligned} a^+_{i+1} = \tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})\Big[&\gamma^-_{\kappa_{i+1}}(\lambda)(-\omega_{i+1}) - \gamma^+_{\kappa_i}(\lambda)(\omega_{i+1}) - (id - P^+_{\kappa_i}(\lambda,\omega_{i+1}))v_i^+(\boldsymbol{\omega},\lambda,\kappa)(\omega_{i+1}) \\ &+ (id - P^-_{\kappa_{i+1}}(\lambda,-\omega_{i+1}))v^-_{i+1}(\boldsymbol{\omega},\lambda,\kappa)(-\omega_{i+1})\Big]. \end{aligned}$$

Recalling that $\alpha(\gamma^-_{\kappa_{i+1}}) = \omega(\gamma^+_{\kappa_i}) = p_{\kappa_{i+1}}$ and taking the scalar product with ψ_{κ_i} leads to

$$
\begin{aligned}
\boldsymbol{T^1_{\kappa_i}} &= \big\langle \psi_{\kappa_i}, P^+_{\kappa_i}(\lambda,0)v^+_i(\boldsymbol{\omega},\lambda,\kappa)(0)\big\rangle = \\
&\Big\langle \Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})^T P^+_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}, \tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})\Big[\big(\gamma^-_{\kappa_{i+1}}(\lambda)(-\omega_{i+1}) - p_{\kappa_{i+1}}\big) - \big(\gamma^+_{\kappa_i}(\lambda)(\omega_{i+1}) - p_{\kappa_{i+1}}\big) \\
&\qquad + (id - P^-_{\kappa_{i+1}}(\lambda,-\omega_{i+1}))v^-_{i+1}(\boldsymbol{\omega},\lambda,\kappa)(-\omega_{i+1}) \\
&\qquad - (id - P^+_{\kappa_i}(\lambda,\omega_{i+1}))v^+_i(\boldsymbol{\omega},\lambda,\kappa)(\omega_{i+1})\Big]\Big\rangle \\
&- \Big\langle \psi_{\kappa_i}, \int_0^{\omega_{i+1}} \Phi^+_{\kappa_i}(\lambda)(0,s)P^+_{\kappa_i}(\lambda,s)h^+_{\kappa_i}(s,v^+_i(\boldsymbol{\omega},\lambda,\kappa)(s),\lambda)ds\Big\rangle.
\end{aligned}
$$

Thereby we denote the adjoint of A with respect to $\langle\cdot,\cdot\rangle$ by A^T and we take into consideration that $\Phi^\pm$ and $P^\pm$ commute.

We get a similar expression for $\boldsymbol{T^2_{\kappa_i}} = -\langle \psi_{\kappa_i}, P^-_{\kappa_i}(\lambda,0)v^-_i(\boldsymbol{\omega},\lambda,\kappa)(0)\rangle$ by following the same lines. Eventually we plug these expressions into (3.63) and finally obtain:

$$
\left.
\begin{aligned}
&\langle \psi_{\kappa_i}, \xi_i(\boldsymbol{\omega},\lambda,\kappa)\rangle = \boldsymbol{T^1_{\kappa_i}} + \boldsymbol{T^2_{\kappa_i}} = \\
&\quad\Big\langle \Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})^T P^+_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ \tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})\Big[\big(\gamma^-_{\kappa_{i+1}}(\lambda)(-\omega_{i+1}) - p_{\kappa_{i+1}}\big) \\
&\qquad\qquad -\big(\gamma^+_{\kappa_i}(\lambda)(\omega_{i+1}) - p_{\kappa_{i+1}}\big) \\
&\qquad\qquad +(id - P^-_{\kappa_{i+1}}(\lambda,-\omega_{i+1}))v^-_{i+1}(\boldsymbol{\omega},\lambda,\kappa)(-\omega_{i+1}) \\
&\qquad\qquad -(id - P^+_{\kappa_i}(\lambda,\omega_{i+1}))v^+_i(\boldsymbol{\omega},\lambda,\kappa)(\omega_{i+1})\Big]\Big\rangle \\
&- \Big\langle \psi_{\kappa_i}, \int_0^{\omega_{i+1}} \Phi^+_{\kappa_i}(\lambda)(0,s)P^+_{\kappa_i}(\lambda,s)h^+_{\kappa_i}(s,v^+_i(\boldsymbol{\omega},\lambda,\kappa)(s),\lambda)ds\Big\rangle \\
&- \Big\langle \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ (id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))\Big[\big(\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) - p_{\kappa_i}\big) \\
&\qquad\qquad -\big(\gamma^-_{\kappa_i}(\lambda)(-\omega_i) - p_{\kappa_i}\big) \\
&\qquad\qquad +(id - P^+_{\kappa_{i-1}}(\lambda,\omega_i))v^+_{i-1}(\boldsymbol{\omega},\lambda,\kappa)(\omega_i) \\
&\qquad\qquad -(id - P^-_{\kappa_i}(\lambda,-\omega_i))v^-_i(\boldsymbol{\omega},\lambda,\kappa)(-\omega_i)\Big]\Big\rangle \\
&- \Big\langle \psi_{\kappa_i}, \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)h^-_{\kappa_i}(s,v^-_i(\boldsymbol{\omega},\lambda)(s),\lambda,\kappa)ds\Big\rangle.
\end{aligned}
\right\} \quad (3.65)
$$

Note that the first two summand on the right-hand side of (3.65) arise from $\boldsymbol{T^1_{\kappa_i}}$ while the remaining two arise from $\boldsymbol{T^2_{\kappa_i}}$.

Within the following sections we derive suitable estimates of the single components of the above representation of the jump. To this end we introduce for all hyperbolic equilibria p_{κ_i} the constants α^s and α^u such that

$$\mathrm{Re}(\mu_{\kappa_i}(\lambda)) < \alpha^s(\kappa_i) < 0 < \alpha^u(\kappa_i) < \ \mathrm{Re}(\tilde{\mu}_{\kappa_i}(\lambda)) \tag{3.66}$$

for all $\mu_{\kappa_i}(\lambda) \in \sigma_s(D_1 f(p_{\kappa_i},\lambda))$ and $\tilde{\mu}_{\kappa_i}(\lambda) \in \sigma_u(D_1 f(p_{\kappa_i},\lambda))$. For the sake of convenience we will spare the dependency on κ_i in the notations. The context will show which $\alpha^{s/u}$ we are talking about. This can be seen on the one hand due to the equilibrium we focus on and on the other hand due to the index of the transition time. The index i of ω_i correspond to the equilibrium p_{κ_i} and hence to $\alpha^{s/u}(\kappa_i)$.

Further we introduce the constant $\nu \in \mathbb{N}$ as follows:

Definition 3.4.2. *For all $i \in \mathbb{Z}$ we define the constant $\nu = \nu(i) \in \mathbb{N}$, $\nu \geq 2$ such that for all $k \in \{0,\dots,\nu-1\} \setminus \{1\}$ it holds $D_1^k f(p_{\kappa_i},\lambda) = 0$ and $D_1^\nu f(p_{\kappa_i},\lambda) \neq 0$.*

For the same reasoning as above in case of $\alpha^{s/u}$ we omit the dependency on i in the notations.

Since p_{κ_i} is a hyperbolic equilibrium of the vector field f, that is $f(p_{\kappa_i}(\lambda),\lambda) = 0$ and $D_1 f(p_{\kappa_i}(\lambda),\lambda) \neq 0$, we find in general $\nu = 2$. For $\nu > 2$ Definition 3.4.2 demands the vanishing of the derivatives of the vector field f at the equilibrium point p_{κ_i} from the second up to the $(\nu-1)$th order. When estimating the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$ we need to distinguish different cases characterised by the value of ν since it effects the convergence rates of some terms in the representation of the jump (3.65). Note that the vanishing of the derivatives of the vector field may be caused by symmetry. In this regard we refer to Section 4.1, where we introduce symmetric vector fields, and to Section 5 where we consider D_{4m}-symmetric vector fields in $\mathbb{R}^4$ containing a homoclinic cycle. Due to the geometric setting introduced in 5.1 we find $\nu = 3$.

Now we continue in the following way. At first we see to the projections $\tilde{P}_{\kappa_i}$, $P^+_{\kappa_i}$ and $P^-_{\kappa_i}$. Especially we are interested in the impact on their composition. Based on these results we estimate all those terms of (3.65) containing $\gamma^\pm_{\kappa_i}$. Then we see to the remaining terms, that is we estimate the $v_i^\pm$ and the integral terms. Finally we summarise the results we gained so far.

3.4.3 Basic estimates of the projections

Looking at the representation of the jump ξ_i in (3.65) we find in most scalar products on the right hand side a composition of the projections $\tilde{P}$ or $(id - \tilde{P})$ and one of the projections $(id - P^+)$ or $(id - P^-)$, respectively. Since the image of $\tilde{P}_{\kappa_i}(\omega_i)$ is equal to the image of $P^+_{\kappa_{i-1}}(\omega_i)$ and the kernel of $\tilde{P}_{\kappa_i}(\omega_i)$ is equal to the kernel of $P^-_{\kappa_i}(-\omega_i)$ it is to be expected that the compositions $\tilde{P}_{\kappa_i}(\omega_i)(id - P^+_{\kappa_{i-1}}(\omega_i))$ and $(id - \tilde{P}_{\kappa_i}(\omega_i))(id - P^-_{\kappa_i}(-\omega_i))$ will become very small. In the following we examine these compositions more closely. Thereby we omit the dependency on λ in our notation.

Lemma 3.4.3. *Assume Hypotheses (H3.1)-(H3.6). Let $P^\pm_{\kappa_i}$ be the projections associated with the exponential dichotomies of the variational equation (3.8) along $\gamma^\pm_{\kappa_i}(\lambda)(\cdot)$ as introduced in (3.17) and let $\tilde{P}_{\kappa_i}$ be the projection introduced in Lemma 3.3.2. Then there exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $\|\lambda\| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ the following estimates apply:*

$$\begin{aligned}
\|\tilde{P}_{\kappa_i}(\omega_i)(id - P^-_{\kappa_i}(-\omega_i)) - P^+_{\kappa_{i-1}}(\omega_i)(id - P^-_{\kappa_i}(-\omega_i))\| &= O(e^{\max\{\nu\alpha^s,\alpha^s-\alpha^u\}\omega_i})\\
\|\tilde{P}_{\kappa_i}(\omega_i)(id - P^+_{\kappa_{i-1}}(\omega_i))\| &= O(e^{1/2(\alpha^s-\alpha^u)\omega_i})\\
\|(id - \tilde{P}_{\kappa_i}(\omega_i))(id - P^-_{\kappa_i}(-\omega_i))\| &= O(e^{\max\{(\nu-1)\alpha^s,1/2(\alpha^s-\alpha^u)\}\omega_i})\\
\|(id - \tilde{P}_{\kappa_i}(\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i)) - P^-_{\kappa_i}(-\omega_i)(id - P^+_{\kappa_{i-1}}(\omega_i))\| &= O(e^{\max\{\nu\alpha^s,\alpha^s-\alpha^u\}\omega_i})
\end{aligned}$$

Proof. Recall that the named projections are defined in a surrounding area of the equilibrium point p_{κ_i}. Hence the exponential rates $\alpha^{s/u}$ correspond to the leading eigenvalues of the linearisation $D_1 f(p_{\kappa_i},\lambda)$. First we provide information about the composition of projections $P^\pm_{\kappa_i}(\cdot)$ and the spectral projection P_{κ_i} of $D_1 f(p_{\kappa_i},\lambda)$, cf. (3.26). The definition of $P^+_{\kappa_i}$ and $P^-_{\kappa_i}$, cf. equation (3.17), provides

$$\ker P^-_{\kappa_i}(-t) = T_{\gamma^-_{\kappa_i}(\lambda)(-t)} W^u(p_{\kappa_i}) \qquad \text{and} \qquad \ker P^+_{\kappa_{i-1}}(t) = T_{\gamma^+_{\kappa_{i-1}}(\lambda)(t)} W^s(p_{\kappa_i}).$$

For t sufficiently large we can invoke Hypothesis (H3.3) that flattens the stable and unstable manifold

$W^s(p_{\kappa_i})$, $W^u(p_{\kappa_i})$ locally around p_{κ_i}. That is for t sufficiently large we have for all λ close to zero

$$T_{\gamma^-_{\kappa_i}(\lambda)(-t)}W^u(p_{\kappa_i}) = T_{p_{\kappa_i}}W^u_{\lambda=0}(p_{\kappa_i}) \qquad \text{and} \qquad T_{\gamma^+_{\kappa_{i-1}}(\lambda)(t)}W^s(p_{\kappa_i}) = T_{p_{\kappa_i}}W^s_{\lambda=0}(p_{\kappa_i})$$

and therefore, cf. (3.26) for the definition of P_{κ_i},

$$\begin{aligned} \operatorname{im}(id - P^-_{\kappa_i}(-t)) &= \ker P^-_{\kappa_i}(-t) &= \ker P_{\kappa_i} &= \operatorname{im}(id - P_{\kappa_i}), \\ \operatorname{im}(id - P^+_{\kappa_{i-1}}(t)) &= \ker P^+_{\kappa_{i-1}}(t) &= \operatorname{im} P_{\kappa_i} &= \ker(id - P_{\kappa_i}). \end{aligned}$$

Thus for $t > 0$ sufficiently large and $\|\lambda\|$ sufficiently small we find

$$\left.\begin{aligned} P^-_{\kappa_i}(-t)(id - P_{\kappa_i}) &= 0, & P^+_{\kappa_{i-1}}(t)P_{\kappa_i} &= 0, \\ P_{\kappa_i}(id - P^-_{\kappa_i}(-t)) &= 0, & (id - P_{\kappa_i})(id - P^+_{\kappa_{i-1}}(t)) &= 0, \\ (id - P_{\kappa_i})(id - P^-_{\kappa_i}(-t)) &= (id - P^-_{\kappa_i}(-t)), & P_{\kappa_i}(id - P^+_{\kappa_{i-1}}(t)) &= (id - P^+_{\kappa_{i-1}}(t)). \end{aligned}\right\} \tag{3.67}$$

Now, due to the definition of $\tilde{P}$, cf. (3.31), we have

$$\tilde{P}_{\kappa_i}(\omega_i) = S_{\kappa_i}(\omega_i)(id - P_{\kappa_i})S_{\kappa_i}(\omega_i)^{-1}, \qquad (id - \tilde{P}_{\kappa_i}(\omega_i)) = S_{\kappa_i}(\omega_i)P_{\kappa_i}S_{\kappa_i}(\omega_i)^{-1}$$

and in combination with (3.30) and (3.33) in Lemma 3.3.2, we find

$$\left.\begin{aligned} \tilde{P}_{\kappa_i}(\omega_i) &= P^+_{\kappa_{i-1}}(\omega_i)(id - P_{\kappa_i}) \sum_{k=0}^{\infty} \Big[(id - P^-_{\kappa_i}(-\omega_i))P_{\kappa_i} + (id - P^+_{\kappa_{i-1}}(\omega_i))(id - P_{\kappa_i})\Big]^k, \\ (id - \tilde{P}_{\kappa_i}(\omega_i)) &= P^-_{\kappa_i}(-\omega_i)P_{\kappa_i} \sum_{k=0}^{\infty} \Big[(id - P^-_{\kappa_i}(-\omega_i))P_{\kappa_i} + (id - P^+_{\kappa_{i-1}}(\omega_i))(id - P_{\kappa_i})\Big]^k. \end{aligned}\right\} \tag{3.68}$$

Then simple calculations by repeatedly invoking (3.67) into (3.68) yield for λ sufficiently small and $\omega_i > \Omega$, Ω according to Lemma 3.3.2,

$$\left.\begin{aligned} \tilde{P}_{\kappa_i}(\omega_i)(id - P^-_{\kappa_i}(-\omega_i)) &= P^+_{\kappa_{i-1}}(\omega_i)(id - P^-_{\kappa_i}(-\omega_i)) \sum_{k=0}^{\infty} \left[(id - P^+_{\kappa_{i-1}}(\omega_i))(id - P^-_{\kappa_i}(-\omega_i))\right]^k \\ \tilde{P}_{\kappa_i}(\omega_i)(id - P^+_{\kappa_{i-1}}(\omega_i)) &= P^+_{\kappa_{i-1}}(\omega_i) \sum_{k=1}^{\infty} \left[(id - P^-_{\kappa_i}(-\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i))\right]^k \\ (id - \tilde{P}_{\kappa_i}(\omega_i))(id - P^-_{\kappa_i}(-\omega_i)) &= P^-_{\kappa_i}(-\omega_i) \sum_{k=1}^{\infty} \left[(id - P^+_{\kappa_{i-1}}(\omega_i))(id - P^-_{\kappa_i}(-\omega_i))\right]^k \\ (id - \tilde{P}_{\kappa_i}(\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i)) &= P^-_{\kappa_i}(-\omega_i)(id - P^+_{\kappa_{i-1}}(\omega_i)) \sum_{k=0}^{\infty} \left[(id - P^-_{\kappa_i}(-\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i))\right]^k \end{aligned}\right\} \tag{3.69}$$

Exemplarily we present the calculation that leads to the first equation in (3.69). To this end we look at the first equation in (3.68). By applying (3.67) we obtain for $k \geq 2$

$$\begin{aligned} &(id - P_{\kappa_i})\Big[(id - P^-_{\kappa_i}(-\omega_i))P_{\kappa_i} + (id - P^+_{\kappa_{i-1}}(\omega_i))(id - P_{\kappa_i})\Big]^k \\ &= (id - P^-_{\kappa_i}(-\omega_i))P_{\kappa_i}\Big[(id - P^-_{\kappa_i}(-\omega_i))P_{\kappa_i} + (id - P^+_{\kappa_{i-1}}(\omega_i))(id - P_{\kappa_i})\Big]^{k-1} \\ &= (id - P^-_{\kappa_i}(-\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i))(id - P_{\kappa_i})\Big[(id - P^-_{\kappa_i}(-\omega_i))P_{\kappa_i} + (id - P^+_{\kappa_{i-1}}(\omega_i))(id - P_{\kappa_i})\Big]^{k-2}. \end{aligned}$$

Hence we find by repeatedly invoking these relations

$$
\begin{aligned}
&(id-P_{\kappa_i})\left[(id-P^-_{\kappa_i}(-\omega_i))P_{\kappa_i}+(id-P^+_{\kappa_{i-1}}(\omega_i))(id-P_{\kappa_i})\right]^k\\
&\qquad=\begin{cases}\left[(id-P^-_{\kappa_i}(-\omega_i))(id-P^+_{\kappa_{i-1}}(\omega_i))\right]^{(k-1)/2}P_{\kappa_i}, & k \text{ odd},\\ \left[(id-P^-_{\kappa_i}(-\omega_i))(id-P^+_{\kappa_{i-1}}(\omega_i))\right]^{k/2}(id-P_{\kappa_i}), & k \text{ even}.\end{cases}
\end{aligned}
$$

Now, multiplying $(id-P^-_{\kappa_i}(-\omega_i))$ from the right side leads to

$$
\begin{aligned}
&(id-P_{\kappa_i})\left[(id-P^-_{\kappa_i}(-\omega_i))P_{\kappa_i}+(id-P^+_{\kappa_{i-1}}(\omega_i))(id-P_{\kappa_i})\right]^k(id-P^-_{\kappa_i}(-\omega_i))\\
&\qquad=\begin{cases}0, & k \text{ odd},\\ \left[(id-P^-_{\kappa_i}(-\omega_i))(id-P^+_{\kappa_{i-1}}(\omega_i))\right]^{k/2}(id-P^-_{\kappa_i}(-\omega_i)), & k \text{ even}\end{cases}\\
&\qquad=\begin{cases}0, & k \text{ odd},\\ (id-P^-_{\kappa_i}(-\omega_i))\left[(id-P^+_{\kappa_{i-1}}(\omega_i))(id-P^-_{\kappa_i}(-\omega_i))\right]^{k/2}, & k \text{ even}.\end{cases}
\end{aligned}
$$

Therefore we can conclude from this and the first equation in (3.68)

$$
\begin{aligned}
&\tilde{P}_{\kappa_i}(\omega_i)(id-P^-_{\kappa_i}(-\omega_i))\\
&\quad= P^+_{\kappa_{i-1}}(\omega_i)(id-P_{\kappa_i})\sum_{k=0}^{\infty}\left[(id-P^-_{\kappa_i}(-\omega_i))P_{\kappa_i}+(id-P^+_{\kappa_{i-1}}(\omega_i))(id-P_{\kappa_i})\right]^k(id-P^-_{\kappa_i}(-\omega_i))\\
&\quad= P^+_{\kappa_{i-1}}(\omega_i)(id-P^-_{\kappa_i}(-\omega_i))\sum_{k=0}^{\infty}\left[(id-P^+_{\kappa_{i-1}}(\omega_i))(id-P^-_{\kappa_i}(-\omega_i))\right]^k
\end{aligned}
$$

The remaining equations in (3.69) can be gained analogously.

After we have generated the representations (3.69) we go on to estimate the compositions of the projections $(id-P^-_{\kappa_i}(\cdot))$ and $(id-P^+_{\kappa_{i-1}}(\cdot))$. To this end we first recall that both $\|(id-P^+_{\kappa_{i-1}}(\omega_i))-P_{\kappa_i}\|$ and $\|(id-P^-_{\kappa_i}(-\omega_i))-(id-P_{\kappa_i})\|$ tend exponentially fast to zero as ω_i tends to infinity. This follows from Lemma 2.1.14. Recall that in Lemma 2.1.14 the image of $P^+(\cdot)$ and here the kernel of $P^+_{\kappa_{i-1}}(\cdot)$ is determined to be the stable subspace, cf. (2.2) versus (3.16).

As a direct consequence of Lemma 2.1.14 we find that there exist positive constants $\mathcal{K}$ and ϑ such that $\|(id-P^+_{\kappa_{i-1}})(t)-P_{\kappa_i}\|\le \mathcal{K}e^{-\vartheta^+_{\kappa_i}t}$. The constant $\vartheta^+_{\kappa_i}$ is constraint only through the two conditions

$$
\vartheta^+_{\kappa_i}<\frac{\beta-\operatorname{Re}(\mu^s_{\kappa_i})}{2},\ \forall\mu^s_{\kappa_i}\in\sigma_s(D_1f(p_{\kappa_i},\lambda))\qquad\text{and}\qquad \vartheta^+_{\kappa_i}<\delta^+_{\kappa_i}, \tag{3.70}
$$

cf. Remark 2.1.15, where

$$
\operatorname{Re}(\mu^s_{\kappa_i})<\alpha^s\le\alpha<\beta\le\alpha^u<\operatorname{Re}(\mu^u_{\kappa_i})
$$

and

$$
\|B^+_{\kappa_i}(t)\|=\|D_1f(\gamma^+_{\kappa_{i-1}}(t))-D_1f(p_{\kappa_i})\|\le K_Be^{-\delta^+_{\kappa_i}t}.
$$

With the Definition 3.4.2 of ν we find

$$\begin{aligned} D_1 f(\gamma^+_{\kappa_{i-1}}(t)) &= \sum_{k=1}^{\nu-1} \tfrac{D^k f(p_{\kappa_i})}{(k-1)!} (\gamma^+_{\kappa_{i-1}}(t) - p_{\kappa_i})^{k-1} + O(\|\gamma^+_{\kappa_{i-1}}(t) - p_{\kappa_i}\|^{\nu-1}) \\ &= D_1 f(p_{\kappa_i}) + O(\|\gamma^+_{\kappa_{i-1}}(t) - p_{\kappa_i}\|^{\nu-1}) \end{aligned}$$

and hence

$$\|B^+_{\kappa_i}(t)\| = \|D_1 f(\gamma^+_{\kappa_{i-1}}(t)) - D_1 f(p_{\kappa_i})\| \leq K \|\gamma^+_{\kappa_{i-1}}(t) - p_{\kappa_i}\|^{\nu-1} \leq KCe^{(\nu-1)\,\mathrm{Re}(\mu^s_{\kappa_i})t}. \tag{3.71}$$

Therefore we find on one hand with

$$\vartheta^+_{\kappa_i} \leq -(\nu-1)\alpha^s < -(\nu-1)\,\mathrm{Re}(\mu^s_{\kappa_i}) =: \delta^+_{\kappa_i}$$

that $\vartheta^+_{\kappa_i}$ satisfies the second inequality in (3.70). The first inequality of (3.70) on the other hand can be satisfied with $\beta = \alpha^u$:

$$\vartheta^+_{\kappa_i} \leq \frac{\alpha^u - \alpha^s}{2} < \frac{\alpha^u - \mathrm{Re}(\mu^s_{\kappa_i})}{2} =: \frac{\beta - \mathrm{Re}(\mu^s_{\kappa_i})}{2}.$$

Analogously we find for $t < 0$ that $\|P^-_{\kappa_i}(t) - P_{\kappa_i}\| \leq \mathcal{K} e^{\vartheta^-_{\kappa_i} t}$ where $\vartheta^-_{\kappa_i} > 0$ has to satisfy the conditions

$$\vartheta^-_{\kappa_i} < \frac{\mathrm{Re}(\mu^u_{\kappa_i}) - \alpha}{2}, \ \forall \mu^u_{\kappa_i} \in \sigma_u(D_1 f(p_{\kappa_i}, \lambda)) \qquad \text{and} \qquad \vartheta^-_{\kappa_i} < \delta^-_{\kappa_i}$$

with

$$\|B^-_{\kappa_i}(t)\| = \|D_1 f(\gamma^-_{\kappa_{i-1}}(t)) - D_1 f(p_{\kappa_i})\| \leq KCe^{(\nu-1)\alpha^u t} = K_B e^{\delta^-_{\kappa_i} t}. \tag{3.72}$$

Hence with

$$\vartheta^-_{\kappa_i} \leq \frac{\alpha^u - \alpha^s}{2} < (\nu-1)\alpha^u$$

both inequalities are fulfilled for all $\nu \geq 2$.

This leads to the estimates

$$\left.\begin{aligned} \left\|P^+_{\kappa_{i-1}}(t) - (id - P_{\kappa_i})\right\| &= \left\|(id - P^+_{\kappa_{i-1}}(t)) - P_{\kappa_i}\right\| &&\leq \mathcal{K} e^{\max\{(\nu-1)\alpha^s, 1/2(\alpha^s - \alpha^u)\}t}, \\ \left\|P^-_{\kappa_i}(-t) - P_{\kappa_i}\right\| &= \left\|(id - P^-_{\kappa_i}(-t)) - (id - P_{\kappa_i})\right\| &&\leq \mathcal{K} e^{\max\{-(\nu-1)\alpha^u, 1/2(\alpha^s - \alpha^u)\}t}. \end{aligned}\right\} \tag{3.73}$$

With this we now can estimate the composition of the projections $(id - P^+_{\kappa_{i-1}}(\cdot))$ and $(id - P^-_{\kappa_i}(\cdot))$. From (3.67) we find for ω_i sufficiently large

$$\begin{aligned} \left\|(id - P^+_{\kappa_{i-1}}(\omega_i))(id - P^-_{\kappa_i}(-\omega_i))\right\| &= \left\|(id - P^+_{\kappa_{i-1}}(\omega_i))(id - P^-_{\kappa_i}(-\omega_i)) - P_{\kappa_i}(id - P^-_{\kappa_i}(-\omega_i))\right\| \\ &\leq \left\|(id - P^+_{\kappa_{i-1}}(\omega_i)) - P_{\kappa_i}\right\| \cdot \left\|(id - P^-_{\kappa_i}(-\omega_i))\right\| \end{aligned}$$

and

$$\begin{aligned} \left\|(id - P^-_{\kappa_i}(-\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i))\right\| &= \left\|(id - P^-_{\kappa_i}(-\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i)) - (id - P_{\kappa_i})(id - P^+_{\kappa_{i-1}}(\omega_i))\right\| \\ &\leq \left\|(id - P^-_{\kappa_i}(-\omega_i)) - (id - P_{\kappa_i})\right\| \cdot \left\|(id - P^+_{\kappa_{i-1}}(\omega_i))\right\|. \end{aligned}$$

Since $(id - P^-_{\kappa_i}(\cdot))$ and $(id - P^+_{\kappa_{i-1}}(\cdot))$ are projections of the exponential dichotomy their norm is bounded and with estimate (3.73) we find

$$\left.\begin{array}{rcl}\left\|(id - P^+_{\kappa_{i-1}}(\omega_i))(id - P^-_{\kappa_i}(-\omega_i))\right\| & = & O\left(e^{\max\{(\nu-1)\alpha^s, 1/2(\alpha^s-\alpha^u)\}\omega_i}\right)\\ \left\|(id - P^-_{\kappa_i}(-\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i))\right\| & = & O\left(e^{\max\{-(\nu-1)\alpha^u, 1/2(\alpha^s-\alpha^u)\}\omega_i}\right)\end{array}\right\} \tag{3.74}$$

Combining (3.69) and (3.74) we obtain that the composition $\tilde{P}_{\kappa_i}(\omega_i)(id - P^-_{\kappa_i}(-\omega_i))$ basically acts as $P^+_{\kappa_{i-1}}(\omega_i)(id - P^-_{\kappa_i}(-\omega_i))$ apart from higher order terms. Analogously $(id - \tilde{P}_{\kappa_i}(\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i))$ equals $P^-_{\kappa_i}(-\omega_i)(id - P^+_{\kappa_{i-1}}(\omega_i))$ plus terms of higher order. The exponential rates follow immediately from (3.74). Only in case of the first and the last estimate in Lemma 3.4.3 we wrote for simplicity $e^{\max\{\nu\alpha^s, \alpha^s-\alpha^u\}\omega_i}$ instead of $e^{\max\{(\nu-1/2)\alpha^s - 1/2\alpha^u, \alpha^s-\alpha^u\}\omega_i}$. □

Remark 3.4.4. *Applying Lemma 2.5.6 instead of 2.1.14 the exponential rate of the projections presented in Lemma 3.4.3 might be improved, cf. Remark 2.5.7. There we find that the following exponential extensions hold*

$$\begin{array}{rcl}\left\|P^+_{\kappa_{i-1}}(t) - (id - P_{\kappa_i})\right\| & \leq & Ce^{\max\{\alpha^s_{ad}, (\nu-1)\alpha^s\}t},\\ \left\|P^-_{\kappa_i}(-t) - P_{\kappa_i}\right\| & \leq & Ce^{\max\{\alpha^s_{ad}, -(\nu-1)\alpha^u\}t},\end{array}$$

for some constant α^s_{ad} which is determined by the inequality

$$\alpha^s_{ad} > -\min\{|\ Re(\mu) - \ Re(\tilde{\mu})|\ |\ \mu, \tilde{\mu} \in \sigma(D_1 f(p_{\kappa_i}, \lambda)),\ Re(\mu) \neq \ Re(\tilde{\mu})\},$$

cf. (2.53). *Now, if $\alpha^s_{ad} < (\alpha^s - \alpha^u)/2$ we obtain an improvement to estimate* (3.73).

According to Remark 3.4.4 there is one special case we would like to mention explicitly. This case is simply determined by the absence of strong stable and strong unstable eigenvalues, cf. Hypothesis (H3.7). Then α^s_{ad} simply has to satisfy $\alpha^s_{ad} > -\operatorname{Re}(\mu^u_{\kappa_i}(\lambda) - \mu^s_{\kappa_i}(\lambda))$ for any stable eigenvalue $\mu^s_{\kappa_i}$ and any unstable eigenvalue $\mu^u_{\kappa_i}$ of $D_1 f(p_{\kappa_i}, \lambda)$. Hence α^s_{ad} can be chosen as $\alpha^s_{ad} := \alpha^s - \alpha^u$ and we find the following exponential extensions

$$\left.\begin{array}{rcl}\left\|P^+_{\kappa_{i-1}}(t) - (id - P_{\kappa_i})\right\| & \leq & Ce^{\max\{\alpha^s-\alpha^u, (\nu-1)\alpha^s\}t},\\ \left\|P^-_{\kappa_i}(-t) - P_{\kappa_i}\right\| & \leq & Ce^{\max\{\alpha^s-\alpha^u, -(\nu-1)\alpha^u\}t}.\end{array}\right\} \tag{3.75}$$

Now, using this estimate instead of (3.73) we obtain

$$\left.\begin{array}{rcl}\left\|(id - P^+_{\kappa_{i-1}}(\omega_i))(id - P^-_{\kappa_i}(-\omega_i))\right\| & = & O\left(e^{\max\{(\nu-1)\alpha^s, (\alpha^s-\alpha^u)\}\omega_i}\right)\\ \left\|(id - P^-_{\kappa_i}(-\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i))\right\| & = & O\left(e^{\max\{\alpha^s-\alpha^u, -(\nu-1)\alpha^u\}\omega_i}\right)\end{array}\right\} \tag{3.76}$$

which results in the following lemma.

Lemma 3.4.5. *Assume Hypotheses (H3.1)-(H3.6). Let $P^\pm_{\kappa_i}$ be the projections associated with the exponential dichotomies of the variational equation* (3.8) *along $\gamma^\pm_{\kappa_i}(\lambda)(\cdot)$ as introduced in* (3.17) *and let $\tilde{P}_{\kappa_i}$ be the projection introduced in Lemma 3.3.2.*
Further assume Hypothesis (H3.7). Then there exist constants Ω and c in accordance to Theorem 3.2.2

such that for all $\|\lambda\| < c$ *and* $\boldsymbol{\omega}$ *with* $\inf \boldsymbol{\omega} > \Omega$ *the following estimates apply:*

$$\begin{aligned}
\|\tilde{P}_{\kappa_i}(\omega_i)(id - P^-_{\kappa_i}(-\omega_i)) - P^+_{\kappa_{i-1}}(\omega_i)(id - P^-_{\kappa_i}(-\omega_i))\| &= O(e^{\max\{\nu\alpha^s - \alpha^u, \min\{2,\nu-1\}(\alpha^s-\alpha^u)\}\omega_i})\\
\|\tilde{P}_{\kappa_i}(\omega_i)(id - P^+_{\kappa_{i-1}}(\omega_i))\| &= O(e^{\max\{\alpha^s-\alpha^u, -(\nu-1)\alpha^u\}\omega_i})\\
\|(id - \tilde{P}_{\kappa_i}(\omega_i))(id - P^-_{\kappa_i}(-\omega_i))\| &= O(e^{\max\{(\nu-1)\alpha^s, \alpha^s-\alpha^u\}\omega_i})\\
\|(id - \tilde{P}_{\kappa_i}(\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i)) - P^-_{\kappa_i}(-\omega_i)(id - P^+_{\kappa_{i-1}}(\omega_i))\| &= O(e^{\max\{\nu\alpha^s - \alpha^u, \min\{2,\nu-1\}(\alpha^s-\alpha^u)\}\omega_i})
\end{aligned}$$

Proof. The proof follows along the same line as the proof of Lemma 3.4.3. We only apply Estimate (3.75) instead of (3.73). □

In the following we will often take advantage of Lemma 3.4.5 and present alternative estimates in the special case where we find no strong stable and strong unstable eigenvalues, cf. Hypothesis (H3.7).

3.4.4 Basic estimates involving $\gamma^\pm$

On the basis of Lemma 3.4.3 or Lemma 3.4.5, respectively, we estimate in the following each term within (3.65) containing one of the expressions $\gamma^\pm_{\kappa_i}$.

Lemma 3.4.6. *Assume Hypotheses (H3.1)-(H3.6). Let* $\tilde{P}_{\kappa_i}$ *be the projection introduced in Lemma 3.3.2. Then there exist constants* Ω *and* c *in accordance to Theorem 3.2.2 such that for all* $\|\lambda\| < c$ *and* $\boldsymbol{\omega}$ *with* $\inf \boldsymbol{\omega} > \Omega$ *the following estimates apply:*

$$\left.\begin{aligned}
\tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})\big(\gamma^-_{\kappa_{i+1}}(\lambda)(-\omega_{i+1}) - p_{\kappa_{i+1}}\big) &= O\big(e^{-\alpha^u\omega_{i+1}}\big),\\
\tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})\big(\gamma^+_{\kappa_i}(\lambda)(\omega_{i+1}) - p_{\kappa_{i+1}}\big) &= O\big(e^{(3\alpha^s-\alpha^u)/2\ \omega_{i+1}}\big),\\
(id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))\big(\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) - p_{\kappa_i}\big) &= O\big(e^{\alpha^s\omega_i}\big),\\
(id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))\big(\gamma^-_{\kappa_i}(\lambda)(-\omega_i) - p_{\kappa_i}\big) &= O\left(e^{\max\{(\nu-1)\alpha^s-\alpha^u, (\alpha^s-3\alpha^u)/2\}\omega_i}\right).
\end{aligned}\right\}\tag{3.77}$$

If we additionally assume Hypothesis (H3.7), the second and fourth estimate can be improved by

$$\left.\begin{aligned}
\tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})\big(\gamma^+_{\kappa_i}(\lambda)(\omega_{i+1}) - p_{\kappa_{i+1}}\big) &= O\big(e^{\max\{(2\alpha^s-\alpha^u), \alpha^s-(\nu-1)\alpha^u\}\omega_{i+1}}\big)\\
(id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))\big(\gamma^-_{\kappa_i}(\lambda)(-\omega_i) - p_{\kappa_i}\big) &= O\big(e^{\max\{(\nu-1)\alpha^s-\alpha^u, (\alpha^s-2\alpha^u)\}\omega_i}\big)
\end{aligned}\right\}\tag{3.78}$$

Proof. The terms $\gamma^-_{\kappa_i}(\lambda)(\cdot)$ and $\gamma^+_{\kappa_{i-1}}(\lambda)(\cdot)$ are solutions within the unstable manifold $W^u(p_{\kappa_i})$ and the stable manifold $W^s(p_{\kappa_i})$, respectively. Therefore we find for $t \geq 0$ due to Lemma 2.2.2 the estimates

$$\|\gamma^-_{\kappa_i}(\lambda)(-t) - p_{\kappa_i}\| \leq \tilde{K}e^{-\alpha^u t} \quad \text{and} \quad \|\gamma^+_{\kappa_{i-1}}(\lambda)(t) - p_{\kappa_i}\| \leq \tilde{K}e^{\alpha^s t}.\tag{3.79}$$

Additionally we find due to Hypothesis (H3.3) that $\gamma^-_{\kappa_i}(\lambda)(-\omega_i) - p_{\kappa_i} \in T_{p_{\kappa_i}}W^u(p_{\kappa_i})$ and $\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) - p_{\kappa_i} \in T_{p_{\kappa_i}}W^s(p_{\kappa_i})$ for ω_i sufficiently large. The definition of the projections $P^-_{\kappa_i}$ and $P^+_{\kappa_{i-1}}$, cf. (3.17), further provides $\ker P^-_{\kappa_i}(-\omega_i) = T_{p_{\kappa_i}}W^u(p_{\kappa_i})$ and $\ker P^+_{\kappa_{i-1}}(\omega_i) = T_{p_{\kappa_i}}W^s(p_{\kappa_i})$. Hence we find for ω_i sufficiently large

$$\begin{aligned}
\gamma^-_{\kappa_i}(\lambda)(-\omega_i) - p_{\kappa_i} &= (id - P^-_{\kappa_i}(-\omega_i))\big(\gamma^-_{\kappa_i}(\lambda)(-\omega_i) - p_{\kappa_i}\big),\\
\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) - p_{\kappa_i} &= (id - P^+_{\kappa_{i-1}}(\omega_i))\big(\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) - p_{\kappa_i}\big)
\end{aligned}$$

and in combination with Lemma 3.4.3 and (3.79) this provides the first four estimates.

Applying Lemma 3.4.5 instead of Lemma 3.4.3, for that we need Hypothesis (H3.7), we obtain the remaining two estimates. □

3.4.5 Estimates regarding $v_i^\pm$

Before starting to estimate $v^\pm$ we take a closer look on the non linearities $h^\pm_{\kappa_i}$, given in (3.14). To this end recall the Definition 3.3.4 for the shortened notation of $h^{\pm,s}$, $h^{\pm,u}$, $v^{\pm,s}$ and $v^{\pm,u}$. We extend this notation to the function f when writing for short

$$\left.\begin{array}{ll} f^s(\gamma^+_{\kappa_i}(t)+\ldots) := (id - P^+_{\kappa_i}(t))f(\gamma^+_{\kappa_i}(t)+...), & f^u(\gamma^+_{\kappa_i}(t)+\ldots) := P^+_{\kappa_i}(t)f(\gamma^+_{\kappa_i}(t)+...), \\ f^s(\gamma^-_{\kappa_i}(t)+\ldots) := (id - P^-_{\kappa_i}(t))f(\gamma^-_{\kappa_i}(t)+...), & f^u(\gamma^-_{\kappa_i}(t)+\ldots) := P^-_{\kappa_i}(t)f(\gamma^-_{\kappa_i}(t)+...). \end{array}\right\} \tag{3.80}$$

The following lemma was inspired by [San93, Lemma 3.13].

Lemma 3.4.7. *Assume Hypotheses (H3.1)-(H3.6). There exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $\|\lambda\| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ there exists a constant $M > 0$ such that the non-linearities $h_i^{\pm,s}$ satisfy*

$$\begin{array}{lcll} \|h^{+,s}_{\kappa_i}(t, v_i^+(t),\lambda)\| & \le & M\big(e^{\alpha^s t}\|v_i^{+,u}(t)\| + \|v_i^{+,s}(t)\|\big)\big(\|v_i^{+,s}(t)\| + \|v_i^{+,u}(t)\|\big), & t\in[0,\omega], \\ \|h^{-,s}_{\kappa_i}(t, v_i^-(t),\lambda)\| & \le & M\big(e^{1/2(\alpha^u-\alpha^s)t}\|v_i^{-,u}(t)\| + \|v_i^{-,s}(t)\|\big)\big(\|v_i^{-,s}(t)\| + \|v_i^{-,u}(t)\|\big), & t\in[-\omega,0]. \end{array}$$

If in addition Hypothesis (H3.7) holds, we find

$$\|h^{-,s}_{\kappa_i}(t, v_i^-(t),\lambda)\| \le M(e^{\alpha^u t}\|v_i^{-,u}(t)\| + \|v_i^{-,s}(t)\|)(\|v_i^{-,s}(t)\| + \|v_i^{-,u}(t)\|), \quad t\in[-\omega,0].$$

Proof. For the sake of convenience we omit the dependency on λ in the notations below. First we prove the validity of the assertion for $t \ge \Omega$ or $t \le -\Omega$, respectively. To this end we call in the definition of $h^\pm_{\kappa_i}$, cf. (3.14) and apply the Mean-value-theorem. This provides

$$\left.\begin{array}{rl} \multicolumn{2}{l}{\|h^{\pm,s}_{\kappa_i}(t,(v_i^{\pm,s}+v_i^{\pm,u})(t))\|} \\ = & \|f^s(\gamma^\pm_{\kappa_i}(t)+v_i^{\pm,s}(t)+v_i^{\pm,u}(t)) - f^s(\gamma^\pm_{\kappa_i}(t)) - Df^s(\gamma^\pm_{\kappa_i}(t))[v_i^{\pm,s}(t)+v_i^{\pm,u}(t)]\| \\ = & \left\|\left[\int\limits_0^1 Df^s(\gamma^\pm_{\kappa_i}(t)+\tau(v_i^{\pm,s}(t)+v_i^{\pm,u}(t))) - Df^s(\gamma^\pm_{\kappa_i}(t))d\tau\right][v_i^{\pm,s}(t)+v_i^{\pm,u}(t)]\right\| \\ \le & \int\limits_0^1 \|\frac{\partial}{\partial v^s}f^s(\gamma^\pm_{\kappa_i}(t)+\tau(v_i^{\pm,s}(t)+v_i^{\pm,u}(t))) - \frac{\partial}{\partial v^s}f^s(\gamma^\pm_{\kappa_i}(t))\|d\tau\|v_i^{\pm,s}(t)\| \\ & +\int\limits_0^1 \|\frac{\partial}{\partial v^u}f^s(\gamma^\pm_{\kappa_i}(t)+\tau(v_i^{\pm,s}(t)+v_i^{\pm,u}(t))) - \frac{\partial}{\partial v^u}f^s(\gamma^\pm_{\kappa_i}(t))\|d\tau\|v_i^{\pm,u}(t)\|. \end{array}\right\} \tag{3.81}$$

The first summand in the last line is already of order $O\big(\|v_i^{\pm,s}(t)\|(\|v_i^{\pm,s}(t)\|+\|v_i^{\pm,u}(t)\|)\big)$ and applying again the Mean-value-theorem to the second summand yields

$$\left.\begin{aligned} &\|h^{\pm,s}_{\kappa_i}(t,(v^{\pm,s}_i+v^{\pm,u}_i)(t))\| \\ &\quad\le\; C\|v^{\pm,s}_i(t)\|(\|v^{\pm,s}_i(t)\|+\|v^{\pm,u}_i(t)\|) \\ &\qquad+\Bigg(\int_0^1\int_0^1\|\tfrac{\partial^2}{\partial v^u\partial v^s}f^s(\gamma^{\pm}_{\kappa_i}(t)+\tau_1\tau_2(v^{\pm,s}_i(t)+v^{\pm,u}_i(t)))\|\tau_1 d\tau_2 d\tau_1\|v^{\pm,s}_i(t)\| \\ &\qquad+\int_0^1\int_0^1\|\tfrac{\partial^2}{\partial (v^u)^2}f^s(\gamma^{\pm}_{\kappa_i}(t)+\tau_1\tau_2(v^{\pm,s}_i(t)+v^{\pm,u}_i(t)))\|\tau_1 d\tau_2 d\tau_1\|v^{\pm,u}_i(t)\|\Bigg)\|v^{\pm,u}_i(t)\|. \end{aligned}\right\} \quad (3.82)$$

The second addend is of the order $O\big(\|v^{\pm,s}_i(t)\|\|v^{\pm,u}_i(t)\|\big)$ For further investigation of the last summand we make use of Hypothesis (H3.3): For $t\ge\Omega$ or $t\le-\Omega$, respectively, we are close to the hyperbolic equilibrium p_{κ_i} or $p_{\kappa_{i+1}}$, respectively, and here the local stable and unstable manifolds $W^s_{loc,\lambda}(p_{\kappa_i})$ and $W^u_{loc,\lambda}(p_{\kappa_i})$ coincide with their tangent spaces $T_{p_{\kappa_i}}W^s_{\lambda=0}(p_{\kappa_i})$ and $T_{p_{\kappa_i}}W^u_{\lambda=0}(p_{\kappa_i})$. Thus, calling the definitions (3.26) of the projection P_{κ_i} and (3.17) of the projection $P^{\pm}_{\kappa_i}(t)$ we find that

$$(id-P_{\kappa_{i+1}})v^{+,u}_i(t)\in W^u_{loc}(p_{\kappa_{i+1}}) \quad\text{and}\quad P_{\kappa_i}v^{-,u}_i(t)\in W^s_{loc}(p_{\kappa_i}).$$

Hence

$$\begin{aligned} f((id-P_{\kappa_{i+1}})v^{+,u}_i(t)) &\in T_{p_{\kappa_{i+1}}}W^u_{\lambda=0}(p_{\kappa_{i+1}}) \quad\text{and} \\ f(P_{\kappa_i}v^{-,u}_i(t)) &\in T_{p_{\kappa_i}}W^s_{\lambda=0}(p_{\kappa_i}) \end{aligned}$$

and we can therefore write

$$\left.\begin{aligned} f((id-P_{\kappa_{i+1}})v^{+,u}_i(t)) &= (id-P_{\kappa_{i+1}})f((id-P_{\kappa_{i+1}})v^{+,u}_i(t)) \quad\text{and} \\ f(P_{\kappa_i}v^{-,u}_i(t)) &= P_{\kappa_i}f(P_{\kappa_i}v^{-,u}_i(t)). \end{aligned}\right\} \quad (3.83)$$

Further we find, cf. (3.67), $P_{\kappa_i}(id-P^-_{\kappa_i}(-t))=0$ and $(id-P_{\kappa_i})(id-P^+_{\kappa_{i-1}}(t))=0$ and hence

$$\begin{aligned} (id-P_{\kappa_{i+1}})f^s((id-P_{\kappa_{i+1}})v^{+,u}_i(t)) &\equiv 0, \quad t\ge\Omega \\ P_{\kappa_i}f^s(P_{\kappa_i}v^{-,u}_i(t)) &\equiv 0, \quad t\le-\Omega. \end{aligned}$$

This provides that each partial derivative of $(id-P)\circ f^s\circ(id-P)$ or $P\circ f^s\circ P$ with respect to $v^{\pm,u}_i$, respectively, vanishes, especially

$$\begin{aligned} \tfrac{\partial^2}{\partial(v^u)^2}(id-P_{\kappa_{i+1}})f^s((id-P_{\kappa_{i+1}})v^{+,u}_i(t)) &= 0, \\ \tfrac{\partial^2}{\partial(v^u)^2}P_{\kappa_i}f^s(P_{\kappa_i}v^{-,u}_i(t)) &= 0. \end{aligned}$$

Therefore we can again apply the Mean-value-theorem on the last summand in (3.82), by adding a zero-term. We do this exemplarily for $h_{\kappa_i}^{-,s}$ where we add $\frac{\partial^2}{\partial(v^u)^2}(-id + (id - P_{\kappa_i}))f^s(P_{\kappa_i} v_i^{-,u}(t))$.

$$
\begin{aligned}
&\|h_{\kappa_i}^{-,s}(t, (v_i^{-,s} + v_i^{-,u})(t))\| \\
&\quad \leq C_1 \|v_i^{-,s}(t)\|(\|v_i^{-,s}(t)\| + \|v_i^{-,u}(t)\|) + C_2 \|v_i^{-,s}(t)\|\|v_i^{-,u}(t)\| \\
&\qquad + \int_0^1\int_0^1 \| \tfrac{\partial^2}{\partial(v^u)^2} f^s(\gamma_{\kappa_i}^-(t) + \tau_1\tau_2(v_i^{-,s}(t) + (id - P_{\kappa_i})v_i^{-,u}(t) + P_{\kappa_i} v_i^{-,u}(t))) \\
&\qquad\qquad - \tfrac{\partial^2}{\partial(v^u)^2} f^s(P_{\kappa_i} v_i^{-,u}(t)) + \tfrac{\partial^2}{\partial(v^u)^2}(id - P_{\kappa_i}) f^s(P_{\kappa_i} v_i^{-,u}(t))\| \tau_1 d\tau_2 d\tau_1 \|v_i^{-,u}(t)\|^2 \\
&\quad \leq (C_1 + C_2)\|v_i^{-,s}(t)\|(\|v_i^{-,s}(t)\| + \|v_i^{-,u}(t)\|) \\
&\qquad + \int_0^1\int_0^1\int_0^1 \| \tfrac{\partial^3}{\partial(v^u)^2\partial\gamma} f^s(\tau_3\gamma_{\kappa_i}^-(t) + \tau_1\tau_2(\tau_3(v_i^{-,s}(t) + (id - P_{\kappa_i})v_i^{-,u}(t)) + P_{\kappa_i} v_i^{-,u}(t)))\| \tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1 \\
&\qquad \cdot \|v_i^{-,u}(t)\|^2 \|\gamma_{\kappa_i}^-(t)\| \\
&\qquad + \int_0^1\int_0^1\int_0^1 \| \tfrac{\partial^3}{\partial(v^u)^2\partial v^s} f^s(\tau_3\gamma_{\kappa_i}^-(t) + \tau_1\tau_2(\tau_3(v_i^{-,s}(t) + (id - P_{\kappa_i})v_i^{-,u}(t)) + P_{\kappa_i} v_i^{-,u}(t)))\| \tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1 \\
&\qquad \cdot \left(\|v_i^{-,u}(t)\|^2 \cdot \|v_i^{-,s}(t)\|\right) \\
&\qquad + \int_0^1\int_0^1\int_0^1 \| \tfrac{\partial^3}{\partial(v^u)^2\partial(id-P)v^u} f^s(\tau_3\gamma_{\kappa_i}^-(t) + \tau_1\tau_2(\tau_3(v_i^{-,s}(t) + (id - P_{\kappa_i})v_i^{-,u}(t)) + P_{\kappa_i} v_i^{-,u}(t)))\| \tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1 \\
&\qquad \cdot \left(\|v_i^{-,u}(t)\|^2 \cdot \|(id - P_{\kappa_i})v_i^{-,u}(t)\|\right) \\
&\qquad + \int_0^1\int_0^1 \| \tfrac{\partial^2}{\partial(v^u)^2}(id - P_{\kappa_i}) f^s(P_{\kappa_i} v_i^{-,u}(t))\| \tau_1 d\tau_2 d\tau_1 \|v_i^{-,u}(t)\|^2
\end{aligned}
$$

First we focus on the last term. With the definition of f^s and (3.83) we find

$$
\begin{aligned}
\tfrac{\partial^2}{\partial(v^u)^2}(id - P_{\kappa_i}) f^s(P_{\kappa_i} v_i^{-,u}(t)) &= \tfrac{\partial^2}{\partial(v^u)^2}(id - P_{\kappa_i})(id - P_{\kappa_i}^-(t)) f(P_{\kappa_i} v_i^{-,u}(t)) \\
&= \tfrac{\partial^2}{\partial(v^u)^2}(id - P_{\kappa_i})(id - P_{\kappa_i}^-(t)) P_{\kappa_i} f(P_{\kappa_i} v_i^{-,u}(t)) \\
&= (id - P_{\kappa_i})(id - P_{\kappa_i}^-(t)) P_{\kappa_i} \tfrac{\partial^2}{\partial(v^u)^2} f(P_{\kappa_i} v_i^{-,u}(t)),
\end{aligned}
$$

where $(id - P_{\kappa_i})(id - P_{\kappa_i}^-(t)) = (id - P_{\kappa_i}^-(t))$, cf. (3.67).

Now, since the vector field f is at least of differentiability class C^3 the partial derivatives are all bounded. Hence we end up with

$$
\left.
\begin{aligned}
\|h_{\kappa_i}^{-,s}(t, (v_i^{-,s} + v_i^{-,u})(t))\| \;\leq\; & (C_1 + C_2)\|v_i^{-,s}(t)\|(\|v_i^{-,s}(t)\| + \|v_i^{-,u}(t)\|) \\
& + C_3(\|\gamma_{\kappa_i}^-(t)\| + \|v_i^{-,s}(t)\| + \|(id - P_{\kappa_i})v_i^{-,u}(t)\|)\|v_i^{-,u}(t)\|^2 \\
& + C_4 \|(id - P_{\kappa_i}^-(t))P_{\kappa_i}\| \; \|v_i^{-,u}(t)\|^2.
\end{aligned}
\right\} \tag{3.84}
$$

We find from (3.79) that $\|\gamma_{\kappa_i}^-(t)\| \leq K e^{\alpha^u t}$ and from (3.73) or (3.75), respectively, that

$$
\begin{aligned}
\|(id - P_{\kappa_i}^-(t))P_{\kappa_i}\| &\leq \mathcal{K} \begin{cases} e^{\alpha^u t}, & \text{if (H3.7) applies,} \\ e^{\frac{1}{2}(\alpha^u - \alpha^s)t}, & \text{else,} \end{cases} \\
\|(id - P_{\kappa_i})v_i^{-,u}(t)\| = \|(id - P_{\kappa_i})P_{\kappa_i}^-(t)v_i^{-,u}(t)\| &\leq \mathcal{K}\|v_i^{-,u}(t)\| \begin{cases} e^{\alpha^u t}, & \text{if (H3.7) applies,} \\ e^{\frac{1}{2}(\alpha^u - \alpha^s)t}, & \text{else.} \end{cases}
\end{aligned}
$$

These estimates in combination with (3.84) provides the claimed estimate for $t \leq -\Omega$.

Analogously we obtain for $h^{+,s}_{\kappa_i}$:

$$\begin{aligned}\|h^{+,s}_{\kappa_i}(t,(v^{+,s}_i+v^{+,u}_i)(t))\| &\le (C_1+C_2)\|v^{+,s}_i(t)\|(\|v^{+,s}_i(t)\|+\|v^{+,u}_i(t)\|)\\ &\quad +C_3(\|\gamma^{+}_{\kappa_i}(t)\|+\|v^{+,s}_i(t)\|+\|P_{\kappa_i}v^{+,u}_i(t)\|)\|v^{+,u}_i(t)\|^2\\ &\quad +C_4\|(id-P_{\kappa_i})(id-P^{+}_{\kappa_i}(t))\|\ \|v^{+,u}_i(t)\|^2\\ &\le M(e^{\alpha^s t}\|v^{+,u}_i(t)\|+\|v^{+,s}_i(t)\|)(\|v^{+,s}_i(t)\|+\|v^{+,u}_i(t)\|)\end{aligned}$$

since $\|\gamma^{+}_{\kappa_i}(t)\|\le Ke^{\alpha^s t}$ and $\|(id-P^{+}_{\kappa_i}(t))(id-P_{\kappa_{i+1}})\|=\|P_{\kappa_{i+1}}P^{+}_{\kappa_i}(t)\|\le \mathcal{K}e^{\alpha^s t}$ due to (3.79) and (3.73). It remains to consider $h^{+,s}_{\kappa_i}$ for $t\in[0,\Omega]$ and $h^{-,s}_{\kappa_i}$ for $t\in[-\Omega,0]$. We confine ourselves with showing the estimate for $h^{+,s}_{\kappa_i}$. For $h^{-,s}_{\kappa_i}$ the proof goes analog. Since Ω is fixed this is simply done by choosing the constants adequately. To be more precise we have for $t\in[0,\Omega]$, since $h^{+,s}_{\kappa_i}(t,v^{+}_i(t),\lambda)=O(\|v^{+}_i(t)\|^2)$,

$$\begin{aligned}\|h^{+,s}_{\kappa_i}(t,v^{+}_i(t),\lambda)\| &\le K(\|v^{+,s}_i(t)\|+\|v^{+,u}_i(t)\|)^2\\ &\le Ke^{-\alpha^s(\Omega-t)}(\|v^{+,s}_i(t)\|+\|v^{+,u}_i(t)\|)^2\\ &\le \tilde{K}(e^{\alpha^s t}\|v^{+,u}_i(t)\|+\|v^{+,s}_i(t)\|)(\|v^{+,s}_i(t)\|+\|v^{+,u}_i(t)\|)\end{aligned}$$

with $\tilde{K}=Ke^{-\alpha^s\Omega}$. □

To gain a similar property for $h^{\pm,u}_{\kappa_i}$ we follow the idea in [Kno04] and decompose $v^{\pm,s}_i(t)$ into

$$v^{\pm,s}_i(t)=v^{\pm,ss}_i(t)+v^{\pm,su}_i(t)$$

where

$$v^{\pm,ss}_i(t)\in\Phi^{\pm}_{\kappa_i}(\lambda)(t,0)W^{\pm}_{\kappa_i}\quad\text{and}\quad v^{\pm,su}_i(t)\in\Phi^{\pm}_{\kappa_i}(\lambda)(t,0)\ \mathrm{span}\{f(\gamma^{\pm}_{\kappa_i}(\lambda)(0),\lambda)\}. \tag{3.85}$$

With this it was shown in [Kno04, Equation (5.25)] that

$$\|h^{\pm,u}_{\kappa_i}(t,v^{\pm}_i(t),\lambda)\|\ \le\ M(\|v^{\pm,u}_i(t)\|+\|v^{\pm,su}_i(t)\|)\|v^{\pm}_i(t)\|.$$

However for our further analysis we need to state this estimate more sophisticated.

Lemma 3.4.8. *Assume Hypotheses (H3.1)-(H3.5). There exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $|\lambda|<c$ and $\boldsymbol{\omega}$ with $\inf\boldsymbol{\omega}>\Omega$ there exists a constant $M>0$ such that the non-linearities $h^{\pm,u}_{\kappa_i}$ satisfy for $t\in[0,\omega]$ or $t\in[-\omega,0]$, respectively,*

$$\|h^{\pm,u}_{\kappa_i}(t,v^{\pm}_i(t),\lambda)\|\ \le\ M(\|v^{\pm,u}_i(t)\|+\|v^{\pm,su}_i(t)\|)\|v^{\pm}_i(t)\|\big(\|v^{\pm}_i(t)\|+\|\gamma^{\pm}_{\kappa_i}(t)-p_{\kappa_{i+1}/\kappa_i}\|\big)^{\nu-2}.$$

Moreover we find for $h^{-,u}_{\kappa_i}(t,v^{-}_i(t),\lambda)$, $t\in[-\omega,0]$

$$\begin{aligned}&\|h^{-,u}_{\kappa_i}(t,v^{-}_i(t),\lambda)-\tfrac12 D_1^2f^u(\gamma^{-}_{\kappa_i}(t),\lambda)[v^{-,u}_i(t),v^{-,u}_i(t)]-\tfrac16 D^3f^u(\gamma^{-}_{\kappa_i}(\lambda)(t),\lambda)[v^{-,u}_i(t),v^{-,u}_i(t),v^{-,u}_i(t)]\|\\ &\le\ M\Big((\|v^{-,u}_i(t)\|+\|v^{-,su}_i(t)\|)\|v^{-,s}_i(t)\|\big(\|v^{-}_i(t)\|+\|\gamma^{-}_{\kappa_i}(t)-p_{\kappa_i}\|\big)^{\nu-2}\\ &\qquad +\|v^{-,u}_i(t)\|^4\big(\|v^{-}_i(t)\|+\|\gamma^{-}_{\kappa_i}(t)-p_{\kappa_i}\|\big)^{\max\{0,\nu-4\}}\Big).\end{aligned}$$

The two isolated terms on the left-hand side of the second estimate of $h^{-,u}_{\kappa_i}$ will play an important role when it comes to find terms of leading exponential rates within the residual term $R_i(\boldsymbol{\omega}, \lambda, \kappa)$.

Proof. We prove these statements again by invoking the Mean-value-theorem. To this end recall the definition of $h^{\pm,u}_{\kappa_i}$ given in (3.14) and Definition 3.3.4 as well as the definition of f^u in (3.80). As before we suppress the dependency on λ.

$$\begin{aligned}
&h^{\pm,u}_{\kappa_i}(t, (v^{\pm,s}_i + v^{\pm,u}_i)(t)) \\
&\quad = f^u(\gamma^{\pm}_{\kappa_i}(t) + v^{\pm,s}_i(t) + v^{\pm,u}_i(t)) - f^u(\gamma^{\pm}_{\kappa_i}(t)) - Df^u(\gamma^{\pm}_{\kappa_i}(t))[v^{\pm,s}_i(t) + v^{\pm,u}_i(t)] \\
&\quad = \left[\int_0^1 \tfrac{\partial}{\partial v^{ss}} f^u(\gamma^{\pm}_{\kappa_i}(t) + \tau(v^{\pm,ss}_i(t) + v^{\pm,su}_i(t) + v^{\pm,u}_i(t))) - \tfrac{\partial}{\partial v^{ss}} f^u(\gamma^{\pm}_{\kappa_i}(t)) d\tau\right] v^{\pm,ss}_i(t) \\
&\qquad + \left[\int_0^1 \tfrac{\partial}{\partial v^{su}} f^u(\gamma^{\pm}_{\kappa_i}(t) + \tau(v^{\pm,ss}_i(t) + v^{\pm,su}_i(t) + v^{\pm,u}_i(t))) - \tfrac{\partial}{\partial v^{su}} f^u(\gamma^{\pm}_{\kappa_i}(t)) d\tau\right] v^{\pm,su}_i(t) \\
&\qquad + \left[\int_0^1 \tfrac{\partial}{\partial v^{u}} f^u(\gamma^{\pm}_{\kappa_i}(t) + \tau(v^{\pm,s}_i(t) + v^{\pm,u}_i(t))) - \tfrac{\partial}{\partial v^{u}} f^u(\gamma^{\pm}_{\kappa_i}(t)) d\tau\right] v^{\pm,u}_i(t)
\end{aligned}$$

Applying again the Mean-value-theorem yields

$$\left.\begin{aligned}
&h^{\pm,u}_{\kappa_i}(t, (v^{\pm,s}_i + v^{\pm,u}_i)(t)) \\
&\quad = \left[\int_0^1\int_0^1 \tfrac{\partial^2}{\partial (v^{ss})^2} f^u(\gamma^{\pm}_{\kappa_i}(t) + \tau_1\tau_2(v^{\pm,s}_i(t) + v^{\pm,u}_i(t)))\tau_1 d\tau_2 d\tau_1\right] [v^{\pm,ss}_i(t), v^{\pm,ss}_i(t)] \\
&\quad + 2\left[\int_0^1\int_0^1 \tfrac{\partial^2}{\partial v^{ss}\partial v^{su}} f^u(\gamma^{\pm}_{\kappa_i}(t) + \tau_1\tau_2(v^{\pm,s}_i(t) + v^{\pm,u}_i(t)))\tau_1 d\tau_2 d\tau_1\right] [v^{\pm,ss}_i(t), v^{\pm,su}_i(t)] \\
&\quad + \left[\int_0^1\int_0^1 \tfrac{\partial^2}{\partial (v^{su})^2} f^u(\gamma^{\pm}_{\kappa_i}(t) + \tau_1\tau_2(v^{\pm,s}_i(t) + v^{\pm,u}_i(t)))\tau_1 d\tau_2 d\tau_1\right] [v^{\pm,su}_i(t), v^{\pm,su}_i(t)] \\
&\quad + 2\left[\int_0^1\int_0^1 \tfrac{\partial^2}{\partial v^{u}\partial v^{s}} f^u(\gamma^{\pm}_{\kappa_i}(t) + \tau_1\tau_2(v^{\pm,s}_i(t) + v^{\pm,u}_i(t)))\tau_1 d\tau_2 d\tau_1\right] [v^{\pm,u}_i(t), v^{\pm,s}_i(t)] \\
&\quad + \left[\int_0^1\int_0^1 \tfrac{\partial^2}{\partial (v^{u})^2} f^u(\gamma^{\pm}_{\kappa_i}(t) + \tau_1\tau_2(v^{\pm,s}_i(t) + v^{\pm,u}_i(t)))\tau_1 d\tau_2 d\tau_1\right] [v^{\pm,u}_i(t), v^{\pm,u}_i(t)].
\end{aligned}\right\} \tag{3.86}$$

For further investigation of the first summand of (3.86) we make use of the Hypothesis (H3.5): Let $\|v^{\pm,s}_i(t)\| < \varepsilon$. Then we find $v^{+,ss}_i(t) \in W^s_{\Sigma_{\lambda,t}}(p_{\kappa_{i+1}}) \cap B(\gamma^{\pm}_{\kappa_i}(\lambda)(t), \varepsilon)$ and $v^{-,ss}_i(t) \in W^u_{\Sigma_{\lambda,t}}(p_{\kappa_i}) \cap B(\gamma^{\pm}_{\kappa_i}(\lambda)(t), \varepsilon)$. Hence we find

$$P^{\pm}_{\kappa_i}(t) \underbrace{f(\gamma^{\pm}_{\kappa_i}(\lambda)(t) + v^{\pm,ss}_i(t), \lambda)}_{\in TW^{s/u}} \equiv 0.$$

Therefore we have $\frac{\partial^2}{\partial (v^{ss})^2} f^u(\gamma^{\pm}_{\kappa_i}(\lambda)(t) + v^{\pm,ss}_i(t), \lambda) = 0$. So we can rewrite the first summand on the right-hand side of (3.86) by adding a zero and apply the Mean-value-theorem again.

$$\left.\begin{aligned}
&\int_0^1\int_0^1 \tfrac{\partial^2}{\partial(v^{ss})^2} f^u(\gamma^\pm_{\kappa_i}(t)+\tau_1\tau_2(v_i^{\pm,ss}(t)+v_i^{\pm,su}(t)+v_i^{\pm,u}(t)))\tau_1 d\tau_2 d\tau_1 [v_i^{\pm,ss}(t),v_i^{\pm,ss}(t)]\\
&= \Bigg[\int_0^1\int_0^1 \Big(\tfrac{\partial^2}{\partial(v^{ss})^2} f^u(\gamma^\pm_{\kappa_i}(t)+\tau_1\tau_2(v_i^{\pm,ss}(t)+v_i^{\pm,su}(t)+v_i^{\pm,u}(t)))\\
&\quad -\tfrac{\partial^2}{\partial(v^{ss})^2} f^u(\gamma^\pm_{\kappa_i}(t)+\tau_1\tau_2 v_i^{\pm,ss}(t))\Big)\tau_1 d\tau_2 d\tau_1\Bigg][v_i^{\pm,ss}(t),v_i^{\pm,ss}(t)]\\
&= \Bigg[\int_0^1\int_0^1\int_0^1 \tfrac{\partial^3}{\partial(v^{ss})^2\partial v^{su}} f^u(\gamma^\pm_{\kappa_i}(t)+\tau_1\tau_2(v_i^{\pm,ss}(t)+\tau_3(v_i^{\pm,su}(t)+v_i^{\pm,u}(t))))\tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1\Bigg]\\
&\quad [v_i^{\pm,ss}(t),v_i^{\pm,ss}(t),v_i^{\pm,su}(t)]\\
&\quad +\Bigg[\int_0^1\int_0^1\int_0^1 \tfrac{\partial^3}{\partial(v^{ss})^2\partial v^{u}} f^u(\gamma^\pm_{\kappa_i}(t)+\tau_1\tau_2(v_i^{\pm,ss}(t)+\tau_3(v_i^{\pm,su}(t)+v_i^{\pm,u}(t))))\tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1\Bigg]\\
&\quad [v_i^{\pm,ss}(t),v_i^{\pm,ss}(t),v_i^{\pm,u}(t)]
\end{aligned}\right\}\quad (3.87)$$

Since the vector field f is smooth the partial derivatives are all bounded. What is even more we find for $k\geq 2$ with Definition 3.4.2 of the constant $\nu\geq 2$ that

$$\begin{aligned}
D^k f(\gamma^+_{\kappa_i}(t)+v_i^+(t)) &= \sum_{l=k}^{\nu-1}\frac{D^l f(p_{\kappa_{i+1}})}{(l-k)!}(\gamma^+_{\kappa_i}(t)+v_i^+(t)-p_{\kappa_{i+1}})^{l-k}+O(\|\gamma^+_{\kappa_i}(t)+v_i^+(t)-p_{\kappa_i}\|^{\nu-k})\\
&= O\big((\|v_i^+(t)\|+\|\gamma^+_{\kappa_i}(t)-p_{\kappa_{i+1}}\|)^{\nu-k}\big),
\end{aligned}$$

and analogously

$$D^k f(\gamma^-_{\kappa_i}(t)+v_i^-(t)) = O\big((\|v_i^-(t)\|+\|\gamma^-_{\kappa_i}(t)-p_{\kappa_i}\|)^{\nu-k}\big).$$

Summarizing we find from equation (3.86) with the estimates for the partial derivatives of f

$$\begin{aligned}
&h^{\pm,u}_{\kappa_i}(t,v_i^\pm(t))\\
&= O\big((\|v_i^{\pm,ss}(t)\|^2+(\|v_i^{\pm,su}(t)\|+\|v_i^{\pm,ss}(t)\|)\|v_i^{\pm,su}(t)\|+\|v_i^{\pm,u}(t)\|(\|v_i^{\pm,u}(t)\|+\|v_i^{\pm,s}(t)\|))\\
&\quad (\|v_i^\pm(t)\|+\|\gamma^\pm_{\kappa_i}(t)-p_{\kappa_{i+1}/\kappa_i}\|)^{\nu-2}\big)\\
&= O\big((\|v_i^{\pm,ss}(t)\|^2+\|v_i^{\pm,s}(t)\|\|v_i^{\pm,su}(t)\|+\|v_i^{\pm,u}(t)\|\|v_i^\pm(t)\|)(\|v_i^\pm(t)\|+\|\gamma^\pm_{\kappa_i}(t)-p_{\kappa_{i+1}/\kappa_i}\|)^{\nu-2}\big).
\end{aligned}$$

Replacing $\|v_i^{\pm,ss}(t)\|^2$ by the estimates we gain from (3.87) we finally obtain

$$\left.\begin{aligned}
&h^{\pm,u}_{\kappa_i}(t,v_i^\pm(t))\\
&= O\left((\|v_i^{\pm,s}(t)\|\|v_i^{\pm,su}(t)\|+\|v_i^{\pm,u}(t)\|\|v_i^\pm(t)\|)(\|v_i^\pm(t)\|+\|\gamma^\pm_{\kappa_i}(t)-p_{\kappa_{i+1}/\kappa_i}\|)^{\nu-2}\right)\\
&\quad +O\left(\|v_i^{\pm,ss}(t)\|^2(\|v_i^{\pm,su}(t)\|+\|v_i^{\pm,u}(t)\|)(\|v_i^\pm(t)\|+\|\gamma^\pm_{\kappa_i}(t)-p_{\kappa_{i+1}/\kappa_i}\|)^{\max\{0,\nu-3\}}\right)\\
&= O\left((\|v_i^{\pm,su}(t)\|+\|v_i^{\pm,u}(t)\|)\|v_i^\pm(t)\|(\|v_i^\pm(t)\|+\|\gamma^\pm_{\kappa_i}(t)-p_{\kappa_{i+1}/\kappa_i}\|)^{\nu-2}\right).
\end{aligned}\right\}\quad (3.88)$$

Here we used $\|v_i^{\pm,ss}(t)\|,\|v_i^{\pm,s}(t)\|\leq\|v_i^\pm(t)\|$ and $\|v_i^\pm(t)\|\leq\|\gamma^\pm_{\kappa_i}(t)-p_{\kappa_{i+1}/\kappa_i}\|+\|v_i^\pm(t)\|$.

It remains to show the second estimate. To this end we have a closer look on the last term of (3.86).

Again we add an appropriately zero-term and apply the Mean-value-theorem:

$$
\begin{aligned}
&\int_0^1\int_0^1 \tfrac{\partial^2}{\partial(v^u)^2} f^u(\gamma^-_{\kappa_i}(t) + \tau_1\tau_2(v_i^{-,s}(t) + v_i^{-,u}(t)))\tau_1 d\tau_2 d\tau_1 [v_i^{-,u}(t), v_i^{-,u}(t)] \\
= &\left[\int_0^1\int_0^1 \left(\tfrac{\partial^2}{\partial(v^u)^2} f^u(\gamma^-_{\kappa_i}(t) + \tau_1\tau_2(v_i^{-,s}(t) + v_i^{-,u}(t))) - \tfrac{\partial^2}{\partial(v^u)^2} f^u(\gamma^-_{\kappa_i}(t)) + \tfrac{\partial^2}{\partial(v^u)^2} f^u(\gamma^-_{\kappa_i}(t))\right)\tau_1 d\tau_2 d\tau_1\right] \\
&\quad [v_i^{-,u}(t), v_i^{-,u}(t)] \\
= &\left[\int_0^1\int_0^1\int_0^1 \tfrac{\partial^3}{\partial(v^u)^2\partial v^s} f^u(\gamma^-_{\kappa_i}(t) + \tau_1\tau_2\tau_3(v_i^{-,s}(t) + v_i^{-,u}(t)))\tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1\right] \\
&\quad [v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,s}(t)] \\
&\quad + \left[\int_0^1\int_0^1\int_0^1 \tfrac{\partial^3}{\partial(v^u)^3} f^u(\gamma^-_{\kappa_i}(t) + \tau_1\tau_2\tau_3(v_i^{-,s}(t) + v_i^{-,u}(t)))\tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1\right] \\
&\quad [v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,u}(t)] \\
&\quad + \underbrace{\int_0^1\int_0^1 \tau_1 d\tau_2 d\tau_1}_{=1/2} D_1^2 f^u(\gamma^-_{\kappa_i}(t))[v_i^{-,u}(t), v_i^{-,u}(t)]
\end{aligned}
$$

The extraction of the term $D^3 f^u(\gamma^-_{\kappa_i})[v_i^{-,u}, v_i^{-,u}, v_i^{-,u}]$ takes place in an analogously way as the extraction of $D_1^2 f^u(\gamma^-_{\kappa_i})[v_i^{-,u}, v_i^{-,u}]$. We start from the middle summand of the latest equation and repeat the procedure:

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^1 \tfrac{\partial^3}{\partial(v^u)^3} f^u(\gamma^-_{\kappa_i}(t) + \tau_1\tau_2\tau_3(v_i^{-,s}(t) + v_i^{-,u}(t)))\tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1 [v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,u}(t)] \\
= &\left[\int_0^1\int_0^1 \left(\int_0^1 \tfrac{\partial^3}{\partial(v^u)^3} f^u(\gamma^-_{\kappa_i}(t) + \tau_1\tau_2\tau_3(v_i^{-,s}(t) + v_i^{-,u}(t)))\right.\right. \\
&\qquad \left.\left. - \tfrac{\partial^3}{\partial(v^u)^3} f^u(\gamma^-_{\kappa_i}(t)) + \tfrac{\partial^3}{\partial(v^u)^3} f^u(\gamma^-_{\kappa_i}(t))\right)\tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1\right] [v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,u}(t)] \\
= &\left[\int_0^1\int_0^1\int_0^1\int_0^1 \tfrac{\partial^4}{\partial(v^u)^3\partial v^s} f^u(\gamma^-_{\kappa_i}(t) + \tau_1\tau_2\tau_3\tau_4(v_i^{-,s}(t) + v_i^{-,u}(t)))\tau_1^3\tau_2^2\tau_3 d\tau_4 d\tau_3 d\tau_2 d\tau_1\right] \\
&\quad [v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,s}(t)] \\
&\quad + \left[\int_0^1\int_0^1\int_0^1\int_0^1 \tfrac{\partial^4}{\partial(v^u)^4} f^u(\gamma^-_{\kappa_i}(t) + \tau_1\tau_2\tau_3\tau_4(v_i^{-,s}(t) + v_i^{-,u}(t)))\tau_1^3\tau_2^2\tau_3 d\tau_4 d\tau_3 d\tau_2 d\tau_1\right] \\
&\quad [v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,u}(t)] \\
&\quad + \underbrace{\int_0^1\int_0^1\int_0^1 \tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1}_{=1/6} D^3 f^u(\gamma^-_{\kappa_i}(t))[v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,u}(t)].
\end{aligned}
$$

Finally we simply have to collect the single terms. From the two latter equations we find

$$\begin{aligned}
&\int_0^1\int_0^1 \frac{\partial^2}{\partial(v^u)^2} f^u(\gamma_{\kappa_i}^-(t)+\tau_1\tau_2(v_i^{-,s}(t)+v_i^{-,u}(t)))\tau_1 d\tau_2 d\tau_1 [v_i^{-,u}(t),v_i^{-,u}(t)]\\
&\quad= D_1^2 f^u(\gamma_{\kappa_i}^-(t))[v_i^{-,u}(t),v_i^{-,u}(t)] + D^3 f^u(\gamma_{\kappa_i}^-(t))[v_i^{-,u}(t),v_i^{-,u}(t),v_i^{-,u}(t)]\\
&\qquad+O\big(\|v_i^{-,u}(t)\|^2\|v_i^{-,s}(t)\|(\|v_i^-(t)\|+\|\gamma_{\kappa_i}^-(t)-p_{\kappa_i}\|)^{\max\{0,\nu-3\}}\big)\\
&\qquad+O\big((\|v_i^{-,u}(t)\|^3(\|v_i^{-,s}(t)\|+\|v_i^{-,u}(t)\|)(\|v_i^-(t)\|+\|\gamma_{\kappa_i}^-(t)-p_{\kappa_i}\|)^{\max\{0,\nu-4\}}\big)
\end{aligned}$$

And this together with (3.86) and (3.87) yields

$$\left.\begin{aligned}
&h_{\kappa_i}^{-,u}(t,v_i^-(t))\\
&\quad= D_1^2 f^u(\gamma_{\kappa_i}^-(t))[v_i^{-,u}(t),v_i^{-,u}(t)] + D^3 f^u(\gamma_{\kappa_i}^-(t))[v_i^{-,u}(t),v_i^{-,u}(t),v_i^{-,u}(t)]\\
&\qquad+O\left((\|v_i^{-,su}(t)\|+\|v_i^{-,u}(t)\|)\|v_i^{-,s}(t)\|(\|v_i^-(t)\|+\|\gamma_{\kappa_i}^-(t)-p_{\kappa_i}\|)^{\nu-2}\right)\\
&\qquad+O\left((\|v_i^{-,u}(t)\|^2\|v_i^{-,s}(t)\|+\|v_i^{-,ss}(t)\|^2(\|v_i^{-,su}(t)\|+\|v_i^{-,u}(t)\|)\right)\\
&\qquad\quad(\|v_i^-(t)\|+\|\gamma_{\kappa_i}^-(t)-p_{\kappa_i}\|)^{\max\{0,\nu-3\}}\Big)\\
&\qquad+O\left((\|v_i^{-,u}(t)\|^3(\|v_i^{-,s}(t)\|+\|v_i^{-,u}(t)\|))(\|v_i^-(t)\|+\|\gamma_{\kappa_i}^-(t)-p_{\kappa_i}\|)^{\max\{0,\nu-4\}}\right)
\end{aligned}\right\} \tag{3.89}$$

In the end we only want to simplify this expression by using the inequality chain

$$\left.\begin{aligned}\|v_i^{-,ss}(t)\|\le\|v_i^{-,s}(t)\|\\ \|v_i^{-,u}(t)\|\end{aligned}\right\}\le\|v_i^-(t)\|\le\|v_i^-(t)\|+\|\gamma_{\kappa_i}^-(t)-p_{\kappa_i}\|.$$

With this we find

$$\begin{aligned}
&\|v_i^{-,u}(t)\|^2\|v_i^{-,s}(t)\|+\|v_i^{-,ss}(t)\|^2(\|v_i^{-,su}(t)\|+\|v_i^{-,u}(t)\|)\\
&\qquad\le(\|v_i^{-,u}(t)\|+\|v_i^{-,su}(t)\|)\|v_i^{-,s}(t)\|(\|v_i^-(t)\|+\|\gamma_{\kappa_i}^-(t)-p_{\kappa_i}\|),
\end{aligned}$$

and

$$\|v_i^{-,u}(t)\|^3\|v_i^{-,s}(t)\|\le\|v_i^{-,u}(t)\|\|v_i^{-,s}(t)\|(\|v_i^-(t)\|+\|\gamma_{\kappa_i}^-(t)-p_{\kappa_i}\|)^2,$$

Thus the second O-term in (3.89) is included in the first O-term as well as the first part of the last O-term. This concludes the proof. □

Now we have everything we need to take a closer look on $v_i^\pm$. To this end we define, in view of Lemma 3.4.7 and for the sake of shortness, the following constant:

$$\alpha^w := \begin{cases} -\alpha^u, & \text{if additionally (H3.7) applies,}\\ (\alpha^s-\alpha^u)/2, & \text{else.}\end{cases} \tag{3.90}$$

Lemma 3.4.9. *Assume Hypotheses (H3.1)-(H3.6). Let $v_i^\pm$ be the solutions of the boundary value problem ((3.12),(3.13)) and denote by $v_i^{\pm,s}$ and $v_i^{\pm,u}$ their projections as defined in Definition 3.3.4. Then there exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $|\lambda|<c$ and $\boldsymbol{\omega}$ with $\inf\boldsymbol{\omega}>\Omega$*

the following estimates apply with some $\bar K>0$*:*

$$\begin{aligned}
\|v_i^{+,s}(t)\| &\le \bar K e^{\alpha^s t}(e^{2\alpha^w(\omega_{i+1}-t)}e^{2\alpha^w\omega_{i+1}}+e^{\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}}+e^{2\alpha^s\omega_i}), && t\in[0,\omega_{i+1}],\\
\|v_i^{-,s}(t)\| &\le \bar K e^{-\alpha^w t}(e^{2\alpha^s(\omega_i+t)}e^{2\alpha^s\omega_i}+e^{2\alpha^s\omega_i}e^{\alpha^w\omega_{i+1}}+e^{2\alpha^w\omega_{i+1}}), && t\in[-\omega_i,0],\\
\|v_i^{+,u}(t)\| &\le \bar K (e^{\alpha^w(\omega_{i+1}-t)}e^{\alpha^w\omega_{i+1}}+e^{4\alpha^s\omega_i}), && t\in[0,\omega_{i+1}],\\
\|v_i^{-,u}(t)\| &\le \bar K (e^{\alpha^s(\omega_i+t)}e^{\alpha^s\omega_i}+e^{4\alpha^w\omega_{i+1}}), && t\in[-\omega_i,0].
\end{aligned}$$

Proof. At first we want to show that v_i^+ and v_i^-, $i\in\mathbb{Z}$, are exponentially small for Ω sufficiently large. To this end recall (3.21) with $\boldsymbol{g}$ replaced by $\boldsymbol{\mathcal{H}}(\boldsymbol{v},\lambda,\kappa)$:

$$\|\boldsymbol{v}\|_{V_{\boldsymbol{\omega}}}\le \hat C(\|\boldsymbol{\mathcal{H}}\|_{V_{\boldsymbol{\omega}}}+\|\boldsymbol{d}\|_{l^\infty_{\mathbb{R}^n}}). \tag{3.91}$$

From Definition 3.3.5 of $\boldsymbol{\mathcal{H}}(\boldsymbol{v},\lambda,\kappa)=(H_i^+(v_i^+,\lambda,\kappa),H_i^-(v_i^-,\lambda,\kappa))_{i\in\mathbb{Z}}$ it follows that $H_i^\pm(0,\lambda,\kappa)=0$ and $D_1H_i^\pm(0,\lambda,\kappa)=0$. Hence we have uniformly in λ

$$H_i^\pm(v_i^\pm,\lambda,\kappa)=O(\|v_i^\pm\|^2),$$

and thus

$$\|\boldsymbol{\mathcal{H}}\|_{V_{\boldsymbol{\omega}}}=O(\|\boldsymbol{v}\|^2_{V_{\boldsymbol{\omega}}}).$$

Therefore there is an $\varepsilon>0$ such that for $\|\boldsymbol{v}\|_{V_{\boldsymbol{\omega}}}<\varepsilon$

$$\tilde C\|\boldsymbol{\mathcal{H}}\|_{V_{\boldsymbol{\omega}}}\le \frac12\|\boldsymbol{v}\|_{V_{\boldsymbol{\omega}}}.$$

Together with (3.91) this shows that there is a constant C such that

$$\|\boldsymbol{v}\|_{V_{\boldsymbol{\omega}}}\le C\|\boldsymbol{d}\|_{l^\infty_{\mathbb{R}^n}}.$$

Now, the definition of $\boldsymbol{d}=(d_i)_{i\in\mathbb{Z}}$, cf. (3.13), finally proves that $\|\boldsymbol{v}\|_{V_{\boldsymbol{\omega}}}$ is exponentially small for Ω sufficiently large.

Let us now go into the estimates in more detail. From the first line in (3.42) we gain with $-\alpha^+_{\kappa_i}=\alpha^s-\delta$ for some positive δ such that $\mathrm{Re}\mu^s_{\kappa_{i+1}}(\lambda)<\alpha^s-\delta$, cf. (3.18) and (3.66), by invoking Lemma 3.4.7

$$\begin{aligned}
\|v_i^{+,s}(t)\| &\le KLe^{(\alpha^s-\delta)t}\|v_i^{-,u}(0)\|+K\int_0^t e^{(\alpha^s-\delta)(t-s)}\|h_i^{+,s}(s,v_i^+(s),\lambda)\|ds\\
&\le KLe^{\alpha^s t}\|v_i^{-,u}(0)\|+KM\int_0^t e^{(\alpha^s-\delta)(t-s)}(e^{\alpha^s s}\|v_i^{+,u}(s)\|+\|v_i^{+,s}(s)\|)(\|v_i^{+,u}(s)\|+\|v_i^{+,s}(s)\|)ds\\
&\le e^{\alpha^s t}\left(KL\|v_i^{-,u}(0)\|+KM(\|v_i^{+,u}\|+\sup_{s\in[0,t]}e^{-\alpha^s s}\|v_i^{+,s}(s)\|)(\|v_i^{+,u}\|+\|v_i^{+,s}\|)\int_0^t e^{-\delta(t-s)}ds\right).
\end{aligned}$$

Multiplying by $e^{-\alpha^s t}$ and applying the supreme norm we obtain

$$\sup_{t\in[0,\omega_{i+1}]}(e^{-\alpha^s t}\|v_i^{+,s}(t)\|)\le KL\|v_i^{-,u}(0)\|+KM\left(\|v_i^{+,u}\|+\sup_{s\in[0,\omega_{i+1}]}(e^{-\alpha^s s}\|v_i^{+,s}(s)\|)\right)(\|v_i^{+,u}\|+\|v_i^{+,s}\|).$$

Since $\|v_i^+\|$ is exponentially small we find for $\Omega<\inf\boldsymbol{\omega}$ large enough that $1-KM(\|v_i^{+,u}\|+\|v_i^{+,s}\|)>\frac12$

and thus there is a constant $\tilde{K}_1$ such that

$$\sup_{t\in[0,\omega_{i+1}]}(e^{-\alpha^s t}\|v_i^{+,s}(t)\|) \leq \tilde{K}_1(\|v_i^{-,u}(0)\| + \|v_i^{+,u}\|(\|v_i^{+,u}\| + \|v_i^{+,s}\|)). \tag{3.92}$$

Analogously we find from (3.42) with $\beta_{\kappa_i}^- = -\alpha^w + \delta$ for $v_i^{-,s}$ that

$$\sup_{t\in[-\omega_i,0]}(e^{\alpha^w t}\|v_i^{-,s}(t)\|) \leq \tilde{K}_1(\|v_i^{+,u}(0)\| + \|v_i^{-,u}\|(\|v_i^{-,u}\| + \|v_i^{-,s}\|)). \tag{3.93}$$

This finally leads to

$$\left.\begin{aligned} \|v_i^{+,s}(t)\| &\leq \tilde{K}_1 e^{\alpha^s t}(\|v_i^{-,u}(0)\| + \|v_i^{+,u}\|(\|v_i^{+,u}\| + \|v_i^{+,s}\|)), \\ \|v_i^{-,s}(t)\| &\leq \tilde{K}_1 e^{-\alpha^w t}(\|v_i^{+,u}(0)\| + \|v_i^{-,u}\|(\|v_i^{-,u}\| + \|v_i^{-,s}\|)), \end{aligned}\right\} \tag{3.94}$$

and, since $1 - \tilde{K}_1\|v_i^{\pm,u}\| > \frac{1}{2}$,

$$\|v_i^{\pm,s}\| \leq 2\tilde{K}_1(\|v_i^{\mp,u}(0)\| + \|v_i^{\pm,u}\|^2). \tag{3.95}$$

As a direct consequence of estimate (3.94) we obtain

$$\left.\begin{aligned} \|a_i^+\| &\leq Ce^{\alpha^w\omega_i}, \\ \|a_i^-\| &\leq Ce^{\alpha^s\omega_i}. \end{aligned}\right\} \tag{3.96}$$

This can be seen by looking at the equation of a^+ and a^- given in (3.48). Calling in the definition of $\boldsymbol{d}$ in (3.13) provides

$$\begin{aligned} \|a_i^+\| \leq\ & \|\tilde{P}_{\kappa_i}(\lambda,\omega_i)(\gamma_{\kappa_i}^-(\lambda)(-\omega_i) - p_{\kappa_i})\| + \|\tilde{P}_{\kappa_i}(\lambda,\omega_i)(\gamma_{\kappa_{i-1}}^+(\lambda)(\omega_i) - p_{\kappa_i})\| \\ & + \|\tilde{P}_{\kappa_i}(\lambda,\omega_i)(id - P_{\kappa_{i-1}}^+(\lambda,\omega_i))\|\|(id - P_{\kappa_{i-1}}^+(\lambda,\omega_i))v_{i-1}^+(\lambda)(\omega_i)\| \\ & + \|\tilde{P}_{\kappa_i}(\lambda,\omega_i)(id - P_{\kappa_i}^-(\lambda,-\omega_i))\|\|(id - P_{\kappa_i}^-(\lambda,-\omega_i))v_i^-(\lambda)(-\omega_i)\|; \end{aligned}$$

$$\begin{aligned} \|a_i^-\| \leq\ & \|(id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))(\gamma_{\kappa_i}^-(\lambda)(-\omega_i) - p_{\kappa_i})\| + \|(id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))(\gamma_{\kappa_{i-1}}^+(\lambda)(\omega_i) - p_{\kappa_i})\| \\ & + \|(id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))(id - P_{\kappa_{i-1}}^+(\lambda,\omega_i))\|\|(id - P_{\kappa_{i-1}}^+(\lambda,\omega_i))v_{i-1}^+(\lambda)(\omega_i)\| \\ & + \|(id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))(id - P_{\kappa_i}^-(\lambda,-\omega_i))\|\|(id - P_{\kappa_i}^-(\lambda,-\omega_i))v_i^-(\lambda)(-\omega_i)\|. \end{aligned}$$

Applying the estimates presented in Lemmata 3.4.3 and 3.4.6 and equation (3.94) we see that each term in $\|a_i^-\|$ can be estimated by $e^{\alpha^s\omega_i}$. In case of $\|a_i^+\|$ the term with the smallest exponential rate is given by $(id - P_{\kappa_i}^-(-\omega_i))v_i^-(-\omega_i) = v_i^{-,s}(-\omega_i)$ with $e^{\frac{1}{2}(\alpha^s - \alpha^u)\omega_i}$. In case that Hypothesis (H3.7) applies we even find that $\|a_i^+\|$ can be estimated by $e^{-\alpha^u\omega_i}$. To this end we apply Lemma 3.4.5 instead of Lemma 3.4.3. This finally leads to the estimate (3.96).

Let us now consider $v_i^{\pm,u}$. Due to (3.40) we have with $\beta^+_{\kappa_i} = \alpha^u$ and $\alpha^-_{\kappa_i} = -\alpha^s$, cf. (3.18) and (3.66),

$$\left.\begin{array}{rcl} \|v_i^{+,u}(t)\| & \leq & Ke^{-\alpha^u(\omega_{i+1}-t)}\|a^+_{i+1}\| + K\int\limits_t^{\omega_{i+1}} e^{\alpha^u(t-s)}\|h_i^{+,u}(s, v_i^+(s), \lambda)\|ds \\ & \leq & KCe^{-\alpha^u(\omega_{i+1}-t)}e^{\alpha^w\omega_{i+1}} + KM(\|v_i^{+,s}\| + \|v_i^{+,u}\|)^2; \\ \|v_i^{-,u}(t)\| & \leq & Ke^{\alpha^s(\omega_i+t)}\|a_i^-\| + K\int\limits_{-\omega_i}^{t} e^{\alpha^s(t-s)}\|h_i^{-,u}(s, v_i^-(s), \lambda)\|ds \\ & \leq & KCe^{\alpha^s(\omega_i+t)}e^{\alpha^s\omega_i} + KM(\|v_i^{-,s}\| + \|v_i^{-,u}\|)^2; \end{array}\right\} \quad (3.97)$$

which leads to

$$\left.\begin{array}{rcl} \|v_i^{+,u}\| & \leq & \tilde{K}_2(e^{\alpha^w\omega_{i+1}} + \|v_i^{+,s}\|^2); \\ \|v_i^{-,u}\| & \leq & \tilde{K}_2(e^{\alpha^s\omega_i} + \|v_i^{-,s}\|^2). \end{array}\right\} \quad (3.98)$$

Again we have used that $\|v_i^\pm\|$ is exponentially small such that for $\inf\boldsymbol{\omega}$ sufficiently large we find $1 - KM(2\|v_i^{\pm,s}\| + \|v_i^{\pm,u}\|) > 1/2$. equations (3.98) and (3.97) for $t = 0$ in combination with (3.95) then yield

$$\begin{array}{rcl} \|v_i^{+,s}\| & \leq & 2\tilde{K}_1(KCe^{2\alpha^s\omega_i} + KM(\|v_i^{-,s}\| + \|v_i^{-,u}\|)^2 + \tilde{K}_2^2(e^{\alpha^w\omega_{i+1}} + \|v_i^{+,s}\|^2)^2) \\ & \leq & 2\tilde{K}_1(KCe^{2\alpha^s\omega_i} + KM(\|v_i^{-,s}\| + \|v_i^{-,u}\|)^2 + \tilde{K}_2^2e^{2\alpha^w\omega_{i+1}} + \varepsilon^+(\boldsymbol{\omega})\|v_i^{+,s}\|); \\ \|v_i^{-,s}\| & \leq & 2\tilde{K}_1(KCe^{(\alpha^w-\alpha^u)\omega_{i+1}} + KM(\|v_i^{+,s}\| + \|v_i^{+,u}\|)^2 + \tilde{K}_2^2(e^{\alpha^s\omega_i} + \|v_i^{-,s}\|^2)^2) \\ & \leq & 2\tilde{K}_1(KCe^{(\alpha^w-\alpha^u)\omega_{i+1}} + KM(\|v_i^{+,s}\| + \|v_i^{+,u}\|)^2 + \tilde{K}_2^2e^{2\alpha^s\omega_i} + \varepsilon^-(\boldsymbol{\omega})\|v_i^{-,s}\|); \end{array}$$

where $\varepsilon^\pm(\boldsymbol{\omega})$ are exponentially small. Hence we find a constant $\tilde{K}_3$ such that

$$\left.\begin{array}{rcl} \|v_i^{+,s}\| & \leq & \tilde{K}_3(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i} + (\|v_i^{-,s}\| + \|v_i^{-,u}\|)^2); \\ \|v_i^{-,s}\| & \leq & \tilde{K}_3(e^{(\alpha^w-\alpha^u)\omega_{i+1}} + e^{2\alpha^s\omega_i} + (\|v_i^{+,s}\| + \|v_i^{+,u}\|)^2). \end{array}\right\} \quad (3.99)$$

Successively plugging the estimates of (3.98) and (3.99) into each other yields in a first step

$$\left.\begin{array}{rcl} \|v_i^{+,s}\| & \leq & \tilde{K}_3\tilde{K}_2^2(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i} + \hat{\varepsilon}^-(\boldsymbol{\omega})\|v_i^{-,s}\|); \\ \|v_i^{-,s}\| & \leq & \tilde{K}_3\tilde{K}_2^2(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i} + \hat{\varepsilon}^+(\boldsymbol{\omega})\|v_i^{+,s}\|); \end{array}\right\} \quad (3.100)$$

with $\hat{\varepsilon}^\pm(\boldsymbol{\omega})$ exponentially small for Ω sufficiently large. Here we have used that $\alpha^w - \alpha^u \leq 2\alpha^w$. In a second step we plug the estimates (3.100) successively into each other and these solutions we apply in (3.98). Finally we obtain

$$\left.\begin{array}{rcl} \|v_i^{+,s}\| & \leq & \tilde{K}_4(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}); \\ \|v_i^{-,s}\| & \leq & \tilde{K}_4(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}); \\ \|v_i^{+,u}\| & \leq & \tilde{K}_4(e^{\alpha^w\omega_{i+1}} + e^{4\alpha^s\omega_i}); \\ \|v_i^{-,u}\| & \leq & \tilde{K}_4(e^{4\alpha^w\omega_{i+1}} + e^{\alpha^s\omega_i}). \end{array}\right\} \quad (3.101)$$

Plugging these estimates into (3.97) and (3.94) gives

$$\|v_i^{+,u}(t)\| \leq \tilde{K}_5(e^{-\alpha^u(\omega_{i+1}-t)}e^{\alpha^w\omega_{i+1}} + (e^{\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i})^2);$$

$$\|v_i^{-,u}(t)\| \leq \tilde{K}_5(e^{\alpha^s(\omega_i+t)}e^{\alpha^s\omega_i} + e^{\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}} + e^{4\alpha^w\omega_{i+1}});$$

and

$$\left.\begin{aligned} \|v_i^{+,s}(t)\| &\leq \tilde{K}_5 e^{\alpha^s t}(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}); \\ \|v_i^{-,s}(t)\| &\leq \tilde{K}_5 e^{-\alpha^w t}(e^{2\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}). \end{aligned}\right\} \tag{3.102}$$

We want to simplify the estimate of $v_i^{+,u}(t)$ by applying $-\alpha^u \leq \alpha^w$ and end up with

$$\left.\begin{aligned} \|v_i^{+,u}(t)\| &\leq \tilde{K}_5(e^{\alpha^w(\omega_{i+1}-t)}e^{\alpha^w\omega_{i+1}} + e^{\alpha^w\omega_{i+1}}e^{2\alpha^s\omega_i} + e^{4\alpha^s\omega_i}); \\ \|v_i^{-,u}(t)\| &\leq \tilde{K}_5(e^{\alpha^s(\omega_i+t)}e^{\alpha^s\omega_i} + e^{\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}} + e^{4\alpha^w\omega_{i+1}}). \end{aligned}\right\} \tag{3.103}$$

With (3.102) and (3.103) the main part of finding suitable estimations of $v_i^{\pm,s}(t)$ and $v_i^{\pm,u}(t)$ is accomplished. But still we wish to specify the estimates further.

We start with $v_i^{\pm,s}$. To this end we look again at the integral term in (3.41), invoke Lemma 3.4.7 and make use of the estimates (3.103) and (3.102). We do this exemplarily for $v_i^{-,s}$. Here we set again $\beta_{\kappa_i}^- = -\alpha^w + \delta$ for some $\delta > 0$ close to zero.

$$\left.\begin{aligned} &\int_t^0 \|\Phi_{\kappa_i}^-(t,s)(id - P_{\kappa_i}^-(s))\|\|h_{\kappa_i}^{-,s}(s, v_i^-(s), \lambda)\|ds \\ &\leq \int_t^0 e^{(-\alpha^w+\delta)(t-s)}\|h_{\kappa_i}^{-,s}(s, v_i^-(s), \lambda)\|ds \\ &\leq M\int_t^0 e^{(-\alpha^w+\delta)(t-s)}(-e^{\alpha^w s}\|v_i^{-,u}(s)\| + \|v_i^{-,s}(s)\|)(\|v_i^{-,u}(s)\| + \|v_i^{-,s}(s)\|)ds \\ &\leq \tilde{K}_5^2 M\ e^{-\alpha^w t}\int_t^0 e^{\delta(t-s)}(e^{\alpha^s(2\omega_i+s)} + e^{2\alpha^w\omega_{i+1}}) \\ &\qquad\qquad \cdot(e^{\alpha^s(2\omega_i+s)} + (e^{\alpha^u s} + e^{\alpha^s\omega_i})e^{2\alpha^w\omega_{i+1}} + e^{4\alpha^w\omega_{i+1}})ds \\ &\leq \tilde{K}_5^2 M\ e^{-\alpha^w t}\int_t^0 e^{\delta(t-s)}(e^{\alpha^s(2\omega_i+s)} + e^{2\alpha^w\omega_{i+1}})^2 ds \\ &\leq \tilde{K}_5^2 M\ e^{-\alpha^w t}\left(e^{4\alpha^s\omega_i}\int_t^0 e^{2\alpha^s s}ds + 2e^{2\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}}\int_t^0 e^{\alpha^s s}ds + e^{4\alpha^w\omega_{i+1}}\int_t^0 e^{\delta(t-s)}ds\right) \\ &\leq \tilde{K}_5^2 M\ e^{-\alpha^w t}\left(e^{\alpha^s(\omega_i+t)}e^{\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}\right)^2. \end{aligned}\right\} \tag{3.104}$$

Analogously we obtain

$$\left.\begin{aligned} \int_0^t \|\Phi_{\kappa_i}^+(t,s)(id - P_{\kappa_i}^+(s))h_{\kappa_i}^{+,s}(s, v_i^+(s), \lambda)\|ds &\leq K\int_0^t e^{(\alpha^s-\delta)(t-s)}\|h_{\kappa_i}^{+,s}(s, v_i^+(s), \lambda)\|ds \\ &\leq K\tilde{K}_5^2 M\ e^{\alpha^s t}\left(e^{\alpha^w(2\omega_{i+1}-t)} + e^{2\alpha^s\omega_i}\right)^2. \end{aligned}\right\} \tag{3.105}$$

In combination with the linear term this finally leads to

$$\begin{aligned}
\|v_i^{-,s}(t)\| &\le KLe^{-\alpha^w t}\|v_i^{+,u}(0)\| + K\int_t^0 e^{(-\alpha^w+\delta)(t-s)}\|h_{\kappa_i}^{-,s}(s, v_i^-(s), \lambda)\|ds \\
&\le \tilde{K}_6 e^{-\alpha^w t}\left(e^{2\alpha^s(\omega_i+t)}e^{2\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^w\omega_{i+1}}\right), \\
\|v_i^{+,s}(t)\| &\le KLe^{\alpha^s t}\|v_i^{-,u}(0)\| + K\int_0^t e^{(\alpha^s-\delta)(t-s)}\|h_{\kappa_i}^{+,s}(s, v_i^+(s), \lambda)\|ds \\
&\le \tilde{K}_6 e^{\alpha^s t}\left(e^{2\alpha^w(\omega_{i+1}-t)}e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^w\omega_{i+1}}e^{\alpha^s\omega_i} + e^{2\alpha^s\omega_i}\right),
\end{aligned}$$

so that we are finished with the estimate of $v_i^{\pm,s}$.

In case of $v_i^{\pm,u}$ we also see again to the integral term. But before we can do this we need to estimate the term $v_i^{\pm,su}$.

To this end recall that we have decomposed $v_i^{\pm}$ by means of the projection $P_{\kappa_i}^{\pm}(t)$ where $v_i^{\pm,u}(t) \in \mathrm{im}P_{\kappa_i}^{\pm}(t)$ and $v_i^{\pm,s}(t) \in \mathrm{im}(id - P_{\kappa_i}^{\pm}(t)) = T_{\gamma_{\kappa_i}^{\pm}(t)}W^{s/u}(0)$. This decomposition we refined by decomposing $v^{\pm,s}(t) = v^{\pm,ss}(t) + v^{\pm,su}(t)$ where $v^{\pm,ss} \in \Phi^{\pm}(\lambda)(t,0)W^{\pm}$ and $v^{\pm,su} \in \Phi^{\pm}(\lambda)(t,0)\,\mathrm{span}\{f(\gamma^{\pm}(\lambda)(0),\lambda)\}$. Note that

$$v_i^{\pm,su}(0) = 0.$$

Therefore we can estimate $v^{\pm,su}(t)$ in the same way as $v^{\pm,s}(t)$ but without the linear term, which drops out. Hence we find, cf. (3.104) and (3.105),

$$\left.\begin{aligned}
\|v_i^{+,su}(t)\| &\le K\int_0^t e^{(\alpha^s-\delta)(t-s)}\|h_{\kappa_i}^{+,s}(s, v_i^+(s), \lambda)\|ds \\
&\le KM\tilde{K}_5^2\, e^{\alpha^s t}\left(e^{\alpha^w(\omega_{i+1}-t)}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}\right)^2, \\
\|v_i^{-,su}(t)\| &\le K\int_t^0 e^{(-\alpha^w+\delta)(t-s)}\|h_{\kappa_i}^{-,s}(s, v_i^-(s), \lambda)\|ds \\
&\le KM\tilde{K}_5^2\, e^{-\alpha^w t}\left(e^{\alpha^s(\omega_i+t)}e^{\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}\right)^2.
\end{aligned}\right\} \tag{3.106}$$

Comparing these estimates with (3.103) we see that the sum of $\|v_i^{\pm,su}(t)\|$ and $\|v_i^{\pm,u}(t)\|$ can be estimated by the same terms as $\|v_i^{\pm,u}(t)\|$, that is

$$\begin{aligned}
\|v_i^{+,su}(t)\| + \|v_i^{+,u}(t)\| &\le \tilde{C}\tilde{K}_5(e^{\alpha^w(\omega_{i+1}-t)}e^{\alpha^w\omega_{i+1}} + e^{\alpha^w\omega_{i+1}}e^{2\alpha^s\omega_i} + e^{4\alpha^s\omega_i}), \\
\|v_i^{-,su}(t)\| + \|v_i^{-,u}(t)\| &\le \tilde{C}\tilde{K}_5(e^{\alpha^s(\omega_i+t)}e^{\alpha^s\omega_i} + e^{\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}} + e^{4\alpha^w\omega_{i+1}}).
\end{aligned}$$

for some constant $\tilde{C} > 1$. Now with this we can estimate the integral term in (3.97) for $v_i^{\pm,u}(t)$ again by invoking Lemma 3.4.8. Additionally we make use of the estimates (3.103) and (3.102) which yield

$$\begin{aligned}
\|v_i^-(s)\| &\le \|v_i^{-,s}(s)\| + \|v_i^{-,u}(s)\| &&\le \tilde{K}_5 e^{\alpha^s(\omega_i+s)}e^{\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}, \\
\|v_i^+(s)\| &\le \|v_i^{+,s}(s)\| + \|v_i^{+,u}(s)\| &&\le \tilde{K}_5 e^{\alpha^w(\omega_{i+1}+s)}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}.
\end{aligned}$$

We see to the integral in (3.97) exemplarily for $v_i^{-,u}$. Here we set $-\alpha^- = \alpha^s - \delta$ for some $\delta > 0$ sufficiently small such that $\alpha^s - \delta > \text{Re}\mu^s(\lambda)$ is still satisfied, cf. (3.18) and (3.66).

$$\left.\begin{aligned}
&\int_{-\omega_i}^{t} \|\Phi^-_{\kappa_i}(\lambda)(t,s)P^-_{\kappa_i}(\lambda,s)h^-_{\kappa_i}(s,v_i^-(s),\lambda)\|ds \\
&\quad\leq \int_{-\omega_i}^{t} e^{(\alpha^s-\delta)(t-s)}\|h_i^{-,u}(s,v_i^-(s),\lambda)\|ds \\
&\quad\leq M\int_{-\omega_i}^{t} e^{(\alpha^s-\delta)(t-s)}(\|v_i^{-,u}(s)\|+\|v_i^{-,su}(s)\|)\|v_i^-(s)\|ds \\
&\quad\leq \tilde{C}\tilde{K}_5^2 M\int_{-\omega_i}^{t} e^{(\alpha^s-\delta)(t-s)}(e^{\alpha^s(\omega_i+s)}e^{\alpha^s\omega_i}+e^{2\alpha^w\omega_{i+1}})^2 ds \\
&\quad\leq \tilde{C}\tilde{K}_5^2 M\, e^{\alpha^s t}\left[e^{3\alpha^s\omega_i}\int_{-\omega_i}^{t} e^{\alpha^s(\omega_i+s)}ds + 2e^{2\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}}\int_{-\omega_i}^{t} e^{-\delta(t-s)}ds\right] \\
&\qquad +\tilde{C}\tilde{K}_5^2 M\, e^{4\alpha^w\omega_{i+1}}\int_{-\omega_i}^{t} e^{(\alpha^s-\delta)(t-s)}ds \\
&\quad\leq \tilde{C}\tilde{K}_5^2 M\left(e^{\alpha^s t}\left[e^{3\alpha^s\omega_i}+2e^{2\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}}\right]+e^{4\alpha^w\omega_{i+1}}\right).
\end{aligned}\right\} \tag{3.107}$$

Analogously we find,

$$\begin{aligned}
&\int_{t}^{\omega_{i+1}} \|\Phi^+_{\kappa_i}(\lambda)(t,s)P^+_{\kappa_i}(\lambda,s)h^+_{\kappa_i}(s,v_i^+(s),\lambda)\|ds \\
&\quad\leq \int_{t}^{\omega_{i+1}} e^{(-\alpha^w+\delta)(t-s)}\|h_i^{+,u}(s,v_i^+(s),\lambda)\|ds \\
&\quad\leq \tilde{C}\tilde{K}_5^2 M\left(e^{-\alpha^w t}\left[e^{3\alpha^w\omega_{i+1}}+2e^{2\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}}\right]+e^{4\alpha^s\omega_i}\right).
\end{aligned}$$

Finally in combination with the linear term we obtain

$$\begin{aligned}
\|v_i^{+,u}(t)\| &\leq Ke^{\alpha^w(\omega_{i+1}-t)}\|a^+_{i+1}\| + K\int_{t}^{\omega_{i+1}} e^{-\alpha^w(t-s)}\|h_i^{+,u}(s,v_i^+(s),\lambda)\|ds \\
&\leq \tilde{K}_7(e^{\alpha^w(\omega_{i+1}-t)}e^{\alpha^w\omega_{i+1}}+e^{4\alpha^s\omega_i}), \\
\|v_i^{-,u}(t)\| &\leq Ke^{\alpha^s(\omega_i+t)}\|a^-_i\| + K\int_{-\omega_i}^{t} e^{\alpha^s(t-s)}\|h_i^{-,u}(s,v_i^-(s),\lambda)\|ds \\
&\leq \tilde{K}_7(e^{\alpha^s(\omega_i+t)}e^{\alpha^s\omega_i}+e^{4\alpha^w\omega_{i+1}}).
\end{aligned}$$

□

Corollary 3.4.10. *Assume Hypotheses (H3.1)-(H3.6), then there exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $\|\lambda\| < c$ and $\boldsymbol{\omega}$ with $\inf\boldsymbol{\omega} > \Omega$ the following estimates apply with some $\bar{K} > 0$:*

$$\begin{aligned}
\|v_i^{+,su}(t)\| + \|v_i^{+,u}(t)\| &\leq \bar{K}(e^{\alpha^w(\omega_{i+1}-t)}e^{\alpha^w\omega_{i+1}}+e^{4\alpha^s\omega_i}), \\
\|v_i^{-,su}(t)\| + \|v_i^{-,u}(t)\| &\leq \bar{K}(e^{\alpha^s(\omega_i+t)}e^{\alpha^s\omega_i}+e^{4\alpha^w\omega_{i+1}}).
\end{aligned} \tag{3.108}$$

Proof. This estimate follows from Lemma 3.4.9 and the Estimate (3.106). □

Now we have collected nearly every information we need to estimate each single term appearing in the representation of the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$, cf. (3.65). What remains to consider are the integral terms $\int \Phi^\pm P^\pm h^{\pm,u}$.

3.4.6 Estimates regarding the integral terms

We start with the integral $\int \Phi^+_{\kappa_i}(0,s)P^+_{\kappa_i}(s)h^+_{\kappa_i}(s,v^+_i(s))ds$.

Lemma 3.4.11. *Assume Hypotheses (H3.1)-(H3.6). Then there exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $\|\lambda\| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ the following estimates apply:*

$$\left\| \int_0^{\omega_{i+1}} \Phi^+_{\kappa_i}(\lambda)(0,s)P^+_{\kappa_i}(\lambda,s)h^+_{\kappa_i}(s,v^+_i(s),\lambda)ds \right\| = O\left(e^{6\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}} + e^{\max\{(\nu-2)\alpha^s+3\alpha^w,4\alpha^w\}\omega_{i+1}}\right),$$

with α^w as defined in (3.90).

Proof. With $h^{+,u}_{\kappa_i} := P^+_{\kappa_i}h^+_{\kappa_i}$ we find due to Lemma 3.4.8

$$\begin{aligned}&\left\| \int_0^{\omega_{i+1}} \Phi^+_{\kappa_i}(\lambda)(0,s)P^+_{\kappa_i}(\lambda,s)h^+_{\kappa_i}(s,v^+_i(s),\lambda)ds \right\| \\ &\le M \int_0^{\omega_{i+1}} \|\Phi^+_{\kappa_i}(\lambda)(0,s)P^+_{\kappa_i}(\lambda,s)\| \left(\|v^{+,u}_i(s)\| + \|v^{+,su}_i(s)\|\right)\|v^+_i(s)\| \left(\|v^+_i(s)\| + \|\gamma^+_{\kappa_i}(s) - p_{\kappa_{i+1}}\|\right)^{\nu-2} ds.\end{aligned}$$

The exponential dichotomy (3.16) provides $\|\Phi^+_{\kappa_i}(\lambda)(0,s)P^+_{\kappa_i}(\lambda,s)\| \le Ke^{-\beta^+_i s}$. Here β^+_i can be chosen as $\beta^+_i = \alpha^u + \delta$ for some $\delta > 0$ such that the inequality $\alpha^u + \delta < \operatorname{Re}(\mu^u(\lambda))$ is still satisfied for all $\|\lambda\|$ sufficiently small, cf. (3.18) and (3.66). Indeed we choose $\beta^+_i = -\alpha^w + \delta$, since $-\alpha^w \le \alpha^u$. Further we find $\|\gamma^+_{\kappa_i}(s) - p_{\kappa_{i+1}}\| \le Ke^{\alpha^s s}$, cf. (3.79). Combining this result with Lemma 3.4.9 we find $\|v^+_i(s)\| + \|\gamma^+_{\kappa_i}(s) - p_{\kappa_{i+1}}\| \le \tilde{K}(e^{\alpha^s s} + e^{4\alpha^s\omega_i})$. Then we obtain by additionally applying the binomial theorem

$$\left.\begin{aligned}&\left\| \int_0^{\omega_{i+1}} \Phi^+_{\kappa_i}(\lambda)(0,s)P^+_{\kappa_i}(\lambda,s)h^+_{\kappa_i}(s,v^+_i(s),\lambda)ds \right\| \\ &\quad\le M \int_0^{\omega_{i+1}} Ke^{(\alpha^w-\delta)s}(\|v^{+,u}_i(s)\| + \|v^{+,su}_i(s)\|)\|v^+_i(s)\| \cdot \tilde{K}^{\nu-2}\left(e^{4\alpha^s\omega_i} + e^{\alpha^s s}\right)^{\nu-2} ds \\ &\quad\le MK\tilde{K}^{\nu-2}\sum_{k=0}^{\nu-2}\binom{\nu-2}{k}e^{4k\alpha^s\omega_i}\int_0^{\omega_{i+1}} e^{((\nu-2-k)\alpha^s+\alpha^w)s}e^{-\delta s}(\|v^{+,u}_i(s)\| + \|v^{+,su}_i(s)\|)\|v^+_i(s)\|ds.\end{aligned}\right\} \tag{3.109}$$

Now we focus on the integral term in the last line. Invoking the estimates for $\|v^{+,s}_i(s)\|$, $\|v^{+,u}_i(s)\|$ and $\|v^{+,u}_i(s)\| + \|v^{+,su}_i(s)\|$ given in Lemma 3.4.9 and Corollary 3.4.10 leads with

$$\|v^+_i(s)\| \le \|v^{+,s}_i(s)\| + \|v^{+,u}_i(s)\| \le \bar{K}(e^{\alpha^w(\omega_{i+1}-s)}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i})$$

to

$$\begin{aligned}&\int_0^{\omega_{i+1}} e^{((\nu-2-k)\alpha^s+\alpha^w)s}e^{-\delta s}(\|v^{+,u}_i(s)\| + \|v^{+,su}_i(s)\|)\|v^+_i(s)\|ds \\ &\le 2\bar{K}^2 \int_0^{\omega_{i+1}} e^{((\nu-2-k)\alpha^s+\alpha^w)s}e^{-\delta s}\left(e^{-\alpha^w s}e^{2\alpha^w\omega_{i+1}} + e^{4\alpha^s\omega_i}\right)\left(-e^{\alpha^w s}e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}\right) ds \\ &\le 2\bar{K}^2 \int_0^{\omega_{i+1}} e^{((\nu-2-k)\alpha^s+\alpha^w)s}e^{-\delta s}\left[e^{-2\alpha^w s}e^{4\alpha^w\omega_{i+1}} + 2e^{-\alpha^w s}e^{2\alpha^w\omega_{i+1}}e^{2\alpha^s\omega_i} + e^{6\alpha^s\omega_i}\right] ds \\ &\le 2\bar{K}^2 \Bigg[e^{4\alpha^w\omega_{i+1}} \int_0^{\omega_{i+1}} e^{((\nu-2-k)\alpha^s-\alpha^w)s}ds + 2e^{2\alpha^w\omega_{i+1}}e^{2\alpha^s\omega_i} \int_0^{\omega_{i+1}} e^{((\nu-2-k)\alpha^s)s}e^{-\delta s}ds \\ &\qquad\qquad + e^{6\alpha^s\omega_i} \int_0^{\omega_{i+1}} e^{((\nu-2-k)\alpha^s+\alpha^w)s}ds\Bigg]\end{aligned}$$

Most of the integral terms in the last line can be estimated by a constant $c > 0$. First we find that

$e^{-\delta s} < 1$ for $s > 0$ so that term dropped out in the first and the last integral. Then we find for all $k = 0, \dots, \nu - 2$, $\nu \geq 2$ that $(\nu - 2 - k)\alpha^s \leq 0$ and $(\nu - 2 - k)\alpha^s + \alpha^w < 0$ and hence

$$\int_0^{\omega_{i+1}} e^{(\nu-2-k)\alpha^s s} e^{-\delta s} ds \leq c \qquad \text{and} \qquad \int_0^{\omega_{i+1}} e^{((\nu-2-k)\alpha^s + \alpha^w)s} ds \leq c.$$

Additionally we also get for the first integral

$$\int_0^{\omega_{i+1}} e^{((\nu-2-k)\alpha^s - \alpha^w)s} ds < c\left(1 + e^{((\nu-2-k)\alpha^s - \alpha^w)\omega_{i+1}}\right)$$

Therefore we obtain

$$\begin{aligned}&\int_0^{\omega_{i+1}} e^{((\nu-2-k)\alpha^s+\alpha^w)s} e^{-\delta s}(\|v_i^{+,u}(s)\| + \|v_i^{+,su}(s)\|)\|v_i^+(s)\|ds \\ &\qquad \leq 2c\bar{K}^2\left(2e^{2\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}} + e^{6\alpha^s\omega_i} + e^{4\alpha^w\omega_{i+1}} + e^{((\nu-2-k)\alpha^s+3\alpha^w)\omega_{i+1}}\right)\end{aligned}$$

which finally leads with (3.109) to

$$\begin{aligned}&\left\|\int_0^{\omega_{i+1}} \Phi^+_{\kappa_i}(\lambda)(t,s)P^+_{\kappa_i}(\lambda,s)h^+_{\kappa_i}(s,v_i^+(s),\lambda)ds\right\| \\ &\qquad \leq C\left(e^{2\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}} + e^{6\alpha^s\omega_i}\right) + C\begin{cases} e^{4\alpha^w\omega_{i+1}} + e^{((\nu-2)\alpha^s+3\alpha^w)\omega_{i+1}}, & \nu > 2, \\ e^{3\alpha^w\omega_{i+1}}, & \nu = 2. \end{cases}\end{aligned}$$

□

It remains to estimate the other integral $\int \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)h^-_{\kappa_i}(s,v_i^-(s))ds$. Following along the lines of the proof of Lemma 3.4.11 we obtain the estimate

$$\left\|\int_{-\omega_i}^0 \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)h^-_{\kappa_i}(s,v_i^-(s))ds\right\| = O(e^{6\alpha^w\omega_{i+1}} + e^{2\alpha^w\omega_{i+1}}e^{2\alpha^s\omega_i}) + \begin{cases} O(e^{3\alpha^s\omega_i}), & \nu = 2, \\ O(e^{4\alpha^s\omega_i}), & \nu > 2. \end{cases} \tag{3.110}$$

However, in view of the determination of the leading terms within the residuals $R_i(\boldsymbol{\omega}, \lambda, \kappa)$ in the determination equation (1.7) we will decompose the integral into several parts which we estimate separately. The separation of the integral is based on the separation of $h^-_{\kappa_i}(s, v_i^-(s))$ presented in Lemma 3.4.8.

Lemma 3.4.12. *Assume Hypotheses (H3.1)-(H3.6). Then there exist constants Ω and c in accordance with Theorem 3.2.2 such that for all $\|\lambda\| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ the following estimates apply:*

$$\begin{aligned}&\left\|\int_{-\omega_i}^0 \Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v^{-,u}(s),v^{-,u}(s)]ds\right\| \\ &\qquad = \begin{cases} O\left(e^{3\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}} + e^{8\alpha^w\omega_{i+1}}\right), & \nu = 2, \\ O\left(e^{4\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}} + e^{8\alpha^w\omega_{i+1}}\right), & \nu \geq 3, \end{cases}\end{aligned}$$

$$\left\| \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)D_1^3 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v^{-,u}(s),v^{-,u}(s),v^{-,u}(s)]ds \right\|$$
$$= \begin{cases} O\left(e^{4\alpha^s\omega_i} + e^{3\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}} + e^{12\alpha^w\omega_{i+1}}\right), & \nu=2,\nu=3 \\ O\left(e^{\max\{4\alpha^s+\alpha^w,6\alpha^s\}\omega_i} + e^{4\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}} + e^{12\alpha^w\omega_{i+1}}\right), & \nu\ge 4, \end{cases}$$

and

$$\Big\| \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)\Big(h^-_{\kappa_i}(s,v_i^-(s),\lambda) - \tfrac{1}{2}D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v^{-,u}(s),v^{-,u}(s)]$$
$$-\tfrac{1}{6}D_1^3 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v^{-,u}(s),v^{-,u}(s),v^{-,u}(s)]\Big)ds\Big\|$$
$$= O\left(e^{5\alpha^s\omega_i} + e^{4\alpha^s\omega_i}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}} + e^{6\alpha^w\omega_{i+1}}\right),$$

with α^w *as defined in* (3.90).

Proof. We start with the estimation of $\int \Phi^-_{\kappa_i}(t,s)P^-_{\kappa_i}(s)D_1^2 f(\gamma^-_{\kappa_i}(s))[v_i^{-,u}(s),v_i^{-,u}(s)]ds$. Due to the exponential dichotomy (3.16) we find $\|\Phi^-_{\kappa_i}(t,s)P^-_{\kappa_i}(s)\| \le Ke^{-\alpha^-_{\kappa_i}(t-s)}$ where we can choose $-\alpha^-_{\kappa_i} := \alpha^s - \delta$ for some $\delta > 0$ such that the inequality $\alpha^s - \delta > \mu^s(\lambda)$ is still satisfied, cf. (3.18). Further we have $\|D_1^2 f(\gamma^-_{\kappa_i}(s))[v_i^{-,u}(s),v_i^{-,u}(s)]\| = O(\|v_i^{-,u}(s)\|^2 \cdot \|\gamma^-_{\kappa_i}(s) - p_{\kappa_i}\|^{\nu-2})$. Equation (3.79) provides $\|\gamma^-_{\kappa_i}(s) - p_{\kappa_i}\| \le \tilde{K}e^{\alpha^u s} \le \tilde{K}e^{-\alpha^w s}$ for $s<0$ and from Lemma 3.4.9 we obtain $\|v_i^{-,u}(s)\| \le \bar{K}(e^{\alpha^s(\omega_i+s)}e^{\alpha^s\omega_i} + e^{4\alpha^w\omega_{i+1}})$. Now this yield

$$\left\| \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v_i^{-,u}(s),v_i^{-,u}(s)]ds \right\|$$
$$\le MK\int_{-\omega_i}^{0} e^{-(\alpha^s-\delta)s}\|v_i^{-,u}(s)\|^2\|\gamma^-_{\kappa_i}(s)-p_{\kappa_i}\|^{\nu-2}ds$$
$$\le MK\bar{K}^2\tilde{K}^{\nu-2}\int_{-\omega_i}^{0} e^{(-\alpha^s-(\nu-2)\alpha^w)s}e^{\delta s}\left(e^{\alpha^s s}e^{2\alpha^s\omega_i} + e^{4\alpha^w\omega_{i+1}}\right)^2 ds$$
$$\le MK\bar{K}^2\tilde{K}^{\nu-2}\Bigg(e^{4\alpha^s\omega_i}\int_{-\omega_i}^{0} e^{(\alpha^s-(\nu-2)\alpha^w)s}ds + 2e^{2\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{-(\nu-2)\alpha^w s}e^{\delta s}ds$$
$$+e^{8\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{(-\alpha^s-(\nu-2)\alpha^w)s}ds\Bigg)$$

The second and the third integral are bounded for all $\nu \ge 2$, the first integral is bounded for $\nu \ge 3$. For $\nu = 2$ we find $\int_{-\omega_i}^{0} e^{(\alpha^s-(\nu-2)\alpha^w)s}ds \le ce^{-\alpha^s\omega_i}$. Hence we get

$$\left\| \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v_i^{-,u}(s),v_i^{-,u}(s)]ds \right\|$$
$$\le Ce^{4\alpha^w\omega_{i+1}}(e^{4\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}) + C\begin{cases} e^{3\alpha^s\omega_i}, & \nu=2, \\ e^{4\alpha^s\omega_i}, & \nu>2. \end{cases}$$

Analogously we estimate

$$\begin{aligned}&\left\|\int\limits_{-\omega_i}^{0}\Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)D_1^3f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v_i^{-,u}(s),v_i^{-,u}(s),v_i^{-,u}(s)]ds\right\|\\ &\qquad\leq MK\int\limits_{-\omega_i}^{0}e^{-(\alpha^s-\delta)s}\|v_i^{-,u}(s)\|^3\|\gamma^-_{\kappa_i}(s)-p_{\kappa_i}\|^{\max\{0,\nu-3\}}ds\end{aligned}$$

only here we use $\|D_1^3f(\gamma^-_{\kappa_i}(s))[v_i^{-,u}(s),v_i^{-,u}(s),v_i^{-,u}(s)]\| = O(\|v_i^{-,u}(s)\|^3\cdot\|\gamma^-_{\kappa_i}(s)-p_{\kappa_i}\|^{\max\{0,\nu-3\}})$. Hence we obtain for $\nu=2$ and $\nu=3$

$$\begin{aligned}&\left\|\int\limits_{-\omega_i}^{0}\Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)D_1^3f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v_i^{-,u}(s),v_i^{-,u}(s),v_i^{-,u}(s)]ds\right\|\\ &\leq\quad MK\bar{K}^2\int\limits_{-\omega_i}^{0}e^{-\alpha^s s}e^{\delta s}\left(e^{\alpha^s s}e^{2\alpha^s\omega_i}+e^{4\alpha^w\omega_{i+1}}\right)^3ds\\ &\leq\quad MK\bar{K}^2\Bigg(e^{4\alpha^s\omega_i}\int\limits_{-\omega_i}^{0}e^{2\alpha^s(\omega_i+s)}ds+3e^{3\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}\int\limits_{-\omega_i}^{0}e^{\alpha^s(\omega_i+s)}ds\\ &\qquad\qquad+3e^{2\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}}\int\limits_{-\omega_i}^{0}e^{\delta s}ds+e^{12\alpha^w\omega_{i+1}}\int\limits_{-\omega_i}^{0}e^{-\alpha^s s}ds\Bigg)\\ &\leq\quad C\left(e^{4\alpha^s\omega_i}+e^{3\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}+e^{2\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}}+e^{12\alpha^w\omega_{i+1}}\right),\end{aligned}$$

since each single integral term is bounded. For $\nu>3$ we get

$$\begin{aligned}&\left\|\int\limits_{-\omega_i}^{0}\Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)D_1^3f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v_i^{-,u}(s),v_i^{-,u}(s),v_i^{-,u}(s)]ds\right\|\\ &\leq\quad MK\bar{K}^2\tilde{K}^{\nu-3}\int\limits_{-\omega_i}^{0}e^{(-\alpha^s-(\nu-3)\alpha^w)s}e^{\delta s}\left(e^{\alpha^s s}e^{2\alpha^s\omega_i}+e^{4\alpha^w\omega_{i+1}}\right)^3ds\\ &\leq\quad MK\bar{K}^2\tilde{K}^{\nu-3}\Bigg(e^{6\alpha^s\omega_i}\int\limits_{-\omega_i}^{0}e^{(2\alpha^s-(\nu-3)\alpha^w)s}ds+3e^{4\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}\int\limits_{-\omega_i}^{0}e^{(\alpha^s-(\nu-3)\alpha^w)s}ds\\ &\qquad\qquad+3e^{2\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}}\int\limits_{-\omega_i}^{0}e^{-(\nu-3)\alpha^w s}e^{\delta s}ds+e^{12\alpha^w\omega_{i+1}}\int\limits_{-\omega_i}^{0}e^{(-\alpha^s-(\nu-3)\alpha^w)s}ds\Bigg).\end{aligned}$$

Again the last three integral terms are bounded for all $\nu>3$. The first integral is bounded for $\nu>4$. For $\nu=4$ we get $\int\limits_{-\omega_i}^{0}e^{(2\alpha^s-(\nu-3)\alpha^w)s}ds\leq ce^{(-2\alpha^s+\alpha^w)\omega_i}$. This finally leads to

$$\begin{aligned}&\left\|\int\limits_{-\omega_i}^{0}\Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)D_1^3f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v_i^{-,u}(s),v_i^{-,u}(s),v_i^{-,u}(s)]ds\right\|\\ &\leq C\left(e^{12\alpha^w\omega_{i+1}}+e^{2\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}}\right)+C\begin{cases}e^{4\alpha^s\omega_i}+e^{3\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}, & \nu=2,\ \nu=3\\ e^{\max\{4\alpha^s+\alpha^w,6\alpha^s\}\omega_i}+e^{4\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}, & \nu>3.\end{cases}\end{aligned}$$

Now let us consider the rest of the integral. With Lemma 3.4.8 and the exponential dichotomy (3.16) we find

$$\begin{aligned}
&\Big\| \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)\big(h^{-,u}_{\kappa_i}(s,v^-_i(s),\lambda) - \tfrac{1}{2}D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v_i^{-,u}(s),v_i^{-,u}(s)] \\
&\qquad\qquad - \tfrac{1}{6}D_1^3 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v_i^{-,u}(s),v_i^{-,u}(s),v_i^{-,u}(s)]\big)ds\Big\| \\
&\quad \le MK \int_{-\omega_i}^{0} e^{-(\alpha^s-\delta)s}\left(\|v_i^{-,u}(s)\| + \|v_i^{-,su}(s)\|\right)\|v_i^{-,s}(s)\|\left(\|v_i^-(s)\| + \|\gamma^-_{\kappa_i}(s) - p_{\kappa_i}\|\right)^{\nu-2} ds \\
&\qquad + MK \int_{-\omega_i}^{0} e^{-(\alpha^s-\delta)s}\|v_i^{-,u}(s)\|^4\left(\|v_i^-(s)\| + \|\gamma^-_{\kappa_i}(s) - p_{\kappa_i}\|\right)^{\max\{0,\nu-4\}} ds.
\end{aligned}$$

This time we simply estimate $\|v_i^-(s)\| + \|\gamma^-_{\kappa_i}(s) - p_{\kappa_i}\|$ with a constant. This leads to

$$\left.\begin{aligned}
&\Big\| \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)\left(h^{-,u}_{\kappa_i}(s,v^-_i(s),\lambda) - \tfrac{1}{2}D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v_i^{-,u}(s),v_i^{-,u}(s)]\right. \\
&\qquad\qquad \left. - \tfrac{1}{6}D_1^3 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[v_i^{-,u}(s),v_i^{-,u}(s),v_i^{-,u}(s)]\right) ds\Big\| \\
&\le C\int_{-\omega_i}^{0} e^{-(\alpha^s-\delta)s}\|v_i^{-,s}(s)\|(\|v_i^{-,u}(s)\| + \|v_i^{-,su}(s)\|)ds + C\int_{-\omega_i}^{0} e^{-(\alpha^s-\delta)s}\|v_i^{-,u}(s)\|^4 ds
\end{aligned}\right\} \quad (3.111)$$

To begin with we focus on the first of the two integrals and obtain by invoking Lemma 3.4.9 and Corollary 3.4.10

$$\begin{aligned}
&\int_{-\omega_i}^{0} e^{-\alpha^s}e^{\delta s}\|v_i^{-,s}(s)\|(\|v_i^{-,u}(s)\| + \|v_i^{-,su}(s)\|)ds \\
&\quad \le \bar{K}^2 \int_{-\omega_i}^{0} e^{-\alpha^s s}\cdot e^{-\alpha^w s}\left(e^{2\alpha^s s}e^{4\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^w\omega_{i+1}}\right)\left(e^{\alpha^s s}e^{2\alpha^s\omega_i} + e^{4\alpha^w\omega_{i+1}}\right)ds \\
&\quad \le \bar{K}^2 \int_{-\omega_i}^{0} e^{-(\alpha^w+\alpha^s)s}\Big[e^{3\alpha^s s}e^{6\alpha^s\omega_i} + e^{2\alpha^s s}e^{4\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}} + e^{\alpha^s s}e^{2\alpha^s\omega_i}(e^{2\alpha^s\omega_i}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^w\omega_{i+1}}) \\
&\qquad\qquad + (e^{2\alpha^s\omega_i}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^w\omega_{i+1}})e^{4\alpha^w\omega_{i+1}}\Big] ds \\
&\quad \le \bar{K}^2\Bigg(e^{6\alpha^s\omega_i}\int_{-\omega_i}^{0} e^{(2\alpha^s-\alpha^w)s}ds + e^{4\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{(\alpha^s-\alpha^w)s}ds \\
&\qquad + e^{2\alpha^s\omega_i}(e^{2\alpha^s\omega_i}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^w\omega_{i+1}})\int_{-\omega_i}^{0} e^{-\alpha^w s}ds + e^{5\alpha^w\omega_{i+1}}(e^{2\alpha^s\omega_i} + e^{\alpha^w\omega_{i+1}})\int_{-\omega_i}^{0} e^{-(\alpha^w+\alpha^s)s}ds\Bigg) \\
&\quad \le \bar{K}^2\left(e^{\max\{6\alpha^s,4\alpha^s+\alpha^w\}\omega_i} + e^{2\alpha^s\omega_i}(e^{2\alpha^s\omega_i}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^w\omega_{i+1}}) + e^{6\alpha^w\omega_{i+1}}\right).
\end{aligned}$$

Finally we turn towards the second integral on the right-hand-side of (3.111) and invoke again Lemma 3.4.9.

$$\begin{aligned}
&\int_{-\omega_i}^{0} e^{-(\alpha^s-\delta)s}\|v_i^{-,u}(s)\|^4 ds \\
&\le \bar{K}^4 \int_{-\omega_i}^{0} e^{-\alpha^s s}e^{\delta s}(e^{\alpha^s s}e^{2\alpha^s\omega_i}+e^{4\alpha^w\omega_{i+1}})^4 ds \\
&\le \bar{K}^4 \int_{-\omega_i}^{0} e^{-\alpha^s s}e^{\delta s}\left[e^{4\alpha^s s}e^{8\alpha^s\omega_i}+4e^{3\alpha^s s}e^{6\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}+6e^{2\alpha^s s}e^{4\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}}\right. \\
&\qquad\qquad \left.+4e^{\alpha^s s}e^{2\alpha^s\omega_i}e^{12\alpha^w\omega_{i+1}}+e^{16\alpha^w\omega_{i+1}}\right] ds \\
&\le \bar{K}^4 \Bigg(e^{5\alpha^s\omega_i}\int_{-\omega_i}^{0} e^{3\alpha^s(\omega_i+s)}ds + 4e^{4\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{2\alpha^s(\omega_i+s)}ds + 6e^{3\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{\alpha^s(\omega_i+s)}ds \\
&\qquad +4e^{2\alpha^s\omega_i}e^{12\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{\delta s}ds + e^{16\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{-\alpha^s s}ds\Bigg) \\
&\le 6\bar{K}^4\left(e^{5\alpha^s\omega_i}+e^{4\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}+e^{3\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}}+e^{2\alpha^s\omega_i}e^{12\alpha^w\omega_{i+1}}+e^{16\alpha^w\omega_{i+1}}\right).
\end{aligned}$$

Combining the last to estimates concludes the proof. □

3.4.7 Summarising the estimates of $T^1_{\kappa_i}$ and $T^2_{\kappa_i}$

Here we collect the estimates of the forgoing sections to finally estimate each single term appearing in the expression of the jump (3.65). To this end we first consider the left-hand sides of the scalar products. Due to exponential dichotomies (3.16) we find with $\beta^+_{\kappa_i}=\alpha^u$ and $-\alpha^-_{\kappa_i}=\alpha^s$, cf. (3.18) and (3.66)

$$\left.\begin{aligned}
\|\Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})^T P^+_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\| &= O(e^{-\alpha^u\omega_{i+1}}), \\
\|\Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\| &= O(e^{\alpha^s\omega_i}).
\end{aligned}\right\} \tag{3.112}$$

With this we obtain the following estimates of the single terms in (3.65).

$\boldsymbol{T^{11}_{\kappa_i}} := \left\langle \Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})^T P^+_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ \tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})\big(\gamma^-_{\kappa_{i+1}}(\lambda)(-\omega_{i+1})-p_{\kappa_{i+1}}\big)\right\rangle$

The estimate of this terms simply follows from Lemma 3.4.6 in combination with the first equation in (3.112):

$$\boldsymbol{T^{11}_{\kappa_i}} = O\left(e^{-2\alpha^u\omega_{i+1}}\right). \tag{3.113}$$

$\boldsymbol{T^{12}_{\kappa_i}} := -\left\langle \Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})^T P^+_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ \tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})\big(\gamma^+_{\kappa_i}(\lambda)(\omega_{i+1})-p_{\kappa_{i+1}}\big)\right\rangle$

Again we may immediately adopt Lemma 3.4.6 and combine it with (3.112). This yields

$$\boldsymbol{T^{12}_{\kappa_i}} = \begin{cases} O\left(e^{\max\{2(\alpha^s-\alpha^u),\alpha^s-\nu\alpha^u\}\omega_{i+1}}\right), & \text{if (H3.7) applies,} \\ O\left(e^{3/2(\alpha^s-\alpha^u)\omega_{i+1}}\right), & \text{else.} \end{cases} \tag{3.114}$$

$\boldsymbol{T^{13}_{\kappa_i}} := \left\langle \Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})^T P^+_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ \tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})(id-P^-_{\kappa_{i+1}}(\lambda,-\omega_{i+1}))v^-_{i+1}(\lambda)(-\omega_{i+1})\right\rangle$

Recall the shortened notation $v^{-,s}_{i+1}(-\omega_{i+1}) := (id-P^-_{\kappa_{i+1}}(-\omega_{i+1}))v^-_{i+1}(-\omega_{i+1})$, cf. Definition 3.3.4. Then we obtain from Lemma 3.4.9 in combination with Estimate (3.112)

$$\boldsymbol{T}^{\mathbf{13}}_{\boldsymbol{\kappa_i}} = \begin{cases} O\left(e^{-2\alpha^u\omega_{i+1}}\left[e^{2\alpha^s\omega_{i+1}} + e^{-2\alpha^u\omega_{i+2}}\right]\right), & \text{if (H3.7) applies,} \\ O\left(e^{1/2(\alpha^s-3\alpha^u)\omega_{i+1}}\left[e^{2\alpha^s\omega_{i+1}} + e^{(\alpha^s-\alpha^u)\omega_{i+2}}\right]\right), & \text{else,} \end{cases} \tag{3.115}$$

since the projection $\tilde{P}_{\kappa_{i+1}}(\lambda, \omega_{i+1})$ is bounded.

$\boldsymbol{T}^{\mathbf{14}}_{\boldsymbol{\kappa_i}} := -\left\langle \Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})^T P^+_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i} \; , \; \tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})(id - P^+_{\kappa_i}(\lambda,\omega_{i+1}))v^+_i(\lambda)(\omega_{i+1})\right\rangle$

From Lemmata 3.4.3 and 3.4.5 we find the estimates

$$\tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})(id - P^+_{\kappa_i}(\lambda,\omega_{i+1})) = \begin{cases} O\left(e^{\max\{\alpha^s-\alpha^u, -(\nu-1)\alpha^u\}\omega_{i+1}}\right), & \text{if (H3.7) applies,} \\ O\left(e^{1/2(\alpha^s-\alpha^u)\omega_{i+1}}\right), & \text{else.} \end{cases}$$

This in combination with the estimate of $v^{+,s}_i(\omega_{i+1}) := (id - P^+_{\kappa_i}(\omega_{i+1}))v^+_i(\omega_{i+1})$ in Lemma 3.4.9 and Estimate (3.112) provides

$$\boldsymbol{T}^{\mathbf{14}}_{\boldsymbol{\kappa_i}} = \begin{cases} O\left(e^{\max\{2(\alpha^s-\alpha^u), \alpha^s-\nu\alpha^u\}\omega_{i+1}}\left[e^{2\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}\right]\right), & \text{if (H3.7) applies,} \\ O\left(e^{3/2(\alpha^s-\alpha^u)\omega_{i+1}}\left[e^{2\alpha^s\omega_i} + e^{(\alpha^s-\alpha^u)\omega_{i+1}}\right]\right), & \text{else.} \end{cases} \tag{3.116}$$

$\boldsymbol{T}^{\mathbf{15}}_{\boldsymbol{\kappa_i}} := -\left\langle \psi_{\kappa_i} \; , \; \int\limits_0^{\omega_{i+1}} \Phi^+_{\kappa_i}(\lambda)(0,s)P^+_{\kappa_i}(\lambda,s)h^+_{\kappa_i}(s, v^+_i(\lambda)(s),\lambda)ds\right\rangle$

The estimate of this term follows immediately from Lemma 3.4.11:

$$\boldsymbol{T}^{\mathbf{15}}_{\boldsymbol{\kappa_i}} = \begin{cases} O\left(e^{6\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{-2\alpha^u\omega_{i+1}} + e^{\max\{(\nu-2)\alpha^s-3\alpha^u, -4\alpha^u\}\omega_{i+1}}\right), & \text{if (H3.7) applies,} \\ O\Big(e^{6\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{(\alpha^s-\alpha^u)\omega_{i+1}} & \\ \quad + e^{\max\{(\nu-\frac{1}{2})\alpha^s-\frac{3}{2}\alpha^u, 2(\alpha^s-\alpha^u)\}\omega_{i+1}}\Big), & \text{else.} \end{cases} \tag{3.117}$$

Summarising we obtain for the first two scalar products on the right-hand side in (3.65) the estimate

$$\boldsymbol{T}^{\mathbf{1}}_{\boldsymbol{\kappa_i}} = \begin{cases} O\left(e^{6\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}\right), & \text{if (H3.7) applies,} \\ O\big(e^{6\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{(\alpha^s-\alpha^u)\omega_{i+1}} + e^{\max\{3/2(\alpha^s-\alpha^u), -2\alpha^u\}\omega_{i+1}} & \\ \quad + e^{1/2(\alpha^s-3\alpha^u)\omega_{i+1}}e^{(\alpha^s-\alpha^u)\omega_{i+2}}\big), & \text{else.} \end{cases} \tag{3.118}$$

We continue with estimating the terms in the remaining two scalar products of (3.65).

$\boldsymbol{T}^{\mathbf{21}}_{\boldsymbol{\kappa_i}} := -\left\langle \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i} \; , \; (id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))\big(\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) - p_{\kappa_i}\big)\right\rangle$

Applying Lemma 3.4.6 in combination with the second equation in (3.112) yields:

$$\boldsymbol{T}^{\mathbf{21}}_{\boldsymbol{\kappa_i}} = O\left(e^{2\alpha^s\omega_i}\right). \tag{3.119}$$

$\boldsymbol{T}^{\mathbf{22}}_{\boldsymbol{\kappa_i}} := \left\langle \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i} \; , \; (id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))\big(\gamma^-_{\kappa_i}(\lambda)(-\omega_i) - p_{\kappa_i}\big)\right\rangle$

Analogously combining Lemma 3.4.6 and (3.112) provides

$$\boldsymbol{T}^{\mathbf{22}}_{\boldsymbol{\kappa_i}} = \begin{cases} O\left(e^{\max\{\nu\alpha^s-\alpha^u, 2(\alpha^s-\alpha^u)\}\omega_i}\right), & \text{if (H3.7) applies,} \\ O\left(e^{\max\{\nu\alpha^s-\alpha^u, 3/2(\alpha^s-\alpha^u)\}\omega_i}\right), & \text{else.} \end{cases} \tag{3.120}$$

$$\boldsymbol{T^{23}_{\kappa_i}} := -\left\langle \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ (id-\tilde{P}_{\kappa_i}(\lambda,\omega_i))(id-P^+_{\kappa_{i-1}}(\lambda,\omega_i))v^+_{i-1}(\lambda)(\omega_i)\right\rangle$$

From Lemma 3.4.9 with $v^{+,s}_{i-1}(-\omega_i) := (id-P^+_{\kappa_{i-1}}(\omega_i))v^+_{i-1}(\omega_i)$ in combination with Estimate (3.112) we obtain

$$\boldsymbol{T^{23}_{\kappa_i}} = \begin{cases} O\left(e^{2\alpha^s\omega_i}\left[e^{2\alpha^s\omega_{i-1}}+e^{-2\alpha^u\omega_i}\right]\right), & \text{if (H3.7) applies,}\\ O\left(e^{2\alpha^s\omega_i}\left[e^{2\alpha^s\omega_{i-1}}+e^{(\alpha^s-\alpha^u)\omega_i}\right]\right), & \text{else.}\end{cases} \tag{3.121}$$

$$\boldsymbol{T^{24}_{\kappa_i}} := \left\langle \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ (id-\tilde{P}_{\kappa_i}(\lambda,\omega_i))(id-P^-_{\kappa_i}(\lambda,-\omega_i))v^-_i(\lambda)(-\omega_i)\right\rangle$$

The Lemmata 3.4.3 and 3.4.5 provide the estimates

$$(id-\tilde{P}_{\kappa_i}(\lambda,\omega_i)(id-P^-_{\kappa_i}(\lambda,-\omega_i)) = \begin{cases} O\left(e^{\max\{(\nu-1)\alpha^s,\alpha^s-\alpha^u\}\omega_i}\right), & \text{if (H3.7) applies,}\\ O\left(e^{\max\{(\nu-1)\alpha^s,(\alpha^s-\alpha^u)/2\}\omega_i}\right), & \text{else.}\end{cases}$$

In combination with the estimate of $v^{-,s}_i(-\omega_i) := (id-P^-_{\kappa_i}-(\omega_i))v^-_i(-\omega_i)$ in Lemma 3.4.9 and Estimate (3.112) this yields

$$\boldsymbol{T^{24}_{\kappa_i}} = \begin{cases} O\left(e^{\max\{\nu\alpha^s-\alpha^u,2(\alpha^s-\alpha^u)\}\omega_i}\left[e^{2\alpha^s\omega_i}+e^{-2\alpha^u\omega_{i+1}}\right]\right), & \text{if (H3.7) applies,}\\ O\left(e^{\max\{(\nu+\frac{1}{2})\alpha^s-\frac{1}{2}\alpha^u,(3\alpha^s-\alpha^u)\}\omega_i}\left[e^{2\alpha^s\omega_i}+e^{(\alpha^s-\alpha^u)\omega_{i+1}}\right]\right), & \text{else.}\end{cases} \tag{3.122}$$

$$\boldsymbol{T^{25}_{\kappa_i}} := -\left\langle \psi_{\kappa_i}\ ,\ \int\limits_{-\omega_i}^{0}\Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)h^-_{\kappa_i}(s,v^-_i(\lambda)(s),\lambda,\kappa)ds\right\rangle$$

Due to equation (3.110) we find

$$\boldsymbol{T^{25}_{\kappa_i}} = \begin{cases} O\left(e^{-6\alpha^u\omega_{i+1}}+e^{-2\alpha^u\omega_{i+1}}e^{2\alpha^s\omega_i}+e^{\min\{\nu+1,4\}\alpha^s\omega_i}\right), & \text{if (H3.7) applies,}\\ O\left(e^{3(\alpha^s-\alpha^u)\omega_{i+1}}+e^{(\alpha^s-\alpha^u)\omega_{i+1}}e^{2\alpha^s\omega_i}+e^{\min\{\nu+1,4\}\alpha^s\omega_i}\right), & \text{else.}\end{cases} \tag{3.123}$$

With this we have estimated each term of the representation (3.65) of the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$.

3.5 The derivative of the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$

In addition to the exponential convergence rates of the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$, the convergence rates of the derivatives of the jump with respect to the transition times ω_j, $D_{\omega_j}\xi_i(\boldsymbol{\omega},\lambda,\kappa)$, $j\in\mathbb{Z}$, will be of importance in the further course of this thesis. Due to the representation (3.61) of the jump its differentiability with respect to $\boldsymbol{\omega}$ and hence ω_j is given thanks to Lemma 3.2.7. With this the derivative with respect to $\boldsymbol{\omega}$ is actually a derivative with respect to $\boldsymbol{\beta}$. However, we will spare ourselves this paraphrase.

It is not surprising that the derivatives of the single components of the jump, that we have considered in the Sections 3.4.3 - 3.4.5, have exactly the same convergence rates as the components themselves. In Section 3.5.2, however, we will verify this in detail and summarize the results in Section 3.5.3.

But beforehand we study in the subsequent section the differentiability of the jump with respect to $\boldsymbol{\omega}\in l^\infty$. As well as the estimates of $D_{\omega_j}\xi_i(\boldsymbol{\omega},\lambda,\kappa)$ this will be needed during the proof of our main result in Chapter 5.

3.5.1 The derivative of $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ with respect to $\boldsymbol{\omega}$

The, so far unpublished, statements and proofs provided in this subsection are the work of Jürgen Knobloch (personal contact, June 2021). Since Theorem 3.5.1 will be important in the course of this thesis and in order to keep the thesis self contained, they were made available for this purpose.

Theorem 3.5.1. *Let i be fixed and let $\boldsymbol{\omega}$ be a sequence with a lower bound $\Omega > 0$. Then $\big(D_j\xi_i(\omega)\big)_{j\in\mathbb{Z}} \in l^1$ with $D_j\xi_i(\omega) = O\big(1/2^{|j-i|}\big)$ and*

$$D\xi_i(\boldsymbol{\omega})\boldsymbol{h} = \sum_j D_j\xi_i(\boldsymbol{\omega})h_j,$$

$\boldsymbol{h} = (h_j) \in l^\infty$.

Before we turn towards the proof of Theorem 3.5.1 we start with some preliminary statements. At first we present a technical lemma that is inspired by [HJKL11, Lemma 4.4].

Lemma 3.5.2. *Let $(a_i^\pm)_{i\in\mathbb{Z}}$ be a sequence of positive numbers such that for all $j \in \mathbb{Z}$*

$$a_j^- + a_j^+ \leq \frac{1}{2q}(a_{j-1}^- + a_{j+1}^+) + \varepsilon,$$

for some $q > 1$ and some additional term ε. Then, for any $i \in \mathbb{N}$ the following holds true

$$a_j^- + a_j^+ \leq \frac{1}{q^i}\|a\| + \frac{4q^2}{(q-1)^2}\varepsilon.$$

Proof. First, we prove by induction with respect to i that

$$(a_{j-i}^- + a_{j+i}^+) + (a_{j-i}^+ + a_{j+i}^-) \leq \frac{1}{q}(a_{j-i-1}^- + a_{j+i+1}^+) + \frac{4q}{q-1}\varepsilon. \tag{3.124}$$

So, let $i = 0$. According to the assumption of the lemma we have $2(a_j^- + a_j^+) \leq \frac{1}{q}(a_{j-1}^- + a_{j+1}^+) + 2\varepsilon(N)$, where $2 < 4q/(q-1)$. Now assume that (3.124) holds true for some i. Again applying the assumption of the lemma we obtain

$$\begin{aligned}
(a_{j-i-1}^- + a_{j+i+1}^+) + (a_{j-i-1}^+ + a_{j+i+1}^-) &\leq \tfrac{1}{2q}(a_{j-i-2}^- + a_{j-i}^+) + \tfrac{1}{2q}(a_{j+i}^- + a_{j+i+2}^+) + 2\varepsilon \\
&= \tfrac{1}{2q}(a_{j-i-2}^- + a_{j+i+2}^+) + \tfrac{1}{2q}(a_{j+i}^- + a_{j-i}^+) + 2\varepsilon \\
&\leq \tfrac{1}{2q}(a_{j-i-2}^- + a_{j+i+2}^+) + \tfrac{1}{2q}[(a_{j+i}^- + a_{j-i}^+) + (a_{j+i}^+ + a_{j-i}^-)] \\
&\quad +2\varepsilon \\
&\leq \tfrac{1}{2q}(a_{j-i-2}^- + a_{j+i+2}^+) + \tfrac{1}{2q^2}(a_{j-i-1}^- + a_{j+i+1}^+) \\
&\quad +(2 + \tfrac{1}{2q}\tfrac{4q}{q-1})\varepsilon.
\end{aligned}$$

Therefore

$$(1 - \frac{1}{2q^2})(a_{j-i-1}^- + a_{j+i+1}^+) + (a_{j-i-1}^+ + a_{j+i+1}^-) \leq \frac{1}{2q}(a_{j-i-2}^- + a_{j+i+2}^+) + \frac{2q}{q-1}\varepsilon$$

and hence

$$(a_{j-i-1}^- + a_{j+i+1}^+) + (a_{j-i-1}^+ + a_{j+i+1}^-) \leq \frac{1}{q}(a_{j-i-2}^- + a_{j+i+2}^+) + \frac{4q}{q-1}\varepsilon.$$

This finally proves (3.124) which in particular yields

$$(a^-_{j-i} + a^+_{j+i}) \leq \frac{1}{q}(a^-_{j-i-1} + a^+_{j+i+1}) + \frac{4q}{q-1}\varepsilon. \tag{3.125}$$

For $i = 0$ we then find by repeatedly invoking (3.125)

$$\begin{aligned}(a^-_j + a^+_j) &\leq \tfrac{1}{q}(a^-_{j-1} + a^+_{j+1}) + \tfrac{4q}{q-1}\varepsilon \\ &\leq \tfrac{1}{q^2}(a^-_{j-2} + a^+_{j+2}) + (\tfrac{1}{q} + 1)\tfrac{4q}{q-1}\varepsilon \\ &\vdots \\ &\leq \tfrac{1}{q^i}(a^-_{j-i} + a^+_{j+i}) + \sum_{k=0}^{i-1} \tfrac{1}{q^k}\tfrac{4q}{q-1}\varepsilon \\ &\leq \tfrac{1}{q^i}(a^-_{j-i} + a^+_{j+i}) + \tfrac{4q^2}{(q-1)^2}\varepsilon\end{aligned}$$

which proves the lemma. □

Lemma 3.5.3. *Let $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2$ be two sequences with a common lower bound Ω for which $\omega^1_i = \omega^2_i$ for all $i \in [-N, N] \cap \mathbb{Z}$. If Ω is sufficiently large, then there exists a constant C such that*

$$\|v_0(\boldsymbol{\omega}^1, \lambda, \kappa)(0) - v_0(\boldsymbol{\omega}^2, \lambda, \kappa)(0)\| \leq C\left({}^1\!/_2\right)^N.$$

Proof. The proof basically runs along similar lines as the proofs of [HJKL11, Lemma 4.5], [Kno00, Lemma 5.1] or [Lin90, Lemma 3.4]. For the sake of simplicity we suppress the dependency on λ and κ. By (3.22) we get

$$v_0(\boldsymbol{\omega})(0) = \hat{v}_{\boldsymbol{\omega},0}(\boldsymbol{\mathcal{H}}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{\omega}))(0).$$

Recall that the banach spaces in which the solutions of the fixed point equations (3.22) are searched depend explicitly on $\boldsymbol{\omega}$. Therefore we define $\tilde{\omega}_i := \min\{\omega^1_i, \omega^2_i\}$ and $\tilde{\boldsymbol{\omega}} := (\tilde{\omega}_i)_{i\in\mathbb{Z}}$. Then of course

$$\|v_0(\boldsymbol{\omega}^1)(0) - v_0(\boldsymbol{\omega}^2)(0)\| \leq \|v_0(\boldsymbol{\omega}^1)(0) - v_0(\tilde{\boldsymbol{\omega}})(0)\| + \|v_0(\tilde{\boldsymbol{\omega}})(0) - v_0(\boldsymbol{\omega}^2)(0)\|.$$

In order to study $\|v_0(\boldsymbol{\omega}^1)(0) - v_0(\tilde{\boldsymbol{\omega}})(0)\|$ we define $\boldsymbol{v}^R := \boldsymbol{v}(\boldsymbol{\omega}^1)|_{\tilde{\boldsymbol{\omega}}}$ that is

$$v^{R,+}_i(\cdot) = v^+_i(\boldsymbol{\omega}^1)(\cdot)|_{[0,\tilde{\omega}_i]} \qquad \text{and} \qquad v^{R,-}_{i+1}(\cdot) = v^-_{i+1}(\boldsymbol{\omega}^1)(\cdot)|_{[-\tilde{\omega}_{i+1},0]}.$$

So we have in particular $v^{R,\pm}_i(0) = v^\pm_i(\boldsymbol{\omega}^1)(0)$. With this we obtain

$$\|v_0(\boldsymbol{\omega}^1)(0) - v_0(\tilde{\boldsymbol{\omega}})(0)\| = \|v^R_0(0) - v_0(\tilde{\boldsymbol{\omega}})(0)\|. \tag{3.126}$$

Now, it is obvious that $\boldsymbol{v}(\tilde{\boldsymbol{\omega}}) = \hat{\boldsymbol{v}}_{\tilde{\boldsymbol{\omega}}}(\boldsymbol{\mathcal{H}}(\boldsymbol{v}(\tilde{\boldsymbol{\omega}})), \boldsymbol{d}(\tilde{\boldsymbol{\omega}}))$. Further, we consider the fixed point equation

$$\boldsymbol{v} = \hat{\boldsymbol{v}}_{\tilde{\boldsymbol{\omega}}}(\boldsymbol{\mathcal{H}}(\boldsymbol{v}), \boldsymbol{d}^R), \tag{3.127}$$

where $\boldsymbol{d}^R = (d^R_i)_{i\in\mathbb{Z}}$ with

$$d^R_i := v^+_{i-1}(\boldsymbol{\omega}^1)(\tilde{\omega}_i) - v^-_i(\boldsymbol{\omega}^1)(-\tilde{\omega}_i). \tag{3.128}$$

Recall in this respect, cf. (3.13), that

$$\left.\begin{aligned} d_{i+1}(\tilde{\boldsymbol{\omega}}) &:= v_i^+(\tilde{\boldsymbol{\omega}})(\tilde{\omega}_{i+1}) - v_{i+1}^-(\tilde{\boldsymbol{\omega}})(-\tilde{\omega}_{i+1}) \\ &= \gamma_{\kappa_{i+1}}^-(\lambda)(-\tilde{\omega}_{i+1}) - \gamma_{\kappa_i}^+(\lambda)(\tilde{\omega}_{i+1}). \end{aligned}\right\} \tag{3.129}$$

The unique solution of (3.127) is $\boldsymbol{v}^R$.

Now we compare $\boldsymbol{v}(\tilde{\boldsymbol{\omega}})$ and $\boldsymbol{v}^R$. To this end we define $\Delta \boldsymbol{v} := (\Delta v_i^+, \Delta v_i^-)_{i\in\mathbb{Z}}$ where

$$\Delta v_i^\pm := v_i^{R,\pm}(t) - v_i^\pm(\tilde{\boldsymbol{\omega}})(t).$$

By linearity of $\hat{\boldsymbol{v}}_{\boldsymbol{\omega}}(\boldsymbol{\mathcal{H}}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{\omega}))$ with respect to the first and second component, cf. Lemma 3.2.5, we obtain

$$\Delta \boldsymbol{v} = \hat{\boldsymbol{v}}_{\tilde{\boldsymbol{\omega}}}(\Delta\boldsymbol{\mathcal{H}}, \Delta\boldsymbol{d}),$$

with $\Delta\boldsymbol{\mathcal{H}} := \boldsymbol{\mathcal{H}}(\boldsymbol{v}^R) - \boldsymbol{\mathcal{H}}(\boldsymbol{v}(\tilde{\boldsymbol{\omega}}))$ and $\Delta\boldsymbol{d} := \boldsymbol{d}^R - \boldsymbol{d}(\tilde{\boldsymbol{\omega}})$. Invoking (3.51) we find

$$\hat{\boldsymbol{v}}_{\tilde{\boldsymbol{\omega}}}(\Delta\boldsymbol{\mathcal{H}}, \Delta\boldsymbol{d}) = \boldsymbol{v}_{\tilde{\boldsymbol{\omega}}}(\Delta\boldsymbol{\mathcal{H}}, \hat{\boldsymbol{a}}_{\tilde{\boldsymbol{\omega}}}(\Delta\boldsymbol{\mathcal{H}}, \Delta\boldsymbol{d})).$$

Henceforth we will use the short hand notation $\Delta\boldsymbol{a} := \hat{\boldsymbol{a}}_{\tilde{\boldsymbol{\omega}}}(\Delta\boldsymbol{\mathcal{H}}, \Delta\boldsymbol{d})$. In particular we find for the term to estimate

$$\|v_0(\boldsymbol{\omega}^1)(0) - v_0(\tilde{\boldsymbol{\omega}})(0)\| = \|\Delta v_0(0)\| = \|v_{\tilde{\boldsymbol{\omega}},0}(\Delta\boldsymbol{\mathcal{H}}, \Delta\boldsymbol{a})(0)\|.$$

From now on we proceed as in the proof of [Kno00, Lemma 5.1]. Recall that $v_{\tilde{\boldsymbol{\omega}},0}^\pm(\boldsymbol{g}, \boldsymbol{a})(\cdot)$ solves the variational equation $\dot{v} = D_1 f(\gamma_{\kappa_0}(t))v + g_0^\pm(t)$ and $a_1^+ = P_{\kappa_0}^+(\tilde{\omega}_1)v_0^+(\tilde{\omega}_1)$ and $a_0^- = P_{\kappa_0}^-(\tilde{\omega}_0)v_0^-(-\tilde{\omega}_0)$, cf. (3.34). Exploiting the asymptotic behaviour of the variational equation, we find, similar to the estimate in (3.46) for $i = 0$, that there are constants $C, M > 0$ and $\alpha < 0$ such that

$$\|\Delta v_0^\pm(0)\| \le Ce^{\alpha\tilde{\Omega}}(\|\Delta a_1^+\| + \|\Delta a_0^-\|) + M\|\Delta\mathcal{H}_0\|,$$

where $\tilde{\Omega}$ is the lower bound of $\tilde{\boldsymbol{\omega}}$. Parallel to the proof of Lemma 3.4.9 we find for $\tilde{\Omega}$ sufficiently large

$$M\|\Delta\mathcal{H}_0\| \le \frac{1}{2}\|\Delta v_0^\pm(0)\|.$$

Putting this together yields

$$\|\Delta v_0^\pm(0)\| \le Ce^{\alpha\tilde{\Omega}}(\|\Delta a_1^+\| + \|\Delta a_0^-\|), \tag{3.130}$$

The fixed point equation (3.49) of $\boldsymbol{a}$ provides the existence of a constant C_1 such that

$$\|\Delta a_i^+\| + \|\Delta a_i^-\| \le C_1(\|\Delta d_i\| + \|(id - P_{\kappa_{i-1}}^+(\tilde{\omega}_i))\Delta v_{i-1}^+(\tilde{\omega}_i)\| + \|(id - P_{\kappa_i}^-(-\tilde{\omega}_i))\Delta v_i^-(-\tilde{\omega}_i)\|).$$

The second and third term at the right-hand side of the last inequality can be expressed as in equation (3.41). Once more exploiting exponential dichotomies and taking into account that the g-terms arising there are $O(\|\Delta\boldsymbol{v}\|^2)$ we end up with

$$\|\Delta a_i^+\| + \|\Delta a_i^-\| \le C_1(\|\Delta d_i\| + C_2 e^{-\alpha\tilde{\Omega}}(\|\Delta v_{i-1}^+(0)\| + \|\Delta v_i^-(0)\|) + \varepsilon(\|\Delta v_{i-1}^+\| + \|\Delta v_i^-\|)). \tag{3.131}$$

Here ε can be chosen arbitrarily small (by taking $\inf(\boldsymbol{\omega})$ and $\inf(\hat{\boldsymbol{\omega}})$ and hence $\tilde{\Omega}$ sufficiently large).

Similar to the proof of estimate (3.45) we can show that

$$\|\Delta v_i^{\pm}\| \leq C_3((\|\Delta a_{i+1}^+\| + \|\Delta a_i^-\|) + (\|\Delta h_i^+\| + \|\Delta h_i^-\|)), \tag{3.132}$$

where $\Delta h_i^{\pm}(t) = h_{\kappa_i}^{\pm}(t, v_i^{R,\pm}(t)) - h_{\kappa_i}^{\pm}(t, v_i^{\pm}(\tilde{\boldsymbol{\omega}})(t)))$. Using the mean-value-theorem in combination with $h_{\kappa_i}^{\pm}(t,0) \equiv 0$ and $D_2 h_{\kappa_i}^{\pm}(t,0) \equiv 0$, cf. (3.14), it yields $\|\Delta h_i^{\pm}(t)\| \leq \varepsilon \|\Delta v_i^{\pm}(t)\|$. Again by choosing ε sufficiently small we obtain from (3.132)

$$\|\Delta v_i^{\pm}\| \leq C_4(\|\Delta a_{i+1}^+\| + \|\Delta a_i^-\|). \tag{3.133}$$

Combining the inequalities (3.131) and (3.133) then yields, again with $\tilde{\Omega}$ sufficiently large,

$$\|\Delta a_i^+\| + \|\Delta a_i^-\| \leq K\|\Delta d_i\| + \frac{1}{4}(\|\Delta a_{i+1}^+\| + \|\Delta a_{i-1}^-\|). \tag{3.134}$$

Recall that for $i \in [-N,N] \cap \mathbb{Z}$ we have $\omega_i^1 = \tilde{\omega}_i$. Hence

$$\begin{aligned} d_i^R &= v_{i-1}^+(\boldsymbol{\omega}^1)(\tilde{\omega}_i) - v_i^-(\boldsymbol{\omega}^1)(-\tilde{\omega}_i) = v_{i-1}^+(\boldsymbol{\omega}^1)(\omega_i^1) - v_i^-(\boldsymbol{\omega}^1)(-\omega_i^1) \\ &= \gamma_{\kappa_i}^-(-\omega_i^1) - \gamma_{\kappa_{i-1}}^+(\omega_i^1) = \gamma_{\kappa_i}^-(-\tilde{\omega}_i) - \gamma_{\kappa_{i-1}}^+(\tilde{\omega}_i) \\ &= d_i(\tilde{\boldsymbol{\omega}}), \qquad \text{for } i \in [-N,N] \cap \mathbb{Z}. \end{aligned}$$

So, $\|\Delta d_i\| = 0$ for $i \in [-N,N] \cap \mathbb{Z}$ and therefore

$$\|\Delta a_i^+\| + \|\Delta a_i^-\| \leq \frac{1}{4}(\|\Delta a_{i+1}^+\| + \|\Delta a_{i-1}^-\|), \quad \text{for } i \in [-N,N] \cap \mathbb{Z}.$$

Now, we can apply Lemma 3.5.2 with $\varepsilon = 0$ that results in

$$\begin{aligned} \|\Delta a_0^+\| + \|\Delta a_0^-\| &\leq \tfrac{1}{2^N}(\|\Delta a_N^+\| + \|\Delta a_{-N}^-\|), \\ \|\Delta a_1^+\| + \|\Delta a_1^-\| &\leq \tfrac{1}{2^{N-1}}(\|\Delta a_N^+\| + \|\Delta a_{2-N}^-\|). \end{aligned}$$

Finally estimating $\|\Delta a_N^+\| + \|\Delta a_{-N}^-\|$ and $\|\Delta a_N^+\| + \|\Delta a_{2-N}^-\|$ by $\|\Delta \boldsymbol{a}\|_{l^\infty}$ and inserting in (3.133) for $i = 0$ yields the lemma. □

Remark 3.5.4. *The statement of Lemma 3.5.3 is also true for v_i with sequences that coincide on a block centred at i.*

Remark 3.5.5. *The statement of Lemma 3.5.3 can be extended to the situation that $|\omega_i^1 - \omega_i^2| < 1/2^N$ for all $i \in [-N,N] \cap \mathbb{Z}$.*

Proof. The proof runs parallel the proof of Lemma 3.5.3 until we reach the Estimate 3.134. Then, we can apply Lemma 3.5.2 that results in

$$\begin{aligned} \|\Delta a_0^+\| + \|\Delta a_0^-\| &\leq 16K\|\Delta d\| + \tfrac{1}{2^N}(\|\Delta a_N^+\| + \|\Delta a_{-N}^-\|), \\ \|\Delta a_1^+\| + \|\Delta a_1^-\| &\leq 16K\|\Delta d\| + \tfrac{1}{2^{N-1}}(\|\Delta a_N^+\| + \|\Delta a_{2-N}^-\|), \end{aligned}$$

with $\|\Delta d\| := \max_{i \in [-N,N]} \|\Delta d_i\|$. Together with (3.130) this yields the existences of two constants C_4 and C_5 such that

$$\|\Delta v_0^{\pm}(0)\| \leq e^{\alpha\tilde{\Omega}}(C_4 \frac{1}{2^{N-1}} + C_5\|\Delta d\|). \tag{3.135}$$

Hence, it remains to consider Δd_i. We start with d_{i+1}^R, cf. the definition in (3.128) and make use of the first statement in (3.13).

$$\begin{aligned} d_{i+1}^R &= v_i^+(\boldsymbol{\omega})(\tilde{\omega}_{i+1}) - v_{i+1}^-(\boldsymbol{\omega})(-\tilde{\omega}_{i+1}) \\ &= v_i^+(\boldsymbol{\omega})(\omega_{i+1}) - v_{i+1}^-(\boldsymbol{\omega})(-\omega_{i+1}) \\ &\quad +(v_i^+(\boldsymbol{\omega})(\tilde{\omega}_{i+1}) - v_i^+(\boldsymbol{\omega})(\omega_{i+1})) + (v_{i+1}^-(\boldsymbol{\omega})(-\omega_{i+1}) - v_{i+1}^-(\boldsymbol{\omega})(-\tilde{\omega}_{i+1})) \\ &= \gamma_{\kappa_{i+1}}^-(-\omega_{i+1}) - \gamma_{\kappa_i}^+(\omega_{i+1}) \\ &\quad +(v_i^+(\boldsymbol{\omega})(\tilde{\omega}_{i+1}) - v_i^+(\boldsymbol{\omega})(\omega_{i+1})) + (v_{i+1}^-(\boldsymbol{\omega})(-\omega_{i+1}) - v_{i+1}^-(\boldsymbol{\omega})(-\tilde{\omega}_{i+1})). \end{aligned}$$

With this we then find by using the relation (3.129)

$$\begin{aligned} \Delta d_{i+1} &= d_{i+1}^R - d_{i+1}(\tilde{\boldsymbol{\omega}}) \\ &= (\gamma_{\kappa_{i+1}}^-(-\omega_{i+1}) - \gamma_{\kappa_{i+1}}^-(-\tilde{\omega}_{i+1})) + (\gamma_{\kappa_i}^+(\tilde{\omega}_{i+1}) - \gamma_{\kappa_i}^+(\omega_{i+1})) \\ &\quad +(v_i^+(\boldsymbol{\omega})(\tilde{\omega}_{i+1}) - v_i^+(\boldsymbol{\omega})(\omega_{i+1})) + (v_{i+1}^-(\boldsymbol{\omega})(-\omega_{i+1}) - v_{i+1}^-(\boldsymbol{\omega})(-\tilde{\omega}_{i+1})). \end{aligned}$$

In the following we apply the mean value theorem on each of the four differences above. In the first two cases we obtain for some intermediate points $\bar{\omega}_{i+1}^{\pm} \in [\tilde{\omega}_{i+1}, \omega_{i+1}]$

$$\|\gamma_{\kappa_{i/i+1}}^{\pm}(\pm\omega_{i+1}) - \gamma_{\kappa_{i/i+1}}^{\pm}(\pm\tilde{\omega}_{i+1})\| \leq \|\dot{\gamma}_{\kappa_{i/i+1}}^{\pm}(\pm\bar{\omega}_{i+1}^{\pm})\| \cdot |\tilde{\omega}_{i+1} - \omega_{i+1}|.$$

For t tending towards infinity the term $\dot{\gamma}^{\pm}(\pm t) = f(\gamma^{\pm}(\pm t))$ is tending towards $f(p) = 0$. In case of the other two terms we find

$$\|v_{i/i+1}^{\pm}(\boldsymbol{\omega})(\pm\omega_{i+1}) - v_{i/i+1}^{\pm}(\boldsymbol{\omega})(\pm\tilde{\omega}_{i+1})\| \leq \|\dot{v}_{i/i+1}^{\pm}(\boldsymbol{\omega})(\pm\bar{\omega}_{i+1}^{\pm})\| \cdot |\tilde{\omega}_{i+1} - \omega_{i+1}|.$$

The $v_i^{\pm}$ satisfy the differential equation $\dot{v}_i^{\pm}(t) = D_1 f(\gamma_{\kappa_i}^{\pm}(t))v_i^{\pm}(t) + h_{\kappa_i}^{\pm}(t, v_i^{\pm}(t))$, cf. (3.12). Due to Lemma 3.4.9 we find that $v_i^{\pm}(\omega)$ tends to zero for ω tending to infinity. Hence $h_{\kappa_i}^{\pm}(\omega, v_i^{\pm}(\omega))$ also tends to zero while $D_1 f(\gamma_{\kappa_i}^{\pm}(\omega))$ tends to $D_1 f(p)$. Thus $\dot{v}_i^{\pm}(\omega)$ tends to zero.
Summarizing we find for Ω sufficiently large and for $|i+1| < N$ with $|\tilde{\omega}_{i+1} - \omega_{i+1}| < 1/2^N$ the existence of a constant c such that

$$\|\Delta d_{i+1}^R\| \leq c\frac{1}{2^N}$$

with $c = c(\tilde{\Omega})$ tending to zero as $\tilde{\Omega}$ tends to infinity. Hence we find from (3.135) that $\|\Delta v_0^{\pm}(0)\| < C_6 \frac{1}{2^N}$ and therefore, cf. (3.126) $\|v_0(\boldsymbol{\omega})(0) - v_0(\tilde{\boldsymbol{\omega}})(0)\| < C_6 \frac{1}{2^N}$. Analogously we find $\|v_0(\tilde{\boldsymbol{\omega}})(0) - v_0(\hat{\boldsymbol{\omega}})(0)\| < C_6 \frac{1}{2^N}$ wich finally results in $\|v_0(\boldsymbol{\omega})(0) - v_0(\hat{\boldsymbol{\omega}})(0)\| < 2C_6 \frac{1}{2^N}$. □

Corollary 3.5.6. *Let i be fixed and let $\boldsymbol{\omega}$ be a sequence with lower bound $\Omega > 0$. Then $\big(D_j \xi_i(\omega)\big)_{j \in \mathbb{Z}} \in l^1$.*

Proof. We prove the statement only for ξ_0 by using Lemma 3.5.3. Due to Remark 3.5.4 the statement can be transferred to ξ_i, $i \neq 0$.

We define

$$\boldsymbol{h_N} := (h_{N,j})_{j \in \mathbb{Z}} : \qquad h_{N,j} = \begin{cases} h_N, & j = N \\ 0, & \text{else.} \end{cases}$$

Write

$$\begin{aligned} D_N\xi_0(\boldsymbol{\omega})h_N + \hat{r}_N(h_N) &= \xi_0(\boldsymbol{\omega}+\boldsymbol{h_N}) - \xi_0(\boldsymbol{\omega}) \\ &= v_0^+(\boldsymbol{\omega}+\boldsymbol{h_N})(0) - v_0^-(\boldsymbol{\omega}+\boldsymbol{h_N})(0) - \big(v_0^+(\boldsymbol{\omega})(0) - v_0^-(\boldsymbol{\omega})(0)\big). \end{aligned}$$

Then the proof of Lemma 3.5.3 reveals that

$$\|D_N\xi_0(\boldsymbol{\omega})h_N + \hat{r}_N(h_N)\| \leq C(1/2)^N \|\Delta\boldsymbol{a}\|_{l^\infty}.$$

Then, applying the mean-value-theorem on the $\|\Delta\boldsymbol{a}\|_{l^\infty}$ we obtain a constant $\tilde{C}$ such that

$$\|D_N\xi_0(\boldsymbol{\omega})h_N + \hat{r}_N(h_N)\| = \|D_N\xi_0(\boldsymbol{\omega}) + \frac{\hat{r}_N(h_N)}{|h_N|}\||h_N| \leq \tilde{C}(1/2)^N|h_N|.$$

Now, the limit $h_N \to 0$ gives

$$\|D_N\xi_0(\boldsymbol{\omega})\| \leq \tilde{C}(1/2)^N.$$

This proves the corollary. □

Remark 3.5.7. *The last two inequalities in the proof of Corollary 3.5.6 imply that there is a constant C such that*

$$\frac{\|\hat{r}_N(h_N)\|}{|h_N|} \leq C(1/2)^N.$$

Now we turn towards the proof of Theorem 3.5.1.

Proof of Theorem 3.5.1. Again we confine to prove the statement for ξ_0. The statement for $i \neq 0$ follows analogously. We define

$$\boldsymbol{h^N} := (h_j^N)_{j\in\mathbb{Z}}: \qquad h_j^N = \begin{cases} h_j, & \in [-N,N]\cap\mathbb{Z} \\ 0, & \text{else.} \end{cases}$$

and write with this

$$\xi_0(\boldsymbol{\omega}+\boldsymbol{h}) - \xi_0(\boldsymbol{\omega}) = \big(\xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}) - \xi_0(\boldsymbol{\omega})\big) + \big(\xi_0(\boldsymbol{\omega}+\boldsymbol{h}) - \xi_0(\boldsymbol{\omega}+\boldsymbol{h^N})\big). \tag{3.136}$$

The sequences $\boldsymbol{\omega}+\boldsymbol{h}$ and $\boldsymbol{\omega}+\boldsymbol{h^N}$ satisfy the assumption of Lemma 3.5.3. Via (3.61) the result of Lemma 3.5.3 is transferred to ξ_0, resulting in

$$\big(\xi_0(\boldsymbol{\omega}+\boldsymbol{h}) - \xi_0(\boldsymbol{\omega}+\boldsymbol{h^N})\big) = O(1/2^N).$$

Hence the limit $N\to\infty$ in (3.136) yields

$$\xi_0(\boldsymbol{\omega}+\boldsymbol{h}) - \xi_0(\boldsymbol{\omega}) = \lim_{N\to\infty}\big(\xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}) - \xi_0(\boldsymbol{\omega})\big). \tag{3.137}$$

As in the finite dimensional case the term on the right-hand side can be written as

$$\xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}) - \xi_0(\boldsymbol{\omega}) = D\xi_0(\boldsymbol{\omega})\boldsymbol{h^N} + r_N(\boldsymbol{h^N}) = \sum_{j=-N}^{N} D_j\xi_0(\boldsymbol{\omega})h_j + r_N(\boldsymbol{h^N}). \tag{3.138}$$

Note that $r_N(\boldsymbol{h^N}) = o(\|\boldsymbol{h^N}\|_{l^\infty})$ and hence also $r_N(\boldsymbol{h^N}) = o(\|\boldsymbol{h}\|_{l^\infty})$. According to Corollary 3.5.6 the limit $\lim\limits_{N\to\infty} \sum\limits_{j=-N}^{N} D_j\xi_0(\boldsymbol{\omega})h_j$ does exist. So, because of (3.137) and (3.138), also the limit $\lim\limits_{N\to\infty} r_N(\boldsymbol{h^N}) =: r(\boldsymbol{h})$ does exist. Altogether this yields

$$\xi_0(\boldsymbol{\omega}+\boldsymbol{h}) - \xi_0(\boldsymbol{\omega}) = \sum_{j\in\mathbb{Z}} D_j\xi_0(\boldsymbol{\omega})h_j + r(\boldsymbol{h}).$$

So it remains to show that $r(\boldsymbol{h}) = o(\|\boldsymbol{h}\|_{l^\infty})$. To this end we write $\xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}) - \xi_0(\boldsymbol{\omega})$ as follows:

$$\xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}) - \xi_0(\boldsymbol{\omega}) \;=\; \underbrace{\xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}) - \xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}-\boldsymbol{h^0})}_{=D_0\xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}-\boldsymbol{h^0})h_0+\hat{r}_0(h_0)} + \xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}-\boldsymbol{h^0}) - \xi_0(\boldsymbol{\omega}).$$

And $\xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}-\boldsymbol{h^0}) - \xi_0(\boldsymbol{\omega})$ we write as

$$\begin{aligned}
\xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}-\boldsymbol{h^0}) - \xi_0(\boldsymbol{\omega}) \;&=\; \xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}-\boldsymbol{h^0}) - \xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}-\boldsymbol{h^0}-(\ldots,0,h_1,0,\ldots)) \\
&\quad + \xi_0\big(\boldsymbol{\omega}+\boldsymbol{h^N}-\boldsymbol{h^0}-(\ldots,0,h_1,0,\ldots)\big) - \xi_0(\boldsymbol{\omega}) \\
&=\; \underbrace{\xi_0\big(\boldsymbol{\omega}+\boldsymbol{h^N}-(\ldots,0,h_0,0,\ldots)\big) - \xi_0\big(\boldsymbol{\omega}+\boldsymbol{h^N}-(\ldots,0,h_0,h_1,0,\ldots)\big)}_{=D_1\xi_0\left(\boldsymbol{\omega}+\boldsymbol{h^N}-(\ldots,0,h_0,h_1,0,\ldots)\right)h_1+\hat{r}_1(h_1)} \\
&\quad + \xi_0\big(\boldsymbol{\omega}+\boldsymbol{h^N}-(\ldots,0,h_0,h_1,0,\ldots)\big) - \xi_0(\boldsymbol{\omega}).
\end{aligned}$$

Proceeding along that line we finally obtain

$$\begin{aligned}
\xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}) - \xi_0(\boldsymbol{\omega}) \;&=\; D_0\xi_0(\boldsymbol{\omega}+\boldsymbol{h^N}-\boldsymbol{h^0})h_0 + \hat{r}_0(h_0) \\
&\quad + D_1\xi_0\big(\boldsymbol{\omega}+\boldsymbol{h^N}-(\ldots,0,h_0,h_1,0,\ldots)\big)h_1 + \hat{r}_1(h_1) \\
&\quad + D_{-1}\xi_0\big(\boldsymbol{\omega}+\boldsymbol{h^N}-(\ldots,0,h_{-1},h_0,h_1,0,\ldots)\big)h_{-1} + \hat{r}_{-1}(h_{-1}) \\
&\quad + \ldots + D_N\xi_0\big(\boldsymbol{\omega}+\boldsymbol{h^N}-(\ldots,0,h_{-(N-1)},\ldots,h_N,0,\ldots)\big)h_N + \hat{r}_N(h_N) \\
&\quad + D_{-N}\xi_0(\boldsymbol{\omega})h_{-N} + \hat{r}_{-N}(h_{-N}),
\end{aligned}$$

where $\hat{r}_i(h_i) = o(h_i)$. Further, we rewrite the terms in the last equation as

$$D_i\xi_0(\boldsymbol{\omega}+\ldots) = D_i\xi_0(\boldsymbol{\omega}) + \underbrace{D_i\xi_0(\boldsymbol{\omega}+\ldots) - D_i\xi_0(\boldsymbol{\omega})}_{=:R_i(\boldsymbol{h})}.$$

Due to the continuity of the partial derivatives of ξ_0 we have $R_i(\boldsymbol{h}) \to 0$ as $\|\boldsymbol{h}\|_{l^\infty} \to 0$. This shows

$$r_N(\boldsymbol{h^N}) = \sum_{i=-(N-1)}^{N} R_i(\boldsymbol{h})h_i + \sum_{i=-N}^{N} \hat{r}_i(h_i).$$

According to Corollary 3.5.6 and Remark 3.5.7 both $\sum_{i=-(N-1)}^{N} \|R_i(\boldsymbol{h})\|$ and $\sum_{i=-N}^{N} \frac{\|\hat{r}_i(h_i)\|}{|h_i|}$ converge uniformly. Hence

$$\begin{aligned}\lim_{\|\boldsymbol{h}\|_{l^\infty}\to 0}\frac{r(\boldsymbol{h})}{\|\boldsymbol{h}\|_{l^\infty}} &\leq \lim_{\|\boldsymbol{h}\|_{l^\infty}\to 0}\lim_{N\to\infty}\left(\sum_{i=-(N-1)}^{N}\|R_i(\boldsymbol{h})\|+\sum_{i=-N}^{N}\frac{\|\hat{r}_i(h_i)\|}{|h_i|}\right)\\ &= \lim_{N\to\infty}\lim_{\|\boldsymbol{h}\|_{l^\infty}\to 0}\left(\sum_{i=-(N-1)}^{N}\|R_i(\boldsymbol{h})\|+\sum_{i=-N}^{N}\frac{\|\hat{r}_i(h_i)\|}{|h_i|}\right)\\ &= 0.\end{aligned}$$

□

3.5.2 Estimates of the derivative of $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$ with respect to ω_j

In what follows we study the derivatives of the single components within the representation of the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$ given in (3.65). For convenience we drop the dependency of λ in the notation. At first we start with the projections $P^{\pm}_{\kappa_i}(t)$ and $\tilde{P}_{\kappa_i}$.

Lemma 3.5.8. *Assume Hypotheses (H3.1)-(H3.7). There exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $|\lambda|<c$ and $\boldsymbol{\omega}$ with $\inf\boldsymbol{\omega}>\Omega$ the following estimates apply:*

$$\begin{aligned}\|D_{\omega_j}(id-P^+_{\kappa_i}(\omega_{i+1}))\| &= \|D_{\omega_j}P^+_{\kappa_i}(\omega_{i+1})\| &&= O(e^{\max\{(\nu-1)\alpha^s,\alpha^s-\alpha^u\}\omega_{i+1}}),\\ \|D_{\omega_j}(id-P^-_{\kappa_{i+1}}(-\omega_{i+1}))\| &= \|D_{\omega_j}P^-_{\kappa_{i+1}}(-\omega_{i+1})\| &&= O(e^{\max\{-(\nu-1)\alpha^u,\alpha^s-\alpha^u\}\omega_{i+1}}),\\ \|D_{\omega_j}(id-\tilde{P}_{\kappa_{i+1}}(\omega_{i+1}))\| &= \|D_{\omega_j}\tilde{P}_{\kappa_{i+1}}(\omega_{i+1})\| &&= O(e^{\max\{(\nu-1)\alpha^s,\alpha^s-\alpha^u\}\omega_{i+1}}),\end{aligned}$$

and

$$\begin{aligned}\|D_{\omega_j}\big((\tilde{P}_{\kappa_{i+1}}(\omega_{i+1})-P^+_{\kappa_i}(\omega_{i+1}))(id-P^-_{\kappa_{i+1}}(-\omega_{i+1}))\big)\| &= O(e^{\max\{\nu\alpha^s-\alpha^u,\min\{2,\nu-1\}(\alpha^s-\alpha^u)\}\omega_{i+1}}),\\ \|D_{\omega_j}\big(\tilde{P}_{\kappa_{i+1}}(\omega_{i+1})(id-P^+_{\kappa_i}(\omega_{i+1}))\big)\| &= O(e^{\max\{-(\nu-1)\alpha^u,\alpha^s-\alpha^u\}\omega_{i+1}}),\\ \|D_{\omega_j}\big((id-\tilde{P}_{\kappa_{i+1}}(\omega_{i+1}))(id-P^-_{\kappa_{i+1}}(-\omega_{i+1}))\big)\| &= O(e^{\max\{(\nu-1)\alpha^s,\alpha^s-\alpha^u\}\omega_{i+1}}),\\ \|D_{\omega_j}\big((id-\tilde{P}_{\kappa_{i+1}}(\omega_{i+1})-P^-_{\kappa_{i+1}}(-\omega_{i+1}))(id-P^+_{\kappa_i}(\omega_{i+1}))\big)\| &= O(e^{\max\{\nu\alpha^s-\alpha^u,\min\{2,\nu-1\}(\alpha^s-\alpha^u)\}\omega_{i+1}}).\end{aligned}$$

Proof. The projections under consideration only depend on ω_{i+1}. So the partial derivatives with respect to ω_j, $j\neq i+1$ are zero. It remains to consider the partial derivative with respect to ω_{i+1}. Exemplarily we do this for $P^+_{\kappa_i}(t)$. From (2.51) we infer

$$D_{\omega_{i+1}}P^+_{\kappa_i}(\omega_{i+1})=\dot{P}^+_{\kappa_i}(\omega_{i+1})=Df(\gamma^+_{\kappa_i}(\omega_{i+1}))P^+_{\kappa_i}(\omega_{i+1})-P^+_{\kappa_i}(\omega_{i+1})Df(\gamma^+_{\kappa_i}(\omega_{i+1})). \tag{3.139}$$

Now we make use of the estimates $\|Df(\gamma^+_{\kappa_i}(\omega_{i+1}))-Df(p)\|\leq e^{(\nu-1)\alpha^s\omega_{i+1}}$, cf. (3.71), and (3.75) and exploit that the projection corresponding to the spectral decomposition of $Df(p)$, $P_{\kappa_{i+1}}\equiv P$, commutes with $Df(p)$:

$$\begin{aligned}D_{\omega_{i+1}}P^+_{\kappa_i}(\omega_{i+1}) &= Df(\gamma^+_{\kappa_i}(t))[P^+_{\kappa_i}(\omega_{i+1})-(id-P)]+[Df(\gamma^+_{\kappa_i}(t))-Df(p)](id-P)\\ &\quad +(id-P)[Df(p)-Df(\gamma^+_{\kappa_i}(t))]+[(id-P)-P^+_{\kappa_i}(\omega_{i+1})]Df(\gamma^+_{\kappa_i}(t))\\ &= O(e^{\max\{(\nu-1)\alpha^s,\alpha^s-\alpha^u\}\omega_{i+1}}).\end{aligned}$$

Analogously we find

$$D_{\omega_{i+1}} P^-_{\kappa_{i+1}}(-\omega_{i+1}) = O(e^{\max\{(\nu-1)\alpha^u, \alpha^s-\alpha^u\}\omega_{i+1}}).$$

For the proof of the estimate for $\tilde{P}_{\kappa_{i+1}}(\omega_{i+1}) = S_{\kappa_{i+1}}(\omega_{i+1})(id - P_{\kappa_{i+1}})S_{\kappa_{i+1}}(\omega_{i+1})^{-1}$ we use Definition (3.31) via the mapping $S_{\kappa_{i+1}}$ and its inverse $S^{-1}_{\kappa_{i+1}}$. By differentiating the relation $S_{\kappa_{i+1}}S^{-1}_{\kappa_{i+1}} = id$ with respect to ω_{i+1} we obtain

$$D_{\omega_{i+1}}(S^{-1}_{\kappa_{i+1}}) = -S^{-1}_{\kappa_{i+1}}(D_{\omega_{i+1}}S_{\kappa_{i+1}})S^{-1}_{\kappa_{i+1}}.$$

Thus, the derivatives of S_{κ_i} and $S^{-1}_{\kappa_i}$ satisfy the same estimation. Calling in the definition of $S_{\kappa_{i+1}} = P^-_{\kappa_{i+1}}(-\omega_{i+1})P_{\kappa_{i+1}} + P^+_{\kappa_i}(\omega_{i+1})(id - P_{\kappa_{i+1}})$, cf. (3.30), and differentiating it with respect to ω_{i+1} then yields with the above estimates of $D_{\omega_{i+1}}P^{\pm}_{\kappa_{i+1}}$ the claimed estimate.

In order to verify the remaining four estimates of the lemma we first study the derivative of the composition of $(id - P^-_{\kappa_{i+1}})$ and $(id - P^+_{\kappa_i})$. To this end we look at the second equation on the right-hand side in (3.67). With this we find

$$\begin{aligned}
&D_{\omega_{i+1}}\big((id - P^-_{\kappa_{i+1}}(-\omega_{i+1}))(id - P^+_{\kappa_i}(\omega_{i+1}))\big) \\
&\quad = \; D_{\omega_{i+1}}\big((id - P^-_{\kappa_{i+1}}(-\omega_{i+1}) - (id - P))(id - P^+_{\kappa_i}(\omega_{i+1}))\big) \\
&\quad = \; \dot{P}^-_{\kappa_{i+1}}(-\omega_{i+1})(id - P^+_{\kappa_i}(\omega_{i+1})) - (id - P^-_{\kappa_{i+1}}(-\omega_{i+1}) - (id - P))\dot{P}^+_{\kappa_i}(\omega_{i+1}).
\end{aligned}$$

Due to (3.75) and the estimates of the derivative of $P^-_{\kappa_{i+1}}$ we then conclude

$$D_{\omega_{i+1}}\big((id - P^-_{\kappa_{i+1}}(-\omega_{i+1}))(id - P^+_{\kappa_i}(\omega_{i+1}))\big) = O(e^{\max\{-(\nu-1)\alpha^u, \alpha^s-\alpha^u\}\omega_{i+1}}). \tag{3.140}$$

Along similar lines we also find

$$D_{\omega_{i+1}}\big((id - P^+_{\kappa_i}(\omega_{i+1}))(id - P^-_{\kappa_{i+1}}(-\omega_{i+1}))\big) = O(e^{\max\{(\nu-1)\alpha^s, \alpha^s-\alpha^u\}\omega_{i+1}}). \tag{3.141}$$

Then the stated estimates follow from differentiation the equations in (3.69) and invoking estimates (3.140), (3.141) and (3.76). □

Remark 3.5.9. *If Hypothesis (H3.7) does not apply, we have to make use of the weaker estimate* (3.73) *instead of* (3.75)*. With this we find the estimates*

$$\begin{aligned}
\|D_{\omega_j}(id - P^+_{\kappa_i}(\omega_{i+1}))\| &= \|D_{\omega_j}P^+_{\kappa_i}(\omega_{i+1})\| &&= O(e^{\max\{(\nu-1)\alpha^s, 1/2(\alpha^s-\alpha^u)\}\omega_{i+1}}), \\
\|D_{\omega_j}(id - P^-_{\kappa_{i+1}}(-\omega_{i+1}))\| &= \|D_{\omega_j}P^-_{\kappa_{i+1}}(-\omega_{i+1})\| &&= O(e^{\max\{-(\nu-1)\alpha^u, 1/2(\alpha^s-\alpha^u)\}\omega_{i+1}}), \\
\|D_{\omega_j}(id - \tilde{P}_{\kappa_{i+1}}(\omega_{i+1}))\| &= \|D_{\omega_j}\tilde{P}_{\kappa_{i+1}}(\omega_{i+1})\| &&= O(e^{\max\{(\nu-1)\alpha^s, 1/2(\alpha^s-\alpha^u)\}\omega_{i+1}})
\end{aligned}$$

and in case of the other four estimates of the derivatives that are stated in the above lemma we find the same estimates as for the terms themselves, cf. Lemma 3.4.3.

Lemma 3.5.10. *Assume Hypotheses (H3.1)-(H3.6). There exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ the following estimates apply:*

$$\begin{aligned}
D_{\omega_j}\gamma^-_{\kappa_{i+1}}(-\omega_{i+1}) &= O(e^{-\alpha^u\omega_{i+1}}), \\
D_{\omega_j}\gamma^+_{\kappa_i}(\omega_{i+1}) &= O(e^{\alpha^s\omega_{i+1}}).
\end{aligned}$$

Proof. The estimate for the derivative of $\gamma^+_{\kappa_i}$ and $\gamma^-_{\kappa_{i+1}}$ follow from (3.79) and the fact that $\gamma^\pm$ solve the differential equation $\dot{x} = f(x)$ which yield via taylor expansion

$$\dot{\gamma}^\pm(t) = f(\gamma^\pm(t)) = f(p) + Df(p)(\gamma^\pm(t) - p) + O\big((\gamma^\pm(t) - p)^2\big)$$

with $f(p) = 0$. So, the derivatives of $\gamma^+_{\kappa_i}$ and $\gamma^-_{\kappa_{i+1}}$ with respect to ω_{i+1} are of order $O(e^{\alpha^s\omega_{i+1}})$ and $O(e^{-\alpha^u\omega_{i+1}})$, respectively. The derivative with respect to ω_j, $j \neq i+1$ are zero. □

Lemma 3.5.11. *Assume Hypotheses (H3.1)-(H3.6). There exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ the following estimates apply:*

$$\begin{aligned} \|D_{\omega_j}\Phi^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T \psi_{\kappa_i}\| &= O(e^{-\alpha^u\omega_{i+1}}), \\ \|D_{\omega_j}\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T \psi_{\kappa_i}\| &= O(e^{\alpha^s\omega_i}). \end{aligned}$$

Proof. Exemplarily we consider $F^1_{\kappa_i}(\boldsymbol{\omega}) := \Phi^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T \psi_{\kappa_i}$. Since this term only depends on ω_{i+1} we have $D_{\omega_j}F^1_{\kappa_i}(\boldsymbol{\omega}) = 0$, for all $j \neq i+1$. To study the derivative $D_{\omega_{i+1}}F^1_{\kappa_i}(\boldsymbol{\omega})$ we recall that $\Phi^+_{\kappa_i}(0,\cdot)^T$ is the transition matrix of $\dot{x} = -[Df(\gamma^+_{\kappa_i}(t))]^T x$. Hence

$$\begin{aligned} D_{\omega_{i+1}}F^1_{\kappa_i}(\boldsymbol{\omega}) &= \dot{\Phi}^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T \psi_{\kappa_i} \\ &= -[Df(\gamma^+_{\kappa_i}(\omega_{i+1}))]^T \Phi^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T \psi_{\kappa_i} \quad = \quad -[Df(\gamma^+_{\kappa_i}(\omega_{i+1}))]^T F^1_{\kappa_i}(\boldsymbol{\omega}). \end{aligned}$$

Incorporating the estimate of $F^1_{\kappa_i}(\boldsymbol{\omega})$ that is given by (3.112) yields

$$D_{\omega_{i+1}}F^1_{\kappa_i}(\boldsymbol{\omega}) \le Ce^{-\alpha^u\omega_{i+1}}.$$

That is both $F^1_{\kappa_i}(\boldsymbol{\omega})$ and its derivatives $D_{\omega_j}F^1_{\kappa_i}(\boldsymbol{\omega})$ are of order $O(e^{-\alpha^u\omega_{i+1}})$. The second statement of the lemma follows accordingly. □

We continue with the partial derivatives of $h^{\pm,s}$ and $h^{\pm,u}$ before moving on to $v_i(\boldsymbol{\omega})(\cdot)$. Note in this respect that $\bar{\boldsymbol{v}}^\pm(\boldsymbol{\omega})$ is differentiable with respect to $\boldsymbol{\omega}$, cf. Lemma 3.2.7. Hence $v_i(\boldsymbol{\omega})(\cdot)$ is differentiable with respect to $\boldsymbol{\omega}$, which can be seen by introducing the linear and bounded projection $Q^i : \bar{\boldsymbol{v}}^\pm(\boldsymbol{\omega}) \mapsto v^\pm_i(\boldsymbol{\omega})$. Then $Q^i \circ D\bar{\boldsymbol{v}}^\pm(\boldsymbol{\omega}) = D\big(Q^i\bar{\boldsymbol{v}}^\pm\big)(\boldsymbol{\omega})$. Therefore the partial derivatives $D_j v_i(\boldsymbol{\omega})(\cdot)$ do exist.
Further recall that $h^\pm_{\kappa_i}(t, v^+_i(\boldsymbol{\omega})(t))$ are Nemyzki operators. Their differentiability with respect to $v^\pm_i$ was shown in Lemma 3.3.6.

Recall Definition 3.3.4 and (3.80) for the notation of the superscripts s and u.

Lemma 3.5.12. *Assume Hypotheses (H3.1)-(H3.6). There exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ there exists a constant $M > 0$ such that $D_{\omega_j}h^{\pm,s}_i$ satisfy*

$$\begin{aligned} \|D_{\omega_j}h^{+,s}_{\kappa_i}(t, v^+_i(\boldsymbol{\omega})(t))\| &\le M\big((e^{\alpha^s t}\|v^{+,u}_i(\boldsymbol{\omega})(t)\| + \|v^{+,s}_i(\boldsymbol{\omega})(t)\|)\|D_j v^{+,u}_i(\boldsymbol{\omega})(t)\| \\ &\quad + \|v^+_i(\boldsymbol{\omega})(t)\|\|D_j v^{+,s}_i(\boldsymbol{\omega})(t)\|\big), \qquad t \in [0,\omega], \\ \|D_{\omega_j}h^{-,s}_{\kappa_i}(t, v^-_i(\boldsymbol{\omega})(t))\| &\le M\big((e^{1/2(\alpha^u - \alpha^s)t}\|v^{-,u}_i(\boldsymbol{\omega})(t)\| + \|v^{-,s}_i(\boldsymbol{\omega})(t)\|)\|D_j v^{-,u}_i(\boldsymbol{\omega})(t)\| \\ &\quad + \|v^-_i(\boldsymbol{\omega})(t)\|\|D_j v^{-,s}_i(\boldsymbol{\omega})(t)\|\big), \qquad t \in [-\omega, 0]. \end{aligned}$$

If additionally Hypothesis (H3.7) applies then

$$\begin{aligned}\|D_{\omega_j} h^{-,s}_{\kappa_i}(t, v^-_i(\boldsymbol{\omega})(t))\| \quad &\leq \quad M\big((e^{\alpha^u t}\|v^{-,u}_i(\boldsymbol{\omega})(t)\| + \|v^{-,s}_i(\boldsymbol{\omega})(t)\|)\|D_j v^{-,u}_i(\boldsymbol{\omega})(t)\| \\ &\quad + \|v^-_i(\boldsymbol{\omega})(t)\|\|D_j v^{-,s}_i(\boldsymbol{\omega})(t)\|\big), \qquad t \in [-\omega, 0].\end{aligned}$$

Proof. Recall (3.14) for the definition of $h^\pm$. According to the chain rule we then find

$$D_{\omega_j} h^\pm_{\kappa_i}(t, v^\pm_i(\boldsymbol{\omega})(t)) = \left[D_1 f(\gamma^\pm_{\kappa_i}(t) + v^\pm_i(\boldsymbol{\omega})(t)) - D_1 f(\gamma^\pm_{\kappa_i}(t))\right] D_j v^\pm_i(\boldsymbol{\omega})(t).$$

Applying the projection $(id - P^\pm_{\kappa_i}(t))$ and separating $v^\pm_i$ by means of this projection yields:

$$\begin{aligned}&\|D_{\omega_j} h^{\pm,s}_{\kappa_i}(t, (v^{\pm,s}_i(\boldsymbol{\omega}) + v^{\pm,u}_i(\boldsymbol{\omega}))(t))\| \\ &\leq \quad \|\tfrac{\partial}{\partial v^s} f^s(\gamma^\pm_{\kappa_i}(t) + v^{\pm,s}_i(\boldsymbol{\omega})(t) + v^{\pm,u}_i(\boldsymbol{\omega})(t)) - \tfrac{\partial}{\partial v^s} f^s(\gamma^\pm_{\kappa_i}(t))\|\|D_j v^{\pm,s}_i(\boldsymbol{\omega})(t)\| \\ &\quad + \|\tfrac{\partial}{\partial v^u} f^s(\gamma^\pm_{\kappa_i}(t) + v^{\pm,s}_i(\boldsymbol{\omega})(t) + v^{\pm,u}_i(\boldsymbol{\omega})(t)) - \tfrac{\partial}{\partial v^u} f^s(\gamma^\pm_{\kappa_i}(t))\|\|D_j v^{\pm,u}_i(\boldsymbol{\omega})(t)\|.\end{aligned}$$

From here we can proceed as in the proof of Lemma 3.4.7 subsequent to equation (3.81). □

In regard to the following lemma recall the decomposition of $v^{\pm,s}_i(t)$ into

$$v^{\pm,s}_i(t) = v^{\pm,ss}_i(t) + v^{\pm,su}_i(t)$$

where, cf. (3.85)

$$v^{\pm,ss}_i(t) \in \Phi^\pm_{\kappa_i}(t,0)W^\pm_{\kappa_i} \quad \text{and} \quad v^{\pm,su}_i(t) \in \Phi^\pm_{\kappa_i}(t,0)\ \mathrm{span}\{f(\gamma^\pm_{\kappa_i}(0))\}.$$

Lemma 3.5.13. *Assume Hypotheses (H3.1)-(H3.6). There exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ there exists a constant $M > 0$ such that $D_{\omega_j} h^{\pm,u}_{\kappa_i}$ satisfy for $t \in [0, \omega]$ or $t \in [-\omega, 0]$, respectively,*

$$\begin{aligned}\|D_{\omega_j} h^{\pm,u}_{\kappa_i}(t, v^\pm_i(\boldsymbol{\omega})(t))\| \quad &\leq \quad M\big((\|v^{\pm,u}_i(\boldsymbol{\omega})(t)\| + \|v^{\pm,su}_i(\boldsymbol{\omega})(t)\|)\|D_j v^{\pm,s}_i(\boldsymbol{\omega})(t)\| \\ &\quad + \|v^\pm_i(\boldsymbol{\omega})(t)\|\|D_j v^{\pm,u}_i(\boldsymbol{\omega})(t)\| + \|v^{\pm,ss}_i(\boldsymbol{\omega})(t)\|\|D_j v^{\pm,su}_i(\boldsymbol{\omega})(t)\|\big) \\ &\quad \cdot(\|\gamma^\pm_{\kappa_i}(t) - p_{\kappa_{i+1}/\kappa_i}\| + \|v^\pm_i(\boldsymbol{\omega})(t)\|)^{\nu-2}\end{aligned}$$

Moreover we find for $D_{\omega_j} h^{-,u}_{\kappa_i}(t, v^-_i(t), \lambda)$, $t \in [-\omega, 0]$

$$\begin{aligned}&\|D_{\omega_j}\left(h^{-,u}_{\kappa_i}(t, v^-_i(t)) - \tfrac{1}{2}D^2 f^u(\gamma^-_{\kappa_i}(t))[v^{-,u}_i(t), v^{-,u}_i(t)] - \tfrac{1}{6}D^3 f^u(\gamma^-_{\kappa_i}(t))[v^{-,u}_i(t), v^{-,u}_i(t), v^{-,u}_i(t)]\right)\| \\ &\leq \quad M\left(\left[(\|v^{-,u}_i(t)\| + \|v^{-,su}_i(t)\|)\|D_j v^{-,s}_i(\boldsymbol{\omega})(t)\|\right.\right. \\ &\quad \left. + (\|D_j v^{-,s}_i(\boldsymbol{\omega})(t)\| + \|D_j v^{-,su}_i(\boldsymbol{\omega})(t)\|)\|v^{-,s}_i(t)\|\right](\|v^-_i(t)\| + \|\gamma^-_{\kappa_i}(t) - p_{\kappa_i}\|)^{\nu-2} \\ &\quad \left. + \|v^{-,u}_i(t)\|^3\|D_j v^{-,u}_i(\boldsymbol{\omega})(t)\|(\|\gamma^-_{\kappa_i}(t) - p_{\kappa_i}\| + \|v^-_i(t)\|)^{\max\{0,\nu-4\}}\right).\end{aligned}$$

Proof. Analogously to the previous lemma we find with the additionally decomposition of $v^{\pm,s}_i$ into

$v_i^{\pm,ss} \in \Phi^{\pm}_{\kappa_i}(t,0)W^{\pm}_{\kappa_i}$ and $v_i^{\pm,su} \in \Phi^{\pm}_{\kappa_i}(t,0)\,\mathrm{span}\{f(\gamma^{\pm}_{\kappa_i}(0))$

$$\begin{aligned}
& D_{\omega_j} h^{\pm,u}_{\kappa_i}(t,(v_i^{\pm,s}(\boldsymbol{\omega})+v_i^{\pm,u}(\boldsymbol{\omega}))(t)) \\
&= \left[Df^u(\gamma^{\pm}_{\kappa_i}(t)+v_i^{\pm,s}(\boldsymbol{\omega})(t)+v_i^{\pm,u}(\boldsymbol{\omega})(t)) - Df^u(\gamma^{\pm}_{\kappa_i}(t))\right]\left[D_j v_i^{\pm,s}(\boldsymbol{\omega})(t)+D_j v_i^{\pm,u}(\boldsymbol{\omega})(t)\right] \\
&= \left[\frac{\partial}{\partial v^{ss}} f^u(\gamma^{\pm}_{\kappa_i}(t)+v_i^{\pm,ss}(\boldsymbol{\omega})(t)+v_i^{\pm,su}(\boldsymbol{\omega})(t)+v_i^{\pm,u}(\boldsymbol{\omega})(t)) - \frac{\partial}{\partial v^{ss}} f^u(\gamma^{\pm}_{\kappa_i}(t))\right] D_j v_i^{\pm,ss}(\boldsymbol{\omega})(t) \\
&\quad + \left[\frac{\partial}{\partial v^{su}} f^u(\gamma^{\pm}_{\kappa_i}(t)+v_i^{\pm,ss}(\boldsymbol{\omega})(t)+v_i^{\pm,su}(\boldsymbol{\omega})(t)+v_i^{\pm,u}(\boldsymbol{\omega})(t)) - \frac{\partial}{\partial v^{su}} f^u(\gamma^{\pm}_{\kappa_i}(t))\right] D_j v_i^{\pm,su}(\boldsymbol{\omega})(t) \\
&\quad + \left[\frac{\partial}{\partial v^{u}} f^u(\gamma^{\pm}_{\kappa_i}(t)+v_i^{\pm,s}(\boldsymbol{\omega})(t)+v_i^{\pm,u}(\boldsymbol{\omega})(t)) - \frac{\partial}{\partial v^{u}} f^u(\gamma^{\pm}_{\kappa_i}(t))\right] D_j v_i^{\pm,u}(\boldsymbol{\omega})(t) \qquad (3.142)
\end{aligned}$$

Proceeding along the lines of the proof of Lemma 3.4.8 from (3.86) to (3.88) yields

$$\begin{aligned}
D_{\omega_j} h^{\pm,u}_{\kappa_i}(t, v_i^{\pm}(\boldsymbol{\omega})(t)) \quad \leq \quad & M\big[(\|\gamma^{\pm}_{\kappa_i}(t)-p_{\kappa_{i+1}/\kappa_i}\| + \|v_i^{\pm}(\boldsymbol{\omega})(t)\|)^{\max\{0,\nu-3\}}\|v_i^{\pm,ss}(\boldsymbol{\omega})(t)\| \\
& \cdot\|D_j v_i^{\pm,ss}(\boldsymbol{\omega})(t)\|(\|v_i^{\pm,u}(\boldsymbol{\omega})(t)\| + \|v_i^{\pm,su}(\boldsymbol{\omega})(t)\|) \\
& +(\|\gamma^{\pm}_{\kappa_i}(t)-p_{\kappa_{i+1}/\kappa_i}\| + \|v_i^{\pm}(\boldsymbol{\omega})(t)\|)^{\nu-2}\big(\|v_i^{\pm,ss}(\boldsymbol{\omega})(t)\|\|D_j v_i^{\pm,su}(\boldsymbol{\omega})(t)\| \\
& +\|v_i^{\pm,su}(\boldsymbol{\omega})(t)\|\|D_j v_i^{\pm,s}(\boldsymbol{\omega})(t)\| + \|v_i^{\pm,u}(\boldsymbol{\omega})(t)\|\|D_j v_i^{\pm,s}(\boldsymbol{\omega})(t)\| \\
& +\|v_i^{\pm,s}(\boldsymbol{\omega})(t)\|\|D_j v_i^{\pm,u}(\boldsymbol{\omega})(t)\| + \|v_i^{\pm,u}(\boldsymbol{\omega})(t)\|\|D_j v_i^{\pm,u}(\boldsymbol{\omega})(t)\|\big)\big].
\end{aligned}$$

Using the relations $\|v_i^{\pm,ss}(t)\| \leq \|\gamma^{\pm}_{\kappa_i}(t) - p_{\kappa_{i+1}/\kappa_i}\| + \|v_i^{\pm}(t)\|$, $\|D_j v_i^{\pm,ss}(t)\| \leq \|D_j v_i^{\pm,s}(t)\|$ and $\|v_i^{\pm,s}(t)\|, \|v_i^{\pm,u}(t)\| \leq \|v_i^{\pm}(t)\|$ leads to the first statement of the lemma.

To prove the second estimate we have a closer look on the term (3.142) for the superscript " $-$ " and apply the mean-value theorem:

$$\begin{aligned}
& \tfrac{\partial}{\partial v^u} f^u(\gamma^{-}_{\kappa_i}(t) + (v_i^{-,s}(t)+v_i^{-,u}(t)))D_j v_i^{-,u}(\boldsymbol{\omega})(t) \\
&= \int_0^1 \tfrac{\partial^2}{\partial v^u \partial v^s} f^u(\gamma^{-}_{\kappa_i}(t) + \tau(v_i^{-,s}(t)+v_i^{-,u}(t)))d\tau[D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,s}(t)] \\
&\quad + \int_0^1 \tfrac{\partial^2}{\partial (v^u)^2} f^u(\gamma^{-}_{\kappa_i}(t) + \tau(v_i^{-,s}(t)+v_i^{-,u}(t)))d\tau[D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,u}(t)].
\end{aligned}$$

Analogous to the procedure in the proof of Lemma 3.4.8 subsequent to (3.88) we now add a zero-term to the second summand. Applying again the Mean-value-theorem yields

$$\begin{aligned}
&\int_0^1 \tfrac{\partial^2}{\partial(v^u)^2} f^u(\gamma_{\kappa_i}^-(t) + \tau_1(v_i^{-,s}(t) + v_i^{-,u}(t)))d\tau_1 [D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,u}(t)] \\
&\quad = \left[\int_0^1 \tfrac{\partial^2}{\partial(v^u)^2} f^u(\gamma_{\kappa_i}^-(t) + \tau_1(v_i^{-,s}(t) + v_i^{-,u}(t))) - \tfrac{\partial^2}{\partial(v^u)^2} f^u(\gamma_{\kappa_i}^-(t)) + \tfrac{\partial^2}{\partial(v^u)^2} f^u(\gamma_{\kappa_i}^-(t)) d\tau_1\right] \\
&\qquad [D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,u}(t)] \\
&\quad = \left[\int_0^1\int_0^1 \tfrac{\partial^3}{\partial(v^u)^2\partial v^s} f^u(\gamma_{\kappa_i}^-(t) + \tau_1\tau_2(v_i^{-,s}(t) + v_i^{-,u}(t)))\tau_1 d\tau_2 d\tau_1\right] [D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,u}(t), v_i^{-,s}(t)] \\
&\qquad + \left[\int_0^1\int_0^1 \tfrac{\partial^3}{\partial(v^u)^3} f^u(\gamma_{\kappa_i}^-(t) + \tau_1\tau_2(v_i^{-,s}(t) + v_i^{-,u}(t)))\tau_1 d\tau_2 d\tau_1\right] [D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,u}(t), v_i^{-,u}(t)] \\
&\qquad + D_1^2 f^u(\gamma_{\kappa_i}^-(t))[D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,u}(t)]
\end{aligned}$$

In exactly the same way we find for the middle summand of the latest equation

$$\begin{aligned}
&\int_0^1\int_0^1 \tfrac{\partial^3}{\partial(v^u)^3} f^u(\gamma_{\kappa_i}^-(t) + \tau_1\tau_2(v_i^{-,s}(t) + v_i^{-,u}(t)))\tau_1 d\tau_2 d\tau_1 [v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,u}(t)] \\
&\quad = \left[\int_0^1\int_0^1\int_0^1 \tfrac{\partial^4}{\partial(v^u)^3\partial v^s} f^u(\gamma_{\kappa_i}^-(t) + \tau_1\tau_2\tau_3(v_i^{-,s}(t) + v_i^{-,u}(t)))\tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1\right] \\
&\qquad [D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,s}(t)] \\
&\qquad + \left[\int_0^1\int_0^1\int_0^1 \tfrac{\partial^4}{\partial(v^u)^4} f^u(\gamma_{\kappa_i}^-(t) + \tau_1\tau_2\tau_3(v_i^{-,s}(t) + v_i^{-,u}(t)))\tau_1^2\tau_2 d\tau_3 d\tau_2 d\tau_1\right] \\
&\qquad [D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,u}(t)] \\
&\qquad + \tfrac{1}{2} D^3 f^u(\gamma_{\kappa_i}^-(t))[D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,u}(t), v_i^{-,u}(t)].
\end{aligned}$$

Collecting all terms and using

$$\begin{aligned}
D_{\omega_j}\left(\tfrac{1}{2} D_1^2 f^u(\gamma_{\kappa_i}^-(t))[v_i^{-,u}(t), v_i^{-,u}(t)]\right) &= D_1^2 f^u(\gamma_{\kappa_i}^-(t))[D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,u}(t)], \\
D_{\omega_j}\left(\tfrac{1}{6} D^3 f^u(\gamma_{\kappa_i}^-(t))[v_i^{-,u}(t), v_i^{-,u}(t), v_i^{-,u}(t)]\right) &= \tfrac{1}{2} D^3 f^u(\gamma_{\kappa_i}^-(t))[D_j v_i^{-,u}(\boldsymbol{\omega})(t), v_i^{-,u}(t), v_i^{-,u}(t)],
\end{aligned}$$

concludes the proof. □

It remains to consider the derivatives of $v_i^{\pm}(\boldsymbol{\omega})(t)$. At first we see to the estimates of their time-derivatives, which equal the estimates for $v_i^{\pm}(\boldsymbol{\omega})(t)$ themselves. Recall Definition 3.3.4 for the introduction of the superscripts s and u and (3.90) for the definition of α^w.

Lemma 3.5.14. *Assume Hypotheses (H3.1)-(H3.6). There exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ the following estimates apply with some $\bar{K} > 0$:*

$$\begin{aligned}
\|\tfrac{d}{dt} v_i^{+,s}(\boldsymbol{\omega})(t)\| &\leq \bar{K} e^{\alpha^s t}(e^{2\alpha^w(\omega_{i+1}-t)} e^{2\alpha^w\omega_{i+1}} + e^{\alpha^s\omega_i} e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}), && t \in [0, \omega_{i+1}], \\
\|\tfrac{d}{dt} v_i^{-,s}(\boldsymbol{\omega})(t)\| &\leq \bar{K} e^{-\alpha^w t}(e^{2\alpha^s(\omega_i+t)} e^{2\alpha^s\omega_i} + e^{2\alpha^s\omega_i} e^{\alpha^w\omega_{i+1}} + e^{2\alpha^w\omega_{i+1}}), && t \in [-\omega_i, 0], \\
\|\tfrac{d}{dt} v_i^{+,u}(\boldsymbol{\omega})(t)\| &\leq \bar{K}(e^{\alpha^w(\omega_{i+1}-t)} e^{\alpha^w\omega_{i+1}} + e^{4\alpha^s\omega_i}), && t \in [0, \omega_{i+1}], \\
\|\tfrac{d}{dt} v_i^{-,u}(\boldsymbol{\omega})(t)\| &\leq \bar{K}(e^{\alpha^s(\omega_i+t)} e^{\alpha^s\omega_i} + e^{4\alpha^w\omega_{i+1}}), && t \in [-\omega_i, 0].
\end{aligned}$$

Proof. Since $v_i^{\pm}(\boldsymbol{\omega})(t)$ satisfy the differential equation (3.12) and the projections satisfy the matrix differential equation

$$\dot{P}^{\pm}_{\kappa_i}(t) = Df(\gamma^{\pm}_{\kappa_i}(t))P^{\pm}_{\kappa_i}(t) - P^{\pm}_{\kappa_i}(t)Df(\gamma^{\pm}_{\kappa_i}(t)),$$

cf. (3.139) or (2.51), we find with $v_i^{\pm,s}(t) = (id - P^{\pm}_{\kappa_i}(t))v_i^{\pm}(t)$ and $v_i^{\pm,u}(t) = P^{\pm}_{\kappa_i}(t)v_i^{\pm}(t)$, cf. Definition 3.3.4, due to the product rule

$$\begin{aligned} \|\dot{v}_i^{\pm,s}(\boldsymbol{\omega})(t)\| &\leq \|Df(\gamma^{\pm}_{\kappa_i}(t))\|\|v_i^{\pm,s}(t)\| + \|h^{\pm,s}_{\kappa_i}(t, v_i^{\pm}(t))\|, \\ \|\dot{v}_i^{\pm,u}(\boldsymbol{\omega})(t)\| &\leq \|Df(\gamma^{\pm}_{\kappa_i}(t))\|\|v_i^{\pm,u}(t)\| + \|h^{\pm,u}_{\kappa_i}(t, v_i^{\pm}(t))\|. \end{aligned}$$

The terms $Df(\gamma^{\pm}_{\kappa_i}(t))$ are bounded for all $t \in (-\infty, 0]$ or $t \in [0, \infty)$, respectively. The estimates for $h^{\pm,s}$ and $h^{\pm,u}$ can be found in the Lemmata 3.4.7 and 3.4.8, respectively. Finally, Lemma 3.4.9 provides the estimates of the terms $v_i^{\pm,s}(t)$ and $v_i^{\pm,u}(t)$. Altogether this results in the lemma. □

We continue with the partial derivatives of $v_i^{\pm}(\boldsymbol{\omega})(t)$ with respect to ω_j.

Lemma 3.5.15. *Assume Hypotheses (H3.1)-(H3.6). There exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ the following estimates apply with some $\bar{K} > 0$:*

$$\begin{aligned} \|D_j v_i^{+,s}(\boldsymbol{\omega})(t)\| &\leq \bar{K}e^{\alpha^s t}(e^{2\alpha^w(\omega_{i+1}-t)}e^{2\alpha^w\omega_{i+1}} + e^{\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}), \quad t \in [0, \omega_{i+1}], \\ \|D_j v_i^{-,s}(\boldsymbol{\omega})(t)\| &\leq \bar{K}e^{-\alpha^w t}(e^{2\alpha^s(\omega_i+t)}e^{2\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^w\omega_{i+1}}), \quad t \in [-\omega_i, 0], \\ \|D_j v_i^{+,u}(\boldsymbol{\omega})(t)\| &\leq \bar{K}(e^{\alpha^w(\omega_{i+1}-t)}e^{\alpha^w\omega_{i+1}} + e^{4\alpha^s\omega_i}), \quad t \in [0, \omega_{i+1}], \\ \|D_j v_i^{-,u}(\boldsymbol{\omega})(t)\| &\leq \bar{K}(e^{\alpha^s(\omega_i+t)}e^{\alpha^s\omega_i} + e^{4\alpha^w\omega_{i+1}}), \quad t \in [-\omega_i, 0]. \end{aligned}$$

Proof. To estimate the derivatives of $v_i^{\pm}(\boldsymbol{\omega})(t)$ we use the fact that they solve

$$\left(D_j v_i^{\pm}(\boldsymbol{\omega})\right)^{\dot{}}(t) = Df(\gamma^{\pm}_{\kappa_i}(t))D_j v_i^{\pm}(\boldsymbol{\omega})(t) + D_{\omega_j}h^{\pm}_{\kappa_i}(t, v_i^{\pm}(\boldsymbol{\omega})(t)). \tag{3.143}$$

Due to the same structure of (3.143) and (3.12) we can follow the procedure in the proof of Lemma 3.4.9. Thereby we make use of the Lemmata 3.5.12 and 3.5.13. Since the proofs differ slightly in some substeps, we go into more detail below. In particular we want to point out, that there is no need to estimate $D_j v_i$ in advance, as we have done in the proof of Lemma 3.4.9 with the term v_i itself.

From (3.143) follows with the variation of constants formula that

$$D_j v_i^{+,s}(\boldsymbol{\omega})(t) = \Phi^{+}_{\kappa_i}(t,0)D_j v_i^{+,s}(\boldsymbol{\omega})(0) + \int_0^t \Phi^{+}_{\kappa_i}(t,s)D_{\omega_j}h_i^{+,s}(s, v_i^{+}(\boldsymbol{\omega})(s))ds.$$

Due to (3.39) we find $D_j v_i^{\pm}(\boldsymbol{\omega})(0) = P^{\pm}_{\kappa_i}(0)D_j v_i^{\pm}(\boldsymbol{\omega})(0) + (id - F_{\kappa_i})P^{\mp}_{\kappa_i}(0)D_j v_i^{\mp}(\boldsymbol{\omega})(0)$ and hence

$$D_j v_i^{+,s}(\boldsymbol{\omega})(0) = (id - P^{+}_{\kappa_i}(0))(id - F_{\kappa_i})D_j v_i^{-,u}(\boldsymbol{\omega})(0).$$

Then we find by exploiting exponential dichotomy and invoking Lemma 3.5.12

$$\begin{aligned}\|D_j v_i^{+,s}(\boldsymbol{\omega})(t)\| \leq\ & KLe^{(\alpha^s-\delta)t}\|D_j v_i^{-,u}(\boldsymbol{\omega})(0)\|\\ &+KM\int\limits_0^t e^{(\alpha^s-\delta)(t-s)}\big((e^{\alpha^s s}\|v_i^{+,u}(s)\|+\|v_i^{+,s}(s)\|)\|D_j v_i^{+,u}(\boldsymbol{\omega})(s)\|\\ &\qquad+\|v_i^{+}(s)\|\|D_j v_i^{+,s}(\boldsymbol{\omega})(s)\|\big)ds\\ \leq\ & e^{\alpha^s t}\Bigg(KL\|D_j v_i^{-,u}(\boldsymbol{\omega})(0)\|+KM\big((\|v_i^{+,u}\|+\sup_{s\in[0,t]}e^{-\alpha^s s}\|v_i^{+,s}(s)\|)\|D_j v_i^{+,u}(\boldsymbol{\omega})(s)\|\\ &+\|v_i^{+}\|\sup_{s\in[0,t]}(e^{-\alpha^s s}\|D_j v_i^{+,s}(\boldsymbol{\omega})(s)\|)\big)\int\limits_0^t e^{-\delta(t-s)}ds\Bigg).\end{aligned}$$

Proceeding as in the proof of Lemma 3.4.9 we find by multiplying with $e^{-\alpha^s t}$ and applying the supreme norm

$$\begin{aligned}\sup_{t\in[0,\omega_{i+1}]}(e^{-\alpha^s t}\|D_j v_i^{+,s}(\boldsymbol{\omega})(t)\|) \leq\ & KL\|D_j v_i^{-,u}(\boldsymbol{\omega})(0)\|+KM\big(\|v_i^{+}\|\cdot\sup_{s\in[0,\omega_{i+1}]}(e^{-\alpha^s s}\|D_j v_i^{+,s}(\boldsymbol{\omega})\|)\\ &+(\|v_i^{+,u}\|+\sup_{s\in[0,\omega_{i+1}]}(e^{-\alpha^s s}\|v_i^{+,s}(s)\|))\|D_j v_i^{+,u}(\boldsymbol{\omega})\|\big).\end{aligned}$$

With $\|v_i^{+}\|$ tending to zero for $\Omega < \inf\boldsymbol{\omega}$ tending to infinity we find for $\inf\boldsymbol{\omega}$ large enough that $1-KM\|v_i^{+}\| > 1/2$. So we obtain with some constant $\tilde{K}_1 > 0$

$$\begin{aligned}\sup_{t\in[0,\omega_{i+1}]}(e^{-\alpha^s t}\|D_j v_i^{+,s}(\boldsymbol{\omega})(t)\|) \leq\ & \tilde{K}_1\big(\|D_j v_i^{-,u}(\boldsymbol{\omega})(0)\|\\ &+\|D_j v_i^{+,u}(\boldsymbol{\omega})\|\big(\|v_i^{+,u}\|+\sup_{s\in[0,\omega_{i+1}]}(e^{-\alpha^s s}\|v_i^{+,s}(s)\|)\big)\big).\end{aligned}$$

An analogous statement hold for $\|D_j v_i^{-,s}\|$ and by invoking (3.92) and (3.93) we find analogously to (3.94)

$$\left.\begin{aligned}\|D_j v_i^{+,s}(\boldsymbol{\omega})(t)\| &\leq \tilde{K}_1 e^{\alpha^s t}(\|D_j v_i^{-,u}(\boldsymbol{\omega})(0)\|+\|D_j v_i^{+,u}(\boldsymbol{\omega})\|(\|v_i^{+,u}\|+\|v_i^{-,u}(0)\|)),\\ \|D_j v_i^{-,s}(\boldsymbol{\omega})(t)\| &\leq \tilde{K}_1 e^{-\alpha^w t}(\|D_j v_i^{+,u}(\boldsymbol{\omega})(0)\|+\|D_j v_i^{-,u}(\boldsymbol{\omega})\|(\|v_i^{-,u}\|+\|v_i^{+,u}(0)\|)).\end{aligned}\right\}\tag{3.144}$$

Note that, just as in the proof of Lemma 3.4.9, we have used the convergence of the terms $v_i^{\pm}$, which we had to provide in advance for the proof of Lemma 3.4.9, but not the convergence of their derivatives with respect to ω_j.

Let us now consider $D_j v_i^{\pm,u}(\boldsymbol{\omega})$. With the variation of constants formula we obtain

$$\left.\begin{aligned}D_j v_i^{+,u}(\boldsymbol{\omega})(t) &= \Phi_{\kappa_i}^{+}(t,\omega_{i+1})D_j v_i^{+,u}(\boldsymbol{\omega})(\omega_{i+1})+\int\limits_t^{\omega_{i+1}}\Phi_{\kappa_i}^{+}(t,s)D_{\omega_j}h_i^{+,u}(s,v_i^{+}(\boldsymbol{\omega})(s))ds.\\ D_j v_i^{-,u}(\boldsymbol{\omega})(t) &= \Phi_{\kappa_i}^{-}(t,-\omega_i)D_j v_i^{-,u}(\boldsymbol{\omega})(-\omega_i)-\int\limits_{\omega_i}^{t}\Phi_{\kappa_i}^{-}(t,s)D_{\omega_j}h_i^{-,u}(s,v_i^{-}(\boldsymbol{\omega})(s))ds.\end{aligned}\right\}\tag{3.145}$$

With $v_i^{+,u}(\boldsymbol{\omega})(\omega_{i+1}) = a_{i+1}^{+}$, cf. (3.34) we find $D_j v_i^{+,u}(\boldsymbol{\omega})(\omega_{i+1}) = D_{\omega_j}a_{i+1}^{+}$ for $j \neq i+1$. In case of a differentiation with respect to ω_{i+1} we find

$$D_{\omega_{i+1}}a_{i+1}^{+} = D_{\omega_{i+1}}v_i^{+,u}(\boldsymbol{\omega})(\omega_{i+1}) = D_{i+1}v_i^{+,u}(\boldsymbol{\omega})(\omega_{i+1})+\dot{v}_i^{+,u}(\boldsymbol{\omega})(\omega_{i+1}).$$

Therefore in (3.145) we replace

$$D_j v_i^{+,u}(\boldsymbol{\omega})(\omega_{i+1}) = \begin{cases} D_{\omega_{i+1}} a_{i+1}^{+} - \dot{v}_i^{+,u}(\boldsymbol{\omega})(\omega_{i+1}), & \text{if } j = i+1, \\ D_{\omega_j} a_{i+1}^{+}, & \text{else,} \end{cases}$$

and

$$D_j v_i^{-,u}(\boldsymbol{\omega})(-\omega_i) = \begin{cases} D_{\omega_i} a_i^{-} - \dot{v}_i^{-,u}(\boldsymbol{\omega})(-\omega_i), & \text{if } j = i, \\ D_{\omega_j} a_i^{-}, & \text{else.} \end{cases}$$

Looking at the definition of a^+ and a^- given in (3.48), calling in the definition of $\boldsymbol{d} = (d_i)_{i\in\mathbb{Z}}$ in (3.13) and differentiating with respect to ω_j then yields exemplarily for $D_{\omega_j} a_i^+$

$$\begin{aligned} \|D_{\omega_j} a_i^+\| \quad \leq \quad & \|D_{\omega_j} \tilde{P}_{\kappa_i}(\omega_i)\| \left(\|\gamma_{\kappa_i}^-(-\omega_i) - p_{\kappa_i}\| + \|v_i^{-,s}(\boldsymbol{\omega})(-\omega_i)\|\right) \\ & + \|\tilde{P}_{\kappa_i}(\omega_i)(id - P_{\kappa_i}^-(-\omega_i))\| \left(\|D_{\omega_j}\gamma_{\kappa_i}^-(-\omega_i)\| + \|D_j v_i^{-,s}(\boldsymbol{\omega})(-\omega_i)\| + \|\tfrac{d}{dt} v_i^{-,s}(\boldsymbol{\omega})(-\omega_i)\|\right) \\ & + \|D_{\omega_j}\left(\tilde{P}_{\kappa_i}(\omega_i)(id - P_{\kappa_{i-1}}^+(\omega_i))\right)\| \left(\|\gamma_{\kappa_{i-1}}^+(\omega_i) - p_{\kappa_i}\| + \|v_{i-1}^{+,s}(\boldsymbol{\omega})(\omega_i)\|\right) \\ & + \|\tilde{P}_{\kappa_i}(\omega_i)(id - P_{\kappa_{i-1}}^+(\omega_i))\| \left(\|D_{\omega_j}\gamma_{\kappa_{i-1}}^+(\omega_i)\| + \|D_j v_{i-1}^{+,s}(\boldsymbol{\omega})(\omega_i)\| + \|\tfrac{d}{dt} v_{i-1}^{+,s}(\boldsymbol{\omega})(\omega_i)\|\right) \end{aligned}$$

By invoking the estimates of $\gamma_{\kappa_i}^{\pm}$ and their derivatives (Lemmata 3.4.6 and 3.5.10), the estimates of the projections and their derivatives (Lemma 3.4.3 and Remark 3.5.9 or in case that (H3.7) applies Lemmata 3.4.5 and 3.5.8) as well as the estimates (3.94), (3.144) and Lemma 3.5.14 of $v_i^{\pm,s}(t)$, $D_j v_i^{\pm,s}(\boldsymbol{\omega})(t)$ and $\frac{d}{dt} v_i^{\pm,s}(\boldsymbol{\omega})(t)$ we find analogously to (3.96)

$$\left.\begin{aligned} \|D_{\omega_j} a_i^+\| \quad &\leq \quad C e^{\alpha^w \omega_i}, \\ \|D_{\omega_j} a_i^-\| \quad &\leq \quad C e^{\alpha^s \omega_i}. \end{aligned}\right\} \tag{3.146}$$

Exploiting this estimate together with Lemma 3.5.14, Lemma 3.5.13 and exponential dichotomies we then find from (3.145)

$$\left.\begin{aligned} \|D_j v_i^{+,u}(\boldsymbol{\omega})(t)\| \quad \leq \quad & KC e^{\alpha^w (2\omega_{i+1} - t)} \\ & + KM(\|v_i^{+,s}\| + \|v_i^{+,u}\|)(\|D_j v_i^{+,s}(\boldsymbol{\omega})\| + \|D_j v_i^{+,u}(\boldsymbol{\omega})\|); \\ \|D_j v_i^{-,u}(\boldsymbol{\omega})(t)\| \quad \leq \quad & KC e^{\alpha^s (2\omega_i + t)} \\ & + KM(\|v_i^{-,s}\| + \|v_i^{-,u}\|)(\|D_j v_i^{-,s}(\boldsymbol{\omega})\| + \|D_j v_i^{-,u}(\boldsymbol{\omega})\|); \end{aligned}\right\} \tag{3.147}$$

which results analogously to (3.98) in

$$\left.\begin{aligned} \|D_j v_i^{+,u}(\boldsymbol{\omega})\| \quad &\leq \quad \tilde{K}_2(e^{\alpha^w \omega_{i+1}} + \|v_i^+\| \|D_j v_i^{+,s}(\boldsymbol{\omega})\|); \\ \|D_j v_i^{-,u}(\boldsymbol{\omega})\| \quad &\leq \quad \tilde{K}_2(e^{\alpha^s \omega_i} + \|v_i^-\| \|D_j v_i^{-,s}(\boldsymbol{\omega})\|). \end{aligned}\right\} \tag{3.148}$$

Here again only the convergence of the terms $\|v_i^{\pm}\|$ is needed, but not the convergence of $\|D_j v_i^{\pm}(\boldsymbol{\omega})\|$.

Equations (3.148), (3.147) for $t = 0$ and the estimates of $\|v_i^{\pm,u}\|, \|v_i^{\pm,u}(0)\|$ given in Lemma 3.4.9 in combination with (3.144) then yield

$$\left.\begin{array}{rcl} \|D_j v_i^{+,s}(\boldsymbol{\omega})\| & \leq & \tilde{K}_3(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i} + \|v_i^-\|(\|D_j v_i^{-,s}(\boldsymbol{\omega})\| + \|D_j v_i^{-,u}(\boldsymbol{\omega})\|)); \\ \|D_j v_i^{-,s}(\boldsymbol{\omega})\| & \leq & \tilde{K}_3(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i} + \|v_i^+\|(\|D_j v_i^{+,s}(\boldsymbol{\omega})\| + \|D_j v_i^{+,u}(\boldsymbol{\omega})\|)). \end{array}\right\} \quad (3.149)$$

Successively plugging the estimates of (3.148) and (3.149) into each other yields analogously to (3.101)

$$\left.\begin{array}{rcl} \|D_j v_i^{+,s}(\boldsymbol{\omega})\| & \leq & \tilde{K}_4(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}); \\ \|D_j v_i^{-,s}(\boldsymbol{\omega})\| & \leq & \tilde{K}_4(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}); \\ \|D_j v_i^{+,u}(\boldsymbol{\omega})\| & \leq & \tilde{K}_4(e^{\alpha^w\omega_{i+1}} + e^{4\alpha^s\omega_i}); \\ \|D_j v_i^{-,u}(\boldsymbol{\omega})\| & \leq & \tilde{K}_4(e^{4\alpha^w\omega_{i+1}} + e^{\alpha^s\omega_i}). \end{array}\right\} \quad (3.150)$$

Thereby we again apply that $v_i^\pm$ tends to zero if $\inf \boldsymbol{\omega}$ tends to infinity.

Plugging these estimates into (3.147) and (3.144) gives

$$\left.\begin{array}{rcl} \|D_j v_i^{+,s}(\boldsymbol{\omega})(t)\| & \leq & \tilde{K}_5 e^{\alpha^s t}(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}); \\ \|D_j v_i^{-,s}(\boldsymbol{\omega})(t)\| & \leq & \tilde{K}_5 e^{-\alpha^w t}(e^{2\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}); \\ \|D_j v_i^{+,u}(\boldsymbol{\omega})(t)\| & \leq & \tilde{K}_5(e^{\alpha^w(\omega_{i+1}-t)} e^{\alpha^w\omega_{i+1}} + e^{\alpha^w\omega_{i+1}} e^{2\alpha^s\omega_i} + e^{4\alpha^s\omega_i}); \\ \|D_j v_i^{-,u}(\boldsymbol{\omega})(t)\| & \leq & \tilde{K}_5(e^{\alpha^s(\omega_i+t)} e^{\alpha^s\omega_i} + e^{\alpha^s\omega_i} e^{2\alpha^w\omega_{i+1}} + e^{4\alpha^w\omega_{i+1}}). \end{array}\right\} \quad (3.151)$$

These estimates for the derivatives agree with those from (3.102) and (3.103). The refinement of the estimate for $D_j v_i^{\pm,s}$ is now completely analogous to that of $v_i^{\pm,s}$ in the proof of Lemma 3.4.9, since the estimates of the terms $h_{\kappa_i}^{\pm,s}$ and $D_{\omega_j} h_{\kappa_i}^{\pm,s}$ are identical.

To refine the estimate of $D_j v_i^{\pm,u}$ we still need the estimate of the term $D_j v_i^{\pm,su}$. To this end recall that $v^{\pm,su} \in \Phi^\pm(t,0)\,\mathrm{span}\{f(\gamma^\pm(0))\}$. Then $v_i^{\pm,su}(\boldsymbol{\omega})(0) = 0$. Hence also $D_j v_i^{\pm,su}(\boldsymbol{\omega})(0) = 0$. Therefore we can estimate $D_j v_i^{\pm,su}(\boldsymbol{\omega})(t)$ along the same lines as $v_i^{\pm,su}(\boldsymbol{\omega})(t)$ and obtain with

$$\left.\begin{array}{rcl} \|D_j v_i^{+,su}(\boldsymbol{\omega})(t)\| & \leq & \tilde{K}_6\; e^{\alpha^s t}\left(e^{\alpha^w(\omega_{i+1}-t)} e^{\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}\right)^2, \\ \|D_j v_i^{-,su}(\boldsymbol{\omega})(t)\| & \leq & \tilde{K}_6\; e^{-\alpha^w t}\left(e^{\alpha^s(\omega_i+t)} e^{\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}\right)^2. \end{array}\right\} \quad (3.152)$$

the same estimate as for $v^{\pm,su}$ in (3.106). Proceeding as in the proof of Lemma 3.4.9 then yields the final estimates. □

Corollary 3.5.16. *There exist constants Ω and c in accordance to Theorem 3.2.2 such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ the following estimates apply:*

$$\begin{array}{rcl} \|D_{\omega_{i+1}}\big(v_i^{+,s}(\boldsymbol{\omega})(\omega_{i+1})\big)\| & = & O\big(e^{\alpha^s\omega_{i+1}}(e^{2\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i})\big) \\ \|D_{\omega_i}\big(v_i^{-,s}(\boldsymbol{\omega})(-\omega_i)\big)\| & = & O\big(e^{\alpha^w\omega_i}(e^{2\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}})\big) \end{array}$$

Proof. We prove this exemplarily for $v_i^{+,s}$. The chain rule yields

$$D_{\omega_{i+1}}\big(v_i^{+,s}(\boldsymbol{\omega})(\omega_{i+1})\big) = D_{i+1} v_i^{+,s}(\boldsymbol{\omega})(\omega_{i+1}) + \frac{d}{dt} v_i^{+,s}(\boldsymbol{\omega})(t)|_{t=\omega_{i+1}}.$$

Then the statement follows with Lemmata 3.5.15 and 3.5.14. □

3.5.3 Summarising the estimates of $D_{\omega_j}\boldsymbol{T^1_{\kappa_i}}$ and $D_{\omega_j}\boldsymbol{T^2_{\kappa_i}}$

Recall the representation (3.65), where we have introduced the partitioning

$$\langle\psi_{\kappa_i},\xi_i(\boldsymbol{\omega},\lambda,\kappa)\rangle = \boldsymbol{T^1_{\kappa_i}}+\boldsymbol{T^2_{\kappa_i}}.$$

First we consider the scalar product $\boldsymbol{T^{11}_{\kappa_i}}+\boldsymbol{T^{12}_{\kappa_i}}+\boldsymbol{T^{13}_{\kappa_i}}+\boldsymbol{T^{14}_{\kappa_i}}$ which we differentiate by means of the product rule. The first factor in the scalar product is $\Phi^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T\psi_{\kappa_i}$. Due to Lemma 3.5.11 and estimate (3.112) both $\Phi^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T\psi_{\kappa_i}$ and its derivatives with respect to ω_j are of order $O(e^{-\alpha^u\omega_{i+1}})$.

The second factor in the scalar product reads

$$\tilde P_{\kappa_{i+1}}(\omega_{i+1})\big[(\gamma^-_{\kappa_{i+1}}(-\omega_{i+1})-p_{\kappa_{i+1}})-(\gamma^+_{\kappa_i}(\omega_{i+1})-p_{\kappa_{i+1}})+(id-P^-_{\kappa_{i+1}}(-\omega_{i+1}))v^-_{i+1}(\boldsymbol{\omega},\kappa)(-\omega_{i+1})$$
$$-(id-P^+_{\kappa_i}(\omega_{i+1}))v^+_i(\boldsymbol{\omega},\kappa)(\omega_{i+1})\big].$$

Due to Lemma 3.5.10 and (3.79) we know that $\gamma^\pm$ and their derivatives are of the same order. Hence we find

$$\left.\begin{aligned} D_{\omega_j}\boldsymbol{T^{11}_{\kappa_i}} &= D_{\omega_j}\left\langle\Phi^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T\psi_{\kappa_i},\tilde P_{\kappa_{i+1}}(\omega_{i+1})(\gamma^-_{\kappa_{i+1}}(-\omega_{i+1})-p_{\kappa_{i+1}})\right\rangle \\ &= O(e^{-2\alpha^u\omega_{i+1}}),\end{aligned}\right\}\tag{3.153}$$

and by additionally invoking Lemmata 3.4.5 and 3.5.8 (in case that (H3.7) applies), or Lemma 3.4.3 and Remark 3.5.9, respectively,

$$\left.\begin{aligned} D_{\omega_j}\boldsymbol{T^{12}_{\kappa_i}} &= D_{\omega_j}\left\langle\Phi^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T\psi_{\kappa_i},-\tilde P_{\kappa_{i+1}}(\omega_{i+1})(\gamma^+_{\kappa_i}(\omega_{i+1})-p_{\kappa_{i+1}})\right\rangle \\ &= \begin{cases} O\left(e^{\max\{2(\alpha^s-\alpha^u),\alpha^s-\nu\alpha^u\}\omega_{i+1}}\right), & \text{if (H3.7) applies,}\\ O\left(e^{3/2(\alpha^s-\alpha^u)\omega_{i+1}}\right), & \text{else.}\end{cases}\end{aligned}\right\}\tag{3.154}$$

The estimates for $D_{\omega_j}\boldsymbol{T^{13}_{\kappa_i}}$ and $D_{\omega_j}\boldsymbol{T^{14}_{\kappa_i}}$ follow from Lemma 3.4.9 and Corollary 3.5.16 and in case of $D_{\omega_j}\boldsymbol{T^{14}_{\kappa_i}}$ additionally from Lemmata 3.4.5 and 3.5.8, or Lemma 3.4.3 and Remark 3.5.9, respectively:

$$\left.\begin{aligned} D_{\omega_j}\boldsymbol{T^{13}_{\kappa_i}} &= D_{\omega_j}\left\langle\Phi^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T\psi_{\kappa_i},\tilde P_{\kappa_{i+1}}(\omega_{i+1})v^{-,s}_{i+1}(\boldsymbol{\omega},\kappa)(-\omega_{i+1})\right\rangle \\ &= \begin{cases} O\left(e^{-2\alpha^u\omega_{i+1}}[e^{2\alpha^s\omega_{i+1}}+e^{-2\alpha^u\omega_{i+2}}]\right), & \text{if (H3.7) applies,}\\ O\left(e^{1/2(\alpha^s-3\alpha^u)\omega_{i+1}}[e^{2\alpha^s\omega_{i+1}}+e^{(\alpha^s-\alpha^u)\omega_{i+2}}]\right), & \text{else,}\end{cases}\\ D_{\omega_j}\boldsymbol{T^{14}_{\kappa_i}} &= D_{\omega_j}\left\langle\Phi^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T\psi_{\kappa_i},-\tilde P_{\kappa_{i+1}}(\omega_{i+1})v^{+,s}_{i}(\boldsymbol{\omega},\kappa)(\omega_{i+1})\right\rangle \\ &= \begin{cases} O\left(e^{\max\{2(\alpha^s-\alpha^u),\alpha^s-\nu\alpha^u\}\omega_{i+1}}[e^{2\alpha^s\omega_i}+e^{-2\alpha^u\omega_{i+1}}]\right), & \text{if (H3.7) applies,}\\ O\left(e^{3/2(\alpha^s-\alpha^u)\omega_{i+1}}[e^{2\alpha^s\omega_i}+e^{(\alpha^s-\alpha^u)\omega_{i+1}}]\right), & \text{else.}\end{cases}\end{aligned}\right\}\tag{3.155}$$

Now we investigate the integral term $\boldsymbol{T^{15}_{\kappa_i}} = -\left\langle\psi_{\kappa_i}\,,\int\limits_0^{\omega_{i+1}}\Phi^+_{\kappa_i}(0,s)P^+_{\kappa_i}(s)h^+_{\kappa_i}(s,v^+_i(\boldsymbol{\omega})(s))ds\right\rangle$. For $j\neq$

$i+1$ we have

$$\|D_{\omega_j}\boldsymbol{T^{15}_{\kappa_i}}\| = C\int_0^{\omega_{i+1}} \|\Phi^+_{\kappa_i}(0,s)P^+_{\kappa_i}(s)\|\|D_{\omega_j}h^{+,u}_{\kappa_i}(s,v^+_i(\boldsymbol{\omega})(s))\|ds \tag{3.156}$$

Since $D_j v_i^{\pm,s/u}(\boldsymbol{\omega})(t)$ satisfy the same estimates as $v_i^{\pm,s/u}(\boldsymbol{\omega})(t)$, cf. Lemmata 3.4.9 and 3.5.15, we find due to Lemmata 3.4.8 and 3.5.13 that $D_{\omega_j}h^{+,u}_{\kappa_i}$ satifies the same estimates as $h^{+,u}_{\kappa_i}$. Hence we can estimate (3.156) in exactly the same way as $\int_0^{\omega_{i+1}} \|\Phi^+_{\kappa_i}(0,s)P^+_{\kappa_i}(s)\|\|h^{+,u}_{\kappa_i}(s,v^+_i(\boldsymbol{\omega})(s))\|ds$ in Lemma 3.4.11. This results in

$$D_{\omega_j}\boldsymbol{T^{15}_{\kappa_i}} = \begin{cases} O\left(e^{6\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{-2\alpha^u\omega_{i+1}} \right. \\ \quad \left. + e^{\max\{(\nu-2)\alpha^s-3\alpha^u,-4\alpha^u\}\omega_{i+1}}\right), & \text{if (H3.7) applies,} \\ O\left(e^{6\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{(\alpha^s-\alpha^u)\omega_{i+1}} \right. \\ \quad \left. + e^{\max\{(\nu-\frac{1}{2})\alpha^s-\frac{3}{2}\alpha^u,2(\alpha^s-\alpha^u)\}\omega_{i+1}}\right), & \text{else.} \end{cases} \tag{3.157}$$

$D_{\omega_{i+1}}\boldsymbol{T^{15}_{\kappa_i}}$ additionally comprises the term $\Phi^+_{\kappa_i}(0,\omega_{i+1})P^+_{\kappa_i}(\omega_{i+1})h^{+,u}_{\kappa_i}(\omega_{i+1},v^+_i(\boldsymbol{\omega})(\omega_{i+1}))$. But also this term is of the above order, due to Lemmata 3.4.8, 3.4.9 and the estimate from the exponential dichotomy (3.16) with $\beta^+_{\kappa_i}=\alpha^u$.

Note that all estimates of $D_{\omega_j}\boldsymbol{T^{1k}_{\kappa_i}}$ equal the corresending estimates (3.113)-(3.117) for the terms $\boldsymbol{T^{1k}_{\kappa_i}}$, $k=1,\dots,5$. Summarizing we find

$$D_{\omega_j}\boldsymbol{T^{1}_{\kappa_i}} = \begin{cases} O\left(e^{6\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}\right), & \text{if (H3.7) applies,} \\ O\big(e^{6\alpha^s\omega_i} + e^{2\alpha^s\omega_i}e^{(\alpha^s-\alpha^u)\omega_{i+1}} + e^{\max\{3/2(\alpha^s-\alpha^u),-2\alpha^u\}\omega_{i+1}} \\ \quad + e^{1/2(\alpha^s-3\alpha^u)\omega_{i+1}}e^{(\alpha^s-\alpha^u)\omega_{i+2}}\big), & \text{else.} \end{cases} \tag{3.158}$$

Analogously to the consideration for $D_{\omega_j}\boldsymbol{T^{1}_{\kappa_i}}$ we find for $D_{\omega_j}\boldsymbol{T^{2k}_{\kappa_i}}$, $k=1,\dots,5$ the same estimates as listed in (3.119)-(3.123) for $\boldsymbol{T^{2k}_{\kappa_i}}$. That is in case that Hypothesis (H3.7) applies

$$\left.\begin{aligned} D_{\omega_j}\boldsymbol{T^{21}_{\kappa_i}} &= -D_{\omega_j}\left\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}\ ,\ (id-\tilde{P}_{\kappa_i}(\omega_i))\big(\gamma^+_{\kappa_{i-1}}(\omega_i)-p_{\kappa_i}\big)\right\rangle \\ &= O\big(e^{2\alpha^s\omega_i}\big), \\ D_{\omega_j}\boldsymbol{T^{22}_{\kappa_i}} &= D_{\omega_j}\left\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}\ ,\ (id-\tilde{P}_{\kappa_i}(\omega_i))\big(\gamma^-_{\kappa_i}(-\omega_i)-p_{\kappa_i}\big)\right\rangle \\ &= O\big(e^{\max\{\nu\alpha^s-\alpha^u,2(\alpha^s-\alpha^u)\}\omega_i}\big), \\ D_{\omega_j}\boldsymbol{T^{23}_{\kappa_i}} &= -D_{\omega_j}\left\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}\ ,\ (id-\tilde{P}_{\kappa_i}(\omega_i))(id-P^+_{\kappa_{i-1}}(\omega_i))v^+_{i-1}(\omega_i)\right\rangle \\ &= O\big(e^{2\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}}+e^{-2\alpha^u\omega_i}]\big), \\ D_{\omega_j}\boldsymbol{T^{24}_{\kappa_i}} &= D_{\omega_j}\left\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}\ ,\ (id-\tilde{P}_{\kappa_i}(\omega_i))(id-P^-_{\kappa_i}(-\omega_i))v^-_i(-\omega_i)\right\rangle \\ &= O\big(e^{\max\{\nu\alpha^s-\alpha^u,2(\alpha^s-\alpha^u)\}\omega_i}[e^{2\alpha^s\omega_i}+e^{-2\alpha^u\omega_{i+1}}\big), \\ D_{\omega_j}\boldsymbol{T^{25}_{\kappa_i}} &= -D_{\omega_j}\left\langle \psi_{\kappa_i}\ ,\ \int_{-\omega_i}^{0}\Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)h^-_{\kappa_i}(s,v^-_i(s),\kappa)ds\right\rangle \\ &= O\big(e^{-6\alpha^u\omega_{i+1}}+e^{-2\alpha^u\omega_{i+1}}e^{2\alpha^s\omega_i}+e^{4\alpha^s\omega_i}\big), \end{aligned}\right\} \tag{3.159}$$

otherwise we find

$$
\begin{aligned}
D_{\omega_j}\boldsymbol{T}^{\mathbf{21}}_{\boldsymbol{\kappa_i}} &= O\big(e^{2\alpha^s\omega_i}\big),\\
D_{\omega_j}\boldsymbol{T}^{\mathbf{22}}_{\boldsymbol{\kappa_i}} &= O\big(e^{\max\{\nu\alpha^s-\alpha^u,\,3/2(\alpha^s-\alpha^u)\}\omega_i}\big),\\
D_{\omega_j}\boldsymbol{T}^{\mathbf{23}}_{\boldsymbol{\kappa_i}} &= O\big(e^{2\alpha^s\omega_i}\big[e^{2\alpha^s\omega_{i-1}}+e^{(\alpha^s-\alpha^u)\omega_i}\big]\big),\\
D_{\omega_j}\boldsymbol{T}^{\mathbf{24}}_{\boldsymbol{\kappa_i}} &= O\big(e^{\max\{(\nu+1/2)\alpha^s-1/2\alpha^u,\,3\alpha^s-\alpha^u\}\omega_i}\big[e^{2\alpha^s\omega_i}+e^{(\alpha^s-\alpha^u)\omega_{i+1}}\big),\\
D_{\omega_j}\boldsymbol{T}^{\mathbf{25}}_{\boldsymbol{\kappa_i}} &= O\big(e^{3(\alpha^s-\alpha^u)\omega_{i+1}}+e^{(\alpha^s-\alpha^u)\omega_{i+1}}e^{2\alpha^s\omega_i}+e^{\min\{\nu+1,4\}\alpha^s\omega_i}\big).
\end{aligned}
$$

4 Examination of the jump Ξ in case of G-equivariant homoclinic cycles

Within this section we want to study the jump Ξ in the context of a homoclinic network in G-equivariant vector fields. Thereby we understand under the expression homoclinic network Γ a hyperbolic equilibrium p and a finite number of homoclinic trajectories γ_i that connect p to itself, $\Gamma = \bigcup_i \gamma_i \cup p$. The symmetric homoclinic network under consideration is a special case of the in [HJKL11] discussed homoclinic cycles.

Basically we focus on the component $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$. After we have presented estimates of the single terms ξ_i consists of in Section 3.4.7 we now will extract explicit expressions of those terms which have leading exponential rates.
To this end we first provide the setting for the symmetric homoclinic cycles in Section 4.1. In Section 4.2 we give the justification of the Hypotheses (H3.2)-(H3.5), which did apply in the proof of the existence of Lin trajectories and in the estimation of some components of $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$. We see to the justification especially in the case of symmetric vector fields. Then Section 4.3 is dedicated to finding an appropriate representation of the jump $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$. We conclude this chapter by examining the derivatives of the jump $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ with respect to the transition times ω_j in Section 4.4.

Now, before we start to present the setting we recall some notions from the theory of group actions. Compare [GSS88] for the following definitions.

Definition 4.0.1. *Let $(G, *)$, $G \neq \emptyset$, be a finite group. A family of vector fields $f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ is called G-`equivariant` (G-`symmetric`), if the following condition holds for all $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^d$ and for all $g \in G$:*

$$gf(x, \lambda) = f(gx, \lambda). \tag{4.1}$$

Indeed, instead of G we use a linear representation of the group G:

Definition 4.0.2. *A* `linear representation`*, also called a* `linear action`*, of G on $\mathbb{R}^n$ is given by a homomorphism $\vartheta : (G, *) \to \big(GL(n, \mathbb{R}), \cdot\big)$ that assigns every $g \in G$ to an invertible linear transformation $\vartheta(g) \in GL(n, \mathbb{R})$, where $GL(n, \mathbb{R})$ denotes the general linear group of $\mathbb{R}^n$. For simplicity we just write $gx := \vartheta(g)x$ and say that G is* `acting` *linearly on $\mathbb{R}^n$.*
A representation of G on a subspace $V \subseteq \mathbb{R}^n$ is called

- `irreducible`*, if the only G-invariant linear subspaces of V are $\{0\}$ and V itself.*
- `absolutely irreducible`*, if the equality $A\vartheta(g) = \vartheta(g)A$ for a linear mapping $A \in \mathbb{L}(V)$ implies that A is a scalar multiple of the identity.*

Note, that such an acting satisfies

(i) $id(x) = x \quad \forall x \in \mathbb{R}^n$ and

(ii) $(gh)x = g(hx) \quad \forall x \in \mathbb{R}^n, \forall g, h \in G$.

Here id denotes the neutral element of the group.

Now, let $G \neq \emptyset$ be a finite group that acts linearly on $\mathbb{R}^n$. In the following we recall the definitions of isotropy group, fixed point space and group orbit, cf. [Fie07].

Definition 4.0.3. *The* ***isotropy group*** G_q *of a point* $q \in \mathbb{R}^n$ *is defined by*

$$G_q = \{g \in G | gq = q\}.$$

The ***fixed point space*** *of a subgroup* $H \subset G$ *is defined as*

$$FixH = \{x \in \mathbb{R}^n | gx = x \quad for\ g \in H\}.$$

The group G *defines a relation* $\sim$ *on* $\mathbb{R}^n$*, where two elements* x *and* y *are related, if there exists an element* $g \in G$ *such that* $gx = y$*. The* ***group orbit*** $G(x)$ *of an element* $x \in \mathbb{R}^n$ *is then given by*

$$G(x) = \{y \in \mathbb{R}^n | y \sim x\}$$

and the group orbit of a subset $A \subset \mathbb{R}^n$ *is given by*

$$G(A) = \bigcup_{x \in A} G(x).$$

Note that it makes sense to speak of an isotropy subgroup G_γ of a trajectory γ, since each point of a trajectory has the same isotropy subgroup.
For $x \in \mathrm{Fix}H$ we find, due to (4.1), that $f(x, \lambda) \in \mathrm{Fix}H$. That means that $\mathrm{Fix}H$ is invariant under the flow $\{\varphi^t_\lambda\}$ of the differential equation.
Due to the definition, every group orbit is a G-invariant subset of $\mathbb{R}^n$.

A special kind of subgroups are the cyclic subgroups.

Definition 4.0.4. *A* ***cyclic subgroup*** $H \subset G$ *is a subgroup that is generated by only one element* $h \in G$*, that is* $H = \{h^k | h \in G, k \in \mathbb{N}\}$*. The* ***order*** *of a cyclic subgroup is given by the smallest integer* $k > 0$ *with* $h^k = id$*. We denote these subgroups by* $H = \mathbb{Z}_k(h)$*.*

To conclude this preface we state:

Remark 4.0.5. *The scalar product* $\langle \cdot, \cdot \rangle$ *we use to define orthogonal complements is chosen in such a way, that it is invariant with respect to the representation of the group* G*. This can be done because* G *is finite and hence compact. To be precise one can set*

$$\langle\langle \cdot, \cdot \rangle\rangle := \frac{1}{|G|} \sum_{g \in G} \langle g\cdot, g\cdot \rangle$$

for any given scalar product $\langle \cdot, \cdot \rangle$*. Then* $\langle\langle \cdot, \cdot \rangle\rangle$ *is obviously a scalar product and group-invariant. For simplicity we write again* $\langle \cdot, \cdot \rangle$*.*

4.1 Setting

During Chapter 3 we considered Lin's method in the context of a heteroclinic network, cf. Hypothesis (H3.1). Indeed we want to apply Lin's method on the special case of a codimension-one homoclinic network. That is to say the heteroclinic network under consideration consists of only one hyperbolic equilibrium p which is connected to itself via a fixed number of homoclinic trajectories. Recall that in the case of a homoclinic trajectory γ asymptotic to p the condition (H3.1)(iv), $\dim T_{\gamma(0)} W^u(p) + \dim T_{\gamma(0)} W^s(p) = n$, is trivially fulfilled.

In order to be able to consider the homoclinic network within a one-dimensional parameter space, we assume that each homoclinic trajectory is of codimension-one, cf. Section 2.6. However, if the unfolding of one homoclinic trajectory is observable in a one-parameter family, it takes in general at least k different parameters to describe the dynamic near a homoclinic network consisting of k different homoclinic trajectories. By introducing a discrete symmetry, the dimension of the corresponding parameter space can be reduced. To this end the context we assume from now on is that of a parameter-dependent differential equation

$$\dot{x} = f(x, \lambda), \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \tag{4.2}$$

that is equivariant under the linear representation of a finite group $G \neq \emptyset$, cf. Definition 4.0.1. Then $x(t)$ is a solution of (4.2) if and only if $gx(t)$ is a solution of (4.2) for all $g \in G$.

Let the homoclinic network Γ be generated by a single homoclinic trajectory γ, that is $\Gamma = G(\overline{\gamma})$ is the group orbit of the closure of γ, cf. Definition 4.0.3. Then each homoclinic trajectory in the network is related to each other by symmetry and the codimension of the whole network equals the codimension of the single trajectory γ. With this the homoclinic network Γ is a special case of a relative homoclinic cycle, cf. [HJKL11] and (1.3).

In the following we give a precise declaration of the kind of homoclinic cycle we will consider further on. To this end we itemize the required hypotheses.

(H4.1).

(i) The vector field $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is smooth, i.e. $f \in C^{l+3}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$, $l \geq \max\{3, \nu\}$, and $f(\cdot, \lambda)$ is equivariant with respect to a finite group $G \neq \emptyset$ for all $\lambda \in \mathbb{R}$.

(ii) At $\lambda = 0$ there is a homoclinic cycle $\Gamma = G(\overline{\gamma})$ equal to the closure of the group orbit of a homoclinic trajectory γ asymptotic to a hyperbolic equilibrium p. We demand that the isotropy group of p, G_p is equal to the whole group G and the isotropy group of γ is not trivial, $G_\gamma \neq \{id\}, G$.

Recall Definition 3.4.2 for the introduction of the constant ν.

The homoclinic cycle $\Gamma = G(\overline{\gamma})$ declared in Hypothesis (H4.1) consists of one hyperbolic equilibrium p and $|G/G_\gamma|$ many homoclinic trajectories, each of them an image of γ. Because γ has a non-trivial isotropy group G_γ it is situated within the fixed point space $\mathrm{Fix}G_\gamma$. Therefore each of the homoclinic trajectories γ_i - let us enumerate them with $i \in \{1, \ldots, |G/G_\gamma|\}$ - is situated within the fixed point space of the corresponding isotropy group G_{γ_i}, i.e. $\gamma_i \subset \mathrm{Fix}G_{\gamma_i}$. For the sake of convenience we use the shortened notation

$$\mathrm{Fix}_i := \mathrm{Fix}G_{\gamma_i}.$$

Then we postulate further that the intersection of two pairwise different fixed point spaces Fix_i is trivial.

(H4.2). *Assume that for all i, j with $i \neq j$ we have $Fix_i \cap Fix_j = \{0\}$.*

By the prescribed linear action of G_p, cf. (H4.1), we find $gp = p$ for all $g \in G$. Hence p is situated in the fixed point space of the whole group G. Due to (H4.2) this fixed point space is trivial and thus $p = 0$.

As a spectral condition, we assume that:

(H4.3). *The isotropy group $G_p = G$ acts absolutely irreducible on the eigenspace $E(\mu^s(\lambda))$ of the leading stable eigenvalue $\mu^s(\lambda)$ of $D_1 f(p, \lambda)$. Further $0 < -\,Re(\mu^s(\lambda)) < \,Re(\mu^u(\lambda))$ for each leading unstable eigenvalue $\mu^u(\lambda)$.*

Due to this hypothesis the leading eigenvalues are not resonant, cf. Hypothesis (H2.4). Especially we can choose the constants α^s and α^u that way that $0 < -\alpha^s < -\operatorname{Re}(\mu^s(\lambda)) < \alpha^u$. Since the whole group G acts absolutely irreducible on the eigenspace $E(\mu^s(\lambda))$ corresponding to $\mu^s(\lambda)$ we can conclude:

Lemma 4.1.1. *Assume Hypothesis (H4.3). Then the leading stable eigenvalue $\mu^s(\lambda)$ is real and semisimple.*

Proof. Due to the equivariance of the vector field and $G_p = G$ we find that the representation of g commutes with the linearisation of the vector field at p for all $g \in G$:

$$gD_1 f(p, \lambda) = D_1 f(gp, \lambda)g = D_1 f(p, \lambda)g.$$

Considering $D_1 f(p, \lambda)$ restricted to $E(\mu^s(\lambda))$ then implies due to the absolutely irreducible representation of G, cf. Definition 4.0.2, that $D_1 f(p, \lambda)_{|E(\mu^s(\lambda))}$ is a scalar multiple of the identity. Hence, looking at the jordan matrix of $D_1 f(p, \lambda)$ we find that the Jordan block corresponding to the eigenvalue $\mu^s(\lambda)$ is equal to a scalar multiple of the identity. Therefore $\mu^s(\lambda)$ has to be real and semisimple. □

As we have mentioned before, the homoclinic trajectory γ and hence the resulting homoclinic cycle $\Gamma = G(\overline{\gamma})$ shall be of codimension-one. To this end it takes adequate premises following the Hypotheses (H2.1), (H2.2) and (H2.3). Since γ lies within $\text{Fix}G_\gamma$, which is a flow-invariant subspace, it suffices to avoid inclination flip and orbit flip and to ensure the splitting of the manifolds within this subspace.
In the following we recall the hypotheses in the context of symmetric vector fields. To this end let the subscript $\text{Fix}G_\gamma$ denote a restriction to $\text{Fix}G_\gamma$. We start with an analogue to (H2.1).

(H4.4). *The homoclinic trajectory γ is non-degenerate, that is $T_\gamma W^s(p) \cap T_\gamma W^u(p) = span\{\dot{\gamma}\}$. Further the restriction of the manifolds to the fixed point space $FixG_\gamma$, $W^s_{FixG_\gamma}(p)$ and $W^u_{FixG_\gamma}(p)$ split with non-zero speed in λ.*

In vector fields that possess a discrete symmetry a homoclinic trajectory γ may occur robustly. This happens when the intersection of the stable and unstable manifolds restricted to the fixed point space $\text{Fix}G_\gamma$ is transversal. Hypothesis (H4.4) excludes such constellations.
Further Hypothesis (H4.4) allows the following stipulation, cf. Section 3.4.1:

Remark 4.1.2. *We choose the parameter λ in that way that λ measures the distance between the stable and unstable manifolds, that is*

$$\lambda = \langle \psi, \gamma^+(\lambda)(0) - \gamma^-(\lambda)(0) \rangle = \langle \psi_i, \xi_i^\infty(\lambda) \rangle.$$

To this end note that $\langle \psi_i, \xi_i^\infty(\lambda) \rangle$ is independent of i due to the equivariance of the vector field and the invariance of the scalar product with respect to the group G.

Hypothesis (H2.2) gives a non-orbit flip condition. With γ also the directions that γ approaches the equilibrium, $\lim_{t \to \pm\infty} \gamma(t)/\|\gamma(t)\|$, lie within $\text{Fix}G_\gamma$.

(H4.5). *The connecting trajectory γ approaches the equilibrium along leading directions, that is*

$$\gamma \not\subset W^{ss}_{\lambda=0, FixG_\gamma}(p) \quad \text{and} \quad \gamma \not\subset W^{uu}_{\lambda=0, FixG_\gamma}(p).$$

At this point we want to mention, that the demand of (H4.5) excludes an interesting class of codimension-one homoclinic cycles where the symmetry forces the connections to approach the equilibrium along non-leading directions. For the derivation of the representation of the determination equations, the first of the two conditions, $\gamma \not\subset W^{ss}_{\lambda=0,\mathrm{Fix}G_\gamma}(p)$, is indeed sufficient.

Next we give the non-inclination flip condition, cf. (H2.3), inside the flow-invariant fixed point space $\mathrm{Fix}G_\gamma$.

(H4.6). *Within $FixG_\gamma$ the leading eigenvalues $\mu^s(\lambda)$ and $\mu^u(\lambda)$ are real and simple. Further for all $x_u \in W^u_{loc}(p) \cap \gamma$ and all $x_s \in W^s_{loc}(p) \cap \gamma$ applies*

$$W^s_{FixG_\gamma}(p) \pitchfork_{x_u} W^{ls,u}_{FixG_\gamma}(p) \quad \text{and} \quad W^u_{FixG_\gamma}(p) \pitchfork_{x_s} W^{s,lu}_{FixG_\gamma}(p).$$

The transversality expressed in Hypothesis (H4.6) is meant in such a way that the sum of the tangent spaces $T_{x_u}W^s_{\mathrm{Fix}G_\gamma}(p)$ and $T_{x_u}W^{ls,u}_{\mathrm{Fix}G_\gamma}(p)$ span a subspace of the same dimension as $\mathrm{Fix}G_\gamma$.

Not only γ is situated within the fixed point space $\mathrm{Fix}G_\gamma$ but also the subspace Z, which we show in the following.

Lemma 4.1.3. *Assume Hypotheses (H4.1) and (H4.4) and let $Z = (T_{\gamma(0)}W^s(p) + T_{\gamma(0)}W^u(p))^\perp$. Then $Z \subset FixG_\gamma$.*

Proof. Since $G_\gamma \neq \{id\}$, G_γ contains at least one other group element $h \in G$, $h \neq id$.

In symmetric vector fields the stable and unstable manifold of an equilibrium point p are invariant with respect to the isotropy subgroup G_p, that is for all $g \in G_p$ we find

$$gW^s(p) = W^s(p), \quad gW^u(p) = W^u(p).$$

Recall that $G_p = G$ due to (H4.1). Moreover we find for all $h \in G_\gamma$

$$hT_{\gamma(0)}W^s(p) = T_{\gamma(0)}W^s(p), \quad hT_{\gamma(0)}W^u(p) = T_{\gamma(0)}W^u(p).$$

Now, let $z \in Z = \big(T_{\gamma(0)}W^s(p) + T_{\gamma(0)}W^u(p)\big)^\perp$ and let y be any element in $T_{\gamma(0)}W^s(p) + T_{\gamma(0)}W^u(p)$. Then also $hy \in T_{\gamma(0)}W^s(p) + T_{\gamma(0)}W^u(p)$ and we find due to the invariance of the scalar product with respect to the group action

$$0 = \langle z, y\rangle = \langle hz, hy\rangle.$$

Hence $hz \in Z$. Then also $z - hz \in Z$ and it yields for any $x \in \mathrm{Fix}G_\gamma$ that

$$\langle z - hz, x\rangle = \langle z, x\rangle - \langle hz, x\rangle = \langle hz, hx\rangle - \langle hz, x\rangle = \langle hz, x\rangle - \langle hz, x\rangle = 0.$$

This implies that either $z - hz = 0$ or $z - hz \in (\mathrm{Fix}G_\gamma)^\perp$. If we assume the latter, it also means that $z \in (\mathrm{Fix}G_\gamma)^\perp$ since Z is only one-dimensional. Hence $Z = \big(T_{\gamma(0)}W^s(p) + T_{\gamma(0)}W^u(p)\big)^\perp \subset (\mathrm{Fix}G_\gamma)^\perp$ and therefore $\mathrm{Fix}G_\gamma \subset T_{\gamma(0)}W^s(p) + T_{\gamma(0)}W^u(p)$. This implies that within $\mathrm{Fix}G_\gamma$ the intersection of the stable and the unstable manifold in $\gamma(0)$ is transversal and hence the homoclinic cycle γ appears robustly, a contradiction to Hypothesis (H4.4).

Therefore $z - hz = 0$, that is $hz = z$ holds true for all $z \in Z$ and all $h \in G_\gamma$. Thus we have shown that $Z \subset \mathrm{Fix}G_\gamma$. □

Corollary 4.1.4. *Let $(\gamma^+(\lambda), \gamma^-(\lambda))$ be the unique pair of solutions of (4.2) given in Lemma 3.1.1. Then $\gamma^\pm(\lambda) \subset FixG_\gamma$.*

Proof. Assuming $\gamma^{\pm}(\lambda)$ are not situated within FixG_γ we find an element $h \in G_\gamma$ such that the pair $(h\gamma^+(\lambda), h\gamma^-(\lambda))$ satisfies the Properties (P3.1) and $h\gamma^{\pm}(\lambda)(\cdot) \neq \gamma^{\pm}(\lambda)(\cdot)$. This is a contradiction to the uniqueness of the pair of solutions satisfying (P3.1). □

Remark 4.1.5. *With $Z \subset FixG_\gamma$ we can conclude from Hypothesis (H4.6) and Lemma 2.6.4 that*

$$Z \not\subset E^{ss}_{-[D_1 f(\gamma^+(\cdot),\lambda)]^T}(0) \quad and \quad Z \not\subset E^{uu}_{-[D_1 f(\gamma^-(\cdot),\lambda)]^T}(0).$$

For the derivation of the representation of the determination equations, only $Z \not\subset E^{uu}_{-[D_1 f(\gamma^-(\cdot),\lambda)]^T}(0)$ will be needed.

Remark 4.1.6. *Regarding the derivation of the determination equations there seems to be no need to restrict the leading unstable eigenvalue μ^u to be real or simple. However, in order to be on the safe side, that the homoclinic trajectory γ satisfies a generic unfolding of codimension-1, we bring these restrictive requirements within the fixed point space $FixG_\gamma$, cf. Hypothesis (H4.6). Whether and to what extent this restriction might be omitted was not further investigated in this thesis.*

Now, with Hypotheses (H4.1)-(H4.6) the assumptions (H1) - (H7) in [HJKL11] are fulfilled so that we operate within the context of [HJKL11]. Further we recall Hypotheses (H3.2)-(H3.5) in terms of the considered homoclinic cycle. For the definition of $\mathcal{S}^i_{\lambda,t}$ also consult the section around Hypothesis (H3.4).

(H4.7). *For sufficiently small λ let*

(i) $p(\lambda) \equiv p$,

(ii) $W^s_{loc,\lambda}(p) \subseteq T_p W^s_{\lambda=0}(p)$ *and* $W^u_{loc,\lambda}(p) \subseteq T_p W^u_{\lambda=0}(p)$.

Further there is an $\varepsilon > 0$ such that for sufficiently small λ and all $i \in \{1, \dots, |G/G_\gamma|\}$

(iii)

$$\begin{aligned} W^s_{\mathcal{S}^i_{\lambda,0}}(p) \cap B(\gamma_i(0), \varepsilon) &\subseteq \gamma_i^+(\lambda)(0) + W_i^+, \\ W^u_{\mathcal{S}^i_{\lambda,0}}(p) \cap B(\gamma_i(0), \varepsilon) &\subseteq \gamma_i^-(\lambda)(0) + W_i^-, \end{aligned}$$

(iv)

$$\begin{aligned} W^s_{\mathcal{S}^i_{\lambda,t}}(p) \cap B(\gamma_i^+(\lambda)(t), \varepsilon) &\subseteq \gamma_i^+(\lambda)(t) + \Phi_i(t,0)W_i^+, \quad t \geq 0, \\ W^u_{\mathcal{S}^i_{\lambda,t}}(p) \cap B(\gamma_i^-(\lambda)(t), \varepsilon) &\subseteq \gamma_i^-(\lambda)(t) + \Phi_i(t,0)W_i^-, \quad t \leq 0. \end{aligned}$$

Hypothesis (H4.7) is necessary for applying Lin's method. However, it does not mean a restriction to the considered homoclinic cycle, since there exist certain vector field transformations such that Hypothesis (H4.7) holds true. Yet, after the transformation we find $f \in C^l(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$, that is we lose a certain degree of differentiability.
Obviously condition (H4.7)(iii) is already included in (H4.7)(iv). However, in view of the vector field transformation we listed them separately, since a transformation in the neighbourhood of $\gamma_i(0)$ is first required before it can be continued along $\gamma_i^{\pm}(t)$ for $t > 0$ or $t < 0$, respectively.
In Section 4.2 we present the transformations of the vector field for justifying Hypothesis (H4.7).

Remark 4.1.7. *With Hypotheses (H4.1), (H4.4) and (H4.7) the assumptions for applying Lin's method, Hypotheses (H3.1) - (H3.5), are satisfied.*

Note that (H4.6) automatically follows from (H4.5) if FixG_γ is two dimensional, no matter what the dimension of the entire space is. Also Hypotheses (H4.5) and (H4.6) are trivially satisfied in a system having no strong stable and no strong unstable eigenvalues. This special case comes along with a few advantages. However we do not consider this case exclusively. Therefore the final hypothesis stated here, cf. also Hypothesis (H3.7), is only called occasionally.

(H4.8). *The spectrum of $D_1 f(p,\lambda)$ has neither strong stable nor strong unstable eigenvalues:*

$$\sigma(D_1 f(p,\lambda)) = \{\mu^s(\lambda), \mu^u(\lambda)\}.$$

4.2 Transformations justifying Hypothesis (H4.7)

In order to prove Lin's method we required the Hypotheses (H3.2) - (H3.5). A justification of these hypotheses can be found in [Kno04] in the more general context of a degenerate homoclinic trajectory. Now, Hypothesis (H4.7) presents the Hypotheses (H3.2) - (H3.5) adapted for the context of the G-equivariant homoclinic cycle under consideration. Therefore the validity of this hypothesis is essential for applying Lin's method on the homoclinic cycle. In this section we introduce transformations of the vector field f which justify this hypothesis. Basically we are guided by the transformations presented in [Kno04]. However, since the vector field under consideration in this thesis is equivariant with respect to the action of a finite group G we only benefit from the transformations if they preserve the equivariance. Therefore we first focus on the question how the symmetry can be retained, before we discuss the single transformations in detail.

To begin with we show how we gain a new vector field out of a transformation. Let $f \in C^{l_1}(\mathbb{R}^n \times \mathbb{R}^d, \mathbb{R}^n)$ be a family of vector fields and $\mathcal{T}_\lambda \in C^{l_2}(\mathbb{R}^n, \mathbb{R}^n)$ be a λ-dependend transformation that is of class C^{l_3} in λ. Then we get the transformed vector field $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ from

$$\tilde{f}(x,\lambda) = D\mathcal{T}_\lambda(\mathcal{T}_\lambda^{-1}(x)) f(\mathcal{T}_\lambda^{-1}(x), \lambda), \tag{4.3}$$

cf. Figure 4.1. Then the transformed vector field $\tilde{f}$ is of class $C^l(\mathbb{R}^n \times \mathbb{R}^d, \mathbb{R}^n)$ where $l = \min\{l_1, l_2 - 1, l_3\}$. Further we find for a G-equivariant transformation $\mathcal{T}_\lambda$ that the new vector field $\tilde{f}$ remains G-equivariant:

$$\begin{aligned} g\tilde{f}(x,\lambda) &= gD\mathcal{T}_\lambda(\mathcal{T}_\lambda^{-1}(x)) f(\mathcal{T}_\lambda^{-1}(x),\lambda) \\ &= D\mathcal{T}_\lambda(g\mathcal{T}_\lambda^{-1}(x)) gf(\mathcal{T}_\lambda^{-1}(x),\lambda) \quad = D\mathcal{T}_\lambda(\mathcal{T}_\lambda^{-1}(gx)) f(g\mathcal{T}_\lambda^{-1}(x),\lambda) \\ &\qquad = D\mathcal{T}_\lambda(\mathcal{T}_\lambda^{-1}(gx)) f(\mathcal{T}_\lambda^{-1}(gx),\lambda) = \tilde{f}(gx,\lambda). \end{aligned}$$

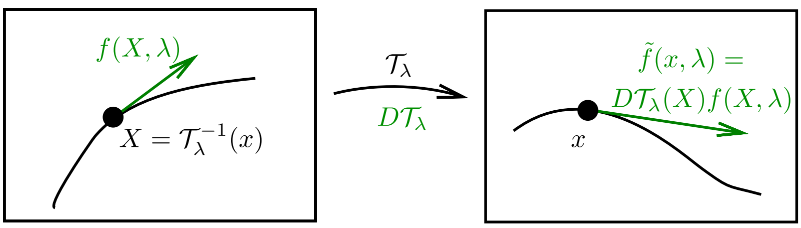


Figure 4.1: The vector field transformation $\mathcal{T}_\lambda$ transports the coordinates $X \in \mathbb{R}^n$ while its derivative in X, $D\mathcal{T}_\lambda(X) \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$, transports the vector field $f(X,\lambda)$, since $\tilde{f}(x,\lambda) = \dot{x} = {}^{d}\!/\!{}_{dt}\mathcal{T}_\lambda(X) = D\mathcal{T}_\lambda(X)\dot{X} = D\mathcal{T}_\lambda(X) f(X,\lambda)$.

Next we see to the question of how to construct such a G-equivariant transformation. For the sake of convenience we omit the dependency on the parameter λ.

Lemma 4.2.1. *Let G be a finite group and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a G-equivariant vector field. For any $q \in \mathbb{R}^n$ there exists a neighbourhood $U(q)$ such that for a G_q-equivariant transformation $\mathcal{T}_q : \mathbb{R}^n \to \mathbb{R}^n$ that acts outside $U(q)$ as identity there exists a transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ that is G-equivariant and acts within $U(q)$ as $\mathcal{T}_q$.*

Proof. Let $V(q)$ be a neighbourhood of any point $q \in \mathbb{R}^n$. Then we get with $U(q) := \{gx | x \in V(q), g \in G_q\} \supseteq V(q)$ a neighbourhood of q that is invariant with respect to the isotropy group G_q. Further consider a transformation $\mathcal{T}_q$ that acts such that outside $U(q)$ the vector field f remains unchanged. Within $U(q)$ the vector field shall remain equivariant with respect to the isotropy subgroup G_q of the point q, that is

$$g\mathcal{T}_q(x) = \mathcal{T}_q(gx) \qquad \forall g \in G_q \ \forall x \in U(q). \tag{4.4}$$

Since $\mathcal{T}_q$ acts outside the neighbourhood $U(q)$ as identity the transformation is G_q-equivariant for all $x \in \mathbb{R}^n$.

If the isotropy subgroup G_q of the point q is equal to G, then q is the only element of the group orbit $G(\{q\}) = \{q\}$ and we see from (4.4) that $\mathcal{T}_q$ and hence the transformed vector field is G-equivariant. Otherwise, if $G_q \neq G$, the group orbit $G(\{q\})$ consists of $k = \frac{|G|}{|G_q|}$ different points $G(\{q\}) = \{q_1, \dots, q_k\}$. Without loss of generality we set $q_1 = q$. Therefore we find k different group elements $h_1, \dots, h_k \in G$ such that $q_i = h_i q$. Each $q_i \in G(\{q\})$ has the isotropy group $G_{q_i} = h_i * G_q * h_i^{-1}$. Note, that the elements h_i, $i \in \{1, \dots, k\}$, are not uniquely defined, but are elements of the left cosets $h_i * G_q := \{h_i * g | g \in G_q\}$. These left cosets present a decomposition of the whole group G.

Now we define for each q_i, $i \in \{1, \dots, k\}$, the transformation

$$\mathcal{T}_{q_i}(x) := h\mathcal{T}_q(h^{-1}x), \qquad \text{where } h \in h_i * G_q. \tag{4.5}$$

Note that $\mathcal{T}_{q_i}$ is well defined since (4.5) is independent of the choice of the representative of the left coset $h_i * G_q$. Namely for any element $h \in h_i * G_q$ there is a $g \in G_q$ such that $h = h_i * g$ and with (4.4) we find

$$h\mathcal{T}_q(h^{-1}x) = (h_i * g)\mathcal{T}_q((g^{-1} * h_i^{-1})x) = h_i\mathcal{T}_q((g * g^{-1} * h_i^{-1})x) = h_i\mathcal{T}_q(h_i^{-1}x).$$

Then $\mathcal{T}_{q_i}$ is a transformation that acts outside the neighbourhood $U(q_i) = h_i U(q)$ as identity. Additionally we find that $\mathcal{T}_{q_i}$ is G_{q_i}-equivariant since for any $g \in G_{q_i}$ there exists a $\tilde{g} \in G_q$ such that $g = h_i * \tilde{g} * h_i^{-1}$ and hence

$$\begin{aligned} g\mathcal{T}_{q_i}(x) \overset{(4.5)}{=} (g * h_i)\mathcal{T}_q(h_i^{-1}x) &= (h_i * \tilde{g} * h_i^{-1} * h_i)\mathcal{T}_q(h_i^{-1}x) \\ &\overset{(4.4)}{=} h_i\mathcal{T}_q((\tilde{g} * h_i^{-1})x) \\ &= h_i\mathcal{T}_q((h_i^{-1} * h_i * \tilde{g} * h_i^{-1})x) = h_i\mathcal{T}_q((h_i^{-1} * g)x) \overset{(4.5)}{=} \mathcal{T}_{q_i}(gx). \end{aligned}$$

Due to the symmetry of the original vector field the transformation $\mathcal{T}_{q_i}$ provides substantially the same result in $U(q_i)$ as the transformation $\mathcal{T}_q$ in $U(q)$.

Finally it remains to combine the transformations $\mathcal{T}_{q_i}$, $q_i \in G(\{q\})$. To this end we have to ensure that the different neighbourhoods $U(q_i)$ are pairwise disjunct: $U(q_i) \cap U(q_j) = \emptyset$ for all $i \neq j$. Since we consider only a finite amount of neighbourhoods this can always be granted by choosing $U(q)$ sufficiently small. Then we define for all $x \in \mathbb{R}^n$ the transformation

$$\mathcal{T} := \mathcal{T}_{q_1} \circ \ldots \circ \mathcal{T}_{q_k}.$$

By construction $\mathcal{T}$ acts outside the neighbourhood $U := U(q_1) \cup \ldots \cup U(q_k)$ as identity and within $U(q_i)$ as $\mathcal{T}_{q_i}$ for all $i \in \{1, \ldots, k\}$. Note that U is G-invariant.

In the following we prove that $\mathcal{T}$ is additionally G-equivariant on $\mathbb{R}^n$. For $x \in \mathbb{R}^n \setminus U$ we find that $gx \in \mathbb{R}^n \setminus U$ and the G-equivariance simply follows from the fact that $\mathcal{T}(x) = x$. Therefore let x be an element of U. Without loss of generality we assume $x \in U(q) = U(q_1)$. Then due to (4.4) we find

$$g \in G_q : \qquad g\mathcal{T}(x) = g\mathcal{T}_q(x) = \mathcal{T}_q(gx) = \mathcal{T}(gx).$$

Now, let $g \notin G_q$. Thus g is an element of one of the left cosets $h_i * G_q$ with $h_i \notin G_q$. Therefore there exists an element $\tilde{g} \in G_q$ such that $g = h_i * \tilde{g}$ and we find

$$gx = (h_i * \tilde{g} * h_i^{-1})h_i x.$$

Since $x \in U(q)$ we find $h_i x \in U(q_i)$ and with $h_i * \tilde{g} * h_i^{-1} \in G_{q_i}$ also gx is an element of $U(q_i)$. This finally leads to

$$g\mathcal{T}(x) = g\mathcal{T}_q(x) \overset{(4.5)}{=} \mathcal{T}_{q_i}(gx) = \mathcal{T}(gx),$$

and the proof of the G-equivariance of $\mathcal{T}$ is completed.

□

Eventually we start with the transformations which justify Hypothesis (H4.7). Due to Lemma 4.2.1 we only construct the transformations for one homoclinic trajectory γ. Therefore we spare the indices i in our notations below. Recall that the vector field $f \in C^{l+3}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$, $l \geq \max\{3, \nu\}$.

Transformation justifying Hypothesis (H4.7)(i)

Let p be a hyperbolic equilibrium of $\dot{x} = f(x, 0)$. Then the equation $0 = f(x, \lambda)$ can be solved near $(p, 0)$ for $x = p(\lambda)$ by applying the implicit function theorem. Then

$$\mathcal{T}_{\lambda,p} : \mathbb{R}^n \to \mathbb{R}^n, \qquad x \mapsto x + (p - p(\lambda))$$

is a transformation on $\mathbb{R}^n$ that obviously leads to Hypothesis (H4.7)(i):

$$\mathcal{T}_{\lambda,p}^{-1}(p) = p - (p - p(\lambda)) = p(\lambda).$$

Further we find that $\mathcal{T}_{\lambda,p}$ is G_p-equivariant.

Now it is necessary to modify the transformation in that way, that it acts only within a G_p-invariant neighbourhood of p. To this end let $\tilde{\chi} : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ cut-off function with the following properties:

(i) $\tilde{\chi}(x) \in [0, 1]$;

(ii) $\tilde{\chi}(x) = 1, \quad \|x\| \leq 1$;

(iii) $\tilde{\chi}(x) = 0, \quad \|x\| \geq 2$.

Here the norm $\|\cdot\|$ is induced by the scalar product $\langle \cdot, \cdot \rangle$ which is invariant with respect to the group action, cf. Remark 4.0.5. Hence $\|\cdot\|$ is G-invariant, that is $\|gx\| = \|x\|$ for all $x \in \mathbb{R}^n$ and for all $g \in G$.

As a direct consequence we find that a δ-neighbourhood of p, $U_\delta(p) = \{x | \|x - p\| \leq \delta\}$ is G-invariant. With that we define another cut-off function χ by

$$\chi(\cdot) = \frac{1}{|G_p|} \sum_{g \in G_p} \tilde{\chi}(g\cdot).$$

Hence χ satisfies not only the conditions (i) - (iii) but in addition

(iv) $\chi(gx) = \chi(x), \quad \forall g \in G_p$.

For $\delta > 0$ we then define

$$\mathcal{T}_{\lambda,\delta,p} : \mathbb{R}^n \to \mathbb{R}^n, \qquad x \mapsto x + \chi\left(\frac{x-p}{\delta}\right) \cdot (p - p(\lambda)).$$

This mapping then acts on a δ-neighbourhood of p as the transformation $\mathcal{T}_{\lambda,p}$ and outside a 2δ-neighbourhood of p the mapping acts as identity. It is easy to see that the transformation $\mathcal{T}_{\lambda,\delta,p}$ is equivariant with respect to the isotropy group G_p, since for all $g \in G_p$ we find

$$g\mathcal{T}_{\lambda,\delta,p}(x) = gx + \chi\left(\frac{gx - gp}{\delta}\right) \cdot (gp - gp(\lambda)) = gx + \chi\left(\frac{gx - p}{\delta}\right) \cdot (p - p(\lambda)) = \mathcal{T}_{\lambda,\delta,p}(gx).$$

Hence the transformation satisfies the assumptions of Lemma 4.2.1.

Due to the implicit function theorem we find that $p(\lambda)$ and hence $\mathcal{T}_{\lambda,\delta,p}$ has the same differentiability class in λ as the vector field f. Further $\mathcal{T}_{\lambda,\delta,p}$ is of class C^∞ in x. Hence the transformed vector field $\tilde{f}$ remains of class C^{l+3}, $l \geq \max\{3, \nu\}$.

Transformation justifying Hypothesis (H4.7)(ii)

In order to verify hypothesis (H4.7)(ii) let $p \equiv p(\lambda)$ be a hyperbolic equilibrium of $\dot{x} = f(x, \lambda)$ for λ sufficiently small. Since $W^u(p)$ and $W^s(p)$ are immersed manifolds, there are C^{l+3}-functions $h^s_{\lambda,p} : T_pW^s_{\lambda=0}(p) \to T_pW^u_{\lambda=0}(p)$ and $h^u_{\lambda,p} : T_pW^u_{\lambda=0}(p) \to T_pW^s_{\lambda=0}(p)$ with $h^{s(u)}_{\lambda,p}(0) = Dh^{s(u)}_{0,p}(0) = 0$ such that the local stable and unstable manifolds can be displayed as $W^s_{loc,\lambda}(p) = p + \text{ graph}(h^s_{\lambda,p})$ and $W^u_{loc,\lambda}(p) = p + \text{ graph}(h^u_{\lambda,p})$, cf. [Kel67, HiPuSh77]. Thus the mapping

$$\begin{array}{rcccl} H_{\lambda,p}: & \mathbb{R}^n = & T_pW^u_{\lambda=0}(p) \oplus T_pW^s_{\lambda=0}(p) & \to & \mathbb{R}^n \\ & & w^u + w^s & \mapsto & w^u + w^s + h^u_{\lambda,p}(w^u) + h^s_{\lambda,p}(w^s) \end{array}$$

maps simultaneously $T_pW^u_{\lambda=0}(p)$ onto graph$(h^u_{\lambda,p})$ and $T_pW^s_{\lambda=0}(p)$ onto graph$(h^s_{\lambda,p})$. Since $Dh^{s(u)}_{0,p}(0) = 0$ we have $DH_{0,p}(0) = id$ and therefore $H_{\lambda,p}$ is indeed a local transformation.

Moreover we find that $H_{\lambda,p}$ is G_p-equivariant. This can be seen as follows. For any $g \in G_p$ we find with $gW^s(p) = W^s(p)$ that

$$gW^s_{loc,\lambda}(p) = gp + g(w^s + h^s_{\lambda,p}(w^s)) = p + gw^s + gh^s_{\lambda,p}(w^s) \stackrel{!}{=} p + \hat{w}^s + h^s_{\lambda,p}(\hat{w}^s) = W^s_{loc,\lambda}(p)$$

where gw^s and $\hat{w}^s$ are both elements of $T_pW^s(p)$ and $gh^s_{\lambda,p}(w^s)$ as well as $h^s_{\lambda,p}(\hat{w}^s)$ are elements of $T_pW^u(p)$. Thus we find

$$gh^s_{\lambda,p}(w^s) = h^s_{\lambda,p}(\hat{w}^s) = h^s_{\lambda,p}(gw^s).$$

Analogously we find $gh^u_{\lambda,p}(w^u) = h^u_{\lambda,p}(gw^u)$ and hence $gH_{\lambda,p}(x) = H_{\lambda,p}(gx)$ for all $g \in G_p$.

Now, a homoclinic or heteroclinic trajectory is a global object and therefore we need a globally defined transformation that acts locally around the equilibrium like $H_{\lambda,p}$. To this end we make again use of the above cut-off function χ. Let $\delta > 0$. Then

$$H_{\lambda,\delta,p}(w^u + w^s) := w^u + w^s + \chi\left(\frac{w^u + w^s}{\delta}\right) h^u_{\lambda,p}(w^u) + \chi\left(\frac{w^u + w^s}{\delta}\right) h^s_{\lambda,p}(w^s)$$

is a G_p-equivariant mapping that acts on a δ-neighbourhood of the origin in $\mathbb{R}^n$ as $H_{\lambda,p}$. Outside a 2δ-neighbourhood of the origin this mapping acts as the identity. If the equilibrium $p = 0$ we are finished. Otherwise we still need the transformation

$$\begin{array}{rccc} J_p: & T_pW^u_{\lambda=0}(p) \oplus T_pW^s_{\lambda=0}(p) & \to & \mathbb{R}^n \\ & w^u + w^s & \mapsto & p + w^u + w^s, \end{array}$$

which is a diffeomorphism of a neighbourhood of the origin in $\mathbb{R}^n$ onto the neighbourhood of $p \in \mathbb{R}^n$. Finally $\mathcal{T}_{\lambda,\delta,p} := J_p \circ H_{\lambda,\delta,p} \circ J_p^{-1}$ transforms a neighbourhood of p onto a neighbourhood of p and satisfies in addition

$$g\mathcal{T}_{\lambda,\delta,p}(x) = \mathcal{T}_{\lambda,\delta,p}(gx)$$

for all $g \in G_p$. Hence it satisfies the assumptions of Lemma 4.2.1.

Note that $\mathcal{T}_{\lambda,\delta,p}$ is of class C^{l+3} in a surrounding area of p. Outside the 2δ-neighbourhood of p it acts as identity and is therefore C^∞. Hence, due to (4.3) the transformed vector field $\tilde{f}$ is of class $C^{l+2}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ within the 2δ-neighbourhood of p and outside that neighbourhood still of class $C^{l+3}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$.

In the end we want to point out that this construction heavily relies on the transversal intersection of $W^s_{loc,\lambda}(p)$ and $W^u_{loc,\lambda}(p)$.

Transformation justifying Hypothesis (H4.7)(iii)

Now we want to perform a similar transformation near $\gamma^\pm(\lambda)(0)$. Of course the intersection of $W^s(p)$ and $W^u(p)$ in $\gamma(0)$ is not transversal. But since the homoclinic trajectory under consideration is non-degenerate, cf. Hypothesis (H4.4), we find at least that within the cross-section $\mathcal{S}$ the traces of the manifolds $W^s(p)$ and $W^u(p)$ have no common tangent. Hypothesis (H4.7)(iii) prescribes the stable and unstable manifolds only within the cross-section $\mathcal{S}$. But the local transformation has to act on an open neighbourhood of $\gamma(0)$ in $\mathbb{R}^n$.

To this end we define $\mathcal{S}_t := \gamma(t) + \Phi(t,0)(W^+ \oplus W^- \oplus Z)$, where $\Phi(\cdot,\cdot)$ denotes again the transition matrix of the variational equation (3.10) along $\gamma(\cdot)$. Then an ε-ball $B(\gamma(0),\varepsilon)$ around $\gamma(0)$ can be thought to be foliated into leaves $\mathcal{S}_t$: $B(\gamma(0),\varepsilon) \subset \bigcup_{t\in(-\varepsilon,\varepsilon)} \mathcal{S}_t$. Then by using appropriate coordinates $(x,t) \in \mathbb{R}^{n-1} \times (-\varepsilon,\varepsilon)$ we construct a local transformation $\mathcal{T}(\cdot,\cdot)$ such that $\mathcal{T}(\cdot,t)$ acts on $\mathcal{S}_t$ and simultaneously flattens within $\mathcal{S}_t$ the stable and unstable manifolds.

We can assume that $\gamma(0)$ lies outside the 2δ-neighbourhood of p that is the effective range of the forgoing transformation. Here the vector field is still of class C^{l+3}. Then there are, just as in the transformation above, C^{l+3} mappings $\hat{h}^s_{\lambda,t,\gamma(0)} : W^+ \to W^- \oplus Z$ and $\hat{h}^u_{\lambda,t,\gamma(0)} : W^- \to W^+ \oplus Z$ such that around $\gamma(0)$ for all small ε the traces of the stable and unstable manifolds within $\mathcal{S}_t$, $t \in (-\varepsilon,\varepsilon)$, can be presented by

$$\begin{aligned} W^s_\lambda(p) \cap \mathcal{S}_t \cap B(\gamma(0),\varepsilon) &= \gamma(t) + \hat{w}^+ + \hat{h}^s_{\lambda,t,\gamma(0)}(\hat{w}^+), \\ W^u_\lambda(p) \cap \mathcal{S}_t \cap B(\gamma(0),\varepsilon) &= \gamma(t) + \hat{w}^- + \hat{h}^u_{\lambda,t,\gamma(0)}(\hat{w}^-), \end{aligned}$$

where $\hat{h}^{s(u)}_{0,0,\gamma(0)}(0) = 0$ and $D\hat{h}^{s(u)}_{0,0,\gamma(0)}(0) = 0$. Further we find in accordance with Lemma 3.1.1

$$\gamma^{\pm}(\lambda)(0) - \gamma(0) = \underbrace{\gamma_1^{\pm}(\lambda)}_{\in W^{\pm}} + \underbrace{\gamma_2^{\pm}(\lambda)}_{\in W^{\mp}\oplus Z}.$$

Using new coordinates

$$w^+ := \hat{w}^+ - \gamma_1^+(\lambda) \in W^+, \qquad w^- := \hat{w}^- - \gamma_1^-(\lambda) \in W^-$$

and new mappings $h^{s(u)}_{\lambda,t,\gamma(0)} : W^{\pm} \to W^{\mp} \oplus Z$,

$$\begin{aligned} h^s_{\lambda,t,\gamma(0)}(w^+) &:= \hat{h}^s_{\lambda,t,\gamma(0)}(w^+ + \gamma_1^+(\lambda)) - \gamma_2^+(\lambda), \\ h^u_{\lambda,t,\gamma(0)}(w^-) &:= \hat{h}^u_{\lambda,t,\gamma(0)}(w^- + \gamma_1^-(\lambda)) - \gamma_2^-(\lambda), \end{aligned}$$

we then find that for $t = 0$ the traces of the stable and unstable manifolds within $\mathcal{S}_0$ can be presented by

$$\begin{aligned} W^s_\lambda(p) \cap \mathcal{S}_0 \cap B(\gamma^+(\lambda)(0), \varepsilon) &= \gamma^+(\lambda)(0) + w^+ + h^s_{\lambda,0,\gamma(0)}(w^+), \\ W^u_\lambda(p) \cap \mathcal{S}_0 \cap B(\gamma^-(\lambda)(0), \varepsilon) &= \gamma^-(\lambda)(0) + w^- + h^u_{\lambda,0,\gamma(0)}(w^-). \end{aligned}$$

Moreover we find $h^{s(u)}_{\lambda,0,\gamma(0)}(0) = 0$ and $Dh^{s(u)}_{0,0,\gamma(0)}(0) = 0$. Actually $(w^+, w^-) \mapsto (w^+ - \gamma_1^+(\lambda), w^- - \gamma_1^-(\lambda))$ is a transformation in $W^+ \oplus W^-$.

Now, with these prearrangements we explain the transformation that leads to Hypothesis (H4.7)(iii). For that we see to the neighbourhood $U(0) = (-\varepsilon, \varepsilon) \times \tilde{U}(0) \subseteq \mathbb{R}^n$, where $\tilde{U}(0) \subseteq W^+ \oplus W^- \oplus Z = \mathbb{R}^{n-1}$ and the neighbourhood $U(\gamma(0)) = \bigcup_{t\in(-\varepsilon,\varepsilon)} \gamma^+(\lambda)(t) + \tilde{U}(0)$. Note that W^+, W^- and Z are $G_{\gamma(0)}$-invariant subspaces each. Hence $\tilde{U}(0)$ can be chosen G_γ-invariant. Then both $U(0)$ and $U(\gamma(0))$ are G_γ-invariant. With

$$\begin{aligned} H_{\lambda,\gamma(0)} : \quad U(0) &\to U(0) \\ (t, w^+ + w^- + z) &\mapsto (t, w^+ + w^- + z + h^s_{\lambda,t,\gamma(0)}(w^+) + h^u_{\lambda,t,\gamma(0)}(w^-)). \end{aligned}$$

we define a mapping which is a diffeomorphism of a neighbourhood of the origin in $\mathbb{R}^n$ onto a neighbourhood of the origin in $\mathbb{R}^n$. Further we define

$$\begin{aligned} J_{\lambda,\gamma(0)} : \quad U(0) &\to U(\gamma(0)) \\ (t, w^+ + w^- + z) &\mapsto \gamma^+(\lambda)(t) + w^+ + w^- + z, \end{aligned}$$

which is a transformation from a neighbourhood of the origin in $\mathbb{R}^n$ onto a neighbourhood of $\gamma^+(\lambda)(t)$ in $\mathbb{R}^n$. Then

$$\mathcal{T}_{\lambda,\gamma(0)} := J_{\lambda,\gamma(0)} \circ H_{\lambda,\gamma(0)} \circ J^{-1}_{\lambda,\gamma(0)}$$

transforms a neighbourhood of $\gamma^+(\lambda)(t)$ onto a neighbourhood of $\gamma^+(\lambda)(t)$. A straightforward calculation shows that $\mathcal{T}_{\lambda,\gamma(0)}(\gamma^+(\lambda)(0) + w^+) = \gamma^+(\lambda)(0) + w^+ + h^s_{\lambda,0,\gamma(0)}(w^+)$ and hence

$$\mathcal{T}_{\lambda,\gamma(0)}(\gamma^+(\lambda)(0) + w^+) \in W^s_\lambda(p) \cap (\gamma^+(\lambda)(0) + (W^+ \oplus W^- \oplus Z)).$$

As a matter of fact $\mathcal{T}_{\lambda,\gamma(0)}$ is defined only for small w^+, w^-, t and λ, but for all z. Further $\mathcal{T}_{\lambda,\gamma(0)}$ acts as the identity in $\gamma^+(\lambda)(0) + Z$. Since $\gamma^+(\lambda)(0) - \gamma^-(\lambda)(0) \in Z$ both $\gamma^+(\lambda)(0)$ and $\gamma^-(\lambda)(0)$ are in the domain of $\mathcal{T}_{\lambda,\gamma(0)}$. By considering

$$\mathcal{T}_{\lambda,\gamma(0)}(\gamma^+(\lambda)(0) + \underbrace{(\gamma^-(\lambda)(0) - \gamma^+(\lambda)(0))}_{=:z\in Z} + w^-) = \underbrace{\gamma^+(\lambda)(0) + z}_{=\gamma^-(\lambda)(0)} + w^- + h^u_{\lambda,0,\gamma(0)}(w^-)$$

we verify

$$\mathcal{T}_{\lambda,\gamma(0)}(\gamma^-(\lambda)(0) + w^-) \in W^u_\lambda(p) \cap (\gamma^-(\lambda)(0) + (W^+ \oplus W^- \oplus Z)).$$

Summarizing these properties we see that in a neighbourhood of $\gamma(0)$ the transformation $\mathcal{T}^{-1}_{\lambda,\gamma(0)}$ maps the trace of the stable manifold within $\mathcal{S} = \mathcal{S}_0$ into $(\gamma^+(\lambda)(0) + W^+)$ and simultaneously the trace of the unstable manifold within $\mathcal{S}$ into $(\gamma^-(\lambda)(0) + W^-)$.

It still remains to globalise $\mathcal{T}_{\lambda,\gamma(0)}$ for fixed λ while preserving its local properties. To this end we define for $\delta > 0$: $\chi_\delta(t, w^+ + w^- + z) := \chi_1\left(\frac{w^+ + w^- + z}{\delta}\right)\chi_2\left(\frac{t}{\delta}\right)$. Here $\chi_1 : W^+ \oplus W^- \oplus Z \subset \mathbb{R}^n \to \mathbb{R}$ and $\chi_2 : \mathbb{R} \to \mathbb{R}$ are two cut-off functions. Further, χ_1 is chosen that way, that $\chi_1(gx) = \chi_1(x)$ for all $g \in G_\gamma$.

Then we can globalise $H_{\lambda,\gamma(0)}$ by:

$$H_{\lambda,\delta,\gamma(0)}(t, w^+ + w^- + z) := (t, w^+ + w^- + z + \chi_\delta(t, w^+ + w^- + z)\left(h^s_{\lambda,t,\gamma(0)}(w^+) + h^u_{\lambda,t,\gamma(0)}(w^-)\right)),$$

and get the desired transformation by

$$\mathcal{T}_{\lambda,\delta,\gamma(0)} := J_{\lambda,\gamma(0)} \circ H_{\lambda,\delta,\gamma(0)} \circ J^{-1}_{\lambda,\gamma(0)}.$$

Finally, following the line of action in the justification of Hypothesis (H4.7)(ii) we find for all $g \in G_{\gamma(0)}$ that $gh^{s(u)}_{\lambda,t,\gamma(0)}(x) = h^{s(u)}_{\lambda,t,\gamma(0)}(gx)$ and hence

$$g\mathcal{T}_{\lambda,\delta,\gamma(0)}(x) = \mathcal{T}_{\lambda,\delta,\gamma(0)}(gx).$$

Thus Lemma 4.2.1 can be applied.

Again we can state that $\mathcal{T}_{\lambda,\delta,\gamma(0)}$ is of class C^{l+3} within the 2δ-neighbourhood of $\gamma(0)$ and outside it is C^∞. Hence the transformed vector field is C^{l+2} in the surrounding areas of p (due to the transformation that leads to Hypothesis (H4.7)(ii)) and $\gamma(0)$ and all there G-images.

Transformation justifying Hypothesis (H4.7)(iv)

Hypothesis (H4.7)(iv) can be seen as a continuation of Hypothesis (H4.7)(iii) along $\gamma^\pm(\lambda)(t)$ as for $t = 0$ (H4.7)(iv) coincides with (H4.7)(iii). Moreover, due to (H4.7)(ii), there is a $T > 0$, such that Hypothesis (H4.7)(iv) is fulfilled for all t with $|t| > T$.

We construct two transformation $\mathcal{T}^+_t$ and $\mathcal{T}^-_t$ to obtain the situation in (H4.7)(iv) for $t \geq 0$ and $t \leq 0$, respectively. These transformation can be gained in quite the same way as the transformation leading to (H4.7)(iii). Due to our comments above we have to consider tubular neighbourhoods of $\{\gamma^+(\lambda)(t), t \in [0, T]\}$ and $\{\gamma^-(\lambda)(t), t \in [-T, 0]\}$. Exemplarily we present the transformation $\mathcal{T}^+_t$, $t \in [0, T]$.

We define $W^+_t := \Phi(t, 0)W^+$, $W^-_t := \Phi(t, 0)W^-$ and $Z_t := \Phi(t, 0)Z$. Then again we find mappings $\hat{h}^s_{\lambda,t,\gamma(t)} : W^+_t \to W^-_t \oplus Z_t$ with $\hat{h}^s_{0,t,\gamma(t)}(0) = D\hat{h}^s_{0,t,\gamma(t)}(0) = 0$ whose graphs locally represent the traces of the stable manifold within $\mathcal{S}_{\lambda,t}$, that is for small ε we have

$$W^s_{\mathcal{S}_{\lambda,t}} \cap B(\gamma^+(\lambda)(t), \varepsilon) = \gamma^+(\lambda)(t) + w^+ + \hat{h}^s_{\lambda,t,\gamma(t)}(w^+).$$

This time $\hat{h}^s$ and $\hat{h}^u$ are of class C^{l+2} since the vector field is only of class C^{l+2} due to the foregoing transformations.

Similar to the justification of Hypothesis (H4.7)(iii) we define:

$$\begin{array}{rlll} H^+_{\lambda,\gamma}: & \mathbb{R}\times W^+_t \oplus W^-_t \oplus Z_t & \to & \mathbb{R}^n \\ & (t, w^+ + w^- + z) & \mapsto & (t, w^+ + w^- + z + \hat{h}^s_{\lambda,t,\gamma(t)}(w^+)), \\ J^+_{\lambda,\gamma}: & \mathbb{R}\times W^+_t \oplus W^-_t \oplus Z_t & \to & \bigcup\limits_{t\in(0,T)} \mathcal{S}_t \\ & (t, w^+ + w^- + z) & \mapsto & \gamma^+(\lambda)(t) + w^+ + w^- + z. \end{array}$$

With that we define $\mathcal{T}^+_{\lambda,\gamma}: \bigcup\limits_{t\in(0,T)} (B(\gamma^+(\lambda)(t),\varepsilon)\cap\mathcal{S}_t) \to \mathbb{R}^n$ by $\mathcal{T}^+_{\lambda,\gamma} := J^+_{\lambda,\gamma}\circ H^+_{\lambda,\gamma}\circ (J^+_{\lambda,\gamma})^{-1}$. Since $H^+_{\lambda,\gamma}$ is a local diffeomorphism, because of $D\hat{h}^s_{0,t,\gamma(t)}(0) = 0$, $\mathcal{T}^+_{\lambda,\gamma}$ is a local transformation. By construction we have $\mathcal{T}^+_{\lambda,\gamma}(\gamma^+(\lambda)(t) + w^+) = \gamma^+(\lambda)(t) + w^+ + \hat{h}^s_{\lambda,t,\gamma(t)}(w^+)$ and hence for sufficiently small ε

$$\mathcal{T}^+_{\lambda,\gamma}\left(\gamma^+(\lambda)(t) + \Phi(t,0)W^+\right) = W^s_{\mathcal{S}_{\lambda,t}} \cap B(\gamma^+(\lambda)(t),\varepsilon).$$

We globalise the transformation $\mathcal{T}^+_{\lambda,\gamma}$ by globalising $H^+_{\lambda,\gamma}$:

$$H^+_{\lambda,\delta,\gamma}(t, w^+ + w^- + z) = (t, w^+ + w^- + z + \chi_1\left(\frac{w^+ + w^- + z}{\delta}\right)\hat{h}^s_{\lambda,t,\gamma(t)}(w^+)).$$

Then $\mathcal{T}^+_{\lambda,\delta,\gamma} = J^+_{\lambda,\gamma}\circ H^+_{\lambda,\delta,\gamma}\circ(J^+_{\lambda,\gamma})^{-1}$ acts locally around a tubular neighbourhood of $\{\gamma^+(\lambda)(t), t\in[0,T]\}$. So in contrast to the previous defined transformations $\mathcal{T}^+_{\lambda,\delta,\gamma}$ does not transform the vector field locally around a single point. However, the isotropy subgroup G_γ of the curve γ coincides with the isotropy group of a single point on this curve. And again we find that the transformation $\mathcal{T}^+_{\lambda,\delta,\gamma}$ is equivariant with respect to that isotropy group G_γ. Additionally, for $t > T$ the transformation coincides with the transformation justifying (H4.7)(ii). Therefore the tubular neighbourhoods can be chosen that way, that they do not intersect each other. So an analogous version of Lemma 4.2.1 formulated for curve segments can be applied.

Finally the resulting vector field looses a differentiability class for each transformation $\mathcal{T}^+_{\lambda,\delta,\gamma}$ and $\mathcal{T}^-_{\lambda,\delta,\gamma}$. Since this time the effective range of the transformations intersect the 2δ-neighbourhoods of the forgoing transformations, the resulting vector field f is of differentiability class C^l.

4.3 Representation of the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$

In this section we finally inspect the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$ in the context of G-equivariant homoclinic cycles as declared in Hypotheses (H4.1) - (H4.7). The aim is to find a suitable representation of the jump in such a way that ξ_i is given as sum of an explicit term plus residual terms. Now, this representation mainly depends on the geometrical location of the fixed point spaces $\mathrm{Fix}_{\kappa_i} := \mathrm{Fix}G_{\gamma_{\kappa_i}}$, $\kappa_i \in \kappa$. To be more precise, it is of great importance whether two consecutive fixed point spaces are orthogonal to each other or not. Recall in this regard that due to Hypothesis (H4.2) the intersection of two different fixed point spaces is trivial. We introduce for fixed κ the index set J_κ with

$$J_\kappa = \{j\in\mathbb{Z} \mid \mathrm{Fix}_{\kappa_{j-1}} \perp \mathrm{Fix}_{\kappa_j}\}. \tag{4.6}$$

In the following we present two theorems that show the appearance of the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$. Basically they present the same representation of $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$ but with slightly different residual terms which depend

on different assumptions to the spectrum of $D_1 f(p, \lambda)$. Theorem 4.3.1 comes with no restriction to the spectrum of $D_1 f(p, \lambda)$. However, in case that the vector field under consideration has no strong stable and unstable eigenvalues, cf. Hypothesis (H4.8), we can improve the estimates of the residual terms. This case is presented in Theorem 4.3.3.
Additionally we assume that $\nu \geq 3$. To this end recall Definition 3.4.2 of ν. This assumption is necessary to determine the leading order terms in case that $i \in J_\kappa$.
The proofs of both theorems follow along same lines and we will mainly focus on the proof of Theorem 4.3.1. We do this in Section 4.3.2 after we have collected some helpful Lemmata in Section 4.3.1.

Theorem 4.3.1. *Assume Hypotheses (H4.1) - (H4.7) and let $\nu \geq 3$. Then there exist constants Ω and c such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ we find the following expression for the jump ξ_i:*

(i) If $i \in \mathbb{Z} \setminus J_\kappa$ then the jump ξ_i can be written as

$$
\begin{aligned}
\xi_i(\boldsymbol{\omega}, \lambda, \kappa) &= -e^{2\mu^s(\lambda)\omega_i} A_i(\lambda, \kappa) + O(e^{2\alpha^s(\omega_{i-1}+\omega_i)}) + O(e^{\max\{\alpha^s+\alpha^{ss}, 3\alpha^s\}\omega_i}) \\
&\quad + \begin{cases} O(e^{(\alpha^s-\alpha^u)\omega_{i+1}}), & i+1 \in \mathbb{Z} \setminus J_\kappa \\ O(e^{2\alpha^s\omega_i} e^{(\alpha^s-\alpha^u)\omega_{i+1}}) + O(e^{3/2(\alpha^s-\alpha^u)\omega_{i+1}}) \\ \quad + O(e^{1/2(\alpha^s-3\alpha^u)\omega_{i+1}} e^{(\alpha^s-\alpha^u)\omega_{i+2}}), & i+1 \in J_\kappa \end{cases}
\end{aligned}
$$

The functions $A_i : \mathbb{R} \to \mathbb{R}$ are smooth and moreover $A_i(0, \kappa) \neq 0$ for all $i \in \mathbb{Z} \setminus J_\kappa$.

(ii) If $i \in J_\kappa$ then the jump ξ_i can be written as

$$
\begin{aligned}
\xi_i(\boldsymbol{\omega}, \lambda, \kappa) &= -e^{4\mu^s(\lambda)\omega_i}[B_i(\lambda, \kappa) + D_i(\lambda, \kappa, \omega_i)] - e^{2\mu^s(\lambda)\omega_{i-1}} e^{2\mu^s(\lambda)\omega_i} C_i(\lambda, \kappa) \\
&\quad + O(e^{2\alpha^s(\omega_{i-1}+\omega_i)}[e^{2\alpha^s\omega_{i-2}} + e^{\max\{\alpha^s, \alpha^{ss}-\alpha^s\}\omega_{i-1}} + e^{\max\{2\alpha^s, 1/2(\alpha^s-\alpha^u), \alpha^{ss}-\alpha^s\}\omega_i}]) \\
&\quad + O(e^{\max\{5\alpha^s, 3\alpha^s+\alpha^{ss}, 3\alpha^s-\alpha^u, 3/2(\alpha^s-\alpha^u)\}\omega_i}) \\
&\quad + O(e^{4\alpha^s\omega_i} e^{1/2(\alpha^s-\alpha^u)\omega_{i+1}}) \\
&\quad + \begin{cases} O(e^{(\alpha^s-\alpha^u)\omega_{i+1}}), & i+1 \in \mathbb{Z} \setminus J_\kappa \\ O(e^{2\alpha^s\omega_i} e^{(\alpha^s-\alpha^u)\omega_{i+1}}) + O(e^{3/2(\alpha^s-\alpha^u)\omega_{i+1}}) \\ \quad + O(e^{1/2(\alpha^s-3\alpha^u)\omega_{i+1}} e^{(\alpha^s-\alpha^u)\omega_{i+2}}), & i+1 \in J_\kappa \end{cases}
\end{aligned}
$$

The functions $B_i, C_i, D_i(\cdot, \cdot, \omega_i) : \mathbb{R} \times \Sigma_{\boldsymbol{C}} \to \mathbb{R}$ are smooth in λ. D_i is equal to zero for $\nu > 3$.

The O-terms are valid for ω_{i-2}, ω_{i-1}, ω_i, ω_{i+1} and ω_{i+2} tending to infinity.

Remark 4.3.2. *Theorem 4.3.1 is consistent with equation (1.7), where $\xi_i(\boldsymbol{\omega}, \lambda, \kappa) = e^{2\mu^s(\lambda)\omega_i} A_i(\lambda, \kappa) + O(e^{2\mu^s(\lambda)\omega_i\delta}) + O(e^{2\mu^s(\lambda)\omega_{i+1}\delta})$ for some $\delta > 1$.*

In case that the vector field under consideration has no strong stable and unstable eigenvalues, cf. Hypothesis (H4.8), the theorem reads as follows.

Theorem 4.3.3. *Assume Hypotheses (H4.1) - (H4.7) and let $\nu \geq 3$. Further assume (H4.8). Then there exist constants Ω and c such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ we find the following expression for the jump ξ_i:*

(i) If $i \in \mathbb{Z} \setminus J_\kappa$ then the jump ξ_i can be written as

$$\begin{aligned}\xi_i(\boldsymbol{\omega},\lambda,\kappa) &= -e^{2\mu^s(\lambda)\omega_i}A_i(\lambda,\kappa) + O(e^{2\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}} + e^{2\alpha^s\omega_i}]) \\ &\quad + \begin{cases} O(e^{-2\alpha^u\omega_{i+1}}), & i+1 \in \mathbb{Z}\setminus J_\kappa \\ O(e^{-2\alpha^u\omega_{i+1}}[e^{2\alpha^s\omega_i} + e^{2\alpha^s\omega_{i+1}} + e^{-2\alpha^u\omega_{i+2}}), & i+1 \in J_\kappa \end{cases}\end{aligned}$$

The functions $A_i(\cdot,\kappa): \mathbb{R} \to \mathbb{R}$ are smooth and moreover $A_i(0,\kappa) \neq 0$ for all $i \in \mathbb{Z}\setminus J_\kappa$.

(ii) If $i \in J_\kappa$ then the jump ξ_i can be written as

$$\begin{aligned}\xi_i(\boldsymbol{\omega},\lambda,\kappa) &= -e^{4\mu^s(\lambda)\omega_i}[B_i(\lambda,\kappa) + D_i(\lambda,\kappa,\omega_i)] - e^{2\mu^s(\lambda)\omega_{i-1}}e^{2\mu^s(\lambda)\omega_i}C_i(\lambda,\kappa) \\ &\quad + O(e^{2\alpha^s(\omega_{i-1}+\omega_i)}[e^{2\alpha^s\omega_{i-2}} + e^{\alpha^s\omega_{i-1}} + e^{\max\{2\alpha^s,-\alpha^u\}\omega_i}]) \\ &\quad + O(e^{\max\{5\alpha^s,\nu\alpha^s-\alpha^u,2(\alpha^s-\alpha^u)\}\omega_i}) + O(e^{4\alpha^s\omega_i}e^{-\alpha^u\omega_{i+1}}) \\ &\quad + \begin{cases} O(e^{-2\alpha^u\omega_{i+1}}), & i+1 \in \mathbb{Z}\setminus J_\kappa \\ O\big(e^{-2\alpha^u\omega_{i+1}}[e^{2\alpha^s\omega_i} + e^{2\alpha^s\omega_{i+1}} + e^{-2\alpha^u\omega_{i+2}}]\big), & i+1 \in J_\kappa \end{cases}\end{aligned}$$

The functions $B_i, C_i, D_i(\cdot,\cdot,\omega_i): \mathbb{R}\times\Sigma_{\boldsymbol{C}} \to \mathbb{R}$ are smooth in λ. D_i is equal to zero for $\nu > 3$.

The O-terms are valid for ω_{i-2}, ω_{i-1}, ω_i, ω_{i+1} and ω_{i+2} tending to infinity.

4.3.1 Properties of G-equivariant vector fields

In preparation of the proofs of Theorems 4.3.1 and 4.3.3 we present some basic characteristics of G-equivariant vector fields. To be precise, we show that the transition matrices $\Phi^{\pm}_{\kappa_i}(\cdot,\cdot)$ of the variational equations (3.8) along $\gamma^{\pm}_{\kappa_i}$ and the transition matrix $\Phi^{\pm}_{\kappa_i}(\cdot,\cdot)^T$ of the adjoint equation (3.9) leave both the fixed point space Fix_{κ_i} and its orthogonal compliment $\mathrm{Fix}^{\perp}_{\kappa_i}$ invariant.
Further we show that also the projections corresponding to the exponential dichotomy of the variational equations (3.8) and (3.9), namely $P^{\pm}_{\kappa_i}$ and $(P^{\pm}_{\kappa_i})^T$, leave Fix_{κ_i} and $\mathrm{Fix}^{\perp}_{\kappa_i}$ invariant.

We start with a property concerning the derivatives of the equivariant vectorfield f.

Lemma 4.3.4. *Let $g \in G$ and $D^k f(x_0) \in \mathbb{L}_k(\mathbb{R}^n \times \ldots \times \mathbb{R}^n, \mathbb{R}^n)$ be the k-th derivative of the G-equivariant vector field f in $x_0 \in \mathbb{R}^n$. Then we find for all $k \in \mathbb{N}$ that*

$$gD^k f(x_0)(\cdot,\ldots,\cdot) = D^k f(gx_0)(g\cdot,\ldots,g\cdot).$$

Proof. This follows by induction for the counter k, where the base case at $k = 0$ is given through the definition of equivariance. □

In case that the fixed point spaces under consideration are generated by a cyclic subgroup of order two, $\mathbb{Z}_2(h)$ for some $h \in G$, we find with the lemma above the following:

Corollary 4.3.5. *Let $h \in G$ with $h = h^{-1}$. Let $x_0 \in Fix\mathbb{Z}_2(h)$ and $\eta \in \big(Fix\mathbb{Z}_2(h)\big)^{\perp}$. Then*

$$D^k f(x_0)[\eta,\ldots,\eta] \in \begin{cases} Fix\mathbb{Z}_2(h), & k \text{ even}, \\ \big(Fix\mathbb{Z}_2(h)\big)^{\perp}, & k \text{ odd}. \end{cases}$$

Proof. First we show that for an element $h \in G$ satisfying $h = h^{-1}$ the assertion $x \in \left(\mathrm{Fix}\mathbb{Z}_2(h)\right)^{\perp}$ is equivalent to $hx = -x$

With $x \in \left(\mathrm{Fix}\mathbb{Z}_2(h)\right)^{\perp}$ we also find that $hx \in \left(\mathrm{Fix}\mathbb{Z}_2(h)\right)^{\perp}$, since for all $y \in \mathrm{Fix}\mathbb{Z}_2(h)$ we have due to the group invariance of the scalar product

$$0 = \langle y, x\rangle = \langle hy, hx\rangle = \langle y, hx\rangle.$$

Thus $x + hx \in \left(\mathrm{Fix}\mathbb{Z}_2(h)\right)^{\perp}$. On the other hand we find

$$h(hx + x) = h^2 x + hx = x + hx$$

and hence $hx + x \in \mathrm{Fix}\mathbb{Z}_2(h)$. Therefore we have $hx + x \in \mathrm{Fix}\mathbb{Z}_2(h) \cap \left(\mathrm{Fix}\mathbb{Z}_2(h)\right)^{\perp} = \{0\}$, that is $hx + x = 0$ and thus $hx = -x$.

Now, let $x = x_1 + x_2$ with $x_1 \in \mathrm{Fix}\mathbb{Z}_2(h)$ and $x_2 \in \left(\mathrm{Fix}\mathbb{Z}_2(h)\right)^{\perp}$ satisfy $hx = -x$. Then x_1 has to be zero and hence $x \in \left(\mathrm{Fix}\mathbb{Z}_2(h)\right)^{\perp}$, since

$$0 = hx + x = h(x_1 + x_2) + x_1 + x_2 = 2x_1 - x_2 + x_2 = 2x_1.$$

Thus $x \in \left(\mathrm{Fix}\mathbb{Z}_2(h)\right)^{\perp}$ is equivalent to $hx = -x$.

With this the statement simply follows from Lemma 4.3.4, since with $x_0 \in \mathrm{Fix}\mathbb{Z}_2(h)$ and $\eta \in \left(\mathrm{Fix}\mathbb{Z}_2(h)\right)^{\perp}$ we find

$$hD^k f(x_0)\left[\eta, \ldots, \eta\right] = D^k f(hx_0)\left[h\eta, \ldots, h\eta\right] = D^k f(x_0)\left[-\eta, \ldots, -\eta\right] = (-1)^k D^k f(x_0)\left[\eta, \ldots, \eta\right].$$

□

In accordance to Chapter 3 we enumerate the homoclinic trajectories with κ_i, $\kappa \in \Sigma_{\boldsymbol{C}}$. So let γ_{κ_i} be one of the homoclinic trajectories within the chain Γ^{κ}. For $g \in G$ we denote the g-image of γ_{κ_i} by

$$\gamma_{g\kappa_i} := g\gamma_{\kappa_i}.$$

Analogously we define

$$\gamma^{\pm}_{g\kappa_i} := g\gamma^{\pm}_{\kappa_i}.$$

The flow invariant stable and unstable manifolds $W^s(p)$ and $W^u(p)$ of the equilibrium p are invariant with respect to the isotropy group $G_p = G$. Therefore the g-image of the tangent space along a solution within such a manifold is a tangent space along the g-image of that solution:

$$gT_{\gamma^{\pm}_{\kappa_i}(t)} W^{s/u}(p) = T_{g\gamma^{\pm}_{\kappa_i}(t)} W^{s/u}(p).$$

Since the subspaces $W^{\pm}_{\kappa_i}, Z_{\kappa_i}$ and U_{κ_i} are defined via the tangent spaces of the stable and unstable manifolds within the point $\gamma_{\kappa_i}(0)$ we then find that the g-images of these subspaces are equal to the subspaces defined for the g-image of $\gamma_{\kappa_i}(0)$, that is $g\gamma_{\kappa_i}(0) = \gamma_{g\kappa_i}(0)$. Therefore we define

$$\begin{aligned} W^{\pm}_{g\kappa_i} &:= gW^{\pm}_{\kappa_i}, \\ Z_{g\kappa_i} &:= gZ_{\kappa_i}, \\ U_{g\kappa_i} &:= gU_{\kappa_i}. \end{aligned}$$

Since $\gamma^{\pm}_{g\kappa_i}$ are situated within the stable or unstable manifolds $W^s(p)$ and $W^u(p)$, respectively, the variational equations

$$\dot{x} = D_1 f(\gamma^{\pm}_{g\kappa_i}(\lambda)(t), \lambda)x \tag{4.7}$$

of f along $\gamma^{\pm}_{g\kappa_i}$ also have an exponential dichotomy either on $\mathbb{R}^+$ or $\mathbb{R}^-$. In accordance with our notation above we denote by $\Phi^{\pm}_{g\kappa_i}(\cdot,\cdot)$ the transition matrices of (4.7) and by $P^{\pm}_{g\kappa_i}(\cdot)$ the corresponding projections of the exponential dichotomy. Analogously we denote by $\Psi^{\pm}_{g\kappa_i}(\cdot,\cdot)$ the transition matrices of the adjoint variational equations

$$\dot{x} = -\left[D_1 f(\gamma^{\pm}_{g\kappa_i}(\lambda)(t), \lambda)\right]^T x \tag{4.8}$$

of f along $\gamma^{\pm}_{g\kappa_i}$. Recall that $\Psi^{\pm}_{g\kappa_i}(t,s) = \Phi^{\pm}_{g\kappa_i}(s,t)^T$ for all $s,t \in \mathbb{R}$.

Now, the following Lemma gives a relation between solutions of the variational equations (3.8) and (4.7) of f along $\gamma^{\pm}_{\kappa_i}$ and its g-image $\gamma^{\pm}_{g\kappa_i}$.

Lemma 4.3.6. *Let $\Phi^{\pm}_{g\kappa_i}(\lambda)(\cdot,\cdot)$ and $\Phi^{\pm}_{\kappa_i}(\lambda)(\cdot,\cdot)$ be the transition matrices of (4.7) and (3.8), respectively. Then*

$$\Phi^{\pm}_{g\kappa_i}(\lambda)(t,\tau) = g\Phi^{\pm}_{\kappa_i}(\lambda)(t,\tau)g^{-1}.$$

Proof. Due to the G-symmetry and Lemma 4.3.4 we have

$$gD_1 f(x(t),\lambda) = D_1 f(gx(t),\lambda)g.$$

Hence we find for a solution $x(\cdot)$ of the Initial Value Problem $\dot{x} = D_1 f(\gamma^{\pm}_{\kappa_i}(\lambda)(t),\lambda)x$, $x(\tau) = \xi$ that $y(\cdot) = gx(\cdot)$ is a solution of $\dot{x} = D_1 f(\gamma^{\pm}_{g\kappa_i}(\lambda)(t),\lambda)x$, $y(\tau) = g\xi$, since

$$(gx)^{\cdot} = g\dot{x} = gD_1 f(\gamma^{\pm}_{\kappa_i}(\lambda)(t),\lambda)x = D_1 f(g\gamma^{\pm}_{\kappa_i}(\lambda)(t),\lambda)gx.$$

Therefore we have for all $\xi \in \mathbb{R}^n$

$$g\Phi^{\pm}_{\kappa_i}(\lambda)(t,\tau)\xi = gx(t) = y(t) = \Phi^{\pm}_{g\kappa_i}(\lambda)(t,\tau)y(\tau) = \Phi^{\pm}_{g\kappa_i}(\lambda)(t,\tau)g\xi.$$

This concludes the proof. □

Since both variational equations (3.8) and (4.7) have an exponential dichotomy we also can give a relation between the corresponding projections.

Lemma 4.3.7. *Let $P^{\pm}_{g\kappa_i}(\lambda,t)$ be a projection of the exponential dichotomy of (4.7) with $\ker P^{\pm}_{g\kappa_i}(\lambda,0) = T_{g\gamma^{\pm}_{\kappa_i}(0)}W^{s/u}(p)$ and $\operatorname{im} P^{\pm}_{g\kappa_i}(\lambda,0) = W^{\mp}_{g\kappa_i} \oplus Z_{g\kappa_i}$. Then we find for all $t \in \mathbb{R}^{\pm}$*

$$P^{\pm}_{g\kappa_i}(\lambda,t) = gP^{\pm}_{\kappa_i}(\lambda,t)g^{-1}.$$

Proof. $P^{\pm}_{\kappa_i}$ is a projection of the exponential dichotomy of (3.8) with $\ker P^{\pm}_{\kappa_i}(\lambda,0) = T_{\gamma^{\pm}_{\kappa_i}(0)}W^{s/u}(p)$ and $\operatorname{im} P^{\pm}_{\kappa_i}(\lambda,0) = W^{\mp}_{\kappa_i} \oplus Z_{\kappa_i}$. Now let $x = x_1 + x_2$ with $x_1 \in \ker P^{\pm}_{g\kappa_i}(\lambda,0)$ and $x_2 \in \operatorname{im} P^{\pm}_{g\kappa_i}(\lambda,0)$. Thus

$$\begin{array}{rclcl} x_1 & \in & T_{g\gamma^{\pm}_{\kappa_i}(0)} W^{s/u}(p) & = & gT_{\gamma^{\pm}_{\kappa_i}(0)} W^{s/u}(p), \\ x_2 & \in & W^{\mp}_{g\kappa_i} \oplus Z_{g\kappa_i} & = & gW^{\mp}_{\kappa_i} \oplus gZ_{\kappa_i}, \end{array}$$

and therefore

$$\begin{array}{rclcl} g^{-1}x_1 & \in & T_{\gamma^{\pm}_{\kappa_i}(0)} W^{s/u}(p) & = & \ker P^{\pm}_{\kappa_i}(\lambda, 0), \\ g^{-1}x_2 & \in & W^{\mp}_{\kappa_i} \oplus Z_{\kappa_i} & = & \operatorname{im} P^{\pm}_{\kappa_i}(\lambda, 0). \end{array}$$

Hence we find that $gP^{\pm}_{\kappa_i}(\lambda, 0)g^{-1}$ has the same kernel and image as $P^{\pm}_{g\kappa_i}(\lambda, 0)$, since $gP^{\pm}_{\kappa_i}(\lambda, 0)g^{-1}x_1 = g0 = 0 = P^{\pm}_{g\kappa_i}(\lambda, 0)x_1$ and $gP^{\pm}_{\kappa_i}(\lambda, 0)g^{-1}x_2 = gg^{-1}x_2 = x_2 = P^{\pm}_{g\kappa_i}(\lambda, 0)x_2$. Thus

$$P^{\pm}_{g\kappa_i}(\lambda, 0) = gP^{\pm}_{\kappa_i}(\lambda, 0)g^{-1}.$$

The equality of these projections for all $t \in \mathbb{R}^{\pm}$ follows from Lemma 4.3.6, since

$$\begin{array}{rcl} P^{\pm}_{g\kappa_i}(\lambda, t) & = & \Phi^{\pm}_{g\kappa_i}(\lambda)(t, 0)P^{\pm}_{g\kappa_i}(\lambda, 0)\Phi^{\pm}_{g\kappa_i}(\lambda)(0, t) \\ & = & g\Phi^{\pm}_{\kappa_i}(\lambda)(t, 0)g^{-1}gP^{\pm}_{\kappa_i}(\lambda, 0)g^{-1}g\Phi^{\pm}_{\kappa_i}(\lambda)(0, t)g^{-1} \\ & = & g\Phi^{\pm}_{\kappa_i}(\lambda)(t, 0)P^{\pm}_{\kappa_i}(\lambda, 0)\Phi^{\pm}_{\kappa_i}(\lambda)(0, t)g^{-1} \\ & = & gP^{\pm}_{\kappa_i}(\lambda, t)g^{-1}. \end{array}$$

□

Recall that $\gamma^{\pm}_{\kappa_i}$ is situated within the fixed point space $\mathrm{Fix}_{\kappa_i} := \mathrm{Fix}G_{\gamma_{\kappa_i}}$ of the isotropy subgroup $G_{\gamma_{\kappa_i}} \subset G$, cf. Corollary 4.1.4. Then Lemma 4.3.6 and 4.3.7 imply that Fix_{κ_i} is invariant under the action of the transition matrix $\Phi^{\pm}_{\kappa_i}$ and the projection $P^{\pm}_{\kappa_i}$, since both objects commute with the linear representation for all elements $h \in G_{\gamma_{\kappa_i}}$. We record this statement in the following corollary.

Corollary 4.3.8. *Let $G_{\gamma_{\kappa_i}}$ be the isotropy subgroup of γ_{κ_i}. Then we find that $\Phi^{\pm}_{\kappa_i}$ and $P^{\pm}_{\kappa_i}$ leave $Fix_{\kappa_i} := FixG_{\gamma_{\kappa_i}}$ invariant.*

Proof. Since $\gamma^{\pm}_{\kappa_i} \subseteq \mathrm{Fix}_{\kappa_i}$ we have $h\gamma^{\pm}_{\kappa_i}(t) = \gamma^{\pm}_{\kappa_i}(t)$ for all $t \in \mathbb{R}$ and for all $h \in G_{\gamma_{\kappa_i}}$ and therefore

$$\begin{array}{rclcl} \Phi^{\pm}_{\kappa_i}(\lambda)(t, \tau) & = & \Phi^{\pm}_{h\kappa_i}(\lambda)(t, \tau) & = & h\Phi^{\pm}_{\kappa_i}(\lambda)(t, \tau)h^{-1}, \\ P^{\pm}_{\kappa_i}(\lambda, t) & = & P^{\pm}_{h\kappa_i}(\lambda, t) & = & hP^{\pm}_{\kappa_i}(\lambda, t)h^{-1}. \end{array}$$

□

In the following we show, that the transition matrices $\Psi^{\pm}_{\kappa_i}(\lambda)(\cdot, \cdot)$ of the adjoint equation (3.9) as well as the transposed projections $(P^{\pm}_{\kappa_i})^T$ also leave Fix_{κ_i} invariant. To this end recall that we denote by $(\cdot)^T$ the adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle$. This scalar product was chosen such that it is invariant in regard to the group action, cf. Remark 4.0.5. Hence we find

Lemma 4.3.9. *For all $h \in G$ we find that $h^T = h^{-1}$.*

Proof. For arbitrary $x, y \in \mathbb{R}^n$ and for all $h \in G$ we find due to the group invariance of the scalar product

$$\langle hx, y \rangle = \langle h^{-1}hx, h^{-1}y \rangle = \langle x, h^{-1}y \rangle.$$

Thus the adjoint of h in regard to the scalar product is $h^T = h^{-1}$. □

With this result we can show:

Lemma 4.3.10. *$\Psi^{\pm}_{\kappa_i}(\lambda)(\cdot,\cdot)$ and $P^{\pm}_{\kappa_i}(\lambda,\cdot)^T$ leave Fix_{κ_i} invariant.*

Proof. Due to Corollary 4.3.8 we find that $(\Phi^{\pm}_{\kappa_i})^T$ and $(P^{\pm}_{\kappa_i})^T$ commute with h^T for all $h \in G_{\gamma_{\kappa_i}}$:

$$\begin{aligned} h^T\Phi^{\pm}_{\kappa_i}(\lambda)(t,\tau)^T &= [\Phi^{\pm}_{\kappa_i}(\lambda)(t,\tau)h]^T &= [h\Phi^{\pm}_{\kappa_i}(\lambda)(t,\tau)]^T &= \Phi^{\pm}_{\kappa_i}(\lambda)(t,\tau)^T h^T, \\ h^T P^{\pm}_{\kappa_i}(\lambda)(t)^T &= [P^{\pm}_{\kappa_i}(\lambda)(t)h]^T &= [hP^{\pm}_{\kappa_i}(\lambda)(t)]^T &= P^{\pm}_{\kappa_i}(\lambda)(t)^T h^T. \end{aligned}$$

Thanks to Lemma 4.3.9 we find $h^T = h^{-1}$ for all $h \in G_{\gamma_{\kappa_i}}$ and hence $G_{\gamma_{\kappa_i}} = \{h^T \mid h \in G_{\gamma_{\kappa_i}}\}$. Thus $(\Phi^{\pm}_{\kappa_i})^T$ and $(P^{\pm}_{\kappa_i})^T$ commute with all elements of $G_{\gamma_{\kappa_i}}$.
Finally with $\Psi^{\pm}_{\kappa_i}(\lambda)(\tau,t) = \Phi^{\pm}_{\kappa_i}(\lambda)(t,\tau)^T$ we find $h^T\Psi^{\pm}_{\kappa_i}(\lambda)(\tau,t) = \Psi^{\pm}_{\kappa_i}(\lambda)(\tau,t)h^T$. □

The following Lemma provides the necessary information to show, that $\Phi^{\pm}_{\kappa_i}$ and $(\Phi^{\pm}_{\kappa_i})^T$ leave $\mathrm{Fix}^{\perp}_{\kappa_i}$ invariant as well.

Lemma 4.3.11. *If the flow of the linear differential equation $\dot{x} = A(t)x$ leaves the subspace U invariant then $U^{\perp}$ is invariant under the flow of the adjoint differential equation $\dot{x} = -A(t)^T x$.*

Proof. Let $\Phi(\cdot,\cdot)$ and $\Psi(\cdot,\cdot)$ denote the transition matrices of $\dot{x} = A(t)x$ and $\dot{x} = -A(t)^T x$, respectively. Then we have for all $x \in U$ and for all $s,t \in \mathbb{R}$ that $\Phi(t,s)x \in U$. Now, with any $y \in U^{\perp}$ we find for all $x \in U$ and for all $s,t \in \mathbb{R}$

$$0 = \langle y, \Phi(t,s)x\rangle = \left\langle \Phi(t,s)^T y, x\right\rangle = \langle \Psi(s,t)y, x\rangle .$$

Hence $\Psi(s,t)y \in U^{\perp}$ for all $s,t \in \mathbb{R}$. □

Corollary 4.3.12.

(i) $\Psi^{\pm}_{\kappa_i}(\lambda)(\cdot,\cdot)$ leaves $Fix^{\perp}_{\kappa_i}$ invariant,

(ii) $\Phi^{\pm}_{\kappa_i}(\lambda)(\cdot,\cdot)$ leaves $Fix^{\perp}_{\kappa_i}$ invariant.

Proof. (i) follows from Lemma 4.3.11 and Corollary 4.3.8, (ii) follows from Lemma 4.3.11 and 4.3.10. □

Finally it remains to show, that the projections $P^{\pm}_{\kappa_i}$ and $(P^{\pm}_{\kappa_i})^T$ leave $\mathrm{Fix}^{\perp}_{\kappa_i}$ invariant.

Lemma 4.3.13.

(i) $P^{\pm}_{\kappa_i}(\lambda,\cdot)$ leaves $Fix^{\perp}_{\kappa_i}$ invariant,

(ii) $P^{\pm}_{\kappa_i}(\lambda,\cdot)^T$ leaves $Fix^{\perp}_{\kappa_i}$ invariant.

Proof. Due to Corollary 4.3.12 it suffices to show the invariance for the time $t = 0$. The rest follows from transporting $P^{\pm}_{\kappa_i}$ and $(P^{\pm}_{\kappa_i})^T$ along $\gamma^{\pm}_{\kappa_i}$ by the transition matrices $\Phi^{\pm}_{\kappa_i}$ and $\Psi^{\pm}_{\kappa_i} = ((\Phi^{\pm}_{\kappa_i})^T)^{-1}$:

$$\begin{aligned} P^{\pm}_{\kappa_i}(\lambda,t) &= \Phi^{\pm}_{\kappa_i}(\lambda)(t,0)P^{\pm}_{\kappa_i}(\lambda,0)\Phi^{\pm}_{\kappa_i}(\lambda)(0,t), \\ P^{\pm}_{\kappa_i}(\lambda,t)^T &= \Phi^{\pm}_{\kappa_i}(\lambda)(0,t)^T P^{\pm}_{\kappa_i}(\lambda,0)^T \Phi^{\pm}_{\kappa_i}(\lambda)(t,0)^T. \end{aligned}$$

At $t = 0$ we have $\mathrm{im}P^{\pm}_{\kappa_i}(0) = W^{\mp}_{\kappa_i} \oplus Z_{\kappa_i}$ and due to Hypothesis (H4.7)(iii) $\ker P^{\pm}_{\kappa_i}(0) = W^{\pm}_{\kappa_i} \oplus U_{\kappa_i}$. Since $U_{\kappa_i} \oplus Z_{\kappa_i} \subseteq \mathrm{Fix}_{\kappa_i}$ we find that $\mathrm{Fix}^{\perp}_{\kappa_i} \subseteq W^{+}_{\kappa_i} \oplus W^{-}_{\kappa_i}$.

Therefore any element in $\mathrm{Fix}^{\perp}_{\kappa_i}$ can be expressed as $w^+ + w^-$, where $w^+ \in (W^{+}_{\kappa_i} \cap \mathrm{Fix}^{\perp}_{\kappa_i})$ and $w^- \in (W^{-}_{\kappa_i} \cap \mathrm{Fix}^{\perp}_{\kappa_i})$. Then we find

$$P^{\pm}_{\kappa_i}(0)(w^+ + w^-) = w^{\mp} \in \mathrm{Fix}^{\perp}_{\kappa_i}.$$

This proves (i).

In general we cannot assume that $W^{+}_{\kappa_i}$ and $W^{-}_{\kappa_i}$ are orthogonal. Thus, in order to prove (ii) we need to introduce two subspaces $V^{+}_{\kappa_i}$ and $V^{-}_{\kappa_i}$ such that $V^{+}_{\kappa_i}$ and $V^{-}_{\kappa_i}$ are subspaces of $W^{+}_{\kappa_i} \oplus W^{-}_{\kappa_i}$ with

$$W^{+}_{\kappa_i} \oplus V^{-}_{\kappa_i} = W^{+}_{\kappa_i} \oplus W^{-}_{\kappa_i} \quad \text{and} \quad W^{+}_{\kappa_i} \perp V^{-}_{\kappa_i},$$
$$W^{-}_{\kappa_i} \oplus V^{+}_{\kappa_i} = W^{+}_{\kappa_i} \oplus W^{-}_{\kappa_i} \quad \text{and} \quad W^{-}_{\kappa_i} \perp V^{+}_{\kappa_i}.$$

Then we find for $t = 0$ with Lemma 2.0.1 that

$$\begin{aligned} \mathrm{im}P^{\pm}_{\kappa_i}(0)^T &= (\ker P^{\pm}_{\kappa_i}(0))^{\perp} &= (W^{\pm}_{\kappa_i} \oplus U_{\kappa_i})^{\perp} &= V^{\mp}_{\kappa_i} \oplus Z_{\kappa_i} \\ \ker P^{\pm}_{\kappa_i}(0)^T &= (\mathrm{im}P^{\pm}_{\kappa_i}(0))^{\perp} &= (W^{\mp}_{\kappa_i} \oplus Z_{\kappa_i})^{\perp} &= V^{\pm}_{\kappa_i} \oplus U_{\kappa_i} \end{aligned}$$

Due to the choice of $V^{\pm}_{\kappa_i}$ we find $\mathrm{Fix}^{\perp}_{\kappa_i} \subseteq V^{+}_{\kappa_i} \oplus V^{-}_{\kappa_i}$. Then any element in $\mathrm{Fix}^{\perp}_{\kappa_i}$ takes the form $v^+ + v^- \in (V^{+}_{\kappa_i} \cap \mathrm{Fix}^{\perp}_{\kappa_i}) \oplus (V^{-}_{\kappa_i} \cap \mathrm{Fix}^{\perp}_{\kappa_i})$ and we obtain

$$P^{\pm}_{\kappa_i}(0)^T(v^+ + v^-) = v^{\mp} \in \mathrm{Fix}^{\perp}_{\kappa_i},$$

which proves (ii). □

4.3.2 Proof of Theorems 4.3.1 and 4.3.3

Now we begin with proving the Theorems 4.3.1 and 4.3.3. We do this in several steps that focus on the single terms $A_i(\lambda, \kappa)$, $B_i(\lambda, \kappa)$, $C_i(\lambda, \kappa)$ and $D_i(\lambda, \kappa, \omega_i)$. These terms can be found within the term $\boldsymbol{T}^{\mathbf{2}}_{\boldsymbol{\kappa_i}}$, cf. (3.64) and (3.65). To be more precise we find, cf. Section 3.4.7

$$\begin{aligned} \boldsymbol{T}^{\mathbf{21}}_{\boldsymbol{\kappa_i}} &= -e^{2\mu^s(\lambda)\omega_i} A_i(\lambda, \kappa) + \boldsymbol{\mathcal{R}}^{\mathbf{21}}_{\boldsymbol{i}}(\boldsymbol{\omega}, \lambda, \kappa), \\ \boldsymbol{T}^{\mathbf{23}}_{\boldsymbol{\kappa_i}} &= -e^{2\mu^s(\lambda)\omega_{i-1}} e^{2\mu^s(\lambda)\omega_i} C_i(\lambda, \kappa) + \boldsymbol{\mathcal{R}}^{\mathbf{23}}_{\boldsymbol{i}}(\boldsymbol{\omega}, \lambda, \kappa), \\ \boldsymbol{T}^{\mathbf{25}}_{\boldsymbol{\kappa_i}} &= -e^{4\mu^s(\lambda)\omega_i}[B_i(\lambda, \kappa) + D_i(\lambda, \kappa, \omega_i)] + \boldsymbol{\mathcal{R}}^{\mathbf{25}}_{\boldsymbol{i}}(\boldsymbol{\omega}, \lambda, \kappa). \end{aligned}$$

First we consider $A_i(\lambda, \kappa)$ in Lemma 4.3.14 which is based on [HJKL11, Proposition 3.7] and [Kno04, Theorem 2.1.9]. This lemma also is a basic step in proving equation (1.7). Lemma 4.3.20 below is dedicated to the terms $B_i(\lambda, \kappa)$ and $D_i(\lambda, \kappa, \omega_i)$ and Lemma 4.3.22 gives the term $C_i(\lambda, \kappa)$.

Lemma 4.3.14. *Assume Hypotheses (H4.1), (H4.3) - (H4.7). Let $P^{\pm}_{\kappa_i}$ be the projections associated with the exponential dichotomies of the variational equation (3.8) along $\gamma^{\pm}_{\kappa_i}(\lambda)(\cdot)$ as introduced in (3.17) and let $\tilde{P}_{\kappa_i}$ be the projection introduced in Lemma 3.3.2. Then there exist constants Ω and c in accordance to*

Theorem 3.2.2 such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ the following estimate applies:

$$\begin{aligned} \boldsymbol{T^{21}_{\kappa_i}} &:= -\Big\langle \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T \psi_{\kappa_i} \ , \ (id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))\big(\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) - p\big)\Big\rangle \\ &= -e^{2\mu^s(\lambda)\omega_i} A_i(\lambda,\kappa) + \boldsymbol{\mathcal{R}^{21}_i}(\boldsymbol{\omega},\lambda,\kappa) \end{aligned}$$

with

$$A_i(\lambda,\kappa) = \langle \eta^-_{\kappa_i}(\lambda), \eta^s_{\kappa_{i-1}}(\lambda)\rangle \qquad \textit{where} \qquad \begin{aligned} \eta^s_{\kappa_{i-1}}(\lambda) &\in Fix_{\kappa_{i-1}} \cap E_{D_1 f(p,\lambda)}(\mu^s(\lambda)), \\ \eta^-_{\kappa_i}(\lambda) &\in Fix_{\kappa_i} \cap [E_{D_1 f(p,\lambda)}(\sigma^c_{\mu^s}(\lambda))]^\perp. \end{aligned}$$

The residual term satisfies

$$\boldsymbol{\mathcal{R}^{21}_i}(\boldsymbol{\omega},\lambda,\kappa) = \begin{cases} O(e^{\max\{(\nu+1)\alpha^s, 2\alpha^s + \min\{2,\nu-1\}(\alpha^s-\alpha^u)\}\omega_i}), & \textit{if additionally (H4.8) applies,} \\ O(e^{\max\{\alpha^{ss}+\alpha^s, (\nu+1)\alpha^s, 3\alpha^s-\alpha^u\}\omega_i}), & \textit{else.} \end{cases}$$

Proof. The function $\gamma^+_{\kappa_{i-1}}(\lambda)(\cdot)$ is a solution within the stable manifold $W^s(p)$ of the hyperbolic equilibrium p. Recall that $p = 0$ and thus $\gamma^+_{\kappa_{i-1}}(t) - p = \gamma^+_{\kappa_{i-1}}(t)$. Further, the leading stable eigenvalue $\mu^s(\lambda) \in \sigma(D_1 f(p,\lambda))$ is real and semisimple and satisfies $\mathrm{Re}(\mu(\lambda)) < \alpha^{ss} < \mu^s(\lambda) < \alpha^s$ for all $\mu(\lambda) \in \sigma_{ss}(D_1 f(p,\lambda))$. Hence the Assumption (A2.1) is satisfied and we can apply the first part of Lemma 2.3.1 and obtain in combination with Remark 2.3.2

$$\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) = e^{\mu^s(\lambda)\omega_i}\eta^s_{\kappa_{i-1}}(\lambda) + \mathcal{R}^{A1}_{\kappa_{i-1}}(\omega_i) \quad \text{with} \quad \mathcal{R}^{A1}_{\kappa_{i-1}}(\omega_i) = O\big(e^{\max\{\alpha^{ss},\nu\alpha^s\}\omega_i}\big) \tag{4.9}$$

where $\eta^s_{\kappa_{i-1}}(\lambda) \in E(\mu^s(\lambda))$. Since $\gamma^+_{\kappa_{i-1}}(\lambda) \subseteq \mathrm{Fix}_{\kappa_{i-1}}$ we also find $\eta^s_{\kappa_{i-1}}(\lambda) \in \mathrm{Fix}_{\kappa_{i-1}}$. Due to Hypothesis (H4.5) we find that $\eta^s_{\kappa_{i-1}}(\lambda) \neq 0$, cf. Corollary 2.3.4.
Note, if Hypothesis (H4.8) applies we do not have strong leading eigenvalues. Hence in that case we find $\mathcal{R}^{A1}_{\kappa_{i-1}}(\omega_i) = O\big(e^{\nu\alpha^s\omega_i}\big)$, cf. Remark 2.3.3.
Additionally for ω_i sufficiently large we find due to Hypothesis (H4.7)(ii) that $\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) \in T_pW^s(p)$. The definition of the projection $P^+_{\kappa_{i-1}}$, cf. (3.17), further provides $\ker P^+_{\kappa_{i-1}}(\omega_i) = T_pW^s(p)$. Hence we find for ω_i sufficiently large

$$\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) = (id - P^+_{\kappa_{i-1}}(\omega_i))\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i).$$

Consulting Lemma 3.4.3 and Lemma 3.4.5 we find

$$(id - \tilde{P}_{\kappa_i}(\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i)) = P^-_{\kappa_i}(-\omega_i)(id - P^+_{\kappa_{i-1}}(\omega_i)) + \mathcal{R}^{A2}_{\kappa_i}(\omega_i) \tag{4.10}$$

with

$$\mathcal{R}^{A2}_{\kappa_i}(\omega_i) = \begin{cases} O\big(e^{\max\{\nu\alpha^s-\alpha^u, \min\{2,\nu-1\}(\alpha^s-\alpha^u)\}\omega_i}\big), & \text{if (H4.8) applies,} \\ O\big(e^{\max\{\nu\alpha^s, \alpha^s-\alpha^u\}\omega_i}\big), & \text{else.} \end{cases} \tag{4.11}$$

Hence we obtain with (4.9) and $p = 0$

$$\begin{aligned}(id - \tilde{P}_{\kappa_i}(\omega_i))\big(\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) - p\big) &= (id - \tilde{P}_{\kappa_i}(\omega_i))(id - P^+_{\kappa_{i-1}}(\omega_i))\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) \\ &= \big[P^-_{\kappa_i}(-\omega_i)(id - P^+_{\kappa_{i-1}}(\omega_i)) + \mathcal{R}^{A2}_{\kappa_i}(\omega_i)\big]\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) \\ &= P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) + \mathcal{R}^{A2}_{\kappa_i}(\omega_i)\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) \\ &= P^-_{\kappa_i}(-\omega_i)\big[e^{\mu^s(\lambda)\omega_i}\eta^s_{\kappa_{i-1}}(\lambda) + \mathcal{R}^{A1}_{\kappa_{i-1}}(\omega_i)\big] \\ &\quad + \mathcal{R}^{A2}_{\kappa_i}(\omega_i)\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i),\end{aligned}$$

that is

$$(id - \tilde{P}_{\kappa_i}(\omega_i))\big(\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) - p\big) = e^{\mu^s(\lambda)\omega_i}P^-_{\kappa_i}(-\omega_i)\eta^s_{\kappa_{i-1}}(\lambda) + \mathcal{R}^{A3}_{\kappa_i}(\omega_i) \tag{4.12}$$

with $\mathcal{R}^{A3}_{\kappa_i}(\omega_i)$ following from $\mathcal{R}^{A1}_{\kappa_{i-1}}(\omega_i)$, $\mathcal{R}^{A2}_{\kappa_i}(\omega_i)$, cf. (4.9) and (4.11), and $\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) = O(e^{\alpha^s\omega_i})$:

$$\left.\begin{aligned}\mathcal{R}^{A3}_{\kappa_i}(\omega_i) &= P^-_{\kappa_i}(-\omega_i)\mathcal{R}^{A1}_{\kappa_{i-1}}(\omega_i) + \mathcal{R}^{A2}_{\kappa_i}(\omega_i)\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) \\ &= \begin{cases} O(e^{\max\{\nu\alpha^s, \alpha^s + \min\{2,\nu-1\}(\alpha^s-\alpha^u)\}\omega_i}), & \text{if (H4.8) applies,} \\ O(e^{\max\{\alpha^{ss}, \nu\alpha^s, 2\alpha^s-\alpha^u\}\omega_i}), & \text{else.}\end{cases}\end{aligned}\right\} \tag{4.13}$$

Considering the left-hand side of the scalar product $\boldsymbol{T^{21}_{\kappa_i}}$ we find that $\Phi^-_{\kappa_i}(\lambda)(0,\cdot)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}$ solves the adjoint variational equation (3.9), that is

$$\dot{x} = -\big[D_1 f(\gamma^-_{\kappa_i}(\lambda)(t),\lambda)\big]^T x = -D_1 f(p,\lambda)^T x - \big[D_1 f(\gamma^-_{\kappa_i}(\lambda)(t),\lambda) - D_1 f(p,\lambda)\big]^T x.$$

Hence we can apply an equivalent assertion of the first part of Lemma 2.4.1 for solutions of linear perturbed equations that tend to zero as $t \to -\infty$. The leading unstable eigenvalue of $-D_1 f(p,\lambda)^T$ we find with $-\mu^s(\lambda)$ which is real and semisimple. The estimate $e^{\delta t}$ of the perturbation $[D_1 f(\gamma^-_{\kappa_i}(\lambda)(t),\lambda) - D_1 f(p,\lambda)]^T$ is given by (3.72) with $\delta = (\nu-1)\alpha^u$. This finally yields in combination with Remark 2.4.2

$$\Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i} = e^{\mu^s(\lambda)\omega_i}\eta^-_{\kappa_i}(\lambda) + \mathcal{R}^{A4}_{\kappa_i}(\omega_i) \tag{4.14}$$

where $\eta^-_{\kappa_i}(\lambda) \in E_{-D_1 f(p,\lambda)^T}(-\mu^s(\lambda)) = [E_{D_1 f(p,\lambda)}(\sigma^c_{\mu^s}(\lambda))]^\perp$, cf. Lemma 2.0.2, and

$$\mathcal{R}^{A4}_{\kappa_i}(\omega_i) = O\left(e^{\max\{\alpha^{ss}, \alpha^s-(\nu-1)\alpha^u\}\omega_i}\right). \tag{4.15}$$

Again, if Hypothesis (H4.8) applies, we obtain the estimate $\mathcal{R}^{A4}_{\kappa_i}(\omega_i) = O\left(e^{(\alpha^s-(\nu-1)\alpha^u)\omega_i}\right)$, cf. Remark 2.4.3.

According to Lemma 4.3.10 we find that the projection $P^-_{\kappa_i}(\lambda,\cdot)^T$ and the transition matrix $\Phi^-_{\kappa_i}(\lambda)(\cdot,\cdot)^T$ leave Fix_{κ_i} invariant. Hence we find with $\psi_{\kappa_i} \in Z_{\kappa_i} \subset \mathrm{Fix}_{\kappa_i}$ that $\Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i} \in \mathrm{Fix}_{\kappa_i}$ and also $\eta^-_{\kappa_i}(\lambda) \in \mathrm{Fix}_{\kappa_i}$. Further we find that $\eta^-_{\kappa_i}(\lambda)$ is different from zero, if $\psi_{\kappa_i} \notin \mathrm{im}(Q^-_{uu}(0)) = E^{uu}_{-D_1 f_{\kappa_i}(\gamma^-(\cdot),\lambda)^T}(0)$, cf. Corollary 2.4.4 applied to the differential equation $\dot{x} = -[D_1 f(\gamma^-_{\kappa_i}(\cdot),\lambda)]^T x$. Thanks to Hypothesis (H4.6) or more precisely Remark 4.1.5 this condition is fulfilled and $\eta^-_{\kappa_i}(\lambda) \neq 0$ holds true.

Combining (4.12) and (4.14) yields

$$\begin{aligned}
-\boldsymbol{T}^{\mathbf{21}}_{\boldsymbol{\kappa_i}} = & \Big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T \psi_{\kappa_i}\ ,\ (id - \tilde{P}_{\kappa_i}(\omega_i))\big(\gamma^+_{\kappa_{i-1}}(\omega_i) - p\big)\Big\rangle \\
= & \ e^{\mu^s(\lambda)\omega_i}\Big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T \psi_{\kappa_i}, P^-_{\kappa_i}(-\omega_i)\eta^s_{\kappa_{i-1}}(\lambda)\Big\rangle + \big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, \mathcal{R}^{A3}_{\kappa_i}(\omega_i)\big\rangle \\
= & \ e^{\mu^s(\lambda)\omega_i}\Big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T \psi_{\kappa_i}, \eta^s_{\kappa_{i-1}}(\lambda)\Big\rangle + \big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, \mathcal{R}^{A3}_{\kappa_i}(\omega_i)\big\rangle \\
= & \ e^{\mu^s(\lambda)\omega_i}\Big\langle e^{\mu^s(\lambda)\omega_i}\eta^-_{\kappa_i}(\lambda) + \mathcal{R}^{A4}_{\kappa_i}(\omega_i), \eta^s_{\kappa_{i-1}}(\lambda)\Big\rangle + \big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, \mathcal{R}^{A3}_{\kappa_i}(\omega_i)\big\rangle \\
= & \ e^{2\mu^s(\lambda)\omega_i}\Big\langle \eta^-_{\kappa_i}(\lambda), \eta^s_{\kappa_{i-1}}(\lambda)\Big\rangle + \boldsymbol{\mathcal{R}}^{\mathbf{21}}_{\boldsymbol{i}}(\boldsymbol{\omega},\lambda,\kappa).
\end{aligned}$$

From the second to the third line we moved $P^-_{\kappa_i}(\lambda,-\omega_i)$ from the right-hand side of the scalar product to the left-hand side and took into consideration that the projection $P^-_{\kappa_i}(\lambda,\cdot)^T$ is idempotent and commutes with $\Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T$.

The residual term $\boldsymbol{\mathcal{R}}^{\mathbf{21}}_{\boldsymbol{i}}(\boldsymbol{\omega},\lambda,\kappa)$ we obtain from $\mathcal{R}^{A3}_{\kappa_i}(\omega_i)$ and $\mathcal{R}^{A4}_{\kappa_i}(\omega_i)$, cf. (4.13) and (4.15), under consideration that $\eta^s_{\kappa_{i-1}}(\lambda)$ is bounded, $\mu^s(\lambda) < \alpha^s$ and $\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i} = O(e^{\alpha^s\omega_i})$:

$$\begin{aligned}
\boldsymbol{\mathcal{R}}^{\mathbf{21}}_{\boldsymbol{i}}(\boldsymbol{\omega},\lambda,\kappa) &= e^{\mu^s(\lambda)\omega_i}\Big\langle \mathcal{R}^{A4}_{\kappa_i}(\omega_i), \eta^s_{\kappa_{i-1}}(\lambda)\Big\rangle + \big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, \mathcal{R}^{A3}_{\kappa_i}(\omega_i)\big\rangle \\
&= \begin{cases} O\big(e^{\max\{(\nu+1)\alpha^s, 2\alpha^s + \min\{2,\nu-1\}(\alpha^s-\alpha^u)\}\omega_i}\big), & \text{if (H4.8) applies,} \\ O\big(e^{\max\{\alpha^s+\alpha^{ss}, (\nu+1)\alpha^s, 3\alpha^s-\alpha^u\}\omega_i}\big), & \text{else.} \end{cases}
\end{aligned}$$

This concludes the proof. □

Lemma 4.3.14 in combination with the representation of the jump ξ_i, cf. (3.65), and the estimates presented in Section 3.4.7 already provides the estimation of the jump ξ_i that is used in the determination equation (1.7) presented in the introduction.

Remark 4.3.15. *If we assume $\mu^u(\lambda)$ being real and semisimple as well we find analogously to $\boldsymbol{T}^{\mathbf{21}}_{\boldsymbol{\kappa_i}}$ that*

$$\begin{aligned}
\boldsymbol{T}^{\mathbf{11}}_{\boldsymbol{\kappa_i}} &:= \Big\langle \Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})^T P^+_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ \tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1})\big(\gamma^-_{\kappa_{i+1}}(\lambda)(-\omega_{i+1}) - p\big)\Big\rangle \\
&= e^{-2\mu^u(\lambda)\omega_{i+1}}\langle \eta^+_{\kappa_i}(\lambda), \eta^u_{\kappa_{i+1}}(\lambda)\rangle + \boldsymbol{\mathcal{R}}^{\mathbf{11}}_{\boldsymbol{i}}(\boldsymbol{\omega},\lambda,\kappa)
\end{aligned}$$

with

$$\eta^u_{\kappa_{i+1}}(\lambda) \in Fix_{\kappa_{i+1}} \cap E(\mu^u(\lambda)) \qquad \text{and} \qquad \eta^+_{\kappa_i}(\lambda) \in Fix_{\kappa_i} \cap E(\sigma^c_{\mu^u}(\lambda))^\perp$$

and

$$\boldsymbol{\mathcal{R}}^{\mathbf{11}}_{\boldsymbol{i}}(\boldsymbol{\omega},\lambda,\kappa) = \begin{cases} O\big(e^{\max\{(\nu-1)\alpha^s-2\alpha^u, -2\alpha^u+\min\{2,\nu-1\}(\alpha^s-\alpha^u)\}\omega_{i+1}}\big), & \text{if (H4.8) applies,} \\ O\big(e^{\max\{-\alpha^{uu}-\alpha^u, (\nu-1)\alpha^s-2\alpha^u, \alpha^s-3\alpha^u\}\omega_{i+1}}\big), & \text{else.} \end{cases}$$

Remark 4.3.16. *Additionally assume Hypothesis (H4.2). If $i \in J_\kappa$, that is Fix_{κ_i} is orthogonal to $Fix_{\kappa_{i-1}}$, we find that not only the term $A_i(\lambda,\kappa) := \langle \eta^-_{\kappa_i}(\lambda), \eta^s_{\kappa_{i-1}}(\lambda)\rangle$ disappears, but the whole expression*

$$\Big\langle \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ P^-_{\kappa_i}(\lambda,-\omega_i)\big(\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i) - p\big)\Big\rangle = 0.$$

This is due to the fact that the left-hand side of the scalar product is situated within Fix_{κ_i}, $\gamma^+_{\kappa_{i-1}}$ is an element of $Fix_{\kappa_{i-1}}$ and the projection $P^-_{\kappa_i}(\lambda,-\omega_i)$ can be shifted from the right-hand side to the other side *of the scalar product where it already stands. Hence we find for $i \in J_\kappa$, cf.* (4.10) *and* (4.11),

$$\begin{aligned}
-\boldsymbol{T^{21}_{\kappa_i}} &= \Big\langle \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ \big((id-\tilde{P}_{\kappa_i}(\lambda,\omega_i)) - P^-_{\kappa_i}(\lambda,-\omega_i)\big)(\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i)-p)\Big\rangle \\
&= \Big\langle \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}, \mathcal{R}^{A2}_{\kappa_i}(\omega_i)\big(\gamma^+_{\kappa_{i-1}}(\lambda)(\omega_i)-p\big)\Big\rangle \\
&= \begin{cases} O\big(e^{\max\{(\nu+2)\alpha^s-\alpha^u, 2\alpha^s+\min\{2,\nu-1\}(\alpha^s-\alpha^u)\}\omega_i}\big), & \text{if (H4.8) applies,} \\ O\big(e^{\max\{(\nu+2)\alpha^s, 3\alpha^s-\alpha^u\}\omega_i}\big), & \text{else.} \end{cases}
\end{aligned}$$

Analogously we find for $i+1 \in J_\kappa$, that is $Fix_{\kappa_i} \perp Fix_{\kappa_{i+1}}$, that

$$\begin{aligned}
\boldsymbol{T^{11}_{\kappa_i}} &= \Big\langle \Phi^+_{\kappa_i}(\lambda)(0,\omega_{i+1})^T P^+_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ \big(\tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1}) - P^+_{\kappa_i}(\lambda,\omega_{i+1})\big)\big(\gamma^-_{\kappa_{i+1}}(\lambda)(-\omega_{i+1})-p\big)\Big\rangle \\
& \begin{cases} O\big(e^{\max\{\nu\alpha^s-3\alpha^u, (\nu-1)\alpha^s-(\nu+1)\alpha^u, 2\alpha^s-4\alpha^u\}\omega_{i+1}}\big), & \text{if (H4.8) applies,} \\ O\big(e^{\max\{\nu\alpha^s-2\alpha^u, \alpha^s-3\alpha^u\}\omega_{i+1}}\big), & \text{else.} \end{cases}
\end{aligned}$$

Before studying the term $B_i(\lambda,\kappa)$ we state a lemma concerning the representation of $\Phi^+_{\kappa_i}(\lambda)(t,s)(id - P^+_{\kappa_i}(\lambda,s))$ and $\Phi^-_{\kappa_i}(\lambda)(s,t)P^-_{\kappa_i}(\lambda,t)$.

Lemma 4.3.17. *Assume Hypotheses (H4.1) and (H4.3). Let $P^\pm_{\kappa_i}$ be the projections associated with the exponential dichotomies of the variational equation* (3.8) *along $\gamma^\pm_{\kappa_i}(\lambda)(\cdot)$ as introduced in* (3.17)*. There exist linear time-depending operators $S^+_{\kappa_i}(\lambda,s)$ and $R^-_{\kappa_i}(\lambda,s)$ with*

$$\begin{aligned}
imS^+_{\kappa_i}(\lambda,s) &= E(\mu^s(\lambda)), & kerS^+_{\kappa_i}(\lambda,s) &= imP^+_{\kappa_i}(\lambda,s) \oplus T_{\gamma^+_{\kappa_i}(s)}\mathcal{F}^{ss}(\gamma^+_{\kappa_i}(s)), \\
kerR^-_{\kappa_i}(\lambda,s) &= E(\sigma^c_{\mu^s}(\lambda)) & imR^-_{\kappa_i}(\lambda,s) &= imP^-_{\kappa_i}(\lambda,s) \cap T_{\gamma^-_{\kappa_i}(s)}W^{ls,u}(p),
\end{aligned}$$

such that

$$\begin{aligned}
\Phi^+_{\kappa_i}(\lambda)(t,s)(id - P^+_{\kappa_i}(\lambda,s)) &= e^{\mu^s(t-s)}S^+_{\kappa_i}(\lambda,s) + \mathcal{R}^S_{\kappa_i}(\lambda)(t,s), & t \geq s \geq 0, \\
\Phi^-_{\kappa_i}(\lambda)(s,t)P^-_{\kappa_i}(\lambda,t) &= e^{-\mu^s(t-s)}R^-_{\kappa_i}(\lambda,s) + \mathcal{R}^R_{\kappa_i}(\lambda)(t,s), & t \leq s \leq 0.
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{R}^S_{\kappa_i}(\lambda)(t,s) &= O(\ e^{\alpha^{ss}(t-s)} + \ e^{\alpha^s(t-s)}e^{(\nu-1)\alpha^s t}), \\
\mathcal{R}^R_{\kappa_i}(\lambda)(t,s) &= O(e^{-\alpha^{ss}(t-s)} + e^{-\alpha^s(t-s)}e^{(\nu-1)\alpha^u t}).
\end{aligned}$$

Proof. Due to Hypothesis (H4.3) the leading stable eigenvalue is real and semisimple. Thus the existence and representation of the operator $S^+_{\kappa_i}(\lambda,s)$ as well as the corresponding estimates immediately follow from Lemma 2.5.2 and Remark 2.5.3(*iii*) with $\delta = (\nu-1)\alpha^s$, cf. (3.71). To this end recall that $\mathrm{im}P^+_{ss}(s) = T_{\gamma^+_{\kappa_i}(s)}\mathcal{F}^{ss}(\gamma^+_{\kappa_i}(s))$, cf. Lemmata 2.2.7 and 2.2.10.
Each assertion regarding the operator $R^-_{\kappa_i}(\lambda,s)$ follow from Remark 2.5.4 with $\delta = (\nu-1)\alpha^u$, cf. (3.72). Again consult Lemmata 2.2.7 and 2.2.10 for $[\mathrm{im}Q^-_{uu}(s)]^\perp = T_{\gamma^-_{\kappa_i}(s)}W^{ls,u}(p)$. □

Remark 4.3.18. *If additionally Hypothesis (H4.8) applies we obtain from Remark 2.5.3(i) that*

$$kerS^+_{\kappa_i}(\lambda,s) = imP^+_{\kappa_i}(\lambda,s), \qquad imR^-_{\kappa_i}(\lambda,s) = imP^-_{\kappa_i}(\lambda,s),$$

and

$$\begin{aligned}\mathcal{R}^S_{\kappa_i}(\lambda)(t,s) &= O(e^{\alpha^s(t-s)}e^{(\nu-1)\alpha^s t}),\\ \mathcal{R}^R_{\kappa_i}(\lambda)(t,s) &= O(e^{-\alpha^s(t-s)}e^{(\nu-1)\alpha^u t}).\end{aligned}$$

Remark 4.3.19. *The properties of the transition matrices $\Phi^{\pm}_{\kappa_i}$ and the projections $P^{\pm}_{\kappa_i}$ we have deduced in Section 4.3.1 pass on to the operators $S^{\pm}_{\kappa_i}$ and $R^{\pm}_{\kappa_i}$. That is to say we find for any $g \in G$*

$$\begin{aligned}S^{+}_{g\kappa_i}(\lambda,s) &= gS^{+}_{\kappa_i}(\lambda,s)g^{-1},\\ R^{-}_{g\kappa_i}(\lambda,s) &= gR^{-}_{\kappa_i}(\lambda,s)g^{-1}.\end{aligned}$$

Hence $S^{+}_{\kappa_i}$ and $R^{-}_{\kappa_i}$ leave the fixed point space Fix_{κ_i} and $Fix^{\perp}_{\kappa_i}$ invariant.

Proof. Due to Lemmata 4.3.6 and 4.3.7 we know that $\Phi^{\pm}_{g\kappa_i}(\lambda)(t,s) = g\Phi^{\pm}_{\kappa_i}(\lambda)(t,s)g^{-1}$ and $P^{\pm}_{g\kappa_i}(\lambda,s) = gP^{\pm}_{\kappa_i}(\lambda,s)g^{-1}$ for all $g \in G$. Hence for fixed $s > 0$ a simple comparing of coefficients

$$\begin{aligned}e^{\mu^s(t-s)}S^{+}_{g\kappa_i}(\lambda,s) + h.o.t. &= \Phi^{+}_{g\kappa_i}(\lambda)(t,s)(id - P^{+}_{g\kappa_i}(\lambda,s))\\ &= g\Phi^{+}_{\kappa_i}(\lambda)(t,s)(id - P^{+}_{\kappa_i}(\lambda,s))g^{-1} \;=\; e^{\mu^s(t-s)}gS^{+}_{\kappa_i}(\lambda,s)g^{1} + h.o.t.\end{aligned}$$

yields $S^{+}_{g\kappa_i}(\lambda,s) = gS^{+}_{\kappa_i}(\lambda,s)g^{-1}$. Analogously we obtain $R^{-}_{g\kappa_i}(\lambda,s) = gR^{-}_{\kappa_i}(\lambda,s)g^{-1}$. Therefore $S^{+}_{\kappa_i}(\lambda,s)$ and $R^{-}_{\kappa_i}(\lambda,s)$ leave Fix_{κ_i} invariant.
Further, since $S^{+}_{\kappa_i}$ and $R^{-}_{\kappa_i}$ commute with all group elements of the subgroup G_{κ_i} we find that $(S^{+}_{\kappa_i})^T$ and $(R^{-}_{\kappa_i})^T$ commute with all elements of G_{κ_i}. Hence $(S^{+}_{\kappa_i})^T$ and $(R^{-}_{\kappa_i})^T$ leave Fix_{κ_i} invariant as well and consequently $S^{+}_{\kappa_i}$ and $R^{-}_{\kappa_i}$ leave $\mathrm{Fix}^{\perp}_{\kappa_i}$ invariant. ☐

Lemma 4.3.20. *Assume Hypotheses (H4.1) - (H4.7) and let $\nu \geq 3$. Let further $i \in J_\kappa$, that is $Fix_{\kappa_{i-1}} \perp Fix_{\kappa_i}$. Then there exist constants Ω and c such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ we find the representation*

$$\begin{aligned}\boldsymbol{T^{25}_{\kappa_i}} &:= -\left\langle \psi_{\kappa_i}\, ,\, \int_{-\omega_i}^{0} \Phi^{-}_{\kappa_i}(\lambda)(0,s)P^{-}_{\kappa_i}(\lambda,s)h^{-}_{\kappa_i}(s,v^{-}_i(s),\lambda)ds \right\rangle\\ &= -e^{4\mu^s(\lambda)\omega_i}[B_i(\lambda,\kappa) + D_i(\lambda,\kappa,\omega_i)] + \boldsymbol{\mathcal{R}^{25}_i}(\boldsymbol{\omega},\lambda,\kappa)\end{aligned}$$

with $B_i(\lambda,\kappa)$ taking the form of the improper integral

$$B_i(\lambda,\kappa) := \frac{1}{2}\int_{-\infty}^{0} e^{2\mu^s(\lambda)s}\left\langle \Phi^{-}_{\kappa_i}(\lambda)(0,s)^T P^{-}_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}, D_1^2 f(\gamma^{-}_{\kappa_i}(\lambda)(s),\lambda)\left[R^{-}_{\kappa_i}(\lambda,s)\eta^{s}_{\kappa_{i-1}}(\lambda)\right]^2\right\rangle ds,$$

where $R^{-}_{\kappa_i}$ is given by Lemma 4.3.17, and $D_i(\lambda,\kappa,\omega_i)$ taking the form

$$D_i(\lambda,\kappa,\omega_i) := \begin{cases} \frac{1}{6}\int_{-\omega_i}^{0} e^{-4\mu^s(\lambda)\omega_i}\langle \Phi^{-}_{\kappa_i}(\lambda)(0,s)^T P^{-}_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}, & \\ \quad D_1^3 f(\gamma^{-}_{\kappa_i}(\lambda)(s),\lambda)\left[\Phi^{-}_{\kappa_i}(\lambda)(s,-\omega_i)P^{-}_{\kappa_i}(\lambda,-\omega_i)\gamma^{+}_{\kappa_{i-1}}(\lambda)(\omega_i)\right]^3\rangle ds, & \text{if } \nu = 3\\ 0, & \text{if } \nu > 3,\end{cases}$$

and the residual term satisfying

$$\boldsymbol{\mathcal{R}_i^{25}}(\boldsymbol{\omega}, \lambda, \kappa) = \begin{cases} \left.\begin{aligned} &O\big(e^{4\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{-\alpha^u\omega_{i+1}}] + e^{(3\alpha^s-(\nu-2)\alpha^u)\omega_i} \\ &\quad + e^{2\alpha^s\omega_i}e^{-2\alpha^u\omega_{i+1}} + e^{-6\alpha^u\omega_{i+1}}\big), \end{aligned}\right\} & \text{if (H4.8) applies} \\ \left.\begin{aligned} &O\big(e^{4\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{(\alpha^s-\alpha^u)/2\omega_{i+1}}] + e^{(3\alpha^s-(\nu-2)\alpha^u)\omega_i} \\ &\quad + e^{(3\alpha^s+\alpha^{ss})\omega_i} + e^{2\alpha^s\omega_i}e^{(\alpha^s-\alpha^u)\omega_{i+1}} + e^{3(\alpha^s-\alpha^u)\omega_{i+1}}\big), \end{aligned}\right\} & \text{else.} \end{cases}$$

Proof. In the following we omit the dependency of λ in our notation. Consulting Lemma 3.4.12 we see that the integral can be expressed as

$$\int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)h^-_{\kappa_i}(s, v_i^-(s))ds = {}^1\!/\!_2 I^2_{\kappa_i}(\boldsymbol{\omega}, \kappa) + {}^1\!/\!_6 I^3_{\kappa_i}(\boldsymbol{\omega}, \kappa) + \mathcal{R}^{B1}_{\kappa_i}(\boldsymbol{\omega}, \kappa)$$

where

$$I^k_{\kappa_i}(\boldsymbol{\omega}, \kappa) = \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D^k f(\gamma^-_{\kappa_i}(s))[v_i^{-,u}(s)]^k ds, \tag{4.16}$$

$k = 2, 3$ and

$$\mathcal{R}^{B1}_{\kappa_i}(\boldsymbol{\omega}, \kappa) = O\left(e^{5\alpha^s\omega_i} + e^{4\alpha^s\omega_i}e^{\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}} + e^{6\alpha^w\omega_{i+1}}\right). \tag{4.17}$$

Hence

$$\begin{aligned} \langle\psi_{\kappa_i}, \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)h^-_{\kappa_i}(s, v_i^-(s))ds\rangle &= {}^1\!/\!_2\langle\psi_{\kappa_i}, I^2_{\kappa_i}(\boldsymbol{\omega}, \kappa)\rangle + {}^1\!/\!_6\langle\psi_{\kappa_i}, I^3_{\kappa_i}(\boldsymbol{\omega}, \kappa)\rangle \\ &\quad + \langle\psi_{\kappa_i}, \mathcal{R}^{B1}_{\kappa_i}(\boldsymbol{\omega}, \kappa)\rangle. \end{aligned} \tag{4.18}$$

Before examining the integrals $I^k_{\kappa_i}$, $k = 2, 3$ we consider the term $v_i^{-,u}$. From (3.36) we obtain by replacing 0 by t

$$v_i^{-,u}(t) := P^-_{\kappa_i}(t)v_i^-(t) = \Phi^-_{\kappa_i}(t, -\omega_i)P^-_{\kappa_i}(-\omega_i)a_i^- + \mathcal{R}^{B2}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(t) \tag{4.19}$$

with, cf. (3.107),

$$\left.\begin{aligned} \mathcal{R}^{B2}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(t) &= \int_{-\omega_i}^{t} \Phi^-_{\kappa_i}(t,s)P^-_{\kappa_i}(s)h^-_{\kappa_i}(s, v_i^-(s))ds \\ &= O(e^{\alpha^s(2\omega_i+t)}(e^{\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}})) + O(e^{4\alpha^w\omega_{i+1}}). \end{aligned}\right\} \tag{4.20}$$

Recall the representation of a_i^- given in (3.48) where d_i is given in (3.13). This provides

$$\Phi^-_{\kappa_i}(t, -\omega_i)P^-_{\kappa_i}(-\omega_i)a_i^- = \Phi^-_{\kappa_i}(t, -\omega_i)P^-_{\kappa_i}(-\omega_i)(id - \tilde{P}_{\kappa_i}(\omega_i))\gamma^+_{\kappa_{i-1}}(\omega_i) + \mathcal{R}^{B3}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(t) \tag{4.21}$$

with

$$\left.\begin{aligned} \mathcal{R}^{B3}_{\kappa_i}(\boldsymbol{\omega}, \lambda, \kappa)(t) = -\Phi^-_{\kappa_i}(t, -\omega_i)P^-_{\kappa_i}(-\omega_i)(id - \tilde{P}_{\kappa_i}(\omega_i))\big[\gamma^-_{\kappa_i}(-\omega_i) + (id - P^-_{\kappa_i}(-\omega_i))v_i^-(-\omega_i) \\ -(id - P^+_{\kappa_{i-1}}(\omega_i))v^+_{i-1}(\omega_i)\big] \end{aligned}\right\} \tag{4.22}$$

Applying (4.10) on the first term on the right-hand side of (4.21) we find

$$\Phi^-_{\kappa_i}(t, -\omega_i)P^-_{\kappa_i}(-\omega_i)(id - \tilde{P}_{\kappa_i}(\omega_i))\gamma^+_{\kappa_{i-1}}(\omega_i) = \Phi^-_{\kappa_i}(t, -\omega_i)P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i) + \mathcal{R}^{B4}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(t) \tag{4.23}$$

where we obtain with $\nu \geq 3$

$$\left.\begin{aligned} \mathcal{R}^{B4}_{\kappa_i}(\boldsymbol{\omega},\kappa)(t) &= \Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)\mathcal{R}^{A2}_{\kappa_i}(\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i) \\ &\leq \begin{cases} O\big(e^{\alpha^s(\omega_i+t)}e^{\max\{(\nu+1)\alpha^s-\alpha^u,3\alpha^s-2\alpha^u\}\omega_i}\big), & \text{if (H4.8) applies,} \\ O\big(e^{\alpha^s(\omega_i+t)}e^{\max\{(\nu+1)\alpha^s,2\alpha^s-\alpha^u\}\omega_i}\big), & \text{else.} \end{cases} \end{aligned}\right\} \tag{4.24}$$

Here we applied the estimates for $\gamma^+_{\kappa_{i-1}}$, (3.79), $\mathcal{R}^{A2}_{\kappa_i}$, (4.11) and the exponential dichotomy (3.16) with $-\alpha^-_{\kappa_i} = \alpha^s$, cf. (3.18). With (4.9) and Lemma 4.3.17 we obtain for the first term on the right-hand side of (4.23)

$$\left.\begin{aligned} \Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i) &= \big[e^{\mu^s(\omega_i+t)}R^-_{\kappa_i}(t) + \mathcal{R}^R_{\kappa_i}(-\omega_i,t)\big]\big[e^{\mu^s\omega_i}\eta^s_{\kappa_{i-1}} + \mathcal{R}^{A1}_{\kappa_{i-1}}(\omega_i)\big] \\ &= e^{\mu^s(2\omega_i+t)}R^-_{\kappa_i}(t)\eta^s_{\kappa_{i-1}} + \mathcal{R}^{B5}_{\kappa_i}(\boldsymbol{\omega},\kappa)(t) \end{aligned}\right\} \tag{4.25}$$

with the residual term satisfying, cf. (4.9) and Lemma 4.3.17 for the residuals $\mathcal{R}^{A1}_{\kappa_{i-1}}$ and $\mathcal{R}^R_{\kappa_i}$,

$$\left.\begin{aligned} \mathcal{R}^{B5}_{\kappa_i}(\boldsymbol{\omega},\kappa)(t) &= \big[e^{\mu^s(\omega_i+t)}R^-_{\kappa_i}(t) + \mathcal{R}^R_{\kappa_i}(-\omega_i,t)\big]\mathcal{R}^{A1}_{\kappa_{i-1}}(\omega_i) + e^{\mu^s\omega_i}\mathcal{R}^R_{\kappa_i}(-\omega_i,t)\eta^s_{\kappa_{i-1}} \\ &= \begin{cases} O\big(e^{\alpha^s(\omega_i+t)}e^{\nu\alpha^s\omega_i}\big), & \text{if (H4.8) applies,} \\ O\big(e^{\alpha^s(\omega_i+t)}e^{\max\{\alpha^{ss},\nu\alpha^s\}\omega_i} + e^{\alpha^{ss}(\omega_i+t)}e^{\alpha^s\omega_i}\big), & \text{else.} \end{cases} \end{aligned}\right\} \tag{4.26}$$

In the following we will estimate the single terms of $\mathcal{R}^{B3}_{\kappa_i}(\boldsymbol{\omega},\kappa)$ in (4.22) by using exponential dichotomies (3.16) with $-\alpha^-_{\kappa_i} = \alpha^s$, Lemma 3.4.3, equation (3.79) and Lemma 3.4.9.

$$\begin{aligned} &\Big\|\Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)(id-\tilde{P}_{\kappa_i}(\omega_i))\gamma^-_{\kappa_i}(-\omega_i)\Big\| \\ &\quad\leq \big\|\Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)\big\|\,\Big\|(id-\tilde{P}_{\kappa_i}(\omega_i))(id-P^-_{\kappa_i}(-\omega_i))\Big\|\,\big\|\gamma^-_{\kappa_i}(-\omega_i)\big\| \\ &\quad\leq e^{\alpha^s(\omega_i+t)}\cdot e^{\max\{(\nu-1)\alpha^s,(\alpha^s-\alpha^u)/2\}\omega_i}\cdot e^{-\alpha^u\omega_i} \end{aligned}$$

$$\begin{aligned} &\Big\|\Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)(id-\tilde{P}_{\kappa_i}(\omega_i))(id-P^+_{\kappa_{i-1}}(\omega_i))v^+_{i-1}(\omega_i)\Big\| \\ &\quad\leq \big\|\Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)\big\|\,\|(id-\tilde{P}_{\kappa_i}(\omega_i))\|\,\Big\|(id-P^+_{\kappa_{i-1}}(\omega_i))v^+_{i-1}(\omega_i)\Big\| \\ &\quad\leq e^{\alpha^s(\omega_i+t)}\cdot e^{\alpha^s\omega_i}\big[e^{2\alpha^w\omega_i} + e^{2\alpha^s\omega_{i-1}}\big] \end{aligned}$$

$$\begin{aligned} &\Big\|\Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)(id-\tilde{P}_{\kappa_i}(\omega_{i1}))(id-P^-_{\kappa_i}(-\omega_i))v^-_i(-\omega_i)\Big\| \\ &\quad\leq \big\|\Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)\big\|\,\Big\|(id-\tilde{P}_{\kappa_i}(\omega_i))(id-P^-_{\kappa_i}(-\omega_i))\Big\|\,\big\|(id-P^-_{\kappa_i}(-\omega_i))v^-_i(-\omega_i)\big\| \\ &\quad\leq e^{\alpha^s(\omega_i+t)}\cdot e^{\max\{(\nu-1)\alpha^s,(\alpha^s-\alpha^u)/2\}\omega_i}\cdot e^{\alpha^w\omega_i}\big[e^{2\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}\big] \end{aligned}$$

Summarizing we have

$$\mathcal{R}^{B3}_{\kappa_i}(\boldsymbol{\omega},\kappa)(t) = O(e^{\alpha^s(2\omega_i+t)}[e^{\max\{(\nu-1)\alpha^s,-\alpha^u\}\omega_i} + e^{2\alpha^s\omega_{i-1}}]). \tag{4.27}$$

Equation (4.19) in combination with (4.21), (4.23) and (4.25) then yields

$$v^{-,u}_i(t) = e^{\mu^s(2\omega_i+t)}R^-_{\kappa_i}(t)\eta^s_{\kappa_{i-1}} + \mathcal{R}^{B6}_{\kappa_i}(\boldsymbol{\omega},\kappa)(t) \tag{4.28}$$

with, cf. (4.20), (4.27), (4.24) and (4.26),

$$\left.\begin{aligned} \mathcal{R}^{B6}_{\kappa_i}(\boldsymbol{\omega},\kappa)(t) &= \mathcal{R}^{B2}_{\kappa_i}(\boldsymbol{\omega},\kappa)(t) + \mathcal{R}^{B3}_{\kappa_i}(\boldsymbol{\omega},\kappa)(t) + \mathcal{R}^{B4}_{\kappa_i}(\boldsymbol{\omega},\kappa)(t) + \mathcal{R}^{B5}_{\kappa_i}(\boldsymbol{\omega},\kappa)(t) \\ &= O(e^{\alpha^s(2\omega_i+t)}[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}]) + O(e^{4\alpha^w\omega_{i+1}}) \\ &\quad + \begin{cases} 0, & \text{if (H4.8) applies,} \\ O(e^{\alpha^{ss}(\omega_i+t)}e^{\alpha^s\omega_i}) + O(e^{\alpha^s(\omega_i+t)}e^{\alpha^{ss}\omega_i}), & \text{else.} \end{cases} \end{aligned}\right\} \quad (4.29)$$

Now we continue with the integralterm $I^2_{\kappa_i}(\boldsymbol{\omega},\kappa)$. From (4.16) we obtain with (4.28)

$$\begin{aligned} I^2_{\kappa_i}(\boldsymbol{\omega},\kappa) &= e^{4\mu^s\omega_i}\int_{-\omega_i}^{0} e^{2\mu^s s}\Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)\left[R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}, R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}\right] ds \\ &\quad + \mathcal{R}^{B7}_{\kappa_i}(\boldsymbol{\omega},\kappa). \end{aligned}$$

For $\nu \geq 3$ the limit of the integral on the right-hand side does exist and we obtain

$$\left.\begin{aligned} I^2_{\kappa_i}(\boldsymbol{\omega},\kappa) = e^{4\mu^s\omega_i}\int_{-\infty}^{0} e^{2\mu^s s}\Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D_1^2 f(\gamma^-_{\kappa_i}(s))\left[R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}, R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}\right] ds \\ + \mathcal{R}^{B7}_{\kappa_i}(\boldsymbol{\omega},\kappa) + \mathcal{R}^{B8}_{\kappa_i}(\boldsymbol{\omega},\kappa) \end{aligned}\right\} \quad (4.30)$$

with

$$\mathcal{R}^{B8}_{\kappa_i}(\boldsymbol{\omega},\kappa) = -e^{4\mu^s\omega_i}\int_{-\infty}^{-\omega_i} e^{2\mu^s s}\Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D_1^2 f(\gamma^-_{\kappa_i}(s))[R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}, R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}]ds.$$

By applying exponential dichotomy (3.16) with $-\alpha^-_{\kappa_i} = \alpha^s$ and $\|D_1^2 f(\gamma^-_{\kappa_i}(s))[\cdot,\cdot]\| = O(e^{(\nu-2)\alpha^u s})$ we get

$$\left.\begin{aligned} \|\mathcal{R}^{B8}_{\kappa_i}(\boldsymbol{\omega},\kappa)\| &\leq Ce^{4\mu^s\omega_i}\int_{-\infty}^{-\omega_i} e^{(2\mu^s-\alpha^s+(\nu-2)\alpha^u)s}ds = O(e^{(2\mu^s+\alpha^s-(\nu-2)\alpha^u)\omega_i}) \\ &= O(e^{(3\alpha^s-(\nu-2)\alpha^u)\omega_i}). \end{aligned}\right\} \quad (4.31)$$

Next we estimate the residual term $\mathcal{R}^{B7}_{\kappa_i}(\boldsymbol{\omega},\kappa)$. Due to the bilinearity of the second differential, we find with (4.28)

$$\begin{aligned} \mathcal{R}^{B7}_{\kappa_i}(\boldsymbol{\omega},\kappa) &= \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D_1^2 f(\gamma^-_{\kappa_i}(s))[\mathcal{R}^{B6}_{\kappa_i}(\boldsymbol{\omega},\kappa)(s), e^{\mu^s(2\omega_i+s)}R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}]ds \\ &\quad + \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D_1^2 f(\gamma^-_{\kappa_i}(s))[e^{\mu^s(2\omega_i+s)}R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}, \mathcal{R}^{B6}_{\kappa_i}(\boldsymbol{\omega},\kappa)(s)]ds \\ &\quad + \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D_1^2 f(\gamma^-_{\kappa_i}(s))[\mathcal{R}^{B6}_{\kappa_i}(\boldsymbol{\omega},\kappa)(s), \mathcal{R}^{B6}_{\kappa_i}(\boldsymbol{\omega},\kappa)(s)]ds. \end{aligned}$$

In case that (H4.8) does not apply this provides with (4.29) the following estimate. Note that we will use the relation $e^{\mu^s(2\omega_i+s)} + \|\mathcal{R}^{B6}_{\kappa_i}(s)\| \leq e^{\alpha^s(2\omega_i+s)} + e^{4\alpha^w\omega_{i+1}}$.

$$\begin{aligned}
\left\|\mathcal{R}^{B7}_{\kappa_i}(\boldsymbol{\omega},\kappa)\right\| &\leq K\int_{-\omega_i}^{0} e^{-\alpha^s s}e^{(\nu-2)\alpha^u s}\left(2e^{\mu^s(2\omega_i+s)}\|\mathcal{R}^{B6}_{\kappa_i}(\boldsymbol{\omega},\kappa)(s)\| + \|\mathcal{R}^{B6}_{\kappa_i}(\boldsymbol{\omega},\kappa)(s)\|^2\right) ds \\
&\leq K\int_{-\omega_i}^{0} e^{-\alpha^s s}e^{(\nu-2)\alpha^u s}\|\mathcal{R}^{B6}_{\kappa_i}(\boldsymbol{\omega},\kappa)(s)\|\left(2e^{\mu^s(2\omega_i+s)} + \|\mathcal{R}^{B6}_{\kappa_i}(\boldsymbol{\omega},\kappa)(s)\|\right) ds \\
&\leq \bar{K}\int_{-\omega_i}^{0} e^{-\alpha^s s}e^{(\nu-2)\alpha^u s}\left(e^{\alpha^s(2\omega_i+s)}[e^{2\alpha^s\omega_{i-1}} + e^{\max\{\alpha^s,\alpha^{ss}-\alpha^s\}\omega_i} + e^{2\alpha^w\omega_{i+1}}]\right. \\
&\qquad\qquad \left. + e^{\alpha^{ss}(\omega_i+s)}e^{\alpha^s\omega_i} + e^{4\alpha^w\omega_{i+1}}\right)\cdot\left(e^{\alpha^s(2\omega_i+s)} + e^{4\alpha^w\omega_{i+1}}\right) ds \\
&\leq \tilde{K}\int_{-\omega_i}^{0} e^{-\alpha^s s}e^{(\nu-2)\alpha^u s}\Big(e^{\alpha^s(4\omega_i+2s)}[e^{2\alpha^s\omega_{i-1}} + e^{\max\{\alpha^s,\alpha^{ss}-\alpha^s\}\omega_i} + e^{2\alpha^w\omega_{i+1}}] \\
&\qquad\qquad + e^{\alpha^s(2\omega_i+s)}e^{4\alpha^w\omega_{i+1}} + e^{\alpha^{ss}(\omega_i+s)}e^{\alpha^s(\omega_i+s)}e^{2\alpha^s\omega_i} \\
&\qquad\qquad + e^{\alpha^{ss}(\omega_i+s)}e^{\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}} + e^{8\alpha^w\omega_{i+1}}\Big) ds \\
&\leq \tilde{K}\Big(e^{4\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}} + e^{\max\{\alpha^s,\alpha^{ss}-\alpha^s\}\omega_i} + e^{2\alpha^w\omega_{i+1}}]\int_{-\omega_i}^{0} e^{\alpha^s s}e^{(\nu-2)\alpha^u s} ds \\
&\quad + e^{2\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{(\nu-2)\alpha^u s} ds + e^{3\alpha^s\omega_i}e^{\alpha^{ss}\omega_i}\int_{-\omega_i}^{0} e^{(\alpha^{ss}+(\nu-2)\alpha^u)s} ds \\
&\quad + e^{\alpha^s\omega_i}e^{\alpha^{ss}\omega_i}e^{4\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{(\alpha^{ss}-\alpha^s)s}e^{(\nu-2)\alpha^u s} ds + e^{8\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{-\alpha^s s}e^{(\nu-2)\alpha^u s} ds\Big).
\end{aligned}$$

Since $\nu \geq 3$ the first two and the last integrals in the latest relation are bounded. From the third and the fourth summand we obtain

$$\begin{aligned}
e^{3\alpha^s\omega_i}e^{\alpha^{ss}\omega_i}\int_{-\omega_i}^{0} e^{(\alpha^{ss}+(\nu-2)\alpha^u)s} ds &= O\big(e^{3\alpha^s\omega_i}e^{\max\{\alpha^{ss},-(\nu-2)\alpha^u\}\omega_i}\big), \\
e^{\alpha^s\omega_i}e^{\alpha^{ss}\omega_i}e^{4\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{(\alpha^{ss}-\alpha^s)s}e^{(\nu-2)\alpha^u s} ds &= O\big(e^{4\alpha^w\omega_{i+1}}e^{\max\{\alpha^s+\alpha^{ss},2\alpha^s-(\nu-2)\alpha^u\}\omega_i}\big).
\end{aligned}$$

Note that these two terms will not appear in the calculation if (H4.8) applies. This finally yields

$$\left.\begin{aligned}
\left\|\mathcal{R}^{B7}_{\kappa_i}(\boldsymbol{\omega},\kappa)\right\| &= O\big(e^{4\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}] + e^{2\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}} + e^{8\alpha^w\omega_{i+1}}\big) \\
&\quad + \begin{cases} 0, & \text{if (H4.8) applies,} \\ e^{3\alpha^s\omega_i}e^{\max\{\alpha^{ss},-(\nu-2)\alpha^u\}\omega_i}, & \text{else.}\end{cases}
\end{aligned}\right\} \tag{4.32}$$

From (4.30) we gain the term $B_i(\lambda,\kappa)$ by commuting the integration and the scalar product and moving the transition matrix $\Phi^-_{\kappa_i}$ and the projection $P^-_{\kappa_i}$ to the left-hand side of the scalar product:

$$\left.\begin{aligned}
&\tfrac{1}{2}\langle\psi_{\kappa_i}, I^2_{\kappa_i}(\boldsymbol{\omega},\kappa)\rangle \\
&= e^{4\mu^s\omega_i}\underbrace{\frac{1}{2}\int_{-\infty}^{0} e^{2\mu^s s}\left\langle \Phi^-_{\kappa_i}(0,s)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, D_1^2 f(\gamma^-_{\kappa_i}(s))\left[R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}\right]^2\right\rangle ds}_{=:B_i(\lambda,\kappa)} \\
&\quad + \langle\psi_{\kappa_i}, \mathcal{R}^{B7}_{\kappa_i}(\boldsymbol{\omega},\kappa) + \mathcal{R}^{B8}_{\kappa_i}(\boldsymbol{\omega},\kappa)\rangle.
\end{aligned}\right\} \tag{4.33}$$

To conclude the proof we need to estimate the remaining scalar product $\langle\psi_{\kappa_i}, I^3_{\kappa_i}(\boldsymbol{\omega},\lambda,\kappa)\rangle$. For $\nu > 3$ we find due to Lemma 3.4.12 that

$$\langle\psi_{\kappa_i}, I^3_{\kappa_i}(\boldsymbol{\omega},\kappa)\rangle = O\left(e^{\max\{4\alpha^s+\alpha^w,6\alpha^s\}\omega_i} + e^{4\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}} + e^{12\alpha^w\omega_{i+1}}\right). \tag{4.34}$$

Hence for $\nu > 3$ the lemma follows from (4.18) by invoking (4.30), (4.32), (4.17), (4.31) and (4.34). That is

$$\boldsymbol{\mathcal{R}_i^{25}}(\boldsymbol{\omega}, \lambda, \kappa) = -\langle \psi_{\kappa_i}, \mathcal{R}^{B1}_{\kappa_i}(\boldsymbol{\omega}, \kappa) + \mathcal{R}^{B7}_{\kappa_i}(\boldsymbol{\omega}, \kappa) + \mathcal{R}^{B8}_{\kappa_i}(\boldsymbol{\omega}, \kappa)\rangle - \langle \psi_{\kappa_i}, I^3_{\kappa_i}(\boldsymbol{\omega}, \kappa)\rangle.$$

To this end recall that α^w was defined as

$$\alpha^w = \begin{cases} -\alpha^u, & \text{if (H4.8) applies,} \\ 1/2(\alpha^s - \alpha^u)\}, & \text{else.} \end{cases}$$

However for $\nu = 3$ Lemma 3.4.12 provides among others the rate $I^3_{\kappa_i}(\boldsymbol{\omega}, \kappa) = O(e^{4\alpha^s \omega_i})$ which is not good enough for our analysis. Therefore we need to discuss this term in more detail for $\nu = 3$. Similar to the procedure above we replace the expression $v_i^{-,u}(s)$ only this time we just use equation (4.19) in combination with (4.21) and (4.23). This yields

$$v_i^{-,u}(t) = \Phi^-_{\kappa_i}(t, -\omega_i) P^-_{\kappa_i}(-\omega_i) \gamma^+_{\kappa_{i-1}}(\omega_i) + \mathcal{R}^{B9}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(t)$$

with, cf. (4.20), (4.24) and (4.27),

$$\left.\begin{array}{rcl} \mathcal{R}^{B9}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(t) & = & \mathcal{R}^{B2}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(t) + \mathcal{R}^{B3}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(t) + \mathcal{R}^{B4}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(t) \\ & = & O(e^{\alpha^s(2\omega_i + t)}[e^{2\alpha^s \omega_{i-1}} + e^{\alpha^s \omega_i} + e^{2\alpha^w \omega_{i+1}}]) + O(e^{4\alpha^w \omega_{i+1}}). \end{array}\right\} \tag{4.35}$$

Then we find from (4.16)

$$\left.\begin{array}{l} \frac{1}{6}\langle \psi_{\kappa_i}, I^3_{\kappa_i}(\boldsymbol{\omega}, \kappa)\rangle \\ = \underbrace{\frac{1}{6}\int\limits_{-\omega_i}^{0} \langle \Phi^-_{\kappa_i}(0, s)^T P^-_{\kappa_i}(0)^T \psi_{\kappa_i}, D^3_1 f(\gamma^-_{\kappa_i}(s), \lambda)[\Phi^-_{\kappa_i}(s, -\omega_i) P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i)]^3\rangle ds}_{=: e^{4\mu^s(\lambda)\omega_i} D_i(\lambda, \kappa, \omega_i)} \\ +\langle \psi_{\kappa_i}, \mathcal{R}^{B10}_{\kappa_i}(\boldsymbol{\omega}, \kappa)\rangle. \end{array}\right\} \tag{4.36}$$

Since $\|\Phi^-_{\kappa_i}(s, -\omega_i) P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i)\| = O(e^{\mu^s(2\omega_i + s)})$, cf. (4.25), we find

$$\begin{array}{rcl} \left\| \int\limits_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0, s) P^-_{\kappa_i}(s) D^3_1 f(\gamma^-_{\kappa_i}(s))[\Phi^-_{\kappa_i}(t, -\omega_i) P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i)]^3 ds \right\| & \leq & K \int\limits_{-\omega_i}^{0} e^{-\mu^s s} e^{3\mu^s(2\omega_i + s)} ds \\ & \leq & K e^{4\mu^s \omega_i} \int\limits_{-\omega_i}^{0} e^{2\mu^s(\omega_i + s)} ds \\ & \leq & \tilde{K} e^{4\mu^s \omega_i}. \end{array}$$

This justifies the assignment of the integral in (4.36) as $e^{4\mu^s(\lambda)\omega_i} D_i(\lambda, \kappa, \omega_i)$.

The residual term $\mathcal{R}^{B10}_{\kappa_i}$ satisfies

$$\begin{array}{rcl} \left\|\mathcal{R}^{B10}_{\kappa_i}(\boldsymbol{\omega}, \kappa)\right\| & \leq & K \int\limits_{-\omega_i}^{0} e^{-(\alpha^s - \delta)s}\big(3(e^{\mu^s(2\omega_i + s)})^2 \cdot \|\mathcal{R}^{B9}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(s)\| \\ & & +3e^{\mu^s(2\omega_i + s)} \cdot \|\mathcal{R}^{B9}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(s)\|^2 + \|\mathcal{R}^{B9}_{\kappa_i}(\boldsymbol{\omega}, \kappa)(s)\|^3\big) ds. \end{array}$$

Here we used that $\|\Phi^-_{\kappa_i}(0, s) P^-_{\kappa_i}(s)\| \leq K e^{-(\alpha^s - \delta)s}$ for some $\delta > 0$ close to zero, cf. (3.16) this time with $-\alpha^-_{\kappa_i} = \alpha^s - \delta < \alpha^s < 0$. The remaining estimates come from $D^3_1 f(\gamma^-_{\kappa_i}(s), \lambda)[v_i^{-,u}(s)]^3 -$

$D_1^3 f(\gamma_{\kappa_i}^-(s),\lambda)[-\Phi_{\kappa_i}^-(s,-\omega_i)P_{\kappa_i}^-(-\omega_i)\gamma_{\kappa_{i-1}}^+(\omega_i)]^3$, where $\|\Phi_{\kappa_i}^-(s,-\omega_i)P_{\kappa_i}^-(-\omega_i)\gamma_{\kappa_{i-1}}^+(\omega_i)\| = O(e^{\mu^s(2\omega_i+s)})$, cf. (4.25). This yields the estimate

$$
\begin{aligned}
&\left\|\mathcal{R}_{\kappa_i}^{B10}(\boldsymbol{\omega},\kappa)\right\| \\
&\le \tilde{K}_1 \int_{-\omega_i}^{0} e^{-(\alpha^s-\delta)s}\big(e^{2\mu^s(2\omega_i+s)}\|\mathcal{R}_{\kappa_i}^{B9}(\boldsymbol{\omega},\kappa)(s)\| + e^{\mu^s(2\omega_i+s)}\|\mathcal{R}_{\kappa_i}^{B9}(\boldsymbol{\omega},\kappa)(s)\|^2 + \|\mathcal{R}_{\kappa_i}^{B9}(\boldsymbol{\omega},\kappa)(s)\|^3\big)ds \\
&\le \tilde{K}_1 \int_{-\omega_i}^{0} e^{-(\alpha^s-\delta)s}\|\mathcal{R}_{\kappa_i}^{B9}(\boldsymbol{\omega},\kappa)(s)\|\big(e^{2\mu^s(2\omega_i+s)} + \|\mathcal{R}_{\kappa_i}^{B9}(\boldsymbol{\omega},\kappa)(s)\|\big(e^{\mu^s(2\omega_i+s)} + \|\mathcal{R}_{\kappa_i}^{B9}(\boldsymbol{\omega},\kappa)(s)\|\big)\big)ds.
\end{aligned}
$$

In the following we use again the relation

$$
e^{\mu^s(2\omega_i+s)} + \|\mathcal{R}_{\kappa_i}^{B9}(s)\| \le C_1(e^{\alpha^s(2\omega_i+s)} + e^{4\alpha^w\omega_{i+1}}),
$$

cf. (4.35) for some positive constant C_1. Consequently we find

$$
\begin{aligned}
e^{2\mu^s(2\omega_i+s)} + \|\mathcal{R}_{\kappa_i}^{B9}(s)\|(e^{\mu^s(2\omega_i+s)} + \|\mathcal{R}_{\kappa_i}^{B9}(s)\|) &\le \big(e^{\mu^s(2\omega_i+s)} + \|\mathcal{R}_{\kappa_i}^{B9}(s)\|\big)^2 \\
&\le C_1^2\big(e^{\alpha^s(2\omega_i+s)} + e^{4\alpha^w\omega_{i+1}}\big)^2.
\end{aligned}
$$

Thus, by additionally invoking the estimate of $\mathcal{R}_{\kappa_i}^{B9}$, cf. (4.35), we obtain

$$
\begin{aligned}
&\left\|\mathcal{R}_{\kappa_i}^{B10}(\boldsymbol{\omega},\kappa)\right\| \\
&\le \tilde{K}_2 \int_{-\omega_i}^{0} e^{-(\alpha^s-\delta)s}\left(e^{\alpha^s(2\omega_i+s)}\big[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}\big] + e^{4\alpha^w\omega_{i+1}}\right) \\
&\qquad\qquad \cdot \left(e^{\alpha^s(2\omega_i+s)} + e^{4\alpha^w\omega_{i+1}}\right)^2 ds \\
&\le \tilde{K}_3 \int_{-\omega_i}^{0} e^{-(\alpha^s-\delta)s}\left(e^{3\alpha^s(2\omega_i+s)}\big[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}\big] + e^{2\alpha^s(2\omega_i+s)}e^{4\alpha^w\omega_{i+1}}\right. \\
&\qquad\qquad \left. + e^{\alpha^s(2\omega_i+s)}e^{8\alpha^w\omega_{i+1}} + e^{12\alpha^w\omega_{i+1}}\right) ds \\
&\le \tilde{K}_3\Big(e^{4\alpha^s\omega_i}\big[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}\big]\int_{-\omega_i}^{0} e^{2\alpha^s(\omega_i+s)}e^{\delta s}ds \\
&\quad + e^{3\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{\alpha^s(\omega_i+s)}e^{\delta s}ds + e^{2\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{\delta s}ds + e^{12\alpha^w\omega_{i+1}}\int_{-\omega_i}^{0} e^{-\alpha^s s}e^{\delta s}ds\Big) \\
&= O\left(e^{4\alpha^s\omega_i}\big[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{2\alpha^w\omega_{i+1}}\big] + e^{3\alpha^s\omega_i}e^{4\alpha^w\omega_{i+1}} + e^{2\alpha^s\omega_i}e^{8\alpha^w\omega_{i+1}} + e^{12\alpha^w\omega_{i+1}}\right).
\end{aligned}
$$

Note, that each of the remaining integrals in the second last line is bounded.

Summarizing, for $\nu = 3$ the lemma follows from (4.18) by using (4.30), (4.17), (4.31), (4.32) and (4.36) with

$$
\boldsymbol{\mathcal{R}_i^{25}}(\boldsymbol{\omega},\lambda,\kappa) = -\langle\psi_{\kappa_i}, \mathcal{R}_{\kappa_i}^{B1}(\boldsymbol{\omega},\kappa) + \mathcal{R}_{\kappa_i}^{B7}(\boldsymbol{\omega},\kappa) + \mathcal{R}_{\kappa_i}^{B8}(\boldsymbol{\omega},\kappa) + \mathcal{R}_{\kappa_i}^{B10}(\boldsymbol{\omega},\kappa)\rangle.
$$

This concludes the proof. ∎

Remark 4.3.21. *As we can see from this lemma the case $\nu = 2$ plays a special role. For $\nu = 2$ the term $I^2_{\kappa_i}$ has with $e^{3\mu^s\omega_i}$ a different rate of convergence. Indeed we find*

$$\int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)[e^{\mu^s(2\omega_i+s)}R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}, e^{\mu^s(2\omega_i+s)}R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}]ds$$
$$= e^{3\mu^s\omega_i}\int_{-\omega_i}^{0} e^{\mu^s s}e^{\mu^s(\omega_i+s)}\Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)\left[R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}, R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}\right]ds,$$

where the integral term is bounded for $\omega_i \to \infty$. However the corresponding coefficient to the leading exponential rate still depends on the transition time ω_i. So, to be precise we find under the assumption of Hypotheses (H4.1) - (H4.7) with $\nu = 2$ that there exist constants Ω and c such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ we find the estimate

$$\left|\left\langle \psi_{\kappa_i}, \int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(\lambda)(0,s)P^-_{\kappa_i}(\lambda,s)h^-_{\kappa_i}(s,v^-_i(s),\lambda)ds\right\rangle - e^{3\mu^s(\lambda)\omega_i}B_i(\lambda,\kappa,\omega_i)\right|$$
$$= O\left(e^{3\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{\alpha^w\omega_{i+1}}] + e^{2\alpha^s\omega_i}e^{2\alpha^w\omega_{i+1}} + e^{6\alpha^w\omega_{i+1}}\right)$$

with $B_i(\lambda,\kappa,\omega_i)$ taking the form of the integral

$$B_i(\lambda,\kappa,\omega_i) := \tfrac{1}{2}\int_{-\omega_i}^{0} e^{\mu^s(\lambda)s}e^{\mu^s(\lambda)(\omega_i+s)}\left\langle \Phi^-_{\kappa_i}(\lambda)(0,s)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i},\right.$$
$$\left. D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)\left[R^-_{\kappa_i}(\lambda,s)\eta^s_{\kappa_{i-1}}(\lambda), R^-_{\kappa_i}(\lambda,s)\eta^s_{\kappa_{i-1}}(\lambda)\right]\right\rangle ds.$$

The following lemma is dedicated to the term $C_i(\lambda,\kappa)$. Recall Lemma 4.3.17 and equation (3.38) for the definition of the terms $S^+_{\kappa_{i-1}}, R^-_{\kappa_{i-1}}$ and $(id - F_{\kappa_{i-1}})$.

Lemma 4.3.22. *Assume Hypotheses (H4.1) - (H4.7). Then there exist constants Ω and c such that for all $|\lambda| < c$ and $\boldsymbol{\omega}$ with $\inf \boldsymbol{\omega} > \Omega$ we find the estimate*

$$\begin{aligned} \boldsymbol{T^{23}_{\kappa_i}} &:= -\left\langle \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ (id - \tilde{P}_{\kappa_i}(\lambda,\omega_i))v^{+,s}_{i-1}(\omega_i)\right\rangle \\ &= -e^{2\mu^s(\lambda)\omega_{i-1}}e^{2\mu^s(\lambda)\omega_i}C_i(\lambda,\kappa) + \boldsymbol{\mathcal{R}^{23}_i}(\boldsymbol{\omega},\lambda,\kappa) \end{aligned}$$

with

$$C_i(\lambda,\kappa) := \left\langle \eta^-_{\kappa_i}(\lambda), S^+_{\kappa_{i-1}}(\lambda,0)(id - F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(\lambda,0)\eta^s_{\kappa_{i-2}}(\lambda)\right\rangle.$$

The residual term satisfies

$$\boldsymbol{\mathcal{R}^{23}_i}(\boldsymbol{\omega},\lambda,\kappa) = \begin{cases} \left.\begin{array}{l} O(e^{2\alpha^s\omega_{i-1}}e^{2\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-2}} + e^{\alpha^s\omega_{i-1}} + e^{\max\{(\nu-1)\alpha^s,-\alpha^u\}\omega_i}] \\ \quad + e^{2(\alpha^s-\alpha^u)\omega_i}), \end{array}\right\} & \text{if (H4.8) applies,} \\ \left.\begin{array}{l} O(e^{2\alpha^s\omega_{i-1}}e^{2\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-2}} + e^{\max\{\alpha^s,\alpha^{ss}-\alpha^s\}\omega_{i-1}} \\ \qquad + e^{\max\{(\nu-1)\alpha^s,(\alpha^s-\alpha^u)/2,\alpha^{ss}-\alpha^s\}\omega_i}] \\ \quad + e^{2\alpha^s\omega_{i-1}}e^{(\alpha^s+\alpha^{ss})\omega_i} + e^{(3\alpha^s-\alpha^u)\omega_i}), \end{array}\right\} & \text{else.} \end{cases}$$

Proof. For the sake of convenience we omit the dependency of λ in our notation. We start with considering the right-hand side of the scalar product, that is $(id - \tilde{P}_{\kappa_i}(\omega_i))v^{+,s}_{i-1}(\omega_i)$. First we inspect the term

$v^{+,s}_{i-1}(\omega_i)$. Equation (3.41) gives the representation of $v_i^{+,s} := (id - P^+_{\kappa_i})v_i^+$ and together with (3.39) that states $v_i^{+,s}(0) = (id - P^+_{\kappa_i}(0))(id - F_{\kappa_i})v_i^{-,u}(0)$ we obtain after an index shift and setting $t = \omega_i$

$$v^{+,s}_{i-1}(\omega_i) = \Phi^+_{\kappa_{i-1}}(\omega_i, 0)(id - P^+_{\kappa_{i-1}}(0))(id - F_{\kappa_{i-1}})v^{-,u}_{i-1}(0) + \mathcal{R}^{C1}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa) \tag{4.37}$$

with, cf. Estimate (3.105),

$$\left.\begin{aligned} \mathcal{R}^{C1}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa) &= \int_0^{\omega_i} \Phi^+_{\kappa_{i-1}}(\omega_i, s)(id - P^+_{\kappa_{i-1}}(s))h^+_{\kappa_{i-1}}(s, v^+_{i-1}(s))ds \\ &= O\left(e^{\alpha^s\omega_i}\left(e^{\alpha^w\omega_i} + e^{2\alpha^s\omega_{i-1}}\right)^2\right). \end{aligned}\right\} \tag{4.38}$$

Recall (3.38) for the definition of the projection F_{κ_i}.
Equation (4.28) together with Lemma 4.3.17 then provides

$$\begin{aligned} &\Phi^+_{\kappa_{i-1}}(\omega_i, 0)(id - P^+_{\kappa_{i-1}}(0))(id - F_{\kappa_{i-1}})v^{-,u}_{i-1}(0) \\ = &\left[e^{\mu^s\omega_i}S^+_{\kappa_{i-1}}(0) + \mathcal{R}^S_{\kappa_i}(\omega_i, 0)\right](id - F_{\kappa_{i-1}})\left[e^{2\mu^s\omega_{i-1}}R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} + \mathcal{R}^{B6}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa)(0)\right] \\ = &\ e^{\mu^s\omega_i}e^{2\mu^s\omega_{i-1}}S^+_{\kappa_{i-1}}(0)(id - F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} + \mathcal{R}^{C2}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa) \end{aligned}$$

with

$$\left.\begin{aligned} \mathcal{R}^{C2}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa) &= \left[e^{\mu^s\omega_i}S^+_{\kappa_{i-1}}(0) + \mathcal{R}^S_{\kappa_i}(\omega_i, 0)\right](id - F_{\kappa_{i-1}})\mathcal{R}^{B6}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa)(0) \\ &\quad + e^{2\mu^s\omega_{i-1}}\mathcal{R}^S_{\kappa_i}(\omega_i, 0)(id - F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} \\ &= O(e^{\alpha^s\omega_i}\|\mathcal{R}^{B6}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa)(0)\|) + O(e^{2\alpha^s\omega_{i-1}}e^{\max\{\alpha^{ss}, \nu\alpha^s\}\omega_i}) \end{aligned}\right\} \tag{4.39}$$

Summarising we find

$$v^{+,s}_{i-1}(\omega_i) = e^{\mu^s\omega_i}e^{2\mu^s\omega_{i-1}}S^+_{\kappa_{i-1}}(0)(id - F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} + \mathcal{R}^{C3}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa) \tag{4.40}$$

where $\mathcal{R}^{C3}_{\kappa_{i-1}}$ results from (4.38) and (4.39) by invoking the estimate (4.29) of $\mathcal{R}^{B6}$ with an index shift from i to $i-1$:

$$\left.\begin{aligned} \mathcal{R}^{C3}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa) &= \mathcal{R}^{C1}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa) + \mathcal{R}^{C2}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa) \\ &= O\left(e^{2\alpha^s\omega_{i-1}}e^{\alpha^s\omega_i}\left[e^{2\alpha^s\omega_{i-2}} + e^{\max\{\alpha^s, \alpha^{ss}-\alpha^s\}\omega_{i-1}} + e^{\max\{(\nu-1)\alpha^s, \alpha^w\}\omega_i}\right]\right) \\ &\quad + O\left(e^{2\alpha^s\omega_{i-1}}e^{\alpha^{ss}\omega_i}\right) + O\left(e^{(\alpha^s+2\alpha^w)\omega_i}\right). \end{aligned}\right\} \tag{4.41}$$

Here again we have

$$\alpha^w = \begin{cases} -\alpha^u, & \text{if (H4.8) applies,} \\ 1/2(\alpha^s - \alpha^u), & \text{else.} \end{cases}$$

and $\alpha^{ss} = -\infty$ if (H4.8) applies.

Now, due to Lemma 3.4.3 the projection $(id - \tilde{P}_{\kappa_i}(\omega_i))$ acts on $(id - P^+_{\kappa_{i-1}}(\omega_i))$ up to higher order terms as $P^-_{\kappa_i}(-\omega_i)$. To be precise we find with Lemma 3.4.3 or 3.4.5, respectively, and Lemma 3.4.9 for $v^{+,s}_{i-1}(\omega_i) := (id - P^+_{\kappa_{i-1}}(\omega_i))v^+_{i-1}(\omega_i)$

$$(id - \tilde{P}_{\kappa_i}(\omega_i))v^{+,s}_{i-1}(\omega_i) = P^-_{\kappa_i}(-\omega_i)v^{+,s}_{i-1}(\omega_i) + \mathcal{R}^{C4}_{\kappa_{i-1}}(\boldsymbol{\omega}, \kappa) \tag{4.42}$$

with

$$\left.\begin{aligned}
&\|\mathcal{R}^{C4}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\| \\
&\leq \|(id-\tilde{P}_{\kappa_i}(\omega_i))(id-P^+_{\kappa_{i-1}}(\omega_i)) - P^-_{\kappa_i}(-\omega_i)(id-P^+_{\kappa_{i-1}}(\omega_i))\|\|v^{+,s}_{i-1}(\omega_i)\| \\
&= \begin{cases} \left.\begin{aligned} &O\big(e^{\max\{(\nu+1)\alpha^s-\alpha^u,\nu\alpha^s-(\nu-1)\alpha^u,3\alpha^s-2\alpha^u\}\omega_i}\big(e^{-2\alpha^u\omega_i} \\ &\qquad\qquad +e^{2\alpha^s\omega_{i-1}}\big)\big), \end{aligned}\right\} & \text{if (H4.8) applies,} \\ O\big(e^{\max\{(\nu+1)\alpha^s,2\alpha^s-\alpha^u\}\omega_i}\big(e^{(\alpha^s-\alpha^u)\omega_i}+e^{2\alpha^s\omega_{i-1}}\big)\big), & \text{else,} \end{cases} \\
&= O\big(e^{\max\{(\nu+1)\alpha^s,2\alpha^s-\alpha^u\}\omega_i}\big(e^{2\alpha^w\omega_i}+e^{2\alpha^s\omega_{i-1}}\big)\big).
\end{aligned}\right\} \quad (4.43)$$

The projection $P^-_{\kappa_i}(-\omega_i)$ in front of $v^{+,s}_{i-1}(\omega_i)$ does not effect the scalar product $\langle(\Phi^-_{\kappa_i})^T(P^-_{\kappa_i})^T\psi_{\kappa_i}, (id-\tilde{P}_{\kappa_i})v^{+,s}_{i-1}\rangle$, since its adjoint $P^-_{\kappa_i}(-\omega_i)^T$ also appears on the left-hand side. So when building the scalar product we shift the projection $P^-_{\kappa_i}(-\omega_i)$ from the right-hand side to the left-hand side and we obtain with (4.42) and $\|\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}\| = O(e^{\alpha^s\omega_i})$, cf. (3.112),

$$\begin{aligned}
-\boldsymbol{T^{23}_{\kappa_i}} &= \Big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, (id-\tilde{P}_{\kappa_i}(\omega_i))v^{+,s}_{i-1}(\omega_i)\Big\rangle \\
&= \big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, P^-_{\kappa_i}(-\omega_i)v^{+,s}_{i-1}(\omega_i)\big\rangle + \Big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, \mathcal{R}^{C4}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\Big\rangle \\
&= \big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, v^{+,s}_{i-1}(\omega_i)\big\rangle + \mathcal{R}^{C5}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa),
\end{aligned}$$

with

$$\|\mathcal{R}^{C5}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\| = O(e^{\alpha^s\omega_i}\|\mathcal{R}^{C4}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\|).$$

The left-hand side of the scalar product we obtain from (4.14) which leads together with (4.40) to

$$\begin{aligned}
&\big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, v^{+,s}_{i-1}(\omega_i)\big\rangle \\
&= \Big\langle e^{\mu^s\omega_i}\eta^-_{\kappa_i} + \mathcal{R}^{A4}_{\kappa_i}(\omega_i)\ ,\ e^{\mu^s\omega_i}e^{2\mu^s\omega_{i-1}}S^+_{\kappa_{i-1}}(0)(id-F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} + \mathcal{R}^{C3}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\Big\rangle \\
&= e^{2\mu^s\omega_i}e^{2\mu^s\omega_{i-1}}\underbrace{\Big\langle \eta^-_{\kappa_i}, S^+_{\kappa_{i-1}}(0)(id-F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}}\Big\rangle}_{C_i(\lambda,\kappa)} + \mathcal{R}^{C6}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa).
\end{aligned}$$

Thereby the residual term follows from (4.15) and (4.41)

$$\begin{aligned}
\mathcal{R}^{C6}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa) &= \Big\langle e^{\mu^s\omega_i}\eta^-_{\kappa_i} + \mathcal{R}^{A4}_{\kappa_i}(\omega_i), \mathcal{R}^{C3}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\Big\rangle \\
&\quad + \Big\langle \mathcal{R}^{A4}_{\kappa_i}(\omega_i), -e^{\mu^s\omega_i}e^{2\mu^s\omega_{i-1}}S^+_{\kappa_{i-1}}(0)(id-F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}}\Big\rangle \\
&= O(e^{\alpha^s\omega_i}\|\mathcal{R}^{C3}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\|) + O(e^{\max\{\alpha^s+\alpha^{ss},2\alpha^s-(\nu-1)\alpha^u\}\omega_i}e^{2\alpha^s\omega_{i-1}}).
\end{aligned}$$

Summarising the last four equations we obtain

$$\boldsymbol{T^{23}_{\kappa_i}} = -e^{2\mu^s\omega_i}e^{2\mu^s\omega_{i-1}}C_i(\lambda,\kappa) + \boldsymbol{\mathcal{R}^{23}_i}(\boldsymbol{\omega},\kappa)$$

with

$$\begin{aligned}\boldsymbol{\mathcal{R}}_{\boldsymbol{i}}^{\mathbf{23}}(\boldsymbol{\omega},\kappa) &= -\mathcal{R}^{C5}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)-\mathcal{R}^{C6}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\\ &= O(e^{\alpha^s\omega_i}(\|\mathcal{R}^{C3}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\|+\|\mathcal{R}^{C4}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\|))+O(e^{\max\{\alpha^s+\alpha^{ss},2\alpha^s-(\nu-1)\alpha^u\}\omega_i}e^{2\alpha^s\omega_{i-1}})\\ &= O(e^{\alpha^s\omega_i}\|\mathcal{R}^{C3}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\|)+O(e^{\max\{\alpha^s+\alpha^{ss},2\alpha^s-(\nu-1)\alpha^u\}\omega_i}e^{2\alpha^s\omega_{i-1}}).\end{aligned}$$

Invoking the estimates (4.41) and (4.43) for the residuals $\mathcal{R}^{C3}_{\kappa_{i-1}}$ and $\mathcal{R}^{C4}_{\kappa_{i-1}}$ concludes the proof. □

Now, with Lemmata 4.3.14, 4.3.20 and 4.3.22 we finally can prove the Theorems 4.3.1 and 4.3.3.

Proof of Theorem 4.3.1.

To begin with, recall the representation of the jump $\langle\psi_{\kappa_i},\xi_i(\boldsymbol{\omega},\lambda,\kappa)\rangle$ given in (3.65). Here we have introduced the partitioning

$$\langle\psi_{\kappa_i},\xi_i(\boldsymbol{\omega},\lambda,\kappa)\rangle = \boldsymbol{T}^{\mathbf{1}}_{\boldsymbol{\kappa_i}}+\boldsymbol{T}^{\mathbf{2}}_{\boldsymbol{\kappa_i}}.$$

Thereby the estimate of $\boldsymbol{T}^{\mathbf{1}}_{\boldsymbol{\kappa_i}}$ depends on $i+1\in\mathbb{Z}\setminus J_\kappa$ or $i+1\in J_\kappa$. On the contrary the structure of $\boldsymbol{T}^{\mathbf{2}}_{\boldsymbol{\kappa_i}}$ is only depending on the index i being an element of $\mathbb{Z}\setminus J_\kappa$ or J_κ.

We start with examining the term $\boldsymbol{T}^{\mathbf{1}}_{\boldsymbol{\kappa_i}} := \sum_{k=1}^{5}\boldsymbol{T}^{\mathbf{1k}}_{\boldsymbol{\kappa_i}}$.

To this end let first $i+1\in\mathbb{Z}\setminus J_\kappa$. Here we simply adopt the Estimate (3.118) of $\boldsymbol{T}^{\mathbf{1}}_{\boldsymbol{\kappa_i}}$ listed in Section 3.4.7. Recall that in Theorem 4.3.1 the Hypothesis (H4.8) and thus (H3.7) does not apply. This yields

$$\left.\begin{aligned} i+1\in\mathbb{Z}\setminus J_\kappa: \quad \boldsymbol{T}^{\mathbf{1}}_{\boldsymbol{\kappa_i}} &= O\big(e^{6\alpha^s\omega_i}+e^{2\alpha^s\omega_i}e^{(\alpha^s-\alpha^u)\omega_{i+1}}+e^{\max\{3/2(\alpha^s-\alpha^u),-2\alpha^u\}\omega_{i+1}}\\ &\quad +e^{1/2(\alpha^s-3\alpha^u)\omega_{i+1}}e^{(\alpha^s-\alpha^u)\omega_{i+2}}\big).\end{aligned}\right\}\tag{4.44}$$

Now, let $i+1\in J_\kappa$. In that case we adopt the estimates of the terms $\boldsymbol{T}^{\mathbf{12}}_{\boldsymbol{\kappa_i}},\dots,\boldsymbol{T}^{\mathbf{15}}_{\boldsymbol{\kappa_i}}$ from Section 3.4.7, (3.114) - (3.117):

$$\begin{aligned}\boldsymbol{T}^{\mathbf{12}}_{\boldsymbol{\kappa_i}} &= O\left(e^{3/2(\alpha^s-\alpha^u)\omega_{i+1}}\right),\\ \boldsymbol{T}^{\mathbf{13}}_{\boldsymbol{\kappa_i}} &= O\left(e^{1/2(\alpha^s-3\alpha^u)\omega_{i+1}}\big[e^{2\alpha^s\omega_{i+1}}+e^{(\alpha^s-\alpha^u)\omega_{i+2}}\big]\right),\\ \boldsymbol{T}^{\mathbf{14}}_{\boldsymbol{\kappa_i}} &= O\left(e^{3/2(\alpha^s-\alpha^u)\omega_{i+1}}\big[e^{2\alpha^s\omega_i}+e^{(\alpha^s-\alpha^u)\omega_{i+1}}\big]\right),\\ \boldsymbol{T}^{\mathbf{15}}_{\boldsymbol{\kappa_i}} &= O\left(e^{6\alpha^s\omega_i}+e^{2\alpha^s\omega_i}e^{(\alpha^s-\alpha^u)\omega_{i+1}}+e^{\max\{(\nu-\frac{1}{2})\alpha^s-\frac{3}{2}\alpha^u,2(\alpha^s-\alpha^u)\}\omega_{i+1}}\right).\end{aligned}$$

Only $\boldsymbol{T}^{\mathbf{11}}_{\boldsymbol{\kappa_i}}$ changes into

$$\boldsymbol{T}^{\mathbf{11}}_{\boldsymbol{\kappa_i}} = O\big(e^{\max\{\nu\alpha^s-2\alpha^u,\alpha^s-3\alpha^u\}\omega_{i+1}}\big),$$

cf. Remark 4.3.16. This leads to

$$\left.\begin{aligned} i+1\in J_\kappa: \quad \boldsymbol{T}^{\mathbf{1}}_{\boldsymbol{\kappa_i}} &= O\big(e^{6\alpha^s\omega_i}+e^{2\alpha^s\omega_i}e^{(\alpha^s-\alpha^u)\omega_{i+1}}+e^{3/2(\alpha^s-\alpha^u)\omega_{i+1}}\\ &\quad +e^{1/2(\alpha^s-3\alpha^u)\omega_{i+1}}e^{(\alpha^s-\alpha^u)\omega_{i+2}}\big).\end{aligned}\right\}\tag{4.45}$$

Next we consider $\boldsymbol{T}^{\mathbf{2}}_{\boldsymbol{\kappa_i}} := \sum_{k=1}^{5}\boldsymbol{T}^{\mathbf{2k}}_{\boldsymbol{\kappa_i}}$.

Let $i\in\mathbb{Z}\setminus J_\kappa$. Due to Lemma 4.3.14 we find that $A_i(\lambda,\kappa):=\langle\eta^-_{\kappa_i}(\lambda),\eta^s_{\kappa_{i-1}}(\lambda)\rangle$ is different from zero.

Therefore we find

$$\boldsymbol{T}^{\mathbf{21}}_{\boldsymbol{\kappa_i}} = -e^{2\mu^s(\lambda)\omega_i} A_i(\lambda, \kappa) + O(e^{\max\{\alpha^{ss}+\alpha^s, (\nu+1)\alpha^s, 3\alpha^s-\alpha^u\}\omega_i}).$$

The estimates of the remaining terms $\boldsymbol{T}^{\mathbf{22}}_{\boldsymbol{\kappa_i}}, \ldots, \boldsymbol{T}^{\mathbf{25}}_{\boldsymbol{\kappa_i}}$ we simply adopt from Section 3.4.7, (3.120) - (3.123):

$$\begin{aligned}
\boldsymbol{T}^{\mathbf{22}}_{\boldsymbol{\kappa_i}} &= O\left(e^{\max\{\nu\alpha^s-\alpha^u, 3/2(\alpha^s-\alpha^u)\}\omega_i}\right), \\
\boldsymbol{T}^{\mathbf{23}}_{\boldsymbol{\kappa_i}} &= O\left(e^{2\alpha^s\omega_i}\left[e^{2\alpha^s\omega_{i-1}} + e^{(\alpha^s-\alpha^u)\omega_i}\right]\right) \\
\boldsymbol{T}^{\mathbf{24}}_{\boldsymbol{\kappa_i}} &= O\left(e^{\max\{(\nu+\frac{1}{2})\alpha^s-\frac{1}{2}\alpha^u, 3\alpha^s-\alpha^u\}\omega_i}\left[e^{2\alpha^s\omega_i} + e^{(\alpha^s-\alpha^u)\omega_{i+1}}\right]\right). \\
\boldsymbol{T}^{\mathbf{25}}_{\boldsymbol{\kappa_i}} &= O\left(e^{3(\alpha^s-\alpha^u)\omega_{i+1}} + e^{(\alpha^s-\alpha^u)\omega_{i+1}}e^{2\alpha^s\omega_i} + e^{\min\{\nu+1,4\}\alpha^s\omega_i}\right)
\end{aligned}$$

This finally gives for $\nu \geq 3$

$$\left.\begin{aligned}
i \in \mathbb{Z} \setminus J_\kappa: \quad \boldsymbol{T}^{\mathbf{2}}_{\boldsymbol{\kappa_i}} &= -e^{2\mu^s(\lambda)\omega_i} A_i(\lambda, \kappa) + O(e^{2\alpha^s\omega_i}e^{2\alpha^s\omega_{i-1}}) + O(e^{2\alpha^s\omega_i}e^{(\alpha^s-\alpha^u)\omega_{i+1}}) \\
&\quad + O(e^{\max\{\alpha^{ss}+\alpha^s, (\nu+1)\alpha^s, 4\alpha^s, 3/2(\alpha^s-\alpha^u)\}\omega_i}) + O(e^{3(\alpha^s-\alpha^u)\omega_{i+1}}).
\end{aligned}\right\} \tag{4.46}$$

Now, let $i \in J_\kappa$. Then we find that $A_i(\lambda, \kappa) = 0$. Therefore we use the estimate given in Remark 4.3.16 and obtain

$$\boldsymbol{T}^{\mathbf{21}}_{\boldsymbol{\kappa_i}} = O\big(e^{\max\{(\nu+2)\alpha^s, 3\alpha^s-\alpha^u\}\omega_i}\big).$$

In case of $\boldsymbol{T}^{\mathbf{22}}_{\boldsymbol{\kappa_i}}$ and $\boldsymbol{T}^{\mathbf{24}}_{\boldsymbol{\kappa_i}}$ we make use of the estimates given in Section 3.4.7, cf. (3.120) and (3.122), see above. Finally Lemmata 4.3.20 and 4.3.22 provide the terms $\boldsymbol{T}^{\mathbf{25}}_{\boldsymbol{\kappa_i}}$ and $\boldsymbol{T}^{\mathbf{23}}_{\boldsymbol{\kappa_i}}$:

$$\begin{aligned}
\boldsymbol{T}^{\mathbf{23}}_{\boldsymbol{\kappa_i}} &= -e^{2\mu^s(\lambda)\omega_{i-1}}e^{2\mu^s(\lambda)\omega_i}C_i(\lambda, \kappa) \quad + O\big(e^{2\alpha^s\omega_{i-1}}e^{2\alpha^s\omega_i}\big[e^{2\alpha^s\omega_{i-2}} + e^{\max\{\alpha^s, \alpha^{ss}-\alpha^s\}\omega_{i-1}} \\
&\qquad + e^{\max\{(\nu-1)\alpha^s, 1/2(\alpha^s-\alpha^u), \alpha^{ss}-\alpha^s\}\omega_i}\big] + e^{(3\alpha^s-\alpha^u)\omega_i}\big), \\
\boldsymbol{T}^{\mathbf{25}}_{\boldsymbol{\kappa_i}} &= -e^{4\mu^s(\lambda)\omega_i}[B_i(\lambda, \kappa) + D_i(\lambda, \kappa, \omega_i)] \quad + O\big(e^{4\alpha^s\omega_i}\big[e^{2\alpha^s\omega_{i-1}} + e^{\max\{\alpha^s, \alpha^{ss}-\alpha^s\}\omega_i} + e^{1/2(\alpha^s-\alpha^u)\omega_{i+1}}\big] \\
&\qquad + e^{(3\alpha^s-(\nu-2)\alpha^u)\omega_i} + e^{2\alpha^s\omega_i}e^{(\alpha^s-\alpha^u)\omega_{i+1}} + e^{3(\alpha^s-\alpha^u)\omega_{i+1}}\big).
\end{aligned}$$

Hence we obtain

$$\left.\begin{aligned}
i \in J_\kappa: \quad \boldsymbol{T}^{\mathbf{2}}_{\boldsymbol{\kappa_i}} &= -e^{4\mu^s(\lambda)\omega_i}[B_i(\lambda, \kappa) + D_i(\lambda, \kappa, \omega_i)] - e^{2\mu^s(\lambda)\omega_{i-1}}e^{2\mu^s(\lambda)\omega_i}C_i(\lambda, \kappa) \\
&\quad + O\big(e^{2\alpha^s\omega_{i-1}}e^{2\alpha^s\omega_i}\left[e^{2\alpha^s\omega_{i-2}} + e^{\max\{\alpha^s, \alpha^{ss}-\alpha^s\}\omega_{i-1}}\right. \\
&\qquad \left. + e^{\max\{2\alpha^s, 1/2(\alpha^s-\alpha^u), \alpha^{ss}-\alpha^s\}\omega_i}\right] \\
&\quad + e^{\max\{5\alpha^s, 3\alpha^s+\alpha^{ss}, 3\alpha^s-\alpha^u, 3/2(\alpha^s-\alpha^u)\}\omega_i} + e^{4\alpha^s\omega_i}e^{1/2(\alpha^s-\alpha^u)\omega_{i+1}} \\
&\quad + e^{2\alpha^s\omega_i}e^{(\alpha^s-\alpha^u)\omega_{i+1}} + e^{3(\alpha^s-\alpha^u)\omega_{i+1}}\big).
\end{aligned}\right\} \tag{4.47}$$

Now we simply need to put the different cases together.

Case 1: $i \in \mathbb{Z} \setminus J_\kappa$ and $i+1 \in \mathbb{Z} \setminus J_\kappa$
Combining the Estimates (4.44) and (4.46) yields for $\nu \geq 3$

$$\begin{aligned} \langle \psi_{\kappa_i}, \xi_i(\boldsymbol{\omega}, \lambda, \kappa)\rangle &= -e^{2\mu^s(\lambda)\omega_i} A_i(\lambda,\kappa) + O(e^{2\alpha^s\omega_i} e^{2\alpha^s\omega_{i-1}}) + O(e^{\max\{\alpha^{ss}+\alpha^s, 4\alpha^s, 3/2(\alpha^s-\alpha^u)\}\omega_i}) \\ &\quad + O(e^{2\alpha^s\omega_i} e^{(\alpha^s-\alpha^u)\omega_{i+1}}) + O(e^{\max\{3/2(\alpha^s-\alpha^u), -2\alpha^u\}\omega_{i+1}}) \\ &\quad + O(e^{1/2(\alpha^s-3\alpha^u)\omega_{i+1}} e^{(\alpha^s-\alpha^u)\omega_{i+2}}) \end{aligned}$$

and more simplified

$$\langle \psi_{\kappa_i}, \xi_i(\boldsymbol{\omega}, \lambda, \kappa)\rangle = -e^{2\mu^s(\lambda)\omega_i} A_i(\lambda,\kappa) + O(e^{2\alpha^s\omega_i} e^{2\alpha^s\omega_{i-1}} + e^{\max\{\alpha^{ss}+\alpha^s, 3\alpha^s\}\omega_i} + e^{(\alpha^s-\alpha^u)\omega_{i+1}}). \quad (4.48)$$

Case 2: $i \in \mathbb{Z} \setminus J_\kappa$ and $i+1 \in J_\kappa$
In this case we combine the Estimates (4.45) and (4.46) and obtain

$$\left.\begin{aligned} \langle \psi_{\kappa_i}, \xi_i(\boldsymbol{\omega}, \lambda, \kappa)\rangle &= -e^{2\mu^s(\lambda)\omega_i} A_i(\lambda,\kappa) + O(e^{2\alpha^s\omega_i} e^{2\alpha^s\omega_{i-1}}) + O(e^{\max\{\alpha^{ss}+\alpha^s, 3\alpha^s\}\omega_i}) \\ &\quad + O(e^{2\alpha^s\omega_i} e^{(\alpha^s-\alpha^u)\omega_{i+1}}) + O(e^{3/2(\alpha^s-\alpha^u)\omega_{i+1}}) \\ &\quad + O(e^{1/2(\alpha^s-3\alpha^u)\omega_{i+1}} e^{(\alpha^s-\alpha^u)\omega_{i+2}}) \end{aligned}\right\} \quad (4.49)$$

Case 3: $i \in J_\kappa$ and $i+1 \in \mathbb{Z} \setminus J_\kappa$
Here we combine (4.44) and (4.47). This yields

$$\left.\begin{aligned} \langle \psi_{\kappa_i}, \xi_i(\boldsymbol{\omega}, \lambda, \kappa)\rangle &= -e^{4\mu^s(\lambda)\omega_i}[B_i(\lambda,\kappa) + D_i(\lambda,\kappa,\omega_i)] - e^{2\mu^s(\lambda)\omega_{i-1}} e^{2\mu^s(\lambda)\omega_i} C_i(\lambda,\kappa) \\ &\quad + O\big(e^{2\alpha^s\omega_{i-1}} e^{2\alpha^s\omega_i}\big[e^{2\alpha^s\omega_{i-2}} + e^{\max\{\alpha^s, \alpha^{ss}-\alpha^s\}\omega_{i-1}} \\ &\quad + e^{\max\{2\alpha^s, \alpha^{ss}-\alpha^s, \frac{1}{2}(\alpha^s-\alpha^u)\}\omega_i}\big] + e^{\max\{5\alpha^s, 3\alpha^s+\alpha^{ss}, 3\alpha^s-\alpha^u, \frac{3}{2}(\alpha^s-\alpha^u)\}\omega_i} \\ &\quad + e^{4\alpha^s\omega_i} e^{1/2(\alpha^s-\alpha^u)\omega_{i+1}} + e^{(\alpha^s-\alpha^u)\omega_{i+1}}\big). \end{aligned}\right\} \quad (4.50)$$

Case 4: $i \in J_\kappa$ and $i+1 \in J_\kappa$
Finally in that case we combine (4.45) and (4.47). This yields

$$\left.\begin{aligned} \langle \psi_{\kappa_i}, \xi_i(\boldsymbol{\omega}, \lambda, \kappa)\rangle &= -e^{4\mu^s(\lambda)\omega_i}[B_i(\lambda,\kappa) + D_i(\lambda,\kappa,\omega_i)] - e^{2\mu^s(\lambda)\omega_{i-1}} e^{2\mu^s(\lambda)\omega_i} C_i(\lambda,\kappa) \\ &\quad + O\big(e^{2\alpha^s\omega_{i-1}} e^{2\alpha^s\omega_i}\big[e^{2\alpha^s\omega_{i-2}} + e^{\max\{\alpha^s, \alpha^{ss}-\alpha^s\}\omega_{i-1}} \\ &\quad + e^{\max\{2\alpha^s, \frac{1}{2}(\alpha^s-\alpha^u), \alpha^{ss}-\alpha^s\}\omega_i}\big] + e^{\max\{5\alpha^s, 3\alpha^s-\alpha^{ss}, 3\alpha^s-\alpha^u, \frac{3}{2}(\alpha^s-\alpha^u)\}\omega_i} \\ &\quad + e^{4\alpha^s\omega_i} e^{1/2(\alpha^s-\alpha^u)\omega_{i+1}} + e^{2\alpha^s\omega_i} e^{(\alpha^s-\alpha^u)\omega_{i+1}} + e^{3/2(\alpha^s-\alpha^u)\omega_{i+1}} \\ &\quad + e^{1/2(\alpha^s+3\alpha^u)\omega_{i+1}} e^{(\alpha^s-\alpha^u)\omega_{i+2}}\big). \end{aligned}\right\} \quad (4.51)$$

Summarising equations (4.48), (4.49), (4.50) and (4.51) concludes the proof. □

Proof of Theorem 4.3.3. The proof of Theorem 4.3.3 follows along the same lines as the proof of Theorem 4.3.1. Only this time we use the estimates that are labelled with the validity of Hypothesis (H3.7) or (H4.8), respectively. That way we obtain, cf. either (3.118) or estimates (3.114) - (3.117) combined with Remark 4.3.16

$$\boldsymbol{T}^{\mathbf{1}}_{\boldsymbol{\kappa_i}} = \begin{cases} O\left(e^{6\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}\right) & \text{, if } i+1 \in \mathbb{Z} \setminus J_\kappa, \\ O\left(e^{6\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}[e^{2\alpha^s\omega_i} + e^{2\alpha^s\omega_{i+1}} + e^{-2\alpha^u\omega_{i+2}}]\right) & \text{, if } i+1 \in J_\kappa. \end{cases}$$

If $i \in \mathbb{Z} \setminus J_\kappa$ we apply again Lemma 4.3.14 to obtain $\boldsymbol{T}^{\mathbf{21}}_{\boldsymbol{\kappa_i}}$ and make use of the estimates of $\boldsymbol{T}^{\mathbf{22}}_{\boldsymbol{\kappa_i}}$ - $\boldsymbol{T}^{\mathbf{25}}_{\boldsymbol{\kappa_i}}$ in (3.120) - (3.123). For $i \in J_\kappa$ we use Remark 4.3.16 to gain the estimate of $\boldsymbol{T}^{\mathbf{21}}_{\boldsymbol{\kappa_i}}$, further we use (3.120) and (3.122) to obtain $\boldsymbol{T}^{\mathbf{22}}_{\boldsymbol{\kappa_i}}$ and $\boldsymbol{T}^{\mathbf{24}}_{\boldsymbol{\kappa_i}}$ and we finally apply the Lemmata 4.3.20 and 4.3.22 to get $\boldsymbol{T}^{\mathbf{25}}_{\boldsymbol{\kappa_i}}$ and $\boldsymbol{T}^{\mathbf{23}}_{\boldsymbol{\kappa_i}}$. This together yield

$$\boldsymbol{T}^{\mathbf{2}}_{\boldsymbol{\kappa_i}} = \begin{cases} \left.\begin{array}{r} -e^{2\mu^s(\lambda)\omega_i}A_i(\lambda,\kappa) + O\big(e^{2\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}} + e^{2\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}] \\ + e^{-6\alpha^u\omega_{i+1}}\big) \end{array}\right\} & \text{, if } i \in \mathbb{Z} \setminus J_\kappa, \\ \left.\begin{array}{r} -e^{4\mu^s(\lambda)\omega_i}[B_i(\lambda,\kappa) + D_i(\lambda,\kappa,\omega_i)] - e^{2\mu^s(\lambda)\omega_{i-1}}e^{2\mu^s(\lambda)\omega_i}C_i(\lambda,\kappa) \\ + O\big(e^{2\alpha^s\omega_{i-1}}e^{2\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-2}} + e^{\alpha^s\omega_{i-1}} + e^{\max\{2\alpha^s,-\alpha^u\}\omega_i}] \\ + e^{\max\{5\alpha^s,\nu\alpha^s-\alpha^u,2(\alpha^s-\alpha^u)\}\omega_i} \\ + e^{4\alpha^s\omega_i}e^{-\alpha^u\omega_{i+1}} + e^{2\alpha^s\omega_i}e^{-2\alpha^u\omega_{i+1}} + e^{-6\alpha^u\omega_{i+1}}\big) \end{array}\right\} & \text{, if } i \in J_\kappa. \end{cases}$$

Finally combining the four cases concludes the proof. □

4.4 The derivative of the jump $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$

Based on the representation of the jump $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ given in Theorems 4.3.1 and 4.3.3, we now consider the derivative of ξ_i with respect to the transition times ω_j, $i, j \in \mathbb{Z}$. For that consider $\langle \xi_i, \psi_{\kappa_i} \rangle$ as a mapping $l^\infty \times \mathbb{R} \times \Sigma_{\boldsymbol{C}} \to \mathbb{R}$. We mainly concentrate on the jump representation given in Theorem 4.3.3, since we will use this for further investigation in Section 5. Analogous results also hold for the representation of ξ_i given in Theorem 4.3.1.

Theorem 4.4.1. *Assume Hypotheses (H4.1) - (H4.7) and let $\nu \geq 3$. Further assume (H4.8). For fixed κ the mapping $\xi_i(\cdot, \lambda, \kappa)$ is smooth, and we find the following expression for the partial derivative of the jump ξ_i with respect to ω_j:*

(i) If $i \in \mathbb{Z} \setminus J_\kappa$ then

$$D_{\omega_j}\langle \xi_i(\boldsymbol{\omega}, \lambda, \kappa), \psi_{\kappa_i} \rangle = -D_{\omega_j}(A_i(\lambda,\kappa)e^{2\mu^s(\lambda)\omega_i}) + D_{\omega_j}R_i(\boldsymbol{\omega}, \lambda, \kappa),$$

where

$$\begin{aligned} D_{\omega_j}R_i(\boldsymbol{\omega}, \lambda, \kappa) &= +O(e^{2\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}} + e^{2\alpha^s\omega_i}]) \\ &\quad + \begin{cases} O(e^{-2\alpha^u\omega_{i+1}}), & i+1 \in \mathbb{Z} \setminus J_\kappa \\ O(e^{-2\alpha^u\omega_{i+1}}[e^{2\alpha^s\omega_i} + e^{2\alpha^s\omega_{i+1}} + e^{-2\alpha^u\omega_{i+2}}), & i+1 \in J_\kappa \end{cases} \end{aligned}$$

(ii) If $i \in J_\kappa$ then

$$\begin{aligned} D_{\omega_j}\langle \xi_i(\boldsymbol{\omega}, \lambda, \kappa), \psi_{\kappa_i} \rangle &= -D_{\omega_j}\big([B_i(\lambda,\kappa) + D_i(\lambda,\kappa,\omega_i)]e^{4\mu^s(\lambda)\omega_i} + C_i(\lambda,\kappa)e^{2\mu^s(\lambda)(\omega_{i-1}+\omega_i)}\big) \\ &\quad + D_{\omega_j}R_i(\boldsymbol{\omega}, \lambda, \kappa), \end{aligned}$$

where

$$\begin{aligned} D_{\omega_j} R_i(\boldsymbol{\omega},\lambda,\kappa) &= (e^{2\alpha^s(\omega_{i-1}+\omega_i)}[e^{2\alpha^s\omega_{i-2}} + e^{\alpha^s\omega_{i-1}} + e^{\max\{2\alpha^s,-\alpha^u\}\omega_i}]) \\ &\quad + O(e^{\max\{5\alpha^s,\nu\alpha^s-\alpha^u,2(\alpha^s-\alpha^u)\}\omega_i}) + O(e^{4\alpha^s\omega_i}e^{-\alpha^u\omega_{i+1}}) \\ &\quad + \begin{cases} O(e^{-2\alpha^u\omega_{i+1}}), & i+1\in\mathbb{Z}\setminus J_\kappa \\ O\big(e^{-2\alpha^u\omega_{i+1}}[e^{2\alpha^s\omega_i} + e^{2\alpha^s\omega_{i+1}} + e^{-2\alpha^u\omega_{i+2}}]\big), & i+1\in J_\kappa \end{cases} \end{aligned}$$

D_i *is equal to zero for* $\nu > 3$.

The O*-terms are valid for* ω_{i-2}, ω_{i-1}, ω_i, ω_{i+1} *and* ω_{i+2} *tending to infinity.*

Comparing Theorem 4.3.3 and Theorem 4.4.1 we see that the partial derivatives of the residual terms with respect to ω_j, $j\in\mathbb{Z}$ satisfy exactly the same estimates as the residual terms themselves.

Remark 4.4.2. *For the estimates of the partial derivatives of the residual terms from Theorem 4.3.1, that is if (H4.8) is omitted, it applies analogously that they correspond to the estimates of the residual terms themselves.*

Proof of Theorem 4.4.1. Note that Theorem 4.4.1 holds under the assumption of Hypothesis (H4.8). The analogue from Section 3 is hypothesis (H3.7). So whenever we use estimates from Section 3, we use those that are valid under Hypothesis (H3.7). Further recall that here we have $\nu \geq 3$.

The statement arises from differentiating the term in Theorem 4.3.3. So it remains to estimate the derivatives of the residual terms $R_i(\boldsymbol{\omega})$. We start from the representation (3.65), where we have introduced the partitioning

$$\langle \psi_{\kappa_i}, \xi_i(\boldsymbol{\omega},\lambda,\kappa)\rangle = \boldsymbol{T^1_{\kappa_i}} + \boldsymbol{T^2_{\kappa_i}}.$$

Recall that the estimate of $\boldsymbol{T^1_{\kappa_i}}$ depends on $i+1\in\mathbb{Z}\setminus J_\kappa$ or $i+1\in J_\kappa$ whereas the estimate of $\boldsymbol{T^2_{\kappa_i}}$ depends on $i\in\mathbb{Z}\setminus J_\kappa$ or $i\in J_\kappa$.

1. We start with examining the term $D_{\omega_j}\boldsymbol{T^1_{\kappa_i}} := \sum_{k=1}^{5} D_{\omega_j}\boldsymbol{T^{1k}_{\kappa_i}}$ for $i+1\in\mathbb{Z}\setminus J_\kappa$.

Here we simply adopt the estimate given in (3.158) in Section 3.5.3 yielding

$$i+1\in\mathbb{Z}\setminus J_\kappa: \qquad D_{\omega_j}\boldsymbol{T^1_{\kappa_i}} = O\left(e^{6\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}\right). \tag{4.52}$$

2. Now we inspect the estimates of $D_{\omega_j}\boldsymbol{T^1_{\kappa_i}} := \sum_{k=1}^{5} D_{\omega_j}\boldsymbol{T^{1k}_{\kappa_i}}$ in case that $i+1\in J_\kappa$.

In this case we have, cf. Remark 4.3.16, $\left\langle \Phi^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T\psi_{\kappa_i}, P^+_{\kappa_i}(\omega_{i+1})\gamma^-_{\kappa_{i+1}}(-\omega_{i+1})\right\rangle = 0$ and thus

$$D_{\omega_j}\boldsymbol{T^{11}_{\kappa_i}} = D_{\omega_j}\left\langle \Phi^+_{\kappa_i}(0,\omega_{i+1})^T P^+_{\kappa_i}(0)^T\psi_{\kappa_i}\ ,\ \big(\tilde{P}_{\kappa_{i+1}}(\lambda,\omega_{i+1}) - P^+_{\kappa_i}(\lambda,\omega_{i+1})\big)\big(\gamma^-_{\kappa_{i+1}}(\lambda)(-\omega_{i+1}) - p\big)\right\rangle.$$

So in addition to the considerations in Section 3.5.3 that result in $D_{\omega_j}\boldsymbol{T^{11}_{\kappa_i}}$, cf. (3.153), we apply Lemma 3.4.5, and the fourth estimate in Lemma 3.5.8 and obtain

$$D_{\omega_j}\boldsymbol{T^{11}_{\kappa_i}} = O\big(e^{\max\{\nu\alpha^s-3\alpha^u,2\alpha^s-4\alpha^u\}\omega_{i+1}}\big),$$

which equals the estimate of $\boldsymbol{T^{11}_{\kappa_i}}$ in Remark 4.3.16 in case of (H4.8).

In case of $D_{\omega_j}\boldsymbol{T^{1k}_{\kappa_i}}$ for $k=2,\dots,5$ we stick to the estimates (3.154), (3.155) and (3.157) presented in Section 3.5.3. Then we end up with

$$i+1\in J_\kappa: \qquad D_{\omega_j}\boldsymbol{T^{1}_{\kappa_i}} = O\left(e^{6\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}[e^{2\alpha^s\omega_i} + e^{2\alpha^s\omega_{i+1}} + e^{-2\alpha^u\omega_{i+2}}]\right). \tag{4.53}$$

3. Next we consider the term $D_{\omega_j}\boldsymbol{T^{2}_{\kappa_i}} := \sum_{k=1}^{5} D_{\omega_j}\boldsymbol{T^{2k}_{\kappa_i}}$ for $i\in\mathbb{Z}\setminus J_\kappa$.

For $D_{\omega_j}\boldsymbol{T^{2k}_{\kappa_i}}$, $k=2,\dots,5$ we use the same estimates as listed in (3.159) in Section 3.5.3. That is

$$\left.\begin{array}{rcl} D_{\omega_j}\boldsymbol{T^{22}_{\kappa_i}} &=& O\big(e^{\max\{\nu\alpha^s-\alpha^u,2(\alpha^s-\alpha^u)\}\omega_i}\big),\\ D_{\omega_j}\boldsymbol{T^{23}_{\kappa_i}} &=& O\big(e^{2\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}} + e^{-2\alpha^u\omega_i}]\big),\\ D_{\omega_j}\boldsymbol{T^{24}_{\kappa_i}} &=& O\big(e^{\max\{\nu\alpha^s-\alpha^u,2(\alpha^s-\alpha^u)\}\omega_i}[e^{2\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}\big),\\ D_{\omega_j}\boldsymbol{T^{25}_{\kappa_i}} &=& O\big(e^{-6\alpha^u\omega_{i+1}} + e^{-2\alpha^u\omega_{i+1}}e^{2\alpha^s\omega_i} + e^{4\alpha^s\omega_i}\big). \end{array}\right\} \tag{4.54}$$

So in remains to consider $\boldsymbol{T^{21}_{\kappa_i}} = -\langle\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}\ ,\ (id-\tilde P_{\kappa_i}(\omega_i))(\gamma^+_{\kappa_{i-1}}(\omega_i)-p)\rangle$. This term contains the term $-e^{2\mu^s\omega_i}\langle\eta^-_{\kappa_i},\eta^s_{\kappa_{i+1}}\rangle$. So actually we are interested in

$$D_{\omega_j}\boldsymbol{R^{21}_i} = D_{\omega_j}\left(\langle\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, (id-\tilde P_{\kappa_i}(\omega_i))(\gamma^+_{\kappa_{i-1}}(\omega_i)-p)\rangle - e^{2\mu^s\omega_i}\langle\eta^-_{\kappa_i},\eta^s_{\kappa_{i+1}}\rangle\right) \tag{4.55}$$

To this end we first consider $D_{\omega_i}\big(\gamma^+_{\kappa_{i-1}}(\omega_i)-p-e^{\mu^s\omega_i}\eta^s_{\kappa_{i+1}}\big)$. The derivative with respect to ω_j, $j\neq i$ is zero.

Since $\gamma^+_{\kappa_{i-1}}$ satisfies the differential equation $\dot x = f(x)$ we find by applying the taylor expansion

$$\begin{array}{rcl} D_{\omega_i}\big(\gamma^+_{\kappa_{i-1}}(\omega_i)-p-e^{\mu^s\omega_i}\eta^s_{\kappa_{i+1}}\big) &=& \dot\gamma^+_{\kappa_{i-1}}(\omega_i) - \mu^s e^{\mu^s\omega_i}\eta^s_{\kappa_{i+1}}\\ &=& f(\gamma^+_{\kappa_{i-1}}(\omega_i)) - \mu^s e^{\mu^s\omega_i}\eta^s_{\kappa_{i+1}}\\ &=& f(p) + Df(p)(\gamma^+_{\kappa_{i-1}}(\omega_i)-p) + O((\gamma^+_{\kappa_{i-1}}(\omega_i)-p)^\nu)\\ && -\mu^s e^{\mu^s\omega_i}\eta^s_{\kappa_{i+1}}. \end{array}$$

With $\gamma^+_{\kappa_{i-1}}(\omega_i)-p\in T_pW^s(p)$ we have $Df(p)(\gamma^+_{\kappa_{i-1}}(\omega_i)-p) = \mu^s(\gamma^+_{\kappa_{i-1}}(\omega_i)-p)$. Then we find with $f(p)=0$ and $\gamma^+_{\kappa_{i-1}}(\omega_i)-p = e^{\mu^s\omega_i}\eta^s_{\kappa_{i-1}} + O(e^{\nu\alpha^s\omega_i})$, cf. (4.9)

$$D_{\omega_i}\big(\gamma^+_{\kappa_{i-1}}(\omega_i)-p-e^{\mu^s\omega_i}\eta^s_{\kappa_{i+1}}\big) = O(e^{\nu\alpha^s\omega_i}). \tag{4.56}$$

Along similar lines we consider $D_{\omega_i}\big(\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i} - e^{\mu^s\omega_i}\eta^-_{\kappa_i}\big)$ and obtain

$$\begin{array}{rl} & D_{\omega_i}\big(\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i} - e^{\mu^s\omega_i}\eta^-_{\kappa_i}\big)\\ = & \dot\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i} - \mu^s e^{\mu^s\omega_i}\eta^-_{\kappa_i}\\ = & -[Df(\gamma^-_{\kappa_i}(-\omega_i))]^T\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i} - \mu^s e^{\mu^s\omega_i}\eta^-_{\kappa_i}\\ = & -[Df(p) + O\big((\gamma^-_{\kappa_i}(-\omega_i)-p)^{\nu-1}\big)]^T\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i} - \mu^s e^{\mu^s\omega_i}\eta^-_{\kappa_i}. \end{array}$$

Applying $\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i} = e^{\mu^s\omega_i}\eta^-_{\kappa_i} + O(e^{(\alpha^s-(\nu-1)\alpha^u)\omega_i})$, cf. (4.14), with $\eta^-_{\kappa_i}\in E_{-Df(p)^T}(-\mu^s)$

yields with $-[Df(p)]^T\eta^-_{\kappa_i} = \mu^s\eta^-_{\kappa_i}$

$$D_{\omega_i}\big(\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i} - e^{\mu^s\omega_i}\eta^-_{\kappa_i}\big) = O(e^{(\alpha^s-(\nu-1)\alpha^u)\omega_i}). \tag{4.57}$$

Then we find from (4.55) by invoking (4.56), (4.57) and the estimates in the Lemmata 3.4.5 and 3.5.8 corresponding to $\big((id-\tilde{P}_{\kappa_i}(\omega_i)) - P^-_{\kappa_i}(-\omega_i)\big)(id - P^+_{\kappa_{i-1}}(\omega_i))$:

$$\begin{aligned} D_{\omega_j}\boldsymbol{R^{21}_i} &= D_{\omega_j}\Big(\langle\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, (id-\tilde{P}_{\kappa_i}(\omega_i))\gamma^+_{\kappa_{i-1}}(\omega_i)\rangle - e^{2\mu^s\omega_i}\langle\eta^-_{\kappa_i},\eta^s_{\kappa_{i+1}}\rangle\Big) \\ &= D_{\omega_j}\Big(\langle\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, \gamma^+_{\kappa_{i-1}}(\omega_i)\rangle - e^{2\mu^s\omega_i}\langle\eta^-_{\kappa_i},\eta^s_{\kappa_{i+1}}\rangle\Big) \\ &\quad + D_{\omega_j}\Big(\langle\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, \big((id-\tilde{P}_{\kappa_i}(\omega_i)) - P^-_{\kappa_i}(-\omega_i)\big)\gamma^+_{\kappa_{i-1}}(\omega_i)\rangle\Big) \\ &= D_{\omega_j}\Big(\langle\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i} - e^{\mu^s\omega_i}\eta^-_{\kappa_i}\ ,\ \gamma^+_{\kappa_{i-1}}(\omega_i)\rangle \\ &\quad + \langle e^{\mu^s\omega_i}\eta^-_{\kappa_i}, \gamma^+_{\kappa_{i-1}}(\omega_i) - e^{\mu^s\omega_i}\eta^s_{\kappa_{i+1}}\rangle\Big) + O\big(e^{\max\{(\nu+2)\alpha^s-\alpha^u, 4\alpha^s-2\alpha^u\}\omega_i}\big) \\ &= O\big(e^{(2\alpha^s-(\nu-1)\alpha^u)\omega_i}\big) + O\big(e^{(\nu+1)\alpha^s\omega_i}\big) + O\big(e^{\max\{(\nu+2)\alpha^s-\alpha^u, 4\alpha^s-2\alpha^u\}\omega_i}\big). \end{aligned}$$

Summarizing we find

$$D_{\omega_j}\big(\boldsymbol{T^{21}_{\kappa_i}} - (-e^{2\mu^s\omega_i}A_i(\lambda))\big) = D_{\omega_j}\boldsymbol{R^{21}_i} = O\big(e^{\max\{(\nu+1)\alpha^s, 4\alpha^s-2\alpha^u\}\omega_i}\big). \tag{4.58}$$

With this the estimate of $D_{\omega_j}\boldsymbol{R^{21}_i}$ equals the estimate of $\boldsymbol{R^{21}_i}$ in Lemma 4.3.14 for $\nu > 3$ in case that (H4.8) applies. Combining all estimates in (4.58) and (4.54) we obtain

$$i\in\mathbb{Z}\setminus J_\kappa:\ D_{\omega_j}\Big(\boldsymbol{T^{2}_{\kappa_i}} + e^{2\mu^s\omega_i}A_i(\lambda)\Big) = O\Big(e^{2\alpha^s\omega_i}\big[e^{2\alpha^s\omega_{i-1}} + e^{2\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}\big] + e^{-6\alpha^u\omega_{i+1}}\Big). \tag{4.59}$$

4. Finally it remains the term $D_{\omega_j}\boldsymbol{T^{2}_{\kappa_i}} := \sum_{k=1}^{5} D_{\omega_j}\boldsymbol{T^{2k}_{\kappa_i}}$ for $i \in J_\kappa$.

Analogously to the considerations in 2. for $\boldsymbol{T^{11}_{\kappa_i}}$ we find $\Big\langle\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i)\Big\rangle = 0$ for $i \in J_\kappa$, cf. Remark 4.3.16, and hence

$$D_{\omega_j}\boldsymbol{T^{21}_{\kappa_i}} = O\big(e^{\max\{(\nu+2)\alpha^s-\alpha^u, 4\alpha^s-2\alpha^u\}\omega_i}\big). \tag{4.60}$$

The term $\boldsymbol{T^{23}_{\kappa_i}} = -\langle\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, (id-\tilde{P}_{\kappa_i}(\omega_i))v^{+,s}_{i-1}(\boldsymbol{\omega})(\omega_i)\rangle$ contains the explicit expression $-e^{2\mu^s\omega_{i-1}}e^{2\mu^s\omega_i}C_i(\kappa)$, cf. Lemma 4.3.22. So we need the estimate of

$$\begin{aligned} D_{\omega_j}\boldsymbol{R^{23}_i} &= D_{\omega_j}\Big(\langle\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}, (id-\tilde{P}_{\kappa_i}(\omega_i))v^{+,s}_{i-1}(\boldsymbol{\omega})(\omega_i)\rangle \\ &\qquad - e^{2\mu^s\omega_{i-1}}e^{2\mu^s\omega_i}\langle\eta^-_{\kappa_i}, S^+_{\kappa_{i-1}}(0)(id-F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}}\rangle\Big). \end{aligned}$$

The derivative of the left-hand side of the scalar product in $\boldsymbol{T^{23}_{\kappa_i}}$ we obtain from (4.57). Now we see to the term $v^{+,s}_{i-1}(\boldsymbol{\omega})(\omega_i)$ on the right-hand side.

In the same way as we arrived at the estimate (4.57) we obtain from Lemma 4.3.17

$$\left.\begin{aligned} &D_{\omega_i}\big(\Phi^+_{\kappa_i}(\omega_i,0)(id-P^+_{\kappa_{i-1}}(0)) - e^{\mu^s\omega_i}S^+_{\kappa_i}(0)\big) = O(e^{\nu\alpha^s\omega_i}), \\ &D_{\omega_i}\big(\Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i) - e^{\mu^s(\omega_i+t)}R^-_{\kappa_i}(t)\big) = O(e^{\alpha^s(\omega_i+t)}e^{-(\nu-1)\alpha^u\omega_i}). \end{aligned}\right\} \tag{4.61}$$

Thereby we take into account that $Df(p)S^+_{\kappa_i}(0) = \mu^s S^+_{\kappa_i}(0)$, since $\mathrm{im}S^+_{\kappa_i}(s) = E(\mu^s(\lambda))$, and $[-Df(p)^T R^-_{\kappa_i}(t)^T]^T = \mu^s R^-_{\kappa_i}(t)$, since $\mathrm{im}R^-_{\kappa_i}(t)^T = [\ker R^-_{\kappa_i}(t)]^\perp = [E(\sigma^c_{\mu^s})]^\perp$, cf. Lemma 4.3.17.

With this we can express the derivative of $v^{+,s}_{i-1}(\boldsymbol{\omega})(\omega_i)$. To do so we trace the deduction of the term C_i that is given in the proof of the Lemmata 4.3.20 and 4.3.22. We start at (4.19) to first obtain the derivative of the term $v^{-,u}_i(\boldsymbol{\omega})(t)$. In the following we simply present the estimates of the derivatives of the residual terms we come across, namely $\mathcal{R}^{B2}_{\kappa_i}$ in (4.20), $\mathcal{R}^{B3}_{\kappa_i}$ in (4.22) and $\mathcal{R}^{B4}_{\kappa_i}$ in (4.24). Due to Lemmata 3.5.8-3.5.15 the partial derivatives always equal the estimates of the residual terms themselves:

$$\begin{aligned}
&D_{\omega_j}\big(v^{-,u}_i(\boldsymbol{\omega})(t) - \Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)a^-_i\big)\\
&\qquad\qquad = D_{\omega_j}\mathcal{R}^{B2}_{\kappa_i}(\boldsymbol{\omega})(t) = O(e^{\alpha^s(2\omega_i+t)}(e^{\alpha^s\omega_i}+e^{-2\alpha^u\omega_{i+1}})) + O(e^{-4\alpha^u\omega_{i+1}}),\\
&D_{\omega_j}\big(\Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)[a^-_i - (id-\tilde{P}_{\kappa_i}(\omega_i))\gamma^+_{\kappa_{i-1}}(\omega_i)]\big)\\
&\qquad\qquad = D_{\omega_j}\mathcal{R}^{B3}_{\kappa_i}(\boldsymbol{\omega})(t) = O(e^{\alpha^s(2\omega_i+t)}[e^{\max\{(\nu-1)\alpha^s,-\alpha^u\}\omega_i} + e^{2\alpha^s\omega_{i-1}}]),\\
&D_{\omega_j}\big(\Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)[(id-\tilde{P}_{\kappa_i}(\omega_i)) - id]\gamma^+_{\kappa_{i-1}}(\omega_i)\big)\\
&\qquad\qquad = D_{\omega_j}\mathcal{R}^{B4}_{\kappa_i}(\boldsymbol{\omega})(t) = O\big(e^{\alpha^s(\omega_i+t)}e^{\max\{(\nu+1)\alpha^s-\alpha^u,3\alpha^s-2\alpha^u\}\omega_i}\big).
\end{aligned}$$

The final estimate we obtain with (4.56) and (4.61):

$$\begin{aligned}
&D_{\omega_j}\big(\Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i) - e^{\mu^s(2\omega_i+t)}R^-_{\kappa_i}(t)\eta^s_{\kappa_{i-1}}\big)\\
&\quad = D_{\omega_j}\big([\Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i) - e^{\mu^s(\omega_i+t)}R^-_{\kappa_i}(t)]\gamma^+_{\kappa_{i-1}}(\omega_i) + e^{\mu^s(\omega_i+t)}R^-_{\kappa_i}(t)[\gamma^+_{\kappa_{i-1}}(\omega_i) - e^{\mu^s\omega_i}\eta^s_{\kappa_{i-1}}]\big)\\
&\quad = O\big(e^{\alpha^s(\omega_i+t)}e^{\nu\alpha^s\omega_i}\big).
\end{aligned}$$

Summarizing we find

$$\left.\begin{aligned}
&D_{\omega_j}\big(v^{-,u}_i(\boldsymbol{\omega})(t) - \Phi^-_{\kappa_i}(t,-\omega_i)P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i)\big)\\
&D_{\omega_j}\big(v^{-,u}_i(\boldsymbol{\omega})(t) - e^{\mu^s(2\omega_i+t)}R^-_{\kappa_i}(t)\eta^s_{\kappa_{i-1}}\big)
\end{aligned}\right\} = \begin{aligned}&O\big(e^{\alpha^s(2\omega_i+t)}[e^{2\alpha^s\omega_{i-1}}+e^{\alpha^s\omega_i}\\ &\quad +e^{-2\alpha^u\omega_{i+1}}] + e^{-4\alpha^u\omega_{i+1}}\big).\end{aligned} \tag{4.62}$$

From this we proceed with estimating the derivative of $v^{+,s}_i(\boldsymbol{\omega})(\omega_i)$ starting from the representation in (4.37):

$$\begin{aligned}
D_{\omega_j}\big(v^{+,s}_{i-1}(\omega_i) - \Phi^+_{\kappa_{i-1}}(\omega_i,0)(id-P^+_{\kappa_{i-1}}(0))(id-F_{\kappa_{i-1}})v^{-,u}_{i-1}(0)\big) &= D_{\omega_j}\mathcal{R}^{C1}_{\kappa_{i-1}}(\boldsymbol{\omega},\kappa)\\
&= O\left(e^{\alpha^s\omega_i}\left(e^{-\alpha^u\omega_i}+e^{2\alpha^s\omega_{i-1}}\right)^2\right),
\end{aligned}$$

and by invoking (4.61) and (4.62) and Lemmata 3.4.9, 3.5.15 for the estimates regarding $v^{-,u}_{i-1}$

$$\begin{aligned}
&D_{\omega_j}\big(\Phi^+_{\kappa_{i-1}}(\omega_i,0)(id-P^+_{\kappa_{i-1}}(0))(id-F_{\kappa_{i-1}})v^{-,u}_{i-1}(0) - e^{\mu^s\omega_i}e^{2\mu^s\omega_{i-1}}S^+_{\kappa_{i-1}}(0)(id-F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}}\big)\\
&\quad = D_{\omega_j}\Big([\Phi^+_{\kappa_{i-1}}(\omega_i,0)(id-P^+_{\kappa_{i-1}}(0)) - e^{\mu^s\omega_i}S^+_{\kappa_{i-1}}(0)](id-F_{\kappa_{i-1}})v^{-,u}_{i-1}(0)\\
&\qquad + e^{\mu^s\omega_i}S^+_{\kappa_{i-1}}(0)(id-F_{\kappa_{i-1}})[v^{-,u}_{i-1}(0) - (e^{2\mu^s\omega_{i-1}}R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}})]\Big)\\
&\quad = O(\big(e^{2\alpha^s\omega_{i-1}}e^{\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-2}} + e^{\alpha^s\omega_{i-1}} + e^{\max\{(\nu-1)\alpha^s,-2\alpha^u\}\omega_i}] + e^{(\alpha^s-4\alpha^u)\omega_i}\big).
\end{aligned}$$

Combining the latest two equations we find

$$\left.\begin{aligned} & D_{\omega_j}\left(v_{i-1}^{+,s}(\omega_i) - e^{\mu^s\omega_i}e^{2\mu^s\omega_{i-1}}S_{\kappa_{i-1}}^{+}(0)(id - F_{\kappa_{i-1}})R_{\kappa_{i-1}}^{-}(0)\eta_{\kappa_{i-2}}^{s}\right) \\ = \quad & O\left(e^{2\alpha^s\omega_{i-1}}e^{\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-2}} + e^{\alpha^s\omega_{i-1}} + e^{\max\{(\nu-1)\alpha^s,-\alpha^u\}\omega_i}]\right) + O\left(e^{(\alpha^s-2\alpha^u)\omega_i}\right). \end{aligned}\right\} \tag{4.63}$$

Recall that $v_{i-1}^{+,s}(\omega_i) = (id - P_{\kappa_{i-1}}^{+}(\omega_i))v_{i-1}^{+,s}(\omega_i)$. Then, due to Lemmata 3.4.5, 3.5.8 and 3.4.9, 3.5.15 we find for $(id - \tilde{P}_{\kappa_i}(\omega_i))v_i^{+,s}(\boldsymbol{\omega})(\omega_i)$ that

$$\begin{aligned} D_{\omega_j}\left(((id - \tilde{P}_{\kappa_i}(\omega_i)) - P_{\kappa_i}^{-}(-\omega_i))v_{i-1}^{+,s}(\boldsymbol{\omega})(\omega_i)\right) \quad = \quad & O\big(e^{\max\{(\nu+1)\alpha^s-\alpha^u,3\alpha^s-2\alpha^u\}\omega_i} \\ & [e^{-2\alpha^u\omega_i} + e^{\alpha^s\omega_{i-1}}e^{-2\alpha^u\omega_i} + e^{2\alpha^s\omega_{i-1}}]\big). \end{aligned} \tag{4.64}$$

Now we have collected every estimate in order to investigate the derivative of the residual $\boldsymbol{R_i^{23}}$ that we write as

$$\begin{aligned} D_{\omega_j}\boldsymbol{R_i^{23}} \quad = \quad & D_{\omega_j}\left(\langle\Phi_{\kappa_i}^{-}(0,-\omega_i)^T P_{\kappa_i}^{-}(0)^T\psi_{\kappa_i}, (id - \tilde{P}_{\kappa_i}(\omega_i))v_{i-1}^{+,s}(\boldsymbol{\omega})(\omega_i)\rangle - e^{2\mu^s\omega_{i-1}}e^{2\mu^s\omega_i}C_i(\kappa)\right) \\ = \quad & D_{\omega_j}\Big(\langle\Phi_{\kappa_i}^{-}(0,-\omega_i)^T P_{\kappa_i}^{-}(0)^T\psi_{\kappa_i}, v_{i-1}^{+,s}(\boldsymbol{\omega})(\omega_i) + \big((id - \tilde{P}_{\kappa_i}(\omega_i)) - P_{\kappa_i}^{-}(-\omega_i)\big)v_{i-1}^{+,s}(\boldsymbol{\omega})(\omega_i)\rangle \\ & -e^{2\mu^s\omega_{i-1}}e^{2\mu^s\omega_i}\langle\eta_{\kappa_i}^{-}, S_{\kappa_{i-1}}^{+}(0)(id - F_{\kappa_{i-1}})R_{\kappa_{i-1}}^{-}(0)\eta_{\kappa_{i-2}}^{s}\rangle\Big) \\ = \quad & D_{\omega_j}\big(\langle\Phi_{\kappa_i}^{-}(0,-\omega_i)^T P_{\kappa_i}^{-}(0)^T\psi_{\kappa_i} - e^{\mu^s\omega_i}\eta_{\kappa_i}^{-} \;,\; v_{i-1}^{+,s}(\boldsymbol{\omega})(\omega_i)\rangle \\ & +\langle e^{\mu^s\omega_i}\eta_{\kappa_i}^{-}, v_{i-1}^{+,s}(\boldsymbol{\omega})(\omega_i) - e^{2\mu^s\omega_{i-1}}e^{\mu^s\omega_i}S_{\kappa_{i-1}}^{+}(0)(id - F_{\kappa_{i-1}})R_{\kappa_{i-1}}^{-}(0)\eta_{\kappa_{i-2}}^{s}\rangle\big) \\ & +D_{\omega_j}\left(\langle\Phi_{\kappa_i}^{-}(0,-\omega_i)^T P_{\kappa_i}^{-}(0)^T\psi_{\kappa_i}, \big((id - \tilde{P}_{\kappa_i}(\omega_i)) - P_{\kappa_i}^{-}(-\omega_i)\big)v_{i-1}^{+,s}(\boldsymbol{\omega})(\omega_i)\rangle\right). \end{aligned}$$

Together with (4.57), (4.63), (4.64) and Lemmata 3.4.9, 3.5.15 this results in

$$\left.\begin{aligned} & D_{\omega_j}\left(\boldsymbol{T_{\kappa_i}^{23}} - (-e^{2\mu^s\omega_{i-1}}e^{2\mu^s\omega_i}C_i(\kappa))\right) = D_{\omega_j}\boldsymbol{R_i^{23}} \\ & = O\left(e^{2\alpha^s\omega_{i-1}}e^{2\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-2}} + e^{\alpha^s\omega_{i-1}} + e^{\max\{(\nu-1)\alpha^s,-\alpha^u\}\omega_i}]\right) + O\left(e^{2(\alpha^s-\alpha^u)\omega_i}\right). \end{aligned}\right\} \tag{4.65}$$

This estimate equals the estimate of $\boldsymbol{R_i^{23}}$ in Lemma 4.3.22 in case that (H4.8) applies.

We continue with $\boldsymbol{T_{\kappa_i}^{25}} = -\langle\psi_{\kappa_i}, \int_{-\omega_i}^{0}\Phi_{\kappa_i}^{-}(0,s)P_{\kappa_i}^{-}(s)h_{\kappa_i}^{-}(s,v_i^{-}(s))ds\rangle$ that contains the term $-e^{4\mu^s\omega_i}(B_i + D_i)$, see Lemma 4.3.20. Hence we need to estimate the residual term $\boldsymbol{R_i^{25}}$. To do so we trace the deduction of the terms B_i and D_i in the proof of Lemma 4.3.20.

Starting from (4.18) we find

$$\left.\begin{aligned} & D_{\omega_j}\Big\langle\psi_{\kappa_i}, \int_{-\omega_i}^{0}\Phi_{\kappa_i}^{-}(0,s)P_{\kappa_i}^{-}(s)h_{\kappa_i}^{-}(s,v_i^{-}(s))ds \\ & \qquad -\tfrac{1}{2}\int_{-\omega_i}^{0}\Phi_{\kappa_i}^{-}(0,s)P_{\kappa_i}^{-}(s)D^2f(\gamma_{\kappa_i}^{-}(s))[v_i^{-,u}(s), v_i^{-,u}(s)]ds \\ & \qquad -\tfrac{1}{6}\int_{-\omega_i}^{0}\Phi_{\kappa_i}^{-}(0,s)P_{\kappa_i}^{-}(s)D^3f(\gamma_{\kappa_i}^{-}(s))[v_i^{-,u}(s), v_i^{-,u}(s), v_i^{-,u}(s)]ds\Big\rangle \\ & = D_{\omega_j}\langle\psi_{\kappa_i}, \mathcal{R}_{\kappa_i}^{B1}(\boldsymbol{\omega})\rangle \\ & = O\left(e^{4\alpha^s\omega_i}[e^{\alpha^s\omega_i} + e^{-\alpha^u\omega_{i+1}}] + e^{2\alpha^s\omega_i}e^{-2\alpha^u\omega_{i+1}} + e^{-6\alpha^u\omega_{i+1}}\right). \end{aligned}\right\} \tag{4.66}$$

Due to Lemmata 3.5.13, 3.4.9 and 3.5.15 the partial derivative of the residual term $\mathcal{R}^{B1}_{\kappa_i}(\boldsymbol{\omega})$, (4.17), with respect to ω_j satisfies the same estimate as the term itself. Note that for $j = i$ additionally the term

$$\begin{aligned}
&\tfrac{d}{dt}\langle \psi_{\kappa_i}, \mathcal{R}^{B1}_{\kappa_i}(\boldsymbol{\omega})(t)\rangle|_{t=\omega_i} \\
&\quad = \langle \psi_{\kappa_i},\ \Phi^-_{\kappa_i}(0,-\omega_i)P^-_{\kappa_i}(-\omega_i)\big[h^-_{\kappa_i}(-\omega_i, v^-_i(-\omega_i)) - {}^1\!/\!_2 D^2 f(\gamma^-_{\kappa_i}(-\omega_i))[v^{-,u}_i(-\omega_i), v^{-,u}_i(-\omega_i)] \\
&\qquad\qquad -{}^1\!/\!_6 D^3 f(\gamma^-_{\kappa_i}(-\omega_i))[v^{-,u}_i(-\omega_i), v^{-,u}_i(-\omega_i), v^{-,u}_i(-\omega_i)]\big]\rangle
\end{aligned}$$

has to be estimated. Also this term is due to Lemmata 3.4.8, 3.4.9 and the estimate from the exponential dichotomy (3.16) with $\alpha^-_{\kappa_i} = -\alpha^s$ at least of the same order as $\mathcal{R}^{B1}_{\kappa_i}(\boldsymbol{\omega})$.

Considering the first subtrahend from the right-hand side of the scalar product in (4.66), also cf. (4.30), we obtain with (4.62)

$$\begin{aligned}
&D_{\omega_j}\left(\int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D^2 f(\gamma^-_{\kappa_i}(s))[v^{-,u}_i(s)]^2 ds - e^{4\mu^s\omega_i}\int_{-\infty}^{0} e^{2\mu^s s}\Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D^2 f(\gamma^-_{\kappa_i}(s))[R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}]^2 ds\right) \\
&= D_{\omega_j}\left(\int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)\left[D^2 f(\gamma^-_{\kappa_i}(s))[v^{-,u}_i(s)]^2 - e^{\mu^s(4\omega_i+2s)}D^2 f(\gamma^-_{\kappa_i}(s))[R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}]^2\right] ds \right. \\
&\qquad \left. -e^{4\mu^s\omega_i}\int_{-\infty}^{-\omega_i} e^{2\mu^s s}\Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D^2 f(\gamma^-_{\kappa_i}(s))[R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}, R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}]ds\right) \\
&= D_{\omega_j}\mathcal{R}^{B7}_{\kappa_i}(\boldsymbol{\omega},\lambda,\kappa)(\omega_i) + D_{\omega_j}\mathcal{R}^{B8}_{\kappa_i}(\boldsymbol{\omega},\lambda,\kappa)(\omega_i) \\
&= O\big(e^{4\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}] + e^{(3\alpha^s-(\nu-2)\alpha^u)\omega_i} + e^{2\alpha^s\omega_i}e^{-4\alpha^u\omega_{i+1}} + e^{-8\alpha^u\omega_{i+1}}\big)
\end{aligned}$$

Thereby we estimated the partial derivatives of the integral terms along the same lines as for the residual terms $\mathcal{R}^{B7}_{\kappa_i}$, (4.32), and $\mathcal{R}^{B8}_{\kappa_i}$, (4.31), in the proof of 4.3.20. The additional terms in case that $j = i$,

$$\frac{d}{dt}\mathcal{R}^{B8}_{\kappa_i}(\boldsymbol{\omega},\lambda,\kappa)(t)|_{t=\omega_i} = e^{2\mu^s\omega_i}\Phi^-_{\kappa_i}(0,-\omega_i)P^-_{\kappa_i}(-\omega_i)D^2 f(\gamma^-_{\kappa_i}(-\omega_i))[R^-_{\kappa_i}(-\omega_i)\eta^s_{\kappa_{i-1}}, R^-_{\kappa_i}(-\omega_i)\eta^s_{\kappa_{i-1}}]$$

and

$$\begin{aligned}
&\tfrac{d}{dt}\mathcal{R}^{B7}_{\kappa_i}(\boldsymbol{\omega},\lambda,\kappa)(t)|_{t=\omega_i} \\
&\quad = \Phi^-_{\kappa_i}(0,-\omega_i)P^-_{\kappa_i}(-\omega_i)\left[D^2 f(\gamma^-_{\kappa_i}(-\omega_i))[v^{-,u}_i(-\omega_i) - e^{\mu^s\omega_i}R^-_{\kappa_i}(-\omega_i)\eta^s_{\kappa_{i-1}}, v^{-,u}_i(-\omega_i)]\right. \\
&\qquad\qquad \left. +D^2 f(\gamma^-_{\kappa_i}(-\omega_i))[e^{\mu^s\omega_i}R^-_{\kappa_i}(-\omega_i)\eta^s_{\kappa_{i-1}}, v^{-,u}_i(-\omega_i) - e^{\mu^s\omega_i}R^-_{\kappa_i}(-\omega_i)\eta^s_{\kappa_{i-1}}]\right]
\end{aligned}$$

can be estimated by applying exponential dichotomy (3.16), 3.4.9 for the estimate of $v^{-,u}_i$, (4.28) and the subsequent estimate of $\mathcal{R}^{B6}_{\kappa_i}$ for the difference between $v^{-,u}_i$ and $R^-_{\kappa_i}\eta^s_{\kappa_{i-1}}$ and the fact that $D^2 f(\gamma^-_{\kappa_i}(-\omega_i)) = O(e^{-(\nu-2)\alpha^u\omega_i})$. With this we obtain the estimates $O\left(e^{(3\alpha^s-(\nu-2)\alpha^u)\omega_i}\right)$ and $O\big(e^{(\alpha^s-(\nu-2)\alpha^u)\omega_i}(e^{2\alpha^s\omega_i}[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}] + e^{\alpha^s\omega_i}e^{-4\alpha^u\omega_{i+1}} + e^{-8\alpha^u\omega_{i+1}})\big)$, respectively.

Analogous considerations for the second subtrahend in (4.66) provides

$$\begin{aligned}
D_{\omega_j}\big(I^3_{\kappa_i}(\boldsymbol{\omega},\lambda,\kappa)(\omega_i)\big) &= D_{\omega_j}\left(\int_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D^3 f(\gamma^-_{\kappa_i}(s))[v^{-,u}_i(s)]^3\right) \\
&= O\big(e^{4\alpha^s\omega_i}[e^{\max\{2\alpha^s,-\alpha^u\}\omega_i} + e^{-4\alpha^u\omega_{i+1}}] + e^{2\alpha^s\omega_i}e^{-8\alpha^u\omega_{i+1}} + e^{-12\alpha^u\omega_{i+1}}\big),
\end{aligned}$$

if $\nu > 3$ and with (4.62)

$$\begin{aligned} & D_{\omega_j}\Bigg(\int\limits_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D^3 f(\gamma^-_{\kappa_i}(s))[v_i^{-,u}(s)]^3 \\ & \quad - \int\limits_{-\omega_i}^{0} \Phi^-_{\kappa_i}(0,s)P^-_{\kappa_i}(s)D^3 f(\gamma^-_{\kappa_i}(s))[\Phi^-_{\kappa_i}(s,-\omega_i)P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i)]^3 ds\Bigg) \\ & = D_{\omega_j}\big(\mathcal{R}^{B10}_{\kappa_i}(\boldsymbol{\omega})\big) \\ & = O\left(e^{4\alpha^s\omega_i}\left[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}\right] + e^{3\alpha^s\omega_i}e^{-4\alpha^u\omega_{i+1}} + e^{2\alpha^s\omega_i}e^{-8\alpha^u\omega_{i+1}} + e^{-12\alpha^u\omega_{i+1}}\right), \end{aligned}$$

in case that $\nu = 3$. Thereby the derivatives are estimated in the same way as the integration term in Lemma 3.4.12 for $\nu > 3$ and as the residual term $\mathcal{R}^{B10}_{\kappa_i}$ in the proof of Lemma 4.3.20 for $\nu = 3$. In case that $j = i$ these estimates also include the estimate of the additional terms

$$\begin{aligned} \tfrac{d}{dt} I^3_{\kappa_i}(\boldsymbol{\omega},\lambda,\kappa)(t)|_{t=\omega_i} &= \Phi^-_{\kappa_i}(0,-\omega_i)P^-_{\kappa_i}(-\omega_i)D^3 f(\gamma^-_{\kappa_i}(-\omega_i))[v_i^{-,u}(-\omega_i)]^3 \\ &= O\big(e^{(\alpha^s-(\nu-3)\alpha^u)\omega_i}(e^{\alpha^s\omega_i} + e^{-4\alpha^u\omega_{i+1}})^3\big), \end{aligned}$$

in case that $\nu > 3$ and

$$\frac{d}{dt}\mathcal{R}^{B10}_{\kappa_i}(\boldsymbol{\omega})(t)|_{t=\omega_i} = O\big(e^{\alpha^s\omega_i}(e^{\alpha^s\omega_i} + e^{-4\alpha^u\omega_{i+1}})^2(e^{\alpha^s\omega_i}\left[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{-2\alpha^u\omega_{i+1}}\right] + e^{-4\alpha^u\omega_{i+1}})\big),$$

if $\nu = 3$.

Summarizing this provides

$$\left.\begin{aligned} & D_{\omega_j}\big(\boldsymbol{T}^{\boldsymbol{25}}_{\boldsymbol{\kappa_i}} - (-e^{4\mu^s\omega_i}(B_i(\kappa) + D_i(\kappa)))\big) = D_{\omega_j}\boldsymbol{R}^{\boldsymbol{25}}_{\boldsymbol{i}} \\ & = \begin{cases} D_{\omega_j}\langle\psi_{\kappa_i}, \mathcal{R}^{B1}_{\kappa_i} + \mathcal{R}^{B7}_{\kappa_i} + \mathcal{R}^{B8}_{\kappa_i} + \mathcal{R}^{B10}_{\kappa_i}\rangle & \text{, if } \nu = 3, \\ D_{\omega_j}\langle\psi_{\kappa_i}, \mathcal{R}^{B1}_{\kappa_i} + \mathcal{R}^{B7}_{\kappa_i} + \mathcal{R}^{B8}_{\kappa_i} + I^3_{\kappa_i}\rangle & \text{, if } \nu > 3, \end{cases} \\ & = O\big(e^{4\alpha^s\omega_i}\left[e^{2\alpha^s\omega_{i-1}} + e^{\alpha^s\omega_i} + e^{-\alpha^u\omega_{i+1}}\right] + e^{(3\alpha^s-(\nu-2)\alpha^u)\omega_i} \\ & \quad + e^{2\alpha^s\omega_i}e^{-2\alpha^u\omega_{i+1}} + e^{-6\alpha^u\omega_{i+1}}\big) \end{aligned}\right\} \tag{4.67}$$

This estimate equals the estimate of $\boldsymbol{R}^{\boldsymbol{25}}_{\boldsymbol{i}}$ in Lemma 4.3.20 in case that (H4.8) applies.

Finally we simply adopt the estimates from 3. in case of $\boldsymbol{T}^{\boldsymbol{22}}_{\boldsymbol{\kappa_i}}$ and $\boldsymbol{T}^{\boldsymbol{24}}_{\boldsymbol{\kappa_i}}$, cf. (4.54). Together with (4.60), (4.65) and (4.67) this results in

$$\left.\begin{aligned} i \in J_\kappa:\quad & D_{\omega_j}\left(\boldsymbol{T}^{\boldsymbol{2}}_{\boldsymbol{\kappa_i}} - \big(-e^{4\mu^s\omega_i}(B_i(\kappa) + D_i(\kappa)) - e^{2\mu^s\omega_{i-1}}e^{2\mu^s\omega_i}C_i(\kappa)\big)\right) \\ = \quad & O\big(e^{2\alpha^s\omega_{i-1}}e^{2\alpha^s\omega_i}\left[e^{2\alpha^s\omega_{i-2}} + e^{\alpha^s\omega_{i-1}} + e^{\max\{2\alpha^s,-\alpha^u\}\omega_i}\right] + e^{4\alpha^s\omega_i}e^{-\alpha^u\omega_{i+1}} \\ & + e^{\max\{5\alpha^s,\nu\alpha^s-\alpha^u,2(\alpha^s-\alpha^u)\}\omega_i} + e^{2\alpha^s\omega_i}e^{-2\alpha^u\omega_{i+1}} + e^{-6\alpha^u\omega_{i+1}}\big). \end{aligned}\right\} \tag{4.68}$$

The estimates in (4.52), (4.53), (4.59) and (4.68) equal those of the terms $\boldsymbol{T}^{\boldsymbol{1}}_{\boldsymbol{\kappa_i}}$ and $\boldsymbol{T}^{\boldsymbol{2}}_{\boldsymbol{\kappa_i}}$ in the proof of Theorem 4.3.3. Hence the estimates that apply for the residual terms of the jump ξ_i also apply for their derivatives. This concludes the proof. □

5 Nonwandering dynamics for D_{4m}-equivariant homoclinic cycles

The prototype system where we find orthogonal fixed point spaces is gained from vector fields which are equivariant with respect to the dihedral group D_k where k is a multiple of 4. Therefore this section is dedicated to the solving of the system of determination equations $\Xi = 0$ in case of D_{4m}-symmetry.

In the subsequent Section 5.1 we present the precise setting of this Chapter. In Section 5.2 we then discuss the sign of the quantities $B_i(\lambda, \kappa)$ and $C_i(\lambda, \kappa)$ in Lemmata 5.2.6 and 5.2.1 and show that $D_i(\lambda, \kappa, \omega_i)$ is equal to zero. Combining these information with the estimates of the jump $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ presented in the last chapter results in Theorem 5.3.1. Based on this representation we then formulate our main results. Theorem 5.3.3 describes the occurrence of shift dynamics under the condition that $|B_i(0, \kappa)| > |C_i(0, \kappa)|$. The proof is given in Section 5.4. In case that $|B_i(0, \kappa)| \leq |C_i(0, \kappa)|$ the behaviour of the nonwandering dynamic completely differs from that described in Theorem 5.3.3. We record this statement in Theorem 5.3.4 that we proof in Section 5.5.

5.1 Setting

At this point we introduce the dihedral group D_{4m} and describe its intended impact on the vector field. Thereby we orientate ourselves to [HJKL11, Table 1, Case 6].

The dihedral group D_{4m} is the symmetry group of a regular $4m$-gon in the plane. The group D_{4m} can be written as semidirect product of the reflection group $\mathbb{Z}_2(\zeta)$ and the rotation group $\mathbb{Z}_{4m}(\theta_{4m})$:

$$D_{4m} = \mathbb{Z}_2(\zeta) \ltimes \mathbb{Z}_{4m}(\theta_{4m}) \tag{5.1}$$

where the reflection ζ and the rotation θ_{4m} denote the generators of the corresponding cyclic subgroups which generate the whole group. This notion further implies that $\mathbb{Z}_2(\zeta) \cap \mathbb{Z}_{4m}(\theta_{4m}) = \{id\}$ and that $\mathbb{Z}_{4m}(\theta_{4m})$ is a normal subgroup of D_{4m} meaning that $g\mathbb{Z}_{4m}(\theta_{4m}) = \mathbb{Z}_{4m}(\theta_{4m})g$ for all $g \in D_{4m}$. Note that θ_{4m} and ζ do not commute. Figure 5.1 exemplarily displays the symmetry group D_4 of the regular 4-gon, the square.

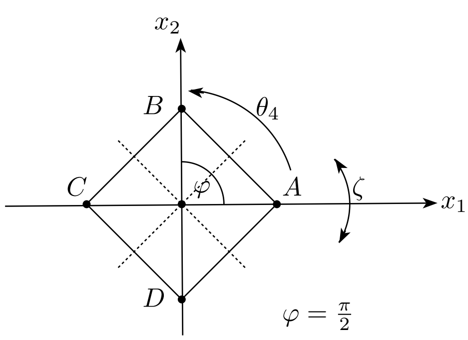


Figure 5.1: The symmetry group D_4 of the square.

Note that the generators ζ and θ_{4m} satisfy

$$\zeta^2 = id, \qquad (\theta_{4m}^{2m})^2 = id \qquad \text{and} \qquad (\theta_{4m}\zeta)^2 = id. \tag{5.2}$$

In the course of our treatment we focus on the particular homoclinic cycle which is characterized by Hypotheses (H4.1) with $G = D_{4m}$:

(H5.1).

(i) The vector field $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *is smooth, i.e.* $f \in C^{l+3}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$, $l \geq \max\{3, \nu\}$, *and* $f(\cdot, \lambda)$ *is equivariant with respect to the dihedral group* D_{4m} *for all* $\lambda \in \mathbb{R}$.

(ii) At $\lambda = 0$ *there is a homoclinic cycle* $\Gamma = G(\overline{\gamma})$ *equal to the closure of the group orbit of a homoclinic trajectory* γ *asymptotic to a hyperbolic equilibrium* p*. We demand that* $G_p = D_{4m}$.

Again recall Definition 3.4.2 for the introduction of the constant ν.

For simplification we assume

(H5.2). *The dimension of the vector field* $n = 4$.

Further we assume

(H5.3).

(i) The leading eigenspace $E(\mu^s(\lambda))$ *is two-dimensional.*

(ii) $G_p = D_{4m}$ *acts on* $E(\mu^s(\lambda))$ *as* D_{4m}.

(iii) $G_p = D_{4m}$ *acts on* $E(\mu^u(\lambda))$ *as* D_{4m}.

(iv) $0 < |\,Re(\mu^s(\lambda))| < \,Re(\mu^u(\lambda))$.

(v) The trajectory γ *has the isotropy group* $G_\gamma = \mathbb{Z}_2(\zeta)$.

Due to Hypothesis (H5.3)(v) the homoclinic cycle consists of $4m$ homoclinic trajectories. Each of them is situated within the fixed point space of a reflection. To be precise, the latter relation in (5.2) implies via induction for the counter k that $(\theta_{4m}^k\zeta)^2 = id$, $k = 1, \dots, 4m$. That is for each k the group element $\theta_{4m}^k\zeta$ defines another reflection. With $\gamma_1 := \gamma \in \mathrm{Fix}\mathbb{Z}_2(\zeta)$ we then find

$$\gamma_i := \theta_{4m}^{i-1}\gamma_1 \in \mathrm{Fix}_i = \theta_{4m}^{(i-1)}\mathrm{Fix}\mathbb{Z}_2(\zeta) = \mathrm{Fix}\mathbb{Z}_2(\theta_{4m}^{2(i-1)}\zeta). \tag{5.3}$$

Here i and $2i$ are considered modulo $4m$.

Since G_p acts on the 2-dimensional $E(\mu^s(\lambda))$ as D_{4m} we find that G_p acts absolutely irreducible on $E(\mu^s(\lambda))$. Hence $\mu^s(\lambda)$ is real and semisimple, cf. Lemma 4.1.1, and

$$\dim(\mathrm{Fix}\,\mathbb{Z}_2(\zeta) \cap E(\mu^s(\lambda))) = 1. \tag{5.4}$$

The situation described by Hypotheses (H5.1) and (H5.3) is reflected in Figure 1.1 in case of D_4-symmetry.

Since $n = 4$, $\dim E(\mu^s(\lambda)) = 2$ and G_p acts on $E(\mu^u(\lambda))$ as D_{4m} we find that also $E(\mu^u(\lambda))$ is two-dimensional, $\mu^u(\lambda)$ is real and semisimple and $\dim(\mathrm{Fix}\,\mathbb{Z}_2(\zeta) \cap E(\mu^u(\lambda))) = 1$. (We wish to point out, that (H5.3)(iii) has no equivalent in [HJKL11].)

In particular that means that $D_1f(p,0)$ has no strong stable or strong unstable eigenvalues: $\sigma(D_1f(p,0)) = \{\mu^s, \mu^u\}$. Hence we do not have to worry about inclination and orbit flip.

Eventually we find due to (H5.2) and (H5.3) that

$$\mathrm{Fix}\mathbb{Z}_2(\theta_{4m}^{2m}) = 0 \quad \text{and} \quad \dim(\mathrm{Fix}\mathbb{Z}_2(\zeta)) = 2. \tag{5.5}$$

Since $U_1 \oplus Z_1 \subseteq \mathrm{Fix}_1 = \mathrm{Fix}\mathbb{Z}_2(\zeta)$, cf. Lemma 4.1.3 we find

$$\mathrm{Fix}_i = U_i \oplus Z_i \quad \text{and} \quad \mathrm{Fix}_i^{\perp} = W_i^+ \oplus W_i^- \tag{5.6}$$

for all $i = 1, \dots, 4m$.

Lemma 5.1.1. *Assume Hypotheses (H5.1) - (H5.3). Then the constant $\nu \geq 3$.*

Proof. From the second relation in (5.2) we find according to the proof of Corollary 4.3.5 that $\theta_{4m}^{2m} = -id$ if and only if $\mathrm{Fix}\mathbb{Z}_2(\theta_{4m}^{2m}) = 0$. Hence (5.5) implies that $\theta_{4m}^{2m}(x) = -x$ for all $x \in \mathbb{R}^n$. Together with the equivariance of the vector field, cf. (4.1), this implies that $f(\cdot, \lambda)$ is odd. Thus there exists a natural number $\nu \geq 3$, cf. Definition 3.4.2, such that

$$D_1^{\nu} f(p, \lambda) \neq 0 \quad \text{and} \quad D_1^k f(p, \lambda) = 0, \ k = 2, \dots, \nu - 1.$$

□

Regarding the homoclinic trajectory we further assume:

(H5.4).

(i) The homoclinic trajectory γ is non-degenerate, that is $T_{\gamma(0)}W^s(p, 0) \cap T_{\gamma(0)}W^u(p, 0) = \mathrm{span}\{\dot{\gamma}(0)\}$.

(ii) Further the restriction of the manifolds to the fixed point space $FixG_\gamma$, $W^s_{FixG_\gamma}(p)$ and $W^u_{FixG_\gamma}(p)$ split with non-zero speed in λ.

In Section 2.6 we discussed the property of γ being twisted or non-twisted, cf. Figure 2.4. The setting here does not allow such a division of the homoclinic trajectory, since our leading eigenvalues are semisimple rather than simple. However, the quantity $\mathcal{O} = \mathrm{sgn}(\langle e^s, e^- \rangle \langle e^u, e^+ \rangle)$ introduced in Corollary 2.6.7 still can be calculated even though we have left the context of Section 2.6.

Lemma 5.1.2. *Assume Hypotheses (H5.1)-(H5.4). Then we find for the homoclinic trajectory γ that constitutes the homoclinic cycle Γ that $\mathcal{O} = 1$.*

Proof. Recall that e^s and e^u denote the directions at which γ is approaching the equilibrium, cf. (2.58), and e^-, e^+ denote the transported directions of $\psi \in Z = (T_{\gamma(0)}W^s(p) + T_{\gamma(0)}W^u(p))^{\perp}$ along the homoclinic trajectory via the transition matrix Ψ of the adjoint variational equation $\dot{x} = -[D_1 f(\gamma, 0)]^T x$ for $t \to \pm\infty$, respectively, cf. (2.59).

Recall from Lemma 4.1.3 that Z is situated in $\mathrm{Fix}G_\gamma$. Further Lemma 4.3.10 implies that the transition matrix Ψ leaves $\mathrm{Fix}G_\gamma$ invariant. Hence $\psi(t) = \Psi(t, 0)\psi$ with $\psi \in Z$ is situated in $\mathrm{Fix}G_\gamma$ for all $t \in \mathbb{R}$. So we find that all four directions e^s, e^u, e^+ and e^- are elements of $\mathrm{Fix}G_\gamma$.

Now, due to Hypotheses (H5.2) and (H5.3) the fixed point space $\mathrm{Fix}G_\gamma$ is two-dimensional, cf. (5.5). Then γ separates this 2-dimensional subspace into a region inside and a region outside γ. When we choose ψ such that $\mathrm{sgn}\langle e^-, e^s \rangle = -1$, cf. (1.11), then e^- points towards the outer region of γ within $\mathrm{Fix}G_\gamma$. Since $\psi(t)$ does not leave the fixed point space, it always has to point towards the outer region of γ for all $t \in \mathbb{R} \cup \{\pm\infty\}$. Hence e^+ points towards the outer region of γ as well and we find $\mathrm{sgn}\langle e^+, e^u \rangle = -1$. □

The symmetry transfers this property to all other homoclinics.

Finally we also stipulate (H4.7) which we do not wish to repeat here.

Remark 5.1.3. *With Hypotheses (H5.1) - (H5.4) and (H4.7) the assumptions for applying Lin's method, Hypotheses (H3.1) - (H3.5), and the Hypotheses of Section 4, (H4.1) - (H4.8), are satisfied.*

Remark 5.1.4. *The above Hypotheses (H5.1) - (H5.4) entail the conditions demanded in [HJKL11].*

5.2 Discussing the Leading terms in case of D_{4m}-symmetry

In this section we study the terms $B_i(\lambda,\kappa), C_i(\lambda,\kappa)$ and $D_i(\lambda,\kappa,\omega_i)$ in case of D_{4m}-symmetry in $\mathbb{R}^4$ when $i \in J_\kappa$. Recall that $i \in J_\kappa$ means that $\mathrm{Fix}_{\kappa_i} \perp \mathrm{Fix}_{\kappa_{i-1}}$, cf. (4.6). Note that the obtained statements concerning $B_i(\lambda,\kappa)$ and $C_i(\lambda,\kappa)$ cannot automatically be transferred to higher-dimensional spaces. In particular, we use in the following the absence of strong stable eigenvalues and the fact that the rotation θ_{4m}^{2m} does not possess a non-trivial fixed point space.

Regarding the quantity $C_i(\lambda,\kappa)$, that is given in Lemma 4.3.22 we know the following:

Lemma 5.2.1. *Let $|\lambda| < c$ and $\inf \boldsymbol{\omega} > \Omega$ according to Lemma 4.3.22. Let further $\kappa \in \Sigma_{4m}$ and assume Hypotheses (H5.1) - (H5.4). If $i-1, i \in J_\kappa$, then $C_i(\lambda,\kappa) \neq 0$. Moreover, there exists a $C(\lambda)$ such that for all $i \in J_\kappa$ for which also $i-1 \in J_\kappa$ either $C_i(\lambda,\kappa) = -C(\lambda)$ or $C_i(\lambda,\kappa) = C(\lambda)$ depending on $\kappa_{i-2} = \kappa_i$ or $\kappa_{i-2} = 2m + \kappa_i$.*

Note that with $i-1, i \in J_\kappa$ we find that $\mathrm{Fix}_{\kappa_{i-2}} = \mathrm{Fix}_{\kappa_i}$. Since the fixed point spaces contain two homoclinic trajectories, we only have the two options that either $\kappa_{i-2} = \kappa_i$ or $\kappa_{i-2} = 2m + \kappa_i$, cf. also the proof below.

The first condition in Lemma 5.2.1, $i \in J_\kappa$, is always satisfied in our considerations. If the second condition $i-1 \in J_\kappa$ is violated we will find, cf. Section 5.4 below, that the term $C_i(\lambda,\kappa)$ has no impact on the determination equation. Hence no further information about the sign of $C_i(\lambda,\kappa)$ in case of $i \in \mathbb{Z} \setminus J_\kappa$ or $i-1 \in \mathbb{Z} \setminus J_\kappa$ is needed.

Proof. First we show that $C_i(\lambda,\kappa) \neq 0$. To this end we start from the representation given in Lemma 4.3.22:

$$C_i(\lambda,\kappa) := \left\langle \eta^-_{\kappa_i}(\lambda), S^+_{\kappa_{i-1}}(\lambda,0)(id - F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(\lambda,0)\eta^s_{\kappa_{i-2}}(\lambda) \right\rangle.$$

From Lemma 4.3.14 we know that $\eta^s_{\kappa_{i-2}} \in \mathrm{Fix}_{\kappa_{i-2}} \cap E(\mu^s(\lambda))$. Therefore it is not situated within $\ker R^-_{\kappa_{i-1}}(\lambda,s) = E(\sigma^c_{\mu^s}(\lambda))$, cf. Lemma 4.3.17. Thus $R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} \neq 0$.

With $i-1 \in J_\kappa$ we find $\eta^s_{\kappa_{i-2}} \in \mathrm{Fix}^\perp_{\kappa_{i-1}}$ and hence $R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} \in \mathrm{Fix}^\perp_{\kappa_{i-1}}$, cf. Remark 4.3.19. More in particular from $\mathrm{im} R^-_{\kappa_{i-1}}(0) \subseteq \mathrm{im} P^-_{\kappa_{i-1}}(0) = W^+_{\kappa_{i-1}} \oplus Z_{\kappa_{i-1}}$, cf. Lemma 4.3.17 and (3.17), we infer $R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} \in W^+_{\kappa_{i-1}}$. Since $W^+_{\kappa_{i-1}} \subseteq \ker F_{\kappa_{i-1}}$, cf. the definition of F_{κ_i} in (3.38), we obtain $(id - F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} = R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} \in W^+_{\kappa_{i-1}}$.

Since there are no strong stable or strong unstable eigenvalues Lemma 4.3.17 and (3.17) yield that $\ker S^+_{\kappa_{i-1}}(0) = \mathrm{im} P^+_{\kappa_{i-1}}(\lambda,0) = W^-_{\kappa_{i-1}} \oplus Z_{\kappa_{i-1}}$ which is complementary to $W^+_{\kappa_{i-1}} \oplus U_{\kappa_{i-1}}$. Hence $S^+_{\kappa_{i-1}}(0)(id - F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} = S^+_{\kappa_{i-1}}(0)R^-_{\kappa_{i-1}}(0)\eta^s_{\kappa_{i-2}} \neq 0$.

So it remains to show that $\eta^-_{\kappa_i}(\lambda)$ is not perpendicular to $\mathrm{im} S^+_{\kappa_{i-1}}(\lambda,0) \cap \mathrm{Fix}^\perp_{\kappa_{i-1}} = E(\mu^s(\lambda)) \cap \mathrm{Fix}^\perp_{\kappa_{i-1}}$, cf. Lemma 4.3.17 and Remark 4.3.19. From its construction it follows that $\eta^-_{\kappa_i}(\lambda) \in \mathrm{Fix}_{\kappa_i} \cap [E(\sigma^c_{\mu^s}(\lambda))]^\perp$, cf. Lemma 4.3.14. But $E(\sigma^c_{\mu^s}(\lambda)) \cap E(\mu^s(\lambda)) = \{0\}$ as well as $\mathrm{Fix}_{\kappa_i} \cap \mathrm{Fix}_{\kappa_{i-1}} = \{0\}$, due to $i \in J_\kappa$. Hence $C_i(\lambda,\kappa) \neq 0$.

Now we turn to the second statement of the lemma. Let, for the moment, $\kappa_{i-1} = 1$. Then cf. (5.3), $\mathrm{Fix}_{\kappa_{i-1}} = \mathrm{Fix}\mathbb{Z}_2(\zeta)$. With $\mathrm{Fix}_{\kappa_{i-2}} \subseteq \mathrm{Fix}\mathbb{Z}_2(\zeta)^\perp$, due to $i-1 \in J_\kappa$, we find

$$\mathrm{Fix}_{\kappa_{i-2}} = \mathrm{Fix}\mathbb{Z}_2(\theta_{4m}^{2m}\zeta) = \theta_{4m}^{m}\mathrm{Fix}\mathbb{Z}_2(\zeta) = \theta_{4m}^{3m}\mathrm{Fix}\mathbb{Z}_2(\zeta),$$

cf. (5.3). Hence either

$$(a)\ \eta^s_{\kappa_{i-2}} = \eta^s_{m+1} \quad \text{or} \quad (b)\ \eta^s_{\kappa_{i-2}} = \eta^s_{3m+1} = \theta_{4m}^{2m}\eta^s_{m+1} = -\eta^s_{m+1}.$$

The latter equality in (b) yields since $\theta_{4m}^{2m} = -id$, cf. the proof of Lemma 5.1.1. Analogously we find $\mathrm{Fix}_{\kappa_i} = \mathrm{Fix}\mathbb{Z}_2(\theta_{4m}^{2m}\zeta)$, due to $i \in J_\kappa$ and hence either

$$(c)\ \eta^-_{\kappa_i} = \eta^-_{m+1} \quad \text{or} \quad (d)\ \eta^-_{\kappa_i} = \eta^-_{3m+1} = \theta_{4m}^{2m}\eta^-_{m+1} = -\eta^-_{m+1}.$$

Inserting this into the representation of C_i given in Lemma 4.3.22 yields the same absolute value in all combinations of the cases (a) or (b) with (c) or (d). Only the sign differs depending on whether $\kappa_{i-2} = \kappa_i$ or $\kappa_{i-2} = 2m + \kappa_i$.

Eventually, let $\kappa_{i-1} = j$, $j \in \{1, \dots, 4m\}$. Then $\mathrm{Fix}_j = \theta_{4m}^{j-1}\mathrm{Fix}\mathbb{Z}_2(\zeta)$ and

$$\begin{array}{llllll} (a) & \eta^s_{\kappa_{i-2}} = \theta_{4m}^{j-1}\eta^s_{m+1} & \text{or} & (b) & \eta^s_{\kappa_{i-2}} = -\theta_{4m}^{j-1}\eta^s_{m+1} & \text{and} \\ (c) & \eta^-_{\kappa_i} = \theta_{4m}^{j-1}\eta^-_{m+1} & \text{or} & (d) & \eta^-_{\kappa_i} = -\theta_{4m}^{j-1}\eta^-_{m+1}. & \end{array}$$

Finally the D_{4m}-invariance of the scalar product, cf. Remark 4.0.5, and the D_{4m}-equivariance of $R^-_{\kappa_{i-1}}$ and $S^+_{\kappa_{i-1}}$, cf. Remark 4.3.19 provide the lemma:

$$\begin{aligned} C_i(\lambda,\kappa) &= \left\langle \eta^-_{\kappa_i}(\lambda), S^+_{\kappa_{i-1}}(\lambda,0)(id - F_{\kappa_{i-1}})R^-_{\kappa_{i-1}}(\lambda,0)\eta^s_{\kappa_{i-2}}(\lambda)\right\rangle \\ &= \pm\left\langle \theta_{4m}^{j-1}\eta^-_{m+1}(\lambda), S^+_j(\lambda,0)R^-_j(\lambda,0)\theta_{4m}^{j-1}\eta^s_{m+1}(\lambda)\right\rangle \\ &= \pm\left\langle \theta_{4m}^{j-1}\eta^-_{m+1}(\lambda), \theta_{4m}^{j-1}S^+_1(\lambda,0)R^-_1(\lambda,0)\eta^s_{m+1}(\lambda)\right\rangle \\ &= \pm\left\langle \eta^-_{m+1}(\lambda), S^+_1(\lambda,0)R^-_1(\lambda,0)\eta^s_{m+1}(\lambda)\right\rangle. \end{aligned}$$

□

At this point we make some considerations about the geometrical interpretation of the term $C(\lambda = 0)$. To this end recall the definition of the fibre bundle $\mathcal{F}(W^s_\gamma)$ along a homoclinic trajectory γ, cf. Remark 2.6.8. Due to the geometric setting and Hypothesis (H4.7) we find that within a tubular neighbourhood of γ the fibre bundle $\mathcal{F}(W^s_\gamma)$ is situated in the stable manifold $W^s(p)$.

Lemma 5.2.2. *Let $|\lambda| < c$ and $\inf \boldsymbol{\omega} > \Omega$ according to Lemma 4.3.22. Let further $\kappa \in \Sigma_{4m}$ and assume Hypotheses (H5.1) - (H5.4). If $i-1, i \in J_\kappa$, then there exists for $\lambda = 0$ a constant $\tilde{c} \neq 0$ such that*

$$C_i(0,\kappa) = \tilde{c}\langle \eta^-_{\kappa_i}(0), \eta^s_{\kappa_{i-2}}(0)\rangle.$$

The sign of $\tilde{c}$ is related to the topological structure of $\mathcal{F}(W^s_\gamma)$. If $\mathcal{F}(W^s_\gamma)$ has the structure of a Möbius band, sgn $\tilde{c} = -1$, if it has the structure of an annulus, sgn $\tilde{c} = 1$.

Proof. We obtain the terms $C_i(\lambda,\kappa)$, cf. Lemma 4.3.22, from

$$\boldsymbol{T^{23}_{\kappa_i}} := \left\langle \Phi^-_{\kappa_i}(\lambda)(0,-\omega_i)^T P^-_{\kappa_i}(\lambda,0)^T\psi_{\kappa_i}\ ,\ (id-\tilde{P}_{\kappa_i}(\lambda,\omega_i))v^{+,s}_{i-1}(\omega_i)\right\rangle.$$

The tracing of the derivation of the leading terms from the right-hand side of the scalarproduct $\boldsymbol{T^{23}_{\kappa_i}}$, cf. the successive application of (4.37), (4.19), (4.21) and (4.23), thereby yields the representation

$$\begin{aligned}&(id-\tilde{P}_{\kappa_i}(\omega_i))v^{+,s}_{i-1}(\omega_i)\\ &\quad = P^-_{\kappa_i}(-\omega_i)\Phi^+_{\kappa_{i-1}}(\omega_i,0)(id-P^+_{\kappa_{i-1}}(0))(id-F_{\kappa_{i-1}})\Phi^-_{\kappa_{i-1}}(0,-\omega_{i-1})P^-_{\kappa_{i-1}}(-\omega_{i-1})\gamma^+_{\kappa_{i-2}}(\omega_{i-1})+h.o.t.\end{aligned}$$

We do not need to pay further attention to the projection $P^-_{\kappa_i}(-\omega_i)$ at the beginning of this expression, since its transposed already appears on the left-hand side of the scalar product $\boldsymbol{T^{23}_{\kappa_i}}$. If we additionally commute $\Phi^-_{\kappa_{i-1}}$ and $P^-_{\kappa_{i-1}}$ with each other we obtain for the scalar product

$$\left.\begin{aligned}&\left\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}\ ,\ (id-\tilde{P}_{\kappa_i}(\omega_i))v^{+,s}_{i-1}(\omega_i)\right\rangle\\ &\quad = \ \big\langle \Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i},\\ &\qquad\qquad \Phi^+_{\kappa_{i-1}}(\omega_i,0)(id-P^+_{\kappa_{i-1}}(0))(id-F_{\kappa_{i-1}})P^-_{\kappa_{i-1}}(0)\Phi^-_{\kappa_{i-1}}(0,-\omega_{i-1})\gamma^+_{\kappa_{i-2}}(\omega_{i-1})\big\rangle\\ &\qquad +h.o.t.\end{aligned}\right\}\qquad(5.7)$$

While the geometrical interpretation of the left-hand side of the scalar product with

$$\Phi^-_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i} = e^{\mu^s\omega_i}\eta^-_{\kappa_i}(\lambda)+h.o.t.$$

is clear, cf. (4.14) in Lemma 4.3.14, we continue to deal with the right-hand side. Thereby we know the following about the images and kernels of the projections in the middle of the term:

$$\begin{aligned}\mathrm{im}(id-P^+_{\kappa_{i-1}}(0)) &= T_{\gamma^+_{\kappa_{i-1}}(0)}W^s(p)\supseteq W^+_{\kappa_{i-1}},\\ \mathrm{im}(id-F_{\kappa_{i-1}}) &= W^+_{\kappa_{i-1}}+W^-_{\kappa_{i-1}},\\ \ker(id-F_{\kappa_{i-1}}) &= Z_{\kappa_{i-1}}+U_{\kappa_{i-1}}\quad\text{and}\\ \mathrm{im}P^-_{\kappa_{i-1}}(0) &= W^+_{\kappa_{i-1}}+Z_{\kappa_{i-1}},\end{aligned}$$

cf. (3.38) and (3.17) in combination with Remark 3.1.2. Consequently

$$\mathrm{im}(id-P^+_{\kappa_{i-1}}(0))(id-F_{\kappa_{i-1}})P^-_{\kappa_{i-1}}(0)=W^+_{\kappa_{i-1}}.\qquad(5.8)$$

At $\lambda=0$ we find $\gamma^+_{\kappa_{i-2}}=\gamma_{\kappa_{i-2}}$ and $\Phi^\pm_{\kappa_{i-1}}(\cdot,\cdot)=\Phi_{\kappa_{i-1}}(\cdot,\cdot)$, where $\Phi_{\kappa_{i-1}}$ denotes the transition matrix of the linear variational equation along the homoclinic solution $\gamma_{\kappa_{i-1}}$, cf. (3.10). With this we now consider the right-hand side of the scalarproduct (5.7) from right to left. At first we find due to $i-1\in J_\kappa$

$$\gamma_{\kappa_{i-2}}(\omega_{i-1})\in T_pW^s(p)\cap\mathrm{Fix}_{\kappa_{i-2}}=T_pW^s(p)\cap\mathrm{Fix}^\perp_{\kappa_{i-1}}.$$

The transition matrix $\Phi_{\kappa_{i-1}}(0,-\omega_{i-1})$ transports elements within the tangent space of the stable manifold at $\gamma_{\kappa_{i-1}}(-\omega_{i-1})$, into elements of the tangent space of the stable manifold at $\gamma_{\kappa_{i-1}}(0)$. In doing this it leaves the orthogonal complement of the fixed point space $\mathrm{Fix}_{\kappa_{i-1}}$ invariant. Hence

$$\Phi_{\kappa_{i-1}}(0,-\omega_{i-1})\gamma_{\kappa_{i-2}}(\omega_{i-1}) \in T_{\gamma_{\kappa_{i-1}}(0)}W^s(p)\cap \mathrm{Fix}^{\perp}_{\kappa_{i-1}} = W^{+}_{\kappa_{i-1}}.$$

Here recall, cf. (3.5), that $W^{+}_{\kappa_{i-1}} = T_{\gamma_{\kappa_{i-1}}(0)}W^s(p)\cap U^{\perp}_{\kappa_{i-1}}$. Applying now the three projections does due to (5.8) not effect the term, resulting in

$$(id - P^{+}_{\kappa_{i-1}}(0))(id - F_{\kappa_{i-1}})P^{-}_{\kappa_{i-1}}(0)\Phi_{\kappa_{i-1}}(0,-\omega_{i-1})\gamma_{\kappa_{i-2}}(\omega_{i-1}) = \Phi_{\kappa_{i-1}}(0,-\omega_{i-1})\gamma_{\kappa_{i-2}}(\omega_{i-1}).$$

Finally the transition matrix $\Phi_{\kappa_{i-1}}(\omega_i,0)$ transports $\Phi_{\kappa_{i-1}}(0,-\omega_{i-1})\gamma_{\kappa_{i-2}}(\omega_{i-1})$ forward in time along the homoclinic trajectory $\gamma_{\kappa_{i-1}}$ close towards the equilibium point p again:

$$\Phi_{\kappa_{i-1}}(\omega_i,0)\Phi_{\kappa_{i-1}}(0,-\omega_{i-1})\gamma_{\kappa_{i-2}}(\omega_{i-1}) = \Phi_{\kappa_{i-1}}(\omega_i,-\omega_{i-1})\gamma_{\kappa_{i-2}}(\omega_{i-1}).$$

Thereby it remains in the tangend space of the stable manifold and the orthogonal complement of the fixed point space $\mathrm{Fix}_{\kappa_{i-1}}$: $\Phi_{\kappa_{i-1}}(\omega_i,-\omega_{i-1})\gamma_{\kappa_{i-2}}(\omega_{i-1}) \in T_{\gamma_{\kappa_{i-1}}(\omega_i)}W^s(p)\cap \mathrm{Fix}^{\perp}_{\kappa_{i-1}}$. This subspace is one-dimensional and equals due to (H4.7) for ω_i sufficiently large $\mathrm{span}\{\eta^s_{\kappa_{i-2}}\}$. Hence there exists a constant $\tilde{c}\neq 0$ such that $\Phi_{\kappa_{i-1}}(\omega_i,-\omega_{i-1})\gamma_{\kappa_{i-2}}(\omega_{i-1}) = \tilde{c}e^{\mu^s\omega_i}e^{2\mu^s\omega_{i-1}}\eta^s_{\kappa_{i-2}}(0)$.

The sign of $\Phi_{\kappa_{i-1}}(\omega_i,-\omega_{i-1})\gamma_{\kappa_{i-2}}(\omega_{i-1})$ depends on whether the 2-dimensional fibre bundle $\mathcal{F}(W^s_{\gamma_{\kappa_{i-1}}})$ along the homoclinic trajectory $\gamma_{\kappa_{i-1}}$, cf. Remark 2.6.8 for definition, has the topological structure of a Möbius band or not, cf. Figure 5.2 below. Due to the symmetry the topological structure of $\mathcal{F}(W^s_\gamma)$ is transferred to all other homoclinic trajectories that belong to the homoclinic cycle. Hence the topological structure of $\mathcal{F}(W^s_{\gamma_{\kappa_{i-1}}})$ equals the structure of $\mathcal{F}(W^s_\gamma)$.

The red arrows in Figure 5.2 display the transportation of the direction $\eta^s_{\kappa_{i-2}}$ along the homoclinic trajectory $\gamma_{\kappa_{i-1}}$ via the transition matrix $\Phi_{\kappa_{i-1}}$. Recall that $\eta^s_{\kappa_{i-2}} \in T_pW^s(p)$ is perpendicular towards the direction $\eta^s_{\kappa_{i-1}}$ of the homoclinic trajectory $\gamma_{\kappa_{i-1}}$ approaching the equilibrium point p, due to $i-1\in J_\kappa$. Hence

$$\mathrm{sgn}\left\langle \Phi_{\kappa_{i-1}}(\omega_i,-\omega_{i-1})\eta^s_{\kappa_{i-2}}, \eta^s_{\kappa_{i-2}}\right\rangle = \begin{cases} 1, & \text{if } \mathcal{F}(W^s_\gamma) \text{ has the structure of an annulus,} \\ -1, & \text{if } \mathcal{F}(W^s_\gamma) \text{ has the structure of a Möbius band.} \end{cases}$$

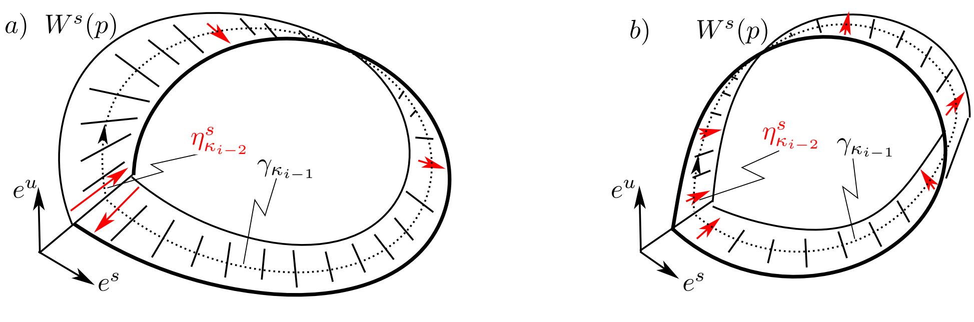


Figure 5.2: Transportation of the direction $\eta^s_{\kappa_{i-2}}$ along the homoclinic trajectory $\gamma_{\kappa_{i-1}}$ via the transition matrix of the variational equation along $\gamma_{\kappa_{i-1}}$ in case that a) $\mathcal{F}(W^s_{\gamma_{\kappa_{i-1}}})$ has the topological structure of a Möbius band and b) $\mathcal{F}(W^s_{\gamma_{\kappa_{i-1}}})$ has the topological structure of an annulus.

Summarising we obtain for $\lambda = 0$

$$\begin{aligned}
\boldsymbol{T^{23}_{\kappa_i}} &= -\left\langle \Phi_{\kappa_i}(0,-\omega_i)^T P^-_{\kappa_i}(0)^T \psi_{\kappa_i} \ , \ \Phi_{\kappa_{i-1}}(\omega_i,-\omega_{i-1})\gamma_{\kappa_{i-2}}(\omega_{i-1})\right\rangle + h.o.t. \\
&= -\left\langle e^{\mu^s\omega_i}\eta^-_{\kappa_i}(0) \ , \ \tilde{c}e^{\mu^s\omega_i}e^{2\mu^s\omega_{i-1}}\eta^s_{\kappa_{i-2}}(0)\right\rangle + h.o.t. \\
&= -\tilde{c}e^{2\mu^s\omega_i}e^{2\mu^s\omega_{i-1}}\left\langle \eta^-_{\kappa_i}(0) \ , \ \eta^s_{\kappa_{i-2}}(0)\right\rangle + h.o.t.,
\end{aligned}$$

with $C_i(0,\kappa) := \tilde{c}\langle\eta^-_{\kappa_i}(0), \eta^s_{\kappa_{i-2}}(0)\rangle$. □

Remark 5.2.3. *In $\mathbb{R}^3$ with underlying simple leading eigenvalues, the topological structure of the two-dimensional homoclinic centre manifold $W^c_{\text{hom}}(\lambda)$, which is indicated via the orientation index $\mathcal{O}$, is correlated with the topological structure of the fibre bundle $\mathcal{F}(W^s_\gamma)$, cf. Remark 2.6.8.*

In the present case, with semisimple eigenvalues still the quantity $\mathcal{O}$ can be calculated. However, it does not provide any information about the orientability of a possible homoclinic centre manifold, since here it is not two-dimensional. The context for introducing the designation of twisted and non-twisted in terms of the homoclinic centre manifold is therefore no longer given. What is even more, the value of $\mathcal{O}$ does not give any information about the topological structure of the still two-dimensional fibre bundle $\mathcal{F}(W^s_\gamma)$ along the homoclinic trajectory $\gamma = \gamma_1$ generating the homoclinic cycle. So although in our geometry Lemma 5.1.2 holds, we see no reason why $\mathcal{F}(W^s_\gamma)$ could not possibly have the structure of a Möbius band.

Corollary 5.2.4. *Let $\kappa \in \Sigma_{4m}$ and assume Hypotheses (H5.1) - (H5.4). If $i-1, i \in J_\kappa$, then the sign of $C_i(\lambda,\kappa)$ can be determined using the following table.*

	$\mathcal{F}(W^s_\gamma)$: Möbius band	$\mathcal{F}(W^s_\gamma)$: annulus
$\kappa_{i-2} = \kappa_i$	$\operatorname{sgn} C_i = \operatorname{sgn}\langle\eta^-_{\kappa_{i-2}}, -\eta^s_{\kappa_{i-2}}\rangle = 1$	$\operatorname{sgn} C_i = \operatorname{sgn}\langle\eta^-_{\kappa_{i-2}}, \eta^s_{\kappa_{i-2}}\rangle = -1$
$\kappa_{i-2} = 2m + \kappa_i$	$\operatorname{sgn} C_i = \operatorname{sgn}\langle-\eta^-_{\kappa_{i-2}}, -\eta^s_{\kappa_{i-2}}\rangle = -1$	$\operatorname{sgn} C_i = \operatorname{sgn}\langle-\eta^-_{\kappa_{i-2}}, \eta^s_{\kappa_{i-2}}\rangle = 1$

Table 5.1: The sign of $C_i(\lambda,\kappa)$.

Proof. If $i-1$ and i are in J_κ we have, cf. Lemma 5.2.1, either $\kappa_i = \kappa_{i-2}$ or $\kappa_{i-2} = 2m + \kappa_i$. In the first case, $\kappa_i = \kappa_{i-2}$, we find $\eta^-_{\kappa_i}(\lambda) = \eta^-_{\kappa_{i-2}}(\lambda)$ and in the other case $\eta^-_{\kappa_i}(\lambda) = \eta^-_{\kappa_i+2m}(\lambda) = -\eta^-_{\kappa_{i-2}}(\lambda)$. Recall that we have chosen the direction of $\psi_1 \in Z_1$ such that $\operatorname{sgn}\langle\eta^-_1(\lambda), \eta^s_1(\lambda)\rangle = -1$, cf. (1.11). With this the corollary follows from Lemma 5.2.2. □

Since in the following the κ, where C_i never changes sign, have a special role, we want to mention them here.

Remark 5.2.5. *Consider those $\kappa \in \Sigma_{4m}$ where $i \in J_\kappa$ holds for all $i \in \mathbb{Z}$ and $C_i(\lambda,\kappa)$ has always the same sign. Then, according to Lemma 5.2.1, cf. also the table above, κ is an element of $\mathcal{K}_2$ or $\mathcal{K}_4$, where*

$$\mathcal{K}_2 := \{\kappa \in \Sigma_{4m} | \forall i \in \mathbb{Z} : i \in J_\kappa \ \textit{and} \ \kappa_{i-2} = \kappa_i\} \tag{5.9}$$

and

$$\mathcal{K}_4 := \{\kappa \in \Sigma_{4m} | \forall i \in \mathbb{Z} : i \in J_\kappa \ \textit{and} \ \kappa_{i-2} = 2m + \kappa_i\}. \tag{5.10}$$

Due to $\kappa_{i-2} = \kappa_i$ for all $i \in \mathbb{Z}$ every second symbol in $\kappa \in \mathcal{K}_2$ is equal which implies that $\mathcal{K}_2$ contains all κ that correspond to those trajectories shadowing alternately two different homoclines. Hence the corresponding trajectories are 2-periodic. Further the traced homoclinic trajectories lie in mutually orthogonal fixed point spaces, since $i \in J_\kappa$ for all $i \in \mathbb{Z}$. In a D_4-equivariant vector field, for example, $\kappa = \overline{12} \in \mathcal{K}_2$.

With $\kappa_{i-2} = 2m + \kappa_i$ for all $i \in \mathbb{Z}$ we find $\kappa_{i-4} = 2m + 2m + \kappa_i = \kappa_i$. So every fourth symbol in κ is equal and we have that $\mathcal{K}_4$ consists of only 4-periodic trajectories. Again the traced homoclinic trajectories lie in mutually orthogonal fixed point spaces. As an example trajectory in a D_4-equivariant vector field consider $\kappa = \overline{1234} \in \mathcal{K}_4$.

The sign of the C_i to the κ from K_2 or K_4 depends on the topological structure of $\mathcal{F}(W^s_\gamma)$, as shown in Table 5.1.

Next we turn towards the quantity $B_i(\lambda, \kappa)$ from Lemma 4.3.20.

Lemma 5.2.6. *Let $|\lambda| < c$ and $\inf \boldsymbol{\omega} > \Omega$ according to Lemma 4.3.20. Let further $\kappa \in \Sigma_{4m}$ and assume Hypotheses (H5.1) - (H5.4). There exists a $B(\lambda)$ such that for all $i \in J_\kappa$ the quantity $B_i(\lambda, \kappa)$ equals $B(\lambda)$: $B_i(\lambda, \kappa) =: B(\lambda)$.*

Proof. The proof uses the same type of arguments as used in the proof of Lemma 5.2.1.

First, let $\kappa_i = 1$. Then, cf. (5.3), $\mathrm{Fix}_1 = \mathrm{Fix}\mathbb{Z}_2(\zeta)$ and we find $\gamma^-_{\kappa_i} = \gamma^-_1 \subset \mathrm{Fix}_1$ and $\psi_{\kappa_i} \in \mathrm{Fix}_1$. Further $\eta_{\kappa_{i-1}} \in \mathrm{Fix}_{\kappa_{i-1}} \subseteq \mathrm{Fix}_1^\perp$ due to $i \in J_\kappa$. Thus $\mathrm{Fix}_{\kappa_{i-1}} = \mathrm{Fix}\mathbb{Z}_2(\theta_{4m}^{2m}\zeta) = \theta_{4m}^{m+1}\mathrm{Fix}\mathbb{Z}_2(\zeta) = \theta_{4m}^{3m+1}\mathrm{Fix}\mathbb{Z}_2(\zeta)$, cf. (5.1). Hence either (a) $\eta^s_{\kappa_{i-1}} = \eta^s_{m+1}$ or (b) $\eta^s_{\kappa_{i-1}} = \eta^s_{3m+1} = \theta_{4m}^{2m}\eta^s_{m+1} = -\eta^s_{m+1}$. Inserting this into the representation of B_i given in Lemma 4.3.20 yields the same value in both cases (a) and (b).

Now, let $\kappa_i = j$, $j \in \{1, \ldots, 4m\}$. Then $\gamma^-_j = \theta_{4m}^{j-1}\gamma^-_1$, $\psi_j = \theta_{4m}^{j-1}\psi_1$ and (a) $\eta^s_{\kappa_{i-1}} = \theta_{4m}^{j-1}\eta^s_{m+1}$ or (b) $\eta^s_{\kappa_{i-1}} = -\theta_{4m}^{j-1}\eta^s_{m+1}$. Finally the D_{4m}-invariance of the scalar product, cf. Remark 4.0.5, and the D_{4m}-equivariance of $R^-_{\kappa_i}$, $\Phi^-_{\kappa_i}$ and $P^-_{\kappa_i}$, cf. Remark 4.3.19, Lemmata 4.3.6 and 4.3.7 provide the lemma. ◻

An analytic proof of $B(\lambda)$ being different from zero we will not be able to give. In Section 7 we will see to a numerical argumentation in case of the example system we introduce in Section 6. However, we still wish to present some results that show, that $B(\lambda)$ does not trivially disappear.

Lemma 5.2.7. *Let $i \in J_\kappa$. Then we find that*

$$D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s), \lambda)\left[R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}, R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}\right] \quad \in \quad \mathit{Fix}_{\kappa_i} \qquad \textit{and}$$

$$D_1^3 f(\gamma^-_{\kappa_i}(\lambda)(s), \lambda)\left[\Phi^-_{\kappa_i}(t, -\omega_i)P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i)\right]^3 \quad \in \quad \mathit{Fix}^\perp_{\kappa_i}.$$

Proof. Without loss of generality we assume $\mathrm{Fix}_{\kappa_i} = \mathrm{Fix}\mathbb{Z}_2(\zeta)$. ζ is a reflection and therefore self inverse. Further we have $R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}} \in \mathrm{Fix}^\perp_{\kappa_i}$, since $\eta^s_{\kappa_{i-1}} \in \mathrm{Fix}_{\kappa_{i-1}} \subseteq \mathrm{Fix}^\perp_{\kappa_i}$, due to $i \in J_\kappa$, and Remark 4.3.19. Also $\Phi^-_{\kappa_i}(t, -\omega_i)P^-_{\kappa_i}(-\omega_i)\gamma^+_{\kappa_{i-1}}(\omega_i) \in \mathrm{Fix}^\perp_{\kappa_i}$ since $\gamma^+_{\kappa_{i-1}}(\omega_i) \in \mathrm{Fix}_{\kappa_{i-1}} \subseteq \mathrm{Fix}^\perp_{\kappa_i}$ and $\Phi^-_{\kappa_i}$ and $P^-_{\kappa_i}$ leave $\mathrm{Fix}^\perp_{\kappa_i}$ invariant, cf. Section 4.3.1. According to that the assertion simply ensues form Corollary 4.3.5. ◻

Corollary 5.2.8. *The quantity $D_i(\lambda, \kappa, \omega_i)$ introduced in Lemma 4.3.20 is equal to zero.*

Proof. For all λ the left-hand side of the scalar product in the term $D_i(\lambda, \kappa, \omega_i)$ is an element of Fix_{κ_i}, the right-hand side is an element of $\mathrm{Fix}^\perp_{\kappa_i}$. ◻

While $D_i(\lambda,\kappa,\omega_i)=0$, we see that $B(\lambda)$ will not vanish due to the D_{4m}-symmetry of the system since both sides of the scalar product lie in the same fixed point space.

Furthermore, the following observation can be made.

Lemma 5.2.9. *(a) There is an $\varepsilon>0$ such that for all $\eta\in Fix_{\kappa_i}^{\perp}\cap T_{\gamma^-_{\kappa_i}(s)}W^u(p)$ with $\|\eta\|<\varepsilon$ we find*

$$\langle \Phi^-_{\kappa_i}(\lambda)(0,s)^T P^-_{\kappa_i}(0,\lambda)^T\psi_{\kappa_i}, D_1^2 f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)\,[\eta,\eta]\rangle = 0.$$

(b) Let $i\in J_\kappa$. Then $R^-_{\kappa_i}(s,\lambda)\eta^s_{\kappa_{i-1}}(\lambda)\notin T_{\gamma^-_{\kappa_i}(\lambda)(s)}W^u(p)$ for all $s\le 0$.

We see that $R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}}$ never satisfies the premise of assertion (a). Therefore $B(\lambda)$ does not vanish for that reason.

Proof. (a) First we show that $D_1^2 f(\gamma_{\kappa_i}(\lambda)(s),\lambda)\,[\eta,\eta]$ is an element of $T_{\gamma^-_{\kappa_i}(s)}W^u(p)$.

Due to Hypothesis (H3.5) there exists an $\varepsilon>0$ such that we can give a parametrisation of the unstable manifold $W^u(p)$ of the following form:

$$W^u_{t,\tau}(p)=\gamma^-_{\kappa_i}(t)+\tau\Phi^-_{\kappa_i}(t,0)w^-_{\kappa_i},\quad t\le 0 \tag{5.11}$$

with $w^-_{\kappa_i}\in W^-_{\kappa_i}$ and $|\tau|<\varepsilon$. Since any $\eta\in T_{\gamma^-_{\kappa_i}(t)}W^u(p)\cap \mathrm{Fix}^{\perp}_{\kappa_i}$ has a representation as $\eta=\tau_0\Phi^-_{\kappa_i}(t,0)w^-_{\kappa_i}$ we find for $\eta\in T_{\gamma^-_{\kappa_i}(t)}W^u(p)\cap\mathrm{Fix}^{\perp}_{\kappa_i}$ sufficiently small that $\gamma^-_{\kappa_i}(t)+\eta\in W^u(p)=W^u_{t,\tau_0}(p)$. Hence $f(\gamma^-_{\kappa_i}(t)+\eta,\lambda)$ is an element of $T_{\gamma^-_{\kappa_i}(t)+\eta}W^u(p)$. Using the parametrisation (5.11) we find

$$T_{\gamma^-_{\kappa_i}(t)+\eta}W^u(p)=\ \mathrm{span}\{\dot\gamma^-_{\kappa_i}(t)+\tau_0\dot\Phi^-_{\kappa_i}(t,0)w^-_{\kappa_i},\Phi^-_{\kappa_i}(t,0)w^-_{\kappa_i}\}.$$

With $\Phi^-_{\kappa_i}(t,0)w^-_{\kappa_i}\in\mathrm{Fix}^{\perp}_{\kappa_i}$ and $\dot\Phi^-_{\kappa_i}(t,0)=D_1f(\gamma^-_{\kappa_i}(t),\lambda)\Phi^-_{\kappa_i}(t,0)$ we also have $\tau_0\dot\Phi^-_{\kappa_i}(t,0)w^-_{\kappa_i}\in\mathrm{Fix}^{\perp}_{\kappa_i}$, since $D_1f(\gamma^-_{\kappa_i}(t),\lambda)$ leaves $\mathrm{Fix}^{\perp}_{\kappa_i}$ invariant, cf. Corollary 4.3.5. On the other hand we have $\dot\gamma^-_{\kappa_i}(t)\in\mathrm{Fix}_{\kappa_i}$. Thus, if we decompose any element $x\in T_{\gamma^-_{\kappa_i}(t)+\eta}W^u(p)$ by means of the projection F_{κ_i} with $\mathrm{im}F_{\kappa_i}=\mathrm{Fix}_{\kappa_i}$ and $\ker F_{\kappa_i}=\mathrm{Fix}^{\perp}_{\kappa_i}$, cf. (3.38) and (5.6), then we find $F_{\kappa_i}x\in\ \mathrm{span}\{\dot\gamma^-_{\kappa_i}(t)\}$.

With this in mind we look at the Taylor expansion of $f(\gamma^-_{\kappa_i}(s)+\eta,\lambda)$ up to the second order

$$f(\gamma^-_{\kappa_i}(s)+\eta,\lambda)=f(\gamma^-_{\kappa_i}(s),\lambda)+D_1f(\gamma^-_{\kappa_i}(s),\lambda)\eta+\frac{1}{2}D_1^2f(\gamma^-_{\kappa_i}(s),\lambda)[\eta,\eta]+O(\|\eta\|^3)$$

and decompose the term by means of the projection F_{κ_i}, cf. Corollary 4.3.5

$$\left.\begin{aligned} F_{\kappa_i}f(\gamma^-_{\kappa_i}(s)+\eta,\lambda) &= f(\gamma^-_{\kappa_i}(s),\lambda)+\tfrac12 D_1^2f(\gamma^-_{\kappa_i}(s),\lambda)[\eta,\eta]+O(\|\eta\|^4),\\ (id-F_{\kappa_i})f(\gamma^-_{\kappa_i}(s)+\eta,\lambda) &= D_1f(\gamma^-_{\kappa_i}(s),\lambda)\eta+O(\|\eta\|^3).\end{aligned}\right\} \tag{5.12}$$

So, the left hand side of the first equation of (5.12) is situated in $\mathrm{span}\{\dot\gamma^-_{\kappa_i}(s)\}$. The same holds for $f(\gamma^-_{\kappa_i}(s),\lambda)$, the first term on the right hand side. Indeed every single term on the right hand side of the first equation of (5.12) lies in $\mathrm{span}\{\dot\gamma^-_{\kappa_i}(s)\}$, since this equation holds for all $\eta\in T_{\gamma^-_{\kappa_i}(s)}W^u(p)\cap\mathrm{Fix}^{\perp}_{\kappa_i}$ with $\|\eta\|<\varepsilon$. Hence we find in particular

$$D_1^2f(\gamma^-_{\kappa_i}(s),\lambda)[\eta,\eta]\in\ \mathrm{span}\{\dot\gamma^-_{\kappa_i}(s)\}.$$

Now, on the other hand the term $\Phi^-_{\kappa_i}(0,s)^T P^-_{\kappa_i}(0)^T\psi_{\kappa_i}$ stands orthogonal to the direction of the vector field $f(\gamma^-_{\kappa_i}(\lambda)(s),\lambda)=\dot\gamma^-_{\kappa_i}(\lambda)(s)$ along $\gamma^-_{\kappa_i}(\lambda)(s)$, since it is transported backwards in time through the adjoint variational equation (3.9). Hence the scalar product disappears.

(b) From Remark 4.3.19 and Lemma 4.3.17 we know that

$$R^-_{\kappa_i}(s)\eta^s_{\kappa_{i-1}} \in \mathrm{Fix}^\perp_{\kappa_i} \cap \mathrm{im} P^-_{\kappa_i}(\lambda, s) \subseteq \Phi^-_{\kappa_i}(s,0)W^+_{\kappa_i}.$$

Now, $W^+_{\kappa_i} \cap T_{\gamma^-_{\kappa_i}(0)}W^u(p) = W^+_{\kappa_i} \cap (W^-_{\kappa_i} \cap U_{\kappa_i}) = \{0\}$. Hence $\Phi^-_{\kappa_i}(s,0)W^+_{\kappa_i} \cap T_{\gamma^-_{\kappa_i}(s)}W^u(p) = \{0\}$ for all $s \le 0$. This concludes the proof. □

5.3 Main result

Representation of the jump $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa)$

Before presenting our main result, we take a brief look at the final representation of the jump $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa)$ in the D_{4m} symmetric case. In Section 3.4.1 we have discussed the first part $\xi^\infty_{\kappa_i}(\lambda)$ and in Section 4.3, Theorem 4.3.3, we examined the second part $\xi_i(\boldsymbol{\omega}, \lambda, \kappa)$. Now we simply collect both results and rewrite the residual terms in a unified form that simplifies the later proof of the main result.

Theorem 5.3.1. *Assume Hypothesis (H5.1)-(H5.4). Let β^s be a constant satisfying $-\alpha^u < \beta^s < \mu^s(\lambda)$. Then there is a $\delta > 1$ such that the jump Ξ_i can be written as one of the following alternatives:*

(i) If $i \in \mathbb{Z} \setminus J_\kappa$ then $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \lambda - A_i(\lambda, \kappa)e^{2\mu^s(\lambda)\omega_i} + R_i(\boldsymbol{\omega}, \lambda, \kappa)$, *where*

$$\begin{aligned} R_i(\boldsymbol{\omega}, \lambda, \kappa) \;&=\; O\big(e^{8/5\mu^s(\lambda)\delta(\omega_{i-1}+\omega_i)}\big) + O\big(e^{16/5\mu^s(\lambda)\delta\omega_i}\big) \\ &+ \begin{cases} O\big(e^{2\mu^s(\lambda)\delta\omega_{i+1}}\big), & i+1 \in \mathbb{Z}\setminus J_\kappa \\ \left.\begin{array}{l} O\big(e^{8/5\mu^s(\lambda)\delta\omega_i}e^{2\mu^s(\lambda)\delta\omega_{i+1}}\big) \\ +O\big(e^{4\mu^s(\lambda)\delta\omega_{i+1}}\big) + O\big(e^{2\mu^s(\lambda)\delta(\omega_{i+1}+\omega_{i+2})}\big) \end{array}\right\}, & i+1 \in J_\kappa. \end{cases} \end{aligned}$$

(ii) If $i \in J_\kappa$ then $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \lambda - B(\lambda)e^{4\mu^s(\lambda)\omega_i} - C_i(\lambda, \kappa)e^{2\mu^s(\lambda)(\omega_{i-1}+\omega_i)} + R_i(\boldsymbol{\omega}, \lambda, \kappa)$, *where*

$$\begin{aligned} R_i(\boldsymbol{\omega}, \lambda, \kappa) \;&=\; O\big(e^{8/5\mu^s(\lambda)\delta(\omega_{i-1}+\omega_i)}[e^{8/5\mu^s(\lambda)\delta\omega_{i-2}} + e^{4/5\mu^s(\lambda)\delta\omega_{i-1}} + e^{\mu^s(\lambda)\delta\omega_i}]\big) \\ &+O\big(e^{4\mu^s(\lambda)\delta\omega_i}\big) + O\big(e^{16/5\mu^s(\lambda)\delta\omega_i}e^{\mu^s(\lambda)\delta\omega_{i+1}}\big) \\ &+ \begin{cases} O\left(e^{2\mu^s(\lambda)\delta\omega_{i+1}}\right), & i+1 \in \mathbb{Z}\setminus J_\kappa \\ \left.\begin{array}{l} O\big(e^{2\frac{-\beta^s}{\alpha^u}\mu^s(\lambda)\delta\omega_i}e^{2\frac{\alpha^u}{-\beta^s}\mu^s(\lambda)\delta\omega_{i+1}}\big) \\ +O\big(e^{4\mu^s(\lambda)\delta\omega_{i+1}}\big) \\ +O\big(e^{2\mu^s(\lambda)\delta(\omega_{i+1}+\omega_{i+2})}\big) \end{array}\right\}, & i+1 \in J_\kappa. \end{cases} \end{aligned}$$

The coefficients $A_i(\lambda, \kappa) := \langle \eta^-_{\kappa_i}(\lambda), \eta^s_{\kappa_{i-1}}(\lambda)\rangle$ are different from zero for all $i \in \mathbb{Z} \setminus J_\kappa$. Further the coefficients $B(\lambda)$ do not depend on $i \in J_\kappa$. Finally there is a $C(\lambda) > 0$ such that $|C_i(\lambda, \kappa)| = C(\lambda)$ for all i with $i-1, i \in J_\kappa$.

Proof. Recall from (3.3) that $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \xi^\infty_i(\lambda) + \xi_i(\boldsymbol{\omega}, \lambda, \kappa)$. Then the theorem basically follows from Theorem 4.3.3. Here we have unified the notation of the O-terms by means of the parameter $\delta > 1$. We will give a justification of this representation below. The assertions concerning the coefficients $B(\lambda)$ and $C_i(\lambda, \kappa)$ follow from Lemma 5.2.6 and Lemma 5.2.1. For the structure of these terms see Lemmata 4.3.20 and 4.3.22. Finally the considerations in Section 3.4.1 show that $\xi^\infty_i(\lambda)$ can be replaced by λ.

Now, we prove the existence of a constant $\delta > 1$ such that the residual terms in Theorem 4.3.3 satisfy the representation above. To this end recall that $-\mu^u(\lambda) < -\alpha^u < \mu^s(\lambda) < \alpha^s < 0$. Then we can find a $\delta_1 > 1$ such that still $-\alpha^u \frac{1}{\delta_1} < \mu^s(\lambda)$. Hence we find

$$O(e^{-k\alpha^u\omega}) = O(e^{-k\mu^s(\lambda)\delta_1\omega}), \tag{5.13}$$

for $k \in \{1,2\}$ and $\omega \in \{\omega_i, \omega_{i+1}, \omega_{i+2}\}$.
The constant α^s can be chosen arbitrarily close to $\mu^s(\lambda)$. Especially we can choose α^s such that $\frac{5}{4}\alpha^s < \mu^s(\lambda)$ and $l\alpha^s - \alpha^u < (l+1)\mu^s(\lambda)$, $l \in \mathbb{N}$. For some $\delta_2 > 1$ sufficiently small we then find

$$O(e^{k\alpha^s\omega}) = O(e^{4k/5\mu^s(\lambda)\delta_2\omega}), \qquad O(e^{k(\alpha^s-\alpha^u)\omega}) = O(e^{2k\mu^s(\lambda)\delta_2\omega}), \tag{5.14}$$

$k \in \{1,2\}$ and $\omega \in \{\omega_{i-1}, \omega_i, \omega_{i+1}\}$. With (5.13) and (5.14) we replace each residual term in Theorem 4.3.3 apart from $O(e^{2\alpha^s\omega_i}e^{2\alpha^u\omega_{i+1}})$ for the second case $i \in J_\kappa$ for which a more difficile estimate is needed.

From $-\alpha^u < \beta^s < \mu^s(\lambda) < 0$ we find that $-\alpha^u < \frac{\alpha^u}{-\beta^s}\mu^s(\lambda)$ where $\frac{\alpha^u}{-\beta^s} > 1$. Further we can chose α^s sufficiently close to $\mu^s(\lambda)$ such that $\frac{\alpha^u}{-\beta^s}\alpha^s\delta_3^{-1} < \mu^s(\lambda)$ for some appropriate $\delta_3 > 1$. With this we obtain

$$O\big(e^{2\alpha^s\omega_i}e^{2\alpha^u\omega_{i+1}}\big) = O\big(e^{2\frac{-\beta^s}{\alpha^u}\mu^s(\lambda)\delta_3\omega_i}e^{2\frac{\alpha^u}{-\beta^s}\mu^s(\lambda)\delta_3\omega_{i+1}}\big). \tag{5.15}$$

Then the estimates listed in Theorem 5.3.1 follow from (5.13), (5.14) and (5.15) with $1 < \delta < \min\{\delta_1, \delta_2, \delta_3\}$. □

Corollary 5.3.2. *Assume Hypothesis (H5.1)-(H5.4). Then the derivatives of the residual terms with respect to ω_j, $D_{\omega_j}R_i(\boldsymbol{\omega}, \lambda, \kappa)$ satisfy exactly the same estimates as the residual terms $R_i(\boldsymbol{\omega}, \lambda, \kappa)$ presented in Theorem 5.3.1.*

Proof. Since the derivatives of the residual terms $D_{\omega_j}R_i(\boldsymbol{\omega}, \lambda, \kappa)$ in Theorem 4.4.1 satisfy the same estimates as the residual terms $R_i(\boldsymbol{\omega}, \lambda, \kappa)$ in Theorem 4.3.3 the corollary can be proved in the same way as Theorem 5.3.1. □

Description of the nonwandering dynamics

Now, we can give a more detailed description of the nonwandering dynamics of the system under consideration. To this end we define the following matrix M by

$$M = (m_{ij})_{i,j\in\{1,\dots,4m\}}, \quad m_{ij} := \begin{cases} \operatorname{sgn}\langle\eta_i^s(\lambda), \eta_j^-(\lambda)\rangle, & \text{if } \operatorname{sgn}\langle\eta_i^s(\lambda), \eta_j^-(\lambda)\rangle \neq 0, \\ \operatorname{sgn}B(\lambda), & \text{if } \operatorname{sgn}\langle\eta_i^s(\lambda), \eta_j^-(\lambda)\rangle = 0. \end{cases} \tag{5.16}$$

In this respect we want to note that $\operatorname{sgn}\langle\eta_i^s(\lambda), \eta_j^-(\lambda)\rangle = 0$ if and only if $\mathrm{Fix}_i \perp \mathrm{Fix}_j$ and that neither $\operatorname{sgn}\langle\eta_i^s(\lambda), \eta_j^-(\lambda)\rangle$ nor $\operatorname{sgn}B(\lambda)$ depends on λ.

In order to characterise the nonwandering dynamics near Γ we need to distinguish two different cases indicated by the size ratio of $|B(0)|$ to $C(0)$. So far, it is not known whether the geometry of the system allows for an arbitrary aspect ratio of $B(0)$ to $C(0)$ or whether a certain ratio is enforced. Therefore, it seems possible to control the ratio of $B(0)$ and $C(0)$ by means of another system parameter. In this sense, the complete description of the local nonwandering dynamics in the neighbourhood of a D_{4m}-symmetric homoclinic cycle might no longer be a pure codimension-1 problem.

Case $|B(0)| > C(0)$:

This case presents itself as a natural extension of Theorem 1.0.2.

Theorem 5.3.3. *Let $\dot{x} = f(x, \lambda)$ be a one parameter family of differential equations equivariant with respect to the finite group D_{4m} which has at $\lambda = 0$ a codimension-one relative homoclinic cycle Γ with hyperbolic equilibrium as defined in Hypotheses (H5.1) - (H5.4). Further assume $|B(0)| > C(0)$.*

With the $4m \times 4m$ matrices $A_- = -\frac{1}{2}(M - |M|)$ and $A_+ = \frac{1}{2}(M + |M|)$, M given by (5.16), the following holds for any generic family unfolding a relative homoclinic cycle as described above:

Take cross sections $\mathcal{S}_i$ transverse to γ_i and write Π_λ for the first return map on the collection of cross sections $\cup_{j=1}^{4m}\mathcal{S}_j$. For $\lambda > 0$ small enough, there is an invariant set $\mathcal{D}_\lambda \subset \cup_{j=1}^{4m}\mathcal{S}_j$ for Π_λ such that for each $\kappa \in \Sigma_{A_+}$ there exists a unique $x \in \mathcal{D}_\lambda$ with $\Pi_\lambda^i(x) \in \mathcal{S}_{\kappa_i}$. Moreover, $(\mathcal{D}_\lambda, \Pi_\lambda)$ is topologically conjugate to (Σ_{A_+}, σ). An analogous statement holds for $\lambda < 0$ with Σ_{A_+} replaced by Σ_{A_-}.

The proof of this theorem is given in Section 5.4. The above description of the dynamics provides a complete picture of the local nonwandering dynamics near Γ in the sense that $A_- + A_+ = \mathbf{1}$ is satisfied. According to the definition of M, cf. (5.16) we find

$$B(\lambda) > 0, \text{ then } \quad m_{ij} = \begin{cases} 1, & |i-j| \geq m, \\ -1, & |i-j| < m, \end{cases} \quad \text{and} \quad B(\lambda) < 0, \text{ then } \quad m_{ij} = \begin{cases} 1, & |i-j| > m, \\ -1, & |i-j| \leq m. \end{cases}$$

The difference $i - j$ is calculated in $\mathbb{Z}_{4m}$, and $|i - j| := \min\{i - j, j - i\}$.

So, for $m = 1$ and $\text{sgn}B(\lambda) = 1$ the matrices A_- and A_+ take the form given in (1.6), whereas for $m = 1$ and $\text{sgn}B(\lambda) = -1$ these matrices read

$$A_- = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \qquad A_+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

For the transition times ω_i it turns out, in the course of proving Theorem 5.3.3, that the following equation holds, cf. (5.40) below:

$$\omega_i(\lambda, \kappa) = \begin{cases} \frac{1}{2\mu^s(0)}(\ln(|\lambda|) + \ln(r_i)), & i \in \mathbb{Z} \setminus J_\kappa, \\ \frac{1}{4\mu^s(0)}(\ln(|\lambda|) + \ln(r_i^2)), & i \in J_\kappa. \end{cases} \tag{5.17}$$

Thereby the r_i are uniformly bounded terms. This means that the bounds of r_i can be chosen independently of λ and, apart from the distinction whether $i \in \mathbb{Z} \setminus J_\kappa$ or $i \in J_\kappa$, also independently of the course of κ. For λ sufficiently small, the terms r_i therefore have a negligible influence on the transition times. Thus we find for fixed λ that it goes about twice as fast to move from a homoclinic trajectory within one fixed point space to another homoclinic trajectory within an orthogonal fixed point space than in the other circumstances. Apart from that the single transition times are nearly about the same size, independent on κ.

Case $|B(0)| \leq C(0)$:

Regarding the case $|B(0)| \leq C(0)$ we find that the above result of Theorem 5.3.3 does not apply. The analysis turns out to be more difficile and requires the distinction of further subcases such as the subdivision according to the topological structure of the fibre bundle $\mathcal{F}(W_\gamma^s)$. In this respect, a complete description of the nonwandering dynamics is not our intention. Further investigations may reveal the need to introduce even further system parameters. We merely restrict the investigation to periodic solutions. Thereby we mean by period length the length of the recurring sequence in κ and not the transition time of the corresponding periodic trajectory.

Theorem 5.3.4. *Let $\dot{x} = f(x, \lambda)$ be a one parameter family of differential equations equivariant with respect to the finite group D_{4m} which has at $\lambda = 0$ a codimension-one relative homoclinic cycle Γ with hyperbolic equilibrium as defined in Hypotheses (H5.1) - (H5.4). Further assume $|B(0)| \leq C(0)$ and let $\mathcal{F}(W^s_\gamma)$ have the topological structure of an annulus.*

With the $4m \times 4m$ matrices $A_- = -\frac{1}{2}(M - |M|)$ and $A_+ = \frac{1}{2}(M + |M|)$, M given by (5.16), the following holds true for any generic family unfolding a relative homoclinic cycle as described above:

(i) Let $B(0) > 0$. For all $N \in \mathbb{N}$ there exists a $\hat{\lambda}(N) > 0$ sufficiently small such that for all $\lambda \in (0, \hat{\lambda})$ and all periodic $\kappa \in \Sigma_{A_+} \setminus \mathcal{K}_2$ with period length smaller or equal to N there is a unique periodic trajectory $x(\lambda, \kappa) : \mathbb{R} \to \mathbb{R}^4$ solution of $\dot{x} = f(x, \lambda)$ being situated in the neighbourhood of the homoclinic cycle Γ.
For $\lambda < 0$ sufficiently small there exists such a unique trajectory for each periodic $\kappa \in \Sigma_{A_-} \cup \mathcal{K}_2$, if $|B(0)| < C(0)$, or $\kappa \in \Sigma_{A_-}$, if $|B(0)| = C(0)$.

(ii) Let $B(0) < 0$. For all $N \in \mathbb{N}$ there exists a $\hat{\lambda}(N) < 0$ sufficiently small such that for all $\lambda \in (\hat{\lambda}, 0)$ and all periodic $\kappa \in \Sigma_{A_-} \setminus \mathcal{K}_4$ with period length smaller or equal to N there is a unique periodic trajectory $x(\lambda, \kappa) : \mathbb{R} \to \mathbb{R}^4$ solution of $\dot{x} = f(x, \lambda)$ being situated in the neighbourhood of the homoclinic cycle Γ.
For $\lambda > 0$ sufficiently small there exists such a unique trajectory for each periodic $\kappa \in \Sigma_{A_+} \cup \mathcal{K}_4$, if $|B(0)| < C(0)$, or $\kappa \in \Sigma_{A_+}$, if $|B(0)| = C(0)$.

If $\mathcal{F}(W^s_\gamma)$ has the topological structure of a Möbius band, analogous statements to (i) and (ii) hold with the sets $\mathcal{K}_2$ and $\mathcal{K}_4$ interchanged.

The result is somewhat similar - in the sense that the existence of a N-periodic trajectory depends on the size of the parameter $\lambda(N)$ with λ tending to zero as N goes to infinity - to the statement in [HomKno06, Theorem 5.1].

The proof of Theorem 5.3.4 can be found in Section 5.5. At this point we just like to say a little about the background of the described behaviour of the dynamics. To this end recall that to prove the existence of a solution trajectory that is determined by κ, we must set the associated jumps $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa)$, $i \in \mathbb{Z}$, equal to zero. Here the form of Ξ_i is given by Theorem 5.3.1. The solvability of the resulting system of determination equations $(\Xi_i(\boldsymbol{\omega}, \lambda, \kappa))_{i \in \mathbb{Z}} = 0$ can then be discussed depending on the choice of κ and λ. In the present context, there can be sequences of successive homoclinic trajectories lying in mutually orthogonal fixed point spaces. The representation of the associated jumps is given by Theorem 5.3.1(ii). The zeros in the corresponding determination equations must basically be generated by the term that contains $B(\lambda)$.

Now, in contrast to the first case, where $|B(0)| > C(0)$, it can be seen here that the sequence κ can have a strong influence on the size of the transition times ω_i that solve the determination equations. In fact, the same equation (5.17) as shown above applies for ω_i. Only with $|B(0)| \leq C(0)$ the terms r_i are not any more uniformly bounded. Particularly problematic are those jumps that satisfy the representation

$$\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \lambda - e^{4\mu^s(\lambda)\omega_i} B(\lambda) - e^{2\mu^s(\lambda)(\omega_{i-1}+\omega_i)} C_i(\lambda, \kappa), + \check{R}_i(\boldsymbol{\omega}, \lambda, \kappa),$$

with $\operatorname{sgn}(C_i(\lambda,\kappa)) \neq \operatorname{sgn}(B(\lambda))$. Assume for example $B(\lambda) > 0$ and let us briefly discuss the solvability of $\Xi_i(\boldsymbol{\omega},\lambda,\kappa) = 0$ for fixed $\lambda > 0$. Thereby we neglect for a moment the residual terms.

If $C_i(\lambda,\kappa) < 0$ the term $e^{4\mu^s(\lambda)\omega_i}B(\lambda)$ has to offset both terms λ and $e^{2\mu^s(\lambda)(\omega_{i-1}+\omega_i)}|C_i(\lambda,\kappa)|$. Since $B(\lambda) \leq |C_i(\lambda,\kappa)|$ this can only be done by decreasing the transition time ω_i in relation to ω_{i-1} and hence increasing $e^{2\mu^s(\lambda)\omega_i}$ in relation to $e^{2\mu^s(\lambda)\omega_{i-1}}$. If $C_{i+1}(\lambda,\kappa) < 0$ applies again in the next jump, anew the corresponding transition time ω_{i+1} has to be reduced in relation to ω_i. This goes on and on until the next index k with either $k \in \mathbb{Z} \setminus J_\kappa$ or $C_k(\lambda,\kappa) > 0$. Then the determination equation $\Xi_k = 0$ can again be solved for a much larger ω_k: $\omega_{i-1} > \omega_i > \omega_{i+1} > \cdots > \omega_{k-1}$ and $\omega_k \gg \omega_{k-1}$.

If the sequences of consecutive jumps holding $C_i(\lambda,\kappa) < 0$ is finite the system of determination equations can be solved at $\lambda > 0$ for the same, at least periodic, κ as in Theorem 5.3.3. However, depending on the length of those sequences the transition times can be very different in size. To ensure that $\inf \boldsymbol{\omega}$ is still large enough to satisfy our analysis the value of λ must become very small. We will find, cf. (5.83) below, that $\lambda < \left({}^{B(0)}/{}_{C(0)}\right)^{2L} e^{4\mu^s(0)\inf \boldsymbol{\omega}}$, where L denotes the length of the longest chain of consecutive indices i in κ satisfying $C_i(\lambda,\kappa) < 0$. Consequently some trajectories do exist only for smaller λ than others.

For those periodic κ that satisfy $C_i(\lambda,\kappa) < 0$ for all $i \in \mathbb{Z}$ there exist no solutions at $\lambda > 0$. Instead the determination equations can be solved for those κ for $\lambda < 0$, if $|B(0)| \lneqq C(0)$. So, whereas all other periodic trajectories that follow homoclinic trajectories in mutually orthogonal fixed point spaces do exist for $\lambda > 0$ we will find this exception for the opposite sign of λ. If $|B(0)| = C(0)$ periodic trajectories corresponding to these κ exist neither for positive nor for negative λ. Hence the existence of these trajectories does not satisfy the rule implied by a Markov chain.

According to Remark 5.2.5 and Corollary 5.2.4 the condition $C_i(\lambda,\kappa) < 0$ for all $i \in \mathbb{Z}$ implies that κ is an element of $\mathcal{K}_2$ if $\mathcal{F}(W^s_\gamma)$ has the topological structure of an annulus and an element of $\mathcal{K}_4$ if $\mathcal{F}(W^s_\gamma)$ has the topological structure of a Möbius band.

If $B(\lambda) < 0$ the critical trajectories are those that satisfy $C_i(\lambda,\kappa) > 0$ for all $i \in \mathbb{Z}$. So in case that $\mathcal{F}(W^s_\gamma)$ has the topological structure of an annulus, $\kappa \in \mathcal{K}_4$ are the exceptions and if $\mathcal{F}(W^s_\gamma)$ has the topological structure of a Möbius band $\kappa \in \mathcal{K}_2$ show the different behaviour.

Remark 5.3.5. *Of course, for any trajectory - periodic or not - that never follows successive homoclinic trajectories lying in mutually orthogonal fixed point spaces, still Theorem 1.0.2 applies. That is, for all $\kappa \in \Sigma_{4m}$ satisfying for all $i \in \mathbb{Z}$ that $i \in \mathbb{Z} \setminus J_\kappa$ the existence of the corresponding trajectory $x(\lambda,\kappa) : \mathbb{R} \to \mathbb{R}^4$ does not depend on any length of a period. This is because in those cases the terms $B(\lambda)$ and $C_i(\lambda,\kappa)$ have no impact on the solvability of the determination equations.*

Especially for $B(\lambda) > 0$ we find at $\lambda < 0$ that all $\kappa \in \Sigma_{A_-}$ are such that $i \in \mathbb{Z} \setminus J_\kappa$ for all $i \in \mathbb{Z}$. Hence for $\lambda < 0$ we do not find any trajectories with transitions from homoclinic trajectories lying in mutually orthogonal fixed point spaces, apart from those belonging to $\mathcal{K}_2$ or $\mathcal{K}_4$, respectively. In the case $B(\lambda) < 0$ and $\lambda > 0$ it is exactly the same, all $\kappa \in \Sigma_{A_+}$ are such that $i \in \mathbb{Z} \setminus J_\kappa$ for all $i \in \mathbb{Z}$.

Remark 5.3.6. *The 2-periodic trajectories corresponding to $\kappa \in \mathcal{K}_2$ and the 4-periodic trajectories corresponding to $\kappa \in \mathcal{K}_4$*

(i) exist for the same sign of λ , if $|B(0)| > C(0)$.

(ii) exist for the opposite sign of λ , if $|B(0)| < C(0)$.

(iii) exclude one another, if $|B(0)| = C(0)$, that is only one type of these trajectories does exist, either for positive or negative λ. The other type does not exist.

The following two sections deal with the detailed proofs of the main Theorems 5.3.3 and 5.3.4.

5.4 Proof of Theorem 5.3.3

We prove the theorem under the assumption that $B(\lambda) > 0$. In principle we thereby follow the idea of the proof of [HJKL11, Theorem 2.4] which is based on the implicit function theorem. However, due to the more complicated structure of the system of determination equations this is not a one to one application.

5.4.1 Solution for fixed κ and $\lambda > 0$

In a first step we rewrite the single jumps Ξ_i given in Theorem 5.3.1 for fixed κ by introducing the shortened notation

$$\hat{r}_i = e^{2\mu^s(\lambda)\omega_i}, \quad \hat{\mathbf{r}} = (\hat{r}_i)_{i\in\mathbb{Z}} \tag{5.18}$$

into

$$i \in \mathbb{Z}\setminus J_\kappa: \quad \Xi_i(\hat{\mathbf{r}},\lambda,\kappa) = \lambda - A_i(\lambda)\hat{r}_i + \hat{R}_i(\hat{\mathbf{r}},\lambda,\kappa), \tag{5.19}$$

$$\hat{R}_i(\hat{\mathbf{r}},\lambda,\kappa) = O\big(\hat{r}_{i-1}^{4/5\delta}\hat{r}_i^{4/5\delta}\big) + O\big(\hat{r}_i^{8/5\delta}\big) + \begin{cases} O\big(\hat{r}_{i+1}^{\delta}\big), & i+1 \in \mathbb{Z}\setminus J_\kappa \\ O\big(\hat{r}_i^{4/5\delta}\hat{r}_{i+1}^{\delta}\big) + O\big(\hat{r}_{i+1}^{2\delta}\big) & \\ +O\big(\hat{r}_{i+1}^{\delta}\hat{r}_{i+2}^{\delta}\big), & i+1 \in J_\kappa. \end{cases}$$

$$i \in J_\kappa: \quad \Xi_i(\hat{\mathbf{r}},\lambda,\kappa) = \lambda - B(\lambda)\hat{r}_i^2 - C_i(\lambda)\hat{r}_{i-1}\hat{r}_i + \hat{R}_i(\hat{\mathbf{r}},\lambda,\kappa), \tag{5.20}$$

$$\begin{aligned} \hat{R}_i(\hat{\mathbf{r}},\lambda,\kappa) \;&=\; O\big(\hat{r}_{i-1}^{4/5\delta}\hat{r}_i^{4/5\delta}[\hat{r}_{i-2}^{4/5\delta} + \hat{r}_{i-1}^{2/5\delta} + \hat{r}_i^{1/2\delta}]\big) + O\big(\hat{r}_i^{2\delta}\big) + O\big(\hat{r}_i^{8/5\delta}\hat{r}_{i+1}^{1/2\delta}\big) \\ &+ \begin{cases} O\left(\hat{r}_{i+1}^{\delta}\right), & i+1 \in \mathbb{Z}\setminus J_\kappa \\ O\big(\hat{r}_i^{\frac{-\beta^s}{\alpha^u}\delta}\hat{r}_{i+1}^{\frac{\alpha^u}{-\beta^s}\delta}\big) + O\big(\hat{r}_{i+1}^{2\delta}\big) & \\ +O\big(\hat{r}_{i+1}^{\delta}\hat{r}_{i+2}^{\delta}\big), & i+1 \in J_\kappa \end{cases} \end{aligned}$$

Our aim is to solve this set of equations near $(\hat{r}_i,\lambda) = (0,0)$ for $\hat{r}_i = \hat{r}_i(\lambda,\kappa)$. Mind that only $\hat{r}_i \geq 0$ makes sense whereas λ can be either positive or negative. Indeed, as considerations in [HJKL11] show, the sign of λ allows for different trajectories, since the sign of the terms $\langle \eta^-_{\kappa_i}(0), \eta^s_{\kappa_{i-1}}(0)\rangle = A_i(0)$ depend on the geometry of the trajectory. In this subsection we consider the case $\lambda > 0$. Section 5.4.2 deals with the case $\lambda < 0$.

For $\lambda > 0$ we introduce the rescaling

$$\hat{r}_i = \begin{cases} \lambda r_i, & i \in \mathbb{Z}\setminus J_\kappa, \\ \sqrt{\lambda} r_i, & i \in J_\kappa. \end{cases} \tag{5.21}$$

With $\mathbf{r} = (r_i)_{i\in\mathbb{Z}}$ it follows from the representations (5.19) and (5.20) of the jumps:

$$i \in \mathbb{Z}\setminus J_\kappa: \quad \Xi_i(\mathbf{r},\lambda,\kappa) = \lambda - A_i(\lambda)\lambda r_i + \tilde{R}_i(\mathbf{r},\lambda,\kappa), \tag{5.22}$$

$$i \in J_\kappa: \quad \Xi_i(\mathbf{r},\lambda,\kappa) = \lambda - B(\lambda)\lambda r_i^2 + \begin{cases} -C_i(\lambda)\lambda r_{i-1}r_i + \tilde{R}_i(\mathbf{r},\lambda,\kappa), & i-1 \in J_\kappa, \\ -C_i(\lambda)\lambda^{3/2} r_{i-1}r_i + \tilde{R}_i(\mathbf{r},\lambda,\kappa), & i-1 \in \mathbb{Z}\setminus J_\kappa. \end{cases} \tag{5.23}$$

The residual terms $\tilde{R}_i$ arise from $\hat{R}_i$ accordingly. Thereby it turns out that for all $i \in \mathbb{Z}$

$$\tilde{R}_i = O(\lambda^\delta), \quad \delta > 1,$$

which can be seen in the following. Note that we replaced $\hat{r}_j = \sqrt{\lambda} r_j$ if it is unknown whether $j \in \mathbb{Z}\setminus J_\kappa$ or $j \in J_\kappa$.

$$\begin{aligned} i \in \mathbb{Z}\setminus J_\kappa: \quad \tilde{R}_i &= O\big(\sqrt{\lambda}^{4/5\delta}\lambda^{4/5\delta}\big) + O\big(\lambda^{8/5\delta}\big) + \begin{cases} O\big(\lambda^\delta\big), & i+1 \in \mathbb{Z}\setminus J_\kappa \\ O\big(\lambda^{4/5\delta}\sqrt{\lambda}^{\delta}\big) + O\big(\sqrt{\lambda}^{2\delta}\big) & \\ +O\big(\sqrt{\lambda}^{\delta}\sqrt{\lambda}^{\delta}\big), & i+1 \in J_\kappa. \end{cases} \\ &= O\big(\lambda^{6/5\delta}\big) + O\big(\lambda^{8/5\delta}\big) + \begin{cases} O\big(\lambda^\delta\big), & i+1 \in \mathbb{Z}\setminus J_\kappa \\ O\big(\lambda^{13/10\delta}\big) + O\big(\lambda^\delta\big) + O\big(\lambda^\delta\big), & i+1 \in J_\kappa. \end{cases} \end{aligned}$$

$$\begin{aligned} i \in J_\kappa: \qquad \tilde{R}_i &= O\big(\sqrt{\lambda}^{4/5\delta}\sqrt{\lambda}^{4/5\delta}[\sqrt{\lambda}^{4/5\delta} + \sqrt{\lambda}^{2/5\delta} + \sqrt{\lambda}^{1/2\delta}]\big) + O\big(\sqrt{\lambda}^{2\delta}\big) + O\big(\sqrt{\lambda}^{8/5\delta}\sqrt{\lambda}^{1/2\delta}\big) \\ &\quad + \begin{cases} O\left(\lambda^\delta\right), & i+1 \in \mathbb{Z}\setminus J_\kappa \\ O\big(\sqrt{\lambda}^{\frac{-\beta^s}{\alpha^u}\delta}\sqrt{\lambda}^{\frac{\alpha^u}{-\beta^s}\delta}\big) + O\big(\sqrt{\lambda}^{2\delta}\big) & \\ +O\big(\sqrt{\lambda}^{\delta}\sqrt{\lambda}^{\delta}\big), & i+1 \in J_\kappa \end{cases} \\ &= O\big(\lambda^{6/5\delta} + \lambda^\delta + \lambda^{21/20\delta}\big) + O\big(\lambda^\delta\big) + O\big(\lambda^{21/20\delta}\big) \\ &\quad + \begin{cases} O\left(\lambda^\delta\right), & i+1 \in \mathbb{Z}\setminus J_\kappa \\ O\big(\lambda^{\frac{(-\beta^s)^2+(\alpha^u)^2}{2(-\beta^s)\alpha^u}\delta}\big) + O\big(\lambda^\delta\big) + O\big(\lambda^\delta\big), & i+1 \in J_\kappa \end{cases} \end{aligned}$$

It is obvious that each displayed exponent is greater or equal to $\delta > 1$. Recall in this regard for the first estimate in the last line that $a^2 + b^2 \geq 2ab$.

This enables to factor out λ on the right-hand sides of (5.22) and (5.23). In the following we interpret the right-hand side as an operator

$$\boldsymbol{\chi}: l^\infty \times \mathbb{R} \times \Sigma_A \to l^\infty, \quad (\boldsymbol{r},\lambda,\kappa) \mapsto \boldsymbol{\chi}(\boldsymbol{r},\lambda,\kappa).$$

So, for $\lambda \neq 0$ all jumps Ξ_i are equal to zero if

$$\boldsymbol{\chi}(\mathbf{r},\lambda,\kappa) = 0, \quad \boldsymbol{\chi} = (\chi_i)_{i\in\mathbb{Z}}, \tag{5.24}$$

where

$$\left.\begin{array}{llll}\chi_i(\mathbf{r},\lambda,\kappa) = & 1 - A_i(\lambda) r_i + O\big(\lambda^{\delta-1}\big), & i \in \mathbb{Z}\setminus J_\kappa, & \\ \chi_i(\mathbf{r},\lambda,\kappa) = & 1 - B(\lambda) r_i^2 - C_i(\lambda) r_{i-1} r_i + O\big(\lambda^{\delta-1}\big), & i \in J_\kappa, & i-1 \in J_\kappa, \\ \chi_i(\mathbf{r},\lambda,\kappa) = & 1 - B(\lambda) r_i^2 - C_i(\lambda)\lambda^{1/2} r_{i-1} r_i + O\big(\lambda^{\delta-1}\big), & i \in J_\kappa, & i-1 \in \mathbb{Z}\setminus J_\kappa. \end{array}\right\} \quad (5.25)$$

By construction $\boldsymbol{\chi}(\cdot,\cdot,\kappa)$ is smooth for $\lambda > 0$ and $r_i > 0$, $i \in \mathbb{Z}$. Note that for $i \in \mathbb{Z}\setminus J_\kappa$ the equation $\chi_i(\mathbf{r},\lambda,\kappa) = 0$ might only be solved for $A_i(\lambda) > 0$. Otherwise $\chi_i(\mathbf{r},\lambda,\kappa) > 0$ for all $\lambda > 0$ close to zero.

In what follows we first consider $\boldsymbol{\chi}(\mathbf{r},0,\kappa) = 0$ and afterwards we continue the corresponding solution for $\lambda > 0$, cf. also (5.21).

A starting solution for $\lambda = 0$

According to (5.25) and using the short-hand notations $B := B(0)$, $C_i := C_i(0)$ and $A_i := A_i(0)$ the equation $\boldsymbol{\chi}(\mathbf{r},0,\kappa) = 0$ translates into

$$\left.\begin{array}{lllll} 0 & = & 1 - A_i r_i, & i \in \mathbb{Z}\setminus J_\kappa, & \\ 0 & = & 1 - B r_i^2 - C_i r_{i-1} r_i, & i \in J_\kappa, & i-1 \in J_\kappa, \\ 0 & = & 1 - B r_i^2, & i \in J_\kappa, & i-1 \in \mathbb{Z}\setminus J_\kappa. \end{array}\right\} \quad (5.26)$$

The equation $0 = 1 - B r_i^2 - C_i r_{i-1} r_i$ can be solved for $r_i > 0$ with

$$r_i = \frac{1}{2B}\left(-C_i r_{i-1} + \sqrt{C_i^2 r_{i-1}^2 + 4B}\right). \quad (5.27)$$

Now, using (5.27) we can rewrite (5.26) as a fixed point equation $\mathbf{r} = \mathbf{F}(\mathbf{r},\kappa)$, $\mathbf{F} = (F_i)_{i\in\mathbb{Z}}$, where

$$\left.\begin{array}{llll} F_i(\mathbf{r},\kappa) = & \frac{1}{A_i}, & i \in \mathbb{Z}\setminus J_\kappa, & \\ F_i(\mathbf{r},\kappa) = & \frac{1}{2B}\left(-C_i r_{i-1} + \sqrt{C_i^2 r_{i-1}^2 + 4B}\right), & i \in J_\kappa, & i-1 \in J_\kappa, \\ F_i(\mathbf{r},\kappa) = & \frac{1}{\sqrt{B}}, & i \in J_\kappa, & i-1 \in \mathbb{Z}\setminus J_\kappa. \end{array}\right\} \quad (5.28)$$

We show that $\mathbf{F}(\cdot,\kappa)$ has a unique fixed point $\mathbf{r}^\kappa = (r_i^\kappa)_{i\in\mathbb{Z}}$ with $r_i^\kappa > 0$. To this end we show that for an appropriate closed set $U \subset l^\infty$ the mapping $\mathbf{F}(\cdot,\kappa) : U \to U$ is a contraction.

Recall from Theorem 5.3.1 that $|C_i(\lambda,\kappa)| = C(\lambda) > 0$ for $i, i-1 \in J_\kappa$. With $C := C(0)$ and

$$q := \frac{C}{B} < 1,$$

we define $U = \underset{i\in\mathbb{Z}}{\times} U_i$, $U_i \subset \mathbb{R}$ by

$$U_i = \left\{\begin{array}{lll} \left\{\frac{1}{A_i}\right\}, & i \in \mathbb{Z}\setminus J_\kappa, & \\ \left[0, \frac{1}{(1-q)\sqrt{B}}\right], & i \in J_\kappa, & i-1 \in J_\kappa, \\ \left\{\frac{1}{\sqrt{B}}\right\}, & i \in J_\kappa, & i-1 \in \mathbb{Z}\setminus J_\kappa. \end{array}\right. \quad (5.29)$$

First we show that $\mathbf{F}(U,\kappa) \subseteq U$. It is obvious that $F_i(U,\kappa) \subseteq U_i$ for $i \in \mathbb{Z}\setminus J_\kappa$ and for $i \in J_\kappa$, $i-1 \in \mathbb{Z}\setminus J_\kappa$. So it remains to verify $F_i(U,\kappa) \subseteq U_i$ for $i-1, i \in J_\kappa$. In doing so we distinguish $i-2 \in J_\kappa$

and $i-2 \in \mathbb{Z} \setminus J_\kappa$. In these cases either, cf. second line in (5.28) and (5.29), $r_{i-1} \in \left[0, \frac{1}{(1-q)\sqrt{B}}\right]$ or $r_{i-1} = \frac{1}{\sqrt{B}}$. Since $1-q<1$ we have $\frac{1}{\sqrt{B}} \in \left[0, \frac{1}{(1-q)\sqrt{B}}\right]$. So we only need to consider

$$F_i(\mathbf{r},\kappa) = \frac{1}{2B}\left(-C_i r_{i-1} + \sqrt{C_i^2 r_{i-1}^2 + 4B}\right), \quad \text{for} \quad r_{i-1} \in \left[0, \frac{1}{(1-q)\sqrt{B}}\right].$$

Using the definition of $C = C(0)$, cf. Theorem 5.3.1, we find for this constellation

$$\begin{aligned} F_i(\mathbf{r},\kappa) &\leq \tfrac{1}{2B}\left(C r_{i-1} + \sqrt{C^2 r_{i-1}^2 + 4B}\right) \\ &< \tfrac{1}{2B}\left(2C r_{i-1} + 2\sqrt{B}\right) = q r_{i-1} + \tfrac{1}{\sqrt{B}} \leq q \tfrac{1}{(1-q)\sqrt{B}} + \tfrac{1}{\sqrt{B}} \\ &= \tfrac{1}{(1-q)\sqrt{B}}. \end{aligned}$$

Altogether this verifies that $\mathbf{F}(U,\kappa) \subset U$.

Next we show that $\mathbf{F}(\cdot,\kappa)$ is contractive on U. To this end we show that q is an appropriate contraction constant. We consider for $\mathbf{r},\mathbf{s} \in U$

$$\|\mathbf{F}(\mathbf{r},\kappa) - \mathbf{F}(\mathbf{s},\kappa)\|_{l^\infty} = \sup_{i\in\mathbb{Z}} |F_i(\mathbf{r},\kappa) - F_i(\mathbf{s},\kappa)|.$$

According to (5.28) we find that $F_i(\mathbf{r},\kappa) - F_i(\mathbf{s},\kappa) = 0$ for $i \in \mathbb{Z} \setminus J_\kappa$ and for $i \in J_\kappa$, $i-1 \in \mathbb{Z} \setminus J_\kappa$. For the remaining possibility $i-1, i \in J_\kappa$ we find:

$$\begin{aligned} |F_i(\mathbf{r},\kappa) - F_i(\mathbf{s},\kappa)| &= \left|\tfrac{1}{2B}\left(-C_i\left(r_{i-1} - s_{i-1}\right) + \sqrt{C_i^2 r_{i-1}^2 + 4B} - \sqrt{C_i^2 s_{i-1}^2 + 4B}\right)\right| \\ &= \left|\tfrac{1}{2B}\left(-C_i\left(r_{i-1} - s_{i-1}\right) + \tfrac{C_i^2\left(r_{i-1}^2 - s_{i-1}^2\right)}{\sqrt{C_i^2 r_{i-1}^2 + 4B} + \sqrt{C_i^2 s_{i-1}^2 + 4B}}\right)\right| \\ &\leq \tfrac{1}{2B}\left(C\|\mathbf{r}-\mathbf{s}\|_{l^\infty} + \tfrac{C^2(r_{i-1}^2 - s_{i-1}^2)}{\sqrt{C^2 r_{i-1}^2} + \sqrt{C^2 s_{i-1}^2}}\right) \\ &\leq \tfrac{1}{2B}\left(C\|\mathbf{r}-\mathbf{s}\|_{l^\infty} + C|r_{i-1} - s_{i-1}|\right) \\ &\leq \tfrac{2C}{2B}\|\mathbf{r}-\mathbf{s}\|_{l^\infty} = q\|\mathbf{r}-\mathbf{s}\|_{l^\infty}. \end{aligned}$$

This proves that $\mathbf{F}(\cdot,\kappa)$ is indeed contractive on U. Finally, the Banach fixed point theorem guarantees the fixed point $\mathbf{r}^\kappa \in U$.

Continuation of the solution for $\lambda > 0$

The above constructions show that

$$\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa) = 0. \tag{5.30}$$

In order to guarantee the intended continuation of the solution we apply the implicit function theorem as stated in [Zei93, Theorem 4B]. At this point we want to stress that $\boldsymbol{\chi}$ can continuously be extended for $\lambda \leq 0$ by defining for $\lambda \leq 0$:

$$\begin{aligned} \chi_i(\mathbf{r},\lambda,\kappa) &= 1 - A_i(\lambda) r_i, && i \in \mathbb{Z} \setminus J_\kappa, \\ \chi_i(\mathbf{r},\lambda,\kappa) &= 1 - B(\lambda) r_i^2 - C_i(\lambda) r_{i-1} r_i, && i \in J_\kappa, \quad i-1 \in J_\kappa, \\ \chi_i(\mathbf{r},\lambda,\kappa) &= 1 - B(\lambda) r_i^2, && i \in J_\kappa, \quad i-1 \in \mathbb{Z} \setminus J_\kappa. \end{aligned}$$

In what follows we consider $\boldsymbol{\chi}(\cdot,\cdot,\kappa)$ as a mapping defined on neighbourhoods of $\mathbf{r}^\kappa$ in l^∞ and 0 in $\mathbb{R}$.

Because of (5.30) it remains to show that $D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa) \in \mathbb{L}(l^\infty, l^\infty)$ is invertible. Note that then the inverse is automatically bounded. We find that

$$D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa)\mathbf{r} = \left(D_1\chi_i(\mathbf{r}^\kappa, 0, \kappa)\mathbf{r}\right)_{i\in\mathbb{Z}},$$

and by using the short-hand notations $B := B(0)$, $C_i := C_i(0)$ and $A_i := A_i(0)$

$$\left.\begin{aligned} D_1\chi_i(\mathbf{r}^\kappa, 0, \kappa)\mathbf{r} &= -A_i r_i, && i \in \mathbb{Z}\setminus J_\kappa, \\ D_1\chi_i(\mathbf{r}^\kappa, 0, \kappa)\mathbf{r} &= -C_i r_i^\kappa r_{i-1} - (2Br_i^\kappa + C_i r_{i-1}^\kappa) r_i, && i \in J_\kappa, \quad i-1 \in J_\kappa, \\ D_1\chi_i(\mathbf{r}^\kappa, 0, \kappa)\mathbf{r} &= -2Br_i^\kappa r_i, && i \in J_\kappa, \quad i-1 \in \mathbb{Z}\setminus J_\kappa. \end{aligned}\right\} \tag{5.31}$$

First we consider $\ker D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa)$. Let $\mathbf{r} \in \ker D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa)$, then:

$$\left.\begin{aligned} r_i &= 0, && i \in \mathbb{Z}\setminus J_\kappa, \\ C_i r_i^\kappa r_{i-1} + (2Br_i^\kappa + C_i r_{i-1}^\kappa) r_i &= 0, && i \in J_\kappa, \quad i-1 \in J_\kappa, \\ r_i &= 0, && i \in J_\kappa, \quad i-1 \in \mathbb{Z}\setminus J_\kappa. \end{aligned}\right\} \tag{5.32}$$

Only the second equation in (5.32), the one which is related to $i \in J_\kappa$, $i-1 \in J_\kappa$, can have non-trivial solutions.

First we consider the case where there is an $i_0 \in J_\kappa$ such that $i_0 - 1 \in \mathbb{Z}\setminus J_\kappa$. Then according to the third equation in (5.32) $r_{i_0} = 0$. From (5.27) we obtain

$$2Br_i^\kappa + C_i r_{i-1}^\kappa = \sqrt{C_i^2 (r_{i-1}^\kappa)^2 + 4B} > 0, \tag{5.33}$$

and hence, according to the second equation in (5.32) it yields

$$r_i = 0, \quad \forall i \in J_\kappa,\, i > i_0.$$

So it remains to consider the case $i_0 \in J_\kappa$ and $i_0 - j \in J_\kappa$, for all $j \in \mathbb{N}$. The second equation in (5.32) yields

$$r_{i_0-1} = -\left(\frac{2B}{C_i} + \frac{r_{i_0-1}^\kappa}{r_{i_0}^\kappa}\right) r_{i_0}. \tag{5.34}$$

So, for $C_i > 0$ we find

$$|r_{i_0-1}| > \frac{2B}{C}\,|r_{i_0}|.$$

Further, from (5.33) we conclude

$$r_i^\kappa = \frac{C_i}{2B} r_{i-1}^\kappa \left(-1 + \operatorname{sgn}(C_i)\sqrt{1 + \frac{4B}{C^2 (r_{i-1}^\kappa)^2}}\right),$$

and hence we find for $C_i < 0$

$$\frac{r_{i_0-1}^\kappa}{r_{i_0}^\kappa} = \frac{2B}{C_i}\left(-1 + \operatorname{sgn}(C_i)\sqrt{1 + \frac{4B}{C^2 (r_{i-1}^\kappa)^2}}\right)^{-1} < \frac{2B}{2C}. \tag{5.35}$$

So, on the other hand for $C_i < 0$ we obtain from (5.34) and (5.35) that

$$r_{i_0-1} > \frac{B}{C}\, r_{i_0}.$$

So, independently of the sign of C_i we find $|r_{i_0-1}| > \frac{B}{C}\,|r_{i_0}|$ and hence

$$|r_{i_0-k}| > \left(\frac{B}{C}\right)^k |r_{i_0}|. \tag{5.36}$$

That means that the sequence $(r_{i_o-k})_{k\in\mathbb{N}}$ is unbounded and therefore it does not belong to l^∞. In the consequence that means that $D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa)\mathbf{r} = 0$ does not have a non-trivial solution in l^∞, or in other words

$$\ker D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa) = \{0\}.$$

Hence $D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa) \in \mathbb{L}(l^\infty, l^\infty)$ is injective.

In order to show that $D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa)$ is surjective let $\mathbf{s} \in l^\infty$ be given and consider

$$D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa)\mathbf{r} = \mathbf{s}.$$

Again we use the Banach fixed point theorem to show that we find for any $\mathbf{s} \in l^\infty$ an appropriate $\mathbf{r} \in l^\infty$. To this end recall that l^∞ equipped with the supremum norm is a Banach space. From (5.31) we can conclude the fixed point equation $\mathbf{r} = \mathbf{F}^{\mathbf{s}}(\mathbf{r}, \kappa)$, $\mathbf{F}^{\mathbf{s}} = (F_i^{\mathbf{s}})_{i\in\mathbb{Z}}$ with

$$\left.\begin{array}{rcllll} F_i^{\mathbf{s}}(\mathbf{r},\kappa) &=& -\frac{s_i}{A_i}, & i \in \mathbb{Z}\setminus J_\kappa, & \\ F_i^{\mathbf{s}}(\mathbf{r},\kappa) &=& \frac{-C_i r_i^\kappa}{2Br_i^\kappa + C_i r_{i-1}^\kappa} r_{i-1} - \frac{s_i}{2Br_i^\kappa + C_i r_{i-1}^\kappa}, & i \in J_\kappa, & i-1 \in J_\kappa, \\ F_i^{\mathbf{s}}(\mathbf{r},\kappa) &=& -\frac{s_i}{2Br_i^\kappa}, & i \in J_\kappa, & i-1 \in \mathbb{Z}\setminus J_\kappa. \end{array}\right\} \tag{5.37}$$

Now we show that for an appropriate closed set $U^{\mathbf{s}} \subset l^\infty$, depending on $\mathbf{s} \in l^\infty$, the mapping $\mathbf{F}^{\mathbf{s}}(\cdot, \kappa) : U^{\mathbf{s}} \to U^{\mathbf{s}}$ is a contraction. With $\mathbf{v} = (v_i)_{i\in\mathbb{Z}} \in l^\infty$,

$$v_i := |s_i| \max\left\{\frac{1}{2Br_i^\kappa}, \frac{1}{2Br_i^\kappa + C_i r_{i-1}^\kappa}\right\}$$

we define $U^{\mathbf{s}} = \underset{i\in\mathbb{Z}}{\times} U_i^{\mathbf{s}}$, $U_i^{\mathbf{s}} \subset \mathbb{R}$ by

$$U_i^{\mathbf{s}} = \left\{\begin{array}{lll} \left\{-\frac{s_i}{A_i}\right\}, & i \in \mathbb{Z}\setminus J_\kappa, & \\ \left[-\frac{B}{B-C}\|\mathbf{v}\|_{l^\infty}, \frac{B}{B-C}\|\mathbf{v}\|_{l^\infty}\right], & i \in J_\kappa, & i-1 \in J_\kappa, \\ \left\{-\frac{s_i}{2Br_i^\kappa}\right\}, & i \in J_\kappa, & i-1 \in \mathbb{Z}\setminus J_\kappa. \end{array}\right. \tag{5.38}$$

Note that all denominator, A_i, $2Br_i^\kappa$ and, due to (5.33), $2Br_i^\kappa + C_i r_{i-1}^\kappa$ are different from zero.

First we show that $\mathbf{F}^{\mathbf{s}}(U^{\mathbf{s}}, \kappa) \subseteq U^{\mathbf{s}}$. It is obvious that $F_i^{\mathbf{s}}(U^{\mathbf{s}}, \kappa) \subseteq U_i^{\mathbf{s}}$ for $i \in \mathbb{Z}\setminus J_\kappa$ and for $i \in J_\kappa$, $i-1 \in \mathbb{Z}\setminus J_\kappa$. So it remains to verify $F_i^{\mathbf{s}}(U^{\mathbf{s}}, \kappa) \subseteq U_i^{\mathbf{s}}$ for $i-1, i \in J_\kappa$. In doing so we distinguish $i-2 \in J_\kappa$ and $i-2 \in \mathbb{Z}\setminus J_\kappa$. In these cases either, cf. the second and third line in (5.38), $r_{i-1} \in \left[-\frac{B}{B-C}\|\mathbf{v}\|_{l^\infty}, \frac{B}{B-C}\|\mathbf{v}\|_{l^\infty}\right]$ or $r_{i-1} = -\frac{s_{i-1}}{2Br_{i-1}^\kappa}$. Due to the definition of v_i we have in the latter case

$|r_{i-1}| = \frac{|s_{i-1}|}{2Br^\kappa_{i-1}} \leq |v_{i-1}| \leq \|\mathbf{v}\|_{l^\infty} \frac{B}{B-C}$. So we only need to consider

$$F^{\mathbf{s}}_i(\mathbf{r},\kappa) = \frac{-C_i r^\kappa_i}{2Br^\kappa_i + C_i r^\kappa_{i-1}} r_{i-1} - \frac{s_i}{2Br^\kappa_i + C_i r^\kappa_{i-1}}, \quad \text{for} \quad r_{i-1} \in \left[-\frac{B}{B-C}\|\mathbf{v}\|_{l^\infty}, \frac{B}{B-C}\|\mathbf{v}\|_{l^\infty}\right].$$

To this end we apply that for $i-1, i \in J_\kappa$, cf. (5.28), $r^\kappa_i = F_i(\mathbf{r}^\kappa,\kappa) = \frac{1}{2B}\left(-C_i r^\kappa_{i-1} + \sqrt{C^2_i (r^\kappa_{i-1})^2 + 4B}\right)$. Further we use $2Br^\kappa_i + C_i r^\kappa_{i-1} = \sqrt{C^2_i (r^\kappa_{i-1})^2 + 4B}$, cf. (5.33). This yield

$$\frac{C r^\kappa_i}{2Br^\kappa_i + C_i r^\kappa_{i-1}} = \frac{C}{2B} \frac{-C_i r^\kappa_{i-1} + \sqrt{C^2_i (r^\kappa_{i-1})^2 + 4B}}{\sqrt{C^2 (r^\kappa_{i-1})^2 + 4B}} \leq \frac{C}{2B}\left(\frac{C r^\kappa_{i-1}}{\sqrt{C^2 (r^\kappa_{i-1})^2}} + 1\right) \leq \frac{C}{B}. \tag{5.39}$$

Applying this we then find

$$\begin{aligned} |F^{\mathbf{s}}_i(\mathbf{r},\kappa)| &\leq \tfrac{C r^\kappa_i}{2Br^\kappa_i + C_i r^\kappa_{i-1}} |r_{i-1}| + \tfrac{|s_i|}{2Br^\kappa_i + C_i r^\kappa_{i-1}} \\ &\leq \tfrac{C}{B}|r_{i-1}| + |v_i| \leq \left(\tfrac{C}{B}\tfrac{B}{B-C} + 1\right)\|\mathbf{v}\|_{l^\infty} = \tfrac{B}{B-C}\|\mathbf{v}\|_{l^\infty} \end{aligned}$$

Altogether this verifies that $\mathbf{F}^{\mathbf{s}}(U^{\mathbf{s}},\kappa) \subset U^{\mathbf{s}}$.

So it remains to show that $\mathbf{F}^{\mathbf{s}}(\cdot,\kappa)$ is contractive. We consider for $\mathbf{r},\mathbf{u} \in l^\infty$

$$\|\mathbf{F}^{\mathbf{s}}(\mathbf{r},\kappa) - \mathbf{F}^{\mathbf{s}}(\mathbf{u},\kappa)\|_{l^\infty} = \sup_{i\in\mathbb{Z}} |F^{\mathbf{s}}_i(\mathbf{r},\kappa) - F^{\mathbf{s}}_i(\mathbf{u},\kappa)|.$$

According to (5.37) we obtain $F^{\mathbf{s}}_i(\mathbf{r},\kappa) - F^{\mathbf{s}}_i(\mathbf{u},\kappa) = 0$ for $i \in \mathbb{Z}\setminus J_\kappa$ and for $i \in J_\kappa$, $i-1 \in \mathbb{Z}\setminus J_\kappa$. For the remaining cases $i-1, i \in J_\kappa$ we find with (5.39):

$$|F^{\mathbf{s}}_i(\mathbf{r},\kappa) - F^{\mathbf{s}}_i(\mathbf{u},\kappa)| = \left|\frac{-C_i r^\kappa_i}{2Br^\kappa_i + C_i r^\kappa_{i-1}}\right| |r_{i-1} - u_{i-1}| \leq \frac{C}{B}\|\mathbf{r}-\mathbf{u}\|_{l^\infty}.$$

Hence $\mathbf{F}^{\mathbf{s}}(\cdot,\kappa)$ is contractive and we find for any $\mathbf{s} \in l^\infty$ a solution $\mathbf{r} \in l^\infty$. This verifies the surjectivity of $D_1\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa)$.

Summarising, for $\lambda > 0$ we can solve $\boldsymbol{\chi}(\boldsymbol{r},\lambda,\kappa) = 0$ with $\mathbf{r} = \mathbf{r}(\lambda,\kappa)$ near $\mathbf{r} = \mathbf{r}^\kappa$ and $\lambda = 0$, if $B > 0$ and $A_i > 0$ for all $i \in \mathbb{Z}\setminus J_\kappa$. According to the definition of $A_i := \langle \eta^-_{\kappa_i}(0), \eta^s_{\kappa_{i-1}}(0)\rangle$, cf. Lemma 4.3.14 and the stipulation that $\langle \eta^-_{\kappa_i}(0), \eta^s_{\kappa_{i-1}}(0)\rangle < 0$, cf. (1.11), we find $\mathrm{sgn}A_i = -\mathrm{sgn}\langle \eta^s_{\kappa_i}(0), \eta^s_{\kappa_{i-1}}(0)\rangle$. Then A_i is positive, if the angle between $\eta^s_{\kappa_i}(0)$ and $\eta^s_{\kappa_{i-1}}(0)$ is greater than 90°. This is equal to the condition that $|\kappa_i - \kappa_{i-1}| := \min\{\kappa_i - \kappa_{i-1}, \kappa_{i-1} - \kappa_i\} > m$, differences calculated in $\mathbb{Z}_{4m}$. Since with $B > 0$ solutions also exist for right angles between $\eta^s_{\kappa_i}(0)$ and $\eta^s_{\kappa_{i-1}}(0)$ we find for $\lambda > 0$ all those trajectories satisfying $|\kappa_i - \kappa_{i-1}| \geq m$ for all $i \in \mathbb{Z}$.

From (5.18) and (5.21) we find the following expression of ω_i in terms of λ and κ:

$$\omega_i(\lambda,\kappa) = \begin{cases} \frac{1}{2\mu^s(0)}(\ln(\lambda) + \ln(r_i(\lambda,\kappa))), & i \in \mathbb{Z}\setminus J_\kappa, \\ \frac{1}{4\mu^s(0)}(\ln(\lambda) + \ln(r^2_i(\lambda,\kappa))), & i \in J_\kappa. \end{cases} \tag{5.40}$$

Due to (5.29) we find that r^κ_i are uniformly bounded. So with $r_i(\lambda,\kappa) = r^\kappa_i + O(\lambda)$ the terms $\ln(r_i(\lambda,\kappa))$ and $\ln(r^2_i(\lambda,\kappa))$ are almost negligible for λ sufficiently small. Therefore the corresponding transition times ω_i basically differ by a factor of two, depending on whether $i \in J_\kappa$ or $i \in \mathbb{Z}\setminus J_\kappa$.

5.4.2 Solution for fixed κ and $\lambda < 0$

We continue from the representations (5.19) and (5.20) of the jumps and introduce for $\lambda < 0$ the rescaling

$$\hat{r}_i = \begin{cases} -\lambda r_i, & i \in \mathbb{Z} \setminus J_\kappa, \\ \sqrt{-\lambda} r_i, & i \in J_\kappa. \end{cases} \tag{5.41}$$

Then we obtain from (5.19) and (5.20) with $\mathbf{r} = (r_i)_{i \in \mathbb{Z}}$

$$i \in \mathbb{Z} \setminus J_\kappa : \quad \Xi_i(\mathbf{r}, \lambda, \kappa) = \lambda + A_i(\lambda)\lambda r_i + \tilde{R}_i(\mathbf{r}, \lambda, \kappa), \tag{5.42}$$

$$i \in J_\kappa : \quad \Xi_i(\mathbf{r}, \lambda, \kappa) = \lambda + B(\lambda)\lambda r_i^2 + \begin{cases} C_i(\lambda)\lambda r_{i-1} r_i + \tilde{R}_i(\mathbf{r}, \lambda, \kappa), & i-1 \in J_\kappa, \\ -C_i(\lambda)(-\lambda)^{3/2} r_{i-1} r_i + \tilde{R}_i(\mathbf{r}, \lambda, \kappa), & i-1 \in \mathbb{Z} \setminus J_\kappa. \end{cases} \tag{5.43}$$

The residual terms $\tilde{R}_i$ arise from $\hat{R}_i$ accordingly. Thereby it turns out that for all $i \in \mathbb{Z}$

$$\tilde{R}_i = O\big((-\lambda)^\delta\big), \quad \delta > 1.$$

Factoring out λ yield the determination equations

$$\left.\begin{array}{llll} 0 = 1 + A_i(\lambda) r_i & +O\big((-\lambda)^{\delta-1}\big), & i \in \mathbb{Z} \setminus J_\kappa, & \\ 0 = 1 + B(\lambda) r_i^2 + C_i(\lambda) r_{i-1} r_i & +O\big((-\lambda)^{\delta-1}\big), & i \in J_\kappa, & i-1 \in J_\kappa, \\ 0 = 1 + B(\lambda) r_i^2 & +O\big((-\lambda)^{\delta-1}\big) + O\big((-\lambda)^{1/2}\big), & i \in J_\kappa, & i-1 \in \mathbb{Z} \setminus J_\kappa. \end{array}\right\} \tag{5.44}$$

An immediate consequence of the rescaling (5.41) is, that the right-hand side of the third equation is due to $B(\lambda) > 0$ positive for λ close to zero. That is, if κ allows for a sequence with $i \in J_\kappa$ and $i-1 \in \mathbb{Z} \setminus J_\kappa$ we cannot find a solution for this κ for $\lambda < 0$.

Moreover, when considering the second equation we find with $C_i(\lambda) > 0$ that the right-hand side also is positive. Hence, if for fixed κ the jump $\boldsymbol{\Xi}(\boldsymbol{\omega}, \lambda, \kappa)$ contains for some i the case $i \in J_\kappa$, $i-1 \in J_\kappa$ and $C_i(\lambda) > 0$ again we do not find a solution for this κ for $\lambda < 0$.

Consequently for $\lambda < 0$ there are only two types of sequences κ we might obtain a solution for: Either we have for all i that $i \in \mathbb{Z} \setminus J_\kappa$ or we have for all i that $i \in J_\kappa$ and $C_i(\lambda) < 0$.

Now, in the first case, $i \in \mathbb{Z} \setminus J_\kappa$ for all i, we obtain the equation

$$\chi_i(\mathbf{r}, \lambda, \kappa) = 1 + A_i(\lambda) r_i + O\big((-\lambda)^{\delta-1}\big)$$

whose solvability follows along the same lines as in Section 5.4.1. Only here we have to assume $A_i(\lambda) < 0$.

In case that $i \in J_\kappa$ and $C_i(\lambda, \kappa) < 0$ for all i the following equation applies

$$\chi_i(\mathbf{r}, \lambda, \kappa) = 1 + B(\lambda) r_i^2 - C(\lambda) r_{i-1} r_i + O\big(\lambda^{\delta-1}\big). \tag{5.45}$$

According to Corollary 5.2.4 the condition $C_i(\lambda, \kappa) = -C(\lambda)$ for all $i \in \mathbb{Z}$ implies that κ is an element of $\mathcal{K}_2$ or $\mathcal{K}_4$, cf. (5.9) and (5.10). In the following we show that the equation $\boldsymbol{\chi}(\mathbf{r}, \lambda, \kappa) = 0$ has no solution for those κ.

Assuming there is a solution $\mathbf{r}(\lambda)$ for λ sufficiently small solving $\boldsymbol{\chi}(\mathbf{r},\lambda,\kappa)=0$. Then there also has to be a solution at $\lambda=0$. Let $\mathbf{r}^\kappa$ be such that $\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa)=0$. Then we conclude from (5.45) that the r_i^κ have to satisfy

$$r_i^\kappa = \frac{1}{2B}\left(Cr_{i-1}^\kappa \pm \sqrt{C^2(r_{i-1}^\kappa)^2-4B}\right) \tag{5.46}$$

for all $i\in\mathbb{Z}_p$, $p=2$ or $p=4$, respectively. Thereby again $B:=B(0)$ and $C:=|C_i(0)|$. This provides the lower bound $r_i^\kappa \geq \frac{2\sqrt{B}}{C}$ for all $i\in\mathbb{Z}_p$, so that the discriminant is greater or equal to zero.

First we discuss equation (5.46) for the minus sign. Here we find the following upper bound for r_i^κ

$$r_i^\kappa = \frac{1}{2B}(Cr_{i-1}^\kappa - \sqrt{C^2(r_{i-1}^\kappa)^2-4B}) = \frac{2}{Cr_{i-1}^\kappa + \sqrt{C^2(r_{i-1}^\kappa)^2-4B}} \leq \frac{2}{Cr_{i-1}^\kappa} \leq \frac{1}{\sqrt{B}}.$$

However, for $B>C$ we find that the upper bound $\frac{1}{\sqrt{B}}$ is smaller than the lower bound $\frac{2\sqrt{B}}{C}$, a contradiction. Hence we cannot find a real solution if at least one equation (5.46) has a negative sign.

Considering equation (5.46) for the plus sign we find the upper bound

$$r_i^\kappa = \frac{Cr_{i-1}^\kappa + \sqrt{C^2(r_{i-1}^\kappa)^2-4B}}{2B} \leq \frac{C}{B}r_{i-1}^\kappa.$$

Since $B>C$ the sequence of the r_i^κ needs to be strictly monotonically decreasing. This leads to the contradictory statement $r_i^\kappa \leq \left(\frac{C}{B}\right)^p r_i^\kappa < r_i^\kappa$. Thus we do not find a solution of $\boldsymbol{\chi}(\mathbf{r},0,\kappa)=0$, which contradicts the assumption.

In summary, for $\lambda<0$ we only find solutions for those κ that satisfy $i\in\mathbb{Z}\setminus J_\kappa$ with $A_i(\lambda)<0$ for all $i\in\mathbb{Z}$. This equals the condition, that the angle between $\eta^s_{\kappa_i}(0)$ and $\eta^s_{\kappa_{i-1}}(0)$ is less than 90° or analogously $|\kappa_i-\kappa_{i-1}|<m$ for all $i\in\mathbb{Z}$.

5.4.3 Topological conjugation

We continue with the proof of Theorem 5.3.3 in the spirit of [HJKL11, Section 4.2]. Let $\lambda>0$. First we define the addressed invariant set $\mathcal{D}_\lambda$ and the related first return map Π_λ.

To this end we consider for given $\kappa\in\Sigma_{A_+}$ the solution $\mathbf{r}(\lambda,\kappa)$ of $\chi(\mathbf{r},\lambda,\kappa)=0$, cf. (5.24). Via the definition in (5.18) and the rescaling (5.21) this solution corresponds to a sequence of transition times $\boldsymbol{\omega}(\lambda,\kappa)$. Note that by construction this sequence has an allocated trajectory $x(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(\cdot)$ of (1.1). Then, following the explanations in [HJKL11] we find

$$x(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(\tau_n)\in S_{\kappa_n}, \quad \text{for} \quad \tau_n = \begin{cases} \sum_{i=1}^{n} 2\omega_i(\lambda,\kappa) & , \quad n\in\mathbb{N}, \\ 0 & , \quad n=0, \\ \sum_{i=n+1}^{0} -2\omega_i(\lambda,\kappa) & , \quad n\in -\mathbb{N}. \end{cases} \tag{5.47}$$

With that we define the set $\mathcal{D}_\lambda$ in the union of cross sections $\cup_{1\leq i\leq 4m} S_i$ by

$$\mathcal{D}_\lambda = \{x(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(\tau_n) \mid n\in\mathbb{Z},\ \kappa\in\Sigma_{A_+}\}.$$

Uniqueness of $\boldsymbol{\omega}$ implies

$$\boldsymbol{\omega}(\lambda,\sigma\kappa)=\hat{\sigma}\boldsymbol{\omega}(\lambda,\kappa),$$

where $\hat{\sigma}$ is the left shift on the set of $\boldsymbol{\omega}$-sequences which is defined in the same way as σ in Definition 1.0.1: $(\hat{\sigma}\boldsymbol{\omega})_i = \omega_{i+1}$. Hence

$$\mathcal{D}_\lambda = \{x(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(0) \mid \kappa \in \Sigma_{A_+}\}.$$

Now, the first return map Π_λ is defined by

$$\Pi_\lambda : \mathcal{D}_\lambda \to \mathcal{D}_\lambda, \quad x(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(0) \mapsto x(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(\tau_1) = x(\boldsymbol{\omega}(\lambda,\sigma\kappa),\lambda,\sigma\kappa)(0).$$

The map

$$\Phi_\lambda : \Sigma_{A_+} \to \mathcal{D}_\lambda, \quad \kappa \mapsto x(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(0) \tag{5.48}$$

is one-to-one. Therefore

$$\Pi_\lambda \circ \Phi_\lambda = \Phi_\lambda \circ \sigma.$$

That is, the systems $(\mathcal{D}_\lambda, \Pi_\lambda)$ and (Σ_{A_+}, σ) are conjugated. For topological conjugacy continuity of Φ_λ and Φ_λ^{-1} must also be established.

For the continuity investigations we consider Σ_{A_+} as being endowed with the product topology, which is, cf. [Shu86, Chap. 10], induced by the metric

$$\varrho(\kappa^1,\kappa^2) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} |\kappa_i^1 - \kappa_i^2|.$$

According to the Tychonoff Theorem, that states that the product of any collection of compact topological spaces is compact with respect to the product topology, cf. [Dug66, Chap. XI, Thm. 1.4], and the fact that Σ_{A_+} is closed in Σ_{4m} the space Σ_{A_+} is compact.

The closure of Σ_{A_+} can easily be seen as follows. Any $\kappa \in \Sigma_{4m} \setminus \Sigma_{A_+}$ is characterised by having an index i such that $a^+_{\kappa_i \kappa_{i+1}} = 0$. Then, $\varepsilon > 0$ can be chosen such that for any $\hat{\kappa}$ satisfying $\varrho(\kappa,\hat{\kappa}) < \varepsilon$ automatically holds that it must equal κ on at least the range $[-(i+1), i+1]$. Thus $\hat{\kappa}$ also lies in $\Sigma_{4m} \setminus \Sigma_{A_+}$. According to this $\Sigma_{4m} \setminus \Sigma_{A_+}$ is open and so Σ_{A_+} is closed in Σ_{4m}. In case of infinite instead of biinfinite sequences $\{1,\dots,4m\}^{\mathbb{N}}$ the proof of the closure of a subset Σ_A can be found in [Dev89, Proposition 13.5]

What follows are some preliminary considerations for the proof of the continuity of Φ_λ. We start with a technical lemma.

Lemma 5.4.1. *Let $(a_i)_{i\in\mathbb{Z}} \in l^\infty$ be a sequence of positive real numbers. Let $l \in \mathbb{Z}$ and $N \in \mathbb{N}$ be arbitrary numbers. If for some K, p with $p > 1$ and $K < (p-1)/2$ the following holds true for all $i \in [l-N, l+N]$*

$$a_i \le K \sum_{\substack{j=i-(N-|i-l|)\\ j\ne i}}^{i+N-|i-l|} \frac{1}{p^{|i-j|}} a_j + \varepsilon, \tag{5.49}$$

then

$$a_i \le \frac{2K}{p} \|a\|_{l^\infty} \left(\frac{2K+1}{p}\right)^{N-|i-l|-1} + \frac{\varepsilon(p-1)}{p-(1+2K)}, \tag{5.50}$$

for all $i \in [l-(N-1), l+(N-1)]$.

Proof. We prove the lemma by induction for the counter N:

Let l be any integer and assume for that l that (5.49) holds true for $N=1$, that is

$$a_l \leq K \sum_{\substack{j=l-1 \\ j\neq l}}^{l+1} \frac{1}{p^{|l-j|}} a_j + \varepsilon = \frac{K}{p}(a_{l-1}+a_{l+1}) + \varepsilon$$

Estimating a_{l-1} and a_{l+1} by $\|a\|_{l^\infty}$ yields

$$a_l \leq \frac{K}{p}(a_{l-1}+a_{l+1}) + \varepsilon \leq \frac{2K}{p}\|a\|_{l^\infty} + \varepsilon \frac{p-1}{p-(1+2K)}.$$

This proves (5.50) for $N=1$ (and any l).

Now, assume that the assertion holds true for N. That is, for any k for which

$$a_i \leq K \sum_{\substack{j=i-(N-|i-k|) \\ j\neq i}}^{i+N-|i-k|} \frac{1}{p^{|i-j|}} a_j + \varepsilon, \tag{5.51}$$

is satisfied for all $i \in [k-N, k+N]$ the following holds true

$$a_i \leq \frac{2K}{p}\|a\|_{l^\infty} \left(\frac{2K+1}{p}\right)^{N-|i-k|-1} + \frac{\varepsilon(p-1)}{p-(1+2K)}, \tag{5.52}$$

for all $i \in [k-(N-1), k+(N-1)]$.

Assume now that for some l estimate (5.49) holds true for $N+1$, that is

$$a_i \leq K \sum_{\substack{j=i-((N+1)-|i-l|) \\ j\neq i}}^{i+(N+1)-|i-l|} \frac{1}{p^{|i-j|}} a_j + \varepsilon, \tag{5.53}$$

$i \in [l-(N+1), l+(N+1)]$. We have to prove that

$$a_i \leq \frac{2K}{p}\|a\|_{l^\infty} \left(\frac{2K+1}{p}\right)^{(N+1)-|i-l|-1} + \frac{\varepsilon(p-1)}{p-(1+2K)}, \tag{5.54}$$

for all $i \in [l-N, l+N]$.

First we verify (5.54) for $i > l$. We exploit that (5.51) implies (5.52) with $k = l+1$. Taking this into consideration we find for all $i \in [l+1, l+N]$

$$a_i \leq K \sum_{\substack{j=i-(N+1-|i-l|) \\ j\neq i}}^{i+N+1-|i-l|} \frac{1}{p^{|i-j|}} a_j + \varepsilon = K \sum_{\substack{j=i-(N-|i-(l+1)|) \\ j\neq i}}^{i+N-|i-(l+1)|} \frac{1}{p^{|i-j|}} a_j + \varepsilon,$$

and hence

$$\begin{aligned} a_i &\leq \tfrac{2K}{p}\|a\|_{l^\infty} \left(\tfrac{2K+1}{p}\right)^{N-(i-(l+1))-1} + \varepsilon \tfrac{p-1}{p-(1+2K)} \\ &= \tfrac{2K}{p}\|a\|_{l^\infty} \left(\tfrac{2K+1}{p}\right)^{N-|i-l|} + \varepsilon \tfrac{p-1}{p-(1+2K)}. \end{aligned}$$

Analogously, reasoning with $k = l - 1$ we find for all $i \in [l - N, l - 1]$ that

$$a_i \leq \frac{2K}{p} \|a\|_{l^\infty} \left(\frac{2K+1}{p}\right)^{N-|i-l|} + \varepsilon \frac{p-1}{p-(1+2K)}.$$

It remains to prove (5.54) for $i = l$:

$$\begin{aligned}
a_l &\leq K \sum_{\substack{j=l-(N+1) \\ j\neq l}}^{l+N+1} \frac{1}{p^{|l-j|}} a_j + \varepsilon \\
&\leq K \sum_{\substack{j=l-N \\ j\neq l}}^{l+N} \frac{1}{p^{|l-j|}} a_j + \frac{K}{p^{N+1}} (a_{l+N+1} + a_{l-N-1}) + \varepsilon \\
&\leq K \sum_{\substack{j=l-N \\ j\neq l}}^{l+N} \frac{1}{p^{|l-j|}} \left(\frac{2K}{p} \|a\|_{l^\infty} \left(\frac{2K+1}{p}\right)^{N-|j-l|} + \varepsilon \frac{p-1}{p-(1+2K)} \right) + \frac{2K}{p^{N+1}} \|a\|_{l^\infty} + \varepsilon \\
&= \frac{2K}{p} \|a\|_{l^\infty} \left(\frac{K}{p^N} 2 \sum_{j=l+1}^{l+N} (2K+1)^{N-j+l} + \frac{1}{p^N} \right) + \varepsilon \left(1 + \frac{K(p-1)}{p-(1+2K)} \sum_{\substack{j=l-N \\ j\neq l}}^{l+N} \frac{1}{p^{|l-j|}} \right) \\
&\leq \frac{2K}{p} \|a\|_{l^\infty} \frac{1}{p^N} \left(2K \sum_{k=0}^{N-1} (2K+1)^{N-1-k} + 1 \right) + \varepsilon \left(1 + \frac{2K(p-1)}{p-(1+2K)} \frac{1}{p(1-1/p)} \right) \\
&= \frac{2K}{p} \|a\|_{l^\infty} \frac{1}{p^N} \left(2K \frac{1-(2K+1)^N}{1-(2K+1)} + 1 \right) + \varepsilon \left(1 + \frac{2K}{p-(1+2K)} \right) \\
&= \frac{2K}{p} \|a\|_{l^\infty} \left(\frac{(2K+1)}{p} \right)^N + \varepsilon \left(\frac{p-1}{p-(1+2K)} \right)
\end{aligned}$$

This concludes the proof. □

We continue with showing the continuity of the three mappings $\boldsymbol{\omega} \mapsto v_0(\boldsymbol{\omega}, \lambda, \kappa)(0)$, $\kappa \mapsto v_0(\boldsymbol{\omega}, \lambda, \kappa)(0)$ and $\kappa \mapsto \boldsymbol{\omega}(\lambda, \kappa)$. To this end recall that we equipped Σ_{A_+} with the metric $\varrho(\kappa^1, \kappa^2) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} |\kappa_i^1 - \kappa_i^2|$. Further we equip l^∞ with the metric $\hat{\varrho}(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} |\omega_i^1 - \omega_i^2|$.

Lemma 5.4.2. *The mapping* $(l^\infty, \hat{\varrho}), \to \mathbb{R}^4 \times \mathbb{R}^4$, $\boldsymbol{\omega} \mapsto v_0(\boldsymbol{\omega}, \lambda, \kappa)(0)$ *is continuous.*

Proof. The statement simply follows from Lemma 3.5.3 and Remark 3.5.5. □

Lemma 5.4.3 ([HJKL11], Lemma 4.5)**.** *The mapping* $(\Sigma_{A_+}, \varrho) \to \mathbb{R}^4 \times \mathbb{R}^4$, $\kappa \mapsto v_0(\boldsymbol{\omega}, \lambda, \kappa)(0)$ *is continuous.*

Proof. Analogously to the proof of Lemma 3.5.3 it is assumed that $\kappa^1, \kappa^2 \in \Sigma_{A+}$ are two sequences that coincide on a block of length $2N + 1$ centered at $j = 0$:

$$\kappa_i^1 = \kappa_i^2, \quad i \in [-N, N] \cap \mathbb{Z}.$$

Then it is established that $\|v_0(\boldsymbol{\omega}, \lambda, \kappa^1)(0) - v_0(\boldsymbol{\omega}, \lambda, \kappa^2)(0)\| = O(1/2^N)$, which implies the lemma.

To this end we start again from (3.22) that shows

$$\|v_0(\boldsymbol{\omega},\kappa^1)(0) - v_0(\boldsymbol{\omega},\kappa^2)(0)\| = \|\hat{v}_{\boldsymbol{\omega},0}(\kappa^1,\boldsymbol{\mathcal{H}}(\boldsymbol{v},\kappa^1),\boldsymbol{d}(\boldsymbol{\omega},\kappa^1))(0) - \hat{v}_{\boldsymbol{\omega},0}(\kappa^2,\boldsymbol{\mathcal{H}}(\boldsymbol{v},\kappa^2),\boldsymbol{d}(\boldsymbol{\omega},\kappa^2))(0)\|.$$

Here once more we suppress the dependency of λ. If $N \geq 1$ we have due to (3.51), (3.37) and (3.52) that

$$\hat{v}_{\boldsymbol{\omega},0}(\kappa^2,\boldsymbol{\mathcal{H}}(\boldsymbol{v},\kappa^2),\boldsymbol{d}(\boldsymbol{\omega},\kappa^2)) = \hat{v}_{\boldsymbol{\omega},0}(\kappa^1,\boldsymbol{\mathcal{H}}(\boldsymbol{v},\kappa^2),\boldsymbol{d}(\boldsymbol{\omega},\kappa^2))$$

and due to the linearity of $\hat{\boldsymbol{v}}_{\boldsymbol{\omega}}$ with respect to the second and third component, recall Lemma 3.2.5, we find

$$\|v_0(\boldsymbol{\omega},\kappa^1)(0) - v_0(\boldsymbol{\omega},\kappa^2)(0)\| = \|\hat{v}_{\boldsymbol{\omega},0}(\kappa^1,\Delta\boldsymbol{\mathcal{H}},\Delta\boldsymbol{d})(0)\|$$

with $\Delta\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}(\boldsymbol{v},\kappa^1) - \boldsymbol{\mathcal{H}}(\boldsymbol{v},\kappa^2)$ and , $\Delta\boldsymbol{d} = \boldsymbol{d}(\boldsymbol{\omega},\kappa^1) - \boldsymbol{d}(\boldsymbol{\omega},\kappa^2)$. Recall that $\Delta d_i = 0$ for $i \in [-N+1, N] \cap \mathbb{Z}$, compare (3.13). Invoking (3.51) we find

$$\hat{\boldsymbol{v}}_{\boldsymbol{\omega},0}(\kappa^1,\Delta\boldsymbol{\mathcal{H}},\Delta\boldsymbol{d}) = \boldsymbol{v}_{\boldsymbol{\omega},0}(\kappa^1,\Delta\boldsymbol{\mathcal{H}},\hat{\boldsymbol{a}}_{\boldsymbol{\omega}}(\kappa^1,\Delta\boldsymbol{\mathcal{H}},\Delta\boldsymbol{d})).$$

Now, proceeding from here as in the proof of Lemma 3.5.3 concludes the proof. □

Corollary 5.4.4. *Let $\kappa^1,\kappa^2 \in \Sigma_{A+}$ be two sequences. Then we find*

$$\|v_i(\boldsymbol{\omega},\lambda,\kappa^1)(0) - v_i(\boldsymbol{\omega},\lambda,\kappa^2)(0)\| = O\big(1/2^N\big) \quad \textit{if} \quad \kappa^1_j = \kappa^2_j, \quad j \in [-|i|-N, |i|+N] \cap \mathbb{Z}.$$

Lemma 5.4.5. *The mapping $(\Sigma_{A_+},\varrho) \to (l^\infty,\hat{\varrho})$, $\kappa \mapsto \boldsymbol{\omega}(\lambda,\kappa)$ is continuous.*

Proof. According to (5.18) and (5.21) we have

$$\omega_i(\lambda,\kappa) = \frac{1}{2\mu^s(\lambda)} \cdot \begin{cases} \ln(\lambda) + \ln r_i(\lambda,\kappa), & i \in \mathbb{Z}\setminus J_\kappa, \\ \ln(\sqrt{\lambda}) + \ln r_i(\lambda,\kappa), & i \in J_\kappa, \end{cases}$$

where $\mathbf{r}(\lambda,\kappa) = (r_i(\lambda,\kappa))_{i\in\mathbb{Z}}$ solves (5.24). So it is enough to prove the corresponding continuity of $\mathbf{r}(\lambda,\cdot)$ (considered as a mapping $(\Sigma_{A_+},\varrho) \to (l^\infty,\hat{\varrho})$). To this end we write (5.24) as a fixed point equation $\mathbf{r} = \mathbf{F}(\mathbf{r},\lambda,\kappa)$, $\mathbf{F} = (F_i)_{i\in\mathbb{Z}}$ similar to (5.28). Further, introducing $\check{R}_i(\mathbf{r},\lambda,\kappa) := \frac{1}{\lambda}\tilde{R}_i(\mathbf{r},\lambda,\kappa)$ we find:

$$\begin{aligned}
F_i(\mathbf{r},\lambda,\kappa) &= \tfrac{1}{A_i(\lambda)}\big(1+\check{R}_i(\mathbf{r},\lambda,\kappa)\big), && i \in \mathbb{Z}\setminus J_\kappa,\\
F_i(\mathbf{r},\lambda,\kappa) &= \tfrac{1}{2B(\lambda)}\left(-C_i(\lambda)r_{i-1} + \sqrt{C_i(\lambda)^2 r_{i-1}^2 + 4B(\lambda)(1+\check{R}_i(\mathbf{r},\lambda,\kappa))}\right), && i \in J_\kappa, i-1 \in J_\kappa,\\
F_i(\mathbf{r},\lambda,\kappa) &= \tfrac{1}{2B(\lambda)}\left(-C_i(\lambda)\lambda^{1/2} r_{i-1} + \sqrt{C_i(\lambda)^2 \lambda r_{i-1}^2 + 4B(\lambda)(1+\check{R}_i(\mathbf{r},\lambda,\kappa))}\right), && i \in J_\kappa, i-1 \in \mathbb{Z}\setminus J_\kappa.
\end{aligned}$$

In order to prove the continuity of $\mathbf{r}(\lambda,\cdot)$ we consider for fixed i

$$\begin{aligned}
&|r_i(\lambda,\kappa^1) - r_i(\lambda,\kappa^2)|\\
&= |F_i(\mathbf{r}(\lambda,\kappa^1),\lambda,\kappa^1) - F_i(\mathbf{r}(\lambda,\kappa^2),\lambda,\kappa^2)|\\
&\leq |F_i(\mathbf{r}(\lambda,\kappa^1),\lambda,\kappa^1) - F_i(\mathbf{r}(\lambda,\kappa^2),\lambda,\kappa^1)| + |F_i(\mathbf{r}(\lambda,\kappa^2),\lambda,\kappa^1) - F_i(\mathbf{r}(\lambda,\kappa^2),\lambda,\kappa^2)|. \quad (5.55)
\end{aligned}$$

With this we will show that $|r_i(\lambda,\kappa^1) - r_i(\lambda,\kappa^2)|$ satisfies

$$|r_i^1 - r_i^2| \leq K(\lambda) \sum_{\substack{j=i-N \\ j\neq i}}^{i+N} \frac{1}{p^{|i-j|}} |r_j^1 - r_j^2| + O\big({}^1/{}_{2^N}\big). \tag{5.56}$$

for some $p > 1$ and a $K(\lambda)$ tending to zero for λ tending to zero. Then applying Lemma 5.4.1 yield

$$|r_i^1 - r_i^2| \leq \frac{2K(\lambda)}{p} \|\boldsymbol{r}\|_{l^\infty} \left(\frac{2K(\lambda)+1}{p}\right)^{N-1} + \frac{(p-1)}{p-(1+2K(\lambda))} O\big({}^1/{}_{2^N}\big),$$

that is $|r_i(\lambda,\kappa^1) - r_i(\lambda,\kappa^2)| = O\Big(\big(\frac{2K(\lambda)+1}{p}\big)^{N-1}\Big) + O\big({}^1/{}_{2^N}\big)$, implying the continuity of the mapping $\kappa \mapsto \boldsymbol{r}(\lambda,\kappa)$.

Now, in what follows we verify the estimate (5.56) by starting from (5.55). Thereby we have to distinguish between the different cases of i and $i-1$ being elements of J_κ or $\mathbb{Z}\setminus J_\kappa$.

For reasons of clarity we omit the λ-dependence in the notation, especially this time we use the short-hand notations $B := B(\lambda)$, $C_i := C_i(\lambda)$ and $A_i := A_i(\lambda)$ and furthermore we use the following short notations

$$\mathbf{r}^j := \mathbf{r}(\lambda,\kappa^j), \quad r_i^j := r_i(\lambda,\kappa^j).$$

First let $\boldsymbol{i} \in \mathbb{Z}\setminus \boldsymbol{J_\kappa}$. We start with the second term of (5.55) and adopt the argumentation from [HJKL11, Proof of Lemma 4.6]:

$$\begin{aligned} |F_i(\boldsymbol{r}^2,\kappa^1) - F_i(\boldsymbol{r}^2,\kappa^2)| &= \tfrac{1}{A_i}\big|\check{R}_i(\boldsymbol{r}^2,\kappa^1) - \check{R}_i(\boldsymbol{r}^2,\kappa^2)\big|. \\ &= \tfrac{1}{|\lambda| A_i} |R_i(\boldsymbol{\omega}(\hat{\boldsymbol{r}}^2),\kappa^1) - R_i(\boldsymbol{\omega}(\hat{\boldsymbol{r}}^2),\kappa^2)| \end{aligned}$$

With Theorem 5.3.1(i), cf. also Theorem 4.3.3(i), we get

$$\begin{aligned} |F_i(\boldsymbol{r}^2,\kappa^1) - F_i(\boldsymbol{r}^2,\kappa^2)| &\leq \frac{1}{|\lambda| A_i} \big|\xi_i(\boldsymbol{\omega}(\kappa^2),\kappa^1) - \xi_i(\boldsymbol{\omega}(\kappa^2),\kappa^2)\big| \\ &= \frac{1}{|\lambda| A_i} \Big(\big|v_i^+(\boldsymbol{\omega}(\kappa^2),\kappa^1)(0) - v_i^+(\boldsymbol{\omega}(\kappa^2),\kappa^2)(0)\big| \\ &\qquad + \big|v_i^-(\boldsymbol{\omega}(\kappa^2),\kappa^1)(0) - v_i^-(\boldsymbol{\omega}(\kappa^2),\kappa^2)(0)\big| \Big). \end{aligned}$$

With Corollary 5.4.4 it follows that

$$|F_i(\boldsymbol{r}^2,\kappa^1) - F_i(\boldsymbol{r}^2,\kappa^2)| = O\big({}^1/{}_{2^N}\big). \tag{5.57}$$

Analogously to the considerations above we obtain for the first term in (5.55)

$$|F_i(\boldsymbol{r}^1,\kappa^1) - F_i(\boldsymbol{r}^2,\kappa^1)| = \frac{1}{|\lambda| A_i} |R_i(\boldsymbol{\omega}(\hat{\boldsymbol{r}}^1),\kappa^1) - R_i(\boldsymbol{\omega}(\hat{\boldsymbol{r}}^2),\kappa^1)|, \tag{5.58}$$

with

$$R_i(\boldsymbol{\omega}(\hat{\boldsymbol{r}}^j),\kappa^1) = \xi_i(\boldsymbol{\omega}(\hat{\boldsymbol{r}}^j),\kappa^1) + \hat{r}_i^j A_i,$$

$j \in \{1,2\}$, cf. Theorem 5.3.1. From Theorem 3.5.1 we find $(D_j \xi_i(\boldsymbol{\omega},\kappa))_{j\in\mathbb{Z}} \in l^1$ and $D_{(\pm N+i)} \xi_i(\boldsymbol{\omega},\kappa) = O({}^1/{}_{2^N})$. Hence $(D_j R_i(\boldsymbol{\omega},\kappa))_{j\in\mathbb{Z}} \in l^1$ and $D_{(\pm N+i)} R_i(\boldsymbol{\omega},\kappa) = O({}^1/{}_{2^N})$ and by applying the mean-value-theorem we obtain

$$
\begin{aligned}
&|R_i(\boldsymbol{\omega}(\hat{\boldsymbol{r}}^1),\kappa^1) - R_i(\boldsymbol{\omega}(\hat{\boldsymbol{r}}^2),\kappa^1)| \\
&\quad\leq \sum_{j\in\mathbb{Z}} |D_j R_i(\bar{\boldsymbol{\omega}},\kappa^1)|\,\big|\omega_j(\hat{r}_j^1)-\omega_j(\hat{r}_j^2)\big| \\
&\quad\leq \sum_{j=i-N}^{i+N} |D_j R_i(\bar{\boldsymbol{\omega}},\kappa^1)|\,\big|\omega_j(\hat{r}_j^1)-\omega_j(\hat{r}_j^2)\big| + \sum_{|j-i|>N+1} |D_j R_i(\bar{\boldsymbol{\omega}},\kappa^1)|\,\big|\omega_j(\hat{r}_j^1)-\omega_j(\hat{r}_j^2)\big| \\
&\quad\leq \sum_{j=i-N}^{i+N} |D_j R_i(\bar{\boldsymbol{\omega}},\kappa^1)|\big(\frac{1}{2|\mu^s(0)\bar{r}_j|}|r_j^1-r_j^2|\big) + O\big(1/2^N\big) \\
&\quad\leq c|D_i R_i(\bar{\boldsymbol{\omega}},\kappa^1)||r_i^1-r_i^2| + \sum_{\substack{j=i-N\\ j\neq i}}^{i+N} c|D_j R_i(\bar{\boldsymbol{\omega}},\kappa^1)||r_j^1-r_j^2| + O\big(1/2^N\big).
\end{aligned} \tag{5.59}
$$

Due to the scalings (5.18) and (5.21) we find from Corollary 5.3.2 that $|D_j R_i(\bar{\boldsymbol{\omega}},\kappa^1)| = O(\lambda^\delta)$ for some $\delta > 1$ for all $j \in \mathbb{Z}$. Further $|D_j R_i(\bar{\boldsymbol{\omega}},\kappa^1)| \leq C \cdot 1/2^{|i-j|}$, for some $C > 0$, cf. Theorem 3.5.1. Since A_i is bounded we finally obtain from (5.58) and (5.59)

$$
|F_i(\boldsymbol{r}^1,\kappa^1) - F_i(\boldsymbol{r}^2,\kappa^1)| \leq c(\lambda)|r_i^1-r_i^2| + c(\lambda)\sum_{\substack{j=i-N\\ j\neq i}}^{i+N} \frac{1}{2^{|i-j|}}|r_j^1-r_j^2| + O(1/2^N),
$$

for some $c(\lambda) \in (0,1)$ that tends to zero as λ tends to zero. For that reason we explicitly indicate the λ-dependence of c. Together with (5.55) and (5.57) we obtain with

$$
|r_i^1-r_i^2| \leq \frac{c(\lambda)}{(1-c(\lambda))}\sum_{\substack{j=i-N\\ j\neq i}}^{i+N} \frac{1}{2^{|i-j|}}|r_j^1-r_j^2| + O\big(1/2^N\big).
$$

a representation that correspond to (5.56).

Now, let $\boldsymbol{i} \in \boldsymbol{J_\kappa}$, $\boldsymbol{i-1} \in \mathbb{Z}\setminus \boldsymbol{J_\kappa}$. Again we start with considering $|F_i(\mathbf{r}^2,\kappa^1) - F_i(\mathbf{r}^2,\kappa^2)|$.

$$
\begin{aligned}
&|F_i(\mathbf{r}^2,\kappa^1) - F_i(\mathbf{r}^2,\kappa^2)| \\
&= \tfrac{1}{2B}\Big|C_i\sqrt{\lambda}(r_{i-1}^2 - r_{i-1}^2) + \sqrt{C_i^2\lambda(r_{i-1}^2)^2 + 4B(1+\check{R}_i(\mathbf{r}^2,\kappa^1))} - \sqrt{C_i^2\lambda(r_{i-1}^2)^2 + 4B(1+\check{R}_i(\mathbf{r}^2,\kappa^2))}\Big| \\
&= \tfrac{1}{2B}\Big|\tfrac{C_i^2\lambda\left((r_{i-1}^2)^2-(r_{i-1}^2)^2\right)+4B\left(\check{R}_i(\mathbf{r}^2,\kappa^1)-\check{R}_i(\mathbf{r}^2,\kappa^2)\right)}{\sqrt{C_i^2\lambda(r_{i-1}^2)^2+4B(1+\check{R}_i(\mathbf{r}^2,\kappa^1))}+\sqrt{C_i^2\lambda(r_{i-1}^2)^2+4B(1+\check{R}_i(\mathbf{r}^2,\kappa^2))}}\Big| \\
&\leq \tfrac{1}{2B}\Big|\tfrac{4B\left(\check{R}_i(\mathbf{r}^2,\kappa^1)-\check{R}_i(\mathbf{r}^2,\kappa^2)\right)}{4\sqrt{B}}\Big| \\
&\leq \tfrac{1}{2\sqrt{B}}\big|\check{R}_i(\mathbf{r}^2,\kappa^1) - \check{R}_i(\mathbf{r}^2,\kappa^2)\big| \\
&= \tfrac{1}{2\lambda\sqrt{B}}\big|R_i(\boldsymbol{\omega}(\mathbf{r}^2),\kappa^1) - R_i(\boldsymbol{\omega}(\mathbf{r}^2),\kappa^2)\big| \\
&= \tfrac{1}{2\lambda\sqrt{B}}\big| - B(r_i^2)^2 - C_i(\kappa^1)r_i^2r_{i-1}^2 + R_i(\boldsymbol{\omega}(\mathbf{r}^2),\kappa^1) - (-B(r_i^2)^2 - C_i(\kappa^2)r_i^2r_{i-1}^2 + R_i(\boldsymbol{\omega}(\mathbf{r}^2),\kappa^2))\big| \\
&= \tfrac{1}{2\lambda\sqrt{B}}\big|\xi_i(\boldsymbol{\omega}(\kappa^2),\kappa^1) - \xi_i(\boldsymbol{\omega}(\kappa^2),\kappa^2)\big|
\end{aligned}
$$

As in the previous case we find due to Corollary 5.4.4 that $\big|\xi_i(\boldsymbol{\omega}(\kappa^2),\kappa^1)-\xi_i(\boldsymbol{\omega}(\kappa^2),\kappa^2)\big| = O\big({}^1\!/_{2^N}\big)$. Hence

$$|F_i(\mathbf{r}^2,\kappa^1)-F_i(\mathbf{r}^2,\kappa^2)| = O\big({}^1\!/_{2^N}\big).$$

Now we turn towards the second term:

$$\begin{aligned}
&|F_i(\mathbf{r}^1,\kappa^1)-F_i(\mathbf{r}^2,\kappa^1)|\\
&\quad= \frac{1}{2B}\Big|C_i\sqrt{\lambda}\big(r^2_{i-1}-r^1_{i-1}\big)+\sqrt{C_i^2\lambda(r^1_{i-1})^2+4B(1+\check{R}_i(\mathbf{r}^1,\kappa^1))}-\sqrt{C_i^2\lambda(r^2_{i-1})^2+4B(1+\check{R}_i(\mathbf{r}^2,\kappa^1))}\Big|\\
&\quad= \frac{1}{2B}\Big|C_i\sqrt{\lambda}\big(r^2_{i-1}-r^1_{i-1}\big)+\frac{C_i^2\lambda\big((r^1_{i-1})^2-(r^2_{i-1})^2\big)+4B\big(\check{R}_i(\mathbf{r}^1,\kappa^1)-\check{R}_i(\mathbf{r}^2,\kappa^1)\big)}{\sqrt{C_i^2\lambda(r^1_{i-1})^2+4B(1+\check{R}_i(\mathbf{r}^1,\kappa^1))}+\sqrt{C_i^2\lambda(r^2_{i-1})^2+4B(1+\check{R}_i(\mathbf{r}^2,\kappa^1))}}\Big|\\
&\quad\le \frac{1}{2B}\Big|-C_i\sqrt{\lambda}\big(r^1_{i-1}-r^2_{i-1}\big)+\frac{C_i^2\lambda\big((r^1_{i-1})^2-(r^2_{i-1})^2\big)}{|C_i|\sqrt{\lambda}\big(r^1_{i-1}+r^2_{i-1}\big)}+\sqrt{B}\big(\check{R}_i(\mathbf{r}^1,\kappa^1)-\check{R}_i(\mathbf{r}^2,\kappa^1)\big)\Big|\\
&\quad\le \frac{|C_i|}{B}\sqrt{\lambda}\big|r^1_{i-1}-r^2_{i-1}\big|+\frac{1}{2\sqrt{B}}\big|\check{R}_i(\mathbf{r}^1,\kappa^1)-\check{R}_i(\mathbf{r}^2,\kappa^1)\big|. \qquad (5.60)
\end{aligned}$$

The latter term in (5.60) can be estimated, as in the case above, by using the mean-value-theorem and $D_{(\pm N+i)}R_i(\boldsymbol{\omega},\kappa) = O\big({}^1\!/_{2^N}\big)$:

$$\begin{aligned}
\tfrac{1}{2\sqrt{B}}\big|\check{R}_i(\mathbf{r}^1,\kappa^1)-\check{R}_i(\mathbf{r}^2,\kappa^1)\big| &= \tfrac{1}{2\sqrt{B}|\lambda|}|R_i(\boldsymbol{\omega}(\hat{\mathbf{r}}^1,\kappa^1))-R_i(\boldsymbol{\omega}(\hat{\mathbf{r}}^2,\kappa^1))|\\
&\le \tfrac{1}{2\sqrt{B}|\lambda|}\sum_{j\in\mathbb{Z}}|D_jR_i(\bar{\boldsymbol{\omega}},\lambda,\kappa^1)|\,\big|\omega_j(\hat{r}^1_j)-\omega_j(\hat{r}^2_j)\big|\\
&\le c(\lambda)|r^1_i-r^2_i|+c(\lambda)\sum_{\substack{j=i-N\\ j\neq i}}^{i+N}\tfrac{1}{2^{|i-j|}}|r^1_j-r^2_j|+O\big({}^1\!/_{2^N}\big).
\end{aligned}$$

In this regard, recall that also for $i\in J_\kappa$, due to the scalings (5.18) and (5.21), $|D_jR_i(\bar{\boldsymbol{\omega}},\lambda,\kappa)| = O(\lambda^\delta)$ for some $\delta>1$, cf. Corollary 5.3.2. Again we want to note that $c(\lambda)$ tends to zero as λ tends to zero. Putting things together and inserting in (5.55) yields with the shortened notation $\Delta r_i := |r^1_i-r^2_i|$:

$$\Delta r_i \le \frac{|C_i|}{B(1-c(\lambda))}\lambda^{1/2}\Delta r_{i-1}+\frac{c(\lambda)}{1-c(\lambda)}\sum_{\substack{j=i-N\\ j\neq i}}^{i+N}\frac{1}{2^{|i-j|}}\Delta r_j+O\big({}^1\!/_{2^N}\big). \qquad (5.61)$$

In the following we include the first term in the sum and obtain

$$\Delta r_i \le \frac{c(\lambda)+2\sqrt{\lambda}|C_i|/B}{1-c(\lambda)}\sum_{\substack{j=i-N\\ j\neq i}}^{i+N}\frac{1}{2^{|i-j|}}\Delta r_j+O\big({}^1\!/_{2^N}\big), \qquad (5.62)$$

which again conforms to (5.56).

Finally, let $\boldsymbol{i\in J_\kappa}$, $\boldsymbol{i-1\in J_\kappa}$. Proceeding as in the previous case we get, instead of (5.61),

$$\Delta r_i \le \frac{|C_i|}{B}\Delta r_{i-1}+c(\lambda)\sum_{j=i-N}^{i+N}\frac{1}{2^{|i-j|}}\Delta r_j+O\big({}^1\!/_{2^N}\big). \qquad (5.63)$$

More generally, if also $i-k \in J_\kappa$, $k=0,\dots,n$ for some $n \in \{1,\dots,N\}$, we get in the same way

$$\Delta r_{i-k} \leq \frac{|C_{i-k}|}{B}\Delta r_{i-k-1} + c(\lambda)\sum_{j=i-N}^{i+N-2k}\frac{1}{2^{|i-k-j|}}\Delta r_j + O\big(1/2^{N-k}\big). \tag{5.64}$$

By replacing Δr_{i-1} in the first summand in (5.63) by (5.64) for $k=1$ we find with $q=q(\lambda):=\max\{\frac{|C_i(\lambda)|}{B(\lambda)},\frac{1}{2}\}$

$$\begin{aligned}\Delta r_i &\leq q\left(q\Delta r_{i-2} + c(\lambda)\sum_{j=i-N}^{i+N-2} q^{|i-1-j|}\Delta r_j + O(q^{N-1})\right) + c(\lambda)\sum_{j=i-N}^{i+N} q^{|i-j|}\Delta r_j + O(q^N)\\ &\leq q^2\Delta r_{i-2} + c(\lambda)\left(\sum_{j=i-N}^{i+N} q^{|i-1-j|+1}\Delta r_j + \sum_{j=i-N}^{i+N} q^{|i-j|}\Delta r_j\right) + 2O(q^N).\end{aligned}$$

Recall that we assume $|C_i(0)|/B(0)<1$ and hence also $|C_i(\lambda)|/B(\lambda)<1$ for sufficiently small λ.

For $j \leq i-1$ we find that $q^{|i-1-j|+1} = q^{i-j}$ and for $j \geq i$ we have $q^{|i-1-j|+1} = q^{j-1+2}$. Therefore we split both sums between $j=i-1$ and $j=i$ and obtain

$$\Delta r_i \leq q^2\Delta r_{i-2} + c(\lambda)\left(\sum_{j=i-N}^{i-1} 2q^{i-j}\Delta r_j + \sum_{j=i}^{i+N}(1+q^2)q^{j-i}\Delta r_j\right) + 2O(q^N). \tag{5.65}$$

Continuing along the same proceeding we find for $i-k \in J_\kappa$, $k=0,\dots,n$

$$\left.\begin{aligned}\Delta r_i \quad\leq\quad & q^n\Delta r_{i-n} + c(\lambda)\left(\sum_{j=i-N}^{i-(n-1)} nq^{i-j}\Delta r_j + \sum_{j=i-(n-2)}^{i-1}\tfrac{1-q^{2(n-(i-j))}}{1-q^2}(i-j+1)q^{i-j}\Delta r_j\right.\\ &\left.+\tfrac{1-q^{2n}}{1-q^2}\sum_{j=i}^{i+N} q^{|i-j|}\Delta r_j\right) + nO\big(q^N\big).\end{aligned}\right\} \tag{5.66}$$

This can be seen by induction for the counter n with (5.65) as base case for $n=2$. In the induction step from n to $n+1$ we replace the Δr_{i-n} in the first summand in (5.66) by (5.64) with $k=n$. To this end we extend the sum in (5.64) to $i+N$ and split in into three parts:

$$\left.\begin{aligned}\Delta r_{i-n} \quad\leq\quad & q\Delta r_{i-n-1} + c(\lambda)\left(\sum_{j=i-N}^{i-n} q^{|i-n-j|}\Delta r_j + \sum_{j=i-n+1}^{i-1} q^{|i-n-j|}\Delta r_j\right.\\ &\left.+\sum_{j=i}^{i+N} q^{|i-n-j|}\Delta r_j\right) + O(q^{N-n}).\end{aligned}\right\} \tag{5.67}$$

Then the combination of the first sum in (5.67) together with the first sum in (5.66) yields

$$q^n\sum_{j=i-N}^{i-n} q^{|i-n-j|}\Delta r_j + \sum_{j=i-N}^{i-(n-1)} nq^{i-j}\Delta r_j = \sum_{j=i-N}^{i-n}(n+1)q^{i-j}\Delta r_j + nq^{n-1}\Delta r_{i-(n-1)}.$$

The last term on the right-hand-side of this equation we add to the second sums in (5.67) and (5.66):

$$\begin{aligned}
&q^n \sum_{j=i-n+1}^{i-1} q^{|i-n-j|}\Delta r_j + \sum_{j=i-(n-2)}^{i-1} \tfrac{1-q^{2(n-(i-j))}}{1-q^2}(i-j+1)q^{i-j}\Delta r_j + nq^{n-1}\Delta r_{i-(n-1)} \\
&= (q^{n+1}+nq^{n-1})\Delta r_{i-(n-1)} + \sum_{j=i-(n-2)}^{i-1} q^n q^{j-i+n}\Delta r_j + \sum_{j=i-(n-2)}^{i-1} \tfrac{1-q^{2(n-(i-j))}}{1-q^2}(i-j+1)q^{i-j}\Delta r_j \\
&= (q^{n+1}+nq^{n-1})\Delta r_{i-(n-1)} + \sum_{j=i-(n-2)}^{i-1} \big(q^{2(n+j-i)} + \tfrac{1-q^{2(n-(i-j))}}{1-q^2}(i-j+1)\big)q^{i-j}\Delta r_j \\
&\le nq^{n-1}(q^2+1)\Delta r_{i-(n-1)} + \sum_{j=i-(n-2)}^{i-1} (i-j+1)\big(q^{2(n+j-i)} + \tfrac{1-q^{2(n-(i-j))}}{1-q^2}\big)q^{i-j}\Delta r_j \\
&= (n-1+1)\tfrac{1-q^{2(n+1-(n-1))}}{1-q^2}q^{n-1}\Delta r_{i-(n-1)} + \sum_{j=i-(n-2)}^{i-1} (i-j+1)\tfrac{1-q^{2(n+1-(i-j))}}{1-q^2}q^{i-j}\Delta r_j \\
&= \sum_{j=i-(n-1)}^{i-1} (i-j+1)\tfrac{1-q^{2(n+1-(i-j))}}{1-q^2}q^{i-j}\Delta r_j .
\end{aligned}$$

Finally we add the third sums in (5.67) and (5.66) and obtain

$$q^n \sum_{j=i}^{i+N} q^{|i-n-j|}\Delta r_j + \frac{1-q^{2n}}{1-q^2}\sum_{j=i}^{i+N} q^{|i-j|}\Delta r_j = \big(q^{2n} + \frac{1-q^{2n}}{1-q^2}\big)\sum_{j=i}^{i+N} q^{|i-j|}\Delta r_j = \frac{1-q^{2(n+1)}}{1-q^2}\sum_{j=i}^{i+N} q^{|i-j|}\Delta r_j$$

With $q^n(q\Delta r_{i-(n+1)} + O(q^{N-n})) = q^{n+1}\Delta r_{i-(n+1)} + O(q^N)$ and the last three estimates the relation (5.66) is proven.

Now, since $q<1$ we have $1-q^{2n} < 1-q^{2(n-(i-j))} < 1$ for $j<i$. Hence from (5.66) we can conclude for all $k=0,\dots,n$ with $i-k \in J_\kappa$, by additionally combining the first two sums

$$\begin{aligned}
\Delta r_i \;&\le\; q^k\Delta r_{i-k} + \tfrac{c(\lambda)}{1-q^2}\left(\sum_{j=i-N}^{i-(k-1)} kq^{i-j}\Delta r_j + \sum_{j=i-(k-2)}^{i-1}(i-j+1)q^{i-j}\Delta r_j + \sum_{j=i}^{i+N} q^{|i-j|}\Delta r_j\right) \\
&\quad + kO\big(q^N\big) \\
&\le\; q^k\Delta r_{i-k} + \tfrac{c(\lambda)}{1-q^2}\left(\sum_{j=i-N}^{i-1}(i-j+1)q^{i-j}\Delta r_j + \sum_{j=i}^{i+N} q^{|i-j|}\Delta r_j\right) + kO\big(q^N\big)
\end{aligned}$$

Recall that for λ tending to zero $c(\lambda)$ tends to zero. Hence for λ sufficiently small we can separate $\frac{c(\lambda)}{1-q^2}\Delta r_i$ from the right-hand-side of the relation, $\frac{c(\lambda)}{1-q^2} < 1$, shift it to the left-hand-side and divide the inequality by $1-\frac{c(\lambda)}{1-q^2}$ leading to

$$\left.\begin{aligned}
\Delta r_i \;&\le\; \tfrac{1-q^2}{1-q^2-c(\lambda)}q^k\Delta r_{i-k} + \tfrac{c(\lambda)}{1-q^2-c(\lambda)}\left(\sum_{j=i-N}^{i-1}(i-j+1)q^{i-j}\Delta r_j + \sum_{j=i+1}^{i+N} q^{|i-j|}\Delta r_j\right) \\
&\quad + kO\big(q^N\big).
\end{aligned}\right\} \tag{5.68}$$

Now there are two alternatives. Either there is an $n\in\mathbb{N}$ such that $i-k\in J_\kappa$, $k=0,\dots,n$ and $i-n-1\in\mathbb{Z}\setminus J_\kappa$, or $i-k\in J_\kappa$ for all $k\in\mathbb{N}$.

According to (5.61) we find in the first case, $i-k \in J_\kappa$, $k=0,\dots,n$ and $i-n-1 \in \mathbb{Z}\setminus J_\kappa$,

$$\begin{aligned}\Delta r_i &\leq \tfrac{1-q^2}{1-q^2-c(\lambda)}\tfrac{|C_{i-n-1}|}{B}\sqrt{\lambda}q^{n-1}\Delta r_{i-n} + \tfrac{c(\lambda)}{1-q^2-c(\lambda)}\left(\sum_{j=i-N}^{i-1}(i-j+1)q^{i-j}\Delta r_j + \sum_{j=i+1}^{i+N} q^{|i-j|}\Delta r_j\right)\\ &\quad + nO\big(q^N\big)\\ &\leq \frac{c(\lambda)+\frac{(1-q^2)|C_{i-n-1}|}{qB(n+1)}\sqrt{\lambda}}{1-q^2-c(\lambda)}\left(\sum_{j=i-N}^{i-1}(i-j+1)q^{i-j}\Delta r_j + \sum_{j=i+1}^{i+N} q^{|i-j|}\Delta r_j\right) + nO\big(q^N\big).\end{aligned}$$

With $q<1$ we have $q<\sqrt{q}<1$ and thus $(i-j+1)q^{(i-j)/2}$ tends to zero for $i-j$ tending to infinity. So by using $(i-j+1)q^{i-j} \leq Kq^{(i-j)/2}$, for some constant K we finally obtain the estimate

$$\Delta r_i \leq K_1(\lambda)\sum_{\substack{j=i-N\\ j\neq i}}^{i+N} q^{|i-j|/2}\Delta r_j + nO\big(q^N\big),$$

where $K_1(\lambda)$ tends to zero for λ tending to zero. This estimate conforms to (5.56).

For the second alternative, $i-k \in J_\kappa$ for all $k\in\mathbb{N}$ we find from (5.68) with the same argumentation as above

$$\begin{aligned}\Delta r_i &\leq \tfrac{1-q^2}{1-q^2-c(\lambda)}q^N\Delta r_{i-N} + \tfrac{c(\lambda)}{1-q^2-c(\lambda)}\left(\sum_{j=i-N}^{i-1}(i-j+1)q^{i-j}\Delta r_j + \sum_{j=i+1}^{i+N} q^{|i-j|}\Delta r_j\right)\\ &\quad + NO\big(q^N\big)\\ &\leq O(q^N) + K_2(\lambda)\sum_{\substack{j=i-N\\ j\neq i}}^{i+N} q^{|i-j|/2}\Delta r_j + O\big(q^{N/2}\big),\end{aligned}$$

with $K_2(\lambda)$ tending to zero for λ tending to zero. Note in this respect that Δr_{i-N} is bounded, cf. Section 5.4.1. So finally also in the last case an estimate corresponding to (5.56) was verified. This concludes the proof. □

Lemma 5.4.6. *The conjugation Φ_λ is a homeomorphism.*

Proof. According to [Dug66, Chap. XI, Thm. 2.1] it is enough to show that Φ_λ defined in (5.48) is continuous. Since Σ_{A^+} is compact and Φ_λ is bijective, it then follows that the inverse of Φ_λ is continuous as well.

Since $x(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)$ are trajectories, we have

$$\Phi_\lambda(\kappa) = x(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(0) = x_0^+(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(0) = \gamma_{\kappa_0}^+(0) + v_0^+(\boldsymbol{\omega}(\lambda,\kappa),\lambda,\kappa)(0).$$

Now, let $\kappa^1,\kappa^2 \in \Sigma_{A_+}$ be two sequences which coincide on a block of length $2N+1$ centered at $i=0$:

$$\kappa_j^1 = \kappa_j^2, \quad j\in[-N,N]\cap\mathbb{Z}.$$

Then, because $\gamma^+_{\kappa_0^1}(0) = \gamma^+_{\kappa_0^2}(0)$,

$$\begin{aligned}\|\Phi_\lambda(\kappa^1) - \Phi_\lambda(\kappa^2)\| \quad \leq \quad & \|v_0^+(\boldsymbol{\omega}(\lambda,\kappa^1),\lambda,\kappa^1)(0) - v_0^+(\boldsymbol{\omega}(\lambda,\kappa^2),\lambda,\kappa^1)(0)\| \\ & +\|v_0^+(\boldsymbol{\omega}(\lambda,\kappa^2),\lambda,\kappa^1)(0) - v_0^+(\boldsymbol{\omega}(\lambda,\kappa^2),\lambda,\kappa^2)(0)\|.\end{aligned}$$

First we consider the first term on the right-hand side. We apply Lemma 5.4.2, which implies that the mapping $(l^\infty, \hat{\varrho}) \to \mathbb{R}^4$, $\boldsymbol{\omega} \mapsto v_0^+(\boldsymbol{\omega},\lambda,\kappa)(0)$ is continuous. Together with Lemma 5.4.5 this yields that this term tends to zero as N tends to infinity.

Further, on the second term on the right-hand side term we can apply Lemma 5.4.3.

Altogether this shows that $\|\Phi_\lambda(\kappa^1) - \Phi_\lambda(\kappa^2)\|$ tends to zero as N tends to infinity. □

5.5 Proof of Theorem 5.3.4

Again we prove the theorem under the assumption that $B(\lambda) > 0$. Further we assume that the fibre bundle $\mathcal{F}(W^s_\gamma)$ has the topological structure of an annulus. Recall that we are only concerned with periodic solutions. Just as in the proof of Theorem 5.3.3 we make use of the implicit function theorem.

5.5.1 Solution for fixed κ and $\lambda > 0$

To begin with we follow the same approach as in the proof of Theorem 5.3.3 that results in the equation (5.25). Again we use the short-hand notations $B := B(0)$, $C_i := C_i(0)$, $C = |C_i|$ and $A_i := A_i(0)$.

A starting solution for $\lambda = 0$

Scanning the single steps of the proof of Theorem 5.3.3 we see that we often make use of the fact that the quotient $\frac{C}{B}$ is smaller than one. When generating a starting solution $\boldsymbol{r}^\kappa$ for $\lambda = 0$ it provides us the bounded areas (5.29) the r_i^κ live in and the contraction of the fixed point equation (5.28). In case that $\frac{C}{B} \geq 1$ we find for $C_i = -C$ with

$$F_i(\mathbf{r},\kappa) = \frac{1}{2B}\left(Cr_{i-1} + \sqrt{C^2 r_{i-1}^2 + 4B}\right) \geq \frac{C}{B} r_{i-1} \tag{5.69}$$

that F_i might rise above any real boundary. Further, the estimate $|F_i(\mathbf{r},\kappa) - F_i(\mathbf{s},\kappa)| \leq \frac{C}{B}|r_{i-1} - s_{i-1}|$ for $i-1, i \in J_\kappa$ provides us no longer the contractivity of $\mathbf{F}$. So we need to find a different way of generating a starting solution $\mathbf{r}^\kappa$ for $\lambda = 0$.

Anyway we start from the fixed point equation (5.28). Since we are restricted to periodic solutions, the index set under consideration is finite. Let N denote the length of the period in κ. Then we denote the index set by $\mathbb{Z}_N$.

Here we define $U = \bigtimes_{i \in \mathbb{Z}_N} U_i$, $U_i \subset \mathbb{R}$ by

$$U_i = \begin{cases} \left\{\frac{1}{A_i}\right\}, & i \in \mathbb{Z}_N \setminus J_\kappa, \\ [0,\infty), & i \in J_\kappa, \qquad i-1 \in J_\kappa, \\ \left\{\frac{1}{\sqrt{B}}\right\}, & i \in J_\kappa, \qquad i-1 \in \mathbb{Z}_N \setminus J_\kappa. \end{cases} \tag{5.70}$$

First we consider the case where there is at least one index $i_0 \in \mathbb{Z}_N \setminus J_\kappa$. Then we trivially get a solution of (5.28) for all indices $i \in \mathbb{Z}_N \setminus J_\kappa$ and $i \in J_\kappa$, $i-1 \in \mathbb{Z}_N \setminus J_\kappa$, if $A_i > 0$. In case that $i, i-1 \in J_\kappa$ we obtain a solution of (5.28) by successive computation from its predecessors.

It remains to consider the case $i \in J_\kappa$ for all $i \in \mathbb{Z}_N$. Here the fixed point equation (5.28) reduces to

$$F_i(\mathbf{r}, \kappa) = \frac{1}{2B}\left(-C_i r_{i-1} + \sqrt{C_i^2 r_{i-1}^2 + 4B}\right) \tag{5.71}$$

for all $i \in \mathbb{Z}_N$.

Now we make use of the special structure of this fixed point problem where the k-th equation of F only depends on r_{k-1} and rewrite the fixed point equation for periodic orbits by repeated invoking of F into the equivalent fixed point problem $\mathbf{r} = \boldsymbol{\mathcal{F}}(\mathbf{r}, \kappa)$, $\boldsymbol{\mathcal{F}} = (\mathcal{F}_i)_{i \in \mathbb{Z}_N}$ with

$$\mathcal{F}_i(\mathbf{r}) := (F_i \circ F_{i-1} \circ \ldots \circ F_{i-N+1})(r_i), \qquad \forall i \in \mathbb{Z}_N. \tag{5.72}$$

Indeed it suffices to solve this equation for one index $i \in \mathbb{Z}_N$. The solution for the remaining indices we obtain by successively plugging in the solution into (5.71).

First we consider the special case where $C_i = -C$ for all $i \in \mathbb{Z}_N$. Recall from Remark 5.2.5 and Corollary 5.2.4 that this condition only applies for $\kappa \in \mathcal{K}_2$, cf. (5.9), since we assume the topological structure of the $\mathcal{F}(W^s_\gamma)$ to be an annulus. In this case we find that equation (5.72) has no solution (for $\lambda > 0$), since estimate (5.69) leads to $r_i^\kappa \geq \left(\frac{C}{B}\right)^N r_i^\kappa$. This is a contradiction for $C > B$. In case that $C = B$ the r_i^κ have to be equal and we are looking for a solution of the equation $r_i = F_i(r_i) = \frac{1}{2B}(Br_i + \sqrt{B^2 r_i^2 + 4B})$ which is equivalent to $r_i = \sqrt{r_i^2 + 4/B}$, also a contradiction. So the trajectories corresponding to $\kappa \in \mathcal{K}_2$ do not exist for $\lambda > 0$. Otherwise we would find a starting solution.

Henceforth consider trajectories with at least one index $j \in \mathbb{Z}_N$ such that $C_j = C$. Now we need to determine each domain D_i such that $\mathcal{F}_i(D_i) \subseteq D_i$. Then the fixed point equation (5.72) is due to the chain rule contractive if

$$\frac{d\mathcal{F}_i}{dr_i}(\zeta_i) = \prod_{k=1}^{N} \frac{dF_k}{dr_{k-1}}(\zeta_{k-1}) < 1 \tag{5.73}$$

for any $\boldsymbol{\zeta} = (\zeta_i)_{i \in \mathbb{Z}_N}$ with $\zeta_i \in D_i$ for all $i \in \mathbb{Z}_N$.

To this end we introduce further index sets. By

$$\left.\begin{array}{rcl} J_\kappa^- & := & \{j \in J_\kappa \mid C_j < 0\} \\ J_\kappa^+ & := & \{j \in J_\kappa \mid C_j > 0\} \end{array}\right\} \tag{5.74}$$

we denote the subsets of J_κ that are determined by either a negative or a positive C_j. Further we define the sequences

$$\left.\begin{array}{rcl} S_\kappa^-(i) & := & \{j \in J_\kappa^- \mid \exists k = k(i) \in \mathbb{N} : i \leq j < i + k(i) \;\; \text{where } i-1, i+k(i) \in J_\kappa^+\} \\ S_\kappa^+(i) & := & \{j \in J_\kappa^+ \mid \exists k = k(i) \in \mathbb{N} : i \leq j < i + k(i) \;\; \text{where } i-1, i+k(i) \in J_\kappa^-\} \end{array}\right\} \tag{5.75}$$

of consecutive indices $j \in J_\kappa^\pm$ with $j \geq i$. The length of such a sequence $S_\kappa^\pm(i)$ is then given by the corresponding $k(i)$.

Then we define the domains $D_j \subset \mathbb{R}$, $j \in \mathbb{Z}_N$ by

$$D_j := \begin{cases} \left\{\frac{1}{\sqrt{B+C}}\right\}, & j \in J_\kappa^+ \ \forall j \in \mathbb{Z}_N \\ \left[0, \frac{1}{\sqrt{B+C}}\right], & \exists i \in J_\kappa^- : \ j \in S_\kappa^+(i+1), \ |i+1-j| \in 2\mathbb{Z} \\ \left[\frac{1}{\sqrt{B+C}}, \frac{1}{\sqrt{B}}\right], & \exists i \in J_\kappa^+ : \ j \in S_\kappa^+(i), \ |i-j| \in 2\mathbb{Z}+1 \\ \left[\frac{1}{\sqrt{B}}\left(\frac{C}{B}\right)^{j-i}, \ \frac{1}{\sqrt{B}}\sum_{l=0}^{j-i+1}\left(\frac{C}{B}\right)^{l}\right], & \exists i \in J_\kappa^- : \ j \in S_\kappa^-(i), \ D_{i-1} = \left[0, \frac{1}{\sqrt{B+C}}\right] \\ \left[\frac{1}{\sqrt{B+C}}\left(\frac{C}{B}\right)^{j-i+1}, \ \frac{1}{\sqrt{B}}\sum_{l=0}^{j-i+1}\left(\frac{C}{B}\right)^{l}\right], & \exists i \in J_\kappa^- : \ j \in S_\kappa^-(i), \ D_{i-1} = \left[\frac{1}{\sqrt{B+C}}, \frac{1}{\sqrt{B}}\right] \end{cases} \tag{5.76}$$

In the following we verify that $\mathcal{F}_j(D_j, \kappa) \subseteq D_j$.

1.) In the case where $j \in J_\kappa^+$ we find due to

$$F_j(\mathbf{r}, \kappa) = \frac{1}{2B}\left(-Cr_{j-1} + \sqrt{C^2 r_{j-1}^2 + 4B}\right) \leq \frac{1}{\sqrt{B}}$$

that $F_j \subset \left[0, \frac{1}{\sqrt{B}}\right]$. Actually the F_j lie alternately above or below the value $\frac{1}{\sqrt{B+C}}$, which we show as follows:

$$F_j(\mathbf{r}, \kappa) = \frac{1}{2B}\left(-Cr_{j-1} + \sqrt{C^2 r_{j-1}^2 + 4B}\right) = \frac{-C^2 r_{j-1}^2 + C^2 r_{j-1}^2 + 4B}{2B\left(Cr_{j-1} + \sqrt{C^2 r_{j-1}^2 + 4B}\right)} = \frac{2}{Cr_{j-1} + \sqrt{C^2 r_{j-1}^2 + 4B}}.$$

Now, if $r_{j-1} > \frac{1}{\sqrt{B+C}}$ then we find

$$F_j(\mathbf{r}, \kappa) = \frac{2}{Cr_{j-1} + \sqrt{C^2 r_{j-1}^2 + 4B}} < \frac{2\sqrt{B+C}}{C + \sqrt{C^2 + 4CB + 4B^2}} = \frac{2\sqrt{B+C}}{C + C + 2B} = \frac{1}{\sqrt{B+C}},$$

whereas for $r_{j-1} < \frac{1}{\sqrt{B+C}}$ we obtain

$$F_j(\mathbf{r}, \kappa) = \frac{2}{Cr_{j-1} + \sqrt{C^2 r_{j-1}^2 + 4B}} > \frac{2\sqrt{B+C}}{C + \sqrt{C^2 + 4CB + 4B^2}} = \frac{1}{\sqrt{B+C}}.$$

Especially we find $F_j\left(\frac{1}{\sqrt{B+C}}\right) = \frac{1}{\sqrt{B+C}}$ for $j \in J_\kappa^+$.

Summarizing it yields for $j \in J_\kappa^+$

$$F_j(\mathbf{r}, \kappa) \in \begin{cases} \left[0, \frac{1}{\sqrt{B+C}}\right], & \text{if } r_{j-1} > \frac{1}{\sqrt{B+C}}, \\ \left[\frac{1}{\sqrt{B+C}}, \frac{1}{\sqrt{B}}\right], & \text{if } r_{j-1} < \frac{1}{\sqrt{B+C}}, \\ \left\{\frac{1}{\sqrt{B+C}}\right\}, & \text{if } r_{j-1} = \frac{1}{\sqrt{B+C}}. \end{cases} \tag{5.77}$$

2.) Now let $j \in J_\kappa^-$. Then we find

$$F_j(\mathbf{r}, \kappa) = \frac{1}{2B}\left(Cr_{j-1} + \sqrt{C^2 r_{j-1}^2 + 4B}\right) \geq \frac{C}{2B} r_{j-1} + \frac{1}{\sqrt{B}} > \frac{1}{\sqrt{B}}. \tag{5.78}$$

Indeed both lower estimates (5.69) and (5.78) apply.

So, let $j \in J_\kappa^-$ be within a sequence $S_\kappa^-(i)$ for some index $i \in J_\kappa^-$. Applying estimate (5.69) $(j-i)$ times then results in $F_j(\mathbf{r}) \geq (C/B)^{j-i} r_i = (C/B)^{j-i} F_i(r_{i-1})$. Since $i-1 \in J_\kappa^+$ we have due to (5.77) either $r_{i-1} = F_{i-1}(\mathbf{r}) \in \left[0, \frac{1}{\sqrt{B+C}}\right]$ or $r_{i-1} \in \left[\frac{1}{\sqrt{B+C}}, \frac{1}{\sqrt{B}}\right]$.

For $r_{i-1} \in \left[0, \frac{1}{\sqrt{B+C}}\right]$ we find by once applying estimate (5.78)

$$F_j(\mathbf{r}, \kappa) \geq \frac{1}{\sqrt{B}} \left(\frac{C}{B}\right)^{j-i},$$

and for $r_{i-1} \in \left[\frac{1}{\sqrt{B+C}}, \frac{1}{\sqrt{B}}\right]$ we obtain by once more applying (5.69) and $r_{i-1} \geq \frac{1}{\sqrt{B+C}}$

$$F_j(\mathbf{r}, \kappa) \geq \frac{1}{\sqrt{B+C}} \left(\frac{C}{B}\right)^{j-i+1}.$$

An upper bound we find with

$$F_j(\mathbf{r}, \kappa) = \frac{1}{2B}\left(C r_{j-1} + \sqrt{C^2 r_{j-1}^2 + 4B}\right) \leq \frac{C}{B} r_{j-1} + \frac{1}{\sqrt{B}}$$

and by applying this $(j-i+1)$ times we obtain with $r_{i-1} < 1/\sqrt{B}$

$$F_j(\mathbf{r}, \kappa) \leq \frac{1}{\sqrt{B}} \sum_{l=0}^{j-i+1} \left(\frac{C}{B}\right)^l.$$

Summarizing we obtain for $j \in J_\kappa^-$

$$F_j(\mathbf{r}, \kappa) \in \begin{cases} \left[\frac{1}{\sqrt{B}}\left(\frac{C}{B}\right)^{j-i}, \frac{1}{\sqrt{B}} \sum\limits_{l=0}^{j-i+1} \left(\frac{C}{B}\right)^l\right], & \exists i \in J_\kappa^- : j \in S_\kappa^-(i),\ r_{i-1} \in \left[0, \frac{1}{\sqrt{B+C}}\right], \\ \left[\frac{1}{\sqrt{B+C}}\left(\frac{C}{B}\right)^{j-i+1}, \frac{1}{\sqrt{B}} \sum\limits_{l=0}^{j-i+1} \left(\frac{C}{B}\right)^l\right], & \exists i \in J_\kappa^- : j \in S_\kappa^-(i),\ r_{i-1} \in \left[\frac{1}{\sqrt{B+C}}, \frac{1}{\sqrt{B}}\right] \end{cases} \tag{5.79}$$

Especially this yields that $F_{i+k(i)}(\mathbf{r}, \kappa) \in \left[0, \frac{1}{\sqrt{B+C}}\right]$ for $i \in S_\kappa^-(i)$, since $i+k(i) \in J_\kappa^+$ and $i+k(i)-1 \in J_\kappa^-$ with $r_{i+k(i)-1} > 1/\sqrt{B+C}$. Hence (5.77) and (5.79) justify the definition of D_j in (5.76).

Now, the considerations above show, that $\mathcal{F}_j(D_j) \subseteq D_j$ if and only if $F_j(r_{j-1}) \in D_j$. To this end the range D_j has to be chosen accordingly.
If for all $i \in \mathbb{Z}_N$ it applies that $i \in J_\kappa^+$ then the choice of D_j is trivially $\{1/\sqrt{B+C}\}$. Otherwise there exist at least one $i \in \mathbb{Z}_N$ with $i \in J_\kappa^-$. If $j \in J_\kappa^+$ and the difference between j and the first index $l < j$ with $l \in J_\kappa^-$ is odd, then $D_j = [0, 1/\sqrt{B+C}]$. If $j \in J_\kappa^+$ and $j - l$ is even, l being the first index $l < j$ with $l \in J_\kappa^-$, then $D_j = [1/\sqrt{B+C}, 1/\sqrt{B}]$.
Finally, if $j \in J_\kappa^-$ the range D_j has to be chosen according to the range D_l of the first index $l < j$ with $l \in J_\kappa^+$. Note, that there has to be such an index $l \in J_\kappa^+$, since we already discussed the non-existence of a solution if $i \in J_\kappa^-$ for all $i \in \mathbb{Z}_N$. The range D_l for $l \in J_\kappa^+$ in turn can be determined as described above from the distance of l to the first index k with $k < l$ and $k \in J_\kappa^-$. Since we are discussing periodic solutions only it might happen that k is equal to $j - N = j$.
Choosing the range D_j according to these considerations ensure that $\mathcal{F}_j(D_j) \subseteq D_j$.

We continue with proving the contractivity of the fixed point equation (5.72) by applying (5.73). The derivative $\frac{dF_j}{dr_{j-1}}$ is given by

$$\frac{dF_j}{dr_{j-1}}(\zeta_{j-1}) = \frac{1}{2B}\left(-C_j + \frac{C_j^2\zeta_{j-1}}{\sqrt{C_j^2\zeta_{j-1}^2+4B}}\right).$$

For $j \in J_\kappa^-$ it is converging from below to $C/B > 1$ for $\zeta_{j-1} \to \infty$. Hence we see that there are single values in the product of the derivatives that will be greater than one. However, for $j \in J_\kappa^+$ we obtain with

$$\begin{aligned}
\left|\frac{dF_j}{dr_{j-1}}(\zeta_{j-1})\right| &= \frac{C}{2B}\left(1-\frac{C\zeta_{j-1}}{\sqrt{C^2\zeta_{j-1}^2+4B}}\right)\\
&= \frac{C}{2B}\frac{\sqrt{C^2\zeta_{j-1}^2+4B}-C\zeta_{j-1}}{\sqrt{C^2\zeta_{j-1}^2+4B}}\\
&= \frac{C}{2B}\frac{C^2\zeta_{j-1}^2+4B-C^2\zeta_{j-1}^2}{C^2\zeta_{j-1}^2+4B+C\zeta_{j-1}\sqrt{C^2\zeta_{j-1}^2+4B}}\\
&< \frac{C}{C^2\zeta_{j-1}^2+2B},
\end{aligned}$$

that the derivative tends to zero for $\zeta_{j-1} \to \infty$. So the product of the derivatives (5.73) has the possibility to become smaller than one.

In the following we show that the product of the derivatives (5.73) for each periodic trajectory with at least one $j \in J_\kappa^+$ is always smaller than one. To this end we use the gained estimates

$$\left|\frac{dF_j}{dr_{j-1}}(\zeta_{j-1})\right| < \begin{cases} \frac{C}{B}, & j \in J_\kappa^-,\\ \frac{C}{C^2\zeta_{j-1}^2+2B}, & j \in J_\kappa^+. \end{cases} \tag{5.80}$$

Further we need to split the periodic trajectory into segments characterised by positive or negative C_j in the following way.

1.) We start with considering the sequences $S_\kappa^-(i)$ of consecutive indices $j \in J_\kappa^-$, cf. (5.75). Then $i-1 \in J_\kappa^+$ and we first demand that $D_{i-1} = [1/\sqrt{B+C}, 1/\sqrt{B}]$, cf. (5.76). Note that in this case the sequence $S_\kappa^+(l)$, $l+k(l) = i$, of foregoing indices $j \in J_\kappa^+$ consists of an even number of elements:

$$\ldots, \underbrace{l, l+1, \ldots, l+k(l)-1 = i-1}_{\in S_\kappa^+(l)\subseteq J_\kappa^+}, \underbrace{i, i+1, \ldots, i+k(i)-1}_{\in S_\kappa^-(i)\subseteq J_\kappa^-}, \underbrace{i+k(i)}_{\in J_\kappa^+}, \ldots$$

Here we see to the product of $\frac{dF_j}{dr_{j-1}}$ for all indices $j \in S_\kappa^-(i)$ as well as the indices $i-1$ and $i+k(i) \in J_\kappa^+$. We make use of (5.80). In case of $i-1 \in J_\kappa^+$ we find $\zeta_{i-2} > 0$ and for $i+k(i) \in J_\kappa^+$ we have $\zeta_{i+k(i)-1} > 1/\sqrt{B+C}(C/B)^{k(i)}$, cf. (5.76). Then we find

$$\begin{aligned}
\prod_{j=i-1}^{i+k(i)} \frac{dF_j}{dr_{j-1}}(\zeta_{j-1}) &< \frac{C}{C^2\zeta_{i-2}^2+2B}\cdot\left(\frac{C}{B}\right)^{k(i)}\cdot\frac{C}{C^2\zeta_{i+k(i)-1}^2+2B}\\
&\le \frac{1}{2}\left(\frac{C}{B}\right)^{k(i)+1}\cdot\frac{C}{\frac{C^2\cdot C^{2k(i)}}{(B+C)\cdot B^{2k(i)}}+2B}\\
&\le \frac{C^{k(i)+2}}{\frac{2C^{2k(i)+2}}{2C\cdot B^{k(i)-1}}+4B^{k(i)+2}} = \frac{1}{\frac{C^{2k(i)+2}}{C^{k(i)+3}\cdot B^{k(i)-1}}+4\frac{B^{k(i)+2}}{C^{k(i)+2}}} = \frac{1}{\left(\frac{C}{B}\right)^{k(i)-1}+4\left(\frac{B}{C}\right)^{k(i)+2}} < 1
\end{aligned}$$

In the special case where $C = B$ we find the product to be smaller than $1/5$. If $C > B$ and $k(i) > 1$

we can simplify this estimate to $\left(\frac{B}{C}\right)^{k(i)-1}$, in case that $C > B$ and $k(i) = 1$ we find at least the upper bound $\frac{C^3}{C^3+4B^3}$.

2.) Next we see to the sequences $S_\kappa^-(i)$ where $D_{i-1} = [0, 1/\sqrt{B+C}]$. In this case the number of elements of the foregoing sequence $S_\kappa^+(l)$, $l + k(l) = i$, is odd. Here we build the product of $\frac{dF_j}{dr_{j-1}}$ only for the indices $j \in S_\kappa^-(i)$ as well as the index $i + k(i) \in J_\kappa^+$. This time we find for $i + k(i) \in J_\kappa^+$ that $\zeta_{i+k(i)-1} > 1/\sqrt{B}(C/B)^{k(i)-1}$, cf. (5.76). Then (5.80) leads to

$$\begin{aligned}
\prod_{j=i}^{i+k(i)} \frac{dF_j}{dr_{j-1}}(\zeta_{j-1}) <& \left(\frac{C}{B}\right)^{k(i)} \cdot \frac{C}{C^2\zeta_{i+k(i)-1}^2+2B} \\
\leq& \left(\frac{C}{B}\right)^{k(i)} \cdot \frac{C}{\frac{C^2C^{2k(i)-2}}{BB^{2k(i)-2}}+2B} \\
=& \frac{C^{k(i)+1}}{\frac{C^{2k(i)}}{B^{k(i)-1}}+2B^{k(i)+1}} = \frac{1}{\frac{C^{2k(i)}}{C^{k(i)+1}.B^{k(i)-1}}+2\frac{B^{k(i)+1}}{C^{k(i)+1}}} = \frac{1}{\left(\frac{C}{B}\right)^{k(i)-1}+2\left(\frac{B}{C}\right)^{k(i)+1}} < 1.
\end{aligned}$$

So in the case $C = B$ we get the upper bound $1/3$, if $C > B$ and $k(i) > 1$ we find again the simpler bound $\left(\frac{B}{C}\right)^{k(i)-1}$ and for $C > B$ and $k(i) = 1$ we get $\frac{C^2}{C^2+2B^2}$.

3.) Eventually there only remain the sequences $S_\kappa^+(i)$. However the first element i of each of these sequences is already considered within the product building above. Additionally the last element $i+k(i)-1$ is also included in the products above, if the sequences $S_\kappa^+(i)$ is even-numbered. Therefore the number of remaining indices of $S_\kappa^+(i)$ is always even. We divide them into pairs of two and obtain from (5.80)

$$\frac{dF_j}{dr_{j-1}}(\zeta_{j-1}) \cdot \frac{dF_{j+1}}{dr_j}(\zeta_j) < \frac{C}{C^2\zeta_{j-1}^2 + 2B} \cdot \frac{C}{C^2\zeta_j^2} = \frac{1}{C^2\zeta_{j-1}^2\zeta_j^2 + 2B\zeta_j^2}$$

Exploiting the second equation in (5.26) yields $C\zeta_{i-1}\zeta_i = 1 - B\zeta_i^2$ and thus

$$\frac{dF_j}{dr_{j-1}}(\zeta_{j-1}) \cdot \frac{dF_{j+1}}{dr_j}(\zeta_j) < \frac{1}{(1 - B\zeta_i^2)^2 + 2B\zeta_i^2} = \frac{1}{B^2\zeta_i^4 + 1} < 1.$$

Now, depending on the sequence κ the whole product (5.73) consists of different combinations of these three subproducts. Since each subproduct is smaller than one we also find (5.73) to be smaller than one. Hence the fixed point equation (5.72) is contractive and therefore has a unique fixed point r_i^κ. This holds for all $i \in \mathbb{Z}_N$.

Continuation of the solution for $\lambda > 0$

With the above constructions we have shown for $\boldsymbol{\chi} : \mathbb{R}^N \times \mathbb{R} \times \Sigma_A \to \mathbb{R}^N$ that

$$\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa) = 0.$$

In order to guarantee the intended continuation of the solution we apply the implicit function theorem. To this end we consider $\boldsymbol{\chi}(\cdot, \cdot, \kappa)$ as a mapping defined on neighbourhoods of $\mathbf{r}^\kappa$ in $\mathbb{R}^N$ and 0 in $\mathbb{R}$ and show that $D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa) \in \mathbb{L}(\mathbb{R}^N, \mathbb{R}^N)$ is invertible. Basically this can be done as in Section 5.4 only here we are restricted to periodic orbits which makes things a lot easier. We find that

$$D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa)\mathbf{r} = \left(D_1\chi_i(\mathbf{r}^\kappa, 0, \kappa)\mathbf{r}\right)_{i\in\mathbb{Z}_N},$$

and

$$
\begin{aligned}
D_1\chi_i(\mathbf{r}^\kappa,0,\kappa)\mathbf{r} = &\quad -A_i r_i, && i\in\mathbb{Z}_N\setminus J_\kappa,\\
D_1\chi_i(\mathbf{r}^\kappa,0,\kappa)\mathbf{r} = &\quad -C_i r_i^\kappa r_{i-1} - (2Br_i^\kappa + C_i r_{i-1}^\kappa) r_i, && i\in J_\kappa, \quad i-1\in J_\kappa,\\
D_1\chi_i(\mathbf{r}^\kappa,0,\kappa)\mathbf{r} = &\quad -2Br_i^\kappa r_i, && i\in J_\kappa, \quad i-1\in\mathbb{Z}_N\setminus J_\kappa.
\end{aligned}
$$

Let $\mathbf{r}\in\ker D_1\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa)$, then:

$$
\left.
\begin{aligned}
r_i &= 0, \quad i\in\mathbb{Z}_N\setminus J_\kappa,\\
C_i r_i^\kappa r_{i-1} + (2Br_i^\kappa + C_i r_{i-1}^\kappa) r_i &= 0, \quad i\in J_\kappa, \quad i-1\in J_\kappa,\\
r_i &= 0, \quad i\in J_\kappa, \quad i-1\in\mathbb{Z}_N\setminus J_\kappa.
\end{aligned}
\right\}
\tag{5.81}
$$

Only the second equation in (5.81), the one which is related to $i\in J_\kappa$, $i-1\in J_\kappa$, can have non-trivial solutions.

First we consider the case where there is an $i_0\in J_\kappa$ such that $i_0-1\in\mathbb{Z}_N\setminus J_\kappa$. Then according to the third equation in (5.81) $r_{i_0}=0$. From (5.71) we obtain with $r_i^\kappa = F_i(\mathbf{r}^\kappa,\kappa)$ that

$$
2Br_i^\kappa + C_i r_{i-1}^\kappa = \sqrt{C_i^2 (r_{i-1}^\kappa)^2 + 4B} > 0, \tag{5.82}
$$

and hence, according to the second equation in (5.81) it yields

$$
r_i = 0, \quad \forall i\in J_\kappa,\, i>i_0.
$$

So it remains to consider the case $i\in J_\kappa$ for all $i\in\mathbb{Z}_N$. In this case we find $D_1\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa)$ explicitly as

$$
D_1\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa) = \begin{bmatrix}
2Br_1^\kappa + C_1 r_N^\kappa & 0 & \dots & 0 & C_1 r_1^\kappa\\
C_2 r_2^\kappa & 2Br_2^\kappa + C_2 r_1^\kappa & 0 & \dots & 0\\
0 & C_3 r_3^\kappa & 2Br_3^\kappa + C_3 r_2^\kappa & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & 0\\
0 & \dots & 0 & C_N r_N^\kappa & 2Br_N^\kappa + C_N r_{N-1}^\kappa
\end{bmatrix}
$$

The determinant of this matrix is given by

$$
\det\left(D_1\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa)\right) = \prod_{i=1}^N (2Br_i^\kappa + C_i r_{i-1}^\kappa) - \prod_{i=1}^N (C_i r_i^\kappa)
$$

and can be estimated by applying (5.82)

$$
\begin{aligned}
\det(D_1\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa)) &= \prod_{i=1}^N \sqrt{C_i^2 (r_{i-1}^\kappa)^2 + 4B} - \prod_{i=1}^N (C_i r_i^\kappa)\\
&> \prod_{i=1}^N \sqrt{C_i^2 (r_{i-1}^\kappa)^2} - \prod_{i=1}^N (C_i r_i^\kappa)\\
&= \prod_{i=1}^N (|C_i| r_{i-1}^\kappa) - \prod_{i=1}^N (C_i r_i^\kappa) \geq 0.
\end{aligned}
$$

Hence $D_1\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa)$ is invertible and we can apply the implicit function theorem.

This concludes the proof of the solvability of the determination equations at $\lambda > 0$ for periodic trajectories. The necessary and sufficient conditions to the corresponding κ are that $A_i > 0$ for all $i \in \mathbb{Z}_N \setminus J_\kappa$, $B > 0$ and, in case that $i \in J_\kappa$ for all $i \in \mathbb{Z}_N$, that there is at least one index $j \in J_\kappa^+$.

Due to (5.18) and the rescaling (5.21), we find that the same expression of ω_i in terms of λ and r_i as presented in (5.40) applies. However, in contrast to the result of Theorem 5.3.3 we find that the r_i for $i \in J_\kappa^-$ are not uniformly bounded but, in case of $B < C$, grow exponentially with the length L of the sequence of successive indices from J_κ^-, $r_i > (C/B)^L$ cf. the fourth and fifth row in (5.76). This reduces the sizes of the corresponding ω_i. In order to satisfy $\inf \boldsymbol{\omega} > \Omega$, as demanded in Theorem 3.2.2, λ has to satisfy

$$\lambda \leq \frac{1}{\max\limits_{i \in J_\kappa}(r_i^2)} e^{4\mu^s(0)\Omega} < \left(\frac{B}{C}\right)^{2L} e^{4\mu^s(0)\Omega}. \tag{5.83}$$

So with L tending to infinity, λ tends to zero.

Remark 5.5.1. *The limiting factor for the existence of a periodic trajectory is therefore not the period length N but rather the length of the sequences $S_\kappa^-(i)$, $i \in J_\kappa^-$. Or in other words, for all $\lambda \in (0, \hat{\lambda}(N))$ there exist not only all periodic trajectories corresponding to $\kappa \in \Sigma_{A_+} \setminus \mathcal{K}_2$ of length N, but all periodic trajectories corresponding to $\kappa \in \Sigma_{A_+} \setminus \mathcal{K}_2$ that have no more than N consecutive determination equations with negative C_i.*

5.5.2 Solution for fixed κ and $\lambda < 0$

The system of equations under consideration are listed in Section 5.4.2 in (5.44). According to the considerations in Section 5.4.2 we find that for $\lambda < 0$ there are only two types of sequences κ we might obtain a solution for: Either we have for all i that $i \in \mathbb{Z} \setminus J_\kappa$ or we have for all i that $i \in J_\kappa$ and $C_i(\lambda) < 0$.

Again the solvability of the first case, $i \in \mathbb{Z} \setminus J_\kappa$ for all i, follows along the same lines as in Section 5.4.1 under the assumption $A_i(\lambda) < 0$.

In the following we discuss the solvability in case that $i \in J_\kappa$ and $C_i(\lambda) < 0$ for all i. Here the following equation applies

$$\chi_i(\mathbf{r}, \lambda, \kappa) = 1 + B(\lambda) r_i^2 - C(\lambda) r_{i-1} r_i + O\big(\lambda^{\delta - 1}\big) \tag{5.84}$$

for all $i \in \mathbb{Z}_N$.

As we have pointed out in Remark 5.2.5 these types of trajectories under consideration belong either to $\kappa \in \mathcal{K}_2$, if the fibre bundle $\mathcal{F}(W_\gamma^s)$ has the topological structure of an annulus, or to $\kappa \in \mathcal{K}_4$, if $\mathcal{F}(W_\gamma^s)$ has the structure of a Möbius band, cf. (5.9) (5.10) and Corollary 5.2.4. Hence they are either 2-periodic or 4-periodic. Here we discuss both cases.

While for $B > C$, using $B := B(0)$ and $C := |C_i(0)|$, there was no solution of the equation $\boldsymbol{\chi}(\mathbf{r}, 0, \kappa) = 0$ we easily find a solution for $B < C$ by $r_i^\kappa = 1/\sqrt{C-B}$ for all $i \in \mathbb{Z}_2$ or $i \in \mathbb{Z}_4$. This solution also can be continued for $\lambda < 0$ as we see in the following.

In case of the 2-periodic trajectory the Jacobian matrix $D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa)$ has the form

$$D_1\boldsymbol{\chi}(\mathbf{r}^\kappa, 0, \kappa) = \frac{1}{\sqrt{C-B}} \begin{bmatrix} 2B - C & -C \\ -C & 2B - C \end{bmatrix}$$

with the determinant

$$\det\left(D_1\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa)\right)=\prod_{i=1}^{2}\frac{2B-C}{\sqrt{C-B}}-\prod_{i=1}^{2}\frac{C}{\sqrt{C-B}}=\frac{(2B-C)^2-C^2}{C-B}=\frac{4B(B-C)}{(C-B)}=-4B<0.$$

In case of the 4-periodic trajectory the Jacobian matrix $D_1\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa)$ has the form

$$D_1\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa)=\frac{1}{\sqrt{C-B}}\begin{bmatrix}2B-C & 0 & 0 & -C\\ -C & 2B-C & 0 & 0\\ 0 & -C & 2B-C & 0\\ 0 & 0 & -C & 2B-C\end{bmatrix}$$

with the determinant

$$\det\left(D_1\boldsymbol{\chi}(\mathbf{r}^\kappa,0,\kappa)\right)=\prod_{i=1}^{4}\frac{2B-C}{\sqrt{C-B}}-\prod_{i=1}^{4}\frac{C}{\sqrt{C-B}}=\frac{(2B-C)^4-C^4}{(C-B)^2}<0.$$

So, no matter which case applies, the determinant is unequal to zero. Hence the implicit function theorem can be applied and we find that these (either 2- or 4-)periodic trajectories exist for negative λ, whereas every other periodic trajectory characterised by existing right-angled transitions does exist for positive λ, cf. Section 5.5.1.

In case that $C=B$ we fail to find a solution of $\boldsymbol{\chi}(\mathbf{r},0,\kappa)=0$ with κ such that $i\in J_\kappa^-$ for all $i\in\mathbb{Z}_p$, p either equal to 2 or 4. This is because equation (5.84) implies $0=1-Br_i(r_{i-1}-r_i)$, which can only be solved for $r_{i-1}>r_i$ for all $i\in\mathbb{Z}_p$, a contradiction. Since the existence of a solution $\mathbf{r}(\lambda)$, λ sufficiently small, of $\boldsymbol{\chi}(\mathbf{r},\lambda,\kappa)=0$ implies a solution to $\boldsymbol{\chi}(\mathbf{r},0,\kappa)=0$, we find that for $\kappa\in\mathcal{K}_2\cup\mathcal{K}_4$ no corresponding trajectory exists.

Summarizing, for $\lambda<0$ we find all trajectories for those κ that satisfy $i\in\mathbb{Z}\setminus J_\kappa$ with $A_i(\lambda)<0$ for all $i\in\mathbb{Z}$. Indeed we are not restricted to periodic trajectories as for these solutions already Theorem 1.0.2 applies. If $B(0)<C(0)$ we additionally find the trajectories belonging to $\kappa\in\mathcal{K}_2$, if $\mathcal{F}(W_\gamma^s)$ has the structure of an annulus, or $\kappa\in\mathcal{K}_4$, if $\mathcal{F}(W_\gamma^s)$ has the structure of a Möbius band. In case that $B(0)=C(0)$ these trajectories do not exist.

6 Construction of a D_k-equivariant homoclinic cycle

In this section we provide an explicit construction of a family of polynomial vector fields f_k in $\mathbb{R}^4$ with D_k-symmetry, $k \in \mathbb{N}$, $k \geq 3$, unfolding a homoclinic cycle. The homoclinic cycle consists of k homoclinic trajectories all connected to the hyperbolic equilibrium at the origin. In case that $k = 4m$, $m \in \mathbb{N}$ the constructed vector field is an example of the in Section 5 considered vector fields.

This chapter is a version of [HKK14]. Only here we omit the description of the precise set-up and background since we have done this in previous sections. We start in Section 6.1 with specifying the properties concerning symmetry and unfolding that the constructed vector fields possess. Section 6.2 then provides the actual construction. In Section 6.3 we give a verification that the homoclinic trajectory that constitute the homoclinic cycle satisfies the non-degeneracy condition (H5.4)(i).

6.1 Demands on the vector field f_k

We present a construction of a D_k-equivariant polynomial vector field $f_k : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$, $(x, \lambda) \mapsto f_k(x, \lambda)$ where we denote the coordinates with $x = (x_1, y_1, x_2, y_2)$. We write $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ and we assume that D_k acts on each $\mathbb{R}^2$ absolutely irreducible, cf. Definition 4.0.2.

For information on group theory in dynamical system contexts, we refer to [Fie07]. Also recall the introduction of Section 4 for basic definitions on group theory and Section 5.1 for the introduction of the group D_{4m} as a representative of D_k. For our purpose the two-dimensional absolutely irreducible representation $\vartheta_k : D_k \to GL(2, \mathbb{R})$ is of interest, [LLHG99] where

$$\zeta := \vartheta_k(\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \theta_k := \vartheta_k(\theta_k) = \begin{pmatrix} \cos(\frac{2\pi}{k}) & \sin(\frac{2\pi}{k}) \\ -\sin(\frac{2\pi}{k}) & \cos(\frac{2\pi}{k}) \end{pmatrix}.$$

On $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$, the state space of the desired vector field, the group D_k acts as $\vartheta_k + \vartheta_k$

$$(\vartheta_k + \vartheta_k)(g)((x_1, y_1), (x_2, y_2)) \equiv (\vartheta_k(g)(x_1, y_1), \vartheta_k(g)(x_2, y_2)),$$

for all $g \in D_k$, $(x_i, y_i) \in \mathbb{R}^2$, $i = 1, 2$. For this representation we have

$$\operatorname{Fix} \mathbb{Z}_2(\zeta) := \{(x_1, 0, x_2, 0) : x_i \in \mathbb{R}, i = 1, 2\}. \tag{6.1}$$

If k is even, this fixed point space is invariant under $\theta_k^{k/2}$.

Then we construct f_k with properties listed in Property (P6.1) below.

(P6.1).

(i) $f_k(\cdot, \lambda)$ *is* D_k*-equivariant with* $k \in \mathbb{N}$, $k \geq 3$, *where* D_k *acts on* $\mathbb{R}^2 \times \mathbb{R}^2$ *as* $\vartheta_k + \vartheta_k$.

(ii) $f_k(\cdot, \lambda)$ *has a hyperbolic equilibrium* $p = 0$.

(iii) $f_k(\cdot, 0)$ *has in Fix*$\mathbb{Z}_2(\zeta)$ *a homoclinic trajectory* γ_k *asymptotic to* p.

(iv) Within Fix $\mathbb{Z}_2(\zeta)$ *the homoclinic trajectory* γ_k *splits up with non-zero speed at* $\lambda = 0$.

Due to Property (P6.1)(i) the vector field $f_k(\cdot, \lambda)$ leaves $\operatorname{Fix}\mathbb{Z}_2(\zeta)$ invariant:

$$\zeta f_k(x_1, 0, x_2, 0, \lambda) = f_k(\zeta(x_1, 0, x_2, 0), \lambda) = f_k(x_1, 0, x_2, 0, \lambda).$$

We denote the restriction of f_k to $\operatorname{Fix}\mathbb{Z}_2(\zeta)$ by $\hat{f}_k$:

$$\hat{f}_k := f_k|_{\operatorname{Fix}\mathbb{Z}_2(\zeta)}.$$

With (P6.1)(ii) the isotropy group of $p = 0$ is obviously equal to the whole group D_k. The trajectory γ_k has due to (P6.1)(iii) the isotropy subgroup $G_\gamma = \mathbb{Z}_2(\zeta)$. It is also a homoclinic trajectory of $\hat{f}_k(\cdot, 0)$.

Note that for k being even, the vector field $\hat{f}_k$ is $\mathbb{Z}_2(\theta_k^{k/2})$-equivariant. This follows from Property (P6.1)(i). In this consideration $\theta_k^{k/2}$ acts on $\operatorname{Fix}\mathbb{Z}_2(\zeta)$ as $-id$. Then the vector field $\hat{f}_k(\cdot, 0)$ has two homoclinic trajectories, γ_k and $-\gamma_k$. Furthermore, $\theta_k^{k/2}$ acts as $-id$ on the whole phase space $\mathbb{R}^4$. Hence the vector field is odd which implies the existence of a natural number $\nu \geq 3$, cf. Definition 3.4.2, such that

$$D_1^\nu f_k(p, \lambda) \neq 0 \quad \text{and} \quad D_1^l f_k(p, \lambda) = 0, \; l = 2, \ldots, \nu - 1.$$

According to (P6.1)(i) the group acts absolutely irreducible on the stable and unstable subspaces $E(\mu^s(\lambda))$ and $E(\mu^u(\lambda))$. Invoking Lemma 4.1.1 then shows that $D_k f(p, 0)$ has two real eigenvalues $\mu^s < 0 < \mu^u$, both of geometric multiplicity two.

Remark 6.1.1. *Any polynomial vector field f_k on $\mathbb{R}^4$ which satisfies (P6.1) satisfies Hypothesis (H4.1) with $G = D_k$, Hypotheses (H5.2), (H5.3) and Hypothesis (H5.4) restricted to Fix$\mathbb{Z}_2(\zeta)$. In case that k is even we also find $\nu \geq 3$.*

For $k = 4m$, $m \in \mathbb{N}$, any polynomial vector field f_k satisfying (P6.1) also satisfies Hypothesis (H5.1).

By Property (P6.1) the leading eigenvalue is not yet determined. In the construction we introduce freely selectable coefficients a and b which can be chosen such that the stable eigenvalue is closer to the imaginary axis than the unstable eigenvalue, cf. Hypothesis (H5.3)(iv).

Note that Hypothesis (H5.4)(i) does not follow from Property (P6.1). This condition is solely satisfied within Fix$\mathbb{Z}_2(\zeta)$. However, we may assume that (H5.4)(i) is also satisfied in the full space since it describes the generic situation. In Section 6.3 we state a range of coefficients for which the constructed vector fields also satisfy (H5.4)(i).

6.2 Construction of D_k-equivariant vector fields in $\mathbb{R}^4$

The desired D_k-equivariant family of vector fields f_k is build in several steps. First a single vector field $\hat{f}_k$ in $\mathbb{R}^2$ possessing a homoclinic trajectory to a hyperbolic equilibrium is constructed. In doing so, we follow the idea of Sandstede [San97] – we construct $\hat{f}_k$ in such a way that a (generalized) Cartesian leaf forms a homoclinic trajectory.

Next, the vector field $\hat{f}_k$ is embedded in a one-parameter family such that by changing the family parameter λ (off the critical value) the homoclinic trajectory splits up with non-zero speed.

In the final step, we extend this family into $\mathbb{R}^4$ and end up with a family as stated in Section 6.1.

6.2.1 Basic construction in $\mathbb{R}^2$

Sandstede used in [San97] the Cartesian leaf to construct a vector field in $\mathbb{R}^2$ having a homoclinic trajectory. In [HKK14] the slightly modified curves

$$\mathcal{C}_k(x_1, x_2) := x_1^2(1 - x_1^{k-2}) - x_2^2$$

were used. Note that $\mathcal{C}_3^{-1}(0)$ is the Cartesian leaf, and $\mathcal{C}_4^{-1}(0)$ is a lemniscate, cf. Figure 6.1. For any odd or even k the curves $\mathcal{C}_k^{-1}(0)$ resemble those for $k=3$ or $k=4$, respectively.

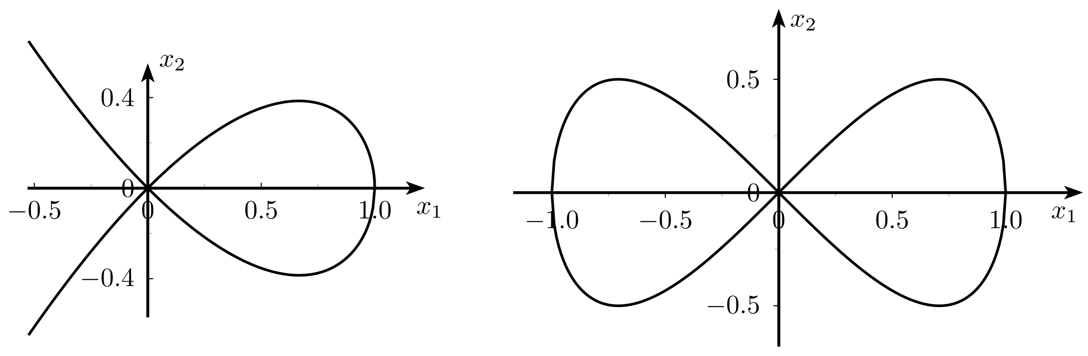


Figure 6.1: The curves $\mathcal{C}_3(x_1,x_2)=0$ (left) and $\mathcal{C}_4(x_1,x_2)=0$ (right).

The zero-level set $\mathcal{C}_k^{-1}(0)$ of $\mathcal{C}_k$ is invariant under the flow of a given vector field $\hat{f}$ if and only if

$$\left\langle \nabla\mathcal{C}_k(x_1,x_2), \hat{f}(x_1,x_2)\right\rangle = 0, \quad \forall (x_1,x_2)\in\mathcal{C}_k^{-1}(0). \tag{6.2}$$

Lemma 6.2.1 ([HKK14] Lemma 4.1)**.** *Let $k\geq 3$ and $a,b\in\mathbb{R}\backslash\{0\}$, $a^2<b^2$. The vector field*

$$\hat{f}_k(x_1,x_2) := \begin{pmatrix} ax_1+bx_2-ax_1^{k-1} \\ bx_1+ax_2-b\frac{k}{2}x_1^{k-1}-a\frac{k}{2}x_1^{k-2}x_2 \end{pmatrix}$$

has a homoclinic trajectory $\hat{\gamma}_k$ which is subset of $\mathcal{C}_k^{-1}(0)\cap\{x_1>0\}$.

The expression of $\hat{f}_k$ follows from a general polynomial ansatz of degree $k-1$ plugged in into equation (6.2). Here we confine to show that the vector field indeed has a homoclinic trajectory to $(x_1,x_2)=(0,0)$.

Proof. First we show that $\hat{f}_k$ satisfies equation (6.2): Let $(x_1,x_2)\in\mathcal{C}_k^{-1}(0)$. Using

$$\nabla\mathcal{C}_k(x_1,x_2) = \begin{pmatrix} 2x_1-kx_1^{k-1} \\ -2x_2 \end{pmatrix},$$

compute

$$\begin{aligned}\left\langle \nabla\mathcal{C}_k(x_1,x_2), \hat{f}_k(x_1,x_2)\right\rangle &= a((2x_1-kx_1^{k-1})(x_1-x_1^{k-1})-2x_2(x_2-\tfrac{k}{2}x_1^{k-2}x_2)) \\ &= a(2\underbrace{(x_1^2(1-x_1^{k-2})-x_2^2)}_{=\mathcal{C}_k(x_1,x_2)}-kx_1^{k-2}\underbrace{(x_1^2-x_1^k-x_2^2)}_{=\mathcal{C}_k(x_1,x_2)}) \\ &= 0.\end{aligned}$$

We must verify that $\hat{f}_k(x_1,x_2)\neq 0$ for all $(x_1,x_2)\in\mathcal{C}_k^{-1}(0)$, $x_1>0$. The first component $\hat{f}_m^1$ evaluated at those points equals

$$\hat{f}_k^1\big(x_1,\pm x_1\sqrt{1-x_1^{k-2}}\big) = x_1\sqrt{1-x_1^{k-2}}\,\big(a\sqrt{1-x_1^{k-2}}\pm b\big),$$

and becomes zero for $x_1=0$, $x_1^{k-2}=1$ or $x_1^{k-2}=1-(b/a)^2$. With the assumption $a^2<b^2$, the right-hand side of the last equation, $1-(b/a)^2$, is negative. Hence, if k is even, this equation has no real solution. If

k is odd the only real solution is negative. Further, the second equation, $x_1^{k-2} = 1$, implies $|x_1| = 1$. But the second component $\hat{f}_k^2(x_1, 0) = b(1 - k/2)$ is different from zero, since $b \neq 0$ and $k \geq 3$. $\square$

Remark 6.2.2. *Because of $a^2 < b^2$ the equilibrium $(0,0)$ is a saddle point with eigenvalues $a + b$ and $a - b$. If one imposes $0 < a^2 < b^2$, then $|a + b| \neq |a - b|$. This implies that the vector field $\hat{f}_k$ is neither Hamiltonian nor reversible. Choosing $a > 0$ implies that the leading stable eigenvalue is closer to the imaginary axis than the unstable eigenvalue*

$$|\mu^s| < \mu^u.$$

Remark 6.2.3. *Let k be even. The vector field $\hat{f}_k$ ist equivariant with respect to $\mathbb{Z}_2(\theta_2)$, where θ_2 acts on $\mathbb{R}^2$ as $-id$. Consequently, $\theta_2(\hat{\gamma}_k)$ is also a homoclinic orbit of $\hat{f}_k$ asymptotic to $(x_1, x_2) = (0, 0)$. Both orbits, $\hat{\gamma}_k$ and $\theta_2(\hat{\gamma}_k)$, together with the equilibrium $(0,0)$ form the "figure-eight" drawn in the right panel of Figure 6.1.*

Remark 6.2.4. *We can find an analytic solution for the homoclinic trajectory $\hat{\gamma}_k = (\hat{\gamma}_k^1, \hat{\gamma}_k^2)$ by choosing the ansatz*

$$\hat{\gamma}_k^1(t) = (1 - u(t)^2)^{\frac{1}{k-2}}, \qquad \hat{\gamma}_k^2(t) = -\hat{\gamma}_k^1(t)u(t)$$

with $u : (-\infty, \infty) \to (-1, 1)$. Then u satisfies the initial value problem

$$\dot{u} = \frac{k-2}{2}(b - au)(1 - u^2), \qquad u(0) = 0.$$

that can be solved by separation of variables. We obtain the inverse function of $u = H(t)$ by

$$u \mapsto t = H^{-1}(u) = \frac{2a\ln(1 - \frac{a}{b}u) - (a+b)\ln(1-u) - (a-b)\ln(1+u)}{(k-2)(b^2 - a^2)}.$$

Next a perturbation term is added to the vector field $\hat{f}_k$, that splits up the homoclinic trajectory $\hat{\gamma}_k$ with non-zero speed (for $\lambda \neq 0$). This results in the family of vector fields $\hat{f}_k : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$.

Lemma 6.2.5 ([HKK14] Lemma 4.2)**.** *Let $k \geq 3$ and $a, b \in \mathbb{R}\backslash\{0\}$, $a^2 < b^2$. Consider the family of vector fields*

$$\hat{f}_k(x, \lambda) := \begin{pmatrix} ax_1 + bx_2 - ax_1^{k-1} \\ bx_1 + ax_2 - b\frac{k}{2}x_1^{k-1} - a\frac{k}{2}x_1^{k-2}x_2 \end{pmatrix} + \lambda\nabla\mathcal{C}_k(x_1, x_2). \tag{6.3}$$

In accordance with Lemma 6.2.1 this vector field has for $\lambda = 0$ the homoclinic trajectory $\hat{\gamma}_k$. This homoclinic trajectory splits up as λ moves off zero. Moreover, let for λ close to zero $d(\lambda)$ denote the distance of the stable and unstable manifolds of the equilibrium $(0,0)$, measured in a direction perpendicular to $\hat{f}_k(\hat{\gamma}_k(0), 0)$. The derivative $d'(0)$ is different from zero.

Proof. The verification that the perturbation splits the homoclinic trajectory in the described way can be done by using the Melnikov integral, cf. Section 3.4.1. It can be shown that, cf. [HomSan10],

$$d'(0) = \int_{-\infty}^{\infty} \left\langle \eta(t), D_\lambda \hat{f}_k(\hat{\gamma}_k(t), 0) \right\rangle dt,$$

where $\eta(t)$ solves the adjoint variational equation $\dot{v} = -[D_1\hat{f}_k(\hat{\gamma}_k(t), 0)]^T v$ with $|\eta(0)| = 1$; $\eta(0) \perp \hat{\gamma}_k(0)$.

Therefore

$$\eta(t) = \phi(t) \begin{pmatrix} -\hat{f}_k^2(\hat{\gamma}_k(t), 0) \\ \hat{f}_k^1(\hat{\gamma}_k(t), 0) \end{pmatrix}$$

with a scalar function ϕ. Simple calculations show that the function ϕ solves

$$\dot{\phi} = -\operatorname{div}(\hat{f}_k)(\hat{\gamma}_k(t), 0)\phi.$$

Combining these results yields

$$d'(0) = \int_{-\infty}^{\infty} \phi(t) \left\langle \begin{pmatrix} -\hat{f}_k^2(\hat{\gamma}_k(t), 0) \\ \hat{f}_k^1(\hat{\gamma}_k(t), 0) \end{pmatrix}, D_\lambda \hat{f}_k(\hat{\gamma}_k(t), 0) \right\rangle dt.$$

By construction the scalar product within this integral is always positive or negative, and as a solution of a scalar linear differential equation $\phi(t)$ does also not change sign. Hence $d'(0) \neq 0$. □

Remark 6.2.6. *If k is even the entire family $\hat{f}_k(\cdot, \lambda)$ is equivariant with respect to representation of $\mathbb{Z}_2(\theta_2)$ which is given in Remark 6.2.3. Consequently, both homoclinic trajectories $\hat{\gamma}_k$ and $\theta_2(\hat{\gamma}_k)$ split up as λ moves off zero.*

Denote the stable and unstable eigenvalues by μ^s and μ^u respectively. Let $a > 0$, then $|\mu^s| < \mu^u$. Applying a first return map, cf. [HomSan10], yields the bifurcation diagram depicted in Figure 6.2. In particular this diagram reveals for which parameter values which periodic orbits do exist.

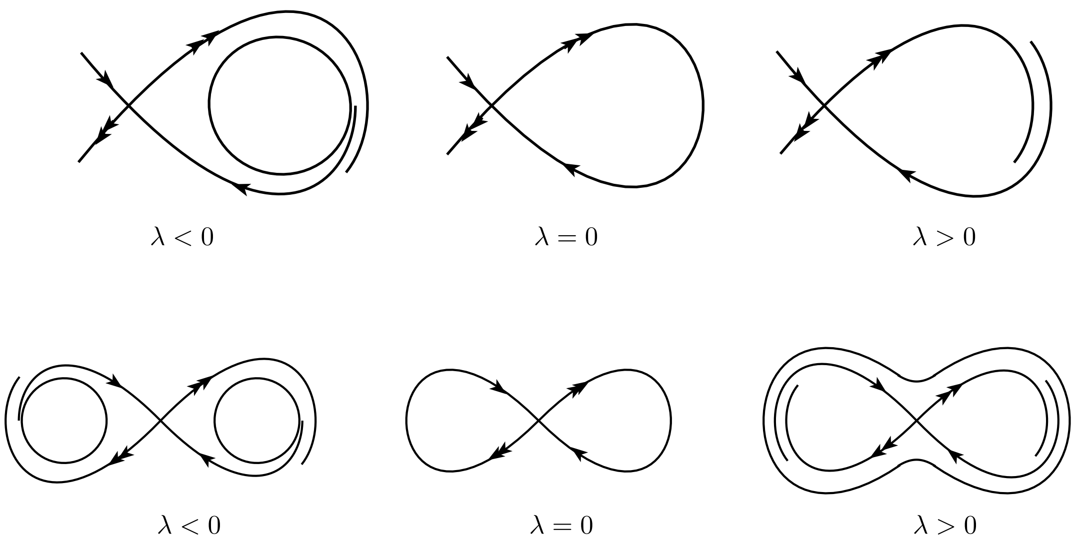


Figure 6.2: Bifurcation diagram of f_k, $a > 0$: m is odd (top); m is even (bottom).

6.2.2 Extending the vector field to $\mathbb{R}^4$

The four-dimensional vector field $f_k = (f_k^1, f_k^2, f_k^3, f_k^4)^T$ is constructed in such a way that "$f_k|_{\mathrm{Fix}\,\mathbb{Z}_2(\zeta)} = \hat{f}_k$", more precisely

$$f_k(x_1, 0, x_2, 0, \lambda) = (\hat{f}_k^1(x_1, x_2, \lambda), 0, \hat{f}_k^2(x_1, x_2, \lambda), 0)^T. \tag{6.4}$$

To this end the perturbed vector field is extended to $\mathbb{R}^4$ (see Theorem 6.2.8) by using a set of generators for D_k-equivariant vector fields. Such generators can be found in a paper by Matthies [Mat99] for the case $k = 3$ and in a paper by Lari-Lavassani, et al. [LLHG99] for $k = 3, 4$.

D_k-equivariant vector fields in $\mathbb{R}^4$

In this section we first describe the structure of D_k-equivariant polynomial vector fields in $\mathbb{R}^4$. To this end we recall the following definitions and results from [ChoLau00, GSS88].

Let G be a finite group acting on the vector space $\mathbb{R}^n$. A (polynomial) function $s : \mathbb{R}^n \to \mathbb{R}$ is called *G-invariant* with respect to the given representation of the group if

$$s = s \circ g \quad \forall g \in G.$$

The set $\mathcal{R}_G$ of G-invariant polynomials forms a ring. This ring is finitely generated, i.e. there is a finite set $s_1, \cdots, s_j$ of G-invariant polynomials, the *generators* for the ring $\mathcal{R}_G$, such that each $s \in \mathcal{R}_G$ has a representation

$$s(x) = B(s_1(x), ..., s_j(x)),$$

where $B : \mathbb{R}^j \to \mathbb{R}$ is polynomial. The set $\{s_1, \cdots, s_j\}$ is called a *Hilbert basis* of $\mathcal{R}_G$. Further, the set of equivariant polynomial vector fields forms a module $\mathcal{M}_G$ over the ring $\mathcal{R}_G$. This module is finitely generated, i.e. there exits a set $\{h_1, \cdots, h_l\}$ of G-equivariant polynomial vector fields such that each $f \in \mathcal{M}_G$ can be written as

$$f(x) = \sum_{i=1}^{l} B_i(s_1(x), ..., s_j(x))h_i(x) \quad \forall x, \tag{6.5}$$

where B_i are polynomials.

Lari-Lavassani et al. [LLHG99] present a generating set for D_3- and D_4-equivariant vector fields $f : \mathbb{R}^4 \to \mathbb{R}^4$, where D_k acts as $\vartheta_k + \vartheta_k$. Matthies [Mat99] also presents a generating set for D_3-equivariant vector fields. Unlike Lari-Lavassani et al., he considered complex vector fields $f_{\mathbb{C}} : \mathbb{C}^2 \to \mathbb{C}^2$. In what follows we identify $\mathbb{R}^2$ with $\mathbb{C}$ since it seems to be adequate to work with complex coordinates in the context of the D_k representations under consideration. We want to remark that the above notions and results spread to representations on $\mathbb{C}^2$.

The coordinates in $\mathbb{C}^2$ we denote by $z = (v, w)$ where $v = x_1+iy_1$ and $w = x_2+iy_2$. With the isomorphism

$$\mathcal{I} : \mathbb{C}^2 \to \mathbb{R}^4, \qquad (v, w) = (x_1 + iy_1, x_2 + iy_2) \mapsto (x_1, y_1, x_2, y_2),$$

we obtain a complex vector field $f_{\mathbb{C}}$ by

$$f_{\mathbb{C}} := \mathcal{I}^{-1} \circ f \circ \mathcal{I}.$$

The vector field $f_{\mathbb{C}}$ is equivariant with respect to the complex representation $(\vartheta_k + \vartheta_k)_{\mathbb{C}}$ of the group D_k defined by

$$g_{\mathbb{C}} := \mathcal{I}^{-1} \circ g \circ \mathcal{I}, \quad g \in D_k.$$

In particular, the corresponding complex representations of ζ and θ_k read

$$\zeta_{\mathbb{C}}(v, w) = (\bar{v}, \bar{w}), \qquad \theta_{k,\mathbb{C}}(v, w) = (e^{i2\pi/k}v, e^{i2\pi/k}w). \tag{6.6}$$

In the following we present sets of D_k-invariant functions and D_k-equivariant vector fields. In the cases

$k = 3$ and $k = 4$ these are generator sets for the corresponding ring $\mathcal{R}_{D_k}$ or the module $\mathcal{M}_{D_k}$, respectively, [LLHG99, Mat99].

Lemma 6.2.7 ([HKK14], Lemma 4.3)**.** *Assume D_k acts on $\mathbb{C}^2$ as defined in (6.6).*

(i) The functions

$$s_0(v,w) = v\bar{v}, \quad s_1(v,w) = w\bar{w}, \quad s_2(v,w) = v\bar{w} + \bar{v}w$$
$$t_j(v,w) = v^j w^{k-j} + \bar{v}^j \bar{w}^{k-j}, \quad j \in \{0,\dots,k\}$$

are D_k-invariant polynomials on $\mathbb{C}^2$.

(ii) The mappings

$$g_0(v,w) = (v,0), \quad g_1(v,w) = (0,v), \quad g_2(v,w) = (w,0), \quad g_3(v,w) = (0,w),$$
$$f_j(v,w) = (\bar{v}^j \bar{w}^{k-1-j}, 0), \quad h_j(v,w) = (0, \bar{v}^j \bar{w}^{k-1-j}), \quad j \in \{0,\dots,k-1\}.$$

are D_k-equivariant polynomial mappings $\mathbb{C}^2 \to \mathbb{C}^2$.

Proof. The invariance or equivariance of the given functions or mappings, respectively, can be verified by straightforward calculations. ☐

The vector field f_k

We use the mappings presented in Lemma 6.2.7 to extend the vector field $\hat{f}_k$ to the desired vector field f_k in $\mathbb{R}^4$. In the course of this we use representations of these vector fields in complex coordinates.

In complex coordinates the fixed space of $\mathbb{Z}_2(\zeta_{\mathbb{C}})$ reads, cf. (6.1) and (6.6)

$$\operatorname{Fix}\mathbb{Z}_2(\zeta_{\mathbb{C}}) := \{(x_1,x_2) : x_i \in \mathbb{R}, i = 1,2\}.$$

According to (6.4) the $\mathbb{R}^2$-vector field $\hat{f}_k$ can be seen as vector field on $\mathbb{Z}_2(\zeta) \subset \mathbb{R}^4$ (we denote this vector field again by $\hat{f}_k$). The related vector field $\hat{f}_{k,\mathbb{C}}$ reads (in complex coordinates)

$$\hat{f}_{k,\mathbb{C}}(x_1,x_2,\lambda) = (\hat{f}_k^1(x_1,x_2,\lambda), \hat{f}_k^2(x_1,x_2,\lambda)).$$

A crucial observation in respect of the intended extension is that the mappings in Lemma 6.2.7(*ii*) leave $\operatorname{Fix}\mathbb{Z}_2(\zeta_{\mathbb{C}})$ invariant, and the polynomials in Lemma 6.2.7(*i*) are real-valued. Further, we find for $(v,w) \in \operatorname{Fix}\mathbb{Z}_2(\zeta_{\mathbb{C}})$, i.e. for $(v,w) = (x_1,x_2)$

$$g_0(x_1,x_2) = (x_1,0), \qquad g_1(x_1,x_2) = (0,x_1), \qquad g_2(x_1,x_2) = (x_2,0), \qquad g_3(x_1,x_2) = (0,x_2),$$
$$f_{k-1}(x_1,x_2) = (x_1^{k-1},0), \qquad h_{k-1}(x_1,x_2) = (0,x_1^{k-1}), \qquad h_{k-2}(x_1,x_2) = (0,x_1^{k-2}x_2).$$

Consequently, the vector field $\hat{f}_{k,\mathbb{C}}$ can be represented by means of merely these vector fields:

$$\begin{aligned}\hat{f}_{k,\mathbb{C}}(x_1,x_2,\lambda) \;&=\; a\left(g_0 + g_3 - f_{k-1} - \tfrac{k}{2}h_{k-2}\right)(x_1,x_2) + b\left(g_1 + g_2 - \tfrac{k}{2}h_{k-1}\right)(x_1,x_2)\\ &\quad + \lambda\left(2g_0 - kf_{k-1} - 2g_3\right)(x_1,x_2).\end{aligned}$$

Theorem 6.2.8 ([HKK14] Theorem 4.1)**.** *The vector field*

$$\begin{aligned} f_{k,\mathbb{C}}(v,w,\lambda) &= a\left(g_0+g_3-f_{k-1}-\tfrac{k}{2}h_{k-2}\right)(v,w)+b\left(g_1+g_2-\tfrac{k}{2}h_{k-1}\right)(v,w)\\ &\quad +\lambda\left(2g_0-kf_{k-1}-2g_3\right)(v,w)\end{aligned}$$

is equivariant with respect to the complex representation $(\vartheta_k+\vartheta_k)_{\mathbb{C}}$.

Moreover, the vector field $f_k := \mathcal{I}\circ f_{k,\mathbb{C}}\circ\mathcal{I}^{-1}$ satisfies (P6.1).

Proof. The first part of the theorem is an immediate consequence of the above representation of $\hat{f}_{k,\mathbb{C}}$.

Further, by construction the vector field f_k leaves Fix $\mathbb{Z}_2(\zeta)$ invariant and its restriction to this fixed point space coincides with $\hat{f}_k$. So the properties concerning the homoclinic trajectory γ_k follow from our considerations carried out in Section 6.2.1. □

Remark 6.2.9. *The construction presented in Theorem 6.2.8 is not the only way to extend the vector field $\hat{f}_k$ to $\mathbb{R}^4$ by using the set of generators listed above. In case that k is even,*

$$\begin{aligned} \tilde{f}_{k,\mathbb{C}}(v,w,\lambda) &= a\left(g_0+g_3-s_0^{(k-2)/2}(g_0+\tfrac{k}{2}g_3)\right)(v,w)+b\left(g_1+g_2-\tfrac{k}{2}s_0^{(k-2)/2}g_1\right)(v,w)\\ &\quad +\lambda\left(2g_0-ks_0^{(k-2)/2}g_0-2g_3\right)(v,w)\end{aligned}$$

is another way to extend the vector field. In this construction we only used the mappings $g_0,\ldots,g_3$ and the functions $s_0,\ldots,s_2$, which have the same structure for all dihedral groups D_k, $k\geq 3$. Hence $\tilde{f}_{k,\mathbb{C}}$ is not just equivariant with respect to D_k but equivariant with respect to D_l for all $l\in\mathbb{N}$, $l\geq 3$.

Remark 6.2.10. *Not any vector field $\hat{f}_{\mathbb{C}}$ on Fix$\mathbb{Z}_2(\zeta_{\mathbb{C}})$ can be extended to an $\vartheta_k\times\vartheta_k$-equivariant vector field $f_{\mathbb{C}}$ on $\mathbb{C}^2$. If k is even, the components of all vector fields in Lemma 6.2.7 are monomials of odd degree, whereas all functions in Lemma 6.2.7 are homogeneous polynomials of even degree. Hence, the components of all polynomial vector fields $f_{\mathbb{C}}$ that can be generated by those functions and vector fields, cf. (6.5), are sums of homogeneous polynomials of odd degree. This must be true already for the restricted vector field $\hat{f}_{\mathbb{C}}$.*

Recall that for $k=4$, Lemma 6.2.7 presents a set of generators for the module $\mathcal{M}_{D_k}$.

For the real vector fields f_k the polynomial structure is getting more and more complicated as k increases. For that reason we confine ourselves to present the representation of f_4:

$$\begin{aligned} f_4(x,\lambda) &= \begin{pmatrix} ax_1 & +bx_2 & -ax_1^3+3ax_1y_1^2 & \\ ay_1 & +by_2 & +3ax_1^2y_1-ay_1^3 & \\ bx_1 & +ax_2 & -2bx_1^3+6bx_1y_1^2 & -2ax_1^2x_2+2ay_1^2x_2+4ax_1y_1y_2 \\ by_1 & +ay_2 & +6bx_1^2y_1-2by_1^3 & +4ax_1y_1x_2+2ay_2(x_1^2-y_1^2) \end{pmatrix}\\ &\quad +\lambda\begin{pmatrix} 2x_1 & -4x_1^3+12x_1y_1^2 \\ 2y_1 & +12x_1^2y_1-4y_1^3 \\ -2x_2 & \\ -2y_2 & \end{pmatrix}. \end{aligned} \tag{6.7}$$

6.3 Verification of Hypothesis (H5.4)(i)

We show that for $|a| \ll |b|$ the constructed vector field f_k satisfies the Hypothesis (H5.4)(i). The perturbation analysis below that shows this yields a strong restriction on the parameters a and b. We do not believe the estimates to be optimal and we expect that condition (H5.4)(i) is still satisfied for many times greater ratios $|a|/|b|$. However, this is not covered by our analysis.

The minimal intersection condition claimed in Hypothesis (H5.4)(i) is equivalent to the fact that the adjoint of the variational equation along γ_k,

$$\dot{\psi} = -\left[D_{(x,y)} f_k(\gamma_k(t), 0)\right]^T \psi, \tag{6.8}$$

has (up to multiples) only one solution which is bounded on $\mathbb{R}$. Note that one such bounded solution lies in Fix $\mathbb{Z}_2(\zeta)$, cf. also the proof of Lemma 6.2.5. With that said it remains to show that equation (6.8) has no bounded solution outside of Fix $\mathbb{Z}_2(\zeta)$. Recall that exactly those solutions of (6.8) are bounded on $\mathbb{R}$ which start in the orthogonal complement of the sum of the tangent spaces of the stable and unstable manifolds of the equilibrium $p = 0$ along γ_k, cf. Lemma 2.2.6.

As before we drop the dependence of the vector field on a and b in our notation.

With $H_i : \mathbb{R}^4 \to \mathbb{R}^4$, $H_i = \left(H_i^1,\, H_i^2,\, H_i^3,\, H_i^4\right)^T$ defined by

$$\begin{aligned} H_1(x,0) =& \left(0,\, a + (k-1)ax_1^{k-2},\, 0,\, b + \frac{k(k-1)}{2}bx_1^{k-2} + \frac{k(k-2)}{2}ax_1^{k-3}x_2\right)^T, \\ H_2(x,0) =& \left(0,\, b,\, 0,\, a + \frac{k}{2}ax_1^{k-2}\right)^T, \end{aligned}$$

we write

$$f_k(x,y,0) = \left(\hat{f}_k^1(x,0), 0, \hat{f}_k^2(x,0), 0\right)^T + y_1 H_1(x,y) + y_2 H_2(x,y).$$

Recall that along $\gamma_k(t)$ the y-components are zero. Therefore the Jacobien of f_k at $\gamma_k(t)$ reads

$$\begin{aligned} D_{(x,y)} f_k(\gamma_k(t), 0) =& D_{(x,y)} \left(\hat{f}_k^1(\gamma_k(t),0), 0, \hat{f}_k^2(\gamma_k(t),0), 0\right)^T \\ &+ H_1(\gamma_k(t))\,(0,1,0,0) + H_2(\gamma_k(t))\,(0,0,0,1). \end{aligned} \tag{6.9}$$

Within Fix $\mathbb{Z}_2(\zeta)$, a solution w of equation (6.8) is bounded on $\mathbb{R}$ if and only if $w(t) \perp \gamma_k(t)$ for all t. Using the inner unit normal ρ of $\mathcal{C}_k^{-1}(0) \cap \{x_1 > 0\}$ within Fix $\mathbb{Z}_2(\zeta)$ we decompose a bounded solution w of (6.8) as follows:

$$w(t) = w_1(t)\rho(t) + w_2(t)(0,1,0,0)^T + w_4(t)(0,0,0,1)^T, \tag{6.10}$$

where $\rho(t) = (\rho^1(t),\, 0\,, \rho^2(t),\, 0)^T$, $|\rho(t)| \equiv 1$, $\langle \gamma_k(t), \rho(t)\rangle = 0$. With (6.10) we find

$$\dot{w}(t) = \dot{w}_1(t)\rho(t) + w_1(t)\dot{\rho}(t) + \dot{w}_2(t)(0,1,0,0)^T + \dot{w}_4(t)(0,0,0,1)^T.$$

We plug in this expression in (6.8) and take the inner product with $(0,1,0,0)^T$ or $(0,0,0,1)^T$ to get differential equations for w_2 or w_4, respectively. Here we take into consideration that, because of $\langle \rho(t), \rho(t)\rangle \equiv 1$, the derivative $\dot{\rho}(t)$ is perpendicular to $\rho(t)$. Moreover, $\dot{\rho}(t)$ is also perpendicular to

$(0,1,0,0)^T$ and $(0,0,0,1)^T$. Further we exploit that $H_i^1(\gamma_k(t)) \equiv H_i^3(\gamma_k(t)) \equiv 0$, $i=1,2$. Thus it follows

$$\begin{aligned}\dot w_2 &= -\,H_1^2(\gamma_k(t))w_2 - H_1^4(\gamma_k(t))w_4\\ \dot w_4 &= -\,H_2^2(\gamma_k(t))w_2 - H_2^4(\gamma_k(t))w_4.\end{aligned}$$

More detailed this equation reads

$$\begin{pmatrix} \dot w_2 \\ \dot w_4 \end{pmatrix} = -\Big(aA(t)+bB(t)\Big)\begin{pmatrix} w_2 \\ w_4 \end{pmatrix} \tag{6.11}$$

with

$$\left.\begin{aligned} A(t) &= \begin{bmatrix} 1+(k-1)\big(\gamma_k^1(t)\big)^{k-2} & \frac{k(k-2)}{2}\big(\gamma_k^1(t)\big)^{k-3}\gamma_k^2(t) \\ 0 & 1+\frac{k}{2}\big(\gamma_k^1(t)\big)^{k-2}\end{bmatrix}\\ B(t) &= \begin{bmatrix} 0 & 1+\frac{k(k-1)}{2}\big(\gamma_k^1(t)\big)^{k-2} \\ 1 & 0\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix} + \begin{bmatrix} 0 & \frac{k(k-1)}{2}\big(\gamma_k^1(t)\big)^{k-2} \\ 0 & 0\end{bmatrix}\end{aligned}\right\} \tag{6.12}$$

First we consider equation (6.11) for $a=0$, that is

$$\begin{pmatrix} \dot w_2 \\ \dot w_4 \end{pmatrix} = -bB(t)\begin{pmatrix} w_2 \\ w_4 \end{pmatrix} \tag{6.13}$$

With $v := w_4$ this equation can be rewritten as second order equation

$$\ddot v = b^2\Big(1+\frac{k(k-1)}{2}\big(\gamma^1(t)\big)^{k-2}\Big)v. \tag{6.14}$$

This equation can be treated as a similar problem in [Yew01, Lemma 2.2, Lemma 2.3]. Recall that $\gamma_k^1(t) > 0$ for all $t \in \mathbb{R}$. Suppose (6.14) has a non-trivial bounded solution v. Then both v and $\dot v$ are square-integrable over $\mathbb{R}$, moreover v decays exponentially fast as $t \to \pm\infty$. Keeping that in mind we find by multiplying (6.14) by v and integrating that

$$0 > -\int_{-\infty}^{\infty}\big(\dot v(t)\big)^2\,dt = \int_{-\infty}^{\infty} b^2\Big(1+\frac{k(k-1)}{2}\big(\gamma_k^1(t)\big)^{k-2}\Big)\big(v(t)\big)^2\,dt > 0.$$

This contradicts the assumption of a bounded solution.

Summarizing, equation (6.13) has no bounded solution. Further, the matrix $B(t)$ can be read as an exponential decaying perturbation of a hyperbolic matrix. By the roughness theorem presented in Lemma 2.1.7 this equation has exponential dichotomies on both $\mathbb{R}^+$ and $\mathbb{R}^-$. Altogether this implies that equation (6.13) has an exponential dichotomy on $\mathbb{R}$.

If we could guarantee that equation (6.11) still has an exponential dichotomy on $\mathbb{R}$ for $a \neq 0$ we are finished. In that case equation (6.11) has no nontrivial bounded solution.

Now, the roughness theorem in [JuWig01, Theorem 3.2] guarantees with the estimate

$$|a|\sup_{t\in\mathbb{R}}\|A(t)\|\left(\frac{K_1}{-\alpha}+\frac{K_2}{\beta}\right) < 1, \tag{6.15}$$

that (6.11) has for sufficiently small $|a|$ still an exponential dichotomy on $\mathbb{R}$. Here K_1, K_2, α and β are constants related to the exponential dichotomy of the truncated equation (6.13):

$$\begin{array}{lll} \|\Phi(t,s)P(s)\| & \leq K_1 e^{\alpha(t-s)}, & t \geq s, \\ \|\Phi(t,s)(id - P(s))\| & \leq K_2 e^{-\beta(s-t)}, & s \geq t, \end{array}$$

where $\Phi(\cdot,\cdot)$ denotes the transition matrix of equation (6.13) and $P(t)$ are the projections related to the exponential dichotomy of this equation, cf. (2.2) .

In what follows we give an estimate of $|a|$ such that (6.15) holds true. To that end we need estimates for the constants K_1, K_2, α and β.

First we consider the exponential dichotomy of (6.13) on subintervalls $[t_0^+,\infty)$ and $(-\infty,t_0^-]$ with corresponding constants $\tilde{K}_1^+$, $\tilde{K}_2^+$ and $\tilde{K}_1^-$, $\tilde{K}_2^-$, respectively. We can show that for

$$\left.\begin{array}{lllll} t_0^+ & = & \frac{1}{(k-2)\mu^s}\ln\left(\frac{-2(k-2)\mu^s}{3k(k-1)|b|}\right) & = & \frac{1}{(k-2)(a-|b|)}\ln\left(\frac{2(k-2)(|b|-a)}{3k(k-1)|b|}\right), \\ t_0^- & = & \frac{1}{-(k-2)\mu^u}\ln\left(\frac{2(k-2)\mu^u}{3k(k-1)|b|}\right) & = & \frac{1}{-(k-2)(a+|b|)}\ln\left(\frac{2(k-2)(a+|b|)}{3k(k-1)|b|}\right), \end{array}\right\} \tag{6.16}$$

one may take constants $\tilde{K}_1^\pm, \tilde{K}_2^\pm \geq \frac{9}{2}$ and $-\alpha = \beta = |b|$. The values of the constants are derived from inspections of the proof of Lemma 2.1.7 which we will explain in the following. To this end we treat $-bB(t)$ as a perturbation of the constant hyperbolic matrix $\begin{pmatrix} 0 & -b \\ -b & 0 \end{pmatrix}$, as indicated in equation (6.12). Then, by Lemma 2.1.7, we find that the exponential rates related to the exponential dichotomy of (6.13) are the same as the rates of the undisturbed equation. Hence we find $-\alpha = \beta = |b|$.
Further we observe that the exponential dichotomy (2.2) of the undisturbed equation hold for a constant $K = 1$ and a constant projection P with $\|P\| = 1$. The perturbation matrix satisfies (2.5) with $K_B = |b|\ \frac{k(k-1)}{2}$ and

$$\delta = \begin{cases} -(k-2)\mu^s, & t \geq 0 \\ (k-2)\mu^u t, & t \leq 0 \end{cases}.$$

With this the choice of $t_0^\pm$ and the constants $\tilde{K}_1^\pm, \tilde{K}_2^\pm$ follow from Remark 2.1.9 and Remark 2.1.10.

In [Cop78, Lecture 2] Coppel shows how an extention of the exponential dichotomy from a subintervall $[t_0,\infty)$ to the halfline $\mathbb{R}^+$ effects the constants. In fact, according to Coppel, any constants K_1^+, K_2^+ and analogously any constants K_1^-, K_2^- satisfying the inequality

$$K_1^\pm \geq \tilde{K}_1^\pm N_\pm^2 e^{\pm|b|t_0^\pm}, \qquad K_2^\pm \geq \tilde{K}_2^\pm N_\pm^2 e^{\pm|b|t_0^\pm} \tag{6.17}$$

with

$$N_\pm = e^{\pm\int_0^{t_0^\pm}|b|\|B(\tau)\|d\tau} \tag{6.18}$$

are suitable constants for the exponential dichotomy of (6.13) on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. The right-hand side in equation (6.18) can be estimated by

$$e^{\pm\int_0^{t_0^\pm}|b|\|B(\tau)\|d\tau} \leq e^{\pm|b|t_0^\pm} e^{\pm\frac{k(k-1)}{2}|b|\int_0^{\pm\infty}(\gamma_k^1(\tau))^{k-2}d\tau}$$

Using the representation of γ_k^1 given in Remark 6.2.4 we can rewrite the integral terms in N_+ and N_-

and get the solution

$$\left.\begin{array}{lllll} \int\limits_{-\infty}^{0} \left(\gamma_k^1(\tau)\right)^{k-2} d\tau & = & -\int\limits_{0}^{1} H^{-1}\left(-\operatorname{sgn}(b)\sqrt{1-s}\right) ds & = & \frac{2\ln(1+\frac{a}{|b|})}{a(k-2)}, \\ \int\limits_{0}^{\infty} \left(\gamma_k^1(\tau)\right)^{k-2} d\tau & = & \int\limits_{0}^{1} H^{-1}\left(\operatorname{sgn}(b)\sqrt{1-s}\right) ds & = & \frac{-2\ln(1-\frac{a}{|b|})}{a(k-2)}. \end{array}\right\} \tag{6.19}$$

For $a > 0$ the eigenvalue $\mu^s = a - |b|$ is the leading one and therefore we find that

$$\int\limits_{-\infty}^{0} \left(\gamma_k^1(\tau)\right)^{k-2} d\tau \leq \int\limits_{0}^{\infty} \left(\gamma_k^1(\tau)\right)^{k-2} d\tau \quad \text{and} \quad -t_0^- \leq t_0^+.$$

For $a < 0$ the relation signs are inversed.

For our further analysis we assume $a > 0$ and define

$$K(a,b) = \frac{9}{2} e^{3|b|t_0^+} e^{k(k-1)|b| \int\limits_0^\infty \left(\gamma_k^1(\tau)\right)^{k-2} d\tau} = \frac{9}{2} e^{\frac{3|b|}{(k-2)(a-|b|)} \ln\left(\frac{2(k-2)(|b|-a)}{3k(k-1)|b|}\right)} e^{\frac{-2k(k-1)|b|}{a(k-2)} \ln(1-\frac{a}{|b|})}.$$

According to our above considerations we may choose

$$K_1^{\pm} = K_2^{\pm} = K(a,b).$$

Further, using $|\gamma_k^i(t)| \leq 1$, $i = 1,2$ we find $\sup\limits_{t\in\mathbb{R}} \|A(t)\| = \frac{k^2}{\sqrt{2}}$. Summarizing, from (6.15) with $\alpha = \beta = |b|$, $K_1 = K_2 = K(a,b)$ and the above estimate of $\|A(t)\|$ we obtain

$$a \leq \frac{|b|}{\sqrt{2}k^2 K(a,b)}. \tag{6.20}$$

From this inequality we gain the desired estimate for a. First we realize that $K(a,b)$ decreases monotonically as $a \to 0$. Consequently we find that for $a \leq \frac{|b|}{r}$, $r \gg 1$

$$\frac{|b|}{\sqrt{2}k^2 K(\frac{|b|}{r}, b)} \leq \frac{|b|}{\sqrt{2}k^2 K(a,b)}.$$

Hence any $a > 0$ with

$$a \leq \min\left\{\frac{|b|}{r}, \frac{|b|}{k^2 K(\frac{|b|}{r}, b)}\right\}, \qquad r \gg 1$$

satisfies the Inequality (6.20). Indeed

$$K(\frac{|b|}{r}, b) = \frac{9}{2} e^{\frac{-3r}{(k-2)(r-1)} \ln(\frac{2(k-2)}{3k(k-1)}(1-\frac{1}{r}))} e^{\frac{-2k(k-1)r\ln(1-\frac{1}{r})}{(k-2)}}$$

does not depend on b any more.

Consider the cases $k = 3$ and $k = 4$. With $r = 1000$ we then find

$$k = 3: \quad a \leq |b| \cdot 0.145 \cdot 10^{-9},$$

$$k = 4: \quad a \leq |b| \cdot 0.221 \cdot 10^{-8}.$$

In order to assess the quality of this estimate, we remark that $\frac{|b|}{\sqrt{2}k^2\mathcal{K}}$, where

$$\mathcal{K} = \lim_{a\to+0} K(a,b) = \frac{9}{2}\left(\frac{3k(k-1)}{2(k-2)}\right)^{\frac{3}{k-2}} e^{\frac{2k(k-1)}{k-2}},$$

is an upper bound for a. For $k=3$ we find $(\sqrt{2}k^2\mathcal{K})^{-1} \approx 0.147\cdot 10^{-9}$ and for $k=4$ we find $(\sqrt{2}k^2\mathcal{K})^{-1} \approx 0.223\cdot 10^{-8}$.

7 Numerical investigation of an example system

In Chapter 6 we have constructed with (6.7) a family of vector fields holding a D_4-equivariant homoclinic cycle. Further we have proven that the system (6.7) satisfies the Hypotheses (H5.1) - (H5.4), apart from (H5.4)(i), whose validity was only proven for extremely small values of $|a|$. Indeed we believe that (H5.4)(i) holds true for much larger values of $|a|$, however this is not covered by our analysis.

During this chapter we wish to investigate system (6.7) numerically in order to check whether it is an example for the statement in Theorem 5.3.3. We start in Section 7.1 with the numerical verification that the leading term $B(\lambda)$, defined in Lemma 4.3.20, is different from zero. In Section 7.2 we try to determine the existence of certain periodic solutions and compare the results with Theorem 5.3.3.

7.1 Checking the sign of $B(\lambda)$ by using Matlab

In Section 5.2 we discussed analytically the sign of the term $B(\lambda)$. Thereby the question of whether the term can possibly disappear was not completely answered. In this section we show numerically that there at least exist example systems, like (6.7), where $B(\lambda)$ is different from zero.

The term $B_i(\lambda,\kappa)$ is given as an improper integral on the interval $(-\infty,0]$, cf. Lemma 4.3.20. Recall in this respect that the value of $B(\lambda)=B_i(\lambda,\kappa)$ is for any $\kappa\in\Sigma_{4m}$ independent of $i\in J_\kappa$, cf. Lemma 5.2.6. Its integrand is the scalar product of two terms multiplied by an converging exponential factor. Now, the idea of the numerical considerations is to approximately calculate the integrand of B_i for $\lambda=0$ and display it over the time $t<0$. We will observe that the integrand does not change sign and hence the integral can not become zero.

For $\lambda=0$ the term B_i reads, cf. Lemma 4.3.20,

$$B_i(0,\kappa):=\frac{1}{2}\int\limits_{-\infty}^{0} e^{2\mu^s s}\left\langle \Phi_{\kappa_i}(0,s)^T P^-_{\kappa_i}(0,0)^T\psi_{\kappa_i}, D_1^2 f(\gamma_{\kappa_i}(0)(s),0)\left[R^-_{\kappa_i}(0,s)\eta^s_{\kappa_{i-1}}(0)\right]^2\right\rangle ds,$$

where γ_{κ_i} is one of the homoclinic solutions, $\Phi_{\kappa_i}(0,s)$ is the transition matrix of the variational equation

$$\dot{x}=D_1 f(\gamma_{\kappa_i}(t),0)x \tag{7.1}$$

and $P^-_{\kappa_i}(0,0)$ is the corresponding projection of the exponential dichotomy mapping onto $W^+_{\kappa_i}\oplus Z_{\kappa_i}$ along $T_{\gamma_{\kappa_i}(0)}W^u(p)$, cf. (3.17).

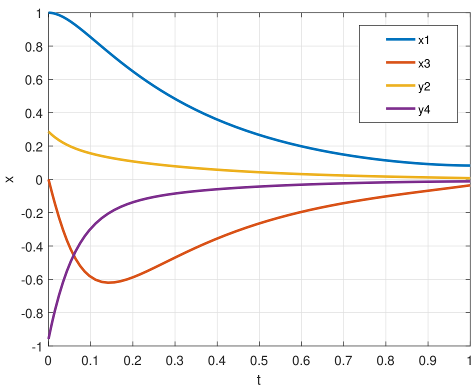
From Lemma 4.3.17 we know that $R^-_{\kappa_i}(0,s)\eta^s_{\kappa_{i-1}}(0)$ is an element of $\mathrm{im}P^-_{\kappa_i}(0,s)\cap\mathrm{Fix}^\perp_{\kappa_i}$ that is transported backwards in time via the transition matrix $\Phi_{\kappa_i}(s,t)$, $t\le s\le 0$. For $s=0$ we therefore find that $R^-_{\kappa_i}(0,0)\eta^s_{\kappa_{i-1}}(0)\in W^+_{\kappa_i}$. Hence $R^-_{\kappa_i}(0,\cdot)\eta^s_{\kappa_{i-1}}(0)$ is a solution of (7.1) with initial value $x(0)\in W^+_{\kappa_i}$. Now, the basic problem in calculating $R^-_{\kappa_i}(0,s)\eta^s_{\kappa_{i-1}}(0)$ is, that we do not have a representation of the subspace $W^+_{\kappa_i}$. However, $W^+_{\kappa_i}$ is a subspace of $T_{\gamma_{\kappa_i}(0)}W^s(p)$ and hence we know that any solution of (7.1) that starts in $W^+_{\kappa_i}$ converges to zero for $t\to\infty$, cf. Lemma 2.2.6.

Now, this leads to the first step for calculating $B(0)$: the determination of $W^+_{\kappa_i}$. Due to the symmetry of the system it suffices to consider the case $\kappa_i=1$. To this end we consider the system of differential equations

$$\left.\begin{aligned}\dot{x} &= f(x,0),\\ \dot{y} &= D_1 f(x,0)y,\end{aligned}\right\} \tag{7.2}$$

with $(x,y):\mathbb{R}\to\mathbb{R}^4\times\mathbb{R}^4$ where the vector field f is given by (6.7). Exemplarily we have chosen the parameters $a=2$ and $b=5$. With this the leading eigenvalues at $\lambda=0$ read $\mu^s(0)=-3$ and $\mu^u(0)=7$. With the initial point $x_0=(1,0,0,0)$ we ensure that the solution $x(\cdot)=\gamma_1(\cdot)$. Thus the second part of the system (7.2) represents the variational equation (7.1) along γ_1. Then, for a certain $\varphi\in[0,\pi]$ the solution $y(\cdot)$ with $y_0=y(0)=(0,\cos(\varphi),0,\sin(\varphi))\in \mathrm{Fix}_1^\perp$ converges to zero for $t\to\infty$. In that case $y(0)$ represents the direction of W_1^+.

When solving the initial value problem (7.2) with $x(0)=(1,0,0,0)$ and $y(0)=(0,\cos(\varphi),0,\sin(\varphi))$ numerically with a Matlab ODE-solver the solution for x will not be exactly the homoclinic one, but only one within a neighbourhood. Also the solution for y will only be an approximation. Therefore, we restrict the numerical investigation to finite intervals in which the solutions still go monotonically towards zero. We have minimized the sum of the squares of the solution components of y at the end point of the interval depending on the starting angle φ. That way we have approximated the value of $\varphi^*=1.86091$ for which the y-components are closest to zero. On the left hand side in Figure 7.1 the non-zero components - namely these are x_1,x_3,y_2 and y_4 - of the obtained solutions for φ^* are displayed over the time interval $[0,1]$. Recall in this respect that the other components remain zero for all time, since we start with (x_0,y_0) in the product space of the fixed point spaces Fix_1 and Fix_2 which is invariant with respect to the flow of (7.2).



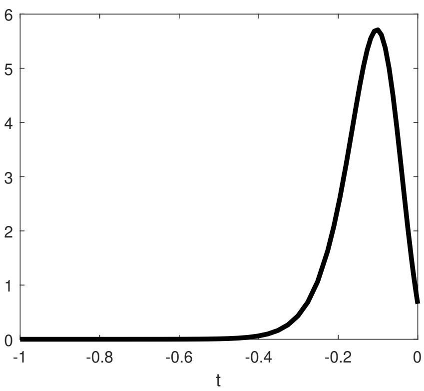


Figure 7.1: Non-zero solution components of the initial value problem (7.2), $x(0)=(1,0,0,0)$ and $y(0)=(0,\cos(\varphi^*),0,\sin(\varphi^*))$ with $\varphi^*=1.86091$ over the time $t\in[0,1]$ (left), integrand term of $B(0)$ displayed over the time $t\in[-1,0]$ (right)

After we have determined the direction of W_1^+ we go on with displaying the integrand term of $B_1(0)$. To this end we now consider the system

$$\left.\begin{aligned}\dot{x} &= f(x,0),\\ \dot{y} &= D_1f(x,0)y,\\ \dot{z} &= -[D_1f(x,0)]^Tz,\end{aligned}\right\}\qquad(7.3)$$

for $t<0$. Again with $x_0=(1,0,0,0)$ we obtain $x(\cdot)=\gamma_1(\cdot)$. With $y_0=(0,\cos(\varphi^*),0,\sin(\varphi^*))$, φ^* as determined above, we obtain with $y(\cdot)$ a solution that lies within $\mathrm{span}\{R_1^-(0,\cdot)\eta^s_{\kappa_{i-1}}(0)\}$, $\kappa_{i-1}\in\{2,4\}$. Note that the solution y and $R_1^-\eta^s_{\kappa_{i-1}}$ may differ in size, since we do not know the precise value of $\|R^-_{\kappa_i}(0,0)\eta^s_{\kappa_{i-1}}(0)\|$. Finally with $z_0=(1,0,0,0)\in Z_1$ we calculate the term $\Phi_1(0,\cdot)^TP_1^-(0,0)^T\psi_1$, since $\psi_1\in Z_1$ is transported backwards in time by the transition matrix $\Psi(t,0)=\Phi(0,t)^T$ of the adjoint variational equation $\dot{x}=-[D_1f(\gamma_1,0)]^Tx$.

Again we solve this initial value problem with Matlab, this time on a negative time interval. After that we build the weighted scalar product from the obtained solution in the following form

$$e^{2\mu^s t}\langle z(t), D_1^2 f(x(t),0)[y(t),y(t)]\rangle,$$

and plot the result over time $t < 0$, cf. the right side in Figure 7.1. It is obvious that the integral of the displayed curve over time is different from zero. In fact, it turns out that for the system (6.7) $B(0)$ has a positive sign. Since $B(\cdot)$ is continuous in $\lambda = 0$, the value $B(\lambda)$ is different from zero for all $|\lambda|$ sufficiently small.

7.2 Finding certain periodic solutions by using AUTO

In this section we introduce a numerical method for looking for periodic trajectories within the neighbourhood of a homoclinic cycle by using the software package AUTO. The idea behind the method is neither new nor innovative. We merely want to confirm the analytic results by verifying the existence of certain periodic trajectories numerically using the example system (6.7). In particular we investigate the existence of the trajectories that are determined by the following periodic sequences κ:

$$\overline{12},\ \overline{123},\ \overline{121},\ \overline{1243},\ \overline{1234} \text{ and } \overline{1214}.$$

To this end we simply used existing AUTO demo-files from [DoeOld12] by adapting them adequately. The scripts were written in collaboration with David Lloyd (personal contact, April 2014). Especially we want to stress that our method is not a numerical implementation of Lin's method. On the contrary it is fitted into the specific situation of a D_4-equivariant vector field having 2-dimensional flow invariant fixed point spaces.

For an introduction to the usage of AUTO we refer to the tutorial [SaLl12]. AUTO is a continuation code that finds solution curves of systems of the form $F(U) = 0$, where $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a given smooth function, starting from an initial value U^* by basically using Newton's method. The problem also can be given as boundary-value problem of the form

$$\left.\begin{aligned} \frac{du}{dt} &= f(u,\lambda), \quad 0 < t < 1, \quad (u,\lambda) \in \mathbb{R}^n \times \mathbb{R}^p \\ g(u(0),u(1),\lambda) &= 0, \qquad g \text{ maps into } \mathbb{R}^{n_{bc}} \\ \int_0^1 h(u(t),\lambda)dt &= 0, \qquad h \text{ maps into } \mathbb{R}^{n_{int}}. \end{aligned}\right\} \tag{7.4}$$

A necessary condition for the correct operating of the continuation - apart from a sufficiently accurate initial guess - is the validity of the following equation

$$n + p - (n_{bc} + n_{int}) = 1. \tag{7.5}$$

That is the dimension of the phase and parameter space minus the number of the boundary conditions needs to be one, so that there remains one extra dimension for continuation.

7.2.1 Idea and Implementation

Our approach is divided into three steps. First we look for initial data as approximation to each homoclinic solution $\gamma_1, \ldots, \gamma_4$ the homoclinic cycle Γ consists of. We need those for both fixed $\lambda > 0$ and fixed $\lambda < 0$, λ sufficiently small.
Proceeding from these initial data we can build the pathway of a trajectory for any periodic sequence κ by attaching the initial data to each other. Thereby we obtain jumps at each coupling point. In contrast to Lin's method the jumps appear near the equilibrium. In the second step we try to get all the jumps equal to zero. The idea for the procedure in this second step is due to Thomas Wagenknecht (personal contact, January 2011).
Finally, if the second step was successful for a fixed κ (for either λ greater or smaller than zero), we need to check that the obtained solution really is a periodic trajectory of the system. We do this by simply continuing this trajectory in the parameters λ and the time period T.

During the whole numerical investigation we consider the vector field (6.7) with the parameters $a = -2$ and $b = -5$ which implies for $\lambda = 0$ the eigenvalues $\mu^s(0) = -7$ and $\mu^u(0) = 3$. Hence, in contrast to our analytic considerations within the previous sections, the unstable eigenvalue μ^u is the leading eigenvalue since it lies closer to the imaginary axis. In fact, we simply achieve a time reversal by this choice of parameters, because $f_4(x, \lambda; -a, -b) = -f_4(x, -\lambda; a, b)$. So this only changes the flow directions of the trajectories which means that our analytical results still apply on this example - appart from a reversal of the sign of λ. The choice of the parameters a and b was made in view of the stability of the homoclinic trajectories for positive time. To be precise, any solution starting in a fixed point space within the inner neighbourhood of a homoclinic trajectory (for $\lambda = 0$) converges towards that homoclinic trajectory. In contrast to the situation depicted in Figure 6.2, we find with this constellation that the 1-periodic trajectories ($\kappa \in \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$) are within the fixed point spaces stable and appear for $\lambda > 0$, the 2-periodic trajectories ($\kappa \in \{\overline{13}, \overline{24}\}$) that trace the figure-eight shape are also stable within the fixed point spaces and appear for $\lambda < 0$, cf. [SSTC01, Theorem 13.11]. Recall on that behalf that the fixed point spaces are 2-dimensional.

Within each of the three steps we consider the task as a boundary value problem (7.4). All of them are solved with the continuation package AUTO in the flavour Auto07p, [DoeOld12]. As is common, we consider the vector field in the time rescaled form

$$\dot{u} = Tf(u, \lambda),$$

where any trajectory segment is parameterized over the unit interval $[0, 1]$. The associated integration time T appears as the separate parameter $PAR(11)$.

Step one - Generating initial data

First we need to find initial data as approximation to the homoclinic solutions γ_1 to γ_4 for fixed λ. To this end we use a simple shooting method that is used for finding saddel-node connections as it can be find in [DoeOld12, Section 18.1 AUTO Demos: fsh]. That is, we solve the differential equation $\dot{u} = Tf(u, \lambda)$, $u \in \mathbb{R}^4$, f given by (6.7), for either $\lambda = 1e^{-5}$ or $\lambda = -1e^{-5}$ starting near the equilibrium within the 1-dimensional intersection of the unstable subspace and one of the fixed point spaces. Then we follow the solution curve with the aim to approach again the equilibrium.

To be more precise, we solve the boundary value problem $\dot{u} = Tf(u, \lambda)$ with

$$(i)\ f(u(1) + \varepsilon v, \lambda) = 0, \qquad (ii)\ D_1 f(u(1) + \varepsilon v, \lambda)v - \mu v = 0 \quad \text{and} \quad (iii)\ \|v\| - 1 = 0, \tag{7.6}$$

with the further parameters $\mu \in \mathbb{R}$ and $v \in \mathbb{R}^4$ and a constant $\varepsilon \in \mathbb{R}$ close to zero. Initially the parameter μ is set equal to the unstable eigenvalue $\mu^u(\lambda)$ and v is set equal to one of the two corresponding eigenvectors v^u lying within one of the fixed point spaces. Further we start with the initial guess $u^* \equiv -\varepsilon v^u$ and a sufficiently small integration time T. Then we ask AUTO to continue the solution in the parameter T, whose absolute value shall be increasing, and the additional parameters μ and v. With the dimension of the phase space $n = 4$, the number of continuation parameters $p = 6 = \dim(T) + \dim(\mu) + \dim(v)$ and the numbers of boundary conditions $n_{bc} = 9$ and $n_{int} = 0$, cf. (7.6), the necessary condition (7.5) is satisfied.

For sufficiently small integration time T and $|\varepsilon|$, the initial guess of $u \equiv -\varepsilon v^u$ is sufficiently accurate to satisfy the boundary-value problem. In the following continuation AUTO is searching for a solution along the same trajectory by increasing the time $|T|$. To this end $u(1)$ remains fixed and finally after some iterations we reach with $u(0)$ the neighbourhood of the equilibrium again. So effectively, only the integration time T changes during the continuation. Due to the invariance of the fixed point spaces with respect to the flow of the differential equation we remain in the same fixed point space we have started in. Hence we have traced the pathway of one of the homoclinic trajectories. By varying the fixed point space we start in and the sign of the constants ε we determine the homoclinic solution we are tracing. Figure 7.2 displays the results of this step for $\lambda < 0$.

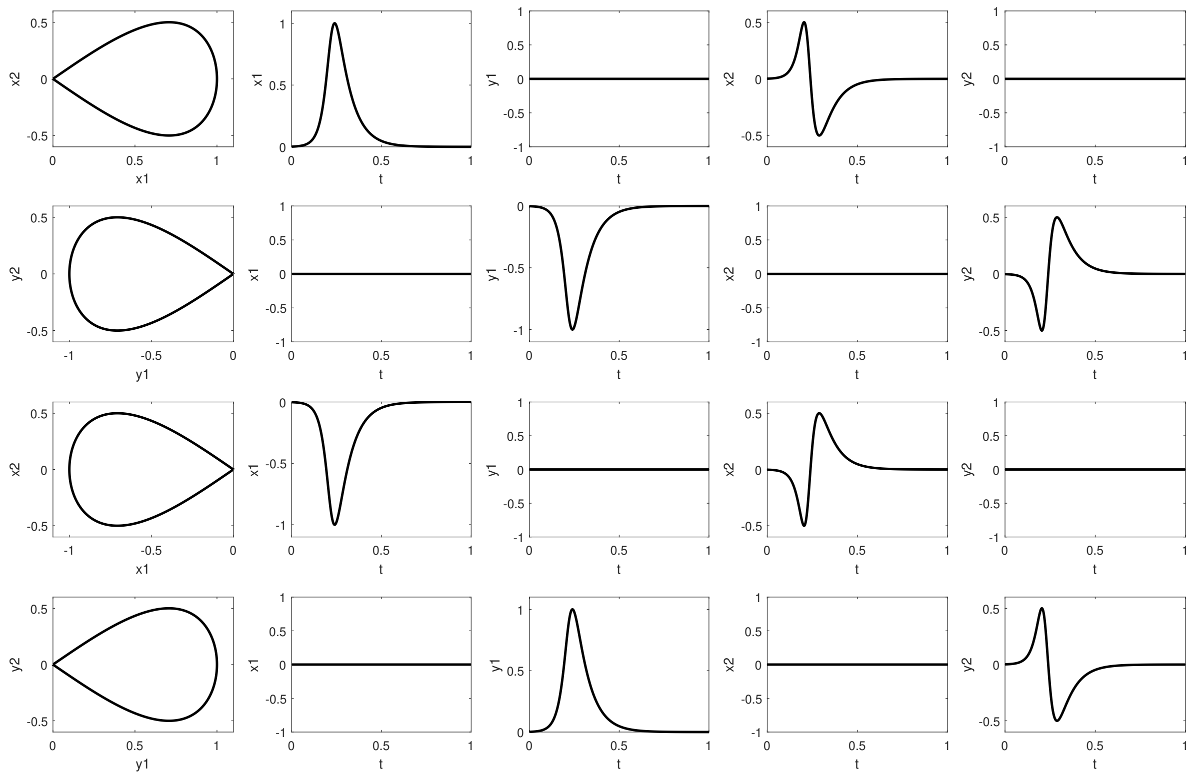


Figure 7.2: Approximations of the homoclinic trajectories $\gamma_1, \ldots, \gamma_4$ for $a = -2$, $b = -5$ and $\lambda = -1e^{-5}$. In the first row the solutions within the fixed point spaces are shown, followed in the other rows by each corresponding component $(x1, y1, x2, y2)$ plotted over the time $t \in [0, 1]$.

Step two - Continuation for finding periodic solutions

We start with the initial data generated in step one. These initial data are given for fixed λ as matrices $[t\ \ x_1^*], \ldots, [t\ \ x_4^*]$ of dimension $D \times (1+4)$, $t \in [0,1]$, where D is the number of the calculated time-steps. Due to the symmetry of the system the time column is identical in all four cases. So is the integration time T^*. Now, to find a N-periodic orbit given by the sequence κ, $\kappa_i \in \{1,2,3,4\}$, we put the starting solutions $[t\ \ x_{\kappa_i}^*]$, $i \in \{1,\ldots,N\}$, together to a $D \times (1+4N)$ matrix of the form $[t\ \ x_{\kappa_1}^*\ \ldots\ x_{\kappa_N}^*]$. Then we continue from this initial solution trying to find the N-periodic orbit. That is, we solve for all $i \in \{1,\ldots,N\}$ the boundary value problem

$$\dot{x}_{\kappa_i} = Tf(x_{\kappa_i}, \lambda)$$

with the $4N$ boundary conditions

$$x_{\kappa_i}(1) = x_{\kappa_{i+1}}(0).$$

Recall that AUTO automatically scales the time to the interval $[0,1]$.

In order to solve this task we set $x_{\kappa_i} = (u_{4(i-1)+1}, \ldots, u_{4(i-1)+4}) \in \mathbb{R}^4$, $i \in \{1,\ldots,N\}$ and set the initial time value $PAR(11) = T^*$. Then we further introduce the $4N$ parameters $PAR(12), \ldots, PAR(4N+11)$, whose initial values are given by

$$\begin{array}{lcl} PAR(12) & = & u_1^*(1) - u_5^*(0) \\ PAR(13) & = & u_2^*(1) - u_6^*(0) \\ & \vdots & \\ PAR(4N-1+11) & = & u_{4N-1}^*(1) - u_3^*(0) \\ PAR(4N+11) & = & u_{4N}^*(1) - u_4^*(0). \end{array}$$

Hence in the beginning the parameters $PAR(12), \ldots, PAR(4N+11)$ are different from zero and we write the $4N$ boundary conditions as follows

$$u_1(1) - u_5(0) - PAR(12) = 0, etc.$$

Note that the dimension of the system here is $n = 4N$. With $n_{bc} = 4N$ and $n_{int} = 0$ we need exactly one parameter ($p = 1$) to satisfy (7.5). Then each parameter can be chosen as single continuation parameter.

Now, through continuation we get the parameters $PAR(12), \ldots, PAR(4N+11)$ step by step to become zero. At first we start from the initial guess $[t\ u^*] = [t\ \ x_{\kappa_1}^*\ \ldots\ x_{\kappa_N}^*]$ and choose one parameter as continuation parameter. After the continuation was successful we use the obtained solution u_{new}^* as new initial guess and replace the initial value of the corresponding parameter with the new value which is now close to zero. After that we choose the next parameter for continuation and repeat the procedure.

Obviously the order of the parameters is very important. The following order did work for our examples

$$\begin{array}{l} PAR(12),\ PAR(16),\ \ldots,\ PAR(4(N-1)+1+11), \\ PAR(13),\ PAR(17),\ \ldots,\ PAR(4(N-1)+2+11), \\ PAR(14),\ PAR(18),\ \ldots,\ PAR(4(N-1)+3+11). \end{array}$$

Now, since we are searching for N-periodic trajectories at fixed λ the time period T needs to be adjusted. That means, when it finally comes to the last N parameters $PAR(15),\ \ldots,\ PAR(4N+11)$ we need to consider the time period $PAR(11)$ as continuation parameter as well. Otherwise we will not get the remaining parameters to become zero. However, adding another continuation parameter requires another boundary or integral condition so that (7.5) still holds true. To this end we include the usual phase condition as integral condition,

$$\int_0^1 \langle \dot{u}_{old}(t), u(t) - u_{old}(t) \rangle dt = 0 \tag{7.7}$$

additionally to the other boundary conditions. Here u_{old} denotes the solution of the previous continuation step, or the initial guess at the beginning of the continuation.
Then we continue in $PAR(11)$ and one of the remaining N parameters $PAR(15), \dots, PAR(4N+11)$ to the end that $PAR(15), \dots, PAR(4N+11)$ become zero. We do this exactly in the same way as before, only this time we also need to update the initial value for the parameter of integration time $PAR(11)$ after each successful continuation.

If each continuation in AUTO was successful and all $4N$ parameters $PAR(12), \dots, PAR(4N+11)$ are nearly zero, we get a solution of the form $[t \;\; x_{\kappa_i} \;\; \dots \;\; x_{\kappa_N}]$ with $t \in [0,1]$ and a corresponding time period T. Now we can rewrite this solution into

$$\begin{bmatrix} T \cdot t & x_{\kappa_1} \\ T(t+1) & x_{\kappa_2} \\ \vdots & \vdots \\ T(t+(N-1)) & x_{\kappa_N} \end{bmatrix} \tag{7.8}$$

and due to the construction it should represent a periodic solution of (6.7), defined by the sequence κ.

Step three - Verification of the solution

Finally we need to verify the existence of the found trajectory. Therefore we continue solution (7.8) in λ and the period NT to make sure the orbit really does exist. To be more precise we consider the problem $\dot{u} = NTf(u, \lambda)$, $u \in \mathbb{R}^4$ with the boundary conditions $u_i(1) - u_i(0) = 0$ for all $i \in \{1,2,3,4\}$. The initial values for T and the function u are given by the final result of step two. Since the dimension of the system is again $n = 4$ and we consider $n_{bc} = 4$ boundary conditions and wish to continue in $p = 2$ parameters we need a further integral condition to satisfy (7.5). Again we use the phase condition (7.7).

If the solution (7.8) of step two is indeed a periodic solution of (6.7) we will be able to continue it in the parameters λ and T and with decreasing $|\lambda|$ the time period T should be increasing and the periodic solution is growing closer to the homoclinic cycle. If (7.8) is not a solution of (6.7), AUTO will break up the continuation immediately.

7.2.2 Results

Figure 7.2 displays the results of the first step for $\lambda < 0$. Also for $\lambda > 0$ we were able to trace the pathway of the homoclinic trajectories in order to generate initial guesses for the second step.

Starting from these solutions we tried to find the periodic trajectories corresponding to the sequences $\kappa \in \{\overline{1},\ \overline{13},\ \overline{12},\ \overline{121},\ \overline{123},\ \overline{1234},\ \overline{1243},\ \overline{1214}\}$.

We did start with the trivial periodic trajectories $\overline{1}$ and $\overline{13}$ whose existence is already known and which are situated in the 2-dimensional fixed point space Fix_1. For $\lambda < 0$ we found the trajectories $\overline{13}$ and for $\lambda > 0$ we found $\overline{1}$, cf. Figure 7.3.

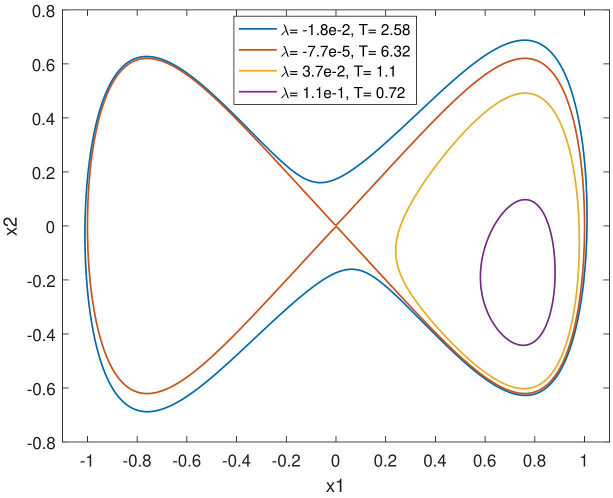


Figure 7.3: 1-periodic trajectories $\kappa = \overline{1}$ (violet and yellow) and 2-periodic trajectories $\kappa = \overline{13}$ (blue and red) for $a = -2$, $b = -5$ displayed for several λ and corresponding transition times T

At $\lambda < 0$ we did find the periodic trajectories $\overline{12}$, $\overline{123}$, $\overline{1234}$, $\overline{1243}$ and $\overline{1214}$. That is, we could in each case complete step two, where the jumps at each coupling point was calculated to be of the order $1e-12$. Further, after feeding the results into step three we were able to continue the data in the parameters λ and T. Therefore we conclude that the named periodic trajectories do exist for $\lambda < 0$. Figure 7.4 exemplarily displays the obtained result for the periodic trajectory $\kappa = \overline{1243}$.

What is more, we failed to find these trajectories for $\lambda > 0$. To be precise, we already had to break up during the continuations in step two. The first $3N-1$ parameters we could continue till they became zero. But after that AUTO failed to satisfy the remaining boundary conditions. We also tried some different ordering of the parameters $PAR(12), \ldots, PAR(4N+11)$ and we even increased the values for the convergence criteria but still AUTO did not get all the parameters to become zero.

The same was true in case of the 3-periodic trajectory given by $\kappa = \overline{121}$, only here we failed to find it for both, $\lambda > 0$ and $\lambda < 0$.

Indeed these results correspond to our expectations based on Theorem 5.3.3. In Section 6 we have already proven that the system (6.7) satisfies the Hypotheses (H5.1) - (H5.4), apart from (H5.4)(i), whose validity was only proven for extremely small values of $|a|$. In Section 7.1 we further verified numerically that $B(0) \neq 0$ for this system. The numeric results now show that additionally the relation $|B(\lambda)| > C(\lambda)$ holds true, since the periodic trajectory $\kappa = \overline{12} \in \mathcal{K}_2$ exists for the same sign of λ as $\kappa = \overline{1234} \in \mathcal{K}_4$, cf. Remark 5.3.6. Further we can once more conclude that the sign of $B(\lambda)$ is positive, since the periodic trajectory $\kappa = \overline{1243}$ exists instead of for example the trajectory $\kappa = \overline{121}$. Therefore we obtain for the in Theorem 5.3.3 mentioned matrices A_- and A_+

$$A_- = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Due to the fact that in our numerical considerations the unstable eigenvalue is the leading one the assignment of the matrices A_- and A_+ is reversed to that in (1.6).

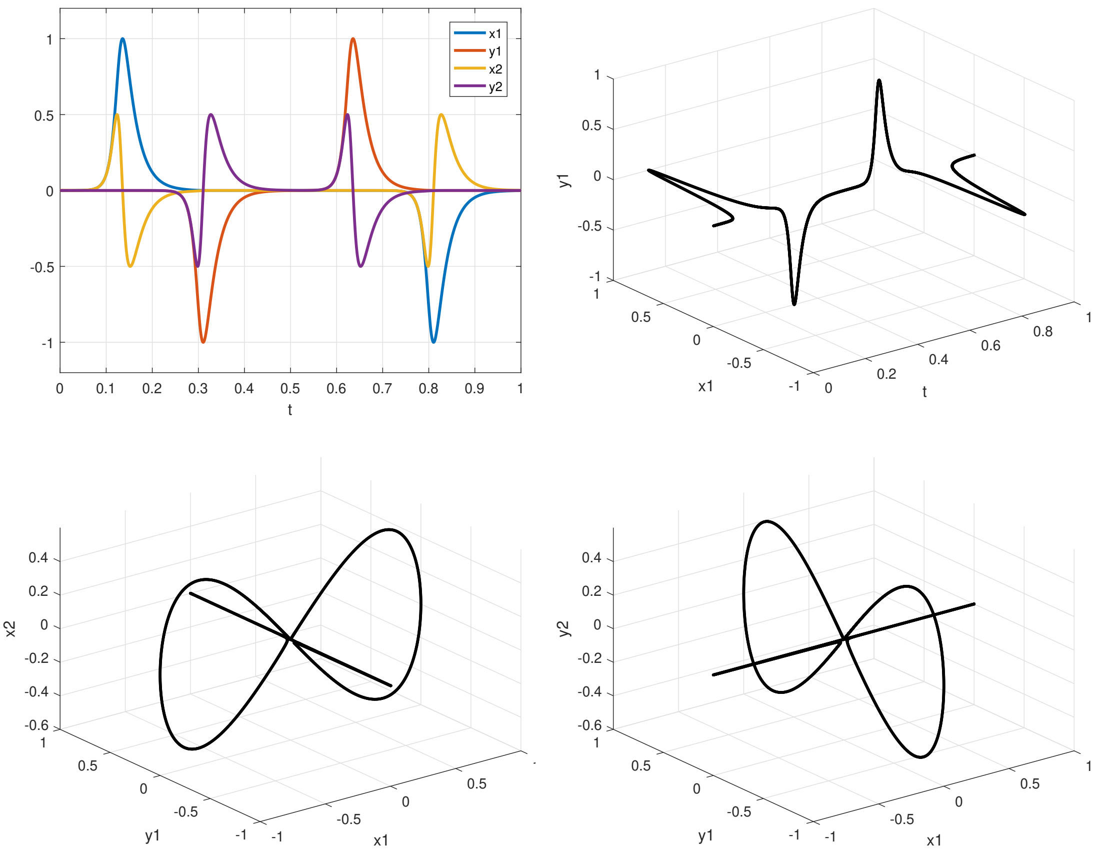


Figure 7.4: 4-periodic trajectory $\kappa = \overline{1243}$ for $a = -2$, $b = -5$ and $\lambda = -1e^{-5}$

There is another result that is verified by the numeric calculations: when examining the top left panel of Figure 7.4 one easily notices the different transition times that are needed to move from one homoclinic trajectory to another one. We see that it takes around twice the time when staying in the same fixed point space than to move to a homoclinic trajectory lying in an orthogonal fixed point space. This conforms with the calculated transition times in (5.40).

List of Notations

$\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$	set of all complex, real, natural numbers and integer
$\mathbb{L}(\mathbb{R}^n, \mathbb{R}^n)$	set of all linear mappings $A : \mathbb{R}^n \to \mathbb{R}^n$
$C^l(X, Y)$	set of all l-times continuous differentiable functions $f : X \to Y$, $l \in \mathbb{N} \cup \{0\}$
$\langle \cdot, \cdot \rangle$	scalar product in $\mathbb{R}^n$, invariant with respect to G
$\vert \cdot \vert$	absolute value of an element in $\mathbb{R}$
$\Vert \cdot \Vert$	norm of an element in $\mathbb{R}^n$ induced by the scalar product, or matrix norm in $\mathbb{R}^{n\times n}$ induced by the vector norm in $\mathbb{R}^n$, or supremum norm of an elements $x \in C^0(I \subset \mathbb{R}, \mathbb{R}^n)$ over a finite interval $I \subset \mathbb{R}$: $\Vert x\Vert := \sup_{t\in I} \Vert x(t)\Vert$
A^T	transposed of a linear mapping A with respect to $\langle \cdot, \cdot \rangle$
$U^\perp$	orthogonal complement of a linear subspace $U \subseteq \mathbb{R}^n$ with respect to $\langle \cdot, \cdot \rangle$
$\mathrm{im} A, \ker A$	image and kernel of a linear mapping A
$\sigma(A)$	spectrum of A, that is set of all (complex) eigenvalues μ of A
σ^c_μ	complementary spectrum of A with respect to μ, $\sigma^c_\mu := \sigma(A) \setminus \{\mu\}$
$E_A(\mu), E_A(\sigma)$	generalized eigenspace of A with respect to the eigenvalue μ or the specrum σ, respectively, sometimes $E(\mu), E(\sigma)$ for short

λ	system parameter
$f(\cdot, \lambda)$	vector field
$D_1 f(\cdot, \lambda)$, $D_1^k f(\cdot, \lambda)$	first and k-th derivativ of f with respect to the first argument, $k \in \mathbb{N}, k > 2$, sometimes $Df(\cdot)$ and $D^k f(\cdot)$ for short
p	hyperbolic equilibrium
ν	natural number satisfying $D_1^k f(p, \lambda) = 0$ and $D_1^\nu f(p, \lambda) \neq 0$ for all $k \in \{0, \ldots, \nu - 1\} \setminus \{1\}$, Definition 3.4.2
$\mu^s(\lambda)$, $\mu^u(\lambda)$	leading stable and unstable eigenvalue of $D_1 f(p, \lambda)$
α^s, α^u, α^{ss}, α^{uu}, β^s, β^u	real constants satisfying $\alpha^{ss} < \beta^s < \mathrm{Re}(\mu^s(\lambda)) < \alpha^s < 0 < \alpha^u < \mathrm{Re}(\mu^u(\lambda)) < \beta^u < \alpha^{uu}$
α^w	real constant, (3.90)

$W^s(p)$, $W^u(p)$	stable and unstable manifold of equilibrium p, (2.33)
$W^s_{loc}(p)$, $W^u_{loc}(p)$	local stable and unstable manifold of equilibrium p, $W^{s/u}_{loc}(p) := W^{s/u}(p) \cap U(p, \varepsilon)$ for some neighbourhood U of p, $\varepsilon > 0$

$W^{ss}(p)$, $W^{uu}(p)$	strong stable and strong unstable manifold of equilibrium p, (2.42)
$W^{s,lu}(p)$, $W^{ls,u}(p)$	extended stable and unstable manifold of equilibrium p, Remark 2.2.8
$\mathcal{F}^{ss}(x_0)$	strong stable fibre in $x_0 \in W^s(p)$ within the strong stable foliation F^{ss}, Remark 2.2.9, Figure 2.1
$T_q\mathcal{M}$	tangent space of the manifold $\mathcal{M}$ in point $q \in \mathcal{M}$
$\mathcal{M} \pitchfork \mathcal{P}$	the manifolds $\mathcal{M}$ and $\mathcal{P}$ intersect transversally, Definition 2.6.1
$\mathcal{O}$	orientation index, Corollary 2.6.7, Remark 2.6.8
$\mathcal{F}(W^s_\gamma)$	fibre bundle of tangent directions of the stable manifold, that are complementary to $\dot{\gamma}$, along the homoclinic trajectory γ, Remark 2.6.8
———	———
$\{\varphi^t(\cdot)\}$	the flow of $\dot{x} = f(x)$
$\Phi(\cdot,\cdot)$, $\Psi(\cdot,\cdot)$	transition matrices of the linear equation $\dot{x} = A(t)x$ or $\dot{x} = -[A(t)]^T x$, respectively, possibly $A(t) = Df(\varphi^t(x_0))$ for some $x_0 \in \mathbb{R}^n$
$E^s_{A(\cdot)}(\tau)$, $E^u_{A(\cdot)}(\tau)$	stable and unstable subspace at time τ of $\dot{x} = A(t)x$ defined via a projection corresponding to an exponential dichotomy on $\mathbb{R}^+$ or $\mathbb{R}^-$, respectively, Definition 2.1.5, Lemma 2.2.3
$E^{ss}_{A(\cdot)}(\tau)$, $E^{uu}_{A(\cdot)}(\tau)$	strong stable and strong unstable subspace at time τ of $\dot{x} = A(t)x$ defined via a projection corresponding to an exponential dichotomy on $\mathbb{R}^+$ or $\mathbb{R}^-$, respectively, Lemma 2.2.7
———	———
$\gamma(\cdot)$	solution of $\dot{x} = f(x,\lambda)$ homoclinic to p
Γ	homoclinic cycle $\Gamma = G(\overline{\gamma})$
κ	biinfinite sequence of natural numbers
Γ^κ	heteroclinic chain defined by κ, Definition 1.0.3
$\alpha(\gamma)$	α-limit of the heteroclinic connection γ: $\alpha(\gamma) = \lim\limits_{t\to-\infty} \gamma(t)$
$\omega(\gamma)$	ω-limit of the heteroclinic connection γ: $\omega(\gamma) = \lim\limits_{t\to\infty} \gamma(t)$
———	———
$\boldsymbol{\omega} = (\omega_i)_{i\in\mathbb{Z}}$	transition time, $\omega_i \in \mathbb{R}^+$
$\Xi_i(\boldsymbol{\omega},\lambda,\kappa)$	jump, $\Xi_i = X_{i+1}(-\omega_{i+1}) - X_i(\omega_i)$
$\xi^\infty_{\kappa_i}(\lambda)$	first part of jump Ξ_i, measures the distance of the stable and unstable manifold
$\xi_i(\boldsymbol{\omega},\lambda,\kappa)$	second part of jump Ξ_i, exponentially small with increasing $\inf \boldsymbol{\omega}$
$\mathcal{M}$	Melnikov integral, (3.60)
$\boldsymbol{T}^{\boldsymbol{j}}_{\boldsymbol{\kappa_i}}$	$j = 1,2$, equation (3.64) et seqq
$R_i(\boldsymbol{\omega},\lambda,\kappa)$	residual terms of $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$
———	———

$\gamma^{\pm}_{\kappa_i}(\lambda)(\cdot)$	particular solution of $\dot{x} = f(x,\lambda)$ within the stable or unstable manifold of $p_{\kappa_{i+1}}$ or p_{κ_i}, respectively, Lemma 3.1.1, Figure 3.1
$\Phi^{\pm}_{\kappa_i}(\lambda)(\cdot,\cdot)$	transition matrix of the variational equation $\dot{x} = D_1 f(\gamma^{\pm}_{\kappa_i}(\lambda)(t),\lambda)x$
$\Psi^{\pm}_{\kappa_i}(\lambda)(\cdot,\cdot)$	transition matrix of the adjoint variational equation $\dot{x} = -[D_1 f(\gamma^{\pm}_{\kappa_i}(\lambda)(t),\lambda)]^T x$
$P^{\pm}_{\kappa_i}(\lambda,\omega_i)$	projection of exponential dichotomy of $\dot{x} = D_1 f(\gamma^{\pm}_{\kappa_i}(\lambda)(t),\lambda)x$, (3.16), (3.17)
$\tilde{P}_{\kappa_i}(\lambda,\cdot)$	projection onto $\mathrm{im}P^{+}_{\kappa_{i-1}}(\lambda,\omega_i)$ along $\mathrm{im}P^{-}_{\kappa_i}(\lambda,-\omega_i)$, Lemma 3.3.2
$S_{\kappa_i}(\lambda,\omega_i)$	invertible linear mapping used for defining $\tilde{P}_{\kappa_i}$, (3.30)
$P_{\kappa_i}(\lambda)$	spectral projection of $D_1 f(p_{\kappa_i},\lambda)$, (3.26)
F_{κ_i}	projection onto $U^{+}_{\kappa_i} \oplus Z^{-}_{\kappa_i}$ along $W^{+}_{\kappa_i} \oplus W^{-}_{\kappa_i}$, (3.38)

$U_{\kappa_i}, Z_{\kappa_i}, W^{+}_{\kappa_i}, W^{-}_{\kappa_i}$	subspaces of the direct sum decomposition of $\mathbb{R}^n$, equations (3.4), (3.5)
ψ_{κ_i}	element in Z_{κ_i}, $\lVert\psi_{\kappa_i}\rVert = 1$
$\mathcal{S}_{\kappa_i}$	cross-section of γ_{κ_i}, (3.7)
$\mathbf{X} = (X_i)_{i\in\mathbb{Z}}$	Lin trajectory, $X_i : [0, 2\omega_i] \to \mathbb{R}^n$ solves $\dot{x} = f(x,\lambda)$, Figure 3.2, (3.56)
$x^{\pm}_i(\boldsymbol{\omega},\lambda,\kappa)(\cdot)$	Lin trajectories $X_i(t) := \begin{cases} x^{+}_{i-1}(t), & t \le \omega_i \\ x^{-}_i(t-2\omega_i), & t \ge \omega_i \end{cases}$, Theorem 3.2.2
$v^{\pm}_i(\boldsymbol{\omega},\lambda,\kappa)(\cdot)$	part of $x^{\pm}_i$, Equations (3.11), (3.12), (3.13)
$h^{\pm}_{\kappa_i}(\cdot,v,\lambda)$	non-linearity, (3.14)
$\boldsymbol{\mathcal{H}} = (H^{+}_i, H^{-}_i)_{i\in\mathbb{Z}}$	sequence of Nemyzki operators defined via $h^{\pm}_{\kappa_i}$, Definition 3.3.5
$\mathbf{g}$	$(g^{+}_i, g^{-}_i)_{i\in\mathbb{Z}} \in V_{\boldsymbol{\omega}}$, (3.15)
$\boldsymbol{v}_{\boldsymbol{\omega}}$	element in $V_{\boldsymbol{\omega}}$, Lemma 3.2.4
$\hat{\boldsymbol{v}}_{\boldsymbol{\omega}}$	element in $V_{\boldsymbol{\omega}}$, Lemma 3.2.5
$\bar{\boldsymbol{v}}_{\boldsymbol{\omega}}$	element in $V_{\boldsymbol{\omega}}$, Lemma 3.2.6
$\bar{\boldsymbol{v}}(\boldsymbol{\omega},\lambda,\kappa)$	$\bar{\boldsymbol{v}}(\boldsymbol{\omega},\lambda,\kappa) := \bar{\boldsymbol{v}}_{\boldsymbol{\omega}}(\lambda,\kappa)$
$\boldsymbol{d} = (d_i(\omega_i,\lambda))_{i\in\mathbb{Z}}$	sequence in $l^{\infty}_{\mathbb{R}^n}$ describing the coupling of $x^{+}_{i-1}(\omega_i)$ and $x^{-}_i(-\omega_i)$ in the neighbourhood of p_{κ_i}, (3.13)
$\boldsymbol{a} = (a_i)_{i\in\mathbb{Z}}$	sequence in $l^{\infty}_{\mathbb{R}^n}$ helping to handle the coupling of $x^{+}_{i-1}(\omega_i)$ and $x^{-}_i(-\omega_i)$, Lemma 3.2.4
a^{+}_i, a^{-}_i	$a^{+}_i = P^{+}_{\kappa_{i-1}}(\omega_i)a_i$, $a^{-}_i = P^{-}_{\kappa_i}(-\omega_i)a_i$, (3.34)
$v^{\pm,s}_i$, $v^{\pm,u}_i$, $h^{\pm,s}_{\kappa_i}$, $h^{\pm,u}_{\kappa_i}$	Definition 3.3.4
$v^{\pm,su}_i$, $v^{\pm,ss}_i$	$v^{\pm,s}_i(t) = v^{\pm,ss}_i(t) + v^{\pm,su}_i(t)$, (3.85)
l^{∞}_U	space of all bounded sequences $\mathbf{x} := (x_i)_{i\in\mathbb{Z}}$, $x_i \in U$, endowed with the supremum norm $\lVert\cdot\rVert_{l^{\infty}_U}$, Definition 3.2.1

$V_{\boldsymbol{\omega}}$	space of all sequences $\mathbf{v} := (v_i^+, v_i^-)_{i\in\mathbb{Z}}$ where $v_i^+ \in C([0,\omega_{i+1}],\mathbb{R}^n)$ and $v_i^- \in C([-\omega_i,0],\mathbb{R}^n)$, equipped with the norm $\|\cdot\|_{V_{\boldsymbol{\omega}}}$, Definition 3.2.1
———	———
$GL(n,\mathbb{R})$	general linear group of $\mathbb{R}^n$
G	finite group
$\vartheta(g)$	linear acting (representation) of group element $g \in G$, Definition 4.0.2
D_k	symmetry group of the regular k-gon in the plane
G_q, G_γ	isotropy subgroup of $q \in \mathbb{R}^n$ or of a solution γ, respectively, Definition 4.0.3
$\mathrm{Fix}H$	fixed point space of subgroup $H \subseteq G$, Definition 4.0.3
$G(q)$, $G(A)$	group orbit of $q \in \mathbb{R}^n$ or of $A \subset \mathbb{R}^n$, respectively, Definition 4.0.3
$\mathbb{Z}_k(h)$	cyclic subgroup of an element $h \in G$ of order k, Definition 4.0.4
θ_k	rotation element of D_k
ζ	reflection element of D_k
———	———
$\gamma_{g\kappa_i}$, $U_{g\kappa_i}$, $Z_{g\kappa_i}$, $W^{\pm}_{g\kappa_i}$	shortened notation for $g\gamma_{\kappa_i}$, $gU_{\kappa_i}, gZ_{\kappa_i}, gW^{\pm}_{\kappa_i}$, g an element of a symmetry group G, Section 4.3.1
$\Phi^{\pm}_{g\kappa_i}$, $\Psi^{\pm}_{g\kappa_i}$, $P^{\pm}_{g\kappa_i}$	transition matrices and projections defined with respect to $\gamma_{g\kappa_i} := g\gamma_{\kappa_i}$, Section 4.3.1
———	———
e_i^s, e_i^u	cf. Definition (1.9) and (2.58)
e_i^+, e_i^-	cf. Definition (1.10) and (2.59)
$\eta^s_{\kappa_i}(\lambda), \eta^-_{\kappa_i}(\lambda)$	$\eta^s_{\kappa_{i-1}}(\lambda) \in \operatorname{span}\{e^s_{\kappa_{i-1}}\}$ and $\eta^-_{\kappa_i}(\lambda) \in \operatorname{span}\{e^-_{\kappa_i}\}$
$\mathrm{Fix}_i := \mathrm{Fix}G_{\gamma_i}$	fixed point space of isotropy subgroup $G_{\gamma_i} \subseteq G$
J_κ	index set $J_\kappa := \{j \in \mathbb{Z} \mid \mathrm{Fix}_{\kappa_{j-1}} \perp \mathrm{Fix}_{\kappa_j}\}$
$A_i(\lambda,\kappa)$	leading term of $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$, $A_i(\lambda,\kappa) = \langle \eta^-_{\kappa_i}(\lambda), \eta^s_{\kappa_{i-1}}(\lambda)\rangle$, Lemma 4.3.14
$B_i(\lambda,\kappa)$	first leading term of residual terms of $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$ in case that $i \in J_\kappa$, Lemma 4.3.20
$C_i(\lambda,\kappa)$	second leading term of residual terms of $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$ in case that $i \in J_\kappa$, Lemma 4.3.22
$B(\lambda)$, B, $C_i(\lambda)$, C	shortened notation for $B_i(\lambda,\kappa)$, $B(0)$, $C_i(\lambda,\kappa)$ and $\lvert C_i(0)\rvert$
$S^{\pm}(s)$, $R^{\pm}(s)$	leading terms of $\Phi(\lambda)(t,s)P^{\pm}(\lambda,s)$ or $\Phi(\lambda)(s,t)(id - P^+(\lambda,t))$, respectively, Lemma 2.5.2, Remark 2.5.4
$S^{\pm}_{\kappa_i}(\lambda,s)$, $R^{\pm}_{\kappa_i}(\lambda,s)$	leading terms of $\Phi^+_{\kappa_i}(\lambda)(t,s)(id - P^+_{\kappa_i}(\lambda,s))$ or $\Phi^-_{\kappa_i}(\lambda)(s,t)P^-_{\kappa_i}(\lambda,t)$, respectively, Lemma 4.3.17
$\mathcal{R}^S_{\kappa_i}$, $\mathcal{R}^R_{\kappa_i}$	residual terms corresponding to $S^{\pm}_{\kappa_i}$, $R^{\pm}_{\kappa_i}$, Lemma 4.3.17

$\mathcal{R}^{A_j}_{\kappa_i}$, $\mathcal{R}^{B_j}_{\kappa_i}$, $\mathcal{R}^{C_j}_{\kappa_i}$	$j \in \mathbb{N}$, residual terms appearing in the proofs of Lemmata 4.3.14, 4.3.20, 4.3.22, respectively
$\boldsymbol{C} = (c_{ij})$	connectivity matrix of a heteroclinic network
Σ_A	Markov chain defined by matrix A, Definition 1.0.1
σ	left shift operator on Σ_A, Definition 1.0.1
A_-, A_+	matrices that describe the nonwandering dynamics near the homoclinic cycle via the corresponding Markov chain Σ_{A_+} or Σ_{A_-}, respectively, $A_- = -\frac{1}{2}(M - \lvert M\rvert)$ and $A_+ = \frac{1}{2}(M + \lvert M\rvert)$
$M = (m_{i,j})_{i,j\in\{1,\dots,4m\}}$	matrix for defining A_- and A_+, (5.16)
$\mathcal{K}_2$, $\mathcal{K}_4$	set of all two- or four-periodic κ defining trajectories having only right-angled transitions where C_i never changes sign, (5.9) and (5.10)
Π_λ	first return map on the collection of cross sections $\cup_{j=1}^{k}\mathcal{S}_j$
$\mathcal{D}_\lambda$	subset of $\cup_{j=1}^{k}\mathcal{S}_j$
Φ_λ	homeomorphism that constitutes the topological conjugation between $(\mathcal{D}_\lambda, \Pi_\lambda)$ and (Σ_{A_+}, σ), Section 5.4.3
ϱ, $\hat{\varrho}$	metric in Σ_A or l^∞, respectively, $\varrho(\kappa^1,\kappa^2) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{\lvert i\rvert}}\lvert\kappa_i^1 - \kappa_i^2\rvert$, $\hat{\varrho}(\boldsymbol{\omega}^1,\boldsymbol{\omega}^2) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{\lvert i\rvert}}\lvert\omega_i^1 - \omega_i^2\rvert$
$\hat{\mathbf{r}} = (\hat{r}_i)_{i\in\mathbb{Z}}$, $\mathbf{r} = (r_i)_{i\in\mathbb{Z}}$	shortened notation $\hat{r}_i := e^{2\mu^s(\lambda)\omega_i}$, (5.18), rescaling (5.21) and (5.41) in case that λ is greater or smaller than zero, respectively
$\hat{R}_i(\hat{\mathbf{r}},\lambda,\kappa)$, $\tilde{R}_i(\mathbf{r},\lambda,\kappa)$, $\check{R}_i(\mathbf{r},\lambda,\kappa)$	residual terms of the jump $\xi_i(\boldsymbol{\omega},\lambda,\kappa)$ that arise from $R_i(\boldsymbol{\omega},\lambda,\kappa)$ accordingly, Sections 5.4.1, 5.4.2, $\check{R}_i(\mathbf{r},\lambda,\kappa) := \frac{1}{\lambda}\tilde{R}_i(\mathbf{r},\lambda,\kappa)$
$\boldsymbol{\chi}$	right hand side of (5.22) and (5.23) after factoring out λ interpreted as operator $\boldsymbol{\chi} : l^\infty \times \mathbb{R} \times \Sigma_A \to l^\infty$, $(\boldsymbol{r},\lambda,\kappa) \mapsto \boldsymbol{\chi}(\boldsymbol{r},\lambda,\kappa)$, (5.25)
$\mathbf{r}^\kappa$	solution of $\boldsymbol{\chi}(\boldsymbol{r},\lambda,\kappa) = 0$ at $\lambda = 0$
J_κ^+, J_κ^-	subsets of J_κ satisfying $C_i(0,\kappa) > 0$ or $C_i(0,\kappa) < 0$, respectively, (5.74)
$S_\kappa^+(i)$, $S_\kappa^-(i)$	sequences of consecutive indices $j \in J_\kappa$, $j \geq i$ with $j \in J_\kappa^+$ or $j \in J_\kappa^-$, respectively, (5.75)
f_k, $f_{k,\mathbb{C}}$	constructed vector field in $\mathbb{R}^4$ that is D_k-equivariant and its related vector field in complex coordinates, Section 6.2
$\hat{f}_k$, $\hat{f}_{k,\mathbb{C}}$	constructed vector fields in $\mathbb{R}^2$ as basis for a D_k-equivariant vector fields f_k in $\mathbb{R}^4$ and its related vector field in complex coordinates, Section 6.2
$\mathcal{C}_k$	curves in $\mathbb{R}^2$ whose zero-level set describes the course of a homoclinic loop, Section 6.2.1 and Figure 6.1

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Zusammenfassung in Deutscher Sprache

Kurzbeschreibung

Das Thema dieser Arbeit ist eine detaillierte Beschreibung der Dynamik in der Nähe von D_{4m}-symmetrischen relativen homoklinen Zykeln mit Hilfe von Lins Methode.

Die homoklinen Zykel haben die Kodimension-1, d.h. wir beobachten ihre generische Entfaltung innerhalb einer einparametrigen Familie. Sie bestehen aus mehreren Trajektorien, die sowohl für positive als auch negative Zeit derselben hyperbolischen Gleichgewichtslage zustreben (Homokline Trajektorien) und die alle durch die von einer endlichen Gruppe induzierten Symmetrie voneinander abhängig sind. Wir nehmen reelle führende Eigenwerte und homokline Trajektorien an, die sich der Gleichgewichtslage entlang führender Richtungen nähern. Die Homoklinen befinden sich in flussinvarianten Unterräumen.

Insbesondere für solche homoklinen Zykel in Differentialgleichungen mit D_k-Symmetrie (D_k ist die Symmetriegruppe eines regelmäßigen k-Ecks in der Ebene), bei denen k ein Vielfaches von 4 ist, stehen einige dieser flussinvarianten Unterräume senkrecht zueinander. Dies impliziert das Verschwinden der typischerweise auftretenden Terme führender exponentieller Konvergenzordnung in einigen der aus Lins Methode gewonnenen Bestimmungsgleichungen. Um eine genaue Beschreibung der nichtwandernden Dynamik eines solchen homoklinen Zykels zu geben, d.h. eine Beschreibung der Lösungen, die in der Umgebung des Zykels sowohl im Phasen- als auch im Parameterraum verbleiben, sind weitere Informationen über die Restterme in den Bestimmungsgleichungen erforderlich.

In dieser Arbeit stellen wir eine verfeinerte Darstellung der Restterme in den Bestimmungsgleichungen vor und identifizieren zwei weitere Terme mit nächsthöheren exponentiellen Konvergenzraten. Darauf aufbauend diskutieren wir die Lösbarkeit der resultierenden Bestimmungsgleichungen für homokline Zykel in $\mathbb{R}^4$. Dabei sind zwei Fälle zu unterscheiden, die vom Größenverhältnis der beiden neuen Terme abhängen. In einem Fall beobachten wir einen endlichen Subshift. Im anderen Fall erweist sich die Analysis als schwieriger, so dass wir die Untersuchung auf periodische Lösungen beschränken.

Einleitung

Die Bifurkationstheorie ist eines der großen Themen in der modernen Theorie dynamischer Systeme. Grob gesagt untersucht sie eine plötzliche Änderung des dynamischen Verhaltens bei Änderung der Parameter in einer Familie dynamischer Systeme. Dies kann von der Veränderung der Anzahl der Gleichgewichtslagen bis hin zum Übergang von zahmer zu wilder (chaotischer) Dynamik reichen. Die Wurzeln der Bifurkationstheorie gehen zurück auf Poincaré, [P1890], aber sie ist immer noch ein lebendiges Forschungsthema mit vielen Anwendungen in verschiedenen wissenschaftlichen Disziplinen.

Dabei spielt die Bifurkationstheorie der heteroklinen und homoklinen Lösungen eine Schlüsselrolle für das Verständnis komplexer (chaotischer) Dynamik. Eine Zusammenfassung der aktuellen Ergebnisse und der Literatur zur Bifurkationstheorie homokliner und heteroklinen Orbits wird von Homburg und Sandstede [HomSan10] gegeben. Eine Einführung in die Welt der chaotischen Dynamik findet sich in [Dev89].

Die moderne Bifurkationstheorie heterokliner und homokliner Lösungen ist maßgeblich von der grundlegenden Arbeit von Shil'nikov aus den 1960er Jahren beeinflusst. Ein Überblick ist in den Monographien [SSTC98, SSTC01] zu finden. Shil'nikovs Ansatz zur Untersuchung von homoklinen Bifurkationsproblemen basiert auf Poincarés erster Rückkehrabbildung. Dies ist zur Standardtechnik der Behandlung dieser Art von Bifurkationsproblemen geworden.

Zu Beginn der 1990er Jahre entwickelte Lin eine Methode zur Konstruktion von Orbits in Umgebungen von heteroklinen Ketten, [Lin90]. Heutzutage ist dieses Verfahren auch als Lins Methode bekannt. Im Wesentlichen basiert es auf einer Liapunov/Schmidt-Reduktion. Im Laufe des Reduktionsprozesses gehen Informationen über die Dynamik verloren (z.B. Stabilitätsaussagen). Allerdings kann in einigen Fällen die Existenz bestimmter Orbits leichter nachgewiesen werden. Für chaotische Dynamiken ist es sogar möglich, mit dieser Methode die Existenz einer invarianten Menge zu beweisen, auf der die Dynamik topologisch konjugiert zu einem endlichen Subshift auf einer endlichen Anzahl von Symbolen ist, [HJKL11].

Von wachsendem Interesse ist die Untersuchung heterokliner Zykel oder, allgemeiner, heterokliner Netzwerke. Im einfachsten Fall bestehen solche Netzwerke aus Gleichgewichtslagen und Orbits, die diese Gleichgewichtslagen verbinden (heterokline Orbits). Solche Netzwerke sind als "Quelle" nicht-trivialer Dynamik identifiziert worden und treten unter anderem in physikalischen Problemen wie Konvektion [GuHo88, Ruc01], in der Populationsdynamik [Hof94, Hof98, MaLe75] oder auch in neuronalen Netzwerken [AOWT07] auf.

Die Untersuchung solcher Netzwerke ist im allgemeinen jedoch nur in entsprechend vielparametrigen Familien von Differentialgleichungen sinnvoll. Anders verhält es sich bei symmetrischen Differentialgleichungen. Die Symmetrie kann flussinvariante Unterräume erzwingen, in denen heterokline Trajektorien robust oder zumindest von geringer Kodimension sind, [Kru97]. Dies kann zu komplizierten heteroklinen Netzwerken mit niedriger Kodimension oder sogar Kodimension-Null führen. Das heißt die Untersuchung solcher Netzwerke benötigt nur wenige bis gar keine Parameter. Insbesondere Netzwerke mit Kodimension-Null sind robust - sie bleiben auch bei Störungen der zugrunde liegenden Differentialgleichung bestehen.

Symmetrien von Differentialgleichungen bzw. Vektorfeldern werden mit Hilfe von Gruppenaktionen beschrieben – ein Vektorfeld hat eine bestimmte Symmetrie oder ist äquivariant unter der (linearen) Wirkung (Darstellung) einer Gruppe G, wenn es mit den Darstellungsoperatoren von G kommutiert. In diesem Sinne wird G auch als Symmetriegruppe des Vektorfeldes bezeichnet. Die Lehrbücher [Van82, GSch85, GSS88, ChoLau00, Fie07] enthalten allgemeine Abhandlungen zur äquivarianten Bifurkationstheorie. In [Fie96] werden u.a. auch symmetrische heterokline Netzwerke betrachtet.

Von besonderem Interesse sind heterokline Netzwerke, die sich als Gruppenorbit der Symmetriegruppe einer einzelnen heteroklinen Trajektorie ergeben. In gewissem Sinne definiert diese Trajektorie das Netzwerk. Dann kann die Kodimension des Netzwerkes mit der Kodimension der definierenden Trajektorie übereinstimmen. Überraschenderweise können solche Netzwerke eine sehr komplexe Dynamik erzeugen.

Robuste heterokline Netzwerke werden seit den 90er Jahren verstärkt untersucht, vgl. [AgCaLa05, AgLaRo10, HomKno10, KLPRS10, KruMel04]. Ein ausführlicher Überblick über robuste heterokline Zykel bietet [Kru97]. Teil II dieser Veröffentlichung beschreibt detailliert den Stand der mathematischen Forschung, während in Teil III Experimente und numerische Anwendungen diskutiert werden.

In jüngerer Zeit sind Bifurkationsprobleme von nichtrobusten symmetrischen heteroklinen Netzwerken in den Fokus gerückt. Selbst einfache Netzwerke dieser Art können eine sehr komplexe Dynamik erzeugen. Dies ist z.B. der Fall bei einem Netzwerk, das aus zwei Homoklinen besteht, die durch Spiegelsymmetrie auseinander hervorgehen und sich einer hyperbolischen Gleichgewichtslage entlang der gleichen Richtung nähern – ein sogenannter $\mathbb{Z}_2$-symmetrischer Bellows. Man beachte, dass $\mathbb{Z}_2$-symmetrische Bellows mindestens einen relativen homoklinen Zykel der Kodimension-1 bilden. Sie erzeugen Shiftdynamik (voller Shift auf zwei Symbolen), [Hom93].

In [Mat99] zeigt Matthies, dass im Verlauf einer D_3-Takens-Bogdanov-Bifurkation D_3- symmetrische relative homokline Zykel entstehen, die einen endlichen Subshift erzeugen. Sowohl in [Hom93] als auch in

[Mat99] wurden Rückkehrabbildungen verwendet, um die Dynamik des Netzwerkes zu untersuchen.

Problemstellung

In [HJKL11] wird die Dynamik in der Nähe von Kodimension-1 homoklinen Zykeln durch Anwendung von Lins Methode betrachtet. Unter offenen Bedingungen und in einer Vielzahl von Fällen wurden Bifurkationsszenarien erstellt, die beschreiben, wie Shift Dynamik in der Bifurkation erscheint oder verschwindet. Es stellte sich heraus, dass die Analysis in [HJKL11] bei homoklinen Zykeln mit bestimmten Symmetrien versagt. Die prototypische Bifurkation, bei der die Analysis versagt, tritt bei homoklinen Zykeln in Differentialgleichungen mit D_k-Symmetrie auf, bei denen k ein Vielfaches von 4 ist.

In dieser Arbeit wird die Analysis in [HJKL11] so erweitert, dass wir eine genauere Beschreibung der nichtwandernden Dynamik in der Umgebung eines homoklinen Zykels erhalten, wo die Analysis in [HJKL11] unzureichend ist. Darüber hinaus geben wir eine Anleitung zur Konstruktion von Beispielvektorfeldern in $\mathbb{R}^4$, die einen homoklinen Zykel mit D_k-Symmetrie enthalten und für numerische Untersuchungen verwendet werden können. Eines dieser Beispiele betrachten wir numerisch, um die analytischen Ergebnisse zu verifizieren.

Zunächst skizzieren wir die zentralen Ergebnisse und Lösungsansätze aus [HJKL11] und beschreiben kurz die Gründe für das Scheitern der Analysis im Fall von D_{4m}-symmetrischen homoklinen Zykeln.

Genau wie in [HJKL11] betrachten wir eine einparametrige Familie von Differentialgleichungen

$$\dot{x} = f(x, \lambda), \tag{D.1}$$

mit $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, wobei f hinreichend glatt sei. Wir nehmen ferner an, dass (D.1) äquivariant (symmetrisch) bezüglich der linearen Darstellung einer endlichen Gruppe G sei, vgl. [GSS88]:

$$gf(x, \lambda) = f(gx, \lambda), \qquad \forall g \in G. \tag{D.2}$$

Zusätzlich verlangen wir, dass die Gleichung (D.1) für $\lambda = 0$ eine heterokline Trajektorie γ besitzt, die zwei hyperbolische Gleichgewichtslagen p und hp mit $h \in G$ verbindet. Das heißt, γ ist eine Lösung von (D.1) für $\lambda = 0$ mit

$$\lim_{t \to -\infty} \gamma(t) = p \quad \text{and} \quad \lim_{t \to \infty} \gamma(t) = hp.$$

Als Folge der Symmetrie finden wir für jede Lösung q von (D.1), dass gq auch eine Lösung von (D.1) für alle $g \in G$ ist. Da γ eine heterokline Trajektorie ist, die p und hp verbindet, ist also auch $g\gamma$ eine heterokline Trajektorie, die gp und $g(hp)$ verbindet. Mit Γ bezeichnen wir den von γ erzeugten relativen homoklinen Zykel:

$$\Gamma = G(\overline{\gamma}). \tag{D.3}$$

Mit anderen Worten: Γ ist der Gruppenorbit des Abschlusses einer einzelnen heteroklinen Trajektorie γ. Er besteht aus der hyperbolischen Gleichgewichtslage p, der heteroklinen Trajektorie γ und allen weiteren G-Bildern von γ und p.

Der Hauptsatz von [HJKL11] verwendet die Notation für topologische Markovketten, die wir nachfolgend aufgreifen, vgl. auch [Shu86, Definition 10.1]. Dabei sei

$$\Sigma_k = \{1, \dots, k\}^{\mathbb{Z}}$$

die Menge der biinfiniten Folgen $\kappa : \mathbb{Z} \to \{1, \dots, k\}$, $i \mapsto \kappa_i$, ausgestattet mit der Produkttopologie.

Ferner sei $A = (a_{ij})_{i,j\in\{1,\ldots,k\}}$ eine 0-1-Matrix, also $a_{i,j} \in \{0,1\}$. Mit

$$\Sigma_A = \{\kappa \in \Sigma_k \mid a_{\kappa_i\kappa_{i+1}} = 1\}$$

bezeichnen wir die durch A definierte topologische Markov-Kette. Die Linksverschiebung σ wirkt auf Σ_k durch

$$\sigma : \Sigma_k \to \Sigma_k, \qquad (\sigma\kappa)_i = \kappa_{i+1}$$

und lässt Σ_A invariant. Das Paar (Σ_A, σ) wird als endlicher Subshift bezeichnet.

Von besonderem Interesse ist die Konnektivitätsmatrix $\boldsymbol{C} = (c_{ij})$ eines heteroklinen Netzes (mit heteroklinen Trajektorien γ_i), eine 0-1 Matrix bei der $c_{ij} = 1$ ist, wenn der Endpunkt (das ω-Limit $\omega(\gamma_i)$) der heteroklinen Verbindung γ_i gleich dem Anfangspunkt (dem α-Limit $\alpha(\gamma_j)$) der heteroklinen Verbindung γ_j ist.

Das folgende Theorem ist die Hauptaussage in [HJKL11].

Theorem D.1 ([HJKL11], Theorem 1.1)**.** *Sei $\dot{x} = f(x,\lambda)$ eine einparametrige Familie von Differentialgleichungen, welche äquivariant bezüglich einer endlichen Gruppe G ist, vgl. (D.2). Bei $\lambda = 0$ habe $\dot{x} = f(x,\lambda)$ einen relativen homoklinen Zykel Γ mit hyperbolischer Gleichgewichtslage wie in (D.3) definiert. Ferner gelten einige allgemeine Bedingungen, die die minimale Schnittmenge der Tangentialräume an den stabilen und instabilen Mannigfaltigkeiten entlang γ sowie die Eigenschaften des Nicht-Orbit-Flips und des Nicht-Inklinations-Flips betreffen. Wir schreiben $\gamma_1, \ldots, \gamma_k$ für die verbindenden Trajektorien, die Γ erzeugen.*

Es gibt eine explizite Konstruktion von $(k\times k)$-Matrizen A_- und A_+ mit Koeffizienten in $\{0,1\}$ und den von Null verschiedenen Koeffizienten an zueinander disjunkten Positionen, so dass für jede generische Familie, die einen relativen homoklinen Zykel wie oben entfaltet, das Folgende gilt.

Man nehme Querschnitte $\mathcal{S}_i$ transversal zu γ_i und schreibe Π_λ für die erste Rückkehrabbildung auf der Vereinigung von Querschnitten $\cup_{j=1}^k \mathcal{S}_j$. Für $\lambda > 0$ gibt es für Π_λ eine invariante Menge $\mathcal{D}_\lambda \subset \cup_{j=1}^k \mathcal{S}_j$ so dass für jedes $\kappa \in \Sigma_{A_+}$ ein eindeutiges $x \in \mathcal{D}_\lambda$ mit $\Pi_\lambda^i(x) \in \mathcal{S}_{\kappa_i}$ existiert. Außerdem ist $(\mathcal{D}_\lambda, \Pi_\lambda)$ topologisch konjugiert zu (Σ_{A_+}, σ). Eine analoge Aussage gilt für $\lambda < 0$ mit Σ_{A_-} anstelle von Σ_{A_+}.

Die obige Beschreibung der Dynamik liefert ein vollständiges Bild der lokalen nichtwandernden Dynamik in der Nähe von Γ genau dann, wenn

$$A_+ + A_- = \boldsymbol{C} \tag{D.4}$$

gilt, wobei $\boldsymbol{C}$ die Konnektivitätsmatrix des relativen homoklinen Zykels bezeichnet.

Aufgrund der topologischen Konjugation zwischen $(\mathcal{D}_\lambda, \Pi_\lambda)$ und dem endlichen Subshift (Σ_{A_+}, σ) spricht man von Shiftdynamik.

Zunächst wollen wir die Aussage des Satzes D.1 am Beispiel eines D_4 symmetrischen relativen homoklinen Zykels erläutern, wie er in Abbildung D.1 dargestellt ist, vgl. auch [HJKL11, Table 1, Case 6]. Hier wird der Gruppenorbit Γ aus einer einzigen homoklinen Trajektorie asymptotisch zu einer G-invarianten hyperbolischen Gleichgewichtslage p gewonnen. Die Konnektivitätsmatrix dieses homoklinen Zykels ist gegeben als $\boldsymbol{C} = \mathbf{1}$, die Matrix, in der alle Einträge gleich eins sind. Da D_k die Symmetriegruppe eines regelmäßigen k-Ecks in der Ebene ist, wird sie durch zwei Elemente erzeugt, der Spiegelung ζ, welche eine zyklische Untergruppe der Ordnung zwei erzeugt, und der Drehung θ_k, die eine zyklische Untergruppe der

Ordnung k erzeugt. Den Ausführungen in [HJKL11] zufolge, sind die Matrizen A_-, A_+ gegeben durch

$$A_- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{D.5}$$

Offensichtlich ist $A_+ + A_- \neq \mathbf{1}$. Das bedeutet, dass Theorem D.1 keine vollständige Beschreibung der nichtwandernden Dynamik in der Umgebung des betrachteten homoklinen Zykels liefert.

In der Tat verifiziert Theorem D.1 für dieses Beispiel lediglich bekannte Tatsachen: Wenn $\lambda < 0$ ist, besteht die nichtwandernde Menge aus vier 1-periodischen Lösungen, die dem Verlauf der einzelnen Homoklinen γ_i $(i = 1, 2, 3, 4)$ folgen. Ist $\lambda > 0$, so besteht die nichtwandernde Dynamik aus zwei 2-periodischen Orbits, die jeweils der Achterkonfiguration folgen, die aus den Paaren homokliner Trajektorien besteht, die sich gegenüberliegen. Dabei ist zu beachten, dass sich diese Paare in invarianten Unterräumen befinden. Daher gilt für jedes Paar [HomSan10, Theorem 5.79].

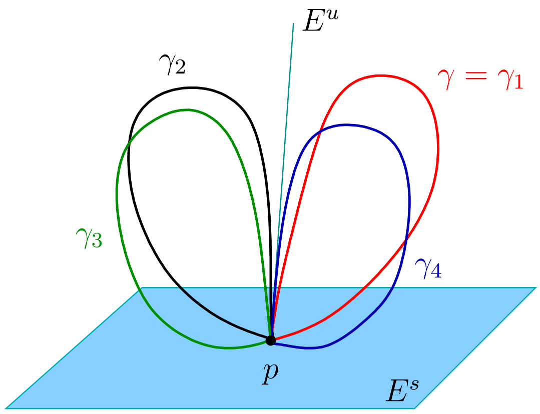

Abbildung D.1: Ein relativer homokliner Zykel, der als Gruppenorbit des Abschlusses der Homoklinen γ aufgebaut ist. Die zugrunde liegende Symmetriegruppe ist D_4. In diesem speziellen Beispiel ist D_4 die Isotropiegruppe der Gleichgewichtslage p, und $\mathbb{Z}_2$ ist die Isotropiegruppe der homoklinen Trajektorie γ. Die einzelnen Homoklinen befinden sich in invarianten Unterräumen. Darüber hinaus sind die Unterräume benachbarter homokliner Trajektorien orthogonal zueinander.

Bevor wir unsere Ergebnisse präsentieren, wollen wir die Ursache dafür aufzeigen, warum die Analysis in [HJKL11] nur eine unvollständige Beschreibung der nicht wandernden Dynamik liefert. Zu diesem Zweck seien $\gamma_1, \ldots, \gamma_k$, wie oben, die verbindenden Trajektorien, die den relativen homoklinen Zykel Γ bilden.

In [HJKL11] werden die Matrizen $A_\pm$ im Wesentlichen dadurch konstruiert, dass man zeigt, dass eine bestimmte Folge von verbindenden Trajektorien $\gamma_i \subset \Gamma$, oder mit anderen Worten eine Reiseroute entlang Γ, von einer tatsächlichen Trajektorie von (D.1) nachverfolgt werden kann oder nicht. Zu diesem Zweck wurde der folgende Begriff eingeführt.

Definition D.2. *Sei $\kappa \in \Sigma_k$ fest. Eine heteroklinische Kette Γ^κ ist eine biinfinite Folge von verbindenden Trajektorien γ_{κ_i}, $i \in \mathbb{Z}$ derart, dass $\omega(\gamma_{\kappa_{i-1}}) = \alpha(\gamma_{\kappa_i})$.*

Mit Hilfe von Lins Methode, [Lin90, San93, Kno04], wird eine stückweise kontinuierliche Trajektorie $\mathbf{X} = (X_i)_{i\in\mathbb{Z}}$, die sogenannte Lin-Trajektorie, konstruiert, die der Folge von Γ^κ, κ fix, folgt. Die $X_i : [-\omega_i, \omega_i] \to \mathbb{R}^n$ selbst sind Lösungen von (D.1) und zwischen dem Endpunkt jedes X_i und dem Anfangspunkt seines Nachfolgers X_{i+1} tritt der endliche Sprung Ξ_i auf. Die Lin Trajektorie $\mathbf{X}$ ist genau dann eine tatsächliche Trajektorie von (D.1), wenn alle Sprünge Ξ_i gleich Null sind. Genauer gesagt, wenn es eine biinfinite Folge von Übergangszeiten $\boldsymbol{\omega} = (\omega_i)_{i\in\mathbb{Z}}$, $\omega_i \in \mathbb{R}^+$ gibt, die für gegebene λ und κ das Gleichungssystem $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = 0$ löst, $i \in \mathbb{Z}$, ist die Existenz einer Trajektorie $\mathbf{X}$, die dem Verlauf von Γ^κ folgt, nachgewiesen.

Wie wir dank Sandstede [San93] wissen, kann der Sprung in zwei Teile unterteilt werden:

$$\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \xi^{\infty}_{\kappa_i}(\lambda) + \xi_i(\boldsymbol{\omega}, \lambda, \kappa).$$

Der erste Teil, $\xi^{\infty}_{\kappa_i}$, ist ein Maß für den Abstand von der stabilen zur instabilen Mannigfaltigkeit und wird somit nur durch den Systemparameter λ beeinflusst. Unter geeigneten Voraussetzungen kann $\xi^{\infty}_{\kappa_i}$ selbst als Systemparameter gewählt werden. Der zweite Teil des Sprungs, ξ_i, wird mit wachsendem ω_i exponentiell klein.

Im Fall des hier diskutierten homoklinen Zykels finden wir, vgl. [HJKL11, Proposition 3.7 und (4.1)],

$$\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) := \lambda - e^{2\mu^s(\lambda)\omega_i}\langle \eta^s_{\kappa_{i-1}}(\lambda), \eta^-_{\kappa_i}(\lambda)\rangle + R_i(\boldsymbol{\omega}, \lambda, \kappa) = 0, \tag{D.6}$$

für alle $i \in \mathbb{Z}$. Dabei genügen die Restterme der Form $R_i(\boldsymbol{\omega}, \lambda, \kappa) = O(e^{2\mu^s(\lambda)\omega_{i+1}\delta}) + O(e^{2\mu^s(\lambda)\omega_i\delta})$ für ein $\delta > 1$. Dieses Gleichungssystem bezeichnen wir als Bestimmungsgleichung.

Wenn nun für gegebenes κ alle Skalarprodukte $\langle \eta^s_{\kappa_{i-1}}, \eta^-_{\kappa_i}\rangle$ von Null verschieden sind, ist das System (D.6) genau dann lösbar, wenn alle diese Produkte das gleiche Vorzeichen wie λ haben. (An dieser Stelle sei erwähnt, dass für festes κ das Vorzeichen des Skalarprodukts nicht von λ abhängt.) Das heißt, mit der Matrix $M = (m_{i,j})_{i,j\in\{1,\ldots,k\}}$ mit $m_{i,j} := \operatorname{sgn}\langle \eta^s_i, \eta^-_j\rangle$ finden wir

$$A_- = \frac{1}{2}(|M| - M) \quad \text{und} \quad A_+ = \frac{1}{2}(|M| + M). \tag{D.7}$$

Wenn κ jedoch zulässt, dass einige Produkte $\langle \eta^s_{\kappa_{i-1}}, \eta^-_{\kappa_i}\rangle$ zu Null werden, ist eine aufwendigere Analysis als die in [HJKL11] notwendig, um die Gleichung (D.6) zu lösen. Genauer gesagt, um zu entscheiden, ob (D.6) lösbar ist oder nicht, braucht man Wissen über die Terme führender Ordnung der Restterme $R_i(\boldsymbol{\omega}, \lambda, \kappa)$.

Wenden wir uns also kurz der Frage zu, ob und wann das Skalarprodukt $\langle \eta^s_{\kappa_{i-1}}, \eta^-_{\kappa_i}\rangle$ Null werden kann. Zu diesem Zweck definieren wir

$$e^s_i := \lim_{t\to\infty} (\gamma_i(t) - p)/\|\gamma_i(t) - p\|,$$

d.h. e^s_i definiert die Richtung, in der sich die Homokline γ_i der Gleichgewichtslage p für positive Zeit nähert. Weiter definieren wir

$$e^-_j := \lim_{t\to-\infty} \psi_j(t)/\|\psi_j(t)\| \tag{D.8}$$

wobei $\psi_j(t)$ eine Lösung der adjungierten Variationsgleichung entlang $\gamma_j(t)$ ist

$$\dot{x} = -[D_1 f(\gamma_j(t), \lambda)]^T x, \quad x(0) = \psi_j.$$

Dabei ist ψ_j ein Einheitsvektor, der der folgenden Bedingung genügt

$$\operatorname{span}\{\psi_j\} = \left(T_{\gamma_j(0)} W^s(\omega(\gamma_j)) + T_{\gamma_j(0)} W^u(\alpha(\gamma_j))\right)^\perp.$$

Das orthogonale Komplement ist durch ein G-invariantes Skalarprodukt $\langle\cdot,\cdot\rangle$ definiert. Aufgrund der im Theorem D.1 genannten minimalen Schnittbedingung, die auf alle verbindenden Trajektorien des Zykels übertragen wird, ist der Vektor ψ_j bis auf skalare Vielfache eindeutig definiert. Die Existenz des Grenzwertes in (D.8) wird durch die Überlegungen in [HJKL11, Abschnitt 3] sichergestellt.

Wenn wir uns nun die Herleitung von (D.6) mit Hilfe von Lins Methode genauer ansehen, stellen wir fest, dass unter geeigneten Annahmen (ohne das Auftreten von Inklination- und Orbit-Flip Konstellationen),

$$\eta^s_{\kappa_{i-1}}(\lambda) \in \operatorname{span}\{e^s_{\kappa_{i-1}}\} \quad \text{und} \quad \eta^-_{\kappa_i}(\lambda) \in \operatorname{span}\{e^-_{\kappa_i}\}$$

gilt und demnach

$$\langle \eta^s_{\kappa_{i-1}}(\lambda), \eta^-_{\kappa_i}(\lambda) \rangle = \tilde{A}_i(\lambda, \kappa) \langle e^s_{\kappa_{i-1}}, e^-_{\kappa_i} \rangle$$

mit $\tilde{A}_i(\lambda, \kappa) > 0$. Das Skalarprodukt $\langle e^s_i, e^-_i \rangle$ ist wegen der Symmetrie gleich $\langle e^s_1, e^-_1 \rangle$. Natürlich hängt das Vorzeichen von $\langle e^s_1, e^-_1 \rangle$ von der Wahl von ψ_1 ab. Wir wählen ψ_1 so, dass

$$\langle e^s_1, e^-_1 \rangle < 0.$$

In Bezug auf den oben eingeführten D_4-symmetrischen homoklinen Zykel, vgl. auch Abbildung D.1, stellen wir nun fest, dass sich die einzelnen Homoklinen γ_i in invarianten Unterräumen befinden. Folglich liegen auch die entsprechenden η^s_i und η^-_i in diesen Unterräumen. Da die D_4 die Symmetriegruppe des Quadrats ist, stehen einige dieser Unterräume senkrecht zueinander. Dies impliziert $\langle \eta^s_i, \eta^-_{i+1} \rangle = 0$, $i = 1, 2, 3, 4$. Dies liefert letztlich die Nullstellen der Nebendiagonalen in der Matrix A_+, die in (D.5) gegeben ist. Mit anderen Worten: Mit Hilfe der Ergebnisse in [HJKL11] kann nicht entschieden werden, ob eine heterokline Kette Γ^κ mit κ derart, dass es für ein $i \in \mathbb{Z}$ ein $j \in \{1, \dots, 4\}$ gibt, so dass $\kappa_i = j$ und $\kappa_{i+1} = j + 1$, eine nachverfolgende tatsächliche Trajektorie hat.

Resultate

Im Folgenden listen wir die Voraussetzungen, unter denen wir die Restterme $R_i(\boldsymbol{\omega}, \lambda, \kappa)$ untersuchen, deren Terme führender Ordnung identifizieren und der Frage nach der Existenz von verfolgenden Trajektorien nachgehen, für die mindestens ein $i \in \mathbb{Z}$ existiert, so dass $\langle \eta^s_{\kappa_{i-1}}(\lambda), \eta^-_{\kappa_i}(\lambda) \rangle = 0$.

Die Gruppe, welche wir dabei betrachten, ist die Diedergruppe D_{4m}, die Symmetriegruppe eines regelmäßigen $4m$-Ecks in der Ebene. Sie wird erzeugt von der Spiegelung ζ und der Drehung θ_{4m}, welche selbst Generatoren der entsprechenden zyklischen Untergruppen, der Spiegelungsgruppe $\mathbb{Z}_2(\zeta)$ und der Rotationsgruppe $\mathbb{Z}_{4m}(\theta_{4m})$, sind.

Wir konzentrieren uns im Wesentlichen auf den homoklinen Zykel, der durch folgende Hypothesen charakterisiert ist.

(H.D.1).

(i) *Das Vektorfeld $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ sei glatt, d.h. $f \in C^{l+3}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$, $l \geq \max\{3, \nu\}$, und $f(\cdot, \lambda)$ ist äquivariant bezüglich der Diedergruppe D_{4m} für alle $\lambda \in \mathbb{R}$.*

(ii) *Für $\lambda = 0$ existiert ein homokliner Zykel $\Gamma = G(\overline{\gamma})$, der gleich dem Abschluss des Gruppenorbits einer homoklinen Trajektorie γ an die hyperbolische Gleichgewichtslage p ist. Wir fordern, dass $G_p = D_{4m}$.*

Die Konstante $\nu \in \mathbb{N}$, $\nu \geq 2$ ist dabei derart definiert, dass für alle $k \in \{0, \dots, \nu-1\} \backslash \{1\}$ gilt $D^k_1 f(p, \lambda) = 0$ und $D^\nu_1 f(p, \lambda) \neq 0$. Da p eine hyperbolische Gleichgewichtslage des Vektorfeldes f ist, gilt $f(p, \lambda) = 0$ und

$D_1 f(p, \lambda) \neq 0$. Damit gilt im Allgemeinen für $\nu = 2$. Für $\nu > 2$ wird das Verschwinden der Ableitungen des Vektorfeldes f an der Gleichgewichtslage p von der zweiten bis zur $(\nu-1)$-ten Ordnung verlangt.

Der Einfachheit halber nehmen wir an

(H.D.2). *Die Dimension des Vektorfeldes sei $n = 4$.*

Weiterhin gelte

(H.D.3).

(i) *Der Eigenraum $E(\mu^s(\lambda))$ zum führenden stabilen Eigenwert $\mu^s(\lambda)$ von $D_1 f(p, \lambda)$ ist zwei-dimensional.*

(ii) *$G_p = D_{4m}$ wirkt auf $E(\mu^s(\lambda))$ als D_{4m}.*

(iii) *$G_p = D_{4m}$ wirkt auf den Eigenraum $E(\mu^u(\lambda))$ zum führenden instabilen Eigenwert $\mu^u(\lambda)$ von $D_1 f(p, \lambda)$ als D_{4m}.*

(iv) $0 < |\ Re(\mu^s(\lambda))| <\ Re(\mu^u(\lambda))$.

(v) *Die Homokline γ besitzt die Isotropiegruppe $G_\gamma = \mathbb{Z}_2(\zeta)$.*

Aufgrund der Hypothese (H.D.3)(v) besteht der homokline Zykel aus $4m$ homoklinen Trajektorien von denen sich jede im Fixraum einer Spiegelung befindet. Aus (H.D.3)(i) und (ii) folgt, dass $\mu^s(\lambda)$ reell und halbeinfach ist und $\dim(\mathrm{Fix}\,\mathbb{Z}_2(\zeta) \cap E(\mu^s(\lambda))) = 1$. Die durch die Hypothesen (H.D.1) und (H.D.3) beschriebene Situation wird in der Abbildung D.1 im Falle der D_4-Symmetrie dargestellt. Weiterhin ist auch $E(\mu^u(\lambda))$ zweidimensional, $\mu^u(\lambda)$ ist reell und halbeinfach und $\dim(\mathrm{Fix}\,\mathbb{Z}_2(\zeta) \cap E(\mu^u(\lambda))) = 1$. (Wir möchten darauf hinweisen, dass (H.D.3)(iii) keine Entsprechung in [HJKL11] hat). Das bedeutet insbesondere, dass $D_1 f(p, 0)$ weder stark stabile noch stark instabile Eigenwerte hat: $\sigma(D_1 f(p, 0)) = \{\mu^s, \mu^u\}$. Daher müssen wir uns keine Gedanken über Inklination und Orbit Flip machen.

Aus den Hypothesen (H.D.1) - (H.D.3) folgt weiterhin, dass $f(-x, \lambda) = -f(x, \lambda)$ gilt für alle $x \in \mathbb{R}^4$. Daraus folgt $D_1^2 f(p, \lambda) = 0$ und somit gilt $\nu \geq 3$. Dies ist eine wichtige Voraussetzung für die Gültigkeit der unten aufgeführten Darstellung des Sprungs Ξ_i.

In Bezug auf die homokline Trajektorie fordern wir weiterhin:

(H.D.4).

(i) *Die Homokline γ ist nicht-degeneriert, das heißt $T_{\gamma(0)} W^s(p, 0) \cap T_{\gamma(0)} W^u(p, 0) =\ span\{\dot{\gamma}(0)\}$.*

(ii) *Ferner entfalten sich die Einschränkungen der stabilen und instabilen Mannigfaltigkeiten auf den Fixraum $FixG_\gamma$, $W^s_{FixG_\gamma}(p)$ und $W^u_{FixG_\gamma}(p)$, generisch in Bezug auf den Parameter λ, das heißt mit einer Geschwindigkeit ungleich Null.*

Die obigen Hypothesen (H.D.1) - (H.D.4) implizieren die in [HJKL11] geforderten Bedingungen.

Unter diesen Voraussetzungen haben die Sprünge die folgende Gestalt. Dafür führen wir für festes κ die Indexmenge J_κ ein, mit

$$J_\kappa = \{j \in \mathbb{Z} \mid \mathrm{Fix}_{\kappa_{j-1}} \perp \mathrm{Fix}_{\kappa_j}\}. \tag{D.9}$$

Dabei bezeichnet Fix_{κ_j} den Fixraum, in welchem die Homokline γ_{κ_j} liegt.

Theorem D.3. *Es gelten die Hypothesen (H.D.1)-(H.D.4). Seien α^u und β^s Konstanten, die $-\mu^u(\lambda) < -\alpha^u < \beta^s < \mu^s(\lambda)$ erfüllen. Dann gibt es ein $\delta > 1$, so dass der Sprung Ξ_i als eine der folgenden Alternativen geschrieben werden kann:*

(i) Ist $i \in \mathbb{Z} \setminus J_\kappa$, dann gilt $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \lambda - A_i(\lambda, \kappa) e^{2\mu^s(\lambda)\omega_i} + R_i(\boldsymbol{\omega}, \lambda, \kappa)$, *mit*

$$R_i(\boldsymbol{\omega}, \lambda, \kappa) = O\big(e^{8/5\mu^s(\lambda)\delta(\omega_{i-1}+\omega_i)}\big) + O\big(e^{16/5\mu^s(\lambda)\delta\omega_i}\big) + \begin{cases} O\big(e^{2\mu^s(\lambda)\delta\omega_{i+1}}\big), & i+1 \in \mathbb{Z} \setminus J_\kappa \\ \left.\begin{array}{l} O\big(e^{8/5\mu^s(\lambda)\delta\omega_i} e^{2\mu^s(\lambda)\delta\omega_{i+1})}\big) \\ +O\big(e^{4\mu^s(\lambda)\delta\omega_{i+1}}\big) + O\big(e^{2\mu^s(\lambda)\delta(\omega_{i+1}+\omega_{i+2})}\big) \end{array}\right\}, & i+1 \in J_\kappa. \end{cases}$$

(ii) Ist $i \in J_\kappa$, dann ist $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \lambda - B(\lambda) e^{4\mu^s(\lambda)\omega_i} - C_i(\lambda, \kappa) e^{2\mu^s(\lambda)(\omega_{i-1}+\omega_i)} + R_i(\boldsymbol{\omega}, \lambda, \kappa)$, *mit*

$$\begin{aligned} R_i(\boldsymbol{\omega}, \lambda, \kappa) &= O\big(e^{8/5\mu^s(\lambda)\delta(\omega_{i-1}+\omega_i)}\big[e^{8/5\mu^s(\lambda)\delta\omega_{i-2}} + e^{4/5\mu^s(\lambda)\delta\omega_{i-1}} + e^{\mu^s(\lambda)\delta\omega_i}\big]\big) \\ &\quad + O\big(e^{4\mu^s(\lambda)\delta\omega_i}\big) + O\big(e^{16/5\mu^s(\lambda)\delta\omega_i} e^{\mu^s(\lambda)\delta\omega_{i+1}}\big) \\ &\quad + \begin{cases} O\left(e^{2\mu^s(\lambda)\delta\omega_{i+1}}\right), & i+1 \in \mathbb{Z} \setminus J_\kappa \\ \left.\begin{array}{l} O\big(e^{2\frac{-\beta^s}{\alpha^u}\mu^s(\lambda)\delta\omega_i} e^{2\frac{\alpha^u}{-\beta^s}\mu^s(\lambda)\delta\omega_{i+1})}\big) \\ +O\big(e^{4\mu^s(\lambda)\delta\omega_{i+1}}\big) \\ +O\big(e^{2\mu^s(\lambda)\delta(\omega_{i+1}+\omega_{i+2})}\big) \end{array}\right\}, & i+1 \in J_\kappa. \end{cases} \end{aligned}$$

Die Koeffizienten $A_i(\lambda, \kappa) := \langle \eta^-_{\kappa_i}(\lambda), \eta^s_{\kappa_{i-1}}(\lambda) \rangle$ sind verschieden von Null für alle $i \in \mathbb{Z} \setminus J_\kappa$. Ferner hängen die Koeffizienten $B(\lambda)$ nicht von $i \in J_\kappa$ ab. Schließlich gibt es ein $C(\lambda) > 0$, so dass $|C_i(\lambda, \kappa)| = C(\lambda)$ für alle i mit $i-1, i \in J_\kappa$.

Für $i, i-1 \in J_\kappa$ ergibt sich die Größe $C_i(0, \kappa)$ aus dem Skalarprodukt von $\eta^-_{\kappa_i}(0)$ auf der einen und einer Richtung innerhalb $\operatorname{span}\{e^s_{\kappa_{i-2}}\}$ auf der anderen Seite. Die zweite Richtung resultiert aus $\eta^s_{\kappa_{i-2}}(0)$ durch den Transport entlang der Homoklinen $\gamma_{\kappa_{i-1}}(t)$ mittels der adjungierten Variationsgleichung $\dot{x} = -[D_1 f(\gamma_{\kappa_{i-1}}(t), 0)]^T x$ von $-\omega_{i-1}$ nach ω_i, vgl. Abbildung D.2. Demnach liegen $\eta^-_{\kappa_i}$ und $\eta^s_{\kappa_{i-2}}$ für $i, i-1 \in J_\kappa$ im selben eindimensionalen Unterraum und es stellt sich heraus, dass alle $C_i(\lambda, \kappa)$ denselben Betrag $C(\lambda) := |C_i(\lambda, \kappa)|$ haben und von Null verschieden sind. Das Vorzeichen von $C_i(\lambda, \kappa)$ hängt von der topologischen Struktur des Faserbündels

$$\mathcal{F}(W^s_\gamma) := \bigcup_{t \in \mathbb{R}} \big(T_{\gamma(t)} W^s(p) \cap [\operatorname{span}\{\dot{\gamma}(t)\}]^\perp\big),$$

welches vereinfacht ausgedrückt der stabilen Mannigfaltigkeit innerhalb einer röhrenförmigen Umgebung der homoklinen Trajektorie γ entspricht, ab und davon, ob $\kappa_i = \kappa_{i-2}$ oder nicht.

Für alle anderen Fälle als $i, i-1 \in J_\kappa$ spielt der Term $C_i(\lambda, \kappa)$ in den Bestimmungsgleichungen keine Rolle.

Die Gestalt des Terms $B(\lambda)$ ist zu kompliziert, als dass wir hierfür eine geometrische Interpretation geben können. Es lässt sich jedoch zeigen, dass $B(\lambda)$ nicht als Folge der Symmetrie des Vektorfeldes verschwindet. Leider konnten wir das mögliche Verschwinden des Terms B aus anderen Gründen als der Symmetrie nicht analytisch ausschließen. Die Annahme $B(\lambda) \neq 0$ ist jedoch möglich und sinnvoll, wie die numerische Untersuchung eines von uns konstruierten Beispielvektorfeldes zeigt. Wir nehmen dies als Rechtfertigung für eine entsprechende Annahme über $B(\lambda)$ in unserer durchgeführten Analysis.

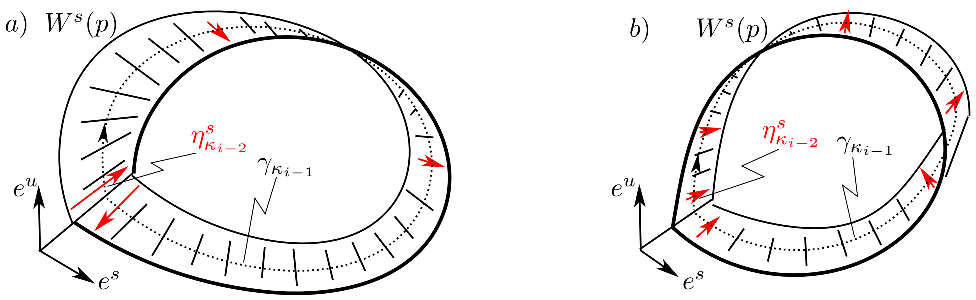

Abbildung D.2: Transport der Richtung $\eta^s_{\kappa_{i-2}}$ entlang der homoklinen Trajektorie $\gamma_{\kappa_{i-1}}$ mittels der Übergangsmatrix der Variationsgleichung entlang $\gamma_{\kappa_{i-1}}$ für den Fall, dass $\mathcal{F}(W^s_{\gamma_{\kappa_{i-1}}})$ a) die topologische Struktur eines Möbiusbandes und b) die topologische Struktur eines Ringes hat.

Mit der genaueren Darstellung des Sprungs können wir eine detailliertere Beschreibung der nichtwandernden Dynamik des betrachteten Systems geben. Dazu erinnern wir uns, dass wir für den Nachweis der Existenz einer Lösungstrajektorie die zugehörigen Sprünge $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa)$, $i \in \mathbb{Z}$, gleich Null setzen müssen. Dabei ist die Form der Ξ_i gegeben durch Theorem D.3. Die Lösbarkeit der so entstandenen Bestimmungsgleichung $(\Xi_i(\boldsymbol{\omega}, \lambda, \kappa))_{i\in\mathbb{Z}} = 0$ kann dann in Abhängigkeit von der Wahl von κ und λ diskutiert werden. Im gegenwärtigen Kontext kann es Sequenzen von aufeinanderfolgenden Homoklinen geben, die in zueinander orthogonalen Fixpunkträumen liegen. Die Darstellung der zugehörigen Sprünge ist gegeben durch Theorem D.3(ii). Die Null in den zugehörigen Bestimmungsgleichungen muss dabei im Grunde von dem Term erzeugt werden, der $B(\lambda)$ enthält. In den Gleichungen, bei denen die aufeinanderfolgenden Homoklinen nicht in zueinander orthogonalen Fixpunkträumen liegen, siehe Theorem D.3(i), wird die Null durch den Term erzeugt, der $A_i(\lambda, \kappa) := \langle \eta^-_{\kappa_i}(\lambda), \eta^s_{\kappa_{i-1}}(\lambda)\rangle$ enthält.

Die Struktur des Systems der Bestimmungsgleichungen ist also erheblich komplizierter geworden. Dies spiegelt sich auch in der resultierende Dynamik wieder, was sich insbesondere an den Übergangszeiten $\boldsymbol{\omega} = (\omega_i)_{i\in\mathbb{Z}}$ zeigt, die das System für gegebene λ und κ lösen. Die Untersuchungen in [HJKL11] zeigen, dass, wenn (D.6) für alle $i \in \mathbb{Z}$ gilt, die lösenden Übergangszeiten ω_i für feste λ ungefähr gleich groß sind. Genauer gesagt erfüllen sie die Gleichung $\omega_i(\lambda, \kappa) = \frac{1}{2\mu^s(0)}(\ln(|\lambda|) + \ln(r_i))$, mit gleichmäßig beschränkten Termen r_i, vgl. [HJKL11, (4.8)].

Bei der Untersuchung der erweiterten Bestimmungsgleichungen aus Theorem D.3 sind zwei grundlegende Fälle zu unterscheiden, die durch das Größenverhältnis von $|B(0)|$ zu $C(0)$ gekennzeichnet sind. In beiden Fällen gilt für die Lösung ω_i die Gleichung

$$\omega_i(\lambda, \kappa) = \begin{cases} \frac{1}{2\mu^s(0)}(\ln(|\lambda|) + \ln(r_i(\lambda, \kappa))), & i \in \mathbb{Z} \setminus J_\kappa, \\ \frac{1}{4\mu^s(0)}(\ln(|\lambda|) + \ln(r_i^2(\lambda, \kappa))), & i \in J_\kappa, \end{cases} \tag{D.10}$$

wie sich bei der Lösungsdiskussion herausstellt. Die Eigenschaften der Terme r_i unterschieden sich jedoch in den beiden Fällen, wie wir im folgenden noch genauer ausführen werden.

Bislang ist nicht bekannt, ob die Geometrie des Systems ein beliebiges Verhältnis von $|B(0)|$ zu $C(0)$ zulässt oder ob ein bestimmtes Verhältnis erzwungen wird. Daher ist es denkbar, das Verhältnis von $|B(0)|$ und $C(0)$ durch einen weiteren Systemparameter zu steuern. In diesem Sinne ist die vollständige Beschreibung der lokalen nichtwandernden Dynamik in der Umgebung eines D_{4m}-symmetrischen homoklinen Zyklus möglicherweise kein reines Kodimension-1 Problem mehr.

1. Fall: $|B(0)| > C(0)$

In diesem Fall, finden wir, ähnlich wie bei [HJKL11], dass die in (D.10) genannten r_i gleichmäßig beschränkt sind. Das bedeutet, dass die Schranken von r_i unabhängig von λ gewählt werden können und, abgesehen von der Unterscheidung, ob $i \in \mathbb{Z} \setminus J_\kappa$ oder $i \in J_\kappa$, auch unabhängig vom Verlauf von κ sind. Für hinreichend kleine λ dominiert $\ln(|\lambda|)$ den anderen Summanden in (D.10), der r_i enthält, und es stellt sich heraus, dass die Übergangszeiten im Wesentlichen gleich groß sind (bei festem λ) und sich nur um den Faktor 2 unterscheiden, je nach Verlauf der Trajektorie:

$$\omega_i(\lambda, \kappa) \approx \begin{cases} \frac{1}{2\mu^s(0)} \ln(|\lambda|), & i \in \mathbb{Z} \setminus J_\kappa, \\ \frac{1}{4\mu^s(0)} \ln(|\lambda|), & i \in J_\kappa. \end{cases}$$

Genauer gesagt benötigt man für den Übergang von der Homoklinen γ_i zur Homoklinen γ_j nur halb so viel Zeit, wenn sie in zueinander orthogonalen Fixräumen liegen, wie in allen anderen Fällen.
Folglich kann das dynamische Verhalten in der Umgebung des homoklinen Zykels für den Fall $|B(0)| > C(0)$ in der gleichen Weise beschrieben werden, wie es in [HJKL11, Theorem 1.1] getan wurde, vgl. Theorem D.1, also durch Shiftdynamik.

Zu diesem Zweck definieren wir die folgende Matrix M durch

$$M = (m_{ij})_{i,j \in \{1,\dots,4m\}}, \quad m_{ij} := \begin{cases} \operatorname{sgn}\langle \eta_i^s(\lambda), \eta_j^-(\lambda) \rangle, & \operatorname{sgn}\langle \eta_i^s(\lambda), \eta_j^-(\lambda) \rangle \neq 0, \\ \operatorname{sgn} B(\lambda), & \operatorname{sgn}\langle \eta_i^s(\lambda), \eta_j^-(\lambda) \rangle = 0. \end{cases} \tag{D.11}$$

In diesem Zusammenhang wollen wir anmerken, dass $\operatorname{sgn}\langle \eta_i^s(\lambda), \eta_j^-(\lambda) \rangle$ genau dann Null ist, wenn $\mathrm{Fix}_i \perp \mathrm{Fix}_j$ und dass weder $\operatorname{sgn}\langle \eta_i^s(\lambda), \eta_j^-(\lambda) \rangle$ noch $\operatorname{sgn} B(\lambda)$ von λ abhängen.

Theorem D.4. *Sei $\dot{x} = f(x, \lambda)$ eine einparametrige Familie von Differentialgleichungen welche äquivariant bezüglich der endlichen Gruppe D_{4m} ist und bei $\lambda = 0$ einen relativen homoklinen Zykel Γ der Kodimension-1 mit hyperbolischer Gleichgewichtslage gemäß der Definition in Hypothesen (H.D.1) - (H.D.4) besitzt. Weiterhin sei $|B(0)| > C(0)$.*

Mit den $(4m \times 4m)$-Matrizen $A_- = -\frac{1}{2}(M - |M|)$ und $A_+ = -\frac{1}{2}(M + |M|)$, M gegeben durch (D.11), gilt für jede generische Familie, die einen relativen homoklinen Zykel wie oben beschrieben entfaltet, Folgendes:

Man nehme Querschnitte $\mathcal{S}_i$ transversal zu γ_i und schreibe Π_λ für die erste Rückkehrabbildung auf der Menge der Querschnitte $\cup_{j=1}^{4m} \mathcal{S}_j$. Für $\lambda > 0$ gibt es eine invariante Menge $\mathcal{D}_\lambda \subset \cup_{j=1}^{4m} \mathcal{S}_j$ für Π_λ so dass für jedes $\kappa \in \Sigma_{A_+}$ ein eindeutiges $x \in \mathcal{D}_\lambda$ mit $\Pi_\lambda^i(x) \in \mathcal{S}_{\kappa_i}$ existiert. Außerdem ist $(\mathcal{D}_\lambda, \Pi_\lambda)$ topologisch konjugiert zu (Σ_{A_+}, σ). Eine analoge Aussage gilt für $\lambda < 0$ mit Σ_{A_-} anstelle von Σ_{A_+}.

Die obige Beschreibung der Dynamik liefert ein vollständiges Bild der lokalen nichtwandernden Dynamik in der Nähe von Γ in dem Sinne, dass $A_- + A_+ = \mathbf{1}$ erfüllt ist. Nach der Definition von M, vgl. (D.11), finden wir

$$B(\lambda) > 0, \text{ dann } \quad m_{ij} = \begin{cases} 1, & |i-j| \geq m, \\ -1, & |i-j| < m, \end{cases} \quad \text{und} \quad B(\lambda) < 0, \text{ dann } \quad m_{ij} = \begin{cases} 1, & |i-j| > m, \\ -1, & |i-j| \leq m. \end{cases}$$

Die Differenz $i - j$ wird in $\mathbb{Z}_{4m}$ berechnet wobei $|i - j| := \min\{i - j, j - i\}$.

Für $m = 1$ und $\mathrm{sgn}B(\lambda) = 1$ haben die Matrizen A_- und A_+ also die Form,

$$A_- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_+ = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{D.12}$$

Nach diesem A_+ für $\lambda > 0$ besteht die nichtwandernde Dynamik aus allen Trajektorien, die vermeiden, zweimal derselben Homoklinen hintereinander zu folgen. Für $m = 1$ und $\mathrm{sgn}B(\lambda) = -1$ sehen diese Matrizen hingegen wie folgt aus

$$A_- = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \qquad A_+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

2. Fall: $|B(0)| \leq C(0)$

Dieser Fall lässt sich nicht in den Kontext von Theorem D.1 einbetten. Die Analysis erweist sich als diffiziler und erfordert die Unterscheidung weiterer Unterfälle wie die Unterteilung nach der topologischen Struktur des Faserbündels $\mathcal{F}(W^s_\gamma)$. In dieser Hinsicht ist eine vollständige Beschreibung der nichtwandernden Dynamik nicht unsere Absicht. Weitere Untersuchungen könnten zeigen, dass es notwendig ist, noch mehr Systemparameter einzuführen. Wir beschränken die Untersuchung daher auf periodische Lösungen. Dabei meinen wir mit Periodenlänge die Länge der wiederkehrenden Folge in κ und nicht die Übergangszeit der entsprechenden periodischen Trajektorie.

In dem nachfolgendem Theorem werden die Mengen $\mathcal{K}_2$ und $\mathcal{K}_4$ genannt, welche wir hier definieren und kurz erklären:

$$\left.\begin{array}{lcl} \mathcal{K}_2 & := & \{\kappa \in \Sigma_{4m} | \forall i \in \mathbb{Z} : i \in J_\kappa \text{ und } \kappa_i = \kappa_{i-2}\}, \\ \mathcal{K}_4 & := & \{\kappa \in \Sigma_{4m} | \forall i \in \mathbb{Z} : i \in J_\kappa \text{ und } \kappa_i = 2m + \kappa_{i-2}\}. \end{array}\right\} \tag{D.13}$$

Da jedes zweite Symbol in $\kappa \in \mathcal{K}_2$ für alle $i \in \mathbb{Z}$ gleich ist, enthält $\mathcal{K}_2$ alle κ, die den Trajektorien entsprechen, die abwechselnd dem Verlauf zweier verschiedener Homoklinen folgen. Die entsprechenden Trajektorien sind also 2-periodisch. Ferner liegen die verfolgten Homoklinen in zueinander orthogonalen Fixpunkträumen, da $i \in J_\kappa$ für alle $i \in \mathbb{Z}$. In einem D_4-äquivarianten Vektorfeld ist z.B. $\kappa = \overline{12} \in \mathcal{K}_2$, also die 2-periodische Trajektorie, die dem Weg der Homoklinen γ_1 und γ_2 folgt, vgl. Abbildung D.1.

Mit $\kappa_i = 2m + \kappa_{i-2}$ für alle $i \in \mathbb{Z}$ finden wir, dass $\kappa_i = 2m + 2m + \kappa_{i-4} = \kappa_{i-4}$. Somit ist jedes vierte Symbol in κ gleich und $\mathcal{K}_4$ besteht nur aus 4-periodischen Trajektorien. Auch hier liegen die nachverfolgten Homoklinen in zueinander orthogonalen Fixpunkträumen. Als Beispiel für eine Trajektorie in einem D_4-äquivarianten Vektorfeld sei $\kappa = \overline{1234} \in \mathcal{K}_4$ genannt, die 4-periodische Trajektorie, die γ_1, γ_2, γ_3 und γ_4 folgt, bevor sie sich schließt.

Theorem D.5. *Sei $\dot{x} = f(x, \lambda)$ eine einparametrige Familie von Differentialgleichungen, welche äquivariant bezüglich der endlichen Gruppe D_{4m} ist und die bei $\lambda = 0$ einen relativen homoklinen Zykel Γ der Kodimension-1 mit hyperbolischer Gleichgewichtslage gemäß der Definition in Hypothesen (H.D.1) - (H.D.4) hat. Weiterhin sei $|B(0)| \leq C(0)$ und $\mathcal{F}(W^s_\gamma)$ habe die topologische Struktur eines Ringes.*

Mit den $(4m \times 4m)$-Matrizen $A_- = -\frac{1}{2}(M - |M|)$ und $A_+ = -\frac{1}{2}(M + |M|)$, M gegeben durch (D.11), gilt für jede generische Familie, die einen relativen homoklinen Zykel wie oben beschrieben entfaltet, *Folgendes:*

(i) *Es sei* $B(0) > 0$. *Für alle* $N \in \mathbb{N}$ *gibt es ein hinreichend kleines* $\hat{\lambda}(N) > 0$, *so dass für alle* $\lambda \in (0, \hat{\lambda})$ *und alle periodischen* $\kappa \in \Sigma_{A_+} \setminus \mathcal{K}_2$ *mit einer Periodenlänge kleiner oder gleich* N *eine eindeutige periodische Trajektorie* $x(\lambda, \kappa) : \mathbb{R} \to \mathbb{R}^4$ *als Lösung von* $\dot{x} = f(x, \lambda)$ *existiert, die sich in der Nähe des homoklinen Zykels* Γ *befindet.*
Für $\lambda < 0$ *hinreichend klein gibt es eine solche eindeutige Trajektorie für jedes periodische* $\kappa \in \Sigma_{A_-} \cup \mathcal{K}_2$, *wenn* $|B(0)| < C(0)$, *oder für* $\kappa \in \Sigma_{A_-}$, *wenn* $|B(0)| = C(0)$.

(ii) *Es sei* $B(0) < 0$. *Für alle* $N \in \mathbb{N}$ *gibt es ein hinreichend kleines* $\hat{\lambda}(N) < 0$, *so dass für alle* $\lambda \in (\hat{\lambda}, 0)$ *und alle periodischen* $\kappa \in \Sigma_{A_-} \setminus \mathcal{K}_4$ *mit einer Periodenlänge kleiner oder gleich* N *eine eindeutige periodische Trajektorie* $x(\lambda, \kappa) : \mathbb{R} \to \mathbb{R}^4$ *als Lösung von* $\dot{x} = f(x, \lambda)$ *existiert, die sich in der Nähe des homoklinen Zykels* Γ *befindet.*
Für $\lambda > 0$ *hinreichend klein gibt es eine solche eindeutige Trajektorie für jedes periodische* $\kappa \in \Sigma_{A_+} \cup \mathcal{K}_4$, *wenn* $|B(0)| < C(0)$, *oder für* $\kappa \in \Sigma_{A_+}$, *wenn* $|B(0)| = C(0)$.

Hat $\mathcal{F}(W^s_\gamma)$ *die topologische Struktur eines Möbiusbandes, so gelten analoge Aussagen zu (i) und (ii) mit der Vertauschung der Mengen* $\mathcal{K}_2$ *und* $\mathcal{K}_4$.

Wir möchten an dieser Stelle kurz auf den Hintergrund des beschriebenen Verhaltens der Dynamik eingehen.

Im Gegensatz zum ersten Fall, wo $|B(0)| > C(0)$, zeigt sich hier, dass die Folge κ einen starken Einfluss auf die Größe der Übergangszeiten ω_i haben kann, welche die Bestimmungsgleichung lösen. Der Grund dafür ist, dass die r_i in (D.10) nicht mehr gleichmäßig beschränkt sind, sondern je nach Verlauf der Trajektorie, die durch κ gegeben ist, enorme Größenunterschiede aufweisen können. Somit kann der zweite Summand in (D.10) einen nicht zu vernachlässigenden Einfluss auf die Übergangszeiten ω_i haben.

Als problematisch erweisen sich die Sprünge, die der Darstellung

$$\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = \lambda - e^{4\mu^s(\lambda)\omega_i} B(\lambda) - e^{2\mu^s(\lambda)(\omega_{i-1}+\omega_i)} C_i(\lambda, \kappa), + \check{R}_i(\boldsymbol{\omega}, \lambda, \kappa),$$

mit $\operatorname{sgn}(C_i(\lambda, \kappa)) \neq \operatorname{sgn}(B(\lambda))$ entsprechen. Nehmen wir zum Beispiel $B(\lambda) > 0$ an und diskutieren wir kurz die Lösbarkeit von $\Xi_i(\boldsymbol{\omega}, \lambda, \kappa) = 0$ (unter Vernachlässigung der Restterme) für feste $\lambda > 0$. Ist $C_i(\lambda, \kappa) < 0$, so muss der Term $e^{4\mu^s(\lambda)\omega_i} B(\lambda)$ beide Terme λ und $e^{2\mu^s(\lambda)(\omega_{i-1}+\omega_i)} |C_i(\lambda, \kappa)|$ ausgleichen. Da $B(\lambda) < |C_i(\lambda, \kappa)|$ kann dies nur geschehen, indem man die Übergangszeit ω_i im Verhältnis zu ω_{i-1} verringert und damit $e^{2\mu^s(\lambda)\omega_i}$ im Verhältnis zu $e^{2\mu^s(\lambda)\omega_{i-1}}$ erhöht. Wenn beim nächsten Sprung wieder $C_{i+1} < 0$ gilt, muss erneut die entsprechende Übergangszeit ω_{i+1} gegenüber ω_i verringert werden. Dies geht immer so weiter bis zum nächsten Index j mit entweder $j \in \mathbb{Z} \setminus J_\kappa$ oder $C_j > 0$. Dann kann die Bestimmungsgleichung $\Xi_k = 0$ wieder für ein viel größeren ω_k gelöst werden: $\omega_{i-1} > \omega_i > \omega_{i+1} > \cdots > \omega_{k-1}$, $\omega_k >> \omega_{k-1}$.

Wenn nun die Sequenzen von aufeinanderfolgenden Sprüngen, für die $C_i(\lambda, \kappa) < 0$ gilt, endlich sind, kann die Bestimmungsgleichung bei $\lambda > 0$ für das gleiche, zumindest periodische, κ wie in Theorem 5.3.3 gelöst werden. Je nach Länge dieser Sequenzen können die Übergangszeiten jedoch stark in ihrer Größe variieren. Um sicherzustellen, dass $\inf(\boldsymbol{\omega})$ noch groß genug ist, um unserer Analysis zu genügen, muss der Wert von λ mitunter sehr klein werden. Es zeigt sich, dass $\lambda < \left({|B(0)|}/{C(0)}\right)^{2L} e^{4\mu^s(0)\inf \boldsymbol{\omega}}$ erfüllen muss, wobei L die Länge der längsten Kette von aufeinanderfolgenden Indizes i in κ bezeichnet, für die $C_i(\lambda, \kappa) < 0$ gilt. Folglich existieren einige Trajektorien nur für kleinere λ als andere.

Für diejenigen periodischen κ, die $C_i(\lambda, \kappa) < 0$ für alle $i \in \mathbb{Z}$ erfüllen, existieren keine Lösungen bei $\lambda > 0$. Für den Fall aber, dass $|B(0)| < C(0)$, kann die Bestimmungsgleichung für diese κ für $\lambda < 0$ gelöst werden. Während also alle anderen periodischen Trajektorien, die Homoklinen in zueinander orthogonalen Fixpunkträumen folgen, für $\lambda > 0$ existieren, finden wir diese Ausnahme für das entgegengesetzte Vorzeichen von λ. Wenn $|B(0)| = |C_i(0, \kappa)|$ gilt, existieren periodische Trajektorien, die diesen κ entsprechen, weder für positive noch für negative λ. Daher erfüllt die Existenz dieser Trajektorien nicht die Regel, die eine Markov-Kette impliziert.

Die periodischen κ für die $\text{sgn}C_i(\lambda, \kappa) \neq \text{sgn}B(\lambda)$ für alle $i \in \mathbb{Z}$ gilt, sind nun gerade die, welche in den in (D.13) definierten Mengen $\mathcal{K}_2$ oder $\mathcal{K}_4$ liegen, je nachdem, welche topologische Struktur $\mathcal{F}(W^s_\gamma)$ besitzt.

Diskussion

Mit dem Vektorfeld

$$f_4(x, \lambda) = \begin{pmatrix} ax_1 & +bx_2 & -ax_1^3 + 3ax_1y_1^2 & \\ ay_1 & +by_2 & +3ax_1^2y_1 - ay_1^3 & \\ bx_1 & +ax_2 & -2bx_1^3 + 6bx_1y_1^2 & -2ax_1^2x_2 + 2ay_1^2x_2 + 4ax_1y_1y_2 \\ by_1 & +ay_2 & +6bx_1^2y_1 - 2by_1^3 & +4ax_1y_1x_2 + 2ay_2(x_1^2 - y_1^2) \end{pmatrix} + \lambda \begin{pmatrix} 2x_1 & -4x_1^3 + 12x_1y_1^2 \\ 2y_1 & +12x_1^2y_1 - 4y_1^3 \\ -2x_2 & \\ -2y_2 & \end{pmatrix}. \tag{D.14}$$

haben wir für $a, b \in \mathbb{R}\backslash\{0\}$, $a^2 < b^2$ ein Beispielsystem konstruiert, welches bei $\lambda = 0$ einen relativen D_4-symmetrischen homoklinen Zykel besitzt, wie er in den Hypothesen (H.D.1) - (H.D.4) beschrieben wird. Die Gültigkeit der Hypothese (H.D.4)(i) haben wir allerdings nur für extrem kleine Werte von $|a|$ analytisch zeigen können. Wir gehen jedoch davon aus, dass (H.D.4)(i) auch für sehr viel größere Werte von $|a|$ gilt.

Für dieses System haben wir numerisch mit MATLAB gezeigt, dass der Term $B(0)$ verschieden von Null ist. Weiterhin haben wir mit der Verfolgungs-Software AUTO die Existenz verschiedener periodischer Orbits in der Umgebung des Homoklinen Zykels untersucht. Aus Konvergenzgründen haben wir hierbei jedoch die Forderung (H.D.3)(iv) invertiert und den führenden instabilen Eigenwert als den dominanten Eigenwert gewählt. Tatsächlich bewirkt dies einfach eine Zeitumkehr, wodurch sich nur die Durchlaufrichtungen aller Trajektorien ändert. Die Gültigkeit der analytischen Ergebnisse bleibt dabei im Wesentlichen - bis auf eine Vertauschung des Vorzeichens von λ - für das Beispielsystem erhalten.

Im Ergebnis konnten wir die durch die Sequenzen $\kappa = \overline{12}, \overline{13}, \overline{123}, \overline{1234}, \overline{1243}$ and $\overline{1214}$ charakterisierten periodischen Trajektorien für $\lambda < 0$ finden. Für $\lambda > 0$ ließen sich diese Trajektorien nicht detektierten, dafür aber die 1-periodische Trajektorie $\kappa = \overline{1}$. Weder für $\lambda > 0$ noch für $\lambda < 0$ ließ sich die Trajektorie $\kappa = \overline{121}$ finden.

Diese Ergebnisse entsprechen unseren Erwartungen, die wir aufgrund des Theorems D.4 hatten. Sie zeigen, dass die Beziehung $|B(0)| > C(0)$ gilt, da die periodischen Trajektorien $\kappa = \overline{12} \in \mathcal{K}_2$ und $\kappa = \overline{1234} \in \mathcal{K}_4$ für dasselbe Vorzeichen von λ existieren. Weiterhin können wir schlussfolgern, dass das Vorzeichen von $B(0)$ positiv ist, da die periodische Trajektorie $\kappa = \overline{1243}$ existiert, anstelle beispielsweise der Trajektorie $\kappa = \overline{121}$. Daher erhalten wir für die in Theorem D.4 genannten Matrizen A_- und A_+

$$A_- = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Da in unseren numerischen Betrachtungen der instabile Eigenwert der führende ist, ist die Zuordnung der Matrizen A_- und A_+ umgekehrt zu derjenigen in (D.12).

Zudem konnten wir die unterschiedlichen Übergangszeiten erkennen, die benötigt werden, um von einer homoklinen Trajektorie zu einer anderen zu gelangen. Es ließ sich ablesen, dass es etwa doppelt so lange dauert, im gleichen Fixraum zu bleiben, als zu einer homoklinen Trajektorie zu wandern, die in einem orthogonalen Fixraum liegt. Dies stimmt mit den berechneten Übergangszeiten in (D.10) überein.

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Erklärung zur Selbstständigkeit

Ich versichere, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.

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Maria Kellner