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Calculus of unbounded spectrahedral shadows and their polyhedral approximation

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Zusammenfassung

Calculus of unbounded spectrahedral shadows and their polyhedral approximation

von Daniel Dörfler

Die vorliegende Dissertation widmet sich der polyedrischen Approximation von nicht notwendigerweise beschränkten spektraedrischen Schatten und dem Rechnen mit ihnen. Diese Mengen sind die Bilder von Spektraedern, den zulässigen Bereichen semidefiniter Programme, unter linearen Transformationen. Zwei benennenswerte Eigenschaften spektraedrischer Schatten sind ihre Abgeschlossenheit bezüglich verschiedener Mengenoperationen sowie die Möglichkeit sie mithilfe linearer Operatoren zwischen symmetrischen Matrizen auf endliche Weise darzustellen. Außerdem bilden polyedrische Mengen eine echte Teilklasse von ihnen. Dadurch eignet sich die Methode der polyedrischen Approximation, um diese Mengen annähernd durch Objekte derselben Klasse mit einfacherer Struktur zu beschreiben.

Grundlagen über affine Räume und lineare Abbildungen, Konvexität und semidefinite Optimierung werden in Kapitel 1 wiederholt. In Kapitel 2 wird ein Kalkül für spektraedrische Schatten entwickelt. Neben dem Herausarbeiten ihrer Abgeschlossenheit unter zahlreichen Mengenoperationen werden insbesondere explizite Darstellungen der resultierenden Menge als spektraedrische Schatten hergeleitet. Dabei stehen Operationen, die auf unbeschränkte Mengen führen, wie z.B. Polarkegel, konische Hülle und Rezessionskegel, im Fokus.

Kapitel 3 widmet sich der Approximation kompakter spektraedrischer Schatten im Hausdorff-Abstand. Es werden zwei Algorithmen zur Approximation durch Polyeder vorgestellt und gezeigt, dass diese terminieren und korrekt sind. In Vorbereitung darauf wird außerdem ein Algorithmus entwickelt, der relativ innere Punkte sowie die affine Hülle eines spektraedrischen Schattens berechnet. Abschließend werden Grenzen der polyedrischen Approximierbarkeit bezüglich des Hausdorff-Abstandes im Allgemeinen untersucht und, aufbauend auf bekannten Resultaten, diejenigen Mengen charakterisiert, die eine solche Approximation erlauben.

In Kapitel 4 werden Approximationskonzepte entwickelt, die mit Unbeschränktheit verträglich sind. Zuerst wird der Begriff der (ε, δ)-Approximation zur äußeren Approximation abgeschlossener spitzer konvexer Mengen eingeführt. Es wird gezeigt, dass Folgen solcher Approximationen bei abnehmenden Approximationsfehlern konvergieren. Im Hinblick auf ein Verfahren für ihre Berechnung wird anschließend ein Algorithmus zur polyedrischen Approximation der Rezessionskegel spektraedrischer Schatten entwickelt, dessen Endlichkeit und Korrektheit ebenfalls bewiesen wird. Durch Kombination der bis dahin vorgestellten Algorithmen wird ein Verfahren zur Berechnung von (ε, δ)-Approximationen abgeleitet. Im letzten Teil der Arbeit wird ein weiterer Approximationsbegriff basierend auf Homogenisierungen konvexer Mengen eingeführt. Die sogenannten homogenen δ-Approximationen erweisen sich als schwächer im Vergleich zu (ε, δ)-Approximationen, lassen sich aber vielfältiger nutzen. Insbesondere beschränken sie sich nicht auf Außenapproximationen und sind verträglich mit Polarität. Um homogene δ-Approximationen zu berechnen, wird der vorhandene Algorithmus zur Approximation von Rezessionskegeln auf Homogenisierungen spektraedrischer Schatten angewandt.

Abstract

Calculus of unbounded spectrahedral shadows and their polyhedral approximation

by Daniel Dörfler

The present thesis deals with the polyhedral approximation and calculus of spectrahedral shadows that are not necessarily bounded. These sets are the images of spectrahedra, the feasible regions of semidefinite programs, under linear transformations. Two remarkable properties of spectrahedral shadows are their closedness under various set operations as well as the possibility to represent them in a finite manner using linear operators between symmetric matrices. Moreover, spectrahedral shadows contain polyhedral sets as a proper subclass. Therefore, the method of polyhedral approximation is a useful device to approximately describe them using members of the same class with a simpler structure.

Basics about affine spaces and linear functions, convexity and semidefinite programming are recapitulated in Chapter 1. In Chapter 2 we develop a calculus for spectrahedral shadows. Besides showing their closedness under numerous set operations, we derive explicit descriptions of the resulting sets as spectrahedral shadows. Special attention is paid to operations that result in unbounded sets, such as the polar cone, conical hull and recession cone.

Chapter 3 is dedicated to the approximation of compact spectrahedral shadows with respect to the Hausdorff distance. We present two algorithms for the computation of polyhedral approximations of such sets. Convergence as well as correctness of both algorithms are proved. As a supplementary tool we also present an algorithm that generates points from the relative interior of a spectrahedral shadow and computes its affine hull. Finally, we investigate the limits of polyhedral approximation in the Hausdorff distance in general and, extending known results, characterize the sets that admit such approximations.

In Chapter 4 we develop concepts and tools for the approximation of spectrahedral shadows that are compatible with unboundedness. We begin with presenting the notion of (ε, δ)-approximation for the outer approximation of closed pointed convex sets. We show that sequences of approximations converge to the true set if the approximation errors diminish. In view of an algorithm for their computation we develop an algorithm for the polyhedral approximation of recession cones of spectrahedral shadows first. Its finiteness and correctness is proved. By combining the algorithms presented so far we derive an algorithm for the computation of (ε, δ)-approximations. In the last part another notion of approximation, called homogeneous δ-approximation, is introduced. It is based on the concept of homogenization of a convex set and is weaker than the notion of (ε, δ)-approximation but addresses two of its shortcomings. In particular, it is not limited to outer approximations and exhibits an elegant behavior under polarity. In order to compute homogeneous δ-approximations we apply the algorithm for the approximation of recession cones to homogenizations of spectrahedral shadows.

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Glossary of Notation

$\mathbb{R}$	set of real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^n_+$	nonnegative orthant in $\mathbb{R}^n$
$\mathcal{M}^\ell$	space of real $\ell \times \ell$-matrices
$\mathcal{S}^\ell$	space of real symmetric $\ell \times \ell$-matrices
$\mathcal{S}^\ell_+$	cone of positive semidefinite $\ell \times \ell$-matrices
$\mathcal{S}^\ell_{++}$	set of positive definite $\ell \times \ell$-matrices
$X \succcurlyeq 0$	X is positive semidefinite
$X \succ 0$	X is positive definite
$(\cdot)^\mathsf{T}$	transpose or adjoint
$x^\mathsf{T} y$	dot product of $x, y \in \mathbb{R}^n$
$X \bullet Y$	inner product of $X, Y \in \mathcal{M}^\ell$
aff M	affine hull of M
conv M	convex hull of M
cone M	conical hull of M
$0^\infty C$	recession cone of C
int C	interior of C
relint C	relative interior of C
cl C	closure of C
C°	polar of C
C^*	polar cone of C
homog C	homogenization of C
ext C	set of extreme points of C
extdir C	set of extreme directions of C
vert P	set of vertices of P
$A, B, \ldots$	linear functions between $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n, m \in \mathbb{N}$, i.e. matrices
$\mathcal{A}, \mathcal{B}, \ldots$	linear functions between $\mathbb{R}^n$ and $\mathcal{S}^\ell$ for $n, \ell \in \mathbb{N}$, i.e. linear pencils of size ℓ
$\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \ldots$	linear functions between $\mathcal{S}^k$ and $\mathcal{S}^\ell$ for $k, \ell \in \mathbb{N}$
$A \otimes B$	Kronecker product between A and B
$A \oplus B$	direct sum between A and B
e_i	vector with 1 at position i and 0 elsewhere
e	vector whose components are all equal to 1
I	identity matrix
E_{ij}	square matrix with 1 at position ij and 0 elsewhere
$\|\cdot\|$	Euclidean norm
$B_r(c)$	Euclidean ball of radius r centered at c
$d(y, C)$	Euclidean distance between y and C
$\pi_C(y)$	projection of y onto C
$e[C_1, C_2]$	excess of C_1 over C_2
$d_\mathsf{H}(C_1, C_2)$	Hausdorff distance between C_1 and C_2
$d_{\mathsf{tH}}(C_1, C_2)$	truncated Hausdorff distance between C_1 and C_2

Chapter 1

Introduction

Archimedes, the Greek mathematician, wrote a treatise titled *Measurements of a Circle* ca. 250 BCE [Kno86]. It contains three propositions, the third of which states: *The ratio of the circumference of any circle to its diameter is less than* $\frac{22}{7}$ *but greater than* $\frac{223}{71}$ [Hea97]. Using modern notation, the proposition is rephrased as

$$\frac{223}{71} < \pi < \frac{22}{7},$$

i.e. Archimedes proved a lower and an upper bound of π that is correct up to two decimal places. He derives the bounds by inscribing and circumscribing 96-sided regular polygons to a circle and measuring their circumference. Then he showed that $\frac{223}{71}$ is a lower bound for the circumference of the inscribed polygon and that $\frac{22}{7}$ is an upper bound for the circumference of the circumscribed polygon, which results in the claim. Archimedes's approach is an example of *polyhedral approximation* of the unit circle. He realized that better polyhedral approximations yield better approximations of π and that π can be computed to any prescribed accuracy using this method. Indeed, in 1630 the astronomer Christoph Grienberger used Archimedes's method to calculate π to 39 decimal places [AH01].

While in the above example polyhedral approximation is merely a tool to approximate π, it has become a profound method in convex analysis with applications ranging through various branches of mathematics in modern literature. In the 1903 article [Min03] Minkowski argued that every compact convex set in three-dimensional Euclidean space can be approximated arbitrarily well by *polyhedra*. Although it is not explicitly stated in the article, his line of argumentation also applies to general n-dimensional Euclidean space. This is later mentioned by Bonnesen and Fenchel in [BF34].

Since then, polyhedral approximation has been applied in mathematical programming, in particular in multiple objective optimization [Ben98; RW05; ESS11; HLR14; LRU14; Dör+22; AUU22], in approximation methods for general convex optimization problems [CG59; Kel60; Vei67; BY11] and in mixed-integer convex optimization [DG86; WP95; KLW16]. Other areas of application include machine learning [Gie+19; YBR08] and large deviation probability theory [NR95].

Despite polyhedral approximation being an interesting problem on its own, interest in it is also driven by the fact that polyhedra form a class of sets that have a simple structure in the sense that they can be described by finitely many points and directions or, equivalently, by finitely many linear inequalities. For that matter, they have already been studied in ancient Greek mathematics, for example by Euclid, who describes the construction of the five platonic solids in his Elements [Euc56].

In this thesis we develop polyhedral approximation algorithms for a special class of convex sets called *spectrahedral shadows* or *semidefinitely representable sets*. They are obtained as linear projections of *spectrahedra* and therefore form a superclass thereof.

Spectrahedra are intersections of the cone of positive semidefinite matrices with affine subspaces and are closely related to semidefinite programming, where they arise as feasible regions. Semidefinite programming is the problem of maximizing a linear function over a spectrahedron. The field has gained a lot of traction with the development of efficient interior point methods in the last decade of the last century, making them computationally tractable. Moreover, semidefinite programming subsumes linear programming as a special case. Hence, semidefinite programs constitute one of the largest class of optimization problems that can be solved in polynomial time to arbitrary precision [VB96]. They have a wide range of applications in combinatorial optimization, where they are used to obtain approximate solutions to NP-hard problems such as the max cut problem [Hel00; GM12], in systems and control theory [VB99], in machine learning, in particular low rank approximation [CR09], and polynomial programming, where semidefinite optimization is used to treat sum-of-squares optimization problems [PT20].
Fueled by the growing popularity of semidefinite programming, spectrahedra and their projections have attracted researchers' attention and efforts were undertaken to gain a better understanding of these classes of sets and their geometrical properties [GR95; Net11]. Understanding which sets can be cast as spectrahedral shadows also means understanding which problems can be modeled as semidefinite optimization problems and therefore solved in polynomial time. Various research has been conducted in this regard [HN10; NS09; HN12; GWZ15; Sch18b].
The literature on polyhedral approximation algorithms and spectrahedral shadows cited so far focuses on compact sets. Indeed, the case of unbounded sets is scarcely discussed. Two important resources are [NR95] by Ney and Robinson, who give a characterization of the sets that can be approximated by polyhedra in Hausdorff distance, and [Ulu18] by Ulus, who gives a similar characterization in the context of convex multiple objective optimization. With the present thesis we investigate not necessarily bounded spectrahedral shadows and provide methods for their polyhedral approximation. This provides means to investigate the geometry of these sets, visualize them in dimensions two and three and, hopefully, drive further research.

1.1 Outline and results of this thesis

In the remainder of this chapter we recapitulate facts about linear algebra, convexity and semidefinite programming, that are used throughout the subsequent chapters. We mostly stick to the notation and terminology in the books by Rockafellar [Roc70] and Boyd and Vandenberghe [BV04].
We develop a calculus for spectrahedral shadows in Chapter 2. Thereby, we show that spectrahedral shadows are closed under a multitude of set operations, such as intersections, linear transformations, Minkowski sums, taking the polar or taking the conical hull, and derive explicit semidefinite representations for these sets. One particular focus is on constructing the recession cone of a spectrahedral shadow, which has not been achieved in the literature before. Our derivation relies on an exact duality theory for semidefinite optimization due to Ramana [Ram97].
Chapter 3 is devoted to the polyhedral approximation of compact spectrahedral shadows in the Hausdorff distance. An inner and an outer approximation algorithm are presented in [Cir19]. We build upon these algorithms and present another outer approximation algorithm based on a norm scalarization. We show that the inner approximation algorithm from [Cir19] and our outer approximation algorithm fit into the framework for the polyhedral approximation of general compact convex

sets presented in [Kam92; Kam93; Kam94] and derive convergence results. We illustrate the functioning of the algorithms and compare their performance on examples. It turns out that the Hausdorff distance is no adequate measure to quantify the quality of a polyhedral approximation if the set to be approximated is unbounded. In the last part of Chapter 3 we investigate the limits of polyhedral approximation in the Hausdorff distance. We compare the characterizations of polyhedral approximability from [NR95] and [Ulu18] and prove relationships between the two.
The development of polyhedral approximation algorithms for unbounded spectrahedral shadows is the focus of Chapter 4. In connection with the results of the preceding chapter we define a notion of polyhedral approximation for convex sets called (ε, δ)-approximation. One important component which makes this notion applicable in the unbounded case in contrast to the Hausdorff distance is the involvement of the recession cones of the polyhedron and the convex set. The representation derived in Chapter 2 turns out to be useful in this context. We prove that (ε, δ)-approximations define a meaningful concept by showing that sequences of approximations converge to the actual set if ε and δ diminish. Thereafter, we present an algorithm for the approximation of recession cones of spectrahedral shadows using a metric on the space of closed convex cones defined in [WW67; RW98; IS10] as the error measure. We combine this algorithm with one presented in Chapter 3 to derive an algorithm for the computation of (ε, δ)-approximations of spectrahedral shadows. Correctness and finiteness of the algorithms are proved and numerical experiments are conducted. In the last part of this thesis we introduce the concept of homogeneous δ-approximation, another notion of polyhedral approximation that is more general than that of (ε, δ)-approximation but converges in the same sense. It arises naturally from the identification of a convex set with a certain convex cone called its *homogenization*. These are commonly used in convex analysis to reduce problems concerning convex sets to ones concerning convex cones, see e.g. [Bri20]. We investigate the relation of both approximation concepts and show that homogeneous δ-approximations exhibit a particularly nice behavior under polarity. Ultimately, we apply the recession cone algorithm to homogenizations of spectrahedral shadows. After an exhaustive search of the literature, we believe that it is the first time such approximation concepts compatible with unboundedness are considered and corresponding algorithms tailored to spectrahedral shadows are developed.

1.2 Affine spaces and linear functions

Throughout this thesis we work with inner product spaces over the field $\mathbb{R}$ of real numbers. Most frequently we use the real n-space $\mathbb{R}^n$, but, since semidefinite optimization will play a role later, we also work with the space $\mathcal{M}^\ell$ of *real* $\ell \times \ell$*-matrices*, i.e. $\mathcal{M}^\ell = \mathbb{R}^{\ell \times \ell}$, and the space $\mathcal{S}^\ell$ of *real symmetric* $\ell \times \ell$*-matrices*. By writing $(\cdot)^\mathsf{T}$ we mean the *transpose* of the object it is applied to, which will typically be vectors, matrices or linear functions. Therefore, it holds $\mathcal{S}^\ell = \left\{ M \in \mathcal{M}^\ell \,\middle|\, M^\mathsf{T} = M \right\}$. The space $\mathcal{M}^\ell$ is isomorphic to $\mathbb{R}^{\ell^2}$ by identifying an element $x \in \mathbb{R}^{\ell^2}$ with the matrix

$$\begin{pmatrix} x_1 & \cdots & x_{(\ell-1)\ell+1} \\ \vdots & \ddots & \vdots \\ x_\ell & \cdots & x_{\ell^2} \end{pmatrix}$$

and vice versa. Likewise, $\mathcal{S}^\ell$ is isomorphic to $\mathbb{R}^N$ for $N = \frac{\ell(\ell+1)}{2}$ due to the symmetry. However, it is convenient to work with $\mathcal{M}^\ell$ and $\mathcal{S}^\ell$ in order to avoid unnecessarily

complicating the notation.
The space $\mathbb{R}^n$ consists of column vectors and is equipped with the *dot product* or *scalar product*. For $x, y \in \mathbb{R}^n$ this is denoted by $x^\mathsf{T} y$. By the above mentioned isomorphism the dot product can be extended to $\mathcal{M}^\ell$ or $\mathcal{S}^\ell$. It is easy to verify that for $X, Y \in \mathcal{M}^\ell$ one obtains the expression

$$\operatorname{tr}\left(X^\mathsf{T} Y\right),$$

written as $X \bullet Y$, where tr denotes the *trace*, i.e. the sum of eigenvalues of the matrix product $X^\mathsf{T} Y$, see [BV04; GM12]. Of course, for $X, Y \in \mathcal{S}^\ell$ the inner product simplifies to $X \bullet Y = \operatorname{tr}(XY)$.
Whenever we are dealing with a set, we will assume that it is nonempty unless otherwise stated. One important type of sets, that are needed to explain the structure of spectrahedra and spectrahedral shadows later on, are affine subspaces. We make the following definitions and statements for $\mathbb{R}^n$, but they can be given with respect to $\mathcal{M}^\ell$ or $\mathcal{S}^\ell$ in exactly the same way.

Definition 1.1. A set $M \subseteq \mathbb{R}^n$ is called an *affine subspace* of $\mathbb{R}^n$ if for every $x, y \in M$ and $\lambda \in \mathbb{R}$ it holds that $\lambda x + (1-\lambda) y \in M$. Thus, M is an affine subspace if every line passing through two points in M is itself contained in M.

Given an arbitrary set $M \subseteq \mathbb{R}^n$, the smallest, with respect to inclusion, affine subspace containing M is called the *affine hull* of M. It can be be expressed as

$$\operatorname{aff} M = \left\{ \sum_{i=1}^{m} \lambda_i x^i \;\middle|\; m \in \mathbb{N}, \lambda \in \mathbb{R}^m, x^i \in M, i = 1, \ldots, m, e^\mathsf{T} \lambda = 1 \right\},$$

where e denotes the vector whose components are all equal to one of appropriate size.

Definition 1.2. For a vector $\omega \in \mathbb{R}^n \setminus \{0\}$ and a scalar $\gamma \in \mathbb{R}$ the set

$$H(\omega, \gamma) = \left\{ x \in \mathbb{R}^n \;\middle|\; \omega^\mathsf{T} x = \gamma \right\}$$

is called a *hyperplane*. The vector ω is called the *normal vector* of $H(\omega, \gamma)$. The sets

$$\begin{aligned} H^-(\omega, \gamma) &= \left\{ x \in \mathbb{R}^n \;\middle|\; \omega^\mathsf{T} x \leqslant \gamma \right\}, \\ H^+(\omega, \gamma) &= \left\{ x \in \mathbb{R}^n \;\middle|\; \omega^\mathsf{T} x \geqslant \gamma \right\} \end{aligned}$$

are called *negative* and *positive halfspace*, respectively. Both sets may also simply be called halfspaces.

Definition 1.3. For a set $M \subseteq \mathbb{R}^n$ a hyperplane $H(\omega, \gamma)$ is called a *supporting hyperplane* of M at x if $x \in M \cap H(\omega, \gamma)$ and $M \subseteq H^-(\omega, \gamma)$ or $M \subseteq H^+(\omega, \gamma)$. The halfspace that contains M is called a *supporting halfspace* of M at x.

The parameters ω and γ uniquely determine a hyperplane up to multiplication with a nonzero factor. The concept of supporting hyperplanes plays a crucial role in the constructions of approximation algorithms in Chapters 3 and 4. While being affine subspaces themselves, hyperplanes are also the building blocks of them as explained in the following theorem.

Theorem 1.4 (cf. [Roc70, Theorem 1.4])**.** *Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then the set $\{x \in \mathbb{R}^n \mid Ax = b\}$ is an affine subspace. Conversely, every affine subspace can be represented in this way.*

Thus, affine subspaces are intersections of hyperplanes or, equivalently, the solutions sets to systems of linear equations. Similarly, the halfspaces associated with a hyperplane can be understood as the building blocks of closed convex sets, which we will discuss in the next section.
Since linear functions play an important role in describing affine subspaces, it is helpful to briefly discuss the type of functions that we need later. Like in the previous theorem, the linear functions between $\mathbb{R}^n$ and $\mathbb{R}^m$ are identified by $m \times n$-matrices. In describing spectrahedral shadows in Chapter 2 we also use linear functions between the spaces $\mathbb{R}^n$ and $\mathcal{S}^\ell$ as well as between $\mathcal{S}^k$ and $\mathcal{S}^\ell$. Indeed, those functions can also be represented by matrices of appropriate size by identifying suitable bases, see [HK71, Ch. 3, Theorem 11]. However, in the context of semidefinite programming it is more convenient to express them in a different way, as it is for example done in [NN94; Boy+94; BPT13].

Definition 1.5. For $A_1, \ldots, A_n \in \mathcal{S}^\ell$ the function $\mathcal{A}\colon \mathbb{R}^n \to \mathcal{S}^\ell$ defined by

$$\mathcal{A}(x) = \sum_{i=1}^{n} A_i x_i$$

is called a *linear pencil* of size ℓ.

When speaking about linear pencils, the value of n is typically apparent from the context or will be mentioned explicitly. Obviously, a linear pencil is a linear function and every linear function from $\mathbb{R}^n$ to $\mathcal{S}^\ell$ can be represented as a linear pencil of size ℓ by considering the ordered basis $\{E_{ij} \in \mathcal{M}^\ell \mid i, j = 1, \ldots, \ell\}$ on $\mathcal{S}^\ell$, where E_{ij} denotes the matrix that has a 1 at position ij and 0 everywhere else. In this way, we can identify every linear function between $\mathbb{R}^n$ and $\mathcal{S}^\ell$ by n matrices belonging to $\mathcal{S}^\ell$. Similarly, every linear map $\boldsymbol{\mathcal{A}}$ between $\mathcal{S}^k$ and $\mathcal{S}^\ell$ can be represented as

$$\boldsymbol{\mathcal{A}}(X) = \sum_{i=1}^{k} \sum_{j=1}^{k} A_{ij} x_{ij}$$

for matrices $A_{ij} \in \mathcal{S}^\ell$, $i, j = 1, \ldots, k$.

Definition 1.6. Let U and W be inner product spaces with inner products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_W$ and let $A\colon U \to W$ be a linear function. A linear function $A^\mathsf{T}\colon W \to U$ is called *adjoint* of A if

$$\langle A(x), y \rangle_W = \langle x, A^\mathsf{T}(y) \rangle_U$$

holds for all $x \in U$ and $y \in W$.

It is well-known that the adjoint of a linear function is unique given that U and W are finite-dimensional, see [HK71, Ch. 3, Theorem 21]. Furthermore, the identity $(A^\mathsf{T})^\mathsf{T} = A$ holds [Rom05, Theorem 3.19]. Adjoint functions are relevant in conic programming, where they arise in the formulation of dual problems [Nem07]. We use the adjoints of the linear functions discussed above frequently in Chapters 2, 3 and 4. Hence, we state explicit representations of them in terms of their defining matrices below.

Proposition 1.7. *Let $A\colon \mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{A}\colon \mathbb{R}^n \to \mathcal{S}^\ell$ and $\boldsymbol{\mathcal{A}}\colon \mathcal{S}^k \to \mathcal{S}^\ell$ be linear functions defined by $A(x) = Ax$ for $A \in \mathbb{R}^{m \times n}$, $\mathcal{A}(x) = \sum_{i=1}^n A_i x_i$ for $A_1, \ldots, A_n \in \mathcal{S}^\ell$ and $\boldsymbol{\mathcal{A}}(X) = \sum_{i=1}^k \sum_{j=1}^k A_{ij} x_{ij}$ for $A_{ij} \in \mathcal{S}^\ell$, $i, j = 1, \ldots, k$, respectively. Then*

(i) $A^\mathsf{T}(y) = A^\mathsf{T} y$,

(ii) $\mathcal{A}^{\mathsf{T}}(Y) = (A_1 \bullet Y, \dots, A_n \bullet Y)^{\mathsf{T}}$,

(iii) for $A'_{ij} = \frac{1}{2}\left(A_{ij} + A_{ji}\right)$, $i, j = 1, \dots, k$, *it holds*

$$\boldsymbol{\mathcal{A}}^{\mathsf{T}}(Y) = \begin{pmatrix} A'_{11} \bullet Y & \cdots & A'_{1k} \bullet Y \\ \vdots & \ddots & \vdots \\ A'_{k1} \bullet Y & \cdots & A'_{kk} \bullet Y \end{pmatrix}.$$

Proof. Assertion (i) is basic knowledge and can be found in [HK71, Ch. 3, Theorem 23], for example. For a proof of (ii) see [GM12, Lemma 4.5.3]. To show the last statement, we compute

$$\begin{aligned} \boldsymbol{\mathcal{A}}(X) \bullet Y &= \left(\sum_{i=1}^{k}\sum_{j=1}^{k} A_{ij}x_{ij}\right) \bullet Y \\ &= \left(\sum_{i=1}^{k}\left(A_{ii}x_{ii} + \sum_{j=i+1}^{k}\left(A_{ij} + A_{ji}\right)x_{ij}\right)\right) \bullet Y \\ &= \left(\sum_{i=1}^{k}\left(A_{ii}x_{ii} + \sum_{j\neq i}\frac{1}{2}\left(A_{ij} + A_{ji}\right)x_{ij}\right)\right) \bullet Y \\ &= \left(\sum_{i=1}^{k}\sum_{j=1}^{k} A'_{ij}x_{ij}\right) \bullet Y \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k}\left(A'_{ij} \bullet Y\right)x_{ij} \\ &= \boldsymbol{\mathcal{A}}^{\mathsf{T}}(Y) \bullet X. \end{aligned}$$

In the second line we use the symmetry of X and in the fifth the linearity of the inner product. □

Note, that we identify both the linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$ and the $m \times n$-matrix representing this function with the symbol A in the above statement.

Definition 1.8. For matrices $A \in \mathbb{R}^{m_1 \times n_1}$ and $B \in \mathbb{R}^{m_2 \times n_2}$ the *Kronecker product* between A and B is the $(m_1 m_2) \times (n_1 n_2)$-matrix

$$\begin{pmatrix} a_{11}B & \cdots & a_{1n_1}B \\ \vdots & \ddots & \vdots \\ a_{m_1}B & \cdots & a_{m_1n_1}B \end{pmatrix}$$

denoted by $A \otimes B$.

It is straightforward to verify that $(A \otimes B)^{\mathsf{T}} = A^{\mathsf{T}} \otimes B^{\mathsf{T}}$.
In order to stay close to the common matrix terminology we adopt the following notation from [GM12], which we mainly utilize in Chapter 2. If $U_1, \dots, U_n$ and $W_1, \dots, W_m$ are inner product spaces and $A_{ij} \colon U_j \to W_i$ are linear functions for all i, j, we write

$$\left(\begin{array}{c|c|c} A_{11} & \cdots & A_{1n} \\ \hline \vdots & \ddots & \vdots \\ \hline A_{m1} & \cdots & A_{mn} \end{array}\right)$$

for the linear function $A' \colon U_1 \oplus \ldots \oplus U_n \to W_1 \oplus \ldots \oplus W_m$ defined by

$$A'(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} A_{1j}(x_j), \ldots, \sum_{j=1}^{n} A_{mj}(x_j) \right).$$

This notation is useful because we have

$$A'^{\mathsf{T}} = \left(\begin{array}{c|c|c} A_{11}^{\mathsf{T}} & \cdots & A_{m1}^{\mathsf{T}} \\ \hline \vdots & \ddots & \vdots \\ \hline A_{1n}^{\mathsf{T}} & \cdots & A_{mn}^{\mathsf{T}} \end{array} \right)$$

just as with matrices, which can be verified by a simple calculation. We omit the horizontal and vertical lines if the A_{ij} are matrices. In the special case $W_i = \mathcal{S}^{\ell_i}$, $i = 1, \ldots, m$, we identify the elements of $\mathcal{S}^{\ell_1} \oplus \ldots \oplus \mathcal{S}^{\ell_m}$ by block diagonal matrices in $\mathcal{S}^{\ell_1 + \cdots + \ell_m}$. Another notation we use for linear functions $A \colon U_1 \to W_1$ and $B \colon U_2 \to W_2$ between inner product spaces U_1, U_2, W_1 and W_2 is $A \oplus B$. It is defined as the function

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right)$$

between $U_1 \oplus U_2$ and $W_1 \oplus W_2$. Thereby, the zeros denote functions that are identically zero between the respective spaces. As above, this definition is in accordance with the definition of direct sum for matrices.

1.3 Convexity

Special convex sets play a central role in this thesis. Hence, we recapitulate some important facts from the literature.

Definition 1.9. A set $C \subseteq \mathbb{R}^n$ is called *convex* if for every $x, y \in C$ and $\lambda \in [0,1]$ it holds that $\lambda x + (1-\lambda)\, y \in C$. Thus, C is a convex set if every line segment between two points in C is itself contained in C.

Given an arbitrary set $M \subseteq \mathbb{R}^n$, the smallest, with respect to inclusion, convex set containing M is called the *convex hull* of M. It can be expressed as

$$\operatorname{conv} M = \left\{ \sum_{i=1}^{m} \lambda_i x^i \;\middle|\; m \in \mathbb{N}, \lambda \in \mathbb{R}^m, x^i \in M, i = 1, \ldots, m, e^{\mathsf{T}} \lambda = 1 \right\}.$$

The field of convex analysis, which entails the study of convex sets, has experienced an upsurge with the formalization and development of linear, and later convex, optimization theory in the middle of the last century. Convexity proved to be a key ingredient to the development of efficient optimization algorithms. This can in part be attributed to the multitude of nice properties convex sets admit. One of these properties is that convexity is preserved under many set operations such as intersection, linear transformations, Minkowski addition, polarity and others, see [BF34; Roc70; Hol75; HL01; AT03].
In the upcoming chapters we mainly work with convex sets that are unbounded. The simplest such sets are cones.

Definition 1.10. A convex set $K \subseteq \mathbb{R}^n$ is called a *convex cone* if for every $x \in K$ and $\mu \geqslant 0$ it holds that $\mu x \in K$. Thus, K is a convex cone if every halfline originating

at the origin through a point in K is itself contained in K. A convex cone K is called *pointed* if it satisfies $K \cap (-K) = \{0\}$.

An example of a pointed convex cone is the nonnegative orthant $\mathbb{R}^n_+$ defined as $\{x \in \mathbb{R}^n \mid x_i \geqslant 0, i = 1, \ldots, n\}$. It is ubiquitous in the formulation of linear optimization problems, see [PB14]. Similarly to the definition of convex hull the *conical hull* of a set $M \subseteq \mathbb{R}^n$ is the smallest convex cone which contains M. It is denoted by cone M and defined as the set

$$\left\{ \sum_{i=1}^{m} \mu_i x^i \;\middle|\; m \in \mathbb{N}, \mu \in \mathbb{R}^m_+, x^i \in M, i = 1, \ldots, m \right\}.$$

If M is convex, then it is easily seen that cone $M = \{\mu x \mid \mu \geqslant 0, x \in M\}$. We use the convention cone $\emptyset = \{0\}$ in order to include edge cases, cf. Theorem 1.16 below.

Definition 1.11. Given a convex set $C \subseteq \mathbb{R}^n$ the *recession cone* $0^\infty C$ of C is the set

$$\{d \in \mathbb{R}^n \mid \forall x \in C, \forall \mu \geqslant 0 \colon x + \mu d \in C\}.$$

An element $d \in 0^\infty C$ with $\|d\| = 1$ is called a *direction* of C or *direction of recession* of C. The set C is called *pointed* if $0^\infty C$ is pointed.

Remark 1.12. One finds literature in which every element of $0^\infty C$ is called a direction of C including the zero vector. We do not adopt this notion and restrict ourselves to vectors of unit length. This is no restriction as every nonzero element of $0^\infty C$ can be obtained from a direction of C through scaling with some positive constant. On the contrary, we can assign to every nonzero $d \in 0^\infty C$ the direction $d/\|d\|$. Moreover, for a convex set that is also closed it is common to call it linefree if it has a pointed recession cone. This resembles the fact that a closed convex set is linefree if and only if it does not contain any nontrivial affine subspace, in particular it does not contain lines, see [Hol75; Bat88].

Remark 1.13. The denotation $0^\infty C$ is due to the fact that directions of $C \subseteq \mathbb{R}^n$ can be interpreted as *points at infinity* in the following sense, cf. [Roc70]. There is a natural correspondence between points $c \in C$ and the halflines $\{\mu(c^\mathsf{T}, 1)^\mathsf{T} \in \mathbb{R}^{n+1} \mid \mu \geqslant 0\}$ of the cone cone$(C \times \{1\})$. The directions d of C then correspond to the halflines $\{\mu(d^\mathsf{T}, 0)^\mathsf{T} \in \mathbb{R}^{n+1} \mid \mu \geqslant 0\}$, which lie in a hyperplane parallel to $C \times \{1\}$. This suggests referring to directions as points at infinity of C. In this way, every convex set in $\mathbb{R}^n$ can be identified by a convex cone in $\mathbb{R}^{n+1}$. These sets are called homogenizations and it is a popular approach in convex analysis to reduce questions concerning convex sets to questions about convex cones, see [Bri20]. We exploit this strategy in Chapter 4 and develop an approximation algorithm based on homogenizations.

If C is a convex set that is also closed and $d \neq 0$, then it is sufficient to require that the halfline $\{x + \mu d \mid \mu \geqslant 0\}$ is contained in C for only one point x in order for d to be a direction of C according to [Roc70, Theorem 8.3]. Moreover, a closed convex set C is bounded if and only if $0^\infty C = \{0\}$, see [Roc70, Theorem 8.4]. Recession cones are important building blocks of unbounded convex sets because they describe their asymptotic behavior. In order to express this formally in Theorem 1.16, we need the concept of extreme points and extreme directions.

Definition 1.14. Let $C \subseteq \mathbb{R}^n$ be convex. A convex set $F \subseteq C$ is called a *face* of C if

$$\forall x, y \in C, \lambda \in (0,1) \colon \lambda x + (1-\lambda) y \in F \implies x, y \in F.$$

A face of dimension $n-1$ is called a *facet* and a zero-dimensional face an *extreme point*. A direction d of C is called an *extreme direction* of C if there exists a point $x \in C$ such that the halfline $\{x + \mu d \mid \mu \geqslant 0\}$ is a face of C.

We denote the set of extreme points of C by $\operatorname{ext} C$ and the set of extreme directions by $\operatorname{extdir} C$. Note that in general the sets $\operatorname{extdir} C$ and $\operatorname{extdir} 0^\infty C$ are not equal.

Example 1.15. Consider the closed convex set $C = \left\{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geqslant 1\right\}$, which is the epigraph of a hyperbola. Then $0^\infty C = \mathbb{R}^2_+$ and its extreme directions are $(1,0)^\mathsf{T}$ and $(0,1)^\mathsf{T}$. However, C does not admit any extreme directions.

Extreme points and directions are important because they entail all the information needed to generate a closed pointed convex set. The following result is called the Klee-Minkowski-Hirsch-Hoffman-Goldman-Tucker theorem.

Theorem 1.16 (cf. [Hol75])**.** *Let $C \subseteq \mathbb{R}^n$ be a closed pointed convex set. Then*

$$C = \operatorname{conv}(\operatorname{ext} C) + \operatorname{cone}(\operatorname{extdir} C).$$

Corollary 1.17 (cf. [Roc70, Corollary 18.5.3])**.** *Let $C \subseteq \mathbb{R}^n$ be a closed pointed convex set. Then $\operatorname{ext} C \neq \emptyset$.*

Corollary 1.18 (cf. [Roc70, Corollary 18.5.2])**.** *Let $K \neq \{0\} \subseteq \mathbb{R}^n$ be a closed pointed convex cone. Then $K = \operatorname{cone}(\operatorname{extdir} K)$.*

The Klee-Minkowski-Hirsch-Hoffman-Goldman-Tucker theorem can be understood as an internal representation of C. In contrast, it is also possible to give an external representation of a closed convex set as hinted at in the remark after Theorem 1.4.

Proposition 1.19 (cf. [BV04])**.** *Let $C \subseteq \mathbb{R}^n$ be a closed convex set. For every boundary point $\bar{x}$ of C there exists a supporting halfspace $H^-(\omega, \gamma)$ of C at $\bar{x}$. If, in addition, C is a cone, then $\gamma = 0$.*

Definition 1.20. Let $C \subseteq \mathbb{R}^n$ be convex and let $\bar{x} \in C$. The *normal cone* of C at $\bar{x}$, denoted by $N_C(\bar{x})$, is defined as the set

$$\left\{\omega \in \mathbb{R}^n \,\middle|\, \forall x \in C\colon \omega^\mathsf{T}(x - \bar{x}) \leqslant 0\right\}.$$

By definition $H^-(\omega, \gamma)$ being a supporting halfspace of C at $\bar{x}$ implies $\omega^\mathsf{T} x \leqslant \gamma$ for all $x \in C$ and $\omega^\mathsf{T}\bar{x} = \gamma$. Thus, a supporting halfspace can be associated with a linear function that attains its maximum on C. The normal cone $N_C(\bar{x})$ collects all the linear functions that achieve their maximum over C at $\bar{x}$.

Theorem 1.21 (cf. [HL01])**.** *A closed convex set $C \subseteq \mathbb{R}^n$ is the intersection of all halfspaces that support C at any point.*

A very important special type of convex sets that we frequently use are polyhedra. They are the convex sets for which the representations according to Theorems 1.16 and 1.21 can be realized in a finite manner.

Definition 1.22 (cf. [Gru07])**.** Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The set

$$P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$$

is called a *polyhedron*. The tuple (A, b) is called an *H-representation* of P. If P is also bounded, it is called a *polytope*.

According to the definition, P is the intersection of the m halfspaces $H^-(a^{i\mathsf{T}}, b_i)$, $i = 1, \ldots, m$, where a^i denotes the i-th row of A. Hence, by eliminating redundancy in an H-representation, it constitutes a finite internal representation in the sense of Theorem 1.21. Analogously, a finite external representation can be given.

Definition 1.23 (cf. [Gru07]). Let $V \in \mathbb{R}^{n\times m}$ and $D \in \mathbb{R}^{n\times r}$. The tuple (V, D) is called a *V-representation* of the polyhedron

$$\left\{ V\lambda + D\mu \;\middle|\; \lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^r_+, e^\mathsf{T}\lambda = 1 \right\}.$$

Calling the set in the definition a polyhedron as well is justified by the celebrated Weyl-Minkowski theorem.

Theorem 1.24 (Weyl-Minkowski theorem, see [Gru07, Theorem 14.3]). *Every polyhedron has a V-representation. Conversely, every set with a V-representation is a polyhedron.*

Given a polyhedron P with V-representation (V, D) and assuming that no column of D is the zero vector, P can be written as

$$P = \operatorname{conv}\left\{v^1, \ldots, v^m\right\} + \operatorname{cone}\left\{ \frac{d^1}{\|d^1\|}, \ldots, \frac{d^r}{\|d^r\|} \right\},$$

where v^i, $i = 1, \ldots, m$, and d^j, $j = 1, \ldots, r$, are the columns of V and D, respectively. Therefore, a polyhedron is the Minkowski sum of the convex hull of finitely many points and the conical hull of finitely many directions. The assumption that every column of D is nonzero is no restriction because every conical hull contains the origin by definition. Hence, every zero column of D can be omitted. The extreme points of P are called *vertices* and the set of all vertices is denoted by $\operatorname{vert} P$.
We have seen that supporting halfspaces to a closed convex set $C \subseteq \mathbb{R}^n$ can be obtained by maximizing a linear function over C. Those functions that yield such a halfspace are determined by the normal cones to C. A relation to $0^\infty C$ is given in Proposition 3.7. Another useful tool for finding supporting halfspaces is the projection onto C. For $y \in \mathbb{R}^n$ we consider the problem

$$\min\{\|x - y\| \mid x \in C\},$$

i.e. we are interested in the points of C closest to y with respect to the Euclidean distance. It is a classical result that for closed convex C the minimum is uniquely attained, hence it is justified, see [HL01]. We denote the above minimum by $d(y, C)$ and define

$$\pi_C(y) = \operatorname{argmin}\{\|x - y\| \mid x \in C\}$$

as the *projection* of y onto C.

Proposition 1.25 (cf. [HL01, Theorem 3.1.1]). *Let $C \subseteq \mathbb{R}^n$ be a closed convex set and let $\bar{x} \in C$, $y \in \mathbb{R}^n$. Then the following are equivalent:*

(i) $\bar{x} = \pi_C(y)$,

(ii) $(y - \bar{x})^\mathsf{T}(x - \bar{x}) \leqslant 0$ *for all* $x \in C$.

If, in addition, C is a cone, then

$$(y - \bar{x})^\mathsf{T}\bar{x} = 0.$$

If $y \in C$, then one clearly has $d(y, C) = 0$ and $\pi_C(y) = y$. Otherwise, $d(y, C) > 0$ and the halfspace $H^-(y - \pi_C(y), \gamma)$ with $\gamma = (y - \pi_C(y))^\mathsf{T}\pi_C(y)$ supports C at $\pi_C(y)$.

1.4 Semidefinite programming

Semidefinite optimization problems play a crucial role in the development of approximation algorithms in Chapters 3 and 4. The algebraic objects needed to describe these problems are positive semidefinite matrices.

Definition 1.26. A matrix $X \in \mathcal{S}^\ell$ is called *positive semidefinite* if $x^\mathsf{T} X x \geqslant 0$ holds for all $x \in \mathbb{R}^\ell$. Matrix X is called *positive definite* if the inequality is strict for all $x \neq 0$. Positive semidefiniteness and positive definiteness of X are denoted by $X \succcurlyeq 0$ and $X \succ 0$, respectively.

The set of positive semidefinite and positive definite matrices of size ℓ are written as $\mathcal{S}^\ell_+$ and $\mathcal{S}^\ell_{++}$, i.e.

$$\mathcal{S}^\ell_+ = \left\{ X \in \mathcal{S}^\ell \;\middle|\; X \succcurlyeq 0 \right\}$$

and

$$\mathcal{S}^\ell_{++} = \left\{ X \in \mathcal{S}^\ell \;\middle|\; X \succ 0 \right\},$$

respectively. It is well-known that $\mathcal{S}^\ell_+$ is a closed pointed convex cone that has nonempty interior. Given symmetric matrices X, Y of the same size, we write $X \succcurlyeq Y$ and $X \succ Y$ if $X - Y \succcurlyeq 0$ and $X - Y \succ 0$, respectively. Moreover, $X \preccurlyeq Y$ and $X \prec Y$ is equivalent to $-X \succcurlyeq -Y$ and $-X \succ -Y$.

Proposition 1.27 (cf. [HJ13])**.** *Let $X \in \mathcal{S}^\ell$. The following statements are equivalent:*

(i) X is positive semidefinite,

(ii) all eigenvalues of X are nonnegative,

(iii) all principal minors of X are nonnegative,

(iv) there exists a matrix $L \in \mathbb{R}^{k \times \ell}$ such that $X = L^\mathsf{T} L$, where k is the rank of X.

Similar results can be formulated for positive definite matrices, see e.g. [LV12; HJ13]. A helpful tool in the treatment of positive semidefinite and definite matrices is the Schur complement.

Proposition 1.28 (cf. [Zha05])**.** *Let $X \in \mathcal{M}^{\ell_1}$, $Y \in \mathbb{R}^{\ell_1 \times \ell_2}$, $Z \in \mathcal{M}^{\ell_2}$ and consider the block matrix*

$$W = \begin{pmatrix} X & Y \\ Y^\mathsf{T} & Z \end{pmatrix}.$$

Then the following hold:

(i) $W \succ 0 \iff X \succ 0$ and $Z - Y^\mathsf{T} X^{-1} Y \succ 0$,

(ii) $W \succ 0 \iff Z \succ 0$ and $X - Y Z^{-1} Y^\mathsf{T} \succ 0$,

(iii) if $X \succ 0$, then $W \succcurlyeq 0 \iff Z - Y^\mathsf{T} X^{-1} Y \succcurlyeq 0$,

(iv) if $Z \succ 0$, then $W \succcurlyeq 0 \iff X - Y Z^{-1} Y^\mathsf{T} \succcurlyeq 0$.

Note that a positive definite matrix is invertible. The expressions $Z - Y^\mathsf{T} X^{-1} Y$ and $X - Y Z^{-1} Y^\mathsf{T}$ are called *Schur complements* of W.
Semidefinite programming is the task of maximizing or minimizing a linear function over the intersection of an affine subspace with the cone of positive semidefinite matrices. It can be seen as a generalization of linear programming which aims to maximize or minimize a linear function over the intersection of an affine subspace with

the cone $\mathbb{R}^n_+$. Both problem types belong to the class of *conic optimization problems*, see [PB14]. A semidefinite program in standard form, also called primal program or primal problem, can be written as

$$\begin{aligned} \min\ C \bullet X \quad \text{s.t.} \qquad & \mathcal{A}^\mathsf{T}(X) = b \\ & X \succcurlyeq 0 \end{aligned} \tag{SDP}$$

for data $C \in \mathcal{S}^\ell$, $b \in \mathbb{R}^m$ and a linear pencil $\mathcal{A}$ of size ℓ. Recall that we identify the map $\mathcal{A}$ with matrices $A_1, \ldots, A_m \in \mathcal{S}^\ell$. Hence, the first constraint can equivalently be expressed as

$$\begin{aligned} A_1 \bullet X &= b_1 \\ &\vdots \\ A_m \bullet X &= b_m \end{aligned}$$

according to Proposition 1.7. Any point satisfying the constraints to an optimization problem is called *feasible point* and the corresponding problem is called *feasible*. The set of all feasible points is referred to as the *feasible region*. A feasible point X of (SDP) that satisfies $X \succ 0$ is a *strictly feasible* point. If the problem (SDP) admits a strictly feasible point, it is called *strictly feasible*. A feasible point X^* of (SDP) with $C \bullet X^* \leqslant C \bullet X$ for all feasible points X is referred to as a *solution* to (SDP). A solution X^* realizes the *optimal value* $C \bullet X^*$, i.e. the objective function value attained at X^*. To every problem of type (SDP) we can assign a dual problem (SDP*) as

$$\max b^\mathsf{T} y \quad \text{s.t.} \quad \mathcal{A}(y) \preccurlyeq C. \tag{SDP*}$$

The dual feasible region consists of all points $y \in \mathbb{R}^m$ for which the linear combination $\sum_{i=1}^m -A_i y_i$ of the matrices $-A_1, \ldots, -A_m$ associated with $\mathcal{A}$ translated by C is positive semidefinite. Strict feasibility of (SDP*) is defined analogously as for (SDP). A problem of the form (SDP*) is again a semidefinite optimization problem and the different formulations (SDP) and (SDP*) can be converted into each other, see [VB96]. The primal and dual problem are connected through an elaborate duality theory that yields weak and strong duality results.

Lemma 1.29 (weak duality, cf. [VB96])**.** *Let X be feasible for (SDP) and y be feasible for (SDP*). Then $b^\mathsf{T} y \leqslant C \bullet X$.*

Theorem 1.30 (strong duality, cf. [BL00])**.** *Let (SDP) be strictly feasible and bounded below, i.e. the set $\{C \bullet X \mid X \text{ is feasible for (SDP)}\}$ has an infimum p^*. Then an optimal solution y^* to (SDP*) exists and*

$$b^\mathsf{T} y^* = p^*.$$

Analogously, if the dual problem (SDP) is strictly feasible and bounded above, i.e. the set $\{b^\mathsf{T} y \mid y \text{ is feasible for (SDP*)}\}$ has a supremum d^*, then an optimal solution X^* to (SDP) exists and*

$$C \bullet X^* = d^*.$$

It is evident that if the primal and dual problem are strictly feasible, then optimal solutions to both exist and the optimal values coincide. The assumptions sufficient for strong duality to hold, namely strict feasibility and boundedness of the objective function values, are called regularity conditions or constraint qualifications. They constitute the main difference to the duality theory for linear programming, where strong duality holds without further assumptions. Some effects of this limitation will become apparent in the following chapter.

Chapter 2

Calculus of spectrahedral shadows

This chapter is concerned with the development of a calculus for *spectrahedra* and *spectrahedral shadows*. We show that the latter are closed under a number of common set operations. Spectrahedral shadows have been studied extensively in the literature and the closedness under some operations that we consider is certainly known. Important references in this regard are [GR95; HN09; NS09; HN10; HN12; Net11; BPT13]. What sets our approach apart is that we derive explicit representations of the resulting sets in terms of the describing data of the spectrahedral shadows. Moreover, we treat unbounded sets in detail and find representations of the polar and the recession cone of a spectrahedral shadow. In particular, recession cones have not been studied before. A similar set calculus for the compact case is due to Ciripoi [Cir19].

2.1 Spectrahedra and their projections

Definition 2.1 (cf. [BPT13])**.** Let $\mathcal{A}$ be a linear pencil of size ℓ and $A_0 \in \mathcal{S}^\ell$. The set

$$C = \{x \in \mathbb{R}^n \mid A_0 + \mathcal{A}(x) \succcurlyeq 0\}$$

is called a *spectrahedron*. The expression $A_0 + \mathcal{A}(x) \succcurlyeq 0$ is referred to as a *linear matrix inequality*, abbreviated as LMI.

Thus, spectrahedra are the feasible regions of the semidefinite optimization problems of type (SDP). They are closed convex sets because an LMI is equivalent to the infinitely many inequalities

$$y^\mathsf{T} \left(A_0 + \mathcal{A}(x)\right) y = yy^\mathsf{T} \bullet \left(A_0 + \mathcal{A}(x)\right) \geqslant 0$$

for all $y \in \mathbb{R}^\ell$, which describe halfspaces in $\mathcal{S}^\ell$. Moreover, defining spectrahedra by a single LMI is no restriction over having finitely many. Given two LMIs $A_0 + \mathcal{A}(x) \succcurlyeq 0$ and $B_0 + \mathcal{B}(x) \succcurlyeq 0$, they can be written as the single LMI

$$\begin{pmatrix} A_0 + \mathcal{A}(x) & 0 \\ 0 & B_0 + \mathcal{B}(x) \end{pmatrix} \succcurlyeq 0$$

with a block diagonal structure.

Remark 2.2. A simple calculation shows that the block diagonal matrix

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix}$$

is positive semidefinite if and only if all the blocks $A_1, \ldots, A_m$ are positive semidefinite.

Spectrahedra, being defined by linear matrix inequalities, can be understood as a generalization of polyhedra, which can be defined by finitely many linear inequalities through an H-representation. Indeed, every polyhedron is also a spectrahedron. To see this, consider the polyhedron $P = \left\{x \in \mathbb{R}^n \,\middle|\, a^{i\mathsf{T}}x \leqslant b_i, i = 1, \ldots, m\right\}$. The defining inequalities of P can be interpreted as positive semidefiniteness of the 1×1-matrices $b_i - a^{i\mathsf{T}}x$, $i = 1, \ldots, m$. Taking the above remark into account P can be written as

$$P = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{pmatrix} b_1 - a^{1\mathsf{T}}x & 0 & \cdots & 0 \\ 0 & b_2 - a^{2\mathsf{T}}x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_m - a^{m\mathsf{T}}x \end{pmatrix} \succcurlyeq 0 \right\},$$

which is the spectrahedron

$$\left\{ x \in \mathbb{R}^n \,\middle|\, \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_m \end{pmatrix} + \sum_{j=1}^{n} \begin{pmatrix} -a_j^1 & & \\ & \ddots & \\ & & -a_j^m \end{pmatrix} x_j \succcurlyeq 0 \right\}.$$

Hence, all spectrahedra that are defined by diagonal matrices are in fact polyhedra. However, verifying polyhedrality of a spectrahedron is not as straightforward as checking whether the spectrahedron is defined by diagonal matrices because it is always possible to add a redundant LMI to the definition of a spectrahedron that does not describe a polyhedron. A thorough investigation of this problem is found in [BRS15]. The above derivation demonstrates that all results about spectrahedra in this chapter also apply to polyhedra. A comprehensive calculus for the class of polyhedra and polyhedral convex functions is developed in [CLW19]. In particular, a stronger version of Proposition 2.28 below holds for the polyhedral case.
It is possible to describe spectrahedra in another way that sets them apart from polyhedra. According to Proposition 1.27 a matrix is positive semidefinite if and only if all its principal minors are nonnegative. Therefore, a point $\bar{x}$ is contained in the spectrahedron

$$C = \{x \in \mathbb{R}^n \mid A_0 + \mathcal{A}(x) \succcurlyeq 0\}$$

if and only if all $2^\ell - 1$ principal minors p_j of $A_0 + \mathcal{A}(x)$ are nonnegative at $\bar{x}$, that is $\bar{x}$ belongs to the set

$$\left\{x \in \mathbb{R}^n \,\middle|\, p_1(x) \geqslant 0, \ldots, p_{2^\ell - 1}(x) \geqslant 0\right\}$$

defined by finitely many polynomial inequalities. Note that the degree of the polynomials p_j is at most ℓ. Any set admitting such a representation is called a *basic closed semialgebraic set*. They are studied as part of the field of real algebraic geometry which is concerned with the solution sets to systems of polynomial equations and inequalities, see e.g. [BCR98]. This suggests to study spectrahedra and their properties within this framework and, indeed, this is done in the literature mentioned at the beginning of this chapter. However, we take a different approach and investigate these sets through the lens of convex analysis and are more focused on the properties of spectrahedra as convex sets rather than solution sets to polynomial inequalities. It is noteworthy that spectrahedra form a proper subset of the class of

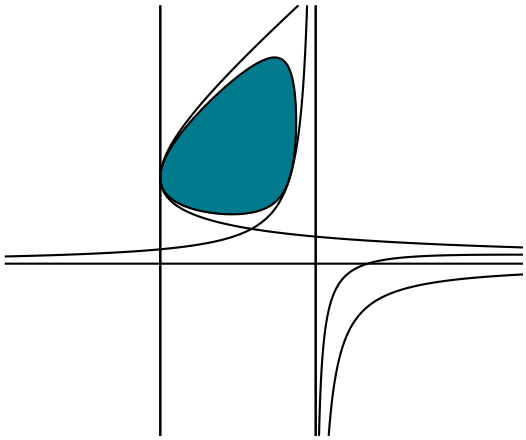

FIGURE 2.1: A spectrahedron defined by a linear pencil of size 3 and the set of zeros of its principal minors.

basic closed semialgebraic sets. This is evident from the fact that the latter do not even have to be convex.

Example 2.3. Consider the following spectrahedron $C \subseteq \mathbb{R}^2$ that is defined by a linear pencil of size 3:

$$\begin{aligned} C &= \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1-x_1 & 0 & 1 \\ 0 & x_1+2 & x_2-1 \\ 1 & x_2-1 & x_2+1 \end{pmatrix} \succcurlyeq 0 \right\} \\ &= \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} x_2 \succcurlyeq 0 \right\}. \end{aligned}$$

It is defined by the region where its 7 principal minors

$$\begin{aligned} p_1(x) &= -x_1^2x_2 + x_1x_2^2 - x_1^2 - x_2^2 - 3x_1x_2 - x_1 + 4x_2 - 1 \\ p_2(x) &= -x_1^2 - x_1 + 2 \\ p_3(x) &= -x_1x_2 - x_1 + x_2 \\ p_4(x) &= -x_2^2 + x_1x_2 + x_1 + 4x_2 + 1 \\ p_5(x) &= -x_1 + 1 \\ p_6(x) &= x_1 + 2 \\ p_7(x) &= x_2 + 1 \end{aligned}$$

are nonnegative. The boundary of C is then contained in the set of points where at least one of the polynomials p_j, $j = 1, \ldots, 7$, vanishes. Figure 2.1 shows spectrahedron C as the shaded region as well as the sets $\{x \in \mathbb{R}^2 \mid p_j(x) = 0\}$, $j = 1, \ldots, 7$. Note that the set of zeros of p_5 and p_6 are contained in that of p_2.

Example 2.4. The Euclidean n-ball

$$B_r(c) = \{x \in \mathbb{R}^n \mid \|x - c\| \leqslant r\}$$

of radius $r > 0$ centered at $c \in \mathbb{R}^n$ is a spectrahedron. This can be seen by applying Proposition 1.28. We reformulate the condition $\|x - c\| \leqslant r$ as

$$\begin{aligned} r \geqslant \|x - c\| &\iff r^2 - (x-c)^\mathsf{T}(x-c) \geqslant 0 \\ &\iff r - (x-c)^\mathsf{T}\frac{1}{r}I(x-c) \geqslant 0, \end{aligned}$$

where I denotes the identity matrix of appropriate size. The expression on the left in the last inequality is a Schur complement of the $(n+1) \times (n+1)$-matrix

$$\begin{pmatrix} rI & x-c \\ (x-c)^\mathsf{T} & r \end{pmatrix}$$

due to the identity $r^{-1}I = (rI)^{-1}$. Therefore, we obtain the representation

$$B_r(c) = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} rI & x-c \\ (x-c)^\mathsf{T} & r \end{pmatrix} \succcurlyeq 0 \right\},$$

which is a spectrahedron. We point out that it is possible to derive a smaller representation of $B_r(c)$ as a spectrahedron using only $n \times n$-matrices, see [Kum16].

Further examples conveying an impression which types of sets spectrahedra encompass are found in [BPT13, Chapter 6].
For a polyhedron $P \subseteq \mathbb{R}^n$ with H-representation (A, b) such that every row of A is nonzero the interior of P, denoted $\operatorname{int} P$, can be expressed as the set

$$\{x \in \mathbb{R}^n \mid Ax < b\}.$$

That is, a point belongs to the interior of P if and only if all the defining inequalities of P are satisfied strictly, see [Zie95, Lemma 2.8]. Naturally, one asks whether a similar result holds for the spectrahedral case, i.e. do we have

$$\operatorname{int} C = \{x \in \mathbb{R}^n \mid A_0 + \mathcal{A}(x) \succ 0\}$$

for the spectrahedron $C = \{x \in \mathbb{R}^n \mid A_0 + \mathcal{A}(x) \succcurlyeq 0\}$? The situation in this case is different. If there exists $\bar{x}$ with $A_0 + \mathcal{A}(\bar{x}) \succ 0$, then $\bar{x}$ is an interior point of C. To see this, note that a positive definite matrix has positive eigenvalues [HJ13] and the minimum eigenvalue function is concave [BV04]. Hence, for every direction $d \in \mathbb{R}^n$ and small enough $\varepsilon > 0$, $A_0 + \mathcal{A}(\bar{x} + \varepsilon d) \succ 0$ holds. On the contrary, consider the spectrahedron

$$C = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} x_1 + x_2 + 1 & -x_1 - x_2 - 1 \\ -x_1 - x_2 - 1 & x_1 + x_2 + 1 \end{pmatrix} \succcurlyeq 0 \right\}. \tag{2.1}$$

Although C is not defined by diagonal matrices, it can easily be seen from a reformulation using principal minors that it is the polyhedron $\{x \in \mathbb{R}^2 \mid x_1 + x_2 \geqslant -1\}$. Hence, the origin is an interior point of C. However, the matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

is not positive definite. This shortcoming is overcome by applying a suitable transformation to the data of C that does not change C itself.

Proposition 2.5. *Let $C = \{x \in \mathbb{R}^n \mid A_0 + \mathcal{A}(x) \succcurlyeq 0\}$ be given. Then there exists a matrix V such that*

$$C = \left\{ x \in \mathbb{R}^n \;\middle|\; V^\mathsf{T} \left(A_0 + \mathcal{A}(x)\right) V \succcurlyeq 0 \right\}$$

and

$$\operatorname{int} C = \left\{ x \in \mathbb{R}^n \;\middle|\; V^\mathsf{T} \left(A_0 + \mathcal{A}(x)\right) V \succ 0 \right\}.$$

Proofs can be found in [GR95, Corollary 5] and [HV07, Lemma 2.3]. The first one also asserts that the columns of the matrix V must span the orthogonal complement of the intersection of the null spaces of the matrices $A_i, i = 0, \ldots, n$, and in the second it is shown that if $0 \in \operatorname{int} C$, then V can be chosen such that $V^\mathsf{T} A_0 V = I$. We will from now on assume that spectrahedra are given in a form according to Proposition 2.5. Moreover, if $\bar{x} \in \operatorname{int} C$, then no generality is lost by assuming that $0 \in \operatorname{int} C$ because we can consider the set $C - \{\bar{x}\}$, which is easily seen to be a spectrahedron again.

Example 2.6. We turn to the spectrahedron defined in Equation (2.1) again. The intersection of the null spaces of the defining matrices is $\{t(1,1)^\mathsf{T} \mid t \in \mathbb{R}\}$. Choosing $V = \frac{1}{2}(1,-1)^\mathsf{T}$ yields

$$\begin{aligned} C &= \left\{ x \in \mathbb{R}^2 \;\middle|\; V^\mathsf{T} \begin{pmatrix} x_1 + x_2 + 1 & -x_1 - x_2 - 1 \\ -x_1 - x_2 - 1 & x_1 + x_2 + 1 \end{pmatrix} V \geqslant 0 \right\} \\ &= \left\{ x \in \mathbb{R}^2 \;\middle|\; x_1 + x_2 \geqslant -1 \right\}. \end{aligned}$$

and $\operatorname{int} C = \left\{ x \in \mathbb{R}^2 \;\middle|\; x_1 + x_2 > -1 \right\}$.

Definition 2.7 (cf. [BPT13])**.** Let $\mathcal{A}$ and $\overline{\mathcal{A}}$ be linear pencils of size ℓ and $A_0 \in \mathcal{S}^\ell$. The set

$$S = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists\, y \in \mathbb{R}^m \colon A_0 + \mathcal{A}(x) + \overline{\mathcal{A}}(y) \succcurlyeq 0 \right\}$$

is called a *spectrahedral shadow* or *projected spectrahedron* or *semidefinitely representable set*.

The name originates from the fact that a spectrahedral shadow S is the orthogonal projection of the spectrahedron

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+m} \;\middle|\; A_0 + \begin{pmatrix} \mathcal{A} \mid \overline{\mathcal{A}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \succcurlyeq 0 \right\}$$

onto the x component. The set C and the vector y are called a *lift* of S and a *lifting variable* in this context. Clearly, every spectrahedron is a spectrahedral shadow with $m = 0$. On the other hand, spectrahedra are not stable under projection, i.e. a projected spectrahedron is not necessarily spectrahedral.

Example 2.8. Consider the set $C = \left\{ x \in \mathbb{R}^2_+ \;\middle|\; x_1 x_2 \geqslant 1 \right\}$. It is described by the LMI

$$\begin{pmatrix} x_1 & 1 \\ 1 & x_2 \end{pmatrix} \succcurlyeq 0.$$

Its projection onto x_1 is the set $S = \mathbb{R}_{++}$ which is not closed and therefore not a spectrahedron. The sets are illustrated in Figure 2.2.

Thus, spectrahedral shadows describe a more general class of convex sets than spectrahedra. This is in contrast to the polyhedral case where it is known that the projection of a polyhedron is again a polyhedron. Fourier-Motzkin elimination

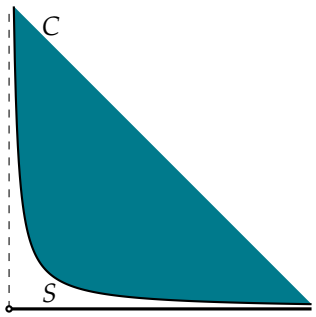

FIGURE 2.2: Spectrahedra are not stable under orthogonal projections.

[Mot36] is an algorithmic procedure for eliminating variables from systems of linear inequalities. It can be used to compute quantifier-free descriptions of polyhedra that are projections of higher dimensional polyhedra in H-representation. Therefore, it certifies that polyhedra are closed under projections. The method is sensitive to the size of the H-representation of the high dimensional polyhedron. In particular, if an H-representation of the polyhedron

$$P = \left\{ x \in \mathbb{R}^n \,\middle|\, \exists\, y \in \mathbb{R}^m \colon Ax + \overline{A}y \leqslant b \right\}$$

with $b \in \mathbb{R}^d$ is sought, then Fourier-Motzkin elimination produces a description of P with at most $4(d/4)^{2^m}$ inequalities, many of which might be redundant, see [Mon10]. Hence, the complexity is doubly exponential and the method is not practical. In [Wei17; CLW19] another method, that is less sensitive to the dimension of the input polyhedron, is proposed to compute both an H- and V-representation of a projected polyhedron using methods from multiple objective linear programming.

Remark 2.9. As mentioned above, spectrahedra are properly contained in the class of basic closed semialgebraic sets. This larger class also lacks the property of being stable under projections because closedness might not be preserved. Analogously to the generalization of spectrahedra to spectrahedral shadows is the transition of basic closed semialgebraic sets to *semialgebraic sets*. Those are the sets that can be generated from basic closed semialgebraic sets by taking finite unions, finite intersections and complements. Every semialgebraic set in $\mathbb{R}^n$ can be written as a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid p_1(x) = \cdots = p_k(x) = 0, p_{k+1} > 0, \ldots, p_m > 0\}$$

for polynomials p_i, $i = 1, \ldots, m$, $k \leqslant m$, see [BCR98, Proposition 2.1.8]. The Tarski-Seidenberg theorem [BPR06, Theorem 2.76] states that the projection of a semialgebraic set is again semialgebraic. This particularly implies that spectrahedral shadows are semialgebraic sets. Explicit descriptions of them in terms of polynomials can be obtained with cylindrical algebraic decomposition [Col75], a quantifier elimination algorithm for semialgebraic sets. Helton and Nie [HN10] conjectured that every convex semialgebraic set is a spectrahedral shadow. Scheiderer proved that the Helton-Nie conjecture is true in the plane [Sch18a] and, recently, showed that it does not hold in dimensions three and above [Sch18b]. Hence, the class of convex semialgebraic sets is in fact larger than the class of spectrahedral shadows.

The modeling power of spectrahedral shadows and their relevance to semidefinite programming make them an active area of research. Applications include system and control theory [Boy+94], structural optimization, wire and transistor design [VB99] and matrix rank minimization [Faz02]. Therefore, deriving explicit semidefinite representations of sets is desired. One popular approach is called Lasserre Relaxation [Las09]. It is a method to construct an approximation to a set S given in terms of polynomial equations and inequalities that is a spectrahedral shadow. The general idea is to introduce a lifting variable for every monomial in the description of S. Thereby, the equations and inequalities become linear. The nonlinearity is then introduced by a positive semidefiniteness condition coming from the lifting variables. Another method for constructing semidefinite representations is called localization, see [BPT13]. The idea is to describe S as the composition of set operations that preserve the semidefinite representable set type applied to simple sets for which descriptions as spectrahedral shadows are known. In [NS09] this is done to derive an explicit representation as a spectrahedral shadow of the convex hull of the union of finitely many spectrahedral shadows. We follow the second approach in this work because it yields exact semidefinite representations given that S is a spectrahedral shadow. As already mentioned in Section 1.4, the problems (SDP) and (SDP*) can be transformed into each other. Hence, the feasible region of (SDP) can be formulated as a LMI. However, this is a tedious process and typically increases the size of the involved matrices. This motivates us to define a representation for spectrahedral shadows that is more flexible than a single LMI as in Definition 2.7.

Definition 2.10. For $A_0 \in \mathcal{S}^\ell$, linear pencils $\mathcal{A}$ and $\overline{\mathcal{A}}$ of size ℓ and $\mathcal{B}$ of size k, $\boldsymbol{\mathcal{A}}\colon \mathcal{S}^k \to \mathcal{S}^\ell$, $B \in \mathbb{R}^{d\times n}$, $\bar{B} \in \mathbb{R}^{d\times m}$ and $b \in \mathbb{R}^d$ the tuple $\left(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^\mathsf{T}, b\right)$ is called a *representation* of the spectrahedral shadow

$$S = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists\, y \in \mathbb{R}^m, Z \in \mathcal{S}^k\colon \begin{array}{r} A_0 + \mathcal{A}(x) + \overline{\mathcal{A}}(y) + \boldsymbol{\mathcal{A}}(Z) \succcurlyeq 0 \\ Bx + \bar{B}y + \mathcal{B}^\mathsf{T}(Z) = b \\ Z \succcurlyeq 0 \end{array} \right\}.$$

It follows from the considerations above that S is indeed a spectrahedral shadow. The advantage of using this representation is flexibility. If one were to apply the results in this chapter to a spectrahedral shadow S, he could do so directly using the representation of S that is available to him, e.g. from an application. Reformulating S to obtain an appearance according to Definition 2.7 is not necessary. Even more general representations are possible, e.g. by allowing upper and lower bounds instead of equalities or projecting only a submatrix of Z, at the cost of additional parameters. Our representation is a middle ground between flexibility and bloat.

Remark 2.11. If any parameter in a representation of a spectrahedral shadow is not present, it is written as $\varnothing$. For example, the tuple $(A_0, \mathcal{A}, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing, \varnothing)$ describes a spectrahedron as defined in Definition 2.1. This is equivalent to the ranges of $\mathcal{A}$, $\overline{\mathcal{A}}$, $\boldsymbol{\mathcal{A}}$, $\mathcal{B}^\mathsf{T}$ containing only the zero element in the respective codomains and B, $\bar{B}$ and b being zero matrices and the zero vector of appropriate sizes, respectively.

2.2 Representations under set operations

We will now derive representations of spectrahedra and spectrahedral shadows under set operations that preserve the set type.

Proposition 2.12 (intersection). *Let $S_1, S_2 \subseteq \mathbb{R}^n$ be spectrahedral shadows represented by*

$$\left(A_0^1, \mathcal{A}^1, \overline{\mathcal{A}^1}, \boldsymbol{\mathcal{A}}^1, B^1, \overline{B^1}, \mathcal{B}^{1\mathsf{T}}, b^1\right) \quad \text{and} \quad \left(A_0^2, \mathcal{A}^2, \overline{\mathcal{A}^2}, \boldsymbol{\mathcal{A}}^2, B^2, \overline{B^2}, \mathcal{B}^{2\mathsf{T}}, b^2\right),$$

respectively. Then the intersection $S_1 \cap S_2 \subseteq \mathbb{R}^n$ is a spectrahedral shadow represented by

$$\left(A_0^1 \oplus A_0^2, \begin{pmatrix} \mathcal{A}^1 \\ \mathcal{A}^2 \end{pmatrix}, \overline{\mathcal{A}^1} \oplus \overline{\mathcal{A}^2}, \boldsymbol{\mathcal{A}}^1 \oplus \boldsymbol{\mathcal{A}}^2, \begin{pmatrix} B^1 \\ B^2 \end{pmatrix}, \overline{B^1} \oplus \overline{B^2}, \left(\mathcal{B}^1 \oplus \mathcal{B}^2\right)^{\mathsf{T}}, \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}\right).$$

Proof. A straightforward calculation yields

$$S_1 \cap S_2 = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, y^1 \in \mathbb{R}^{m_1}, y^2 \in \mathbb{R}^{m_2} \\ Z^1 \in \mathcal{S}^{k_1}, Z^2 \in \mathcal{S}^{k_2} \end{array} : \begin{array}{r} A_0^1 + \mathcal{A}^1(x) + \overline{\mathcal{A}^1}(y^1) + \boldsymbol{\mathcal{A}}^1(Z^1) \succcurlyeq 0 \\ A_0^2 + \mathcal{A}^2(x) + \overline{\mathcal{A}^2}(y^2) + \boldsymbol{\mathcal{A}}^2(Z^2) \succcurlyeq 0 \\ B^1 x + \overline{B^1} y^1 + \mathcal{B}^{1\mathsf{T}}(Z^1) = b^1 \\ B^2 x + \overline{B^2} y^2 + \mathcal{B}^{2\mathsf{T}}(Z^2) = b^2 \\ Z^1 \succcurlyeq 0, Z^2 \succcurlyeq 0 \end{array} \right\}$$

$$= \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, y \in \mathbb{R}^{m_1+m_2}, Z \in \mathcal{S}^{k_1+k_2} : \\ \quad A_0^1 \oplus A_0^2 + \left(\dfrac{\mathcal{A}^1}{\mathcal{A}^2}\right)(x) + \\ \qquad \left(\overline{\mathcal{A}^1} \oplus \overline{\mathcal{A}^2}\right)(y) + \left(\boldsymbol{\mathcal{A}}^1 \oplus \boldsymbol{\mathcal{A}}^2\right)(Z) \succcurlyeq 0 \\ \quad \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} x + \left(\overline{B^1} \oplus \overline{B^2}\right) y + \left(\mathcal{B}^1 \oplus \mathcal{B}^2\right)^{\mathsf{T}}(Z) = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} \\ \qquad\qquad\qquad\qquad\qquad\qquad Z \succcurlyeq 0 \end{array} \right\}.$$

In the second equation we implicitly set

$$y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} Z^1 & V \\ V^{\mathsf{T}} & Z^2 \end{pmatrix}$$

for a matrix $V \in \mathbb{R}^{k_1 \times k_2}$. It is not necessary to explicitly add the condition $V = 0$ because $Z \succcurlyeq 0$ implies $Z^1 \succcurlyeq 0$ and $Z^2 \succcurlyeq 0$ and because $\boldsymbol{\mathcal{A}}^1 \oplus \boldsymbol{\mathcal{A}}^2$ and $(\mathcal{B}^1 \oplus \mathcal{B}^2)^{\mathsf{T}}$ do not act on V. Hence, there is a solution to the system in the first line if and only if there is a solution to the system in the second line with $V = 0$. □

Corollary 2.13. *Let $C_1, C_2 \subseteq \mathbb{R}^n$ be spectrahedra defined by the linear pencils $\mathcal{A}$, $\mathcal{B}$ and matrices A_0, B_0 of sizes ℓ_1 and ℓ_2, respectively. Then the intersection $C_1 \cap C_2 \subseteq \mathbb{R}^n$ is a spectrahedron defined by the linear pencil*

$$\left(\frac{\mathcal{A}}{\mathcal{B}}\right)$$

and the matrix $A_0 \oplus B_0$ of sizes $\ell_1 + \ell_2$.

Proof. Apply Proposition 2.12 with $m_1 = m_2 = k_1 = k_2 = 0$ and $B^1 = B^2 = b^1 = b^2 = \varnothing$ to obtain

$$C_1 \cap C_2 = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} A_0 & 0 \\ 0 & B_0 \end{pmatrix} + \sum_{i=1}^{n} \begin{pmatrix} A_i & 0 \\ 0 & B_i \end{pmatrix} x_i \succcurlyeq 0 \right\}.$$

□

Another operation under which spectrahedra are stable is the Cartesian product.

Proposition 2.14 (Cartesian product). *Let $S_1 \subseteq \mathbb{R}^{n_1}$, $S_2 \subseteq \mathbb{R}^{n_2}$ be spectrahedral shadows represented by*

$$\left(A_0^1, \mathcal{A}^1, \overline{\mathcal{A}^1}, \boldsymbol{\mathcal{A}}^1, B^1, \overline{B^1}, \mathcal{B}^{1\mathsf{T}}, b^1\right) \quad \text{and} \quad \left(A_0^2, \mathcal{A}^2, \overline{\mathcal{A}^2}, \boldsymbol{\mathcal{A}}^2, B^2, \overline{B^2}, \mathcal{B}^{2\mathsf{T}}, b^2\right),$$

respectively. Then the Cartesian product $S_1 \times \mathcal{S}_2 \subseteq \mathbb{R}^{n_1+n_2}$ is a spectrahedral shadow represented by

$$\left(A_0^1 \oplus A_0^2, \mathcal{A}^1 \oplus \mathcal{A}^2, \overline{\mathcal{A}^1} \oplus \overline{\mathcal{A}^2}, \boldsymbol{\mathcal{A}}^1 \oplus \boldsymbol{\mathcal{A}}^2, B^1 \oplus B^2, \overline{B^1} \oplus \overline{B^2}, \left(\mathcal{B}^1 \oplus \mathcal{B}^2\right)^{\mathsf{T}}, \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}\right).$$

Proof. The Cartesian product is defined as

$$\begin{aligned} S_1 \times S_2 &= \left\{ \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in \mathbb{R}^{n_1+n_2} \;\middle|\; \begin{array}{r} \exists\, y^1 \in \mathbb{R}^{m_1}, y^2 \in \mathbb{R}^{m_2}, Z^1 \in \mathcal{S}^{k_1}, Z^2 \in \mathcal{S}^{k_2}: \\ A_0^1 + \mathcal{A}^1(x^1) + \overline{\mathcal{A}^1}(y^1) + \boldsymbol{\mathcal{A}}^1(Z^1) \succcurlyeq 0 \\ A_0^2 + \mathcal{A}^2(x^2) + \overline{\mathcal{A}^2}(y^2) + \boldsymbol{\mathcal{A}}^2(Z^2) \succcurlyeq 0 \\ B^1 x^1 + \overline{B^1} y^1 + \mathcal{B}^{1\mathsf{T}}(Z^1) = b^1 \\ B^2 x^2 + \overline{B^2} y^2 + \mathcal{B}^{2\mathsf{T}}(Z^2) = b^2 \\ Z^1 \succcurlyeq 0, Z^2 \succcurlyeq 0 \end{array} \right\} \\ &= \left\{ x \in \mathbb{R}^{n_1+n_2} \;\middle|\; \begin{array}{r} \exists\, y \in \mathbb{R}^{m_1+m_2}, Z \in \mathcal{S}^{k_1+k_2}: \\ A_0^1 \oplus A_0^2 + \left(\mathcal{A}^1 \oplus \mathcal{A}^2\right)(x) + \\ \left(\overline{\mathcal{A}^1} \oplus \overline{\mathcal{A}^2}\right)(y) + \left(\boldsymbol{\mathcal{A}}^1 \oplus \boldsymbol{\mathcal{A}}^2\right)(Z) \succcurlyeq 0 \\ \left(B^1 \oplus B^2\right) x + \left(\overline{B^1} \oplus \overline{B^2}\right) y + \\ \left(\mathcal{B}^1 \oplus \mathcal{B}^2\right)^{\mathsf{T}}(Z) = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} \\ Z \succcurlyeq 0 \end{array} \right\}. \end{aligned}$$

In the last expression we have set

$$x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \quad y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} Z^1 & V \\ V^{\mathsf{T}} & Z^2 \end{pmatrix}$$

for a matrix $V \in \mathbb{R}^{k_1 \times k_2}$. This definition of Z is valid with the same line of argumentation as in the proof of Proposition 2.12. □

Corollary 2.15. *Let $C_1 \subseteq \mathbb{R}^{n_1}$, $C_2 \subseteq \mathbb{R}^{n_2}$ be spectrahedra defined by the linear pencils $\mathcal{A}$, $\mathcal{B}$ and matrices A_0, B_0 of sizes ℓ_1 and ℓ_2, respectively. Then the Cartesian product $C_1 \times C_2 \subseteq \mathbb{R}^{n_1+n_2}$ is a spectrahedron defined by the linear pencil $\mathcal{A} \oplus \mathcal{B}$ and the matrix $A_0 \oplus B_0$ of sizes $\ell_1 + \ell_2$.*

Proof. Apply Proposition 2.14 with $m_1 = m_2 = k_1 = k_2 = 0$ and $b^1 = B^2 = b^1 = b^2 = \varnothing$ to obtain

$$C_1 \times C_2 = \left\{ \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in \mathbb{R}^{n_1+n_2} \;\middle|\; \begin{pmatrix} A_0 & 0 \\ 0 & B_0 \end{pmatrix} + \sum_{i=1}^{n_1} \begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix} x_i^1 + \sum_{i=1}^{n_2} \begin{pmatrix} 0 & 0 \\ 0 & B_i \end{pmatrix} x_i^2 \succcurlyeq 0 \right\}.$$

□

Proposition 2.16 (Minkowski sum). *Let $S_1, S_2 \subseteq \mathbb{R}^n$ be spectrahedral shadows represented by*

$$\left(A_0^1, \mathcal{A}^1, \overline{\mathcal{A}^1}, \boldsymbol{\mathcal{A}}^1, B^1, \overline{B^1}, \mathcal{B}^{1\mathsf{T}}, b^1\right) \quad \text{and} \quad \left(A_0^2, \mathcal{A}^2, \overline{\mathcal{A}^2}, \boldsymbol{\mathcal{A}}^2, B^2, \overline{B^2}, \mathcal{B}^{2\mathsf{T}}, b^2\right),$$

respectively. Then the Minkowski sum $S_1 + S_2 \subseteq \mathbb{R}^n$ is a spectrahedral shadow represented by

$$\left(A_0^1 \oplus A_0^2, \varnothing, \left(\begin{array}{c|c} \mathcal{A}^1 & \overline{\mathcal{A}^1} \end{array}\right) \oplus \left(\begin{array}{c|c} \mathcal{A}^2 & \overline{\mathcal{A}^2} \end{array}\right), \boldsymbol{\mathcal{A}}^1 \oplus \boldsymbol{\mathcal{A}}^2, M^1, M^2, M^3, \begin{pmatrix} b^1 \\ b^2 \\ 0 \end{pmatrix}\right)$$

with

$$M^1 = \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}, \quad M^2 = \begin{pmatrix} B^1 & \overline{B^1} & & \\ & & B^2 & \overline{B^2} \\ -I & & -I & \end{pmatrix} \quad \text{and} \quad M^3 = \left(\begin{array}{c|c} \mathcal{B}^1 \oplus \mathcal{B}^2 & 0 \end{array}\right)^{\mathsf{T}}.$$

Proof. By definition the Minkowski sum is

$$\begin{aligned} S_1 + S_2 &= \left\{ x \in \mathbb{R}^n \,\middle|\, \exists\, x^1 \in \mathbb{R}^n, x^2 \in \mathbb{R}^n \colon x = x^1 + x^2, x^1 \in S_1, x^2 \in S_2 \right\} \\ &= \left\{ x \in \mathbb{R}^n \,\middle|\, \exists \begin{array}{l} x^1 \in \mathbb{R}^n, y^1 \in \mathbb{R}^{m_1}, Z^1 \in \mathcal{S}^{k_1} \\ x^2 \in \mathbb{R}^n, y^2 \in \mathbb{R}^{m_2}, Z^2 \in \mathcal{S}^{k_2} \end{array} \colon \begin{array}{r} A_0^1 + \mathcal{A}^1(x^1) + \overline{\mathcal{A}^1}(y^1) + \boldsymbol{\mathcal{A}}^1(Z^1) \succcurlyeq 0 \\ A_0^2 + \mathcal{A}^2(x^2) + \overline{\mathcal{A}^2}(y^2) + \boldsymbol{\mathcal{A}}^2(Z^2) \succcurlyeq 0 \\ B^1 x^1 + \overline{B^1} y^1 + \mathcal{B}^{1\mathsf{T}}(Z^1) = b^1 \\ B^2 x^2 + \overline{B^2} y^2 + \mathcal{B}^{2\mathsf{T}}(Z^2) = b^2 \\ x - x^1 - x^2 = 0 \\ Z^1 \succcurlyeq 0, Z^2 \succcurlyeq 0 \end{array} \right\}. \end{aligned}$$

The claim now follows by grouping the projected variables into

$$y = \begin{pmatrix} x^1 \\ y^1 \\ x^2 \\ y^2 \end{pmatrix} \in \mathbb{R}^{2n+m_1+m_2} \quad \text{and} \quad Z = \begin{pmatrix} Z^1 & \\ & Z^2 \end{pmatrix} \in \mathcal{S}^{k_1+k_2}$$

and combining the inequalities and equations. The equality $x - x^1 - x^2 = 0$ is expressed by the bottom rows of M^1, M^2, M^3 and the respective right hand side. □

The Minkowski sum of two spectrahedra is not necessarily a spectrahedron, but always a spectrahedral shadow. To explain this it is relevant to know another property of spectrahedra. In [GR95] it is shown that their faces are *exposed*. This means that every face is either empty, the whole set or the intersection of a supporting hyperplane with the set itself, i.e. if F is a face of a spectrahedron C, then there exist $\omega \neq 0$ and γ such that $F = H(\omega, \gamma) \cap C$ under the condition $\varnothing \neq F \neq C$.

Example 2.17. Consider the sets $C_1 = B_1(0)$ and $C_2 = [-1, 1] \times \{0\}$, both of which are spectrahedra, cf. Example 2.4. In fact, C_2 is polyhedral. Their Minkowski sum is shown in Figure 2.3. It is the convex hull of the union of the sets $B_1((-1, 0)^{\mathsf{T}})$ and

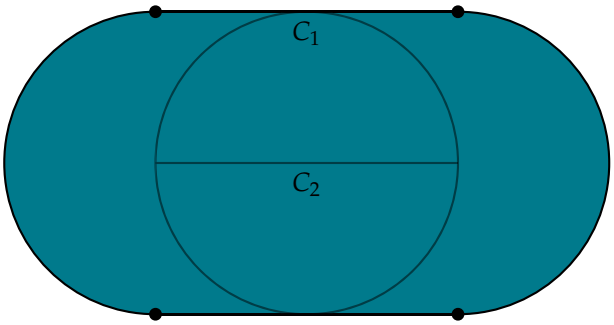

FIGURE 2.3: The Minkowski sum of the unit ball $C_1 = B_1(0)$ and the set $C_2 = [-1,1] \times \{0\}$. Marked by dots are four faces which are not exposed. Thus, the set is not a spectrahedron.

$B_1((0,1)^\mathsf{T})$ and admits four nonexposed faces. Hence, $C_1 + C_2$ cannot a spectrahedron.

Remark 2.18. Every face of a spectrahedron is itself a spectrahedron.

Proposition 2.19 (affine transformation)**.** *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^\mathsf{T}, b)$, $M\colon \mathbb{R}^n \to \mathbb{R}^p$ be a linear transformation and $c \in \mathbb{R}^p$. Then the affine transformation $M[S] + \{c\} \subseteq \mathbb{R}^p$ of S is a spectrahedral shadow represented by*

$$\left(A, \varnothing, \begin{pmatrix} \mathcal{A} \mid \overline{\mathcal{A}} \end{pmatrix}, \boldsymbol{\mathcal{A}}, \begin{pmatrix} 0 \\ I \end{pmatrix}, \begin{pmatrix} B & \bar{B} \\ -M & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{B} \mid 0 \end{pmatrix}^\mathsf{T}, \begin{pmatrix} b \\ c \end{pmatrix}\right).$$

Proof. The set $M[S] + \{c\}$ can be written as

$$\{z \in \mathbb{R}^p \mid \exists\, x \in \mathbb{R}^m \colon z = Mx + c, x \in S\}.$$

Rearranging the equation and expanding the definition of S yields

$$\left\{ z \in \mathbb{R}^p \;\middle|\; \begin{array}{r} x \in \mathbb{R}^n \\ \exists\, y \in \mathbb{R}^m \colon \\ Z \in \mathcal{S}^k \end{array} \;\middle|\; \begin{aligned} A_0 + \begin{pmatrix} \mathcal{A} \mid \overline{\mathcal{A}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \boldsymbol{\mathcal{A}}(Z) &\succcurlyeq 0 \\ \begin{pmatrix} B & \bar{B} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{B}^\mathsf{T}(Z) &= b \\ z - \begin{pmatrix} M & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= c \\ Z &\succcurlyeq 0 \end{aligned} \right\}.$$

Combining the two systems of equations into a single system concludes the proof. □

Since orthogonal projection is a special kind of linear transformation, the image of a spectrahedron under a linear map is not a spectrahedron in general. However, if a bijective affine transformation is applied to a spectrahedron, the property of spectrahedrality is unaltered, see [GR95].

Remark 2.20. The Minkowski sum is a special case of a linear transformation of the Cartesian product. For spectrahedral shadows $S_1, S_2 \subseteq \mathbb{R}^n$ we have the identity

$$S_1 + S_2 = M[S_1 \times S_2],$$

where $M\colon \mathbb{R}^{2n} \to \mathbb{R}^n$, $(x_1, x_2)^\mathsf{T} \mapsto x_1 + x_2$. Therefore, an alternative way to obtain a representation of the Minkowski sum is to combine the results of Propositions 2.14 and 2.19.

It is also possible to represent faces of spectrahedral shadows.

Proposition 2.21 (face of a spectrahedral shadow)**.** *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^\mathsf{T}, b)$ and assume S is the projection of a spectrahedron in the space $\mathbb{R}^{n+m} \times \mathcal{S}^k$. Then every face $F \subseteq \mathbb{R}^n$ of S is a spectrahedral shadow represented by*

$$\left(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, \begin{pmatrix} B \\ \omega^{1\mathsf{T}} \end{pmatrix}, \begin{pmatrix} \bar{B} \\ \omega^{2\mathsf{T}} \end{pmatrix}, \left(\begin{array}{c|c} \mathcal{B} & \Omega \end{array}\right)^\mathsf{T}, \begin{pmatrix} b \\ \gamma \end{pmatrix}\right)$$

for some $\omega^1 \in \mathbb{R}^n$, $\omega^2 \in \mathbb{R}^m$, $\Omega \in \mathcal{S}^k$ and $\gamma \in \mathbb{R}$.

Proof. Let $F \subseteq \mathbb{R}^n$ be a face of S. If $F = S$, then the claim follows immediately by choosing $\omega^1 = 0$, $\omega^2 = 0$, $\Omega = 0$ and $\gamma = 0$. Similarly, for $F = \emptyset$ one can choose the same values for ω^1, ω^2 and Ω and set $\gamma = 1$. Now assume that $\emptyset \neq F \neq S$. Denote by $C \subseteq \mathbb{R}^{n+m} \times \mathcal{S}^k$ the spectrahedron that projects to S and by $\pi\colon \mathbb{R}^{n+m} \times \mathcal{S}^k \to \mathbb{R}^n$ the projection $(x, y, Z) \mapsto x$, i.e. $S = \pi[C]$. Then $\bar{F} = \pi^{-1}[F] \cap C$ is a face of C. Since $\bar{F}$ is an exposed face, there exists a hyperplane $H((\omega^1, \omega^2, \Omega), \gamma) \subseteq \mathbb{R}^{n+m} \times \mathcal{S}^k$ such that

$$\bar{F} = H((\omega^1, \omega^2, \Omega), \gamma) \cap C.$$

The face $\bar{F}$ projects to F under π. Hence, we conclude

$$\begin{aligned} F &= \pi\left[\bar{F}\right] \\ &= \pi\left[\left\{ \begin{pmatrix} x \\ y \\ Z \end{pmatrix} \in \mathbb{R}^{n+m} \times \mathcal{S}^k \;\middle|\; \begin{aligned} A_0 + \mathcal{A}(x) + \overline{\mathcal{A}}(y) + \boldsymbol{\mathcal{A}}(Z) &\succcurlyeq 0 \\ Bx + \bar{B}y + \mathcal{B}^\mathsf{T}(Z) &= b \\ \omega^{1\mathsf{T}} x + \omega^{2\mathsf{T}} y + \Omega \bullet Z &= \gamma \\ Z &\succcurlyeq 0 \end{aligned} \right\}\right] \\ &= \left\{ x \in \mathbb{R}^n \;\middle|\; \exists \begin{matrix} y \in \mathbb{R}^m \\ Z \in \mathcal{S}^k \end{matrix} : \begin{aligned} A_0 + \mathcal{A}(x) + \overline{\mathcal{A}}(y) + \boldsymbol{\mathcal{A}}(Z) &\succcurlyeq 0 \\ Bx + \bar{B}y + \mathcal{B}^\mathsf{T}(Z) &= b \\ \omega^{1\mathsf{T}} x + \omega^{2\mathsf{T}} y + \Omega \bullet Z &= \gamma \\ Z &\succcurlyeq 0 \end{aligned} \right\}. \end{aligned}$$

Combining the equations yields the result. □

The fact that spectrahedral shadows need not be closed allows us to remove faces without losing semidefinite representability.

Proposition 2.22 (removing a face)**.** *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^\mathsf{T}, b)$ that is the projection of a spectrahedron in the space $\mathbb{R}^{n+m} \times \mathcal{S}^k$. Then for every face F of S the set $S \setminus F$ is a spectrahedral shadow represented by*

$$\left(A_0 \oplus \begin{pmatrix} -\gamma & 1 \\ 1 & 0 \end{pmatrix}, \left(\begin{array}{c} \mathcal{A} \\ \hline \omega^{1\mathsf{T}} \\ \hline 0 \end{array}\right), \left(\begin{array}{c} \overline{\mathcal{A}} \\ \hline \omega^{2\mathsf{T}} \end{array}\right) \oplus 1, \left(\begin{array}{c} \boldsymbol{\mathcal{A}} \\ \hline \Omega \\ \hline 0 \end{array}\right), B, \begin{pmatrix} \bar{B} & 0 \end{pmatrix}, \mathcal{B}^\mathsf{T}, b\right)$$

for some $\omega^1 \in \mathbb{R}^n$, $\omega^2 \in \mathbb{R}^m$, $\Omega \in \mathcal{S}^k$ and $\gamma \in \mathbb{R}$.

Proof. First, assume that $\emptyset \neq F \neq S$. Let C, π and $\bar{F}$ be defined as in the proof of Proposition 2.21. Then $C \setminus \bar{F}$ projects to $S \setminus F$, i.e. $\pi\left[C \setminus \bar{F}\right] = S \setminus F$. The set $C \setminus \bar{F}$ is given as

$$\left\{ \begin{pmatrix} x \\ y \\ Z \end{pmatrix} \in C \;\middle|\; \omega^{1\mathsf{T}} x + \omega^{2\mathsf{T}} y + \Omega \bullet Z \neq \gamma \right\}.$$

Without loss of generality the expression $\omega^{1\mathsf{T}} x + \omega^{2\mathsf{T}} y + \Omega \bullet Z - \gamma$ may assumed to be nonnegative on C because $H((\omega^1, \omega^2, \Omega), \gamma)$ is a supporting hyperplane of C. Otherwise we replace ω^1, ω^2, Ω and γ with their negations. Using the Schur complement we can equivalently express the inequality condition by the LMI

$$\begin{pmatrix} \omega^{1\mathsf{T}} x + \omega^{2\mathsf{T}} y + \Omega \bullet Z - \gamma & 1 \\ 1 & t \end{pmatrix} \succcurlyeq 0 \tag{2.2}$$

for some $t \in \mathbb{R}$. If $\omega^{1\mathsf{T}} x + \omega^{2\mathsf{T}} y + \Omega \bullet Z - \gamma$ is zero, then the matrix is not positive semidefinite because the off-diagonal entries are nonzero, see [HJ13]. Otherwise we can choose t large enough such that the LMI is satisfied. We obtain the result by combining the LMIs in the definition of C and (2.2) and applying π to $C \setminus \bar{F}$.
If $F = \emptyset$ or $F = S$, then we set $\gamma = -1$ or $\gamma = 1$ and the other parameters in (2.2) to zero, respectively. In the first case (2.2) is fulfilled for all $t \geqslant 1$, which results in $S \setminus F = S$, in the second case the first diagonal entry is negative. Hence the LMI is inconsistent and we have $S \setminus F = \emptyset$. □

We have shown that we can remove a finite number of faces from a spectrahedral shadow without changing the set type. This result is generalized by Netzer [Net11] to infinitely many faces given certain conditions. For two spectrahedral shadows $S_1 \subseteq S_2$ he defines the set $S_1 \hookleftarrow\!\!\circ\, S_2$ that is obtained by removing all faces of S_2 that are disjoint from S_1 and shows that $S_1 \hookleftarrow\!\!\circ\, S_2$ is a spectrahedral shadow.

Example 2.23. We return to the spectrahedral shadow $S = B_1(0) + ([-1,1] \times \{0\})$ from Example 2.17 again. According to Proposition 2.16 set S is the projection of a spectrahedron $C \subseteq \mathbb{R}^6$ that is defined by the system

$$\begin{aligned} \begin{pmatrix} 1 + x_1 & y_1 \\ y_1 & 1 - x_1 \end{pmatrix} \succcurlyeq 0 \qquad \begin{pmatrix} 1 + x_2 & 0 \\ 0 & 1 - x_2 \end{pmatrix} \succcurlyeq 0 \\ \begin{pmatrix} x - x_1 - x_2 \\ y - y_1 - y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad\qquad y_2 = 0 \end{aligned} \tag{2.3}$$

onto the x and y components. We want to describe the spectrahedral shadow that results when the three faces

$$F_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; -1 \leqslant x \leqslant 1, y = -1 \right\}, \quad F_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad F_3 = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

of S are removed. Denote by π the projection that maps C to S. In order to apply Proposition 2.22 we need to find the supporting hyperplanes of C that generate the faces $\pi^{-1}[F_i] \cap C$, $i = 1, \ldots, 3$, when intersected with C. The face $\pi^{-1}[F_1] \cap C$ of C is the set

$$\left\{ (x, y, x_1, y_2, x_2, y_2)^{\mathsf{T}} \in \mathbb{R}^6 \;\middle|\; x = x_2, x_1 = 0, -1 \leqslant x_2 \leqslant 1, y = y_1 = -1, y_2 = 0 \right\}.$$

This results by plugging $y_1 = -1$ into the system (2.3) which yields $x_1 = 0$. Since the condition $-1 \leqslant x \leqslant 1$ is already implied by $y = -1$, we conclude that $\pi^{-1}[F_1] \cap C$

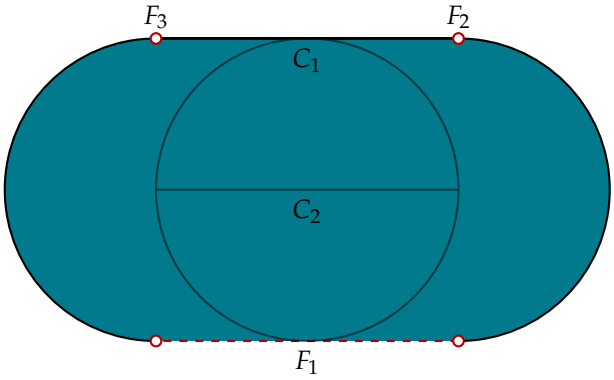

FIGURE 2.4: The spectrahedral shadow from Example 2.17 is the projection of a spectrahedron in $\mathbb{R}^6$. The three faces F_1, F_2 and F_3 indicated in red have been removed. The resulting set is again a spectrahedral shadow that is projected from $\mathbb{R}^9$.

is the intersection of C with the hyperplane $H_1 = H((0,1,0,0,0,0)^\mathsf{T}, -1)$. Similarly, the face $\pi^{-1}[F_2] \cap C$ is the singleton set $\{(1,1,0,1,1,0)^\mathsf{T}\}$ and it is straightforward to verify that

$$\pi^{-1}[F_2] \cap C = H_2 \cap C,$$

where $H_2 = H((0,-1,0,0,-1,0)^\mathsf{T}, -2)$. Analogously, $\pi^{-1}[F_3] \cap C$ corresponds to the hyperplane $H_3 = H((0,-1,0,0,1,0)^\mathsf{T}, -2)$. Observe that $C \subseteq H_i^+$ for every $i \in \{1,\dots,3\}$. Now we can apply Proposition 2.22 for each face F_i. Note that removing a face requires the introduction of a single auxiliary variable. Therefore, the set $S \setminus (F_1 \cup F_2 \cup F_3)$ is the spectrahedral shadow in the variables x and y described by system (2.3) and the LMIs

$$\begin{pmatrix} y+1 & 1 \\ 1 & t_1 \end{pmatrix} \succcurlyeq 0, \quad \begin{pmatrix} -y-x_2+2 & 1 \\ 1 & t_2 \end{pmatrix} \succcurlyeq 0 \quad \text{and} \quad \begin{pmatrix} -y+x_2+2 & 1 \\ 1 & t_3 \end{pmatrix} \succcurlyeq 0,$$

i.e. it is the projection of a spectrahedron from $\mathbb{R}^9$. The set is shown in Figure 2.4.

One can take the procedure of removing faces the furthest and remove all faces of a spectrahedral shadow except the set itself. This yields the relative interior of the set, see [Roc70].

Definition 2.24. Let $C \subseteq \mathbb{R}^n$ be a convex set. The *relative interior* of C, denoted $\operatorname{relint} C$, is defined as $\{x \in \mathbb{R}^n \mid \exists\, \varepsilon > 0 \colon B_\varepsilon(x) \cap \operatorname{aff} C \subseteq C\}$.

Every nonempty convex set has a point in its relative interior and if the set is full dimensional the relative interior coincides with the interior. Moreover, for $\bar{x} \in \operatorname{relint} C$ the identity $\operatorname{relint} C = \{x \in \mathbb{R}^n \mid \exists\, \varepsilon > 0 \colon x + \varepsilon(x - \bar{x}) \in C\}$ holds, cf. [Roc70].

Proposition 2.25 (relative interior)**.** *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^\mathsf{T}, b)$. Then the relative interior of S is a spectrahedral shadow represented by*

$$\left(M^1, \left(\begin{array}{c|c} \mathcal{A} & \\ \hline 0 & \\ \hline 0 & \end{array}\right), M^2, \left(\begin{array}{c|c} \boldsymbol{\mathcal{A}} & \\ \hline 0 & \\ \hline 0 & \end{array}\right), B, \begin{pmatrix} \bar{B} & B\bar{x} - b & 0 \end{pmatrix}, \mathcal{B}^\mathsf{T}, B\bar{x} \right)$$

for a fixed $\bar{x} \in \operatorname{relint} S$ *and*

$$M^1 = -\mathcal{A}(\bar{x}) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M^2 = \left(\begin{array}{c|c|c} \overline{\mathcal{A}} & A_0 + \mathcal{A}\bar{x} & 0 \\ \hline 0 & E_{11} & E_{22} \\ \hline 0 & -E_{11} & E_{22} \end{array}\right).$$

Proof. For a point $\bar{x} \in \operatorname{relint} S$ we can express the relative interior of S as the set $\{x \in \mathbb{R}^n \mid \exists\, \varepsilon > 0 \colon x + \varepsilon\,(x - \bar{x}) \in S\}$. Expanding the set membership yields

$$\begin{aligned}
\operatorname{relint} S &= \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} & \varepsilon > 0 \qquad A_0 + (1+\varepsilon)\mathcal{A}(x) - \varepsilon\mathcal{A}(\bar{x}) + \overline{\mathcal{A}}(y) + \boldsymbol{\mathcal{A}}(Z) \succcurlyeq 0 \\ \exists\, y \in \mathbb{R}^m \colon & (1+\varepsilon)Bx - \varepsilon B\bar{x} + \bar{B}y + \mathcal{B}^{\mathsf{T}}(Z) = b \\ & Z \in \mathcal{S}^k \qquad\qquad Z \succcurlyeq 0 \end{array} \right\} \\
&= \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, \varepsilon > 0, y \in \mathbb{R}^m, Z \in \mathcal{S}^k \colon \\ \quad \dfrac{1}{1+\varepsilon} A_0 + \mathcal{A}(x) - \dfrac{\varepsilon}{1+\varepsilon}\mathcal{A}(\bar{x}) + \overline{\mathcal{A}}(y) + \boldsymbol{\mathcal{A}}(Z) \succcurlyeq 0 \\ \qquad\qquad Bx - \dfrac{\varepsilon}{1+\varepsilon} B\bar{x} + \bar{B}y + \mathcal{B}^{\mathsf{T}}(Z) = \dfrac{1}{1+\varepsilon} b \\ \qquad\qquad Z \succcurlyeq 0 \end{array} \right\} \\
&= \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, \varepsilon \in (0,1), y \in \mathbb{R}^m, Z \in \mathcal{S}^k \colon \\ \quad -\mathcal{A}(\bar{x}) + \mathcal{A}(x) + \overline{\mathcal{A}}(y) + (A_0 + \mathcal{A}(\bar{x}))\,\varepsilon + \boldsymbol{\mathcal{A}}(Z) \succcurlyeq 0 \\ \qquad\qquad Bx + \bar{B}y + (B\bar{x} - b)\,\varepsilon + \mathcal{B}^{\mathsf{T}}(Z) = B\bar{x} \\ \qquad\qquad Z \succcurlyeq 0 \end{array} \right\}.
\end{aligned}$$

In the last equation we have substituted ε for $1/(1+\varepsilon)$. The conditions $\varepsilon > 0$ and $1 - \varepsilon > 0$ can be expressed with the Schur complement as

$$\begin{pmatrix} \varepsilon & 1 \\ 1 & t \end{pmatrix} \succcurlyeq 0 \quad \text{and} \quad \begin{pmatrix} 1-\varepsilon & 1 \\ 1 & t \end{pmatrix} \succcurlyeq 0$$

for an additional variable $t \in \mathbb{R}$. Finally, the relative interior of S is the set

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\; (y, \varepsilon, t)^{\mathsf{T}} \in \mathbb{R}^{m+2}, Z \in \mathcal{S}^k \colon \\ \quad -\mathcal{A}(\bar{x}) + \mathcal{A}(x) + \overline{\mathcal{A}}(y) + (A_0 + \mathcal{A}(\bar{x}))\,\varepsilon + \boldsymbol{\mathcal{A}}(Z) \succcurlyeq 0 \\ \qquad\qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varepsilon + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} t \succcurlyeq 0 \\ \qquad\qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \varepsilon + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} t \succcurlyeq 0 \\ \qquad\qquad Bx + \bar{B}y + (B\bar{x} - b)\,\varepsilon + \mathcal{B}^{\mathsf{T}}(Z) = B\bar{x} \\ \qquad\qquad Z \succcurlyeq 0 \end{array} \right\}.$$

□

In order to explicitly express the relative interior of a spectrahedral shadow S, we need to know a point $\bar{x}$ from the set beforehand. If the representation of S satisfies certain additional conditions, it might be possible to state a suitable $\bar{x}$ immediately. For example, if $A_0 \succ 0$ and $b = 0$, then $\bar{x} = 0$ is an interior point of S. However, it is not trivial to specify such an $\bar{x}$ in general. In the next chapter we approach this problem algorithmically and present a procedure that computes a relative interior point in Algorithm 3.1. This allows us to deal with the drawback of Proposition 2.25.

2.2.1 Operations involving polarity

We now turn to set operations that involve polarity.

Definition 2.26 (cf. [Roc70]). For a convex set $C \subseteq \mathbb{R}^n$ the sets C° and C^* defined as

$$\left\{u \in \mathbb{R}^n \,\middle|\, \forall\, x \in C\colon x^\mathsf{T} u \leqslant 1\right\} \quad \text{and} \quad \left\{u \in \mathbb{R}^n \,\middle|\, \forall\, x \in C\colon x^\mathsf{T} u \leqslant 0\right\}$$

are called the *polar* and *polar cone* of C, respectively. They are the sets of linear functions that are bounded above on C by either 1 or 0. Additionally, if C is a cone, then $C^\circ = C^*$.

Due to the universal quantifier in the definitions, polars of spectrahedral shadows are closely related to semidefinite programming. The main goal of this section is to derive an explicit description of S° for a spectrahedral shadow S. The proof relies on duality results for semidefinite programming. Remember that unlike for linear programming strong duality may fail in the semidefinite case. This is a challenge in the derivation of S° that we deal with by either assuming additional properties about S or by a strong duality theory due to Ramana [Ram97; RTW97] in Theorems 2.32 and 2.35, respectively.
There are differences in the spectrahedral and the projected case that we want to outline. Therefore it is beneficial to consider these cases separately.

Proposition 2.27 (see [GR95]). *Let $C \subseteq \mathbb{R}^n$ be a spectrahedron defined by the linear pencil $\mathcal{A}$ and matrix A_0 of size ℓ and suppose that $A_0 \succcurlyeq 0$. Then the polar C° of C is the closure of the set*

$$C^+ = \left\{u \in \mathbb{R}^n \,\middle|\, \exists\, V \in \mathcal{S}^\ell\colon u = -\mathcal{A}^\mathsf{T}(V), A_0 \bullet V \leqslant 1, V \succcurlyeq 0\right\}.$$

The set C^+ is called the *algebraic polar* of C in [GR95]. It is a spectrahedral shadow represented by

$$\left(1, \varnothing, \varnothing, -\boldsymbol{\mathcal{A}}, I, \varnothing, \mathcal{A}^\mathsf{T}, 0\right)$$

for $\boldsymbol{\mathcal{A}}\colon \mathcal{S}^\ell \to \mathcal{S}^1$, $\boldsymbol{\mathcal{A}}(V) = \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} (A_0)_{ij} v_{ij}$. Note that $S^1 = \mathbb{R}$ and $\boldsymbol{\mathcal{A}}(V) = A_0 \bullet V$. Hence, we can represent the polar of a spectrahedron C as a spectrahedral shadow up to closure. The only assumption on C is that $A_0 \succcurlyeq 0$ or, equivalently, the origin is contained in the set. This requirement is due to the fact that the polar of a set always contains the origin and it holds $(C^\circ)^\circ = \operatorname{cl}\operatorname{conv}(C \cup \{0\})$, see [Roc70]. If C is a spectrahedron containing the origin, the latter set is just C itself. Thus, if C is a spectrahedron containing a point $\bar{x}$, the proposition can at least be applied to the translated set $C - \{\bar{x}\}$. In [GR95] it is shown that the algebraic polar and the polar of a spectrahedron can be different, i.e. its algebraic polar need not be closed. However, we can dispose of the closure operation by a stronger assumption.

Proposition 2.28. *Let $C \subseteq \mathbb{R}^n$ be a spectrahedron defined by the linear pencil $\mathcal{A}$ and matrix A_0 of size ℓ and suppose there exists a point $\bar{x} \in \mathbb{R}^n$ satisfying $A_0 + \mathcal{A}(\bar{x}) \succ 0$. Then $C^\circ = C^+$.*

Proof. The polar of C is defined as

$$\begin{aligned} C^\circ &= \left\{u \in \mathbb{R}^n \,\middle|\, \forall\, x \in C\colon x^\mathsf{T} u \leqslant 1\right\} \\ &= \left\{u \in \mathbb{R}^n \,\middle|\, \sup\left\{x^\mathsf{T} u \,\middle|\, x \in C\right\} \leqslant 1\right\} \\ &= \left\{u \in \mathbb{R}^n \,\middle|\, \sup\left\{x^\mathsf{T} u \,\middle|\, A_0 + \mathcal{A}(x) \succcurlyeq 0\right\} \leqslant 1\right\} \end{aligned}$$

$$
\begin{aligned}
&= \left\{ u \in \mathbb{R}^n \;\middle|\; \inf\left\{ A_0 \bullet V \;\middle|\; -u = \mathcal{A}^{\mathsf{T}}(V), V \succcurlyeq 0 \right\} \leqslant 1 \right\} \\
&= \left\{ u \in \mathbb{R}^n \;\middle|\; \exists\, V \in \mathcal{S}^{\ell} \colon u = -\mathcal{A}^{\mathsf{T}}(V), A_0 \bullet V \leqslant 1, V \succcurlyeq 0 \right\} \\
&= C^{+}.
\end{aligned}
$$

The fourth equality holds because the supremum is bounded above and we have strict feasibility for $\bar{x}$, which allows us to apply Theorem 1.30. For the same reason the fifth equality holds because it implies that the infimum is attained. □

Taking into account the above remark about the interior of spectrahedra, Proposition 2.28 allows us to express the polar of a spectrahedron with nonempty interior as a spectrahedral shadow. In general, one cannot expect to find a description of C° as a spectrahedron. This is illustrated in the next example, thus, establishing that spectrahedra are not closed under polarity.

Example 2.29. Consider the spectrahedron

$$
C = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1 & \frac{1}{2}x_1 & 0 \\ \frac{1}{2}x_1 & -x_2 & 0 \\ 0 & 0 & -x_1 \end{pmatrix} \succcurlyeq 0 \right\} = \left\{ x \in \mathbb{R}^2 \;\middle|\; x_2 \leqslant -\frac{1}{4}x_1^2 \right\} \cap -\mathbb{R}^2_+.
$$

By applying Proposition 2.28 we can calculate C° as the set of all points $u \in \mathbb{R}^2$ that admit the existence of a matrix $V \in \mathcal{S}^3_+$ such that the system

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet V \leqslant 1, \quad -\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \bullet V = u_1 \quad \text{and} \quad -\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet V = u_2
$$

is solved. This is equivalent to $v_{11} \leqslant 1$, $u_1 = v_{33} - v_{12}$ and $u_2 = v_{22}$. Due to the block structure of the data we can, without loss of generality, assume that V has the same structure. Then $V \succcurlyeq 0$ is equivalent to

$$
\begin{aligned}
v_{ii} &\geqslant 0 \quad i = 1, \ldots, 3 \\
v_{11}v_{22} - v_{12}^2 &\geqslant 0.
\end{aligned}
\tag{2.4}
$$

It immediately follows that $u_2 \geqslant 0$ for all $u \in C^{\circ}$. Now, assume that $u \geqslant 0$. Then (2.4) is solved by setting $v_{12} = 0$ which implies $\mathbb{R}^2_+ \subseteq C^{\circ}$. If $u_1^2 \leqslant u_2$, then we can choose $v_{33} = 0$ and $v_{11} = 1$. This yields the inclusion $\left\{ u \in \mathbb{R}^2 \;\middle|\; u_1^2 \leqslant u_2 \right\} + \mathbb{R}^2_+ \subseteq C^{\circ}$. Finally, if $u_1 < 0$ and $u_1^2 > u_2$, then

$$
v_{11}v_{22} - v_{12}^2 = v_{11}u_2 - v_{33}^2 + 2v_{33}u_1 - u_1^2 \leqslant u_2 - u_1^2 < 0,
$$

i.e. (2.4) has no solution. Thus, C° is in fact equal to $\left\{ u \in \mathbb{R}^2 \;\middle|\; u_1^2 \leqslant u_2 \right\} + \mathbb{R}^2_+$. Observe that the singleton $\{0\}$ is a face of this set that is not exposed. Hence, C° can not be a spectrahedron. The sets C and C° are shown in Figure 2.5.

We will now discuss the projected case. The first question we want to give an answer to is whether there is an analogue to Proposition 2.27, i.e. does the closure of the algebraic polar S^{+} of a spectrahedral shadow S coincide with its polar S°? Here, we define S^{+} as the set $\pi^{\mathsf{T}^{-1}}[C^{+}]$ for the spectrahedron C that projects to S and the corresponding projection π such that $S = \pi[C]$. This definition is justified because it is in harmony with the behavior of polarity under linear transformations, cf. [Roc70, Corollary 16.3.2]. Now, it turns out that only a containment is guaranteed.

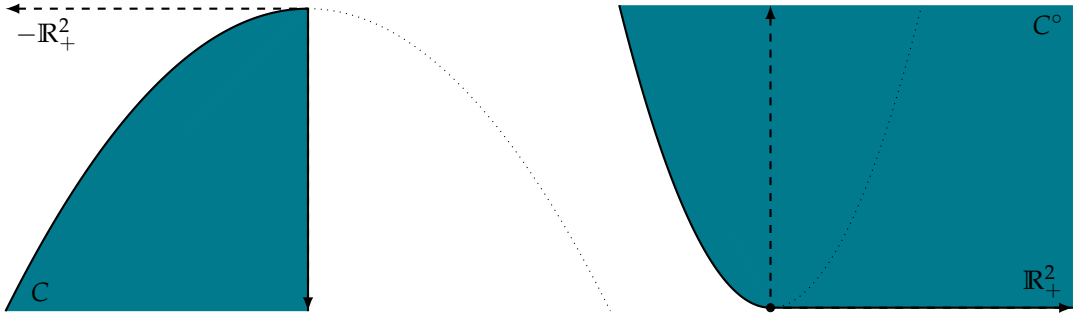

FIGURE 2.5: The dual pair C and C° from Example 2.29. The dotted lines indicate the sets $\{x \in \mathbb{R}^2 \mid 4x_2 \leqslant x_1^2\}$ and $\{x \in \mathbb{R}^2 \mid x_2 \geqslant x_1^2\}$ which are polar to each other. The set C is the intersection of the first set with the cone $-\mathbb{R}^2_+$. Its polar set is the Minkowski sum of the polar of the second set and the cone $\mathbb{R}^2_+$. Note the nonexposed face of C° indicated by a black dot. It certifies that C° is not a spectrahedron.

Proposition 2.30 (algebraic polar). *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^\mathsf{T}, 0)$ and suppose that $A_0 \succcurlyeq 0$. Then its algebraic polar S^+ is a spectrahedral shadow represented by*

$$\left(0, \varnothing, \mathcal{B}, \left(\begin{array}{c|c} -\boldsymbol{\mathcal{A}}^\mathsf{T} & 0 \end{array}\right), \begin{pmatrix} -I \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} B^\mathsf{T} \\ \bar{B}^\mathsf{T} \\ 0 \end{pmatrix}, \left(\begin{array}{c|c} -\mathcal{A}^\mathsf{T} & 0 \\ \hline -\overline{\mathcal{A}}^\mathsf{T} & 0 \\ \hline A_0 & 1 \end{array}\right), \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

and it holds $\operatorname{cl} S^+ \subseteq S^\circ$.

Proof. Denote by $C \subseteq \mathbb{R}^{n+m} \times \mathcal{S}^k$ the spectrahedron that projects to S and by π the corresponding projection such that $S = \pi[C]$. The algebraic polar C^+ of C is given as the set

$$\left\{ \begin{pmatrix} u \\ w \\ U \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{S}^k \;\middle|\; \exists\, v \in \mathbb{R}^d, V \in \mathcal{S}^\ell \colon \begin{array}{r} u = -\mathcal{A}^\mathsf{T}(V) + B^\mathsf{T} v \\ w = -\overline{\mathcal{A}}^\mathsf{T}(V) + \bar{B}^\mathsf{T} v \\ U \preccurlyeq -\boldsymbol{\mathcal{A}}^\mathsf{T}(V) + \mathcal{B}(v) \\ 1 \geqslant A_0 \bullet V \\ V \succcurlyeq 0 \end{array} \right\} \tag{2.5}$$

which is evident by considering the primal-dual pair

$$\begin{aligned} \max\, u^\mathsf{T} x + w^\mathsf{T} y + U \bullet Z \quad \text{s.t.} \quad & A_0 + \mathcal{A}(x) + \overline{\mathcal{A}}(y) + \boldsymbol{\mathcal{A}}(Z) \succcurlyeq 0 \\ & Bx + \bar{B}y + \mathcal{B}^\mathsf{T}(Z) = 0 \\ & Z \succcurlyeq 0 \end{aligned}$$

and

$$\begin{aligned} \min\; A_0 \bullet V \quad \text{s.t.} \quad & u = -\mathcal{A}^\mathsf{T}(V) + B^\mathsf{T} v \\ & w = -\overline{\mathcal{A}}^\mathsf{T}(V) + \bar{B}^\mathsf{T} v \\ & U \preccurlyeq \boldsymbol{\mathcal{A}}^\mathsf{T}(V) + \mathcal{B}(v) \\ & V \succcurlyeq 0 \end{aligned}$$

of semidefinite programs. The conditions $A_0 \succcurlyeq 0$ and $b = 0$ simply mean the origin is contained in C. Hence, by applying Proposition 2.27 we have $C^\circ = \operatorname{cl} C^+$. We now

conclude

$$S^\circ = \pi[C]^\circ = \pi^{\mathsf{T}-1}[C^\circ] = \pi^{\mathsf{T}-1}[\operatorname{cl} C^+] \supseteq \operatorname{cl} \pi^{\mathsf{T}-1}[C^+] = \operatorname{cl} S^+.$$

The second equation holds due to [Roc70, Corollary 16.3.2] and the inclusion merely resembles the fact that linear transformations are continuous. The representation of S^+ follows from rearranging and grouping the equations and LMIs in (2.5) and taking into account that $\pi^{\mathsf{T}-1}[C^+] = \{u \in \mathbb{R}^n \mid (u,0,0) \in C^+\}$ as well as that $A_0 \bullet V \leqslant 1$ is equivalent to $A_0 \bullet V + s = 1$ for some $s \geqslant 0$. □

It is possible for the inclusion in the proposition to be strict.

Example 2.31. Consider the spectrahedral shadow

$$S = \left\{ x \in \mathbb{R} \;\middle|\; \exists\, y \in \mathbb{R}\colon \begin{pmatrix} -y & x \\ x & 0 \end{pmatrix} \succcurlyeq 0 \right\}$$

projected from $\mathbb{R}^2$. Since the bottom right entry is zero, it holds $x = 0$ whenever the defining pencil of S is positive semidefinite. Hence, $S = \{0\}$ and $S^\circ = \mathbb{R}$. According to Proposition 2.30 the set S^+ consists of all points $y \in \mathbb{R}$ such that the system

$$y = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bullet V \quad \text{and} \quad 0 = -\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \bullet V$$

has a solution in $\mathcal{S}^2_+$. The second equation yields $v_{11} = 0$ and the positive semidefiniteness of V implies $v_{12} = 0$. Thus, $S^+ = \{0\} = \operatorname{cl} S^+$. This also demonstrates that the discrepancy between S° and $\operatorname{cl} S^+$ can be arbitrarily bad.

Analogously to Proposition 2.28 we can close the gap between S° and S^+ by imposing stronger assumptions upon S.

Theorem 2.32 (polar set, first version)**.** *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^{\mathsf{T}}, b)$ and assume there exist $\bar{x} \in \mathbb{R}^n$, $\bar{y} \in \mathbb{R}^m$ and $\bar{Z} \in \mathcal{S}^k_{++}$ such that $A_0 + \mathcal{A}(\bar{x}) + \overline{\mathcal{A}}(\bar{y}) + \boldsymbol{\mathcal{A}}(\bar{Z}) \succ 0$ and $B\bar{x} + \bar{B}\bar{y} + \mathcal{B}^{\mathsf{T}}(\bar{Z}) = b$. Then the polar S° of S is a spectrahedral shadow represented by*

$$\left(0, \varnothing, \mathcal{B}, \left(\begin{array}{c|c} -\boldsymbol{\mathcal{A}}^{\mathsf{T}} & 0 \end{array}\right), \begin{pmatrix} -I \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} B^{\mathsf{T}} \\ \bar{B}^{\mathsf{T}} \\ b^{\mathsf{T}} \end{pmatrix}, \left(\begin{array}{c|c} -\mathcal{A}^{\mathsf{T}} & 0 \\ \hline -\overline{\mathcal{A}}^{\mathsf{T}} & 0 \\ \hline A_0 & 1 \end{array}\right), \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Proof. The proof follows the same argumentation as the proof of Proposition 2.28 with the semidefinite programs replaced by the primal-dual pair

$$\begin{aligned} \max\, u^{\mathsf{T}} x \quad \text{s.t.} \quad & A_0 + \mathcal{A}(x) + \overline{\mathcal{A}}(y) + \boldsymbol{\mathcal{A}}(Z) \succcurlyeq 0 \\ & Bx + \bar{B}y + \mathcal{B}^{\mathsf{T}}(Z) = b \\ & Z \succcurlyeq 0 \end{aligned}$$

and

$$\begin{aligned} \min\, A_0 \bullet V + b^{\mathsf{T}} v \quad \text{s.t.} \quad & u = -\mathcal{A}^{\mathsf{T}}(V) + B^{\mathsf{T}} v \\ & 0 = -\overline{\mathcal{A}}^{\mathsf{T}}(V) + \bar{B}^{\mathsf{T}} v \\ & 0 \preccurlyeq -\boldsymbol{\mathcal{A}}^{\mathsf{T}}(V) + \mathcal{B}(v) \\ & V \succcurlyeq 0 \end{aligned}$$

for which strong duality holds. □

Remark 2.33. The theorem reveals another motivation for using the representation of a spectrahedral shadow defined in 2.10. The feasible region of the dual program has essentially the same structure as the feasible region of the primal, i.e. it describes a representation of a spectrahedral shadow. Notice, however, that the data $\mathcal{A}$ and $\overline{\mathcal{A}}$ from the LMI in the primal program appear as part of the equalities in the dual in the form of their adjoints. The operator $\mathcal{B}$ swaps its role in the same way. Thus, if we had considered a spectrahedral shadow defined only by a LMI, then its polar set would not immediately be given by another LMI but by equality constraints and an additional reformulation would be necessary.

We are now able to represent the polar of a spectrahedral shadow given that it satisfies a Slater-like constraint qualification. This raises two problems. Firstly, it is not trivial to determine whether the conditions are satisfied by a given representation. Secondly, if a series of different set operations are applied to a spectrahedral shadow, some of which involve polarity, then the conditions cannot be guaranteed to be satisfied by the intermediate sets even if the original spectrahedral shadow fulfills them.
In the next paragraph we deal with both problems by deriving a stronger version of Theorem 2.32. It is based on an exact duality theory for semidefinite programs due to Ramana [Ram97]. He formulates the following dual program to (SDP*), which he denotes *Extended Lagrange-Slater Dual*.

$$\begin{aligned}
\min\ & C \bullet (V^{p+1} + W^{p+1} + W^{p+1\mathsf{T}}) & & \\
\text{s.t. } & \mathcal{A}^{\mathsf{T}}(V^{p+1} + W^{p+1} + W^{p+1\mathsf{T}}) = b & & \\
& V^{p+1} \succcurlyeq 0 & & \\
& \mathcal{A}^{\mathsf{T}}(V^{i} + W^{i} + W^{i\mathsf{T}}) = 0 & & i = 1, \dots, p \\
& C \bullet (V^{i} + W^{i} + W^{i\mathsf{T}}) = 0 & & i = 1, \dots, p \\
& V^{i-1} \succcurlyeq W^{i\mathsf{T}} W^{i} & & i = 1, \dots, p+1 \\
& W^{1} = 0 & &
\end{aligned} \tag{ELSD}$$

The variables are $V^i \in \mathcal{S}^\ell$ and $W^i \in \mathcal{M}^\ell$, $i = 1, \dots, p+1$, where $p = \min\{m, \ell\}$. Problem (ELSD) is a semidefinite program because the constraints $V^{i-1} \succcurlyeq W^{i\mathsf{T}} W^i$ can equivalently be stated as

$$\begin{pmatrix} V^{i-1} & W^i \\ W^{i\mathsf{T}} & I \end{pmatrix} \succcurlyeq 0$$

using Schur complement for $i = 1, \dots, p+1$. Ramana establishes a strong duality result for the pair (SDP*) and (ELSD). We repeat the main result of his work for conciseness.

Theorem 2.34 ([Ram97, Theorem 6])**.** *The following hold for the problems* (SDP*) *and* (ELSD)*:*

(i) if y is feasible for (SDP*) *and V^i, W^i, $i = 1, \dots, p+1$, are feasible for* (ELSD)*, then $b^{\mathsf{T}} y \leqslant C \bullet (V^{p+1} + W^{p+1} + W^{p+1\mathsf{T}})$,*

(ii) if (SDP*) *is feasible, then its optimal value is finite if and only if* (ELSD) *is feasible,*

(iii) if (SDP*) *and* (ELSD) *are feasible, then their optimal values coincide,*

(iv) whenever the optimal values of (SDP*) *and* (ELSD) *coincide,* (ELSD) *has a solution.*

In [RTW97] it is shown that this duality approach is equivalent to the one in [BW81], which proposes to a given convex program a regularized program that requires the computation of the so-called minimal cone of the problem and satisfies strong duality statements. Moreover, an extended dual program to (SDP) is introduced as well which reads as

$$\begin{aligned} \max\ & b^{\mathsf{T}} v^{p+1} \\ \text{s.t. } & \mathcal{A}(v^{p+1}) + W^{p+1} + W^{p+1\mathsf{T}} \preccurlyeq C \\ & \mathcal{A}(v^i) + W^i + W^{i\mathsf{T}} \succcurlyeq 0 \qquad i = 1, \dots, p \\ & b^{\mathsf{T}} v^i = 0 \qquad i = 1, \dots, p \\ & \mathcal{A}(v^{i-1}) \succcurlyeq W^{i\mathsf{T}} W^i \quad i = 1, \dots, p+1 \\ & W^1 = 0. \end{aligned} \tag{ELSD*}$$

Strong duality analogous to the statements in Theorem 2.34 connects the pair (SDP) and (ELSD*) of semidefinite programs. We can use these duality results to derive a description of the polar set of a spectrahedral shadow. This is already done for spectrahedra in [Ram97].

Theorem 2.35 (polar set, second version)**.** *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^{\mathsf{T}}, b)$. Then a point $u \in \mathbb{R}^n$ belongs to the polar S° if and only if the system*

$$\begin{aligned} -\mathcal{A}^{\mathsf{T}}(V^{p+1} + W_1^{p+1} + W_1^{p+1\mathsf{T}}) + B^{\mathsf{T}} v^{p+1} &= u \\ A_0 \bullet (V^{p+1} + W_1^{p+1} + W_1^{p+1\mathsf{T}}) + b^{\mathsf{T}} v^{p+1} &\leqslant 1 \\ V^{p+1} &\succcurlyeq 0 \\ -\mathcal{A}^{\mathsf{T}}(V^i + W_1^i + W_1^{i\mathsf{T}}) + B^{\mathsf{T}} v^i &= 0 \qquad i = 1, \dots, p \\ -\overline{\mathcal{A}}^{\mathsf{T}}(V^i + W_1^i + W_1^{i\mathsf{T}}) + \bar{B}^{\mathsf{T}} v^i &= 0 \qquad i = 1, \dots, p+1 \\ -\boldsymbol{\mathcal{A}}^{\mathsf{T}}(V^i + W_1^i + W_1^{i\mathsf{T}}) + \mathcal{B}(v^i) + W_2^i + W_2^{i\mathsf{T}} &\succcurlyeq 0 \qquad i = 1, \dots, p+1 \\ A_0 \bullet (V^i + W_1^i + W_1^{i\mathsf{T}}) &= 0 \qquad i = 1, \dots, p \\ b^{\mathsf{T}} v^i &= 0 \qquad i = 1, \dots, p \\ V^{i-1} &\succcurlyeq W_1^{i\mathsf{T}} W_1^i \quad i = 1, \dots, p+1 \\ -\boldsymbol{\mathcal{A}}^{\mathsf{T}}(V^{i-1}) + \mathcal{B}(v^{i-1}) &\succcurlyeq W_2^{i\mathsf{T}} W_2^i \quad i = 1, \dots, p+1 \\ W_1^1 = 0, W_2^1 &= 0 \end{aligned}$$

is consistent for some $V^i \in \mathcal{S}^\ell$, $v^i \in \mathbb{R}^d$, $W_1^i \in \mathcal{M}^\ell$, $W_2^i \in \mathcal{M}^k$, $i = 1, \dots, p+1$, with $p = \min\{n + m + k, \ell + d\}$.

Proof. Denote the system in the claim without the inequality in the second line by $(\star)$. Following the construction in [RTW97, Section 4], the Extended Lagrange-Slater Dual to

$$\max u^{\mathsf{T}} x \quad \text{s.t.} \quad x \in S \tag{P}$$

can be formulated as

$$\begin{aligned} \min\ & A_0 \bullet (V^{p+1} + W_1^{p+1} + W_1^{p+1\mathsf{T}}) + b^{\mathsf{T}} v^{p+1} \\ \text{s.t. } & (V^i, v^i, W_1^i, W_2^i)_{i=1}^{p+1} \text{ solves } (\star). \end{aligned} \tag{ELSD-P}$$

Strong duality holds for the pair (P) and (ELSD-P) according to Theorem 2.34. The remaining part of the proof is identical to the proof of Proposition 2.28 with the corresponding pair of semidefinite programs replaced by (P) and (ELSD-P). □

The theorem allows us to explicitly describe the polar of a spectrahedral shadow S in terms of LMIs and equations without further assumptions on the set. This comes at the cost of a description whose size grows polynomial in the size of the description of S, i.e. if S is projected from $\mathbb{R}^{n+m} \times \mathcal{S}^k$ and described by an LMI of size ℓ and d equations, then S° is projected from $\mathbb{R}^{n+d(p+1)} \times \mathcal{S}^{\ell(p+1)} \times \mathcal{M}^{\ell(p+1)} \times \mathcal{M}^{k(p+1)}$. Hence, although Theorem 2.35 is stronger than Theorem 2.32, it is preferred to apply the latter from a practical perspective to obtain a smaller description whenever possible.
Before we continue with the recession cone in the next subsection, we cover the polar cone, normal cone, closure and convex hull of the union.

Corollary 2.36 (polar cone operation)**.** *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^\mathsf{T}, b)$ and assume there exist $\bar{x} \in \mathbb{R}^n$, $\bar{y} \in \mathbb{R}^m$ and $\bar{Z} \in \mathcal{S}^k_{++}$ such that $A_0 + \mathcal{A}(\bar{x}) + \overline{\mathcal{A}}(\bar{y}) + \boldsymbol{\mathcal{A}}(\bar{Z}) \succ 0$ and $B\bar{x} + \bar{B}\bar{y} + \mathcal{B}^\mathsf{T}(\bar{Z}) = b$. Then the polar cone S^* of S is a spectrahedral shadow represented by*

$$\left(0, \varnothing, \mathcal{B}, \left(\begin{array}{c|c} -\boldsymbol{\mathcal{A}}^\mathsf{T} & 0 \end{array}\right), \begin{pmatrix} -I \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} B^\mathsf{T} \\ \bar{B}^\mathsf{T} \\ b^\mathsf{T} \end{pmatrix}, \left(\begin{array}{c|c} -\mathcal{A}^\mathsf{T} & 0 \\ \hline -\overline{\mathcal{A}}^\mathsf{T} & 0 \\ \hline A_0 & 1 \end{array}\right), 0\right).$$

Proof. The proof is very similar to the proofs of Theorem 2.32 and Proposition 2.28 with the 1 on the right hand side replaced with 0. □

Clearly, an analogous result to Theorem 2.35 can be formulated for the polar cone by setting the right hand side in $A_0 \bullet (V^{p+1} + W_1^{p+1} + W_1^{p+1\mathsf{T}}) + b^\mathsf{T} v^{p+1} \leqslant 1$ to 0. We omit this here and instead hint at another way to represent the polar cone.

Proposition 2.37 (cf. [Roc70])**.** *For spectrahedral shadows $S_1, S_2 \subseteq \mathbb{R}^n$ the following hold true:*

(i) *if $0 \in S_1$, then $\operatorname{cl} S_1 = (S_1{}^\circ)^\circ$,*

(ii) *if $0 \in \operatorname{conv}(S_1 \cup S_2)$, then $\operatorname{cl}\operatorname{conv}(S_1 \cup S_2) = (S_1{}^\circ \cap S_2{}^\circ)^\circ$,*

(iii) *$(\operatorname{cone} S_1)^\circ = S_1^*$,*

(iv) *$N_{S_1}(\bar{x}) = (S_1 - \{\bar{x}\})^*$.*

Remark 2.38. The first part of Proposition 2.37 indicates how one can express the closure of a spectrahedral shadow. Theorems 2.32 or 2.35 have to be applied twice to the set. By (ii) we are able to compute the closure of the convex hull of the sets S_1 and S_2 because we can represent intersections and polars by Proposition 2.12 and Theorem 2.35, respectively. Another approach is found in [HN09; NS09] where the result is generalized to finitely many spectrahedral shadows. Statement (iii) can be easily verified from the definitions of polar set and polar cone. If we can represent the conical hull of a spectrahedral shadow, then the polar cone ensues. We treat the conical hull in the next subsection. Finally, an expression for the normal cone follows from (iii) and Proposition 2.16.

2.2.2 Representing the closure of the recession cone

For a spectrahedron $C = \{x \in \mathbb{R}^n \mid A_0 + \mathcal{A}(x) \succcurlyeq 0\}$ it is straightforward to verify that $0^\infty C = \{x \in \mathbb{R}^n \mid \mathcal{A}(x) \succcurlyeq 0\}$. The same is not true for spectrahedral shadows. In particular, the recession cone of a spectrahedral shadow S does not need to coincide with the projection of the recession cone of the spectrahedron that projects to S.

Example 2.39. Consider the projection $S = \left\{x \in \mathbb{R} \,\middle|\, \exists\, y \in \mathbb{R} \colon y - x^2 \geqslant 0\right\}$ of the epigraph C of the function $x \mapsto x^2$. This set is a spectrahedral shadow because the condition $y - x^2 \geqslant 0$ is equivalent to

$$\begin{pmatrix} y & x \\ x & 1 \end{pmatrix} \succcurlyeq 0.$$

Clearly, S is the whole real line which is its own recession cone. The recession cone $0^\infty C$ is generated by the direction $(0,1)^\mathsf{T}$, which, projected onto the first variable, is just the singleton $\{0\}$.

Hence, a more thorough treatment is necessary to obtain the recession cone of a spectrahedral shadow S. In this subsection we use the results about polar operations to derive an explicit expression for $\operatorname{cl} 0^\infty S$. Our approach relies on the following known relationship between $0^\infty S$ and the polar S°.

Proposition 2.40 ([Roc70, Theorem 14.6]). *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow containing the origin. Then it is true that $(0^\infty S)^\circ = \operatorname{cl} \operatorname{cone} S^\circ$.*

It follows that $\operatorname{cl} 0^\infty S$ is the set $(\operatorname{cone} S^\circ)^\circ$, which, taking Proposition 2.37 (iii) into account, is the same as $(S^\circ)^*$. The result also sheds light on the fact why we fail to determine the recession cone correctly in the above example. Since the assignment $x = 0$ and $y = 1$ is strictly feasible for the shadow S, we can apply Theorem 2.32 to obtain the polar

$$S^\circ = \left\{ u \in \mathbb{R} \,\middle|\, \exists\, V \in \mathcal{S}^2_+ \colon u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bullet V, 0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet V, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \bullet V \leqslant 1 \right\}.$$

The second equality states $v_{11} = 0$ which implies $v_{12} = 0$ and $u = 0$ due to the positive semidefiniteness of V. We would now need to compute the polar cone of S° to obtain $\operatorname{cl} 0^\infty S$. However, there is no positive definite assignment of V in the representation of S° because $v_{11} = 0$. Therefore, Corollary 2.36 is not applicable. This failure of strong duality is the crux of Example 2.39. If we were able to apply the corollary, then, indeed, the recession cone of S would be the projection of $0^\infty C$.

Proposition 2.41 (conical hull). *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^\mathsf{T}, b)$. Then the conical hull $\operatorname{cone} S$ is a spectrahedral shadow represented by*

$$\left(0, \begin{pmatrix} \mathcal{A} \\ M^1 \end{pmatrix}, M^2, \begin{pmatrix} \boldsymbol{\mathcal{A}} \\ 0 \end{pmatrix}, B, \begin{pmatrix} \bar{B} & -b & 0 \end{pmatrix}, \mathcal{B}^\mathsf{T}, 0\right)$$

with

$$M^1 = \left(E_{11} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,\middle|\, \cdots \,\middle|\, E_{nn} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

$$M^2 = \left(\begin{array}{c|c|c} \overline{\mathcal{A}} & A_0 & 0 \\ \hline 0 & I \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & I \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right).$$

Proof. By convexity of S its conical hull is

$$\begin{aligned}
\operatorname{cone} S &= \{\mu x \in \mathbb{R}^n \mid \mu \geqslant 0, x \in S\} \\
&= \{z \in \mathbb{R}^n \mid \exists\, x \in \mathbb{R}^n, \mu \geqslant 0 \colon z = \mu x, x \in S\} \\
&= \left\{ z \in \mathbb{R}^n \;\middle|\; \exists\, \mu > 0 \colon \frac{z}{\mu} \in S \right\} \cup \{0\} \\
&= \left\{ z \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \mu > 0 & A_0\mu + \mathcal{A}(z) + \overline{\mathcal{A}}(\mu y) + \boldsymbol{\mathcal{A}}(\mu Z) \succcurlyeq 0 \\ \exists\, y \in \mathbb{R}^m \colon & B z + \bar{B}\mu y + \mathcal{B}^{\mathsf{T}}(\mu Z) = \mu b \\ Z \in \mathcal{S}^k & \mu Z \succcurlyeq 0 \end{array} \right\} \cup \{0\} \\
&= \left\{ z \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \mu \in \mathbb{R} & \mathcal{A}(z) + \overline{\mathcal{A}}(y') + A_0\mu + \boldsymbol{\mathcal{A}}(Z') \succcurlyeq 0 \\ t \in \mathbb{R} & \begin{pmatrix} \mu & z_i \\ z_i & t \end{pmatrix} \succcurlyeq 0 \quad i = 1, \dots, n \\ \exists\, y' \in \mathbb{R}^m \colon & B z + \bar{B} y' - b\mu + \mathcal{B}^{\mathsf{T}}(Z') = 0 \\ Z' \in \mathcal{S}^k & Z' \succcurlyeq 0 \end{array} \right\}.
\end{aligned}$$

In the fourth step we expand the definition of S and multiply the LMIs and equations by μ. Note that if y and Z are valid assignments for $z/\mu \in S$, then we can choose $y' = \mu y$ and $Z' = \mu Z$ to express $z \in \mu S$. To justify the last equation we first consider the inclusion from left to right. If μ is positive than we can set $t \geqslant \mu^{-1} \max\{z_i^2 \mid i = 1, \dots, n\}$ to satisfy the LMIs involving t. Otherwise the LMIs force all z_i to be equal to 0. For the other inclusion assume that $z \neq 0$. Then $z_i \neq 0$ for some $i \in \{1, \dots, n\}$. From the positive semidefiniteness it follows $\mu t \geqslant z_i^2 > 0$ and, since μ and t are also nonnegative, they must actually both be positive. The case $z = 0$ is trivial. To conclude the proof note that

$$\begin{pmatrix} \mu & z_i \\ z_i & t \end{pmatrix} \succcurlyeq 0 \iff E_{ii} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z_i + I \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mu + I \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} t \succcurlyeq 0$$

for $E_{ii}, I \in \mathcal{S}^n$. □

In general, the conical hull of a spectrahedron is not a spectrahedral cone.

Example 2.42. Consider the spectrahedron $C = \{x \in \mathbb{R}^2 \mid x_2 \geqslant x_1^2\}$. Its conical hull is the set $\{x \in \mathbb{R}^2 \mid x_2 > 0\} \cup \{0\}$, which is not closed. However, it is a projected spectrahedron because

$$\operatorname{cone} C = \left\{ x \in \mathbb{R}^2 \;\middle|\; \exists\, y \in \mathbb{R}^2 \colon \begin{pmatrix} x_2 & x_1 \\ x_1 & y_1 \end{pmatrix} \succcurlyeq 0, \begin{pmatrix} y_1 & x_1 \\ x_1 & y_2 \end{pmatrix} \succcurlyeq 0, \begin{pmatrix} y_1 & x_2 \\ x_2 & y_2 \end{pmatrix} \succcurlyeq 0 \right\}$$

by Proposition 2.41.

Due to Proposition 2.40, it is easier to describe the polar $(0^\infty S)^\circ$ of the recession cone of S than the recession cone of S itself given a representation of S.

Corollary 2.43 (polar of the recession cone)**.** *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^{\mathsf{T}}, b)$ containing the origin and assume there exist $\bar{x} \in \mathbb{R}^n$, $\bar{y} \in \mathbb{R}^m$ and $\bar{Z} \in \mathcal{S}^k_{++}$ such that $A_0 + \mathcal{A}(\bar{x}) + \overline{\mathcal{A}}(\bar{y}) + \boldsymbol{\mathcal{A}}(\bar{Z}) \succ 0$ and $B\bar{x} + \bar{B}\bar{y} + \mathcal{B}^{\mathsf{T}}(\bar{Z}) = b$. Then the polar $(0^\infty S)^\circ$ of the recession cone of S is the closure of the spectrahedral shadow represented by*

$$\left(0, \varnothing, \mathcal{B}, -\boldsymbol{\mathcal{A}}^{\mathsf{T}}, \begin{pmatrix} -I \\ 0 \end{pmatrix}, \begin{pmatrix} B^{\mathsf{T}} \\ \bar{B}^{\mathsf{T}} \end{pmatrix}, \begin{pmatrix} -\mathcal{A}^{\mathsf{T}} \\ \hline -\overline{\mathcal{A}}^{\mathsf{T}} \end{pmatrix}, 0\right).$$

Proof. Since S contains the origin, one has $(0^\infty S)^\circ = \operatorname{cl}\operatorname{cone} S^\circ$ according to Proposition 2.40. Moreover, we can apply Theorem 2.32 because S satisfies Slater-like conditions. Hence, applying Theorem 2.32 to S and Proposition 2.41 to the resulting set yields that the set cone S° is represented by

$$\left(0, \begin{pmatrix} 0 \\ M^1 \end{pmatrix}, M^2, \left(\begin{array}{c|c} -\boldsymbol{\mathcal{A}}^\mathsf{T} & 0 \\ 0 & 0 \end{array}\right), \begin{pmatrix} -I \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} B^\mathsf{T} & 0 & 0 \\ \bar{B}^\mathsf{T} & 0 & 0 \\ b^\mathsf{T} & -1 & 0 \end{pmatrix}, \left(\begin{array}{c|c} -\mathcal{A}^\mathsf{T} & 0 \\ \hline -\bar{\mathcal{A}}^\mathsf{T} & 0 \\ \hline A_0 & 1 \end{array}\right), 0\right)$$

for M^1 as defined in Proposition 2.41 and

$$M^2 = \left(\begin{array}{c|c|c} \mathcal{B} & 0 & 0 \\ \hline 0 & I \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & I \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right).$$

Writing the set explicitly, we get

$$\operatorname{cone} S^\circ = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, y \in \mathbb{R}^d \\ \exists\, \mu, t \in \mathbb{R} \\ \exists\, Z \in \mathcal{S}^{\ell+1} \end{array} : \begin{array}{rl} \mathcal{B}(y) - \left(\begin{array}{c|c} \boldsymbol{\mathcal{A}}^\mathsf{T} & 0 \end{array}\right)(Z) \succcurlyeq 0 & \\ \begin{pmatrix} \mu & x_i \\ x_i & t \end{pmatrix} \succcurlyeq 0 & i = 1, \dots, n \\ B^\mathsf{T} y - \left(\begin{array}{c|c} \mathcal{A}^\mathsf{T} & 0 \end{array}\right)(Z) = x & \\ \bar{B}^\mathsf{T} y - \left(\begin{array}{c|c} \bar{\mathcal{A}}^\mathsf{T} & 0 \end{array}\right)(Z) = 0 & \\ b^\mathsf{T} y - \mu + \begin{pmatrix} A_0 & 0 \\ 0 & 1 \end{pmatrix} \bullet Z = 0 & \\ Z \succcurlyeq 0 & \end{array} \right\}.$$

We can decompose the matrix $Z \in \mathcal{S}_+^{\ell+1}$ into a block diagonal matrix with blocks $Z' \in \mathcal{S}_+^\ell$ and $z \in \mathbb{R}_+$ because the involved operators do not act on the remaining entries of Z. Therefore, the only difference to the claimed representation is the occurrence of the LMIs in the second line and the equality $b^\mathsf{T} y - \mu + A_0 \bullet Z' + z = 0$. We show that these can be omitted. Denote by M the spectrahedral shadow with the claimed representation. Clearly, cone $S^\circ \subseteq M$. If $x \in M$, then there exist $y \in \mathbb{R}^d$ and $Z' \in \mathcal{S}_+^\ell$ that satisfy the conditions imposed by the representation. Assume $b^\mathsf{T} y + A_0 \bullet Z' \leqslant 0$. Choosing $\mu = 1$ and $z = 1 - b^\mathsf{T} y - A_0 \bullet Z'$ satisfies the equality with $z \geqslant 0$. Moreover, we can choose t large enough such that

$$\begin{pmatrix} \mu & x_i \\ x_i & t \end{pmatrix} \succcurlyeq 0$$

holds for all $i \in \{1, \dots, n\}$, i.e. $t \geqslant \max\left\{x_i^2 \,\middle|\, i = 1, \dots, n\right\}$. On the other hand, if $b^\mathsf{T} y + A_0 \bullet Z' > 0$, then we can set $\mu = b^\mathsf{T} y + A_0 \bullet Z'$, $t \geqslant \mu^{-1} \max\left\{x_i^2 \,\middle|\, i = 1, \dots, n\right\}$ and $z = 0$ to fulfill the conditions. □

We can now formulate the main result of this subsection.

Theorem 2.44 (closure of the recession cone). *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \bar{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^\mathsf{T}, b)$ containing the origin and assume there exist $\bar{x} \in \mathbb{R}^n$, $\bar{y} \in \mathbb{R}^m$ and $\bar{Z} \in \mathcal{S}_{++}^k$ such that $A_0 + \mathcal{A}(\bar{x}) + \bar{\mathcal{A}}(\bar{y}) + \boldsymbol{\mathcal{A}}(\bar{Z}) \succ 0$ and $B\bar{x} + \bar{B}\bar{y} + \mathcal{B}^\mathsf{T}(\bar{Z}) = b$. Then a point $u \in \mathbb{R}^n$ belongs to $\operatorname{cl} 0^\infty S$ if and only if the system*

$$\bar{B} v^{p+1} - \mathcal{B}^\mathsf{T}(V^{p+1} + W_1^{p+1} + W_1^{p+1\mathsf{T}}) = Bu$$

$$
\begin{aligned}
-\overline{\mathcal{A}}(v^{p+1}) + \boldsymbol{\mathcal{A}}(V^{p+1} + W_1^{p+1} + W_1^{p+1\mathsf{T}}) + \\
W_2^{p+1} + W_2^{p+1\mathsf{T}} &\succcurlyeq -\mathcal{A}(u) \\
V^{p+1} &\succcurlyeq 0 \\
\bar{B}v^i - \mathcal{B}^{\mathsf{T}}(V^i + W_1^i + W_1^{i\mathsf{T}}) &= 0 && i = 1, \dots, p \\
-\overline{\mathcal{A}}(v^i) + \boldsymbol{\mathcal{A}}(V^i + W_1^i + W_1^{i\mathsf{T}}) + W_2^i + W_2^{i\mathsf{T}} &\succcurlyeq 0 && i = 1, \dots, p \\
V^{i-1} &\succcurlyeq W_1^{i\mathsf{T}} W_1^i && i = 1, \dots, p+1 \\
-\overline{\mathcal{A}}(v^{i-1}) + \boldsymbol{\mathcal{A}}(V^{i-1}) &\succcurlyeq W_2^{i\mathsf{T}} W_2^i && i = 1, \dots, p+1 \\
W_1^1 = 0, W_2^1 &= 0
\end{aligned}
$$

is consistent for some $V^i \in \mathcal{S}^k$, $v^i \in \mathbb{R}^m$, $W_1^i \in \mathcal{M}^k$, $W_2^i \in \mathcal{M}^\ell$, $i = 1, \dots, p+1$, *with* $p = \min\{n + d + \ell, n + m + k\}$.

Proof. From Proposition 2.40 it follows that $\operatorname{cl} 0^\infty S = (\operatorname{cl} \operatorname{cone} S^\circ)^\circ = (\operatorname{cone} S^\circ)^\circ$. The set $\operatorname{cone} S^\circ$ is a spectrahedral shadow represented by

$$
\left(0, \varnothing, \mathcal{B}, -\boldsymbol{\mathcal{A}}^{\mathsf{T}}, \begin{pmatrix} -I \\ 0 \end{pmatrix}, \begin{pmatrix} B^{\mathsf{T}} \\ \bar{B}^{\mathsf{T}} \end{pmatrix}, \left(\begin{array}{c} -\mathcal{A}^{\mathsf{T}} \\ \hline -\overline{\mathcal{A}}^{\mathsf{T}} \end{array}\right), 0\right)
$$

according to Corollary 2.43. We now apply Theorem 2.35 to this set. This yields $u \in \operatorname{cl} 0^\infty S$ if and only if the system

$$
\begin{aligned}
\begin{pmatrix} -I & 0 \end{pmatrix} \bar{v}^{p+1} &= u \\
V^{p+1} &\succcurlyeq 0 \\
\begin{pmatrix} -I & 0 \end{pmatrix} \bar{v}^i &= 0 && i = 1, \dots, p \\
-\mathcal{B}^{\mathsf{T}}(V^i + W_1^i + W_1^{i\mathsf{T}}) + \begin{pmatrix} B & \bar{B} \end{pmatrix} \bar{v}^i &= 0 && i = 1, \dots, p+1 \\
\boldsymbol{\mathcal{A}}(V^i + W_1^i + W_1^{i\mathsf{T}}) - \left(\begin{array}{c|c} \mathcal{A} & \overline{\mathcal{A}} \end{array}\right)(\bar{v}^i) + W_2^i + W_2^{i\mathsf{T}} &\succcurlyeq 0 && i = 1, \dots, p+1 \\
V^{i-1} &\succcurlyeq W_1^{i\mathsf{T}} W_1^i && i = 1, \dots, p+1 \\
\boldsymbol{\mathcal{A}}(V^{i-1}) - \left(\begin{array}{c|c} \mathcal{A} & \overline{\mathcal{A}} \end{array}\right)(\bar{v}^{i-1}) &\succcurlyeq W_2^{i\mathsf{T}} W_2^i && i = 1, \dots, p+1 \\
W_1^1 = 0, W_2^1 &= 0
\end{aligned}
$$

is consistent. The proof is completed by simplifying the system. To this end we decompose the variables $\bar{v}^i \in \mathbb{R}^{n+m}$ into $v'^i \in \mathbb{R}^n$ and $v^i \in \mathbb{R}^m$. From the first and third equalities it follows $v'^{p+1} = -u$ and $v'^i = 0$, $i = 1, \dots, p$, respectively. Substituting these values into the rest of the system concludes the proof. □

Chapter 3

Polyhedral approximation of compact spectrahedral shadows

The tools developed in the preceding chapter allow systematic computations with spectrahedral shadows. Although it is useful to express the result of calculus operations explicitly, a representation gives little insight into the appearance and geometry of a shadow. In this chapter we partially address this shortcoming by developing algorithms that compute polyhedral approximations of spectrahedral shadows in the compact case. On one hand, this allows us to visualize such sets and, on the other hand, it provides an approximate description of a given spectrahedral shadow in terms of linear inequalities. This is desired because polyhedra can be seen as a simpler class of sets than spectrahedra and their projections in the sense that they are described by linear functions rather than polynomials of higher degree. The results in this chapter are partially based on the article [Dör22] titled *On the approximation of unbounded convex sets by polyhedra* and published in *Journal of Optimization Theory and Applications*.

3.1 Historical overview

We want to give an overview of the literature on the development and application of polyhedral approximation algorithms. The following is not restricted to spectrahedra or their projections as most of the literature studies general convex sets or sets defined as intersections of level sets of convex functions which contain spectrahedral shadows as a special case. Since we need to quantify the quality of a polyhedral approximation, we need to measure the distance between sets. The prevalent way of doing this is with respect to the *Hausdorff distance*, which defines a metric on the space of compact subsets of $\mathbb{R}^n$.

Definition 3.1. A compact convex set $C \subseteq \mathbb{R}^n$ containing the origin in its interior is called a *convex body*.

Definition 3.2 (cf. [HL01])**.** For sets $M_1, M_2 \subseteq \mathbb{R}^n$ the *excess of* M_1 *over* M_2, denoted by $e\,[M_1, M_2]$, is defined as

$$\sup_{x \in M_1} d\,(x, M_2)\,.$$

The square brackets emphasize the fact that the excess is not symmetric in its arguments. If the set M_1 is bounded, then $e\,[M_1, M_2]$ is finite, but in general this need not be the case. If it is also closed, then the supremum is attained by some point from M_1 and can be substituted for a maximum. This is true in particular if M_1 is a convex body. The Hausdorff distance can easily be expressed using excesses.

Definition 3.3. For sets $M_1, M_2 \subseteq \mathbb{R}^n$ the *Hausdorff distance between* M_1 *and* M_2 is defined as

$$\max\{e[M_1, M_2], e[M_2, M_1]\}$$

and denoted by $d_{\mathsf{H}}(M_1, M_2)$.

Definition 3.4 (cf. [RW98])**.** A sequence $\{M_k\}_{k\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$ is said to *Hausdorff converge* or *H-converge* to a set $M \subseteq \mathbb{R}^n$ if $\lim_{k\to\infty} d_{\mathsf{H}}(M_k, M) = 0$.

As mentioned in the introduction, the first explicit appearance of polyhedral approximation known to the author is due to Minkowski in his 1903 treatise *Volumen und Oberfläche* [Min03]. He shows that every convex body $C \subseteq \mathbb{R}^3$ can be approximated by a convex set $\mathcal{O}$ that is the level set of a convex function and has differentiable boundary such that $C \subseteq \mathcal{O} \subseteq (1+\varepsilon)C$ for any predefined $\varepsilon > 0$. As part of his proof he constructs a polytope P with the property $C \subseteq P \subseteq \mathcal{O}$ by considering a space-filling tessellation of $\mathbb{R}^3$ made up of cubic cells with edge length depending on ε and C. Polytope P is then defined as the convex hull of the union of all cells having nonempty intersection with C. Bonnesen and Fenchel extract this technique and present it for general dimension in their Book *Theorie der konvexen Körper* [BF34]. Moreover, they conclude that for every convex body $C \subseteq \mathbb{R}^n$ there exists a sequence of polytopes $\{P_k\}_{k\in\mathbb{N}}$ that H-converges to C. Fejes Tóth [Fej48] improves upon this result in 1948 for the cases that $C \subseteq \mathbb{R}^2$ is bounded by a rectifiable Jordan curve that is sufficiently smooth and $C \subseteq \mathbb{R}^3$ being an ellipsoid. For these two cases he shows that there exist sequences $\{P_k\}_{k\in\mathbb{N}}$ of polyhedra with k vertices that are inscribed in C, i.e. their vertices lie on the boundary of C, such that

$$\lim_{k\to\infty} k^2 d_{\mathsf{H}}(P_k, C) < \infty \quad \text{and} \quad \lim_{k\to\infty} k d_{\mathsf{H}}(P_k, C) < \infty,$$

respectively. Moreover, he computes these limits in terms of curvature properties of C. The latter result is extended to arbitrary dimension and convex bodies $C \subseteq \mathbb{R}^n$ with three times continuously differentiable boundary by Schneider [Sch81]. Explicit bounds on the Hausdorff distance were independently obtained by Bronstein and Ivanov [BI75] in 1975 and Dudley [Dud74] in 1974. They show that the estimate

$$d_{\mathsf{H}}(P, C) \leqslant \frac{c(C)}{k^{2/(n-1)}}$$

holds for a polytope P with k vertices inscribed in C. Here, $c(C)$ is a constant depending on C. The same relation holds for a polytope P with k facets that is circumscribed to C, i.e. the facets of P correspond to hyperplanes that support C. Since then, many improvements have been achieved for special cases and metrics other than the Hausdorff distance were considered. An elaborate survey of these result is due to Bronstein [Bro08].

The first usage of polyhedral approximation from an algorithmic perspective appears to be due to Cheney and Goldstein [CG59] and, independently, also Kelley [Kel60]. They consider the problem of minimizing a linear function over a convex body C with differentiable boundary. To find an approximate solution to the problem they start with an initial polytope $P_0 \supseteq C$ and solve the linear program that arises by replacing C with P_0. If the solution is contained in C, the original problem is solved. Otherwise, a hyperplane H is found that separates the solution and C. Assuming H^- is the corresponding halfspace containing C, the polyhedral approximation is refined as $P_1 = P_0 \cap H^-$. This process is called *cutting* because the solution of the linear program is cut off from the set of interest by H. Repeating this process

yields a sequence of solutions of linear programs that converge to the solution of the convex program and a sequence of polytopes containing C that approximate C well around the solution. In 1967, their method is refined by Veinott [Vei67] in the sense that the hyperplanes used throughout the algorithm support C in a boundary point. This advantage is offset by the increased computational effort required to find these points, i.e. for each point of support a subproblems is solved using interval bisection. The development of polyhedral approximation algorithms then continued in different fields of optimization, in particular global optimization, mixed-integer convex optimization and vector optimization. Veinott's algorithm is modified to find the minimum of a concave function over C in [Hof81]. Improvements, where some of the assumptions are relaxed, are due to Thiệu, Tâm and Ba'n [TTB83]. In [BH91] the algorithm is combined with branch-and-bound techniques to solve the same problem but with simpler subproblems and in [Hor+91] it is applied to minimize the difference of two convex functions over C. In mixed-integer optimization the concept of cutting was popularized by Gomory [Gom58], who developed the well-known Gomory cuts to solve integer linear programs. Duran and Grossman [DG86] apply polyhedral outer approximation to solve mixed-integer convex programs. They identify an equivalent mixed-integer linear program that arises by computing a polyhedral approximation of the feasible region of the relaxation of the original problem. To find the halfspaces generating the polyhedral approximation they solve a convex subproblem for every feasible assignment of integer variables. In 1995, Westerlund and Pettersson [WP95] apply Kelley's algorithm to this problem class. An improvement of their algorithm is presented in [KLW16] where the supporting points for the polyhedral approximation are found using a simple line search. Lassez and Lassez [LL92] use polyhedral approximation to compute V-representations of projections of high dimensional polyhedra. A main feature of their algorithms is its generality, i.e. no assumptions are made about the input polyhedron. Similar methods are used in [Löh+19] where bilevel problems with polyhedral constraint sets are solved by solving an associated polyhedral projection problem. Kamenev [Kam92; Kam93; Kam94] develops general approximation schemes for both outer and inner polyhedral approximation of convex bodies. These schemes, called *cutting* and *augmenting schemes*, respectively, subsume many of the approximation algorithms in the literature. We discuss these general procedures in detail in the next section. In 1998, Benson adapted the methods developed in the realm of global optimization to solve linear vector optimization problems [Ben98]. He observes that the image space is typically smaller than the preimage space of such problems and exploits this by approximating a polyhedral set in the image space that is associated with the vector linear program. The term *Benson algorithm* or *Benson-type algorithm* was adopted by the vector optimization community and subsequently various extensions of the original algorithm have come forth, see e.g. [RW05; SE08a]. Alongside dual algorithms [SE08b; ELS12; HLR14] generalizations to the convex case have also been developed [ESS11; LRU14; Dör+22; AUU22; Wag+22].

3.2 Cutting and augmenting schemes

In this section we develop two polyhedral approximation algorithms tailored to compact spectrahedral shadows with nonempty interior. One algorithm computes a circumscribed approximating polyhedron of a given spectrahedral shadow, while the other computes an inscribed approximation. Both algorithms fit into a framework of adaptive approximation schemes due to Kamenev [Kam92] called cutting

and augmenting schemes. These describe a class of outer and inner polyhedral approximation algorithms for convex bodies, respectively, in which, at each iteration, a known polyhedral approximation is refined.
Given a convex body $C \subseteq \mathbb{R}^n$ and a polyhedral approximation P_k with $C \subseteq P_k$, an iteration of a cutting scheme is the execution of the steps

1. Choose a unit direction $u \in \mathbb{R}^n$.
2. Construct $P_{k+1} = P_k \cap H^- (u, \sigma_C(u))$.

Here, $\sigma_C(u)$ denotes the *support function* of C at u, i.e.

$$\sigma_C(u) = \sup \left\{ u^\mathsf{T} x \;\middle|\; x \in C \right\}.$$

Since C is bounded and closed, its support function is finite everywhere and the supremum is always attained, respectively. Hence, $H(u, \sigma_C(u))$ is a supporting hyperplane of C. One iteration of an augmenting scheme for $P_k \subseteq C$ is described by the steps

1. Choose a point p on the boundary of C.
2. Construct $P_{k+1} = \operatorname{conv}(P_k \cup \{p\})$.

Kamenev derives convergence results for a particular subclass of cutting and augmenting schemes called *Hausdorff schemes*. These are algorithms that admit the existence of a constant $\gamma > 0$ such that

$$d_\mathsf{H}(P_k, P_{k+1}) \geqslant \gamma d_\mathsf{H}(P_k, C) \tag{3.1}$$

holds in every iteration k of a cutting or augmenting scheme.

Theorem 3.5 (cf. [Kam92, Theorem 2])**.** *Let $C \subseteq \mathbb{R}^n$ be a convex body and $\{P_k\}_{k \in \mathbb{N}}$ be a sequence of polyhedra generated by a Hausdorff scheme. Then for every $\kappa > 0$ there exists k_0 such that for all $k \geqslant k_0$*

$$d_\mathsf{H}(P_k, C) \leqslant (1 + \kappa) \frac{c(C)}{k^{1/(n-1)}}$$

holds for a constant $c(C)$ depending on C.

Remark 3.6. A similar result holds for the Nikodym metric instead of the Hausdorff metric. Moreover, Kamenev gives an improved upper bound with denominator $k^{2/(n-1)}$ for sets with twice continuously differentiable boundary.

In order to formulate the algorithms in Subsection 3.2.2 we need three semidefinite programs associated with a spectrahedral shadow $S \subseteq \mathbb{R}^n$ and their duals that give insights into the geometry of S. Solving the first problem can be interpreted as determining the maximum shift of a hyperplane with fixed normal vector that is possible within S. The problem is parameterized by a nonzero vector $\omega \in \mathbb{R}^n$ and can be stated as

$$\begin{aligned} \max \omega^\mathsf{T} x \quad \text{s.t.} \quad & A_0 + \mathcal{A}(x) + \bar{A}(y) + \boldsymbol{\mathcal{A}}(Z) \succcurlyeq 0 \\ & Bx + \bar{B}y + \mathcal{B}^\mathsf{T}(Z) = b \\ & Z \succcurlyeq 0, \end{aligned} \tag{$\mathrm{P}_1(\omega, S)$}$$

that is $\max \omega^\mathsf{T} x$ subject to $x \in S$ for a spectrahedral shadow defined according to Definition 2.10. Put differently, solving $(\mathrm{P}_1(\omega, S))$ amounts to the evaluation of the support function of S at ω. The corresponding dual program is given as

$$\begin{aligned} \min\ A_0 \bullet V + b^\mathsf{T} v \quad \text{s.t.} \quad & -\mathcal{A}^\mathsf{T}(V) + B^\mathsf{T} v = \omega \\ & -\overline{\mathcal{A}}^\mathsf{T}(V) + \bar{B}^\mathsf{T} v = 0 \\ & -\boldsymbol{\mathcal{A}}^\mathsf{T}(V) + \mathcal{B}(v) \succcurlyeq 0 \\ & V \succcurlyeq 0. \end{aligned} \tag{$\mathrm{D}_1(\omega, S)$}$$

The existence of a solution to problem $(\mathrm{P}_1(\omega, S))$ is related to the polar of the recession cone of S.

Proposition 3.7. *Let $S \subseteq \mathbb{R}^n$ be a closed spectrahedral shadow and $\omega \in \operatorname{relint}(0^\infty S)^\circ$. Then $(\mathrm{P}_1(\omega, S))$ has an optimal solution (x^*, y^*, Z^*). Moreover, if $\omega \neq 0$, then the hyperplane $H(\omega, \omega^\mathsf{T} x^*)$ supports S at x^*.*

Proof. According to [Roc70, Corollary 14.2.1] the polarity relation

$$(0^\infty S)^\circ = \operatorname{cl}\{u \in \mathbb{R}^n \mid \sigma_S(u) < \infty\}$$

holds and the convexity of S implies

$$\operatorname{relint}(0^\infty S)^\circ = \operatorname{relint}\{u \in \mathbb{R}^n \mid \sigma_S(u) < \infty\},$$

see [Roc70, Theorem 6.3]. Hence, $\sigma_S(\omega) < \infty$, i.e. the optimal value of $(\mathrm{P}_1(\omega, S))$ is finite. Now, since ω belongs to the relative interior of the effective domain of the support function σ_S, the subdifferential $\partial\sigma_S(\omega)$ is nonempty, see [Roc70, Theorem 23.4]. The set $\partial\sigma_S(\omega)$, however, is exactly the set of solutions to $(\mathrm{P}_1(\omega, S))$ according to [Roc70, Corollary 23.5.3]. Finally, the supporting property follows for $\omega \neq 0$ because $\omega^\mathsf{T} x \leqslant \sigma_S(\omega) = \omega^\mathsf{T} x^*$ for $x^* \in \partial\sigma_S(\omega)$ and all $x \in S$. □

Clearly, if S is compact, then $(\mathrm{P}_1(\omega, S))$ has a solution for every $\omega \in \mathbb{R}^n$. Solutions to $(\mathrm{P}_1(\omega, S))$ and $(\mathrm{D}_1(\omega, S))$ also reveal some information about the polar S° under additional assumptions on S.

Proposition 3.8. *Let $S \subseteq \mathbb{R}^n$ be a closed spectrahedral shadow and assume that the problem $(\mathrm{P}_1(\omega, S))$ is strictly feasible for $\omega \in \operatorname{relint}(0^\infty S)^\circ$. Then solutions (x^*, y^*, Z^*) to $(\mathrm{P}_1(\omega, S))$ and (V^*, v^*) to $(\mathrm{D}_1(\omega, S))$ exist and the following hold:*

(i) if $A_0 \bullet V^* + b^\mathsf{T} v^* = 0$, then $\omega \in 0^\infty S^\circ$,

(ii) if $A_0 \bullet V^* + b^\mathsf{T} v^* > 0$, then the hyperplane $H(x^*, 1)$ supports S° at $\frac{\omega}{A_0 \bullet V^* + b^\mathsf{T} v^*}$.

Proof. Since $\omega \in \operatorname{relint}(0^\infty S)^\circ$ and S is closed, a solution (x^*, y^*, Z^*) to $(\mathrm{P}_1(\omega, S))$ exists according to Proposition 3.7. Strict feasibility implies strong duality. Hence, a dual solution (V^*, v^*) exists as well. Now, assume $A_0 \bullet V^* + b^\mathsf{T} v^* = 0$. Since $0 \in S^\circ$ and S° is closed, it suffices to show $\mu\omega \in S^\circ$ for all $\mu > 0$, see [Roc70, Theorem 8.3]. Define $V_\mu = \mu V^*$ and $v_\mu = \mu v^*$. Then $V_\mu \succcurlyeq 0$ because $V^* \succcurlyeq 0$. Moreover, feasibility of (V^*, v^*) for $(\mathrm{D}_1(\omega, S))$ yields $\mu\omega = -\mathcal{A}^\mathsf{T}(V_\mu) + B^\mathsf{T} v_\mu$, $0 = -\overline{\mathcal{A}}^\mathsf{T}(V_\mu) + \bar{B}^\mathsf{T} v_\mu$ and $-\boldsymbol{\mathcal{A}}^\mathsf{T}(V_\mu) + \mathcal{B}(v_\mu) \succcurlyeq 0$. It also holds $A_0 \bullet V_\mu + b^\mathsf{T} v_\mu = \mu\left(A_0 \bullet V^* + b^\mathsf{T} v^*\right) = 0 \leqslant 1$. Finally, since $(\mathrm{P}_1(\omega, S))$ is strictly feasible, we can apply Theorem 2.32 and conclude $\mu\omega \in S^\circ$.

It remains to show the second assertion. Therefore, assume $A_0 \bullet V^* + b^\mathsf{T} v^* > 0$. We conclude $x^* \neq 0$ because $\omega^\mathsf{T} x^* = A_0 \bullet V^* + b^\mathsf{T} v^* > 0$ holds due to strong duality. Let

$\bar{\omega} = (A_0 \bullet V^* + b^\mathsf{T} v^*)^{-1}\omega$, $\bar{V} = (A_0 \bullet V^* + b^\mathsf{T} v^*)^{-1} V^*$ and $\bar{v} = (A_0 \bullet V^* + b^\mathsf{T} v^*)^{-1} v^*$. Then $(\bar{V}, \bar{v})$ is feasible for $(\mathrm{D}_1(\bar{\omega}, S))$ and it holds $A_0 \bullet \bar{V} + b^\mathsf{T}\bar{v} = 1$, i.e. $\bar{\omega} \in S^\circ$. By definition of the polar it holds ${x^*}^\mathsf{T} u \leqslant 1$ for all $u \in S^\circ$. Moreover, we have

$$ {x^*}^\mathsf{T}\bar{\omega} = \frac{{x^*}^\mathsf{T}\omega}{A_0 \bullet V^* + b^\mathsf{T} v^*} = \frac{A_0 \bullet V^* + b^\mathsf{T} v^*}{A_0 \bullet V^* + b^\mathsf{T} v^*} = 1, $$

where the second equality is validated by strong duality again. □

Remark 3.9. There is not a similar polarity relation for the case that $A_0 \bullet V^* + b^\mathsf{T} v^*$ is negative in the previous proposition. The reason is that this implies that the origin is not contained in S. In that case it can happen that the hyperplane $H(x^*, 1)$ does not support S° at all. However, since $S^\circ = (\operatorname{cl}\operatorname{conv}(S \cup \{0\}))^\circ$, we can assume without loss of generality that S contains the origin in which case the optimal values of neither $(\mathrm{P}_1(\omega, S))$ nor $(\mathrm{D}_1(\omega, S))$ can be negative.

Remark 3.10. Taking Proposition 2.5 into account, the assumption that $(\mathrm{P}_1(\omega, S))$ is strictly feasibly can be interpreted as the spectrahedron projecting to S having nonempty interior.

The second pair of primal-dual semidefinite programs we consider is

$$ \begin{aligned} \max\ t \quad \text{s.t.} \quad A_0 + \mathcal{A}(x) + \bar{\mathcal{A}}(y) + \boldsymbol{\mathcal{A}}(Z) &\succcurlyeq 0 \\ Bx + \bar{B}y + \mathcal{B}^\mathsf{T}(Z) &= b \\ Z &\succcurlyeq 0 \\ x &= p + td, \end{aligned} \tag{$\mathrm{P}_2(p,d,S)$} $$

i.e. $\max t$ subject to $p + td \in S$, and

$$ \begin{aligned} \min\ (A_0 + \mathcal{A}(p)) \bullet V + (b - Bp)^\mathsf{T} v \quad \text{s.t.} \quad -\mathcal{A}^\mathsf{T}(V) + B^\mathsf{T} v &= w \\ -\bar{\mathcal{A}}^\mathsf{T}(V) + \bar{B}^\mathsf{T} v &= 0 \\ -\boldsymbol{\mathcal{A}}^\mathsf{T}(V) + \mathcal{B}(v) &\succcurlyeq 0 \\ V &\succcurlyeq 0 \\ d^\mathsf{T} w &= 1 \end{aligned} \tag{$\mathrm{D}_2(p,d,S)$} $$

for parameters $p, d \in \mathbb{R}^n$. Typically, we assume that $p \in S$ and d is a direction. Then problem $(\mathrm{P}_2(p,d,S))$ can be motivated as follows. Starting at the point $p \in S$ the maximum distance is to be determined that can be moved in direction d without leaving the set. If S is compact, then the maximum distance will be attained in the point where halfline $\{p + td \mid t \geqslant 0\}$ intersects the boundary of S.

Proposition 3.11. *Let $S \subseteq \mathbb{R}^n$ be a closed spectrahedral shadow and $p \in S$. Then the following hold:*

(i) if $(\mathrm{P}_2(p,d,S))$ is unbounded, then $d \in 0^\infty S$,

(ii) if $(\mathrm{P}_2(p,d,S))$ is strictly feasible with finite optimal value t^*, then an optimal solution (V^*, v^*, w^*) to $(\mathrm{D}_2(p,d,S))$ exists and the hyperplane $H(w^*, {w^*}^\mathsf{T} p + t^*)$ supports S at $p + t^* d$.

Proof. Since $p \in S$, there exist $y_p \in \mathbb{R}^m$ and $Z_p \in \mathcal{S}^k$ such that the point (p, y_p, Z_p, t_p) is feasible for $(\mathrm{P}_2(p,d,S))$ with $t_p = 0$. Now, the unboundedness of the problem implies the existence of feasible points (x_t, y_t, Z_t, t) for every $t \geqslant 0$. The last constraint

asserts that $p + td \in S$ for every $t \geqslant 0$ and the closedness of S implies $d \in 0^\infty S$. The existence of a dual solution in the second statement is guaranteed by the strict feasibility of the primal and the finiteness of t^* according to Theorem 1.30. Let $x \in S$. Then there exist $y \in \mathbb{R}^m$ and $Z \in \mathcal{S}_+^k$ such that $A_0 + \mathcal{A}(x) + \overline{\mathcal{A}}(y) + \boldsymbol{\mathcal{A}}(Z) \succcurlyeq 0$ and $Bx + \bar{B}y + \mathcal{B}^\mathsf{T}(Z) = b$. We compute

$$\begin{aligned}
w^{*\mathsf{T}}x - \left(w^{*\mathsf{T}}p + t^*\right) &= w^{*\mathsf{T}}(x-p) - (A_0 + \mathcal{A}(p)) \bullet V^* + (b - Bp)^\mathsf{T} v^* \\
&= \left(-\mathcal{A}^\mathsf{T}(V^*) + B^\mathsf{T}v^*\right)^\mathsf{T}(x-p) - \\
&\qquad (A_0 + \mathcal{A}(p)) \bullet V^* + (b - Bp)^\mathsf{T} v^* \\
&= -V^* \bullet (A_0 + \mathcal{A}(x)) + v^{*\mathsf{T}}(Bx - b) + 0^\mathsf{T} y \\
&= -V^* \bullet \left(A_0 + \mathcal{A}(x) + \overline{\mathcal{A}}(y)\right) + v^{*\mathsf{T}}\left(Bx + \bar{B}y - b\right) + 0 \bullet Z \\
&\leqslant -V^* \bullet \left(A_0 + \mathcal{A}(x) + \overline{\mathcal{A}}(y) + \boldsymbol{\mathcal{A}}(Z)\right) + \\
&\qquad v^{*\mathsf{T}}\left(Bx + \bar{B}y + \mathcal{B}^\mathsf{T}(Z) - b\right) \\
&\leqslant 0.
\end{aligned}$$

The first equality is true due to strong duality. In the second, fourth and fifth line we have used the first, second and third constraint of (D$_2$ (p, d, S)), respectively. The last inequality holds because $x \in S$ and $V^* \succcurlyeq 0$. Hence, we conclude $w^{*\mathsf{T}}x \leqslant w^{*\mathsf{T}}p + t^*$ for all $x \in S$. Finally, the supporting property at $p + t^*d$ follows from $d^\mathsf{T}w^* = 1$. □

Clearly, if the direction is chosen as $d = v - p$ for some point $v \notin S$, then the optimal value t^* of (P$_2$ (p, d, S)) is finite and satisfies $t^* \in (0, 1)$. Similar to Proposition 3.8 we can infer information about S° from solutions to the two problems.

Proposition 3.12. *Let (x^*, y^*, Z^*, t^*) and (V^*, v^*, w^*) be solutions to (P$_2$ (p, d, S)) and (D$_2$ (p, d, S)), respectively. Given strong duality the following hold:*

(i) if $A_0 \bullet V^* + b^\mathsf{T}v^* = 0$, then $w^* \in 0^\infty S^\circ$,

(ii) if $A_0 \bullet V^* + b^\mathsf{T}v^* > 0$, then the hyperplane $H(x^*, 1)$ supports S° at $\frac{w^*}{A_0 \bullet V^* + b^\mathsf{T}v^*}$.

Proof. The proof of the first part is similar to the corresponding part in the proof of Proposition 3.8. Regarding the second statement we obtain the relation

$$w^{*\mathsf{T}}x^* = w^{*\mathsf{T}}(p + t^*d) = w^{*\mathsf{T}}p + t^* = A_0 \bullet V^* + b^\mathsf{T}v^* > 0$$

from which we conclude $x^* \neq 0$. For the second equation we invoke $d^\mathsf{T}w^* = 1$ and for the third we use strong duality and the first constraint of (D$_2$ (p, d, S)). Moreover, for $\overline{w} = (A_0 \bullet V^* + b^\mathsf{T}v^*)^{-1}w^*$ we have

$$\begin{aligned}
x^{*\mathsf{T}}\overline{w} &= (p + t^*d)^\mathsf{T}\overline{w} \\
&= \left(A_0 \bullet V^* + b^\mathsf{T}v^*\right)^{-1}\left(p^\mathsf{T}w^* + t^*\right) \\
&= \left(A_0 \bullet V^* + b^\mathsf{T}v^*\right)^{-1}\left(p^\mathsf{T}w^* + (A_0 + \mathcal{A}(p)) \bullet V^* + (b - Bp)^\mathsf{T}v^*\right) \\
&= \left(A_0 \bullet V^* + b^\mathsf{T}v^*\right)^{-1}\left(p^\mathsf{T}w^* + p^\mathsf{T}\left(\mathcal{A}^\mathsf{T}(V^*) - B^\mathsf{T}v^*\right) + A_0 \bullet V^* + b^\mathsf{T}v^*\right) \\
&= \left(A_0 \bullet V^* + b^\mathsf{T}v^*\right)^{-1}\left(p^\mathsf{T}w^* - p^\mathsf{T}w^* + A_0 \bullet V^* + b^\mathsf{T}v^*\right) \\
&= 1.
\end{aligned}$$

To conclude the proof, we need to show $\overline{w} \in S^\circ$. This is true because $(\overline{V}, \bar{v}, \overline{w})$ is feasible for $(\mathrm{D}_2(p,d,S))$ where $\overline{V} = (A_0 \bullet V^* + b^\mathsf{T} v^*)^{-1} V^*$, $\bar{v} = (A_0 \bullet V^* + b^\mathsf{T} v^*)^{-1} v^*$ and $A_0 \bullet \overline{V} + b^\mathsf{T} \bar{v} = 1$. □

The last problem we discuss is closely related to the projection operator defined in Chapter 1. It is given as

$$\begin{aligned} \min \ \|p - x\| \quad \text{s.t.} \quad & A_0 + \mathcal{A}(x) + \overline{\mathcal{A}}(y) + \boldsymbol{\mathcal{A}}(Z) \succcurlyeq 0 \\ & Bx + \bar{B}y + \mathcal{B}^\mathsf{T}(Z) = b \\ & Z \succcurlyeq 0 \end{aligned} \qquad (\mathrm{P}_3(p,S))$$

for a parameter $p \in \mathbb{R}^n$ typically not belonging to S. Hence, part of a solution to $(\mathrm{P}_3(p,S))$ is just the projection $\pi_S(p)$ of p onto S. Proposition 1.25 outlines how a solution to the problem generates a supporting hyperplane to S. The dual program reads as

$$\begin{aligned} \max \ -(A_0 + \mathcal{A}(p)) \bullet V - (b - Bp)^\mathsf{T} v \quad \text{s.t.} \quad & -\mathcal{A}^\mathsf{T}(V) + B^\mathsf{T} v = w \\ & -\overline{\mathcal{A}}^\mathsf{T}(V) + \bar{B}^\mathsf{T} v = 0 \\ & -\boldsymbol{\mathcal{A}}^\mathsf{T}(V) + \mathcal{B}(v) \succcurlyeq 0 \\ & V \succcurlyeq 0 \\ & \|w\| \leqslant 1. \end{aligned} \qquad (\mathrm{D}_3(p,S))$$

Proposition 3.13. *Let (x^*, y^*, Z^*) and (V^*, v^*, w^*) be solutions to the problems $(P_3(p,S))$ and $(D_3(p,S))$, respectively. Given strong duality the following hold:*

(i) if $A_0 \bullet V^* + b^\mathsf{T} v^* = 0$, then $w^* \in 0^\infty S^\circ$,

(ii) if $A_0 \bullet V^* + b^\mathsf{T} v^* > 0$, then the hyperplane $H(x^*, 1)$ supports S° at $\frac{w^*}{A_0 \bullet V^* + b^\mathsf{T} v^*}$.

Proof. The proof is similar to the proofs of Propositions 3.8 and 3.12. □

Remark 3.14. Problems similar to $(\mathrm{P}_1(\omega,S))$ and $(\mathrm{P}_2(p,d,S))$ are commonly used in multiple objective optimization as scalarization techniques, see e.g. [Eic08; Löh11]. Recently, problems involving norms have also been considered [Dör+22; AUU22].

3.2.1 Computing relative interior points

One of the assumptions of the algorithms will be that the spectrahedral shadow S to be approximated has nonempty interior in order to make them fit into the setting of Kamenev's schemes. We present an algorithm that verifies this assumption for a given input $S \subseteq \mathbb{R}^n$. The algorithm computes the affine hull of S as well as a point $\bar{x} \in \operatorname{relint} S$. Thus, if the affine hull of S is the whole space, $\bar{x}$ is actually an interior point of S, which gives a certificate whether $\operatorname{int} S \neq \emptyset$. Another important motivation for the algorithm is that it closes the gap in Proposition 2.25. Remember, that we can express $\operatorname{relint} S$ as a spectrahedral shadow given that a fixed point $\bar{x}$ of the relative interior is known. The algorithm below will provide such a point. A similar algorithm that computes an interior point of a compact convex set is found in [Löh11, Section 5.5].

In the following, we assume that S is bounded. This is no restriction because for every $x \in S$ the set $\operatorname{relint} S \cap B_1(x)$ is nonempty, see e.g. [Roc70, Theorem 6.1]. Moreover, $S \cap B_1(x)$ is again a spectrahedral shadow according to Example 2.4 and Proposition 2.12.

ALGORITHM 3.1: Relative interior point algorithm

```
Input: a compact spectrahedral shadow S ⊆ R^n
Output: a point x̄ ∈ relint S, finite sets L and L^⊥ generating the subspace
        aff(S − {x̄}) and its orthogonal complement (aff(S − {x̄}))^⊥
1  X ← ∅
2  L ← ∅
3  L^⊥ ← ∅
4  while |L ∪ L^⊥| < n do
5  |  compute ω ≠ 0 such that d^T ω = 0 for all d ∈ L ∪ L^⊥
6  |  compute a solution (x_ω, y_ω, Z_ω) to (P_1(ω, S))
7  |  compute a solution (x_{−ω}, y_{−ω}, Z_{−ω}) to (P_1(−ω, S))
8  |  if ω^T x_ω = ω^T x_{−ω} then
9  |  |  L^⊥ ← L^⊥ ∪ {ω}
10 |  else
11 |  |  L ← L ∪ {x_ω − x_{−ω}}
12 |  end
13 |  X ← X ∪ {x_ω, x_{−ω}}
14 end
15 x̄ ← (1/|X|) Σ_{x∈X} x
```

Theorem 3.15. *Algorithm 3.1 works correctly, i.e. the point $\bar{x}$ satisfies $\bar{x} \in \operatorname{relint} S$ and for the linear hulls $\operatorname{lin} L$ and $\operatorname{lin} L^\perp$ it holds $\operatorname{aff} S = \{\bar{x}\} + \operatorname{lin} L$ and $(\operatorname{lin} L)^\perp = \operatorname{lin} L^\perp$, respectively.*

Proof. The algorithm terminates after exactly n iterations of the loop in lines 4–14 because in every iteration either one element is added to $L^\perp$ in line 9 or L in line 11. Solutions to the problems in lines 6 and 7 exist for every ω according to Proposition 3.7 because S is compact. The elements of $L \cup L^\perp$ are linearly independent. To see this, first consider an element $\ell \in L$ and observe that it holds $\ell \neq 0$ because L is updated if and only if $\omega^\mathsf{T}\ell \neq 0$. Now, assume that during some iteration of the algorithm $x_\omega - x_{-\omega}$ is to be added to L and $x_\omega - x_{-\omega} \in \operatorname{lin}\left(L \cup L^\perp\right)$. Then

$$x_\omega - x_{-\omega} = \sum_{\ell \in L \cup L^\perp} \lambda_\ell \ell$$

for some $\lambda_\ell \in \mathbb{R}$. According to line 5 we have $\omega^\mathsf{T}\ell = 0$ for all $\ell \in L \cup L^\perp$, but $\omega^\mathsf{T}(x_\omega - x_{-\omega}) \neq 0$ because L is updated. This is a contradiction as $\omega \neq 0$. The elements of $L^\perp$ are linearly independent in $L \cup L^\perp$ due to the choice of ω in line 5. This implies $(\operatorname{lin} L)^\perp = \operatorname{lin} L^\perp$ at termination. Next, we show that $\operatorname{lin} L = \operatorname{aff}(S - \{\bar{x}\})$. The point $\bar{x}$ belongs to S because $\operatorname{conv} X \subseteq S$. Hence, $\operatorname{aff}(S - \{\bar{x}\}) = \operatorname{lin}(S - \{\bar{x}\})$. For $x_\omega - x_{-\omega} \in L$ we have

$$x_\omega - x_{-\omega} = (x_\omega - \bar{x}) - (x_{-\omega} - \bar{x}) \in \operatorname{lin}(S - \{\bar{x}\}).$$

Therefore, $\operatorname{lin} L \subseteq \operatorname{lin}(S - \{\bar{x}\})$. The other inclusion follows because $L \cup L^\perp$ spans $\mathbb{R}^n$. It remains to prove that $\bar{x} \in \operatorname{relint} S$. Since $\bar{x}$ belongs to $\operatorname{relint} \operatorname{conv} X$ by its definition in line 15, it suffices to show that $\operatorname{conv} X$ and S have the same dimension. Then $\operatorname{conv} X \subseteq S$ implies $\operatorname{relint} \operatorname{conv} X \subseteq \operatorname{relint} S$, see e.g. [Roc70, Corollaries 6.3.3 and 6.5.2]. Assume that the dimension of $\operatorname{conv} X$ is smaller than the dimension of S. Then there exists a direction $\ell \in \operatorname{lin} L$ with $\ell \in (\operatorname{lin} \operatorname{conv} X - \{\bar{x}\})^\perp$. Then for

all points x_ω and $x_{-\omega}$ added to X in line 13 it holds $\ell^\mathsf{T}(x_\omega - x_{-\omega}) = 0$, i.e. $\ell \in \operatorname{lin} L^\perp$. This is a contradiction to $\ell \in \operatorname{lin} L$. Therefore, $\operatorname{conv} X$ and S have the same dimension. □

Remark 3.16. Algorithm 3.1 is formulated for spectrahedral shadows S, but it would suffice to state it for spectrahedra. In particular, if C is a spectrahedron that maps to S under the projection π, i.e. $S = \pi[C]$, then $\operatorname{relint} S = \pi[\operatorname{relint} C]$ holds, see [Roc70, Theorem 6.6]. However, since the algorithm operates in the typically lower dimensional ambient space $\mathbb{R}^n$ of the spectrahedral shadow S, working with S directly can be beneficial.

We illustrate the algorithm on an example.

Example 3.17. Consider the disk $S = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leqslant 1\} \times \{0\}$ embedded in $\mathbb{R}^3$. We apply Algorithm 3.1 to S. At the beginning of the first iteration of the while loop in lines 4–14 the sets L and $L^\perp$ are empty. Hence, any nonzero ω is valid in line 5. Here, $\omega = (0.5, 0, 0.5)^\mathsf{T}$ is chosen, which yields $x_\omega = (1, 0, 0)^\mathsf{T}$ and $x_{-\omega} = (-1, 0, 0)^\mathsf{T}$. Consequently, the point $(2, 0, 0)^\mathsf{T}$ is added to L in line 11. In the next iteration a nonzero point ω with $\omega_1 = 0$ needs to be computed, e.g. $\omega = (0, 1, 0)^\mathsf{T}$. For this assignment the problems $(\mathrm{P}_1(\omega, S))$ and $(\mathrm{P}_1(-\omega, S))$ yield the points $(0, 1, 0)^\mathsf{T}$ and $(0, -1, 0)^\mathsf{T}$, respectively. Again, the set L is updated. In the last loop run the direction $\omega = (0, 0, 1)^\mathsf{T}$ is computed. Since S is contained in the hyperplane $H(\omega, 0)$, the direction is appended to $L^\perp$. Moreover, any point in S is a valid part of a solution in lines 6 and 7 because ω has a constant value of zero as a linear function over S. For illustration we choose $x_\omega = (0.5, 0.5, 0)^\mathsf{T}$ and $x_{-\omega} = (-0.5, 0, 0)^\mathsf{T}$. At termination the set X consists of the points

$$\begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

and the returned relative interior point is

$$\bar{x} = \frac{1}{12}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The steps of the algorithm are shown in Figure 3.1.

3.2.2 Outer and inner polyhedral approximation algorithms

In order to state the approximation algorithms we need to concretize the steps for the augmenting and cutting scheme outlined at the beginning of this section. In particular, we need to provide rules for choosing a direction in every iteration of a cutting scheme and for choosing a boundary point in every iteration of an augmenting scheme. For these steps the semidefinite programs introduced earlier are vital. Moreover, we need to construct initial outer respectively inner approximations as a starting point for the schemes.

Polyhedral approximation algorithms for spectrahedral shadows are also studied in [Cir19]. In fact, the augmenting scheme presented as Algorithm 3.3 below appears therein. However, there is a discrepancy between this algorithm and the cutting scheme that is presented in [Cir19] in the sense that convergence properties are

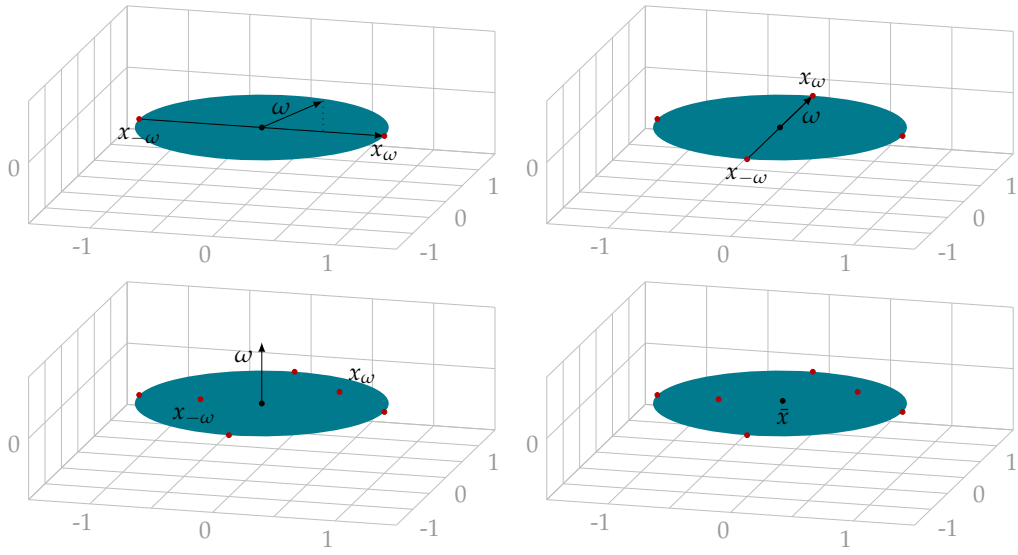

FIGURE 3.1: Algorithm 3.1 applied to the set from Example 3.17. The different iterations are shown from left to right and top to bottom.

proved for the former but could not be obtained for the latter. We eliminate this gap in this section and present a different cutting scheme that admits the same convergence properties as Algorithm 3.3, see Theorems 3.20 and 3.22 below.
For the remainder of this section we assume that the spectrahedral shadow S is a convex body. This assumption is not overly restrictive. Indeed, if the origin is not contained in the interior of S, but S is full dimensional, then we can use Algorithm 3.1 to compute a point $\bar{x} \in \operatorname{int} S$ and work with the set $S - \{\bar{x}\}$ instead, which has the origin in its interior. If S is not full dimensional, we can restrict the ambient space to the affine hull of S. The closedness of S is merely of theoretical importance, i.e. to ensure that solutions to the discussed semidefinite programs exist. In practice, an interior-point method would be employed to solve the problems to within a given accuracy, which is possible regardless of whether S is closed. Hence, the only real limitation is the boundedness of S. We account for that in Remark 3.19.

Definition 3.18. For a convex set $C \subseteq \mathbb{R}^n$ and $\varepsilon \geqslant 0$ a polyhedron $P \subseteq \mathbb{R}^n$ is called an *ε-Hausdorff-approximation* or *ε-H-approximation* of C if $d_{\mathsf{H}}(P, C) \leqslant \varepsilon$.

Clearly, if P is an ε-H-approximation of C, then this is also true for $\operatorname{cl} C$ because polyhedra are closed sets. The property of being an ε-H-approximation is used as a termination condition in the algorithms, which we are now prepared to formulate. We begin with stating a cutting scheme for S. For a predefined error tolerance ε it computes an outer ε-H-approximation P of S. Algorithm 3.2 describes the method in pseudo code.

The algorithm starts by initializing the polyhedral approximation as the whole space, which trivially contains S. Next, an initial compact outer approximation is computed in lines 2–5. Therefore, problem $(\mathrm{P}_1(\omega, S))$ is solved for every standard unit vector $\omega \in \{e_i \mid i = 1, \ldots, n\}$ and the negative ones vector $\omega = -e$. These solutions generate supporting halfspaces to S according to Proposition 3.7. The initial outer approximation is set as the intersection of all such halfspaces. In the main loop in lines 7–14 of the algorithm the current approximation is successively refined. To this end $(\mathrm{P}_3(v, S))$ is solved for every vertex v of the current approximation, i.e. for every vertex its projection $\pi_S(v) = x_v$ onto S is computed. Clearly, knowledge of the

ALGORITHM 3.2: Cutting scheme algorithm for compact spectrahedral shadows

Input: a spectrahedral shadow and convex body $S \subseteq \mathbb{R}^n$, error tolerance $\varepsilon > 0$

Output: an outer ε-H-approximation P of S

1 $P \leftarrow \mathbb{R}^n$
2 for $\omega \in \{-e, e_1, \ldots, e_n\}$ do
3 | compute a solution $(x_\omega, y_\omega, Z_\omega)$ to $(\mathrm{P}_1(\omega, S))$
4 | $P \leftarrow P \cap H^-(\omega, \omega^\mathsf{T} x_\omega)$
5 end
6 $\kappa \leftarrow \infty$
7 while $\kappa > \varepsilon$ do
8 | for $v \in \operatorname{vert} P$ do
9 | | compute a solution (x_v, y_v, Z_v) to $(\mathrm{P}_3(v, S))$
10 | end
11 | $\bar{v} \leftarrow \operatorname{argmax}\{\|v - x_v\| \mid v \in \operatorname{vert} P\}$
12 | $P \leftarrow P \cap H^-(\bar{v} - x_{\bar{v}}, \gamma)$ where $\gamma = (\bar{v} - x_{\bar{v}})^\mathsf{T} x_{\bar{v}}$
13 | $\kappa \leftarrow \|\bar{v} - x_{\bar{v}}\|$
14 end

set vert P or, equivalently, a V-representation of P is required to perform these computations. However, during the algorithm only an H-representation of P is known because the initial approximation as well as the refinement in line 12 are defined as an intersection of halfspaces. The task of computing a V-representation from an H-representation of a polyhedron is called *vertex enumeration* and the reverse problem is called *facet enumeration*. The package `bensolve tools` [LW16; CLW18] or the library `cddlib` are available to solve these problems. It has been pointed out recently that the methods implemented therein are prone to numerical problems caused by using floating point arithmetic. Hence, it might be beneficial to use alternative methods that only compute approximate V-representations but are compatible with floating point arithmetic such as the one presented for $n = 2$ and $n = 3$ in [Löh20]. Among all vertices v of P one yielding the largest distance $d(v, S)$ is chosen. This vertex is denoted by $\bar{v}$ and the current approximation quality is set to $\|\bar{v} - x_{\bar{v}}\| = d(\bar{v}, S)$ in line 13. If this value is positive than we obtain a hyperplane supporting S at $x_{\bar{v}}$ and separating S and $\bar{v}$ according to Proposition 1.25. In this case we cut off $\bar{v}$ of the current approximation, i.e. we refine P as

$$P \cap H^- \left(\bar{v} - x_{\bar{v}}, (\bar{v} - x_{\bar{v}})^\mathsf{T} x_{\bar{v}}\right).$$

On the other hand, if $\bar{v} = x_{\bar{v}}$, we do not need to update the polyhedral approximation anymore because all vertices are already contained in S. This scenario occurs if and only if S is itself a polyhedron and $P = S$. In this case the current approximation is simply intersected with the whole space in line 12. These steps are iterated until the current approximation error κ is certifiably not larger than the predefined error tolerance ε. Note that a solution to the problem $(\mathrm{P}_3(v, S))$ needs to be computed only once and can be used in subsequent iterations if v remains a vertex of P. Since in every iteration κ is set to $\max\{d(v, S) \mid v \in \operatorname{vert} P\}$ and $S \subseteq P$, in fact one has $\kappa = d_\mathsf{H}(P, S)$, see [Bat86, Theorem 3.3]. Hence, the algorithm terminates if the Hausdorff distance between the approximating polyhedron P and the spectrahedral

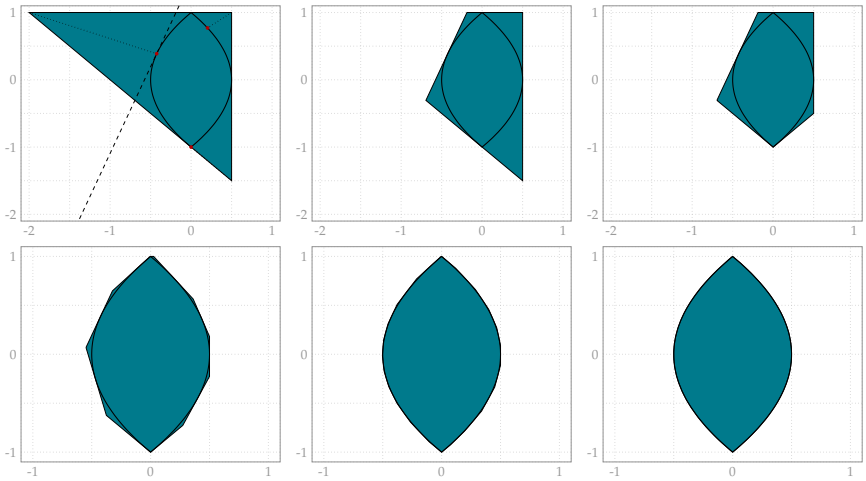

FIGURE 3.2: Illustration of Algorithm 3.2 applied to the set S which is the polar of the set from Example 2.17. The first image shows the initial approximation computed by lines 2–5 in blue. The dotted lines and the red points visualize the distances from the vertices of the outer approximation to S and the projections of the vertices onto S, respectively. The vertex $\bar{v} = (-2, 1)^\mathsf{T}$ is the furthest from the set and selected in line 11. The dashed line is the supporting hyperplane containing $\pi_S(\bar{v})$ that is obtained from a solution of $(\mathrm{P}_3(\bar{v}, S))$. The second picture shows the approximation after $\bar{v}$ is cut off. It is the set that is obtained after the first iteration of lines 7–14. After another iteration the set in the top right is the current approximation. The bottom row depicts the outer approximations for $\varepsilon = 0.1$, $\varepsilon = 0.01$ and $\varepsilon = 0.001$ with 10, 23 and 63 facets, respectively.

shadow S is at most ε. The functioning of Algorithm 3.2 is depicted in Figure 3.2.

Remark 3.19. The initialization step of Algorithm 3.2 in lines 2–5 can be used to verify whether the set S that is input to the algorithm is actually bounded. If S is unbounded, then there exists an index $i \in \{1, \dots, n\}$ and a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq S$ such that either $\lim_{k\to\infty} (x_k)_i = \infty$ or $\lim_{k\to\infty} (x_k)_i = -\infty$. In the former case problem $(\mathrm{P}_1(e_i, S))$ is unbounded and in the latter case $(\mathrm{P}_1(-e, S))$ is unbounded. Hence, if S is unbounded, then at least one of the programs in line 3 is unbounded, and, on the contrary, if one of the problems $(\mathrm{P}_1(\omega, S))$ is unbounded, so is S.

We are able to obtain a convergence result for Algorithm 3.2 that is compatible with Theorem 3.5.

Theorem 3.20. *Algorithm 3.2 works correctly, in particular it terminates with an outer ε-H-approximation P of a compact spectrahedral shadow S containing the origin in its interior. Moreover, for the sequence $\{P_k\}$ of outer approximations computed by Algorithm 3.2, the value of $d_\mathsf{H}(P_k, S)$ decreases with order $\mathcal{O}\left(\frac{1}{k^{1/(n-1)}}\right)$.*

Proof. Since S is compact, optimal solutions $(x_\omega, y_\omega, Z_\omega)$ to the problems $(\mathrm{P}_1(\omega, S))$ in line 3 exist for every $\omega \in \{-e, e_1, \dots, e_n\}$ according to Proposition 3.7. Furthermore, the hyperplanes $H(\omega, \omega^\mathsf{T} x_\omega)$ support S at x_ω. Hence, $S \subseteq H^-(\omega, \omega^\mathsf{T} x_\omega)$ and the initial approximation P computed in lines 1–5 contains S. Moreover, P is bounded because $0^\infty P = \left\{x \in \mathbb{R}^n \,\middle|\, \omega^\mathsf{T} x \leqslant 0, \omega = -e, e_1, \dots, e_n\right\} = \{0\}$, cf. [Roc70]. Let P_k be the approximation computed after k iterations of the loop in lines 7–14. A

solution (x_v, y_v, Z_v) to $(\mathrm{P}_3(v, S))$ always exists, see [HL01]. Proposition 1.25 guarantees that the hyperplane $H(\bar{v} - x_{\bar{v}}, (\bar{v} - x_{\bar{v}})^\mathsf{T} x_{\bar{v}})$ in line 12 supports S at $x_{\bar{v}}$ and $S \subseteq H^-(\bar{v} - x_{\bar{v}}, (\bar{v} - x_{\bar{v}})^\mathsf{T} x_{\bar{v}})$. Hence, the updated polyhedral approximation contains S. Suppose the algorithm terminates after K iterations with polyhedron P_K. Then for κ defined in line 13 it holds $\kappa \leqslant \varepsilon$. By definition we have

$$\kappa = \max\{\|v - x_v\| \mid v \in \operatorname{vert} P_K\}. \tag{3.2}$$

Since $S \subseteq P_K$, the Hausdorff distance $d_\mathsf{H}(P_K, S)$ is attained as $e[P_K, S]$. It follows from [Bat86, Theorem 3.3] that $e[P_K, S]$ is attained at a vertex of P_K, i.e.

$$e[P_K, S] = \max\{\|v - x_v\| \mid v \in \operatorname{vert} P_K\} = \kappa. \tag{3.3}$$

Therefore, P_K is an outer ε-H-approximation of S. To prove the second part of the theorem let P_k be the approximation after iteration k of the algorithm. Following the same argumentation that lead to Equations (3.2) and (3.3) we obtain $d_\mathsf{H}(P_k, S) = \kappa = \|\bar{v} - x_{\bar{v}}\|$. For the updated polyhedron P_{k+1} in line 12 it holds

$$d_\mathsf{H}(P_{k+1}, P_k) = d_\mathsf{H}\left(H(\bar{v} - x_{\bar{v}}, (\bar{v} - x_{\bar{v}})^\mathsf{T} x_{\bar{v}}), P_k\right) = \|\bar{v} - x_{\bar{v}}\| = d_\mathsf{H}(P_k, S).$$

Thus, Algorithm 3.2 constitutes a Hausdorff scheme with constant $\gamma = 1$ as defined in (3.1). The rate of $\mathcal{O}\left(\frac{1}{k^{1/(n-1)}}\right)$ now follows from Theorem 3.5. In particular, Algorithm 3.2 terminates after finitely many iterations. □

Using polarity we also obtain a polyhedral approximation of the polar S° from Algorithm 3.2. If $\{P_k\}_{k=1}^K$ is the sequence of polyhedral outer approximations computed throughout one execution of the algorithm, then one can investigate how the sequence $\{P_k^\circ\}_{k=1}^K$ of polar approximations relates to S°. First of all, there is the relation $P_k^\circ \subseteq S^\circ$ for all $k \in \{1, \ldots, K\}$ because the polyhedra P_k contain S and the polarity operation is inclusion-inverting, see [Roc70]. Moreover, the values $d_\mathsf{H}(P_k^\circ, S^\circ)$ decrease with the same rate as $d_\mathsf{H}(P_k, S)$ in the number k of iterations. This follows from the fact that $\{P_k\}_{k=1}^K$ is generated by a Hausdorff scheme according to Theorem 3.20 and [Kam08, Theorem 7]. In particular, Inequality (3.1) holds for the sets P_k° and S° but with a possibly different constant γ. Depending on the scales of the sets S and S° the termination criterion of Algorithm 3.2 might behave badly with the approximation of S°, i.e. $d_\mathsf{H}(P_K, S) \leqslant \varepsilon$ holds due to Theorem 3.20 but only the upper bound

$$d_\mathsf{H}(P_K^\circ, S^\circ) \leqslant \frac{d_\mathsf{H}(P_K, S)}{r_0(P_K) r_0(S)},$$

where $r_0(P_K)$ and $r_0(S)$ are the radius of the largest Euclidean ball centered at the origin that is contained in P_K and S, respectively, is known, see [Kam02, Lemma 1]. Hence, the approximation quality of P_K° depends on the position of the origin relative to the boundary of S. In order to obtain a description of P_K° Proposition 2.28 can be applied.

We now turn to the presentation of an augmenting scheme for S. For a predefined error tolerance ε it computes an inner $\gamma\varepsilon$-H-approximation P of S for some constant $\gamma \geqslant 1$ that depends on the geometry of S and an initial inner approximation. Unlike the cutting scheme, which is based on problem $(\mathrm{P}_3(v, S))$, the augmenting scheme utilizes $(\mathrm{P}_1(\omega, S))$ in its main loop. The method is presented as pseudo code in Algorithm 3.3.

ALGORITHM 3.3: Augmenting scheme algorithm for compact spectrahedral shadows

Input: a spectrahedral shadow and convex body $S \subseteq \mathbb{R}^n$, error tolerance $\varepsilon > 0$

Output: an inner $\gamma\varepsilon$-H-approximation P of S for some $\gamma \geqslant 1$

1 $P \leftarrow \emptyset$
2 for $d \in \{-e, e_1, \ldots, e_n\}$ do
3 | compute a solution (t_d, x_d, y_d, Z_d) to $(\mathrm{P}_2(0, d, S))$
4 | $P \leftarrow \operatorname{conv}(P \cup \{x_d\})$
5 end
6 $\kappa \leftarrow \infty$
7 while $\kappa > \varepsilon$ do
8 | for *every facet of P defined by a hyperplane $H(\omega, \gamma)$* do
9 | | compute a solution $(x_\omega, y_\omega, Z_\omega)$ to $(\mathrm{P}_1(\omega, S))$
10 | end
11 | $(\bar{\omega}, \bar{\gamma}) \leftarrow \operatorname{argmax}\left\{ \frac{\omega^\mathsf{T} x_\omega - \gamma}{\|\omega\|} \,\middle|\, H(\omega, \gamma) \cap P \text{ is a facet of } P \right\}$
12 | $P \leftarrow \operatorname{conv}(P \cup \{x_{\bar{\omega}}\})$
13 | $\kappa \leftarrow \frac{\bar{\omega}^\mathsf{T} x_{\bar{\omega}} - \bar{\gamma}}{\|\bar{\omega}\|}$
14 end

In the first step of the algorithm the current approximation is initialized as the empty set. Then, analogously to Algorithm 3.2, an initial approximation is computed. However, problem $(\mathrm{P}_2(0, d, S))$ is solved for $d = -e$ and every direction $d = e_i$, $i = 1, \ldots, n$, to obtain a first inner approximation. Since $0 \in S$, every solution (t_d, x_d, y_d, Z_d) yields a point on the boundary of S, in particular $x_d = t_d d$. The initial approximation P is constructed as the convex hull of these $n + 1$ boundary points describing a full dimensional polyhedron contained in S. Lines 7–14 comprise the main loop of Algorithm 3.3, in which the polyhedral approximation is successively enlarged. To this end $(\mathrm{P}_1(\omega, S))$ is solved for every normal direction ω of a facet $H(\omega, \gamma) \cap P$ of P. This step requires the computation of an H-representation of P, which is only known by its vertices during the algorithm, in every iteration. A facet which admits the largest shift within S, i.e. for which the value $\left(\omega^\mathsf{T} x_\omega - \gamma\right) / \|\omega\|$ is maximized among solutions $(x_\omega, y_\omega, Z_\omega)$ to $(\mathrm{P}_1(\omega, S))$, is chosen. The corresponding normal vector is denoted by $\bar{\omega}$ in line 11. Proposition 3.7 implies that the point $x_{\bar{\omega}}$ lies on the boundary of S. It is appended to a V-representation of the current approximation in the update step in line 12 yielding a new inner approximation

$$\operatorname{conv}(P \cup \{x_{\bar{\omega}}\}),$$

which is again contained in S. The approximation error κ is set to $\left(\bar{\omega}^\mathsf{T} x_{\bar{\omega}} - \bar{\gamma}\right) / \|\bar{\gamma}\|$, that is to the largest distance one can shift a facet of P within the set S. This procedure is repeated until no facet of the current approximation can be shifted a distance larger than ε at which point Algorithm 3.3 terminates. Note that for compact convex sets C_1 and C_2 one has the identity

$$d_\mathsf{H}(C_1, C_2) = \max\{|\sigma_{C_1}(\omega) - \sigma_{C_2}(\omega)| \mid \|\omega\| = 1\} \tag{3.4}$$

involving the support functions of C_1 and C_2, see e.g. [Gru07, Proposition 6.3]. In

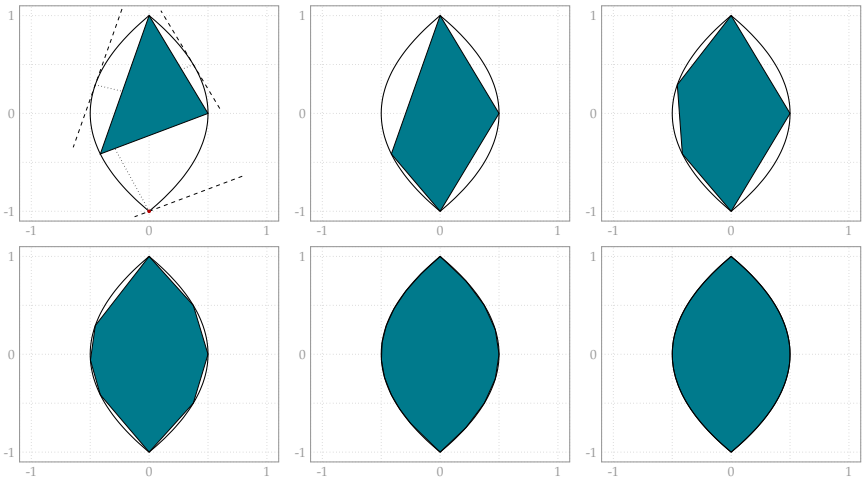

FIGURE 3.3: Illustration of Algorithm 3.3 applied to the set S which is the polar of the set from Example 2.17. The initial approximation computed in lines 2–5 is shown in the first picture. The dotted lines are the possible shifts of facets of the inner approximation. The red point marks the solution of $(\mathrm{P}_1(\bar{\omega}, S))$, which belongs to the facet admitting the largest shift. The updated inner approximation after the first and second iteration of the main loop are portrayed in the second and third picture. The bottom row displays the approximations obtained for $\varepsilon = 0.1$, $\varepsilon = 0.01$ and $\varepsilon = 0.001$ having 8, 19 and 57 vertices, respectively.

every iteration of the algorithm the support function of S is evaluated for the directions ω that are normal to a facet of P. Therefore, the value κ in line 13 is a lower bound for the Hausdorff distance $d_{\mathsf{H}}(P, S)$. According to [Kam94, Theorem 1] one also has the upper bound

$$d_{\mathsf{H}}(P, S) \leqslant \frac{R}{r}\kappa \tag{3.5}$$

where $B_r(0) \subseteq P_0 \subseteq S \subseteq B_R(0)$ and P_0 is the initial inner approximation computed in lines 2–5. Hence, Algorithm 3.3 terminates with $d_{\mathsf{H}}(P, S) \leqslant \gamma\varepsilon$ for some $\gamma \geqslant 1$. Note that this additional factor γ, which does not appear in Algorithm 3.2, arises from the fact that it is not feasible in practice to evaluate the support function σ_S for every direction ω. Otherwise, the equality $\kappa = d_{\mathsf{H}}(P, S)$ would indeed hold. An illustration of the augmenting scheme is shown in Figure 3.3.

Similar to Algorithm 3.2 one obtains a polyhedral outer approximation of S° from the output of Algorithm 3.3 using polarity, i.e. if the inner approximation P_K is returned, then $S^\circ \subseteq P_K^\circ$.

Remark 3.21. We have observed that Algorithm 3.2 can be used to certify whether S is bounded. In fact, the initialization phase of Algorithm 3.3 can be exploited as well to verify one of the assumptions on S. If any of the optimal values t_d in line 3 is less than or equal to zero, then $0 \notin \operatorname{int} S$. For instance, if $t_d = 0$, then $td \notin S$ for every $t > 0$. Hence, the origin must be contained in the boundary of S. Similarly, if $t_d < 0$ or $(\mathrm{P}_2(0, d, S))$ is infeasible for any $d \in \{-e, e_1, \ldots, e_n\}$, then $0 \notin S$ because there do not exist y and Z such that $(0, 0, y, Z)$ is feasible for $(\mathrm{P}_2(0, d, S))$. We conclude that $0 \in \operatorname{int} S$ if and only if every optimal value t_d to $(\mathrm{P}_2(0, d, S))$ in line 3 is positive. Considering the problems $(\mathrm{P}_2(p, d, S))$ for a given point $p \in \mathbb{R}^n$ this method can be used to obtain a containment certificate of $p \in \operatorname{int} S$. We point out that the point p has

to be chosen beforehand whereas in Algorithm 3.1 some interior point is computed given that $\operatorname{int} S \neq \emptyset$.

A similar convergence result like Theorem 3.20 holds for Algorithm 3.3.

Theorem 3.22. *Algorithm 3.3 works correctly, in particular it terminates with an inner $\gamma\varepsilon$-H-approximation P of a compact spectrahedral shadow S containing the origin in its interior for some $\gamma \geqslant 1$. Moreover, for the sequence $\{P_k\}$ of inner approximations computed by Algorithm 3.3, the value of $d_{\mathsf{H}}(P_k, S)$ decreases with order $\mathcal{O}\left(\frac{1}{k^{1/(n-1)}}\right)$.*

Proof. Solutions (t_d, x_d, y_d, Z_d) to the problems $(\mathrm{P}_2(0, d, S))$ in line 3 exist for every $d \in \{-e, e_1, \ldots, e_n\}$ because S is compact and $0 \in \operatorname{int} S$. Moreover, $t_d > 0$ and $x_d = t_d d$ is a point on the boundary of S according to Proposition 3.11. Therefore, the set P computed after the loop in lines 2–5 is a full dimensional inner approximation of S. Let P_k be the approximation computed after k iterations of the loop in lines 7–14. A solution $(x_\omega, y_\omega, Z_\omega)$ to $(\mathrm{P}_1(\omega, S))$ exists due to Proposition 3.7 and the point $x_{\bar{\omega}}$ utilized in line 12 belongs to the boundary of S. Hence, the updated inner approximation satisfies $P \subseteq S$. Suppose the algorithm terminates after K iterations with polyhedron P_K. Then for κ in line 13 it holds

$$\kappa = \max\left\{ \frac{\sigma_S(\omega) - \sigma_{P_K}(\omega)}{\|\omega\|} \,\middle|\, \omega \text{ is normal to a facet of } P_K \right\} \leqslant \varepsilon$$

and according to the upper bound (3.5) it follows that there exist $0 < r < R$ such that

$$d_{\mathsf{H}}(P_K, S) \leqslant \frac{R}{r}\kappa \leqslant \frac{R}{r}\varepsilon.$$

This proves that P_K is an inner $\gamma\varepsilon$-H-approximation of S. The rate follows immediately from [Kam94, Theorem 2] and Theorem 3.5. In particular, this implies that Algorithm 3.3 terminates after finitely many steps. □

Remark 3.23. Algorithms 3.2 and 3.3 do not rely on any unique properties of spectrahedral shadows. In fact, both algorithms can be stated for more general convex bodies, e.g. sets of the form $C = \{x \in \mathbb{R}^n \mid f_i(x) \leqslant 0, i = 1, \ldots, m\}$ for convex functions $f_1, \ldots, f_m$, in exactly the same way. Furthermore, the convergence result in Theorems 3.20 and 3.22 applies as well. It needs to be ensured, however, that the involved optimization problems can be solved for C. These are then general convex programming problems rather than semidefinite ones.

Example 3.24. We apply Algorithms 3.2 and 3.3 to the spectrahedral shadow

$$S = \left\{ x \in \mathbb{R}^3 \,\middle|\, \exists\, V \in \mathcal{S}^4 \colon \begin{pmatrix} I \\ 0 \end{pmatrix} x + \mathcal{B}^{\mathsf{T}}(V) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, V \succcurlyeq 0 \right\},$$

with

$$\mathcal{B}(v) = \begin{pmatrix} v_4 & v_1 & v_2 & 0 \\ v_1 & v_4 & v_3 & 0 \\ v_2 & v_3 & v_4 & 0 \\ 0 & 0 & 0 & v_4 \end{pmatrix}.$$

It is the polar set of the spectrahedron that is defined by the linear matrix inequality

$$\begin{pmatrix} 1 & v_1 & v_2 \\ v_1 & 1 & v_3 \\ v_2 & v_3 & 1 \end{pmatrix} \succcurlyeq 0$$

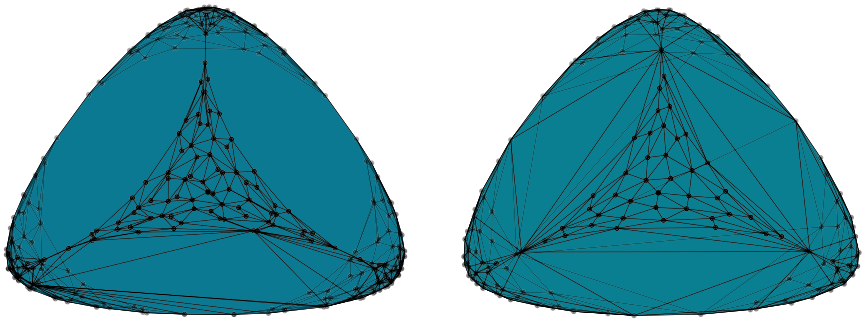

FIGURE 3.4: Outer and inner polyhedral approximation of the set from Example 3.24 computed for $\varepsilon = 0.01$ by Algorithm 3.2 and 3.3, respectively.

according to Theorem 2.32. The table below summarizes various numerical results from the application of the algorithms to the spectrahedral shadow S. For instance, the number of iterations of the main loop and the number of vertices and facets of the approximations are shown. Figure 3.4 displays the approximations for $\varepsilon = 0.01$. From the results it is evident that both algorithms need a comparable number of iterations to reach a given error tolerance. This is surprising because the inner approximations may have a worse approximation quality than the outer approximations whose Hausdorff distance to S is provably at most ε. This is also reflected in the smaller number of vertices and facets for the inner approximations. Note that, for this example, Inequality (3.5) and the outer approximation for $\varepsilon = 0.005$ yield $R \leqslant 1.15876$ and the initial inner approximation yields $r \geqslant 0.26455$, i.e. we obtain the estimate $\gamma \leqslant 4.3801$. It would be interesting to investigate whether Algorithm 3.2 typically needs fewer iterations than Algorithm 3.3 to reach the same approximation quality or whether the upper bound (3.5) can be improved.

TABLE 3.1: Numerical results from Example 3.24.

	ε	0.1	0.05	0.01	0.005
Alg. 3.2	iterations	17	32	154	333
	vertices	38	68	312	670
	facets	53	118	554	1330
Alg. 3.3	iterations	18	35	163	343
	vertices	22	39	167	347
	facets	40	71	330	690

3.3 Limitations of the Hausdorff distance for polyhedral approximation

We have seen that every compact convex set can be approximated by polyhedra in the Hausdorff distance to arbitrary precision and that there are algorithms that compute such approximating polyhedra. Moreover, Section 3.1 emphasizes that there is a vast amount of literature that deals with this problem both from a theoretical and an algorithmic perspective. In the last section of this chapter we investigate the

possibility of approximating convex sets by polyhedra with respect to the Hausdorff distance that are not necessarily bounded. The literature on that matter is scarce. A possible reason for this lack might be explained by properties of the Hausdorff distance. It is well known that the Hausdorff distance constitutes a metric on the space of compact subsets of $\mathbb{R}^n$. This makes it a suitable error measure for approximation in the compact case. However, for unbounded sets the Hausdorff distance does not need to be finite. A first step towards understanding its limitations in view of polyhedral approximation is the following result.

Proposition 3.25. *Let $C_1, C_2 \subseteq \mathbb{R}^n$ be closed convex sets and assume $0^\infty C_1 \neq 0^\infty C_2$. Then $d_H(C_1, C_2) = \infty$.*

Proof. Without loss of generality let $d \in 0^\infty C_1 \setminus 0^\infty C_2$. The Hausdorff distance between C_1 and C_2 can be expressed as

$$d_H(C_1, C_2) = \inf\{\varepsilon > 0 \mid C_1 \subseteq C_2 + B_\varepsilon(0), C_2 \subseteq C_1 + B_\varepsilon(0)\},$$

see e.g. [Gru07, Proposition 6.3]. Choose $\varepsilon > 0$ such that $C_1 \cap (C_2 + B_\varepsilon(0)) \neq \emptyset$ and fix an element x from this set. Then $x + \mu d \in C_1$ for all $\mu \geqslant 0$. The recession cone of $C_2 + B_\varepsilon(0)$ equals $0^\infty C_2$ according to [Roc70, Corollary 9.1.2]. Hence, there exists some μ_ε such that $x + \mu d \notin C_2 + B_\varepsilon(0)$ for all $\mu \geqslant \mu_\varepsilon$. This yields $d_H(C_1, C_2) \geqslant \varepsilon$. Letting $\varepsilon \to \infty$ completes the proof. □

From the proposition it follows that an ε-H-approximation of a closed convex set C must approximate $0^\infty C$ exactly. Thus, a set with nonpolyhedral recession cone is not within the scope of polyhedral approximability with respect to the Hausdorff distance. One may ask whether it is sufficient for a closed convex set C to have a polyhedral recession cone for an ε-H-approximation to exist. The answer to this question is negative. A characterization of the class of sets that admit polyhedral ε-H-approximations for every $\varepsilon > 0$ is due to Ney and Robinson [NR95].

Theorem 3.26 (cf. [NR95, Theorem 2.1])**.** *Let $C \subseteq \mathbb{R}^n$ be a closed set. Then the following are equivalent:*

(i) C is convex, $0^\infty C$ is polyhedral and $e[C, 0^\infty C] < \infty$,

(ii) there exists a polyhedral cone $D \subseteq \mathbb{R}^n$ such that for every $\varepsilon > 0$ there exists a finite set $V \subseteq \mathbb{R}^n$ such that $d_H(\operatorname{conv} V + D, C) \leqslant \varepsilon$.

Further, if (ii) holds, then $D = 0^\infty C$.

Since $0^\infty C \subseteq C - \{x\}$ for any $x \in C$, the quantity $e[0^\infty C, C]$ is always finite. Hence, the condition $e[C, 0^\infty C]$ in the theorem is equivalent to the Hausdorff distance between C and its recession cone being finite. The conditions in Theorem 3.26 constrain the class of sets that can be approximated by polyhedra significantly. Even simple sets are beyond approximation as is illustrated in Figure 3.5. The approximability of certain convex sets by polyhedra is also investigated by Ulus in [Ulu18] but in the context of convex vector optimization problems. In order to formulate the relevant result we need another definition.

Definition 3.27 (cf. [Ulu18])**.** A convex set $C \subseteq \mathbb{R}^n$ is said to be *self-bounded* if there exists $x \in \mathbb{R}^n$ such that $C \subseteq \{x\} + 0^\infty C$.

Ulus gives a sufficient criterion for a closed convex set to be polyhedrally approximable.

Proposition 3.28 (cf. [Ulu18, Proposition 3.7 and Remark 3.8]). *Let $C \subseteq \mathbb{R}^n$ be a closed convex set. If C is self-bounded, then for every $\varepsilon > 0$ there exists a finite set $V \subseteq \mathbb{R}^n$ such that $d_H(\operatorname{conv} V + 0^\infty C, C) \leqslant \varepsilon$.*

If in the proposition $0^\infty C$ is also a polyhedral cone, then, clearly, C can be approximated arbitrarily well by polyhedra. The difference to Theorem 3.26 is the self-boundedness of C instead of the finiteness of $e[C, 0^\infty C]$. The following result points out the connections between these conditions. It shows that a converse statement of Proposition 3.28 can only be obtained under an additional assumption. In this case, self-boundedness and the condition $e[C, 0^\infty C] < \infty$ coincide. We also establish the equivalence to a condition that is a slightly weaker alternative to self-boundedness.

Theorem 3.29. *Let $C \subseteq \mathbb{R}^n$ be a closed convex set. The following statements are equivalent:*

(i) $e[C, 0^\infty C] < \infty$,

(ii) there exists a bounded set M such that $C \subseteq M + 0^\infty C$.

If, additionally, $\operatorname{int} 0^\infty C \neq \emptyset$, *then (i) and (ii) are equivalent to*

(iii) C is self-bounded.

Proof. We begin with the assertion (i) $\Longrightarrow$ (ii). Denote $\kappa = e[C, 0^\infty C]$ and let $x \in C$. Then $\|x - \pi_{0^\infty C}(x)\| \leqslant \kappa$ and we have

$$x = (x - \pi_{0^\infty C}(x)) + \pi_{0^\infty C}(x) \in B_\kappa(0) + 0^\infty C.$$

Since $x \in C$ is arbitrary, this yields $C \subseteq B_\kappa(0) + 0^\infty C$.
To prove (ii) $\Longrightarrow$ (i) assume $C \subseteq M + 0^\infty C$ holds for some bounded set M. We compute

$$\begin{aligned} e[C, 0^\infty C] &\leqslant e[M + 0^\infty C, 0^\infty C] \\ &= \sup_{x \in M,\, d \in 0^\infty C} \|(x + d) - \pi_{0^\infty C}(x + d)\| \\ &\leqslant \sup_{x \in M,\, d \in 0^\infty C} \|(x + d) - d\| \\ &= \sup_{x \in M} \|x\| \\ &< \infty. \end{aligned}$$

The implication (iii) $\Longrightarrow$ (ii) holds trivially even if $\operatorname{int} 0^\infty C = \emptyset$. Hence, it suffices to show (i) $\Longrightarrow$ (iii) whenever $\operatorname{int} 0^\infty C \neq \emptyset$ to complete the proof. Assume (i) holds, i.e. $\kappa = e[C, 0^\infty C] < \infty$. We have already seen this implies

$$C \subseteq B_\kappa(0) + 0^\infty C. \tag{3.6}$$

Let $d \in \operatorname{int} 0^\infty C$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(d) \subseteq 0^\infty C$. It holds

$$B_\kappa\left(\frac{\kappa}{\varepsilon} d\right) = \frac{\kappa}{\varepsilon} B_\varepsilon(d) \subseteq \frac{\kappa}{\varepsilon} 0^\infty C = 0^\infty C.$$

The second equation is valid because $0^\infty C$ is a cone. After translation this yields the relation $B_\kappa(0) \subseteq \left\{-\frac{\kappa}{\varepsilon} d\right\} + 0^\infty C$. Moreover,

$$B_\kappa(0) + 0^\infty C \subseteq \left\{-\frac{\kappa}{\varepsilon} d\right\} + 0^\infty C + 0^\infty C = \left\{-\frac{\kappa}{\varepsilon} d\right\} + 0^\infty C.$$

Again, the equality holds because $0^\infty C$ is a cone. Finally, we invoke (3.6) and conclude that C is self-bounded. □

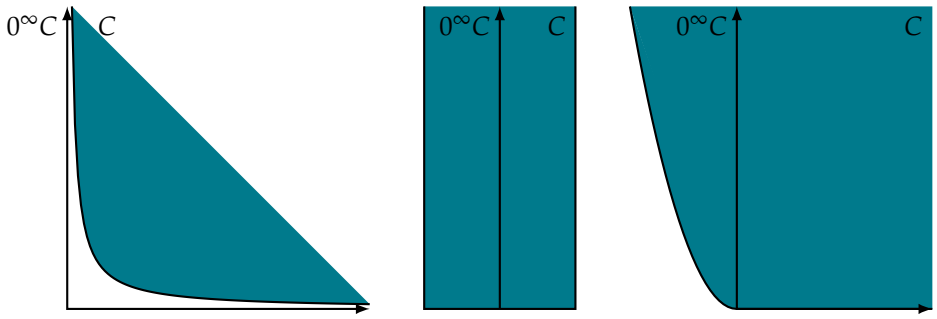

FIGURE 3.5: An illustration of the relationships in Theorem 3.29. The set on the left is contained in its own recession cone. Hence, it is self-bounded and admits polyhedral approximations. Note, that the set ext C of extreme points of C is unbounded, i.e. C cannot be written as the sum of a compact set and its recession cone. The central set is discussed in Example 3.30. The excess of C over its own recession cone is finite but C is not self-bounded. A set that is neither self-bounded nor satisfies $e\,[C, 0^\infty C] < \infty$ is shown on the right. By traversing the parabolic arc the distance to $0^\infty C$ increases without bound.

Example 3.30. A counterexample for the implication (i) $\implies$ (iii) is the set

$$C = \operatorname{conv}\left\{\begin{pmatrix}\pm 1\\ 0\end{pmatrix}\right\} + \operatorname{cone}\left\{\begin{pmatrix}0\\ 1\end{pmatrix}\right\}.$$

It holds $e\,[C, 0^\infty C] = \left\|(1,0)^\mathsf{T}\right\| = 1$ but, since $\operatorname{int} C \neq \emptyset$ and $\operatorname{int} 0^\infty C = \emptyset$, C is not self-bounded.

The relationships from Theorem 3.29 are also illustrated in Figure 3.5. In view of the result we suggest calling a convex set self-bounded if it satisfies Condition (ii) of the theorem. Firstly, this allows for a characterization of polyhedral approximability in terms of self-boundedness and, secondly, makes the notion compatible with the usual notion of boundedness. Note, that a bounded convex set is self-bounded in the sense of Definition 3.27 if and only if it is a singleton. We close this section with a summary of the results.

Corollary 3.31. *Let $C \subseteq \mathbb{R}^n$ be a closed set. Then the following are equivalent:*

(i) for every $\varepsilon > 0$ there exists an ε-H-approximation of C,

(ii) C is convex, $0^\infty C$ is polyhedral and $e\,[C, 0^\infty C] < \infty$,

(iii) C is convex, $0^\infty C$ is polyhedral and $C \subseteq M + 0^\infty C$ for a bounded set M.

Proof. The proof follows immediately from Theorems 3.26 and 3.29. □

Chapter 4

Polyhedral approximation of unbounded spectrahedral shadows

We have demonstrated in the previous chapter that the Hausdorff distance is a good measure of approximation quality for the polyhedral approximation of convex bodies but too restrictive for convex sets that are not necessarily bounded. In this final chapter we develop tools and algorithms that are compatible with unbounded sets in this regard. The results of this chapter are based on the joint work [DL22] with Andreas Löhne and the article [Dör22]. To the best of our knowledge this is the first time that polyhedral approximation of convex sets to such a degree of generality is considered and explicit algorithms are developed.

4.1 A notion of polyhedral approximation for pointed convex sets

The aim of this section is to find a notion of polyhedral approximation that is applicable to a class of convex sets as large as possible. Being required to compute the recession cone exactly whenever we want to approximate a convex set by a polyhedron in the Hausdorff distance is already too constraining for this purpose and this is even disregarding the additional condition we need according to Corollary 3.31. Hence, a way to quantify similarity between convex cones is required as a starting point.

Definition 4.1 (cf. [WW67])**.** Let $K_1, K_2 \subseteq \mathbb{R}^n$ be convex cones. The *truncated Hausdorff distance* between K_1 and K_2, denoted by $d_{\mathsf{tH}}(K_1, K_2)$, is defined as

$$d_{\mathsf{H}}(K_1 \cap B_1(0), K_2 \cap B_1(0)).$$

The truncated Hausdorff distance between the cones K_1 and K_2 considers the usual Hausdorff distance between K_1 and K_2 restricted to the unit ball. Since cones contain the origin, $d_{\mathsf{tH}}(K_1, K_2) \leqslant 1$ always holds. This is of course due to the choice of intersecting the sets with the unit ball, which is arbitrary to a certain degree. Indeed, one might as well intersect K_1 and K_2 with any convex body and define $d_{\mathsf{tH}}(K_1, K_2)$ accordingly without losing the following important property. The truncated Hausdorff distance provides a metric on the space of closed convex cones in $\mathbb{R}^n$, see e.g. [WW67]. While there are other ways for measuring the distance between closed convex cones which are discussed in the survey article [IS10] by Iusem and Seeger, we choose the truncated Hausdorff distance because it is derived from the Hausdorff distance in a natural way and we have used the latter in the preceding chapter already. With it we define a notion of polyhedral approximation for closed pointed convex sets.

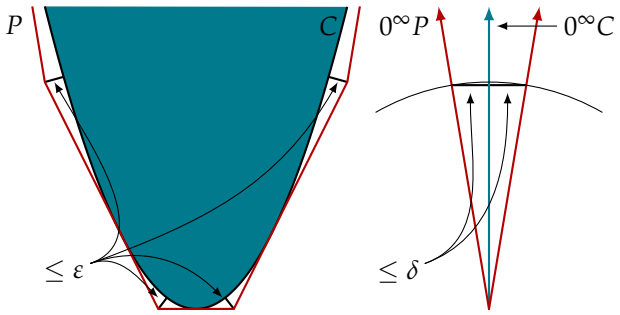

FIGURE 4.1: Illustration of Definition 4.2. Red polyhedron P on the left is an (ε, δ)-approximation of C. The distance of any vertex of P to the set C is at most ε. The recession cones of both sets are shown on the right with the arc indicating the unit ball. The truncated Hausdorff distance between them is bounded above by δ.

Definition 4.2. Let $C \subseteq \mathbb{R}^n$ be a closed pointed convex set and $\varepsilon, \delta \geqslant 0$. A pointed polyhedron P is called an (ε, δ)-*approximation* of C if the following hold:

(i) $e\left[\operatorname{vert} P, C\right] \leqslant \varepsilon$,

(ii) $d_{\mathsf{tH}}\left(0^\infty P, 0^\infty C\right) \leqslant \delta$,

(iii) $C \subseteq P$.

Figure 4.1 illustrates the definition. The notion of (ε, δ)-approximation specifies a concept of polyhedral outer approximation. If P is an (ε, δ)-approximation of a set C it means that every vertex of P is close to C in Euclidean distance and that the recession cones of P and C are close to one another in the sense of Definition 4.1. It is not straightforward to find a similar concept for inner approximations or without a containment criterion altogether. If Condition (iii) was dropped, then (i) and (ii) can be satisfied while P is an arbitrarily bad approximation of C. This, for example, is the case when C is a ball of radius $r > 0$ and P is comprised of just a single point from C. Then P would be a $(0, 0)$-approximation of C, which is certainly nonsensical. Hence, in order to expand the notion to not only outer approximations, it would be required to adapt the condition $e\left[\operatorname{vert} P, C\right] \leqslant \varepsilon$. If, for example, an inner approximation was sought, the roles of P and C would have to be interchanged in that condition resulting in $e\left[\operatorname{ext} C, P\right] \leqslant \varepsilon$ because the analogue of considering vertices of P is to consider extreme points of C. However, this leads to the same complications due to which the Hausdorff distance is insufficient in the unbounded setting. In particular, if $P \subseteq C$ and $e\left[C, 0^\infty C\right] = \infty$, i.e. there do not exist ε-H-approximations of C, then $e\left[\operatorname{ext} C, P\right] = \infty$ as well. This can easily be deduced from the triangle inequality. It holds

$$\infty = e\left[C, 0^\infty C\right] \leqslant e\left[C, 0^\infty P\right] \leqslant e\left[C, P\right] + e\left[P, 0^\infty P\right].$$

Corollary 3.31 implies that $e\left[P, 0^\infty P\right]$ is finite. Therefore, $e\left[C, P\right]$ is not finite, but that is identical to $e\left[\operatorname{ext} C, P\right] = \infty$, see [Roc70, Theorem 32.3].
A comment on the fact that only pointed sets are considered in Definition 4.2 is also expedient. The condition that P is pointed is equivalent to $\operatorname{vert} P \neq \emptyset$, see Corollary 1.17 for sufficiency, necessity is trivial. Hence, the pointedness of P is required for Condition (i) and, since $C \subseteq P$, it also applies to C. In fact, if C contains lines, Condition (i) can not be omitted. For example, if C is the halfspace $H^-(\omega, \gamma)$ and $P = H^-(\omega, \gamma + r)$, then Conditions (ii) and (iii) are satisfied for every $r \geqslant 0$, but the

distance between the boundaries of C and P can be arbitrarily large. However, C can be expressed as the direct sum

$$C = \left(C \cap L^{\perp}\right) + L,$$

where L is the lineality space of C, i.e. the linear subspace $0^{\infty}C \cap (-0^{\infty}C)$, and $L^{\perp}$ is its orthogonal complement, see [Roc70]. The set $C \cap L^{\perp}$ in this expression is pointed. Given C is a spectrahedral shadow, Algorithm 3.1 can be used to compute L or $L^{\perp}$ as it holds

$$L^{\perp} = \operatorname{aff} C^{\circ}$$

according to [Roc70, Theorem 14.6]. To that effect, it is no restriction to consider pointed sets in Definition 4.2 because one can always approximate the pointed part of a convex set C and add its lineality space to the result afterwards.

Remark 4.3. The distinction between ε and δ in the definition of (ε, δ)-approximation is not necessary but it is advisable in two regards. Firstly, these are absolute error measures and depending on the scale of the set C relative to $0^{\infty}C$ it is sensible to allow them to have differing values. Secondly, $d_{\mathsf{tH}}\left(0^{\infty}P, 0^{\infty}C\right)$ is bounded above by 1 by definition. Therefore, reasonable values for δ are smaller than 1, while ε is not restricted in this way.

The notion of (ε, δ)-approximation can be understood as an extension of the notion of ε-H-approximation in the sense that both coincide for outer approximations in the compact case.

Proposition 4.4. *Let $C \subseteq \mathbb{R}^n$ be a convex body and $\delta < 1$. Then the following are equivalent:*

(i) P is an (ε, δ)-approximation of C,

(ii) P is an ε-H-approximation of C and $C \subseteq P$.

Proof. Let P be an (ε, δ)-approximation of C for some $\delta < 1$. By definition it holds that $d_{\mathsf{tH}}\left(0^{\infty}P, 0^{\infty}C\right) < 1$. Moreover, $0^{\infty}C = \{0\}$ because C is bounded. This implies that P is bounded as well. Otherwise there existed some $d \in 0^{\infty}P$ with $\|d\| = 1$ and

$$d_{\mathsf{tH}}\left(0^{\infty}P, 0^{\infty}C\right) \geqslant d\left(d, 0^{\infty}C\right) = d\left(d, \{0\}\right) = 1.$$

This is a contradiction, hence $0^{\infty}P = \{0\}$. Furthermore it is true that

$$d_{\mathsf{H}}\left(P, C\right) = e\left[P, C\right] = e\left[\operatorname{vert} P, C\right] \leqslant \varepsilon.$$

The first equality is valid because $C \subseteq P$. The second one follows from the facts that $P = \operatorname{conv} \operatorname{vert} P$ according to Theorem 1.16 and compactness of P and that $e\left[P, C\right]$ is attained at a vertex of P, see [Bat86, Theorem 3.3]. Consequently, P is an ε-H-approximation of C.
Now, assume the second statement holds. Since $d_{\mathsf{H}}\left(P, C\right) \leqslant \varepsilon$, Proposition 3.25 implies that $0^{\infty}P = 0^{\infty}C$. Therefore, $d_{\mathsf{tH}}\left(0^{\infty}P, 0^{\infty}C\right) = 0$ and P is bounded. The fact that $e\left[\operatorname{vert} P, C\right] \leqslant d_{\mathsf{H}}\left(P, C\right) \leqslant \varepsilon$ implies that P is an $(\varepsilon, 0)$-approximation of C. In particular, P is also an (ε, δ)-approximation of C for every $\delta \geqslant 0$. □

In the remainder of this section we demonstrate that (ε, δ)-approximations provide a meaningful notion of polyhedral approximation. To this end the concept of Painlevé-Kuratowski convergence is used, a theory of convergence for sequences of

sets that applies to a broader class than Hausdorff convergence. The aim is to prove that a sequence of (ε, δ)-approximations of a given set C converges to C in the sense of Painlevé-Kuratowski if ε and δ diminish. We start out by providing necessary definitions, while utilizing the notation from [RW98], which is also the main reference for this section. Let $\mathcal{N}_\infty$ and $\mathcal{N}_\infty^\#$ denote the collections $\{N \subseteq \mathbb{N} \mid \mathbb{N} \setminus N \text{ is finite}\}$ and $\{N \subseteq \mathbb{N} \mid N \text{ is infinite}\}$ of subsets of $\mathbb{N}$, respectively. The set $\mathcal{N}_\infty$ can be regarded as the set of subsequences of $\mathbb{N}$ that contain all natural numbers larger than some $\bar{n} \in \mathbb{N}$ and $\mathcal{N}_\infty^\#$ as the set of all subsequences of $\mathbb{N}$. Obviously, $\mathcal{N}_\infty \subseteq \mathcal{N}_\infty^\#$.

Definition 4.5. For a sequence $\{M_k\}_{k\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$ the *outer limit* is the set

$$\left\{x \in \mathbb{R}^n \,\middle|\, \forall\, \varepsilon > 0,\ \exists\, N \in \mathcal{N}_\infty^\#,\ \forall\, k \in N\colon x \in M_k + B_\varepsilon(0)\right\},$$

denoted by $\limsup_{k\to\infty} M_k$, and the *inner limit* $\liminf_{k\to\infty} M_k$ is the set

$$\left\{x \in \mathbb{R}^n \,\middle|\, \forall\, \varepsilon > 0,\ \exists\, N \in \mathcal{N}_\infty,\ \forall\, k \in N\colon x \in M_k + B_\varepsilon(0)\right\}.$$

The outer limit of a sequence $\{M_k\}_{k\in\mathbb{N}}$ can be paraphrased as the set of all points for which every open neighbourhood intersects infinitely many elements M_k of the sequence. Similarly, the inner limit is the set for which the same is true but for all elements M_k beyond some $\bar{k} \in \mathbb{N}$. In particular, both sets are closed.

Definition 4.6. A sequence $\{M_k\}_{k\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$ is said to *converge to* a set $M \subseteq \mathbb{R}^n$ *in the sense of Painlevé-Kuratowski* or *PK-converge*, written $M_k \to M$, if

$$M = \limsup_{k\to\infty} M_k = \liminf_{k\to\infty} M_k.$$

Example 4.7 (cf. [RW98, p. 118])**.** As already mentioned, PK-convergence is more general than H-convergence. For example, consider the sequence $\{M_k\}_{k\in\mathbb{N}}$ defined by $M_k = \{x, y_k\}$ for points $x, y_k \in \mathbb{R}^n$ where $\|y_k\| \to \infty$. Then $M_k \to \{x\}$ but $d_\mathsf{H}(M_k, \{x\}) = \|y_k - x\| \to \infty$, i.e. $\{M_k\}_{k\in\mathbb{N}}$ PK-converges to $\{x\}$ but fails to converge with respect to the Hausdorff distance.

Although Painlevé-Kuratowski and Hausdorff convergence are different, they are closely related. In fact, if there exists a bounded set $B \subseteq \mathbb{R}^n$ such that $M_k, M \subseteq B$, then $\{M_k\}_{k\in\mathbb{N}}$ PK-converges if and only if it H-converges, see [RW98]. Moreover, PK-convergence can be characterized using the Hausdorff distance for sequences of convex sets.

Proposition 4.8 (cf. [Mos69, Lemma 1.1] and [SW79, Theorem 4])**.** *Let $\{C_k\}_{k\in\mathbb{N}}$ be a sequence of closed convex subsets of $\mathbb{R}^n$, $C \subseteq \mathbb{R}^n$ be a closed convex set and $x \in \mathbb{R}^n$. Then the following are equivalent:*

(i) $C_k \to C$,

(ii) there exists $\bar{r} > 0$ such that for all $r \geqslant \bar{r}$

$$\lim_{k\to\infty} d_\mathsf{H}\left(C_k \cap B_r(x), C \cap B_r(x)\right) = 0.$$

Essentially, for sequences of closed convex sets PK-convergence is equivalent to convergence in the Hausdorff distance of all truncations of sequence elements. Since we are working in a convex setting, Proposition 4.8 provides an accessible way to prove convergence. We need three auxiliary results before we can apply it to a sequence of (ε, δ)-approximations. The first one gives a bound on the Hausdorff

distance between a truncation of a closed pointed convex set C and a truncation of an (ε, δ)-approximation of C.

Proposition 4.9. *Let $C \subseteq \mathbb{R}^n$ be a closed pointed convex set and $P \subseteq \mathbb{R}^n$ be an (ε, δ)-approximation of C. Then for every $r \geqslant \varepsilon$ and $x \in \operatorname{conv} \operatorname{vert} P$ there exists $v \in \operatorname{conv} \operatorname{vert} P$ such that*

$$d_{\mathsf{H}}\left(P \cap B_r(x), C \cap B_r(x)\right) \leqslant 2\left(\varepsilon + \delta\left(r + \|x - v\|\right)\right).$$

Moreover, if the distance $d_{\mathsf{H}}\left(P \cap B_r(x), C \cap B_r(x)\right)$ is attained at $p^* \in P \cap B_r(x)$, then $v \in \{p^*\} - 0^\infty P$.

Proof. The distance function $d(y, C)$ is a convex function of y, see [Roc70]. Therefore, the condition $e[\operatorname{vert} P, C] \leqslant \varepsilon$ in Definition 4.2 implies

$$d(y, C) \leqslant \varepsilon \tag{4.1}$$

for every $y \in \operatorname{conv} \operatorname{vert} P$. Now, because P is an (ε, δ)-approximation of C, $r \geqslant \varepsilon$ and $x \in \operatorname{conv} \operatorname{vert} P$, the set $C \cap B_r(x)$ is nonempty. Thus, $d_{\mathsf{H}}\left(P \cap B_r(x), C \cap B_r(x)\right)$ is attained as $\|p^* - c^*\|$ for some $p^* \in P \cap B_r(x)$ and $c^* \in C \cap B_r(x)$, see e.g. [Bat86, Theorem 3.3]. For $\lambda \in [0,1]$ let $y(\lambda) = \lambda p^* + (1-\lambda)x$. We distinguish the two cases $p^* \in \operatorname{conv} \operatorname{vert} P$ and $p^* \notin \operatorname{conv} \operatorname{vert} P$. First, assume $p^* \in \operatorname{conv} \operatorname{vert} P$. Then $y(\lambda) \in \operatorname{conv} \operatorname{vert} P$ for every $\lambda \in [0,1]$, which yields the existence of $c_\lambda \in C$ with $\|y(\lambda) - c_\lambda\| \leqslant \varepsilon$ according to (4.1). If $\|p^* - x\| \leqslant \varepsilon$, then

$$\begin{aligned} d_{\mathsf{H}}\left(P \cap B_r(x), C \cap B_r(x)\right) = \|p^* - c^*\| &\leqslant \|p^* - c_0\| \\ &\leqslant \|p^* - x\| + \|x - c_0\| \\ &\leqslant 2\varepsilon. \end{aligned} \tag{4.2}$$

The first and third inequality hold because $\|x - c_0\| \leqslant \varepsilon \leqslant r$ and $x = y(0)$. In particular, $c_0 \in C \cap B_r(x)$. If $\|p^* - x\| > \varepsilon$, set $\lambda^* = 1 - \frac{\varepsilon}{\|p^* - x\|}$. A similar estimate yields

$$\begin{aligned} d_{\mathsf{H}}\left(P \cap B_r(x), C \cap B_r(x)\right) = \|p^* - c^*\| &\leqslant \|p^* - c_{\lambda^*}\| \\ &\leqslant \|p^* - y(\lambda^*)\| + \|y(\lambda^*) - c_{\lambda^*}\| \\ &\leqslant 2\varepsilon. \end{aligned} \tag{4.3}$$

To see that the first inequality is valid we need to show that $c_{\lambda^*} \in B_r(x)$. This is true because $\|y(\lambda^*) - x\| = \|p^* - x\| - \varepsilon \leqslant r - \varepsilon$ implies $B_\varepsilon(y(\lambda^*)) \subseteq B_r(x)$. Now, assume $p^* \notin \operatorname{conv} \operatorname{vert} P$. Then there exists $\bar{\lambda} \in (0,1)$ such that $y(\bar{\lambda}) \in \operatorname{conv} \operatorname{vert} P$ and $y(\lambda) \notin \operatorname{conv} \operatorname{vert} P$ for all $\lambda > \bar{\lambda}$. We define

$$v_\lambda = \operatorname{argmin}\left\{\left\|y(\bar{\lambda}) - v\right\| \,\middle|\, v \in (\{y(\lambda)\} - 0^\infty P) \cap \operatorname{conv} \operatorname{vert} P\right\}. \tag{4.4}$$

For $\lambda > \bar{\lambda}$ one can decompose $y(\lambda)$ as $y(\lambda) = v_\lambda + \mu_\lambda d_\lambda$ for some direction $d_\lambda \in 0^\infty P$ and $\mu_\lambda > 0$. Since P is an (ε, δ)-approximation of C, there exist $\bar{c}_\lambda \in C$ and $\bar{d}_\lambda \in 0^\infty C$ such that $\|v_\lambda - \bar{c}_\lambda\| \leqslant \varepsilon$ and $\left\|d_\lambda - \bar{d}_\lambda\right\| \leqslant \delta$. We compute

$$\left\|y(\lambda) - \left(\bar{c}_\lambda + \mu_\lambda \bar{d}_\lambda\right)\right\| \leqslant \|v_\lambda - \bar{c}_\lambda\| + \mu_\lambda \left\|d_\lambda - \bar{d}_\lambda\right\| \leqslant \varepsilon + \mu_\lambda \delta. \tag{4.5}$$

Moreover, it holds $\left\|y(\bar{\lambda}) - v_\lambda\right\| \leqslant \left\|y(\bar{\lambda}) - v_1\right\|$. This is seen by observing that (4.4) implies $p^* - v_1 \in 0^\infty P$ and noting that

$$\frac{1-\lambda}{1-\bar{\lambda}} y(\bar{\lambda}) + \left(1 - \frac{1-\lambda}{1-\bar{\lambda}}\right) v_1 = y(\lambda) - \frac{\lambda - \bar{\lambda}}{1-\bar{\lambda}}\left(p^* - v_1\right),$$

i.e. $y(\lambda) - \frac{\lambda-\bar{\lambda}}{1-\bar{\lambda}}(p^* - v_1)$ lies on the line segment connecting $y(\bar{\lambda})$ and v_1. Since both of these points belong to conv vert P, so does $y(\lambda) - \frac{\lambda-\bar{\lambda}}{1-\bar{\lambda}}(p^* - v_1)$. Hence, the claim follows from the definition of v_λ in (4.4). Considering $\|d_\lambda\| = 1$, this leads to the estimate

$$\begin{aligned} \mu_\lambda = \|y(\lambda) - v_\lambda\| &\leqslant \|y(\lambda) - y(\bar{\lambda})\| + \|y(\bar{\lambda}) - v_\lambda\| \\ &\leqslant \|y(\lambda) - y(\bar{\lambda})\| + \|y(\bar{\lambda}) - v_1\| \\ &\leqslant \|y(\lambda) - y(\bar{\lambda})\| + \|y(\bar{\lambda}) - x\| + \|x - v_1\| \\ &= \|y(\lambda) - x\| + \|x - v_1\| \\ &\leqslant r + \|x - v_1\|. \end{aligned} \tag{4.6}$$

If $\delta \geqslant \frac{r-\varepsilon}{r+\|x-v_1\|}$, then

$$\begin{aligned} d_{\mathsf{H}}(P \cap B_r(x), C \cap B_r(x)) = \|p^* - c^*\| &\leqslant \|p^* - c_0\| \\ &\leqslant \|p^* - x\| + \|x - c_0\| \\ &\leqslant r + \varepsilon \\ &\leqslant 2\varepsilon + \delta(r + \|x - v_1\|), \end{aligned} \tag{4.7}$$

where the first inequality holds because $c_0 \in C \cap B_r(0)$, which we have shown above. Otherwise, the last inequality is violated. In this case we consider the point $y(\lambda^*)$ for $\lambda^* = 1 - \frac{\varepsilon+\delta(r+\|x-v_1\|)}{\|p^*-x\|} \in [0,1)$. This choice yields

$$\{z \in \mathbb{R}^n \mid \|z - y(\lambda^*)\| \leqslant \varepsilon + \delta(r + \|x - v_1\|)\} \subseteq B_r(x) \tag{4.8}$$

because $\|y(\lambda^*) - x\| = \|p^* - x\| - (\varepsilon + \delta(r + \|x - v_1\|))$ and $\|p^* - x\| \leqslant r$. Now, if $\lambda^* \leqslant \bar{\lambda}$, then $y(\lambda^*) \in$ conv vert P and $\|y(\lambda^*) - c_{\lambda^*}\| \leqslant \varepsilon$. We conclude

$$\begin{aligned} d_{\mathsf{H}}(P \cap B_r(x), C \cap B_r(x)) = \|p^* - c^*\| &\leqslant \|p^* - c_{\lambda^*}\| \\ &\leqslant \|p^* - y(\lambda^*)\| + \|y(\lambda) - c_{\lambda^*}\| \\ &\leqslant 2\varepsilon + \delta(r + \|x - v_1\|), \end{aligned} \tag{4.9}$$

where $c_{\lambda^*} \in C \cap B_r(x)$ by (4.8). If $\lambda^* > \bar{\lambda}$, then, according to (4.5) and (4.6),

$$\begin{aligned} d_{\mathsf{H}}(P \cap B_r(x), C \cap B_r(x)) = \|p^* - c^*\| &\leqslant \|p^* - (\bar{c}_{\lambda^*} + \mu_{\lambda^*}\bar{d}_{\lambda^*})\| \\ &\leqslant \|p^* - y(\lambda^*)\| + \\ &\qquad \|y(\lambda^*) - (\bar{c}_{\lambda^*} + \mu_{\lambda^*}\bar{d}_{\lambda^*})\| \\ &\leqslant 2(\varepsilon + \delta(r + \|x - v_1\|)). \end{aligned} \tag{4.10}$$

Finally, we conclude

$$d_{\mathsf{H}}(P \cap B_r(x), C \cap B_r(x)) \leqslant 2(\varepsilon + \delta(r + \|x - v_1\|))$$

by taking into account Estimates (4.2), (4.3), (4.7), (4.9) and (4.10). The other statement of the proposition follows from (4.4), which yields $v_1 \in \{p^*\} - 0^\infty P$. □

The expression $\|x - v\|$ in the estimate can be bounded further by taking the supremum over $x, v \in$ conv vert P. The resulting quantity is called the diameter of the set conv vert P. This gives an upper bound on $d_{\mathsf{H}}(P \cap B_r(x), C \cap B_r(x))$ that does not depend on the point p^* at which the Hausdorff distance is attained. However, we need the stronger bound from Proposition 4.9 for the proof of Theorem 4.12 below.

Lemma 4.10. *Let $C \subseteq \mathbb{R}^n$ be a closed pointed convex set and $\{v_k\}_{k\in\mathbb{N}}$, $\{d_k\}_{k\in\mathbb{N}}$ be sequences such that $\lim_{k\to\infty} d(v_k, C) = 0$, $\lim_{k\to\infty} d(d_k, 0^\infty C) = 0$ and $\{v_k + d_k\}_{k\in\mathbb{N}}$ is bounded. Then $\{v_k\}_{k\in\mathbb{N}}$ is bounded.*

Proof. Assume that $\{v_k\}_{k\in\mathbb{N}}$ is unbounded. Then $\{d_k\}_{k\in\mathbb{N}}$ is unbounded as well since $\{v_k + d_k\}_{k\in\mathbb{N}}$ is bounded. Hence, $0^\infty C \neq \{0\}$ because $\lim_{k\to\infty} d(d_k, 0^\infty C) = 0$. Without loss of generality we can assume that $d_k \neq 0$ for all $k \in \mathbb{N}$ by retreating to a suitable subsequence. Then the sequence $\{r_k\}_{k\in\mathbb{N}}$ given by $r_k = d_k/\|d_k\|$ is well-defined and has a convergent subsequence because it is bounded. Without loss of generality, again, we can assume that $\{r_k\}_{k\in\mathbb{N}}$ is itself convergent to some $r \in 0^\infty C$, where we take $\lim_{k\to\infty} d(d_k, 0^\infty C) = 0$ into account. We will show that $-r \in 0^\infty C$. Therefore, let $c \in C$ and $\mu \geqslant 0$. Since $\{v_k + d_k\}_{k\in\mathbb{N}}$ is bounded, there exists some $\kappa \geqslant 0$ such that $\|v_k + d_k\| \leqslant \kappa$ for every $k \in \mathbb{N}$. Hence,

$$\|c - d_k - \pi_C(v_k)\| = \|c + v_k - \pi_C(v_k) - v_k - d_k\| \leqslant \|c\| + d(v_k, C) + \kappa \leqslant \bar{\kappa} \quad (4.11)$$

holds for some $\bar{\kappa} \in \mathbb{R}$. The last inequality is valid because $\lim_{k\to\infty} d(v_k, C) = 0$. Define

$$y_k = \frac{\mu}{\|d_k\|}\pi_C(v_k) + \left(1 - \frac{\mu}{\|d_k\|}\right) c \in C.$$

This yields

$$\begin{aligned} d(c - \mu r_k, C) &\leqslant \|(c - \mu r_k) - y_k\| \\ &= \left\| \frac{\mu}{\|d_k\|} c - \mu r_k - \frac{\mu}{\|d_k\|}\pi_C(v_k) \right\| \\ &= \frac{\mu}{\|d_k\|}\|c - d_k - \pi_C(v_k)\| \\ &\leqslant \frac{\mu}{\|d_k\|}\bar{\kappa} \end{aligned}$$

for every $k \in \mathbb{N}$. We apply (4.11) to get the inequality in the last line. By passing to the limit we see that $\lim_{k\to\infty} d(c - \mu r_k, C) = 0$ as $\{d_k\}_{k\in\mathbb{N}}$ is unbounded. Utilizing the convergence of $\{r_k\}_{k\in\mathbb{N}}$ and the fact that C is closed we obtain $c - \mu r \in C$ for all $\mu \geqslant 0$. Now, closedness of C also implies $-r \in 0^\infty C$. This contradicts the pointedness of C and we conclude that $\{v_k\}_{k\in\mathbb{N}}$ is bounded. □

The lemma can be interpreted as follows. For a closed pointed convex set C and a point $c \in C$ the set $(\{c\} - 0^\infty C) \cap C$ is compact. In particular, c can be written as $c = v + d$ for suitable $v \in C$ and $d \in 0^\infty C$. This decomposition need not be unique, but the set $\{v \in C \mid \exists\, d \in 0^\infty C\colon c = v + d\}$ is compact. Lemma 4.10 shows that this is also true in an approximate sense, i.e. if the point c is not fixed, but c, v and d are regarded as sequences converging to C and $0^\infty C$, respectively.

Lemma 4.11. *Let $C \subseteq \mathbb{R}^n$ be a closed pointed convex set and $\{P_k\}_{k\in\mathbb{N}}$ be a sequence of $(\varepsilon_k, \delta_k)$-approximations of C. If $\lim_{k\to\infty}(\varepsilon_k, \delta_k) = (0, 0)$, then for every $c \in \operatorname{ext} C$ there exists a sequence $\{v_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} v_k = c$ and $v_k \in \operatorname{conv}\operatorname{vert} P_k$.*

Proof. The set $\operatorname{ext} C$ is nonempty according to Corollary 1.17. Fix a point $c \in \operatorname{ext} C$. Assume that every sequence $\{v_k\}_{k\in\mathbb{N}}$ satisfying the property $v_k \in \operatorname{conv}\operatorname{vert} P_k$ does not converge to c. Then, for every such sequence, there exist $\kappa > 0$ and $N \in \mathcal{N}_\infty^\#$ such that $\|v_k - c\| \geqslant \kappa$ for every $k \in N$. Moreover, since $C \subseteq P_k$, there exist $v_k \in \operatorname{conv}\operatorname{vert} P_k$ and $d_k \in 0^\infty P_k$ such that $c = v_k + d_k$ for every $k \in \mathbb{N}$. Therefore, we can fix a sequence $\{\bar{v}_k\}_{k\in\mathbb{N}}$ with $\bar{v}_k \in \operatorname{conv}\operatorname{vert} P_k$ and $\|\bar{v}_k - c\| \geqslant \kappa > 0$ and a corresponding sequence $\{\bar{d}_k\}_{k\in\mathbb{N}}$ such that $c = \bar{v}_k + \bar{d}_k$ for every $k \in \mathbb{N}$. It is true that

$\lim_{k\to\infty} d(\bar{v}_k, C) = \lim_{k\to\infty} d(\bar{d}_k, 0^\infty C) = 0$ because P_k is an $(\varepsilon_k, \delta_k)$-approximation of C and $\lim_{k\to\infty}(\varepsilon_k, \delta_k) = (0,0)$. Now, according to Lemma 4.10 the sequences $\{\bar{v}_k\}_{k\in\mathbb{N}}$ and $\{\bar{d}_k\}_{k\in\mathbb{N}}$ are bounded. Hence, they have convergent subsequences. Without loss of generality we assume that $\lim_{k\to\infty} \bar{v}_k = \bar{v}$ and $\lim_{k\to\infty} \bar{d}_k = \bar{d}$. It holds $\bar{v} \in C$ and $\bar{d} \in 0^\infty C$ because C is closed. Note that $\bar{d} \neq 0$. This is evident from the fact that $\|\bar{d}_k\| = \|\bar{v}_k - c\| \geqslant \kappa > 0$ for all $k \in \mathbb{N}$. Finally, we conclude

$$c = \bar{v} + \bar{d} = \frac{1}{2}\left(\bar{v} + 2\bar{d}\right) + \frac{1}{2}\bar{v}.$$

This contradicts $c \in \operatorname{ext} C$. □

From a geometric perspective the statement of Lemma 4.11 is intuitive. Given an (ε, δ)-approximation P of C, every extreme point of C will eventually be approximated well by points from the set $\operatorname{conv}\operatorname{vert} P$ if ε and δ are sufficiently small.
We now have all the tools needed to prove the main result of this section.

Theorem 4.12. *Let $C \subseteq \mathbb{R}^n$ be a closed pointed convex set and $\{P_k\}_{k\in\mathbb{N}}$ be a sequence of $(\varepsilon_k, \delta_k)$-approximations of C. If $\lim_{k\to\infty}(\varepsilon_k, \delta_k) = (0,0)$, then $P_k \to C$.*

Proof. We can apply Proposition 4.8 and show that there exist $\bar{r} > 0$ and $c \in \mathbb{R}^n$ such that for all $r \geqslant \bar{r}$ it holds

$$\lim_{k\to\infty} d_{\mathsf{H}}\left(P_k \cap B_r(c), C \cap B_r(c)\right) = 0. \tag{4.12}$$

Choose $r \geqslant \max\{\varepsilon_k \mid k \in \mathbb{N}\}$ and $c \in \operatorname{ext} C$. Such a point exists due to Corollary 1.17. According to Lemma 4.11 there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_k = c$ and $x_k \in \operatorname{conv}\operatorname{vert} P_k$. Using the triangle inequality and Proposition 4.9 yields

$$\begin{aligned} d_{\mathsf{H}}\left(P_k \cap B_r(c), C \cap B_r(c)\right) &\leqslant d_{\mathsf{H}}\left(P_k \cap B_r(c), P_k \cap B_r(x_k)\right) + \\ &\quad d_{\mathsf{H}}\left(P_k \cap B_r(x_k), C \cap B_r(x_k)\right) + \\ &\quad d_{\mathsf{H}}\left(C \cap B_r(x_k), C \cap B_r(c)\right) \\ &\leqslant d_{\mathsf{H}}\left(P_k \cap B_r(c), P_k \cap B_r(x_k)\right) + \\ &\quad 2\left(\varepsilon_k + \delta_k\left(r + \|x_k - v_k\|\right)\right) + \\ &\quad d_{\mathsf{H}}\left(C \cap B_r(x_k), C \cap B_r(c)\right), \end{aligned} \tag{4.13}$$

for some $v_k \in \operatorname{conv}\operatorname{vert} P_k$. The first and third term in the last sum tend to zero as $\{x_k\}_{k\in\mathbb{N}}$ converges to c. Thus, it suffices to show that $\|x_k - v_k\|$ is bounded from above. According to Proposition 4.9 there exists $d_k \in 0^\infty P_k$ such that $p_k = v_k + d_k$ where $p_k \in P_k \cap B_r(x_k)$ is the point at which $d_{\mathsf{H}}\left(P_k \cap B_r(x_k), C \cap B_r(x_k)\right)$ is attained. We compute

$$\begin{aligned} \|v_k + d_k - c\| = \|p_k - c\| &\leqslant \|p_k - x_k\| + \|x_k - c\| \\ &\leqslant r + \max\{\|x_k - c\| \mid k \in \mathbb{N}\} \\ &< \infty. \end{aligned}$$

The last inequality is valid because $\{x_k\}_{k\in\mathbb{N}}$ converges to c. Hence, $\{v_k + d_k\}_{k\in\mathbb{N}}$ is bounded. Since P_k is an $(\varepsilon_k, \delta_k)$-approximation of C and $\lim_{k\to\infty}(\varepsilon_k, \delta_k) = (0,0)$, it holds $\lim_{k\to\infty} d(v_k, C) = \lim_{k\to\infty} d(d_k, 0^\infty C) = 0$. Now, we can apply Lemma 4.10 and deduce that $\{v_k\}_{k\in\mathbb{N}}$ is bounded. Therefore, $\{x_k - v_k\}_{k\in\mathbb{N}}$ is also bounded and Estimate (4.13) implies that Equation (4.12) holds for all $r \geqslant \max\{\varepsilon_k \mid k \in \mathbb{N}\}$ and $c \in \operatorname{ext} C$. □

4.2 Polyhedral approximation of recession cones

Before we present an algorithm for the computation of (ε, δ)-approximations of spectrahedral shadows in Section 4.3, we focus on the second condition in Definition 4.2. We have seen how one can represent the closure of the recession cone of a spectrahedral shadow S in Theorem 2.44. Given that $0^\infty S$ has nonempty interior, we could apply Algorithms 3.2 and 3.3 to a suitable subset of $0^\infty S$ to extract polyhedral outer and inner approximations of the recession cone in theory. However, the representation of $\operatorname{cl} 0^\infty S$ is too complicated to be used efficiently in practice. In view of (ε, δ)-approximations we are also only interested in a polyhedral outer approximation of $\operatorname{cl} 0^\infty S$. In this section we present an algorithm for computing such an approximation that does not rely on $\operatorname{cl} 0^\infty S$ but uses the spectrahedral shadow S directly to infer information about its recession cone. Unlike Algorithms 3.2 and 3.3, which compute either an outer or an inner approximation of a given set S, Algorithm 4.1 below computes an outer and inner approximation simultaneously. More precisely, it computes polyhedral cones K_O and K_I such that for a given spectrahedral shadow S and error tolerance δ it holds $K_\mathsf{I} \subseteq 0^\infty S \subseteq K_\mathsf{O}$, $d_{\mathsf{tH}}(K_\mathsf{O}, 0^\infty S) \leqslant \delta$ and $d_{\mathsf{tH}}(K_\mathsf{I}, 0^\infty S) \leqslant \delta$. A similar algorithm has recently and independently been developed in [Wag+22] but in the framework of convex vector optimization problems.
Throughout this section we make the following assumptions about the spectrahedral shadow $S \subseteq \mathbb{R}^n$ that is given according to Definition 2.10:

(A1) S is closed,

(A2) a point $p \in \mathbb{R}^n$ is known for which there exist $y_p \in \mathbb{R}^m$ and $Z_p \in \mathcal{S}^k_{++}$ such that $A_0 + \mathcal{A}(p) + \bar{\mathcal{A}}(y_p) + \boldsymbol{\mathcal{A}}(Z_p) \succ 0$ and $Bp + \bar{B}y_p + \mathcal{B}^\mathsf{T}(Z_p) = b$,

(A3) a direction $\bar{d} \in \operatorname{int} 0^\infty S$ is known.

Assumption (A1) is already discussed in the previous chapter as part of the cutting and augmenting schemes. The second assumption is equivalent to strict feasibility of a semidefinite program with S as feasible region. Algorithm 4.1 employs the primal-dual pair $(\mathrm{P}_2(p, d, S))$ and $(\mathrm{D}_2(p, d, S))$ and Assumption (A2) ensures the existence of solutions to both problems if the primal is bounded. The last assumption is reminiscent of the assumption that the origin is an interior point in Algorithms 3.2 and 3.3. It is substantial in proving convergence of the algorithm. In order to find a point $\bar{d} \in \operatorname{int} 0^\infty S$ one can apply Algorithm 3.1 to $0^\infty S \cap B_1(0)$ using the representation from Theorem 2.44. This requires solving $2n$ problems of type $(\mathrm{P}_1(\omega, 0^\infty S \cap B_1(0)))$. It remains open whether Assumption (A3) can be relaxed. If $0^\infty S$ has empty interior and $\bar{d}$ is a point from its relative interior, then the computations can be reduced to the affine hull of $0^\infty S$, which is also computed by Algorithm 3.1. However, it would be desiring to investigate if the algorithm still converges if $\bar{d}$ is merely a direction of recession of S. We were unable to find an example where this was not the case. Pseudo code for the method is presented in Algorithm 4.1.

The algorithm begins with a preprocessing step in which problem $(\mathrm{P}_2(p, -\bar{d}, S))$ is investigated for boundedness. If the problem is unbounded, then $-\bar{d} \in 0^\infty S$. Together with the assumption that $\bar{d} \in \operatorname{int} 0^\infty S$ this implies $0 \in \operatorname{int} 0^\infty S$, see [Roc70, Theorem 6.1]. In this case S is the whole space and the algorithm is terminated with $K_\mathsf{O} = K_\mathsf{I} = \mathbb{R}^n$. Otherwise, a solution $(V_{-\bar{d}}, v_{-\bar{d}}, w_{-\bar{d}})$ to $(\mathrm{D}_2(p, -\bar{d}, S))$ gives rise to a supporting hyperplane of S with normal vector $w_{-\bar{d}}$ according to Proposition 3.11. Initial approximations are then defined in lines 7 and 8 as $K_\mathsf{O} = H^-(w_{-\bar{d}}, 0)$ and $K_\mathsf{I} = \operatorname{cone}\{\bar{d}\}$, respectively. The outer and inner approximation are iteratively

ALGORITHM 4.1: Approximation algorithm for recession cones of spectrahedral shadows

Input: a closed spectrahedral shadow $S \subseteq \mathbb{R}^n$, point p satisfying (A2), direction $\bar{d}$ satisfying (A3), error tolerance $\delta > 0$

Output: outer and inner polyhedral approximation K_O and K_I of $0^\infty S$ with $d_{\mathrm{tH}}(K_O, K_I) \leqslant \delta$

1 if $(P_2(p, -\bar{d}, S))$ *is* unbounded then
2 $K_O \leftarrow \mathbb{R}^n$
3 $K_I \leftarrow \mathbb{R}^n$
4 return
5 else
6 compute a solution $(V_{-\bar{d}}, v_{-\bar{d}}, w_{-\bar{d}})$ to $(\mathrm{D}_2(p, -\bar{d}, S))$
7 $K_O \leftarrow H^-(w_{-\bar{d}}, 0)$
8 $K_I \leftarrow \operatorname{cone}\{\bar{d}\}$
9 end
10 repeat
11 stop $\leftarrow$ true
12 for $r \in \operatorname{vert}\{x \in K_O \mid \|x\|_\infty \leqslant 1\} \setminus \{0\}$ do
13 for $k = 1$ to $\left\lceil \log_2\left(\frac{\|r-\bar{d}\|}{\delta}\right)\right\rceil$ do
14 $d \leftarrow \frac{2^k-1}{2^k} r + \frac{1}{2^k}\bar{d}$
15 if $(P_2(p, d, S))$ *is* unbounded then
16 $K_I \leftarrow \operatorname{cone}(K_I \cup \{d\})$
17 else
18 compute a solution (V_d, v_d, w_d) to $(\mathrm{D}_2(p, d, S))$
19 $K_O \leftarrow K_O \cap H^-(w_d, 0)$
20 stop $\leftarrow$ false
21 break
22 end
23 end
24 end
25 until stop

improved in the main loop in lines 10–25. The general approach is to compute a direction $d \in \mathbb{R}^n$ and determine whether $(\mathrm{P}_2(p, d, S))$ is bounded. If it is unbounded, then Proposition 3.11 yields $d \in 0^\infty S$ and the current inner approximation K_I is updated by appending d to its V-representation in line 16. This yields the new inner approximation

$$\operatorname{cone}(K_I \cup \{d\}),$$

which is again contained in $0^\infty S$. Otherwise, the part w_d of a solution to $(\mathrm{D}_2(p, d, S))$ is the normal vector of a supporting halfspace of S. Translating this halfspace such that the origin is contained on its boundary results in a halfspace supporting $0^\infty S$. In particular, if $H^-(w_d, \gamma)$ supports S, then $H^-(w_d, 0)$ supports $0^\infty S$. Note that $w_d^\mathsf{T} d = 1$. Hence, the direction d is cut off of the current outer approximation K_O by intersecting it with $H^-(w_d, 0)$ in line 19.

The search directions d are chosen according to the following rule. First, a V-representation of the polytope

$$K_O \cap \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant 1\}$$

is computed in line 12. This set characterizes K_O in the sense that taking its conical hull yields K_O again. In particular, $r \in \operatorname{extdir} K_O$ implies that $r/\|r\|_\infty$ is a vertex of $K_O \cap \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant 1\}$. Next, for every nonzero vertex r of this set a bisection is performed on the line segment connecting r and the direction $\bar{d}$ to obtain new search directions d that are increasingly closer to r. The bisection is terminated either if the outer approximation is updated, in which case r is cut off of it, or if the current search direction is within a distance less than δ of r. Finally, the algorithm terminates if K_O is not updated during one iteration of the loop in lines 10–25. In this case there exists a direction $d \in K_I$ with $\|r - d\| \leqslant \delta$ for every point r computed in line 12. This gives an upper bound of δ on the truncated Hausdorff distance $d_{tH}(K_O, K_I)$ and thus on the truncated Hausdorff distances between $0^\infty S$ and the approximations. One iteration of the algorithm is illustrated in Figure 4.2.

Remark 4.13. Instead of investigating $(P_2(p, d, S))$ for a given search direction d in line 15 right away it can be beneficial to test whether $d \in K_I$ first. Since K_I is known by a V-representation during execution of the algorithm, membership of d in K_I can be verified by solving the linear feasibility problem

$$\max 0^T \mu \quad \text{s.t.} \quad D\mu = d, \quad \mu \geqslant 0,$$

where D is such that $K_I = \{D\mu \mid \mu \geqslant 0\}$. If this problem is feasible, then the possibly more costly problem $(P_2(p, d, S))$ does not need to be investigated and the next search direction can be considered immediately.
Another possibility is to first determine the point

$$d^* = \lambda^* r + (1 - \lambda^*)\bar{d},$$

where $\lambda^* = \operatorname{argmax}\{\lambda \mid \lambda r + (1 - \lambda)\bar{d} \in K_I\}$, on the boundary of K_I by solving a linear program and performing the bisection procedure on the line segment connecting r and d^*. The corresponding search directions are then guaranteed to lead to an update of one of the approximations in line 16 or 19. This approach requires solving one additional linear program for every point r computed in line 12 but can be beneficial if multiple search directions d in line 14 are already contained in the inner approximation. We have not incorporated these ideas into Algorithm 4.1 in order to keep it simple but want to mention them as a possibility.

Theorem 4.14. *Algorithm 4.1 works correctly, in particular it terminates with a polyhedral outer approximation K_O and a polyhedral inner approximation K_I of $0^\infty S$ satisfying $d_{tH}(K_O, 0^\infty S) \leqslant \delta$ and $d_{tH}(K_I, 0^\infty S) \leqslant \delta$.*

Proof. Assume $(P_2(p, -\bar{d}, S))$ in line 1 is unbounded. Then $-\bar{d} \in 0^\infty S$ according to Proposition 3.11. Since $\bar{d} \in \operatorname{int} 0^\infty S$ by Assumption (A3), this implies $0 \in \operatorname{int} 0^\infty S$, see [Roc70, Theorem 6.1]. Thus, $0^\infty S = S = \mathbb{R}^n$ and Algorithm 4.1 terminates in line 4 with $K_O = K_I = \mathbb{R}^n$. Now, assume that $(P_2(p, -\bar{d}, S))$ is bounded. Due to Assumption (A2), the point $(x, y, Z, t) = (p, y_p, Z_p, 0)$ is strictly feasible for $(P_2(p, -\bar{d}, S))$ and strong duality holds according to Theorem 1.30. In particular, a dual solution $(V_{-\bar{d}}, v_{-\bar{d}}, w_{-\bar{d}})$ in line 6 exists. From Proposition 3.11 it follows that the halfspace $H^-(w_{-\bar{d}}, w_{-\bar{d}}^T p + t_{-\bar{d}})$, where $t_{-\bar{d}}$ is the optimal value of $(P_2(p, -\bar{d}, S))$, supports S. Then $H^-(w_{-\bar{d}}, 0)$ supports $0^\infty S$ and the cone K_O initialized in line 7 contains $0^\infty S$. The initial inner approximation K_I in line 8 is contained in $0^\infty S$ by Assumption (A3). Let r be an element of the set defined in line 12. Such a point exists because $K_O \neq \emptyset$.

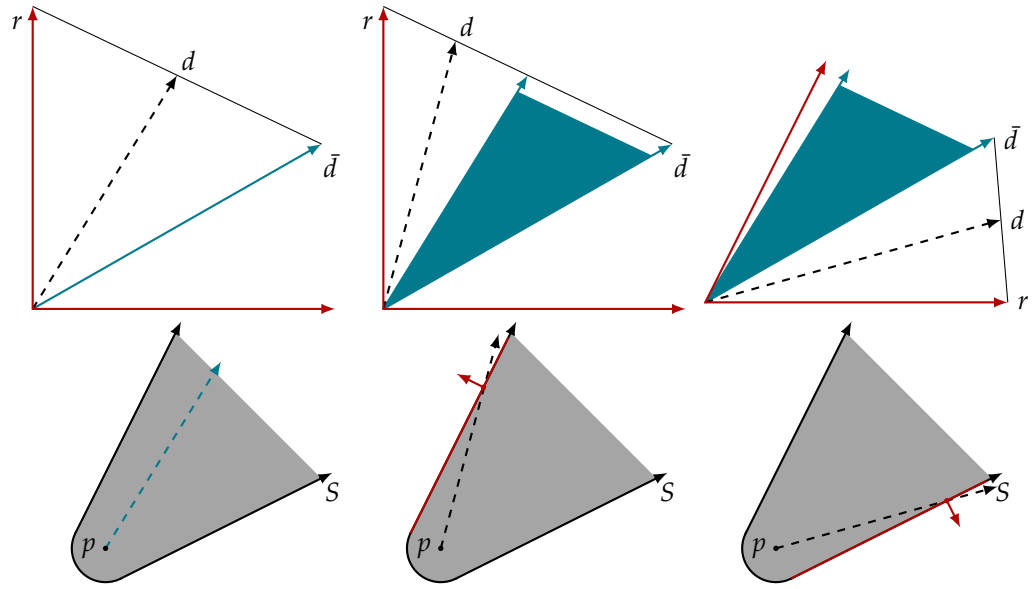

FIGURE 4.2: One iteration of the loop in lines 12–24 of Algorithm 4.1. The top row illustrates how the current outer and inner approximation, which are indicated in red and blue, respectively, are updated during the process and how new search directions d are computed. The spectrahedral shadow S and its interior point p are depicted in the bottom row. The dashed arrows indicate the problem $(\mathrm{P}_2(p,d,S))$. In the leftmost column the direction is chosen as the midpoint on the line segment connecting r and $\bar{d}$. The corresponding problem is unbounded as seen from the blue ray originating from p. Hence, direction d is added to the inner approximation K_I and the new search direction is determined as the midpoint between r and d. This is shown in the top center. The new search direction is not a direction of recession of S and the red hyperplane in the bottom center is obtained from a solution to $(\mathrm{D}_2(p,d,S))$. Therefore, K_O is updated and the loop in lines 13–23 is exited. In the right column another point $r \in K_\mathsf{O}$ is chosen and a new bisection begins.

Also, the set is a polytope because $\{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant 1\} = \{x \in \mathbb{R}^n \mid -e \leqslant x \leqslant e\}$. Without loss of generality we can assume that the loop in lines 13-23 is executed at least once for r. Otherwise,

$$\log_2\left(\frac{\|r-\bar{d}\|}{\delta}\right) \leqslant 0 \tag{4.14}$$

holds for all $r \in \operatorname{vert}\{x \in K_\mathsf{O} \mid \|x\|_\infty \leqslant 1\}$. This implies

$$\begin{aligned} d_{\mathsf{tH}}(K_\mathsf{O}, K_\mathsf{I}) &= e[K_\mathsf{O} \cap B_1(0), K_\mathsf{I}] \\ &\leqslant e[K_\mathsf{O} \cap \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant 1\}, K_\mathsf{I}] \\ &= \max\{d(r, K_\mathsf{I}) \mid r \in \operatorname{vert}(K_\mathsf{O} \cap \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant 1\})\} \\ &\leqslant \max\{\|r-\bar{d}\| \mid r \in \operatorname{vert}(K_\mathsf{O} \cap \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant 1\})\} \\ &\leqslant \delta. \end{aligned} \tag{4.15}$$

The first equality holds true because $K_\mathsf{I} \subseteq K_\mathsf{O}$ and the projection onto a convex cone is a norm-reducing operation, i.e. $\|\pi_{K_\mathsf{I}}(x)\| \leqslant \|x\|$ holds for all $x \in \mathbb{R}^n$, cf. [Mor62].

The inequality in the second line simply resembles the fact that $B_1(0)$ is contained in $\{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant 1\}$. Moreover, the excess in the second line is attained at a vertex of $K_\mathsf{O} \cap \{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant 1\}$ according to [Bat86, Theorem 3.3]. Hence, if Inequality (4.14) is satisfied, the termination criterion is fulfilled.
Next, note that the search direction d defined in line 14 can be the zero vector only if $r = -\bar{d}$. By Assumption (A3) the initial outer approximation K_O is a halfspace containing $\bar{d}$ in its interior and the origin on its boundary. Therefore, $-\bar{d}$ is not contained in the initial outer approximation or in any subsequent outer approximation computed in line 19. In particular, $d \neq 0$ holds throughout the algorithm.
If $(\mathrm{P}_2(p, d, S))$ in line 15 is unbounded, then Proposition 3.11 and the closedness of S imply $d \in 0^\infty S$. Thus, the updated inner approximation K_I in line 16 satisfies $K_\mathsf{I} \subseteq 0^\infty S$. On the other hand, if $(\mathrm{P}_2(p, d, S))$ is bounded, then a solution (V_d, v_d, w_d) to the dual exists by the same reasoning as for the problem $(\mathrm{P}_2(p, -\bar{d}, S))$ above. According to Proposition 3.11 the halfspace $H^-(w_d, w_d^\mathsf{T} p + t_d)$ where t_d is the optimal value of $(\mathrm{P}_2(p, d, S))$ supports S. Therefore, $H^-(w_d, 0)$ supports $0^\infty S$ and the updated outer approximation K_O in line 19 remains a superset of $0^\infty S$ as well as a cone.
Now, assume the algorithm terminates with outer approximation K_O and inner approximation K_I. Termination occurs if and only if lines 18–21 are not executed, i.e. K_O is not updated, during one iteration of the repeat loop in lines 10–25. This is the case if and only if for every $r \in \operatorname{vert}\{x \in K_\mathsf{O} \mid \|x\|_\infty \leqslant 1\}$ the point

$$d_r = \frac{2^\kappa - 1}{2^\kappa} r + \frac{1}{2^\kappa} \bar{d}$$

with $\kappa = \left\lceil \log_2 \left(\frac{\|r - \bar{d}\|}{\delta} \right) \right\rceil$ satisfies $d_r \in K_\mathsf{I}$. Note that the definition of d_r yields $\|r - d_r\| \leqslant \delta$. Using the same derivation as in (4.15) but replacing $\bar{d}$ with d_r in the penultimate line and using $\|r - d_r\| \leqslant \delta$ gives

$$d_{\mathsf{tH}}(K_\mathsf{O}, K_\mathsf{I}) \leqslant \delta.$$

Since $K_\mathsf{I} \subseteq 0^\infty S \subseteq K_\mathsf{O}$, this yields $d_{\mathsf{tH}}(K_\mathsf{O}, 0^\infty S) \leqslant \delta$ and $d_{\mathsf{tH}}(K_\mathsf{I}, 0^\infty S) \leqslant \delta$.
It remains to show that the algorithm terminates after finitely many iterations. Assume that the sets K_O and K_I have been computed and that the algorithm does not halt after the next iteration of the main loop in lines 10–25. Then there exist $r \in \operatorname{vert}\{x \in K_\mathsf{O} \mid \|x\|_\infty \leqslant 1\}$ and $1 \leqslant k \leqslant \left\lceil \log_2 \left(\frac{\|r - \bar{d}\|}{\delta} \right) \right\rceil$ for which lines 18–21 are executed, i.e. for which K_O is modified. Let $H(w, 0)$ be the hyperplane corresponding to the halfspace used to update K_O in line 19. Since $\bar{d} \in \operatorname{int} 0^\infty S$, there exists $\varepsilon > 0$ such that $B_\varepsilon(\bar{d}) \subseteq 0^\infty S$. Moreover, as $H^-(w, 0)$ supports $0^\infty S$, it holds

$$d\left(\bar{d}, H(w, 0)\right) = \frac{|w^\mathsf{T} \bar{d}|}{\|w\|} = -\frac{w^\mathsf{T} \bar{d}}{\|w\|} \geqslant \varepsilon.$$

Taking into account the definition of d in line 14 and the fact that $w^\mathsf{T} d = 1$ we can further conclude

$$\begin{aligned} d(r, H(w, 0)) = \frac{|w^\mathsf{T} r|}{\|w\|} &= \frac{1}{\|w\|} \left| w^\mathsf{T} \left(\frac{2^k}{2^k - 1} d - \frac{1}{2^k - 1} \bar{d} \right) \right| \\ &\geqslant \frac{2^k}{(2^k - 1)\|w\|} + \frac{\varepsilon}{2^k - 1} \end{aligned}$$

$$\begin{aligned} &\geqslant \frac{\varepsilon}{2^k - 1} \\ &\geqslant \frac{\varepsilon}{2\left(\left\|r - \bar{d}\right\| / \delta\right) - 1} \\ &> \frac{\delta\varepsilon}{4\sqrt{n} - \delta}. \end{aligned}$$

The third inequality follows from $k \leqslant \log_2\left(\frac{\left\|r-\bar{d}\right\|}{\delta}\right) + 1$ and the last inequality holds because $k \geqslant 1$, i.e. $\delta < \left\|r - \bar{d}\right\|$, and $\left\|r - \bar{d}\right\| < 2\sqrt{n}$. The latter is evident from $\|r\|_\infty \leqslant 1$ and $\left\|\bar{d}\right\| = 1$. Hence, whenever the outer approximation K_O is refined, a ball of radius at least $\delta\varepsilon/(4\sqrt{n} - \delta)$ around the point r is cut off. But since r belongs to the compact set $\{x \in \mathbb{R}^n \mid \|x\|_\infty \leqslant 1\}$, K_O can only be updated a finite number of times. □

We have seen in the last chapter that applying Algorithm 3.2 or 3.3 to the spectrahedral shadow S also yields a polyhedral inner or outer approximation of its polar set S°. Similar considerations can be done for the polar cones K_O^* and K_I^* of the sets computed by Algorithm 4.1. According to Proposition 2.40 one has the relation

$$K_\mathsf{O}^* \subseteq \operatorname{cl}\operatorname{cone} S^\circ \subseteq K_\mathsf{I}^*,$$

i.e. from an approximation of the recession cone of a spectrahedral shadow S we obtain an approximation of the closure of the cone generated by S°. It turns out that we are even able to find bounds on the approximation error of the polar cones. This is in contrast to the compact case. The following theorem establishes a connections between the truncated Hausdorff distance between two closed convex cones and the truncated Hausdorff distance between their polars. It is known as the *Walkup-Wets Isometry Theorem*.

Theorem 4.15 ([WW67, Theorem 1]). *Let $K_1, K_2 \subseteq \mathbb{R}^n$ be closed convex cones. Then*

$$d_\mathsf{tH}(K_1, K_2) = d_\mathsf{tH}(K_1^*, K_2^*).$$

Therefore, if Algorithm 4.1 returns the sets K_O and K_I, it holds

$$d_\mathsf{tH}(K_\mathsf{O}^*, \operatorname{cl}\operatorname{cone} S^\circ) \leqslant \delta \quad \text{and} \quad d_\mathsf{tH}(K_\mathsf{I}^*, \operatorname{cl}\operatorname{cone} S^\circ) \leqslant \delta.$$

Example 4.16. Consider the set $S = S_1 + S_2$ where $S_1 = \left\{x \in \mathbb{R}^4_+ \,\middle|\, x_1x_2x_3x_4 \geqslant 1\right\}$ and $S_2 = \left\{x \in \mathbb{R}^4 \,\middle|\, 2x_1x_2 \geqslant x_3^2 + x_4^2, x_1, x_2 \geqslant 0\right\}$. Set S is the Minkowski sum of a 4-dimensional hyperboloid and a rotated Lorentz cone. It holds that S_1 is the spectrahedral shadow defined by the LMIs

$$\begin{pmatrix} x_1 & y_1 \\ y_1 & x_2 \end{pmatrix} \succcurlyeq 0, \quad \begin{pmatrix} x_3 & y_2 \\ y_2 & x_4 \end{pmatrix} \succcurlyeq 0 \quad \text{and} \quad \begin{pmatrix} y_1 & 1 \\ 1 & y_2 \end{pmatrix} \succcurlyeq 0$$

for some $y \in \mathbb{R}^2$, cf. [BPT13, Section 6.3]. Furthermore, S_2 is the image of the spectrahedron

$$\left\{ y \in \mathbb{R}^4 \,\middle|\, \begin{pmatrix} y_1 & & & y_2 \\ & y_1 & & y_3 \\ & & y_1 & y_4 \\ y_2 & y_3 & y_4 & y_1 \end{pmatrix} \succcurlyeq 0 \right\}$$

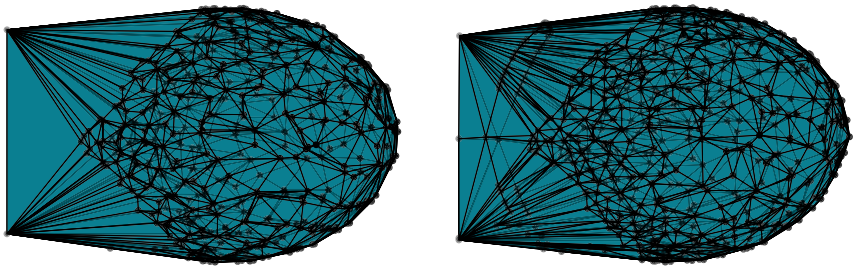

FIGURE 4.3: Bases $K_O \cap H(\omega, -1)$ and $K_I \cap H(\omega, -1)$ of an outer and inner polyhedral approximation of the recession cone of the set S from Example 4.16. The approximations K_O and K_I are obtained from Algorithm 4.1 with tolerance $\delta = 0.01$.

under the rotation

$$x \mapsto \frac{1}{2} \begin{pmatrix} \sqrt{2} & \sqrt{2} & & \\ \sqrt{2} & -\sqrt{2} & & \\ & & 2 & \\ & & & 2 \end{pmatrix} x,$$

which is a spectrahedral shadow by Proposition 2.19. Thus, S is a spectrahedral shadow as the Minkowski sum of spectrahedral shadows according to Proposition 2.16. We apply Algorithm 4.1 with $p \approx (1.6540, 1.6540, 0.8098, 0.8098)^\mathsf{T}$ and $\bar{d} = 1/2e$ to S. The point p is determined from Algorithm 3.1. Numerical data including the number of solved instances of $(\mathrm{P}_2\,(p, d, S))$, the number of extreme directions of K_O and K_I, as well as the computation time relative to the case $\delta = 0.1$ are displayed in the table below for different values of δ. The lack of a linear relationship between the computation time and the number of solved semidefinite programs is due to the influence of the vertex enumeration. Consecutive iterations take increasingly more time because the number of extreme directions and facets of the current approximations, and thus the effort to perform vertex enumerations, increase.
Since the resulting approximations for $\delta = 0.01$ are pointed cones, they admit compact bases, see [Kle57, Proposition 3.2]. Figure 4.3 shows the bases $K_O \cap H(\omega, -1)$ and $K_I \cap H(\omega, -1)$ for $\omega \approx -(0.5616, 0.5604, 0.4388, 0.4219)^\mathsf{T} \in \operatorname{int} K_O^*$. Note that a suitable ω can be found by applying Algorithm 3.1 to K_O^*. Since we have the relation $K_O^* \subseteq (0^\infty S)^\circ$, an interior point of K_O^* also yields a base of $0^\infty S$. A representation of the former set is available via Theorem 2.32.

TABLE 4.1: Numerical results from Example 4.16.

δ	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
solved SDPs	1181	1205	1305	1251	2166	2864	3462	3931	5159	13231
rel. time	1.00	1.04	1.44	1.29	2.19	7.82	10.17	12.50	19.75	86.80
$\lvert\mathrm{extdir}\, K_O\rvert$	74	74	91	74	91	133	173	194	261	534
$\lvert\mathrm{extdir}\, K_I\rvert$	118	118	132	120	132	188	238	263	374	684

Example 4.17. A real polynomial p in n variables of degree at most $2d$ is called a *sum of squares* of polynomials or *sos*-polynomial if there exist polynomials $q_1, \ldots, q_m$ of degree less than or equal to d such that $p = \sum_{i=1}^m q_i^2$. The set of all such sos-polynomials is denoted by $\Sigma_{n,2d}$. It is a closed pointed convex cone with nonempty

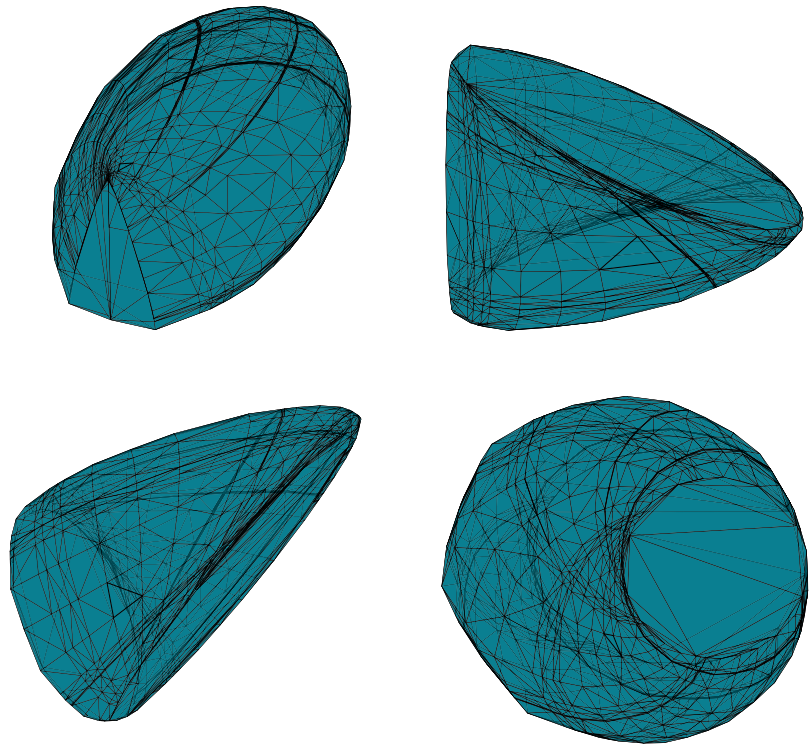

FIGURE 4.4: Projections of the base $K_O \cap H(\omega, -1)$ of an outer approximation of the cone $\Sigma_{1,4}$ from Example 4.17 with error tolerance $\delta = 0.1$ and $\omega \approx -(0.7003, 0.0753, 0.2281, 0.0734, 0.6681)^T$. K_O is a 5-dimensional pointed cone generated by 2358 directions. Shown are the projections onto $(x_1, x_2, x_3)^T$, $(x_1, x_2, x_4)^T$, $(x_1, x_3, x_4)^T$ and $(x_2, x_3, x_4)^T$ from left to right and top to bottom, respectively.

interior in the space of real polynomials, see [BPT13, Theorem 3.26]. If $p \in \Sigma_{n,2d}$, then $p(x) \geqslant 0$ at every $x \in \mathbb{R}^n$, i.e. being a sum of squares is sufficient for a polynomial to be nonnegative. This insight is important because the condition $p \in \Sigma_{n,2d}$ can be modeled using semidefinite programming, whereas deciding nonnegativity of a polynomial in general is NP-hard, cf. [BPT13, Chapter 3]. This allows to approach certain difficult polynomial optimization problems via tractable SDP relaxations. Various applications in which $\Sigma_{n,2d}$ arises are also discussed in [BPT13]. By identifying a polynomial with its coefficients we can regard $\Sigma_{n,2d}$ as a subset of $\mathbb{R}^N$ for $N = \binom{n+2d}{2d}$.

We approximate the cone $\Sigma_{1,4} \subseteq \mathbb{R}^5$ with $\delta = 0.1$ and $p = \bar{d} = 1/\sqrt{3}(1, 0, 1, 0, 1)^T$. It can be described as the set of points $x \in \mathbb{R}^5$ that admit the existence of some $y \in \mathbb{R}$ such that

$$\begin{pmatrix} x_1 & \frac{1}{2}x_2 & \frac{1}{3}x_3 - y \\ \frac{1}{2}x_2 & \frac{1}{3}x_3 + 2y & \frac{1}{2}x_4 \\ \frac{1}{3}x_3 - y & \frac{1}{2}x_4 & x_5 \end{pmatrix} \succcurlyeq 0$$

is satisfied, see [BPT13, Chapter 6]. Algorithm 4.1 solves 7730 problems of type (P$_2$ (p, d, S)) to find approximations K_O and K_I. An illustration of K_O is shown in Figure 4.4.

4.3 Computing (ε, δ)-approximations

We have developed all the tools necessary to formulate an algorithm for the computation of an (ε, δ)-approximation of a given spectrahedral shadow S in this section. The procedure is a combination of Algorithms 3.1, 3.2 and 4.1. Therefore, we assume that Assumptions (A1)–(A3) from the previous section hold. In addition, we assume S to be pointed as (ε, δ)-approximations are defined for pointed sets. The algorithm consists of two phases and is presented as Algorithm 4.2 below. In the first phase a pointed outer approximation K_O of $0^\infty S$ is computed using Algorithm 4.1 and in the second phase Algorithm 3.2 is applied to a certain compact subset of S determined from K_O and S. Finally, the results are combined to obtain an (ε, δ)-approximation of S. We describe both phases in more detail now.

ALGORITHM 4.2: (ε, δ)-approximation algorithm for pointed spectrahedral shadows

`Input:` a closed pointed spectrahedral shadow $S \subseteq \mathbb{R}^n$, point p satisfying (A2), direction $\bar{d}$ satisfying (A3), error tolerances $\varepsilon, \delta > 0$, control parameter $\gamma \in (0,1)$

`Output:` an (ε, δ)-approximation P of S

1 $\bar{\delta} \leftarrow (1-\gamma)\delta$
2 `repeat`
3 compute an outer approximation K_O of $0^\infty S$ with error tolerance $\bar{\delta}$ using Algorithm 4.1
4 $\bar{\delta} \leftarrow (1-\gamma)\delta$
5 compute a direction $\omega \in \operatorname{relint} K_O^*$ and $\operatorname{aff} K_O^*$ using Algorithm 3.1
6 `until` $(\operatorname{aff} K_O^*)^\perp = \{0\}$ `// i.e.` K_O `is pointed`
7 $K_O \leftarrow \operatorname{cone}\left[(K_O \cap H(\omega, -1-\gamma\delta)) + \{x \in \mathbb{R}^n \mid \|x\|_1 \leqslant \gamma\delta\}\right]$
8 $\bar{S} \leftarrow S \cap H^+\left(\omega, \min\left\{\omega^\mathsf{T} x \,\middle|\, x \in \operatorname{conv}\operatorname{ext}(S + K_O)\right\} - 1\right)$
9 compute an outer ε-H-approximation $\bar{P}$ of $\bar{S}$ using Algorithm 3.2
10 $P \leftarrow \bar{P} + K_O$

The algorithm starts by computing an outer approximation K_O of $0^\infty S$ with Algorithm 4.1 and error tolerance $(1-\gamma)\delta$ for preset parameters $\delta > 0$ and $\gamma \in (0,1)$. If K_O is not pointed, the tolerance is decreased by a factor of $1-\gamma$ and a new outer approximation is computed. This process is repeated until K_O is pointed which is eventually the case because $0^\infty S$ is pointed and the truncated Hausdorff distance constitutes a metric on the space of closed convex cones. The pointedness of K_O can be verified by determining its lineality space $K_O \cap (-K_O)$. For that, one can use the relationship

$$K_O \cap (-K_O) = (\operatorname{aff} K_O^*)^\perp,$$

see [Roc70, Theorem 14.6]. The set on the right side of the equation is computed by Algorithm 3.1 along with a direction $\omega \in \operatorname{relint} K_O^*$ if applied to K_O^*. After a pointed outer approximation is found, K_O is enlarged such that the pointedness is preserved and $0^\infty S \setminus \{0\} \subseteq \operatorname{int} K_O$ holds. To achieve this, the direction ω computed in line 5 is used to obtain the base $K_O \cap H(\omega, -1-\gamma\delta)$ of K_O. That is possible because if K_O is pointed, then ω actually belongs to the interior of K_O^*. The base is enlarged by adding a 1-norm ball of radius $\gamma\delta$ to it and the new outer approximation K_O is set as the conical hull of this larger set. Due to this construction, it is ensured that $d_{\mathsf{H}}(K_O, 0^\infty S) \leqslant \delta$ is satisfied. Thus, the control parameter γ steers how fine $0^\infty S$ is approximated with Algorithm 4.1. In particular, a smaller value of γ means more

weight is put towards the approximation with Algorithm 4.1, while a larger value means the approximation is coarser and the result is enlarged to a greater extent.
The second phase of the algorithm is carried out in lines 8–10. A compact subset $\bar{S}$ of S is determined in line 8 that contains the set of extreme points of $S + K_O$. More precisely, S is intersected with the halfspace

$$H^+\left(\omega, \min\left\{\omega^\mathsf{T} x \,\middle|\, x \in \operatorname{conv}\operatorname{ext}(S + K_O)\right\} - 1\right),$$

for the direction ω computed in line 5. Then $S + K_O$ can be decomposed as

$$S + K_O = \bar{S} + K_O.$$

Algorithm 3.2 is applied to $\bar{S}$ in line 9 yielding a polyhedral outer approximation $\bar{P}$. Finally, the approximation K_O of $0^\infty S$ and $\bar{P}$ are combined to receive an (ε, δ)-approximation $\bar{P} + K_O$ of S.
In order to prove that Algorithm 4.2 works correctly and is finite, we need two preliminary results. The following one is well-known in the literature.

Proposition 4.18 (cf. [Roc70, Corollary 14.6.1])**.** *Let $K \subseteq \mathbb{R}^n$ be a closed convex cone. Then K is pointed if and only if* $\operatorname{int} K^* \neq \emptyset$.

The second result is universal enough to be presented on its own.

Proposition 4.19. *Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $K \subseteq \mathbb{R}^n$ be a closed convex cone such that $0^\infty C \setminus \{0\} \subseteq \operatorname{int} K$. Then* $\operatorname{ext}(C + K)$ *is bounded.*

Proof. We may assume that $C + K$ is pointed. Otherwise, $\operatorname{ext}(C + K) = \emptyset$ and the statement is vacuous. Note that $\operatorname{ext}(C + K) \subseteq \operatorname{ext} C$ because for every $x \in C$ and $d \in K \setminus \{0\}$ it is true that

$$x + d = \frac{1}{2}x + \frac{1}{2}(x + 2d),$$

i.e. $x + d$ is not an extreme point of $C + K$. Now, assume that $\operatorname{ext}(C + K)$ is unbounded. Then $\operatorname{ext} C$ is unbounded as well. Let $\{x_k\}_{k\in\mathbb{N}}$ be an unbounded sequence of extreme points of $C + K$. Without loss of generality we assume that $\{\|x_k\|\}_{k\in\mathbb{N}}$ is strictly monotonically increasing. If this condition is not satisfied, we can pass to a suitable subsequence. Define the sequence of radial projections of $x_k - x_1$ as

$$\{d_k\}_{k\geqslant 2} = \left\{\frac{x_k - x_1}{\|x_k - x_1\|}\right\}_{k\geqslant 2}.$$

Since $d_k \in B_1(0)$ for all $k \geqslant 2$, it has a convergent subsequence. Again, without loss of generality, assume $\{d_k\}_{k\geqslant 2}$ is itself convergent with limit $\bar{d}$. According to [Roc70, Theorem 8.2] it holds $\bar{d} \in 0^\infty(C - \{x_1\}) = 0^\infty C$. Since $0^\infty C \setminus \{0\} \subseteq \operatorname{int} K$ and $\|\bar{d}\| = 1$, there exists some $k_0 \in \mathbb{N}$ such that $d_k \in K$ for all $k \geqslant k_0$. This implies $x_k - x_1 \in K$ for all $k \geqslant k_0$ as K is a cone. Therefore, $x_k \in \{x_1\} + K$ for all $k \geqslant k_0$. However, this contradicts the assumption that $x_k \in \operatorname{ext}(C + K)$ for all $k \in \mathbb{N}$. □

Theorem 4.20. *Algorithm 4.2 works correctly, in particular it terminates with an (ε, δ)-approximation P of S.*

Proof. Since S is closed and pointed, so is $0^\infty S$. The truncated Hausdorff distance defines a metric on the space of closed convex cones in $\mathbb{R}^n$, see [IS10, Proposition 2.1]. Therefore, the sequence of cones K_O computed in line 3 PK-converges to $0^\infty S$ if and

only if $\bar{\delta} \to 0$. This also follows from [RW98, Theorem 4.36]. Hence, K_O is pointed for all $\bar{\delta}$ that are sufficiently small. According to [Roc70, Theorem 14.6] the condition that K_O is pointed is equivalent to $(\operatorname{aff} K_O^*)^\perp = \{0\}$. A representation of K_O^* is available from Theorem 2.32. This implies that the loop in lines 2–6 terminates and the polyhedral cone K_O computed in the last iteration is pointed. Moreover, the direction ω determined in line 5 belongs to the interior of K_O^* according to Proposition 4.18. Denote by $\bar{B}$ the set $K_O \cap H(\omega, -1 - \gamma\delta)$ and by B the set $\bar{B} + \{x \in \mathbb{R}^n \mid \|x\|_1 \leqslant \gamma\delta\}$, i.e. $\operatorname{cone} \bar{B}$ is the cone determined after the loop in lines 2–6, which we denote $\bar{K}_O$ from here onwards, and $\operatorname{cone} B$ is its redefinition in line 7. It follows from [Gob+13, Theorem 12] that $\bar{B}$ is compact. Hence, B is compact as well. In addition to that, B is a polyhedron as the Minkowski sum of polyhedra, see e.g. [Roc70, Corollary 19.3.2]. Now, K_O as defined in line 7 is a polyhedral cone. This is evident from [Roc70, Theorem 19.7] and the fact that $K_O = \operatorname{cone} B$ is closed because B is compact. Next, we show that $0^\infty S \setminus \{0\} \subseteq \operatorname{int} K_O$, K_O is pointed and $d_{\mathsf{tH}}(K_O, 0^\infty S) \leqslant \delta$. The first assertion is immediate from the inclusions $0^\infty S \subseteq \operatorname{cone} \bar{B}$, $\bar{B} \subseteq \operatorname{int} B$ and the equality $K_O = \operatorname{cone} B$. Note that the first inclusion is ensured by Theorem 4.14 and the fact that $\bar{B}$ is a base of the outer approximation $\bar{K}_O$ computed by Algorithm 4.1 in line 3. To see that K_O is pointed observe that $x \in \bar{B}$ implies $\|x\| \geqslant 1 + \gamma\delta$ because ω is a direction, i.e. $\|\omega\| = 1$. Consequently, $\|x\| \geqslant 1$ for all $x \in B$. In particular, $0 \notin B$. Therefore, K_O is pointed. Otherwise, the existence of $\pm d \in K_O \setminus \{0\}$ would imply $\pm\mu d \in B$ for some $\mu \geqslant 0$ and $0 \in B$ by convexity. In order to show the third claim, note that

$$d_{\mathsf{tH}}(K_O, 0^\infty S) \leqslant d_{\mathsf{tH}}(K_O, \bar{K}_O) + d_{\mathsf{tH}}(\bar{K}_O, 0^\infty S) \leqslant d_{\mathsf{tH}}(K_O, \bar{K}_O) + (1 - \gamma)\,\delta \tag{4.16}$$

holds because d_{tH} is a metric for closed convex cones and $d_{\mathsf{tH}}(\bar{K}_O, 0^\infty S) \leqslant \bar{\delta}$ for $\bar{\delta} \leqslant (1 - \gamma)\,\delta$. Moreover, it is true that

$$\begin{aligned} d_{\mathsf{tH}}(K_O, \bar{K}_O) &= e\left[K_O \cap B_1(0), \bar{K}_O \cap B_1(0)\right] \\ &= e\left[K_O \cap B_1(0), \bar{K}_O\right] \\ &\leqslant e\left[B, \bar{K}_O\right] \\ &\leqslant e\left[B, \bar{B}\right] \\ &= d_{\mathsf{H}}(B, \bar{B}) \\ &\leqslant \gamma\delta. \end{aligned} \tag{4.17}$$

The first and last equality hold because $\bar{K}_O \subseteq K_O$ and $\bar{B} \subseteq B$, respectively. The second equation resembles the fact that the projection mapping is nonexpansive, that is

$$\left\|\pi_{\bar{K}_O}(x)\right\| \leqslant \|x\|$$

holds for all $x \in \mathbb{R}^n$, see [HL01, Proposition 3.1.3]. Taking into account $\|x\| \geqslant 1$ whenever $x \in B$ and $\mu d(x, \bar{K}_O) = d(\mu x, \bar{K}_O)$ for all $\mu \geqslant 0$ as $\bar{K}_O$ is a cone, justifies the inequality in line 3. Finally, combining Estimates (4.16) and (4.17) yields $d_{\mathsf{tH}}(K_O, 0^\infty S) \leqslant \delta$.

Since $0^\infty S \setminus \{0\} \subseteq \operatorname{int} K_O$, $\operatorname{ext}(S + K_O)$ is bounded according to Proposition 4.19. Therefore, its convex hull is bounded as well. Moreover, it is nonempty because $S + K_O$ is closed according to [Roc70, Corollary 9.1.1] and pointed due to K_O being pointed. Thus, Corollary 1.17 guarantees the existence of an extreme point and Theorem 1.16 the closedness of the set $\operatorname{conv}\operatorname{ext}(S + K_O)$. Together this implies that $\inf\left\{\omega^\mathsf{T} x \,\middle|\, x \in \operatorname{conv}\operatorname{ext}(S + K_O)\right\}$ is finite and attained, i.e. line 8 of the algorithm is

well-defined. The additional shift of -1 on the right hand side of the halfspace ensures that the intersection with S has nonempty interior. Furthermore, $\bar{S}$ is compact, see [Gob+13, Theorem 12]. Hence, it is a valid input to Algorithm 3.2 in line 9 and an ε-H-approximation $\bar{P}$ of $\bar{S}$ is computed correctly.
Now, we show that $P = \bar{P} + K_0$ is an (ε, δ)-approximation of S. Since $\bar{P}$ is compact, it holds $0^\infty P = K_0$ and P is pointed. We have already demonstrated $d_{\mathsf{tH}}(K_0, 0^\infty S) \leqslant \delta$. As vert $P \subseteq$ vert $\bar{P}$, it holds

$$e\left[\operatorname{vert} P, S\right] \leqslant e\left[\operatorname{vert} \bar{P}, S\right] \leqslant e\left[\operatorname{vert} \bar{P}, \bar{S}\right] \leqslant \varepsilon.$$

It remains to show $S \subseteq P$. From the fact that conv ext $(S + K_0) \subseteq \bar{S}$ one obtains the decomposition

$$S + K_0 = \left((S + K_0) \cap H^+\right) + K_0,$$

where H^+ denotes the halfspace utilized in line 8, see [Gob+13, Corollary 2]. Moreover, it is easy to verify that

$$\left((S + K_0) \cap H^+\right) + K_0 = \bar{S} + K_0$$

is true because the normal vector ω of H^+ satisfies $\omega \in K_0^*$. We conclude

$$S \subseteq S + K_0 = \bar{S} + K_0 \subseteq \bar{P} + K_0 = P.$$

This completes the proof of correctness. Finally, the finiteness of Algorithm 4.2 is evident from the respective property of Algorithms 3.1, 3.2 and 4.1. □

The difficulty of Algorithm 4.2 is the determination of the halfspace

$$H^+\left(\omega, \min\left\{\omega^{\mathsf{T}} x \,\middle|\, x \in \operatorname{conv ext}(S + K_0)\right\} - 1\right) \tag{4.18}$$

in line 8. The reason is that, although $\min\left\{\omega^{\mathsf{T}} x \,\middle|\, x \in \operatorname{conv ext}(S + K_0)\right\}$ is a convex program, a representation of conv ext $(S + K_0)$ as a spectrahedral shadow or a more general description in terms of convex functions is not readily available. In fact, it is an open question whether conv ext $(S + K_0)$ is a spectrahedral shadow in the first place. Even if this turns out to be the case, it might still be difficult to give a representation of the set depending on how $S + K_0$ is represented. Consider for instance the special case of $S + K_0$ being a polyhedron in H-representation. Then the obvious way of computing conv ext $(S + K_0)$ is to determine a V-representation via vertex enumeration and obtain the set as the convex hull of its finitely many vertices. However, knowledge of the extreme points of $S + K_0$ may not be necessary if one only wants to represent their convex hull. Thus, it would be interesting to investigate whether this set can be described directly using the data describing $S + K_0$ in the manner done in Chapter 2 with other operations. In particular, if it is possible without performing some algorithmic procedure first.
So far, the only knowledge we have about conv ext $(S + K_0)$ is acquired from Proposition 4.19, which implies its compactness and the existence of the halfspace (4.18). In order to deal with this limitation from a computational perspective, we suggest a modification of Algorithm 4.2. Note that in line 8 of the algorithm it suffices to intersect S with a halfspace H^+ such that their intersection is bounded and the containment

$$\operatorname{conv ext}(S + K_0) \subseteq H^+$$

is satisfied, cf. [Gob+13, Corollary 2]. The following variant of Algorithm 4.2 computes such a halfspace iteratively. It replaces lines 8–10 of the original algorithm.

ALGORITHM 4.3: Variant of lines 8–10 of Algorithm 4.2

```
1  for w ∈ extdir K_O^* do
2  |  compute a solution (x_w, y_w, Z_w) to (P_1(w, S))
3  end
4  α ← min({ω^T x_w | w ∈ extdir K_O^*} ∪ {0}) − 1
5  repeat
6  |  S̄ ← S ∩ H^+(ω, α)
7  |  compute an outer ε-H-approximation P̄ of S̄ using Algorithm 3.2
8  |  α ← 2α
9  |  P ← P̄ + K_O
10 until S ⊆ P
```

First, problem $(\mathrm{P}_1(w, S))$ is solved for every $w \in \operatorname{extdir} K_{\mathrm{O}}^*$ yielding solutions (x_w, y_w, Z_w) which give rise to an initial halfspace $H^+(\omega, \alpha)$, where ω is the direction computed in line 5 of Algorithm 4.2 and

$$\alpha = \min\left(\left\{\omega^{\mathsf{T}} x_w \,\middle|\, w \in \operatorname{extdir} K_{\mathrm{O}}^*\right\} \cup \{0\}\right) - 1.$$

A compact subset $\bar{S}$ of S is obtained as the intersection of S and $H^+(\omega, \alpha)$. It has nonempty interior because $x_w \in \operatorname{int} H^+(\omega, \alpha)$ and $\operatorname{int} S \neq \emptyset$. Moreover, $\alpha < 0$. Now, Algorithm 3.2 is used to compute a polyhedral outer approximation $\bar{P}$ of $\bar{S}$ with tolerance ε and it is checked whether $P = \bar{P} + K_{\mathrm{O}}$ is an (ε, δ)-approximation of S by verifying the containment $S \subseteq P$. If the containment holds, the algorithm is terminated and P is returned as a solution. Otherwise, a new compact subset is obtained by doubling the value of α in the definition of $H^+(\omega, \alpha)$, which corresponds to a shift of $H^+(\omega, \alpha)$ in the direction $-\omega$, and the approximation is repeated.
The containment $S \subseteq P$ can easily be verified using semidefinite programming. Suppose (A, b) is an H-representation of P for $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Let $a^i \in \mathbb{R}^n$ denote the i-th row of A. Then $S \subseteq P$ if and only if $\sigma_S(a^i) \leqslant b_i$ for every $i = 1, \ldots, m$. This follows from Proposition 3.7 because $H(a^i, \sigma_S(a^i))$ is a supporting hyperplane of S whenever $a^i \neq 0$ and $\sigma_S(a^i) < \infty$. Thus, if $\sigma_S(a^i) \leqslant b_i$, then $S \subseteq H^-(a^i, b_i)$. Remember that an evaluation of the support function of S at the point a^i corresponds to solving problem $(\mathrm{P}_1(a^i, S))$. Hence, the containment in line 10 can be verified by computing an H-representation of P and solving m semidefinite programs.

Theorem 4.21. *Algorithm 4.2 with Variant 4.3 works correctly, in particular it terminates with an (ε, δ)-approximation P of S.*

Proof. According to [Gob+13, Corollary 2] it holds

$$S + K_{\mathrm{O}} = \left((S + K_{\mathrm{O}}) \cap H^+\right) + K_{\mathrm{O}}$$

for every halfspace H^+ satisfying that $(S + K_{\mathrm{O}}) \cap H^+$ is nonempty and compact and $\operatorname{ext}(S + K_{\mathrm{O}}) \subseteq H^+$. It suffices to show that these conditions are eventually fulfilled for the halfspace $H^+(\omega, \alpha)$ utilized in line 6. With the observation that

$$\left((S + K_{\mathrm{O}}) \cap H^+(\omega, \alpha)\right) + K_{\mathrm{O}} = \left(S \cap H^+(\omega, \alpha)\right) + K_{\mathrm{O}}$$

holds as $\omega \in K_0^*$, the arguments demonstrating that polyhedron P in line 9 is an (ε, δ)-approximation of S are then identical to the corresponding part of the proof of Theorem 4.20.
Since $0^\infty S \setminus \{0\} \subseteq \operatorname{int} K_0$, it holds $K_0^* \setminus \{0\} \subseteq \operatorname{int}(0^\infty S)^\circ$, cf. [GT56, Lemma 2]. Therefore, solutions to $(\mathrm{P}_1(w, S))$ exist for every $w \in \operatorname{extdir} K_0^*$ according to Proposition 3.7. This implies that the initial assignment of $\bar{S}$ in line 6 is nonempty because $x_w \in \bar{S}$ for every $w \in \operatorname{extdir} K_0^*$. Observe that if α decreases, $\bar{S}$ can only become larger. In line 4 the value of α is initialized as at most -1 and whenever line 8 is executed α decreases. Hence, $\bar{S}$ is nonempty in subsequent iterations as well. Trivially, this also applies to $(S + K_0) \cap H^+(\omega, \alpha)$. Furthermore, sets $(S + K_0) \cap H^+(\omega, \alpha)$ and $\bar{S}$ are compact since $\omega \in \operatorname{int} K_0^* \subseteq \operatorname{int}(0^\infty S)^\circ$, see [Gob+13, Theorem 12].
By Proposition 4.19 $\operatorname{ext}(S + K_0)$ is bounded because $0^\infty S \setminus \{0\} \subseteq \operatorname{int} K_0$. Thus, $\operatorname{ext}(S + K_0) \subseteq H^+(\omega, \alpha)$ for small enough α.
To complete the proof, note that $\operatorname{int} \bar{S} \neq \emptyset$ holds in every iteration. This is evident from $x_w \in \operatorname{int} H^+(\omega, \alpha)$ for every $w \in \operatorname{extdir} K_0^*$, Assumption (A3), which implies $\operatorname{int} S \neq \emptyset$, and [Roc70, Theorem 6.1]. Hence, $\bar{S}$ is a valid input to Algorithm 3.2 in line 7. Finally, the containment in line 10 can be verified as discussed in the paragraph before Theorem 4.21. □

Remark 4.22. It has to be noted that, although Variant 4.3 leads to a finite algorithm, the performance might be poor from a computational perspective. Since it is not known a priori which value of α yields $S \subseteq P$, it is possible that many unusable approximations are computed in line 7. In fact, one can construct examples such that the loop in lines 5–10 runs an arbitrary number of times. For instance, let

$$S = \operatorname{conv}\left((B_1(0) \times \{0\}) \cup \left\{ \begin{pmatrix} 1 \\ 1 \\ 2^k \end{pmatrix} \right\} \right) + \operatorname{cone}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

where $B_1(0) \subseteq \mathbb{R}^2$ and $k \in \mathbb{N}$, and $K_0 = \operatorname{cone}\left(\left\{x \in \mathbb{R}^2 \,\middle|\, \|x\|_\infty \leqslant \varepsilon\right\} \times \{1\}\right)$. If $\omega = (0, 0, -1)^\mathsf{T}$, then the points x_w in line 2 are of the form

$$\begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}$$

and the initial value of α is -1. In order for the point $(1, 1, 2^k)^\mathsf{T}$ to be contained in $S \cap H^+(\omega, \alpha)$, we need $\alpha \leqslant -2^k$. Hence, if ε is sufficiently small such that the containment $S \subseteq P$ is not fulfilled in an earlier iteration, the loop in lines 5–10 needs to be executed $k + 1$ times.

Example 4.23. We close this section with an illustration of Algorithm 4.2 using Variant 4.3. Consider the spectrahedral shadow

$$S = \left\{ x \in \mathbb{R}^3 \,\middle|\, \exists\, Z \in \mathcal{S}^2 \colon I + \mathcal{A}(Z) \succcurlyeq 0, x = \mathcal{B}^\mathsf{T}(Z), Z \succcurlyeq 0 \right\},$$

where

$$\mathcal{A}(Z) = \begin{pmatrix} z_{11} & z_{11} + 2z_{12} + z_{22} \\ z_{11} + 2z_{12} + z_{22} & z_{22} \end{pmatrix} \quad \text{and} \quad \mathcal{B}^\mathsf{T}(Z) = \begin{pmatrix} z_{11} \\ z_{22} \\ 2z_{12} \end{pmatrix}.$$

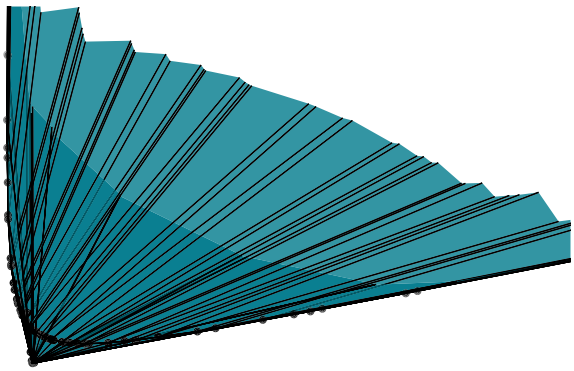

FIGURE 4.5: An (ε, δ)-approximation of the spectrahedral shadow from Example 4.23 for $\varepsilon = 0.01$ and $\delta = 0.05$.

For parameters $\varepsilon = 0.01$, $\delta = 0.05$ and $\gamma = 0.1$ as well as

$$\bar{d} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad p \approx \begin{pmatrix} 0.5088 \\ 0.5678 \\ -0.2098 \end{pmatrix}$$

Algorithm 4.2 solves 35 problems of type $(\mathrm{P}_1(\omega, S))$, 286 of type $(\mathrm{P}_2(p, d, S))$ and 6166 of type $(\mathrm{P}_3(p, S))$ in total. A visualization of the resulting approximation is shown in Figure 4.5.

4.4 Approximation via homogenization

In the last section of this thesis we present another notion of approximation for possibly unbounded convex sets that is weaker than the notion of (ε, δ)-approximation but addresses two of its shortcomings. The first one being that it constitutes a concept of outer approximation only and that there is no obvious extension to inner approximations as discussed in Section 4.1. The second is the lack of a polarity correspondence, which is closely tied to the first. In particular, if P is an (ε, δ)-approximation of a set C, it is unclear whether or how P° is related to C°. Certainly, P° is not an (ε, δ)-approximation of C° merely from the inclusion $P^\circ \subseteq C^\circ$.
Remember that the main reason for the insufficiency of the Hausdorff distance as an error measure for the approximation of unbounded closed convex sets is that it is not a metric on the collection of these sets, cf. Section 3.3. However, we have another metric at our disposal that allows us to work with unbounded sets to a limited degree, namely the truncated Hausdorff distance, which defines a metric on the space of closed convex cones. This property is the motivation for the approach we take in this section.
We need the concept of homogenization, which assigns a closed convex cone from $\mathbb{R}^{n+1}$ to a convex set from $\mathbb{R}^n$, before we can utilize the truncated Hausdorff distance for our purpose.

Definition 4.24 (cf. [Bri20]). Let $C \subseteq \mathbb{R}^n$ be a convex set. The *homogenization* or *conification* of C, denoted by $\operatorname{homog} C$, is the cone

$$\operatorname{cl} \operatorname{cone} (C \times \{1\}) .$$

We can identify every closed convex set $C \subseteq \mathbb{R}^n$ with its homogenization because C can be recovered from homog C. In particular,

$$C = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} x \\ 1 \end{pmatrix} \in \operatorname{homog} C \right\},$$

i.e. intersecting homog C with the hyperplane $H(e_{n+1}, 1) = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\}$ and projecting the result onto the first n variables yields the original set C. Using this correspondence, one can work with convex cones entirely. This is a standard tool in convex analysis in order to reduce problems to a conic setting, see e.g. [Roc70; RW98; Bri20]. Here, we use it to reduce the problem of approximating convex sets to the problem of approximating convex cones.

Definition 4.25. For a convex set $C \subseteq \mathbb{R}^n$ and $\delta \geqslant 0$ a polyhedron $P \subseteq \mathbb{R}^n$ is called a *homogeneous δ-approximation* of C if

$$d_{\mathsf{tH}}(\operatorname{homog} P, \operatorname{homog} C) \leqslant \delta.$$

Remark 4.26. The notion of homogeneous δ-approximation can be understood as a relative error measure between the involved sets. Consider a convex set $C \subseteq \mathbb{R}^n$ and a polyhedron $P \subseteq \mathbb{R}^n$. If $x \in P$ with $d(x, C) \leqslant \varepsilon$, then the distance of the corresponding direction of homog P to homog C obeys the relation

$$d\left(\frac{1}{\sqrt{x^\mathsf{T} x + 1}} \begin{pmatrix} x \\ 1 \end{pmatrix}, \operatorname{homog} C \right) \leqslant \frac{\varepsilon}{\sqrt{x^\mathsf{T} x + 1}}.$$

In particular, the distance depends inversely on $\|x\|$. Hence, if P is a homogeneous δ-approximation of C, then points of one set that are far from the origin are allowed a larger distance to the other set than points closer to the origin.
This observation is explained from the fact that the distinction between points and directions of a closed convex set C collapses when transitioning to its homogenization. One has the identity

$$\operatorname{homog} C = \operatorname{cone}(C \times \{1\}) \cup (0^\infty C \times \{0\}), \tag{4.19}$$

see [Roc70, Theorem 8.2], i.e. points as well as directions of C correspond to directions of homog C. Thus, a convergent sequence $\{(x_k, \mu_k)^\mathsf{T}\}_{k \in \mathbb{N}}$ of directions of homog C does not necessarily yield a convergent sequence in C itself. For instance, if $\{(x_k, \mu_k)^\mathsf{T}\}_{k \in \mathbb{N}}$ converges to a direction $(\bar{x}, 0)^\mathsf{T}$ and provided all μ_k are positive, then the sequence $\left\{ \mu_k^{-1} x_k^\mathsf{T} \right\}_{k \in \mathbb{N}} \subseteq C$ is unbounded. This means that closeness of homogenizations with respect to the truncated Hausdorff distance does not imply closeness of the sets themselves with respect to the Hausdorff distance. It turns out, this is merely a restatement of the fact that H-convergence is weaker than PK-convergence as seen from Proposition 4.27 below. In order to formally describe any such unboundedness behavior exhibited by C, the concept of *cosmic closure* of $\mathbb{R}^n$ is introduced and studied in [RW98].

Homogeneous δ-approximations define a suitable notion of approximation for closed convex sets because PK-convergence can be characterized in terms of homogenizations and the truncated Hausdorff distance. The following result about homogeneous δ-approximations is the counterpart to Theorem 4.12 about (ε, δ)-approximations.

Proposition 4.27 (cf. [RW98, Corollary 4.47]). *Let $\{C_k\}_{k\in\mathbb{N}}$ be a sequence of closed convex subsets of $\mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ be a closed convex set. Then the following are equivalent:*

(i) $C_k \to C$,

(ii) $d_{\mathsf{tH}}(\operatorname{homog} C_k, \operatorname{homog} C) \to 0$.

Unlike for (ε,δ)-approximations no inclusion between P and C is required by Definition 4.25 and it is not limited to pointed sets. In these regards it is in harmony with the definition of ε-H-approximations. Moreover, homogeneous δ-approximations exhibit an elegant behavior under polarity which we discuss after pointing out another relation to (ε,δ)-approximations.

Proposition 4.28. *Let $C \subseteq \mathbb{R}^n$ be a closed pointed convex set. If P is an (ε,δ)-approximation of C, then P is a homogeneous $\bar{\delta}$-approximation of C where*

$$\bar{\delta} = \varepsilon + \max\left\{ \sqrt{v^{\mathsf{T}}v+1} \,\middle|\, v \in \operatorname{vert} P \right\} \delta.$$

Proof. It holds $\operatorname{homog} C \subseteq \operatorname{homog} P$ because $C \subseteq P$. Thus, the truncated Hausdorff distance between $\operatorname{homog} P$ and $\operatorname{homog} C$ is attained as

$$d_{\mathsf{tH}}(\operatorname{homog} P, \operatorname{homog} C) = e\left[\operatorname{homog}(P) \cap B_1(0), \operatorname{homog}(C) \cap B_1(0)\right].$$

The right hand side is identical to $e\left[\operatorname{homog}(P) \cap B_1(0), \operatorname{homog} C\right]$ because the projection mapping onto $\operatorname{homog} C$ is nonexpansive, see [HL01, Proposition 3.1.3]. In particular, if $(x,\mu)^{\mathsf{T}} \in \operatorname{homog}(P) \cap B_1(0)$, then $\left\|\pi_{\operatorname{homog} C}\left((x,\mu)^{\mathsf{T}}\right)\right\| \leqslant 1$. According to [Roc70, Theorem 8.2] $\operatorname{homog} P$ admits the decomposition

$$\operatorname{homog} P = \operatorname{cone}(P \times \{1\}) \cup (0^{\infty}P \times \{0\}). \tag{4.20}$$

Therefore, the excess simplifies to

$$\max\left(\left\{ d\left(\frac{1}{\sqrt{x^{\mathsf{T}}x+1}}\begin{pmatrix}x\\1\end{pmatrix}, \operatorname{homog} C\right) \,\middle|\, x \in P \right\} \cup \left\{ d\left(\begin{pmatrix}d\\0\end{pmatrix}, \operatorname{homog} C\right) \,\middle|\, d \in 0^{\infty}P \cap B_1(0) \right\}\right).$$

If $x \in P$, then there exist $v \in \operatorname{conv}\operatorname{vert} P$, $\mu \geqslant 0$ and $d \in 0^{\infty}P \cap B_1(0)$ such that $x = v + \mu d$. Now, one has

$$\begin{aligned} d\left(\frac{1}{\sqrt{x^{\mathsf{T}}x+1}}\begin{pmatrix}x\\1\end{pmatrix}, \operatorname{homog} C\right) &= \frac{1}{\sqrt{x^{\mathsf{T}}x+1}} d\left(\begin{pmatrix}x\\1\end{pmatrix}, \operatorname{homog} C\right) \\ &\leqslant \frac{1}{\sqrt{x^{\mathsf{T}}x+1}} d\left(\begin{pmatrix}x\\1\end{pmatrix}, \operatorname{homog}(C) \cap H(e_{n+1}, 1)\right) \\ &= \frac{1}{\sqrt{x^{\mathsf{T}}x+1}} d(x, C) \\ &= \frac{1}{\sqrt{(v+\mu d)^{\mathsf{T}}(v+\mu d)+1}} d(v+\mu d, C) \\ &\leqslant \frac{\varepsilon + \mu\delta}{\sqrt{(v+\mu d)^{\mathsf{T}}(v+\mu d)+1}}. \end{aligned} \tag{4.21}$$

The first line is valid because homog C is a cone and the last inequality follows from the fact that P is an (ε, δ)-approximation of C. Using $\sqrt{(v+\mu d)^\mathsf{T}(v+\mu d)+1} \geqslant 1$ and maximizing the coefficient of δ with respect to μ yields

$$\begin{aligned} d\left(\frac{1}{\sqrt{x^\mathsf{T}x+1}}\begin{pmatrix}x\\1\end{pmatrix}, \operatorname{homog} C\right) &\leqslant \varepsilon + \frac{1}{\sqrt{1-\frac{(v^\mathsf{T}d)^2}{1+v^\mathsf{T}v}}}\delta \\ &\leqslant \varepsilon + \sqrt{v^\mathsf{T}v+1}\delta, \end{aligned} \tag{4.22}$$

with the second inequality being obtained by an application of the Cauchy-Schwarz inequality. A similar derivation as in (4.21) shows

$$\begin{aligned} d\left(\begin{pmatrix}d\\0\end{pmatrix}, \operatorname{homog} C\right) &\leqslant d\left(\begin{pmatrix}d\\0\end{pmatrix}, \operatorname{homog}(C) \cap H(e_{n+1}, 0)\right) \\ &= d\left(d, 0^\infty C\right) \\ &\leqslant \delta \end{aligned} \tag{4.23}$$

for $d \in 0^\infty P \cap B_1(0)$. The equality follows by evoking an analogous description of homog C as in Equation (4.20). Finally, combining (4.22) and (4.23) and taking into account that the maximum of $\sqrt{v^\mathsf{T}v+1}$ for $v \in \operatorname{conv}\operatorname{vert} P$ is attained at a vertex of P yields the result. □

Clearly, a reverse statement can not be expected because homogeneous δ-approximations apply to a broader class of sets than (ε, δ)-approximations. Another reason, through which their more general nature comes to light, is that they do not contain direct information about the relation between the recession cones of the involved sets. In fact, a homogeneous δ-approximation may be compact for any $\delta > 0$ although the set to be approximated is unbounded.

Example 4.29. Let $C = [0, \infty)$ and define $P_\delta = \left[0, \sqrt{1-\delta^2}/\delta\right]$ for $\delta \in (0, 1]$. Then, $0^\infty C = C$ and P_δ is compact for every $\delta \in (0, 1]$, but one has

$$\begin{aligned} d_\mathsf{tH}\left(\operatorname{homog} P_\delta, \operatorname{homog} C\right) &= \left\| \begin{pmatrix}1\\0\end{pmatrix} - \pi_{\operatorname{homog} P_\delta}\left(\begin{pmatrix}1\\0\end{pmatrix}\right)\right\| \\ &= \left\| \begin{pmatrix}1\\0\end{pmatrix} - \delta\sqrt{1-\delta^2}\begin{pmatrix}\frac{\sqrt{1-\delta^2}}{\delta}\\1\end{pmatrix}\right\| \\ &= \delta. \end{aligned}$$

The second equality is evident by taking into account that the projection of $(1,0)^\mathsf{T}$ onto homog P_δ is attained on the ray generated by $\left(\sqrt{1-\delta^2}/\delta, 1\right)^\mathsf{T}$. Hence, P_δ is a homogeneous δ-approximation of C with $d_\mathsf{tH}\left(0^\infty P_\delta, 0^\infty C\right) = 1$. Note that fixing δ and interchanging the roles of C and P_δ yields a homogeneous δ-approximation that is a $(0,1)$-approximation in the sense of Definition 4.2.

Surprisingly, the polar set of a homogeneous δ-approximation of C is a homogeneous δ-approximation of C° provided the origin is contained in the sets. This is different from the compact case discussed in Chapter 3, where the quantity $d_\mathsf{H}\left(P^\circ, C^\circ\right)$ for an ε-H-approximation P of C does not only depend on ε but also on constants related to the geometry of C. In order to establish the result, we need the following connection between the polar cone of homog C and C°.

Proposition 4.30. *Let $C \subseteq \mathbb{R}^n$ be a convex set containing the origin. Then*

$$\operatorname{homog}(C)^* = \operatorname{cl}\operatorname{cone}(C^\circ \times \{-1\}) = -\operatorname{homog}(-C^\circ).$$

Proof. The first equality is proved in [BBW22, Theorem 3.1] for C being closed and convex. It also holds for arbitrary convex sets C because $\operatorname{homog} C = \operatorname{homog}\operatorname{cl} C$. The inclusion $\operatorname{homog} C \subseteq \operatorname{homog}\operatorname{cl} C$ is immediate from $C \subseteq \operatorname{cl} C$. For the other direction consider a point

$$\mu \begin{pmatrix} x \\ 1 \end{pmatrix} \in \operatorname{cone}(\operatorname{cl} C \times \{1\}),$$

that means we fix $\mu \geqslant 0$ and $x \in \operatorname{cl} C$. Thus, there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq C$ converging to x. For every $k \in \mathbb{N}$ it holds $\mu\left(x_k^\mathsf{T}, 1\right)^\mathsf{T} \in \operatorname{cone}(C \times \{1\})$. Therefore, $\mu\left(x^\mathsf{T}, 1\right)^\mathsf{T} \in \operatorname{cl}\operatorname{cone}(C \times \{1\})$. This implies

$$\operatorname{cone}(\operatorname{cl} C \times \{1\}) \subseteq \operatorname{cl}\operatorname{cone}(C \times \{1\})$$

and by taking closures

$$\operatorname{homog}\operatorname{cl} C \subseteq \operatorname{homog} C.$$

To show the second equation in the original statement we compute

$$\begin{aligned}
\operatorname{cl}\operatorname{cone}(C^\circ \times \{-1\}) &= \operatorname{cl}\left\{\mu \begin{pmatrix} x \\ -1 \end{pmatrix} \in \mathbb{R}^{n+1} \,\middle|\, \mu \geqslant 0, x \in C^\circ\right\} \\
&= \operatorname{cl} -\left\{\mu \begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1} \,\middle|\, \mu \geqslant 0, x \in -C^\circ\right\} \\
&= -\operatorname{cl}\left\{\mu \begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1} \,\middle|\, \mu \geqslant 0, x \in -C^\circ\right\} \\
&= -\operatorname{homog}(-C^\circ).
\end{aligned}$$

The penultimate line follows from [Roc70, Theorem 9.1]. □

Theorem 4.31. *Let $C \subseteq \mathbb{R}^n$ be a convex set containing the origin. If P is a homogeneous δ-approximation of C containing the origin, then P° is a homogeneous δ-approximation of C°.*

Proof. Let $M\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote the isometry defined by

$$M \begin{pmatrix} x \\ \mu \end{pmatrix} = \begin{pmatrix} x \\ -\mu \end{pmatrix}.$$

Now, we compute

$$\begin{aligned}
d_{\mathsf{tH}}(\operatorname{homog}(P^\circ), \operatorname{homog}(C^\circ)) &= d_{\mathsf{tH}}(M[\operatorname{homog}(P^\circ)], M[\operatorname{homog}(C^\circ)]) \\
&= d_{\mathsf{tH}}(-\operatorname{homog}(-P^\circ), -\operatorname{homog}(-C^\circ)) \\
&= d_{\mathsf{tH}}(\operatorname{homog}(P)^*, \operatorname{homog}(C)^*) \\
&= d_{\mathsf{tH}}(\operatorname{homog} P, \operatorname{homog} C) \\
&\leqslant \delta.
\end{aligned}$$

The first equation is true by [IS10, Proposition 2.1]. The second and third follow from Proposition 4.30 using the fact that $0 \in P \cap C$. Theorem 4.15 yields the equality in line 4. Finally, the inequality holds because P is a homogeneous δ-approximation of C. Thus, P° is a homogeneous δ-approximation of C°. □

For the remainder of this section we turn to the problem of computing homogeneous δ-approximations of a given spectrahedral shadow S. The fact that every convex cone is its own recession cone suggests that we can use the tools developed in Section 4.2 to solve this problem. In particular, we could apply Algorithm 4.1 to homog S and obtain a homogeneous δ-approximation of S by undoing the homogenization. This approach presupposes that S satisfies suitable assumptions such that the requirements of Algorithm 4.1 are fulfilled for homog S and, more importantly, that homog S is a spectrahedral shadow for which a representation is available or can be computed from a representation of S. The following proposition shows that the latter is possible.

Proposition 4.32 (homogenization). *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow represented by $(A_0, \mathcal{A}, \overline{\mathcal{A}}, \boldsymbol{\mathcal{A}}, B, \bar{B}, \mathcal{B}^\mathsf{T}, b)$. Then the homogenization* homog S *is the closure of the spectrahedral shadow represented by*

$$\left(0, \left(\begin{array}{c|c} \mathcal{A} & A_0 \\ \hline M^1 & M^2 \end{array}\right), \overline{\mathcal{A}} \oplus \left(I \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right), \begin{pmatrix} \boldsymbol{\mathcal{A}} \\ 0 \end{pmatrix}, \begin{pmatrix} B & -b \end{pmatrix}, \begin{pmatrix} \bar{B} & 0 \end{pmatrix}, \mathcal{B}^\mathsf{T}, 0\right)$$

with

$$M^1 = \left(E_{11} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\middle|\; \cdots \;\middle|\; E_{nn} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \quad \text{and} \quad M^2 = I \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, homog S *is itself a spectrahedral shadow.*

Proof. The singleton $\{1\}$ can be regarded as a spectrahedral shadow with representation $(\varnothing, \varnothing, \varnothing, \varnothing, 1, \varnothing, \varnothing, 1)$. Applying Proposition 2.14 to S and $\{1\}$ and Proposition 2.41 to the resulting set yields that cone $(S \times \{1\})$ is a spectrahedral shadow represented by

$$\left(0, \left(\begin{array}{c|c} \mathcal{A} & 0 \\ \hline M^1 & T^1 \end{array}\right), \left(\begin{array}{c|c|c} \overline{\mathcal{A}} & A_0 & 0 \\ \hline 0 & M^2 & I \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right), \begin{pmatrix} \boldsymbol{\mathcal{A}} \\ 0 \end{pmatrix}, B \oplus 1, T^2, \left(\begin{array}{c|c} \mathcal{B} & 0 \end{array}\right)^\mathsf{T}, 0\right),$$

where M^1 and M^2 are as defined above and T^1 and T^2 are

$$E_{(n+1)(n+1)} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{B} & -b & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

respectively. Therefore, cone $(S \times \{1\})$ can be written as

$$\left\{ \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \exists \begin{array}{l} y \in \mathbb{R}^m \\ \mu \in \mathbb{R} \\ t \in \mathbb{R} \\ Z \in \mathcal{S}^k \end{array} : \begin{array}{r} \mathcal{A}(x) + \overline{\mathcal{A}}(y) + A_0\mu + \boldsymbol{\mathcal{A}}(Z) \succcurlyeq 0 \\ \begin{pmatrix} \mu & x_i \\ x_i & t \end{pmatrix} \succcurlyeq 0 \quad i = 1, \ldots, n+1 \\ Bx + \bar{B}y - b\mu + \mathcal{B}^\mathsf{T}(Z) = 0 \\ x_{n+1} - \mu = 0 \\ Z \succcurlyeq 0 \end{array} \right\}.$$

Using the equality $x_{n+1} = \mu$ we eliminate μ from the description of the set. The resulting spectrahedral shadow differs from the one defined by the claimed representation only through the occurrence of the LMI

$$\begin{pmatrix} x_{n+1} & x_{n+1} \\ x_{n+1} & t \end{pmatrix} \succcurlyeq 0. \tag{4.24}$$

Remember from the proof of Proposition 2.41 that the reason the LMIs

$$\begin{pmatrix} \mu & x_i \\ x_i & t \end{pmatrix} \succcurlyeq 0, \tag{4.25}$$

$i = 1, \ldots, n$, are introduced in the description of the conical hull of a spectrahedral shadow is to ensure $x = 0$ whenever $\mu = 0$. Since $\mu = x_{n+1}$, LMI (4.24) can be omitted. In addition, note that $x_{n+1} \geqslant 0$ is already implied by the LMIs (4.25).
The second statement now follows from Proposition 2.37 (i) and Theorem 2.35 taking into account $0 \in \operatorname{cone}(S \times \{1\})$. □

Proposition 4.32 and its preceding paragraph motivate the formulation of the following algorithm for the computation of homogeneous δ-approximations of S.

ALGORITHM 4.4: homogeneous δ-approximation algorithm for spectrahedral shadows

`Input:` a spectrahedral shadow $S \subseteq \mathbb{R}^n$, point $\bar{x} \in \operatorname{int} S$ satisfying (A2), error tolerance $\delta > 0$
`Output:` outer and inner homogeneous δ-approximation P_{O} and P_{I} of S

1 $K \leftarrow \operatorname{cone}(S \times \{1\})$
2 compute outer and inner approximation K_{O} and K_{I} of K using Algorithm 4.1 with input parameters $p = (\bar{x}^{\mathsf{T}}, 1)^{\mathsf{T}}$, $\bar{d} = p/\|p\|$ and error tolerance δ
3 $P_{\mathsf{O}} \leftarrow \pi_{\mathbb{R}^n}[K_{\mathsf{O}} \cap H(e_{n+1}, 1)]$
4 $P_{\mathsf{I}} \leftarrow \pi_{\mathbb{R}^n}[K_{\mathsf{I}} \cap H(e_{n+1}, 1)]$

The function $\pi_{\mathbb{R}^n} \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ in lines 3 and 4 denotes the projection given by

$$\pi_{\mathbb{R}^n}\left(\begin{pmatrix} x \\ x_{n+1} \end{pmatrix}\right) = x.$$

Equation (4.19) implies that K in line 1 is closed if and only if S is compact. Thus, Algorithm 4.1 in line 2 may be executed with a set that is not closed. We have already discussed in Subsection 3.2.2 that all presented algorithms are not limited to closed spectrahedral shadows and that the closedness merely guarantees that the optimal values of the primal semidefinite programs from Section 3.2, if finite, are attained.
For Algorithm 4.4 we are more specific in regard to this argument because the closedness of K may not be assumed. Here, only problems of type $(\mathrm{P}_2(p, d, K))$ and $(\mathrm{D}_2(p, d, K))$ are relevant, namely in the execution of Algorithm 4.1 in line 2. According to Proposition 3.11 (i) it holds $d \in 0^\infty K$ whenever $(\mathrm{P}_2(p, d, K))$ is unbounded and K is closed. If K is not closed, then unboundedness of $(\mathrm{P}_2(p, d, K))$ implies $d \in 0^\infty \operatorname{cl} K$, cf. [Roc70, Theorem 8.3]. Furthermore, the existence of a solution to $(\mathrm{D}_2(p, d, K))$ only depends on the boundedness and strict feasibility of the primal problem but not on the existence of a solution to it. In particular, if $p \in K$ and t^* is the optimal value of $(\mathrm{P}_2(p, d, K))$, then the hyperplane $H(w^*, w^{*\mathsf{T}} p + t^*)$ generated from a solution to $(\mathrm{D}_2(p, d, K))$ according to Proposition 3.11 (ii) exists. Since closedness of K is not used in the proof of Proposition 3.11 (ii) and $p + t^* d \in \operatorname{cl} K$, $H(w^*, w^{*\mathsf{T}} p + t^*)$ is a supporting hyperplane of $\operatorname{cl} K$. It follows that Algorithm 4.1 applied to K returns approximations of $0^\infty \operatorname{cl} K$ rather than $0^\infty K$ but is still finite irrespective of whether K is closed or not. Because K is a convex cone, it holds

$$0^\infty \operatorname{cl} K = \operatorname{cl} K = \operatorname{cl} 0^\infty K. \tag{4.26}$$

Thus, the cones K_O and K_I computed in line 2 are approximations of $\operatorname{cl} K = \operatorname{homog} S$.

Remark 4.33. In Equation (4.26) it is crucial that K is a convex cone. As an illustration, consider the convex set

$$C = \left\{ x \in \mathbb{R}^2 \,\middle|\, 0 \leqslant x_1 < 1, 0 \leqslant x_2 \right\} \cup \left\{ (1,0)^\mathsf{T} \right\}.$$

Then $\operatorname{cl} 0^\infty C = 0^\infty C = \{0\}$ but $0^\infty \operatorname{cl} C = \operatorname{cone} \{(0,1)^\mathsf{T}\}$. Hence, if Algorithm 4.1 is applied to a spectrahedral shadow S that is not closed, it returns approximations of $0^\infty \operatorname{cl} S$ but not necessarily of $\operatorname{cl} 0^\infty S$. However, if S is closed, then $\operatorname{cl} 0^\infty S = 0^\infty S$, see [Roc70, Theorem 8.2].

The above considerations enable the following result.

Theorem 4.34. *Algorithm 4.4 works correctly, in particular it terminates with an outer homogeneous δ-approximation P_O and an inner homogeneous δ-approximation P_I of S.*

Proof. Since $\bar{x}$ satisfies Assumption (A2) for S, the point $p = \left(\bar{x}^\mathsf{T}, 1\right)^\mathsf{T}$ satisfies analogous conditions for K. This is seen from a direct calculation using the representation of K derived in Proposition 4.32. Similarly, $\bar{x} \in \operatorname{int} S$ implies $\left(\bar{x}^\mathsf{T}, 1\right)^\mathsf{T} \in \operatorname{int} K$. In particular,

$$\bar{d} = \frac{1}{\sqrt{\bar{x}^\mathsf{T}\bar{x} + 1}} \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}$$

satisfies Assumption (A3) for K because $K = 0^\infty K$. From the preceding considerations and Theorem 4.14 it follows that line 2 of Algorithm 4.4 works correctly and returns polyhedral cones K_O and K_I satisfying

$$K_\mathsf{I} \subseteq 0^\infty \operatorname{cl} K = \operatorname{cl} K = \operatorname{homog} S \subseteq K_\mathsf{O} \tag{4.27}$$

as well as $d_\mathsf{tH}\left(K_\mathsf{O}, \operatorname{homog} S\right) \leqslant \delta$ and $d_\mathsf{tH}\left(K_\mathsf{I}, \operatorname{homog} S\right) \leqslant \delta$. For the sets P_O and P_I in lines 3 and 4 it holds

$$\operatorname{homog} P_\mathsf{O} = K_\mathsf{O} \cap H^+(e_{n+1}, 0) \quad \text{and} \quad \operatorname{homog} P_\mathsf{I} = K_\mathsf{I}.$$

The difference arises from the inclusions in (4.27) and the fact that homogenizations are contained in the halfspace $H^+(e_{n+1}, 0)$ by definition. However,

$$d_\mathsf{tH}\left(K_\mathsf{O} \cap H^+(e_{n+1}, 0), \operatorname{homog} S\right) \leqslant d_\mathsf{tH}\left(K_\mathsf{O}, \operatorname{homog} S\right) \leqslant \delta$$

holds due to the inclusion $\operatorname{homog} S \subseteq K_\mathsf{O} \cap H^+(e_{n+1}, 0)$. Therefore, P_O and P_I are homogeneous δ-approximations of S.
The finiteness of Algorithm 4.4 is evident from the finiteness of Algorithm 4.1. □

Remark 4.35. The inner homogeneous δ-approximation P_I of S computed by Algorithm 4.4 is always compact. Due to the containment of its homogenization K_I in the set cone $(S \times \{1\})$, one has $\mu > 0$ whenever $(x, \mu)^\mathsf{T} \in K_\mathsf{I} \setminus \{0\}$. Thus, every direction of K_I corresponds to a point of P_I, in particular

$$0^\infty P_\mathsf{I} = \pi_{\mathbb{R}^n}\left[K_\mathsf{I} \cap H(e_{n+1}, 0)\right] = \{0\}$$

according to Equation (4.19).

We conclude this section with two examples.

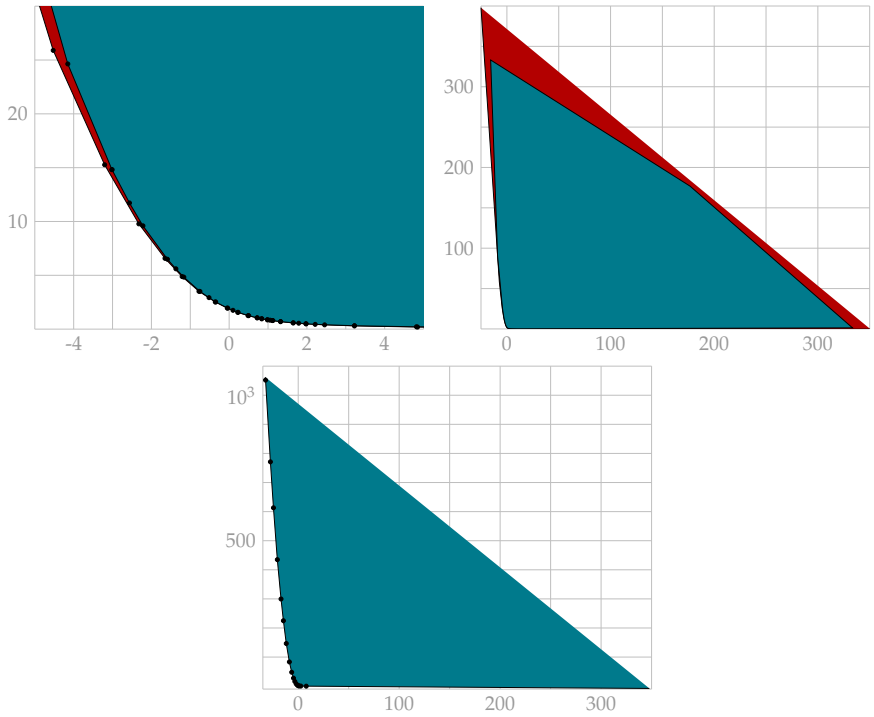

FIGURE 4.6: The top row depicts the outer and inner homogeneous δ-approximation P_O and P_I of the set from Example 4.36 computed by Algorithm 4.4 with $\delta = 0.01$ in red and blue, respectively. On the left is a section around the origin with vertices marked by black dots. Note that the distance between vertices of P_O and P_I increases with their distance to the origin. The right figure shows the scale of P_I, which is compact. An (ε, δ)-approximation with the same guarantees in approximation quality as P_O is displayed in the bottom row.

Example 4.36. Algorithm 4.4 is applied to the spectrahedral shadow

$$S = \left\{ x \in \mathbb{R}^2 \,\middle|\, \exists\, y \in \mathbb{R}^2 \colon \begin{pmatrix} x_1 - y_1 & 1 \\ 1 & x_2 - y_2 \end{pmatrix} \succcurlyeq 0, \begin{pmatrix} y_2 & y_1 \\ y_1 & 1 \end{pmatrix} \succcurlyeq 0 \right\},$$

which is the Minkowski sum of the epigraphs of the functions $x \mapsto x^{-1}$ restricted to $\mathbb{R}_{++}$ and $x \mapsto x^2$. As input parameters we set $\delta = 0.01$ and $\bar{x} \approx (1.8846, 1.8846)^\mathsf{T}$. For the resulting outer approximation P_O it holds

$$e\left[\operatorname{vert} P_O, S\right] \approx 0.1713 \quad \text{and} \quad d_{\mathsf{H}}\left(0^\infty P_O, 0^\infty S\right) \approx 0.0520.$$

For comparison, we use Algorithm 4.2 to compute an (ε, δ)-approximation P of S with ε and δ set to these values. The remaining parameters are $p = \bar{x}$, $\bar{d} = e/\|e\|$ and $\gamma = 0.1$. Figure 4.6 depicts P_O as well as the determined inner homogeneous δ-approximation P_I in the top row and P in the bottom row. In total, 534 semidefinite programs are solved during the execution of Algorithm 4.4 for finding P_O and P_I, while 222 are solved to obtain P with Algorithm 4.2. We point out that the maximum norm of vertices of P_O and P are around 58.8 and 1052.5, respectively. This difference is explained by the scale of the compact component $\bar{S}$ that is approximated as part of Algorithm 4.2.

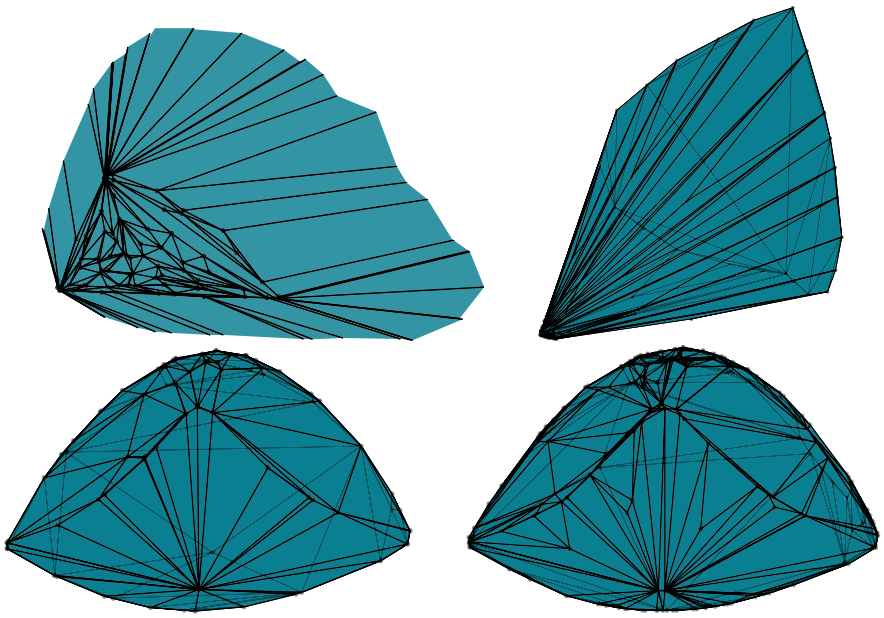

FIGURE 4.7: The top row shows homogeneous δ-approximations P_{O} and P_{I} of the set S from Example 4.37 for $\delta = 0.03$ restricted to $B_7(0)$ and $B_{100}(0)$, respectively. Their polar sets, which are homogeneous δ-approximations of S°, are depicted in the bottom row. Both are contained in a ball of radius 1.02.

Example 4.37. Consider the spectrahedral shadow S consisting of points $x \in \mathbb{R}^3$ that admit the existence of $y^1, y^2 \in \mathbb{R}^3$ and $Z \in \mathcal{S}^3_+$ such that the system

$$\begin{aligned} x - y^1 - y^2 &= 0 & 2z_{13} + z_{22} &= 0 \\ z_{11} - 1 &= 0 & z_{33} - 1 &= 0 \\ \begin{pmatrix} y_1^2 + y_2^2 & 0 & 2\left(y_1^2 - y_2^2\right) \\ 0 & y_1^2 + y_2^2 & 2\sqrt{2}y_3^2 \\ 2\left(y_1^2 - y_2^2\right) & 2\sqrt{2}y_3^2 & y_1^2 + y_2^2 \end{pmatrix} &\succcurlyeq 0 & \begin{pmatrix} z_{11} & y_1^1 & y_2^1 \\ y_1^1 & z_{22} & y_3^1 \\ y_2^1 & y_3^1 & z_{33} \end{pmatrix} &\succcurlyeq 0 \end{aligned}$$

is satisfied. Algorithm 4.4 with $\delta = 0.03$ and $\bar{x} = 0$ requires the solutions to 1989 problems of type $(\mathrm{P}_2(p, d, K))$, where $K = \operatorname{cone}(S \times \{1\})$. The resulting outer approximation P_{O} has 92 vertices and 24 extreme directions, while the compact inner approximation P_{I} is composed of 177 vertices. According to Theorem 4.31 their polar sets P_{O}° and P_{I}° are an inner and outer homogeneous δ-approximation of S°, respectively. They are both compact with 94 and 232 vertices each. Sets P_{O} and P_{I} as well as their polars are shown in Figure 4.7.

Bibliography

[AH01] J. Arndt and C. Haenel. *Pi—unleashed*. Second. Translated from the 1998 German original by Catriona Lischka and David Lischka, With 1 CD-ROM (Windows). Springer-Verlag, Berlin, 2001, pp. xii+270.

[AT03] A. Auslender and M. Teboulle. *Asymptotic cones and functions in optimization and variational inequalities*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003, pp. xii+249.

[AUU22] Ç. Ararat, F. Ulus, and M. Umer. "A norm minimization-based convex vector optimization algorithm". In: *J. Optim. Theory Appl.* 194.2 (2022), pp. 681–712.

[Bat86] R. G. Batson. "Extensions of Radstrom's lemma with application to stability theory of mathematical programming". In: *J. Math. Anal. Appl.* 117.2 (1986), pp. 441–448.

[Bat88] R. G. Batson. "Necessary and sufficient conditions for boundedness of extreme points of unbounded convex sets". In: *J. Math. Anal. Appl.* 130.2 (1988), pp. 365–374.

[BBW22] H. H. Bauschke, T. Bendit, and H. Wang. *The homogenization cone: polar cone and projection*. 2022. URL: https://arxiv.org/abs/2206.02694.

[BCR98] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry*. Vol. 36. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Translated from the 1987 French original, Revised by the authors. Springer-Verlag, Berlin, 1998, pp. x+430.

[Ben98] H. P. Benson. "Hybrid approach for solving multiple-objective linear programs in outcome space". In: *J. Optim. Theory Appl.* 98.1 (1998), pp. 17–35.

[BF34] T. Bonnesen and W. Fenchel. *Theorie der konvexen Körper*. Springer Berlin, Heidelberg, 1934, pp. vii+164.

[BH91] H. P. Benson and R. Horst. "A branch and bound–outer approximation algorithm for concave minimization over a convex set". In: *Comput. Math. Appl.* 21.6-7 (1991), pp. 67–76.

[BI75] E. M. Bronstein and L. D. Ivanov. "The approximation of convex sets by polyhedra". In: *Sibirsk. Mat. Ž.* 16.5 (1975), pp. 1110–1112, 1132.

[BL00] J. M. Borwein and A. S. Lewis. *Convex analysis and nonlinear optimization*. Vol. 3. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Theory and examples. Springer-Verlag, New York, 2000, pp. x+273.

[Boy+94] S. Boyd et al. *Linear matrix inequalities in system and control theory*. Vol. 15. SIAM Studies in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994, pp. xii+193.

[BPR06] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry*. Second. Vol. 10. Algorithms and Computation in Mathematics. Springer-Verlag, Berlin, 2006, pp. x+662.

[BPT13] G. Blekherman, P. A. Parrilo, and R. R. Thomas, eds. *Semidefinite optimization and convex algebraic geometry*. Vol. 13. MOS-SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2013, pp. xx+476.

[Bri20] J. Brinkhuis. *Convex analysis for optimization—a unified approach*. Graduate Texts in Operations Research. Springer, Cham, 2020, pp. xxvii+257.

[Bro08] E. M. Bronstein. "Approximation of convex sets by polytopes". In: *J. Math. Sci.* 153.6 (2008), pp. 727–762.

[BRS15] A. Bhardwaj, P. Rostalski, and R. Sanyal. "Deciding polyhedrality of spectrahedra". In: *SIAM J. Optim.* 25.3 (2015), pp. 1873–1884.

[BV04] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, Cambridge, 2004, pp. xiv+716.

[BW81] J. Borwein and H. Wolkowicz. "Regularizing the abstract convex program". In: *J. Math. Anal. Appl.* 83.2 (1981), pp. 495–530.

[BY11] D. P. Bertsekas and H. Yu. "A unifying polyhedral approximation framework for convex optimization". In: *SIAM J. Optim.* 21.1 (2011), pp. 333–360.

[CG59] E. W. Cheney and A. A. Goldstein. "Newton's method for convex programming and Tchebycheff approximation". In: *Numer. Math.* 1 (1959), pp. 254–268.

[Cir19] D. Ciripoi. "Approximation of spectrahedral shadows and spectrahedral calculus". PhD thesis. Friedrich Schiller University Jena, 2019.

[CLW18] D. Ciripoi, A. Löhne, and B. Weißing. "A vector linear programming approach for certain global optimization problems". In: *J. Global Optim.* 72.2 (2018), pp. 347–372.

[CLW19] D. Ciripoi, A. Löhne, and B. Weißing. "Calculus of convex polyhedra and polyhedral convex functions by utilizing a multiple objective linear programming solver". In: *Optimization* 68.10 (2019), pp. 2039–2054.

[Col75] G. E. Collins. "Quantifier elimination for real closed fields by cylindrical algebraic decomposition". In: *Automata theory and formal languages (Second GI Conf., Kaiserslautern, 1975)*. Lecture Notes in Comput. Sci., Vol. 33. Springer, Berlin, 1975, pp. 134–183.

[CR09] E. J. Candès and B. Recht. "Exact matrix completion via convex optimization". In: *Found. Comput. Math.* 9.6 (2009), pp. 717–772.

[DG86] M. A. Duran and I. E. Grossmann. "An outer-approximation algorithm for a class of mixed-integer nonlinear programs". In: *Math. Programming* 36.3 (1986), pp. 307–339.

[DL22] D. Dörfler and A. Löhne. *A polyhedral approximation algorithm for recession cones of spectrahedral shadows*. 2022. URL: https://arxiv.org/abs/2206.15172.

[Dör+22] D. Dörfler et al. "A Benson-type algorithm for bounded convex vector optimization problems with vertex selection". In: *Optim. Methods Softw.* 37.3 (2022), pp. 1006–1026.

[Dör22] D. Dörfler. "On the approximation of unbounded convex sets by polyhedra". In: *J. Optim. Theory Appl.* 194.1 (2022), pp. 265–287.

[Dud74] R. M. Dudley. "Metric entropy of some classes of sets with differentiable boundaries". In: *J. Approximation Theory* 10 (1974), pp. 227–236.

[Eic08] G. Eichfelder. *Adaptive scalarization methods in multiobjective optimization*. Vector Optimization. Springer-Verlag, Berlin, 2008, pp. xiv+241.

[ELS12] M. Ehrgott, A. Löhne, and L. Shao. "A dual variant of Benson's "outer approximation algorithm" for multiple objective linear programming". In: *J. Global Optim.* 52.4 (2012), pp. 757–778.

[ESS11] M. Ehrgott, L. Shao, and A. Schöbel. "An approximation algorithm for convex multi-objective programming problems". In: *J. Global Optim.* 50.3 (2011), pp. 397–416.

[Euc56] Euclid. *The thirteen books of Euclid's Elements translated from the text of Heiberg. Vol. I: Introduction and Books I, II. Vol. II: Books III–IX. Vol. III: Books X–XIII and Appendix*. Translated with introduction and commentary by Thomas L. Heath, 2nd ed. Dover Publications, Inc., New York, 1956, xi+432 pp., i+436 pp., i+546.

[Faz02] M. Fazel. "Matrix rank minimization with applications". PhD thesis. Stanford University, 2002.

[Fej48] L. Fejes Tóth. "Approximation by polygons and polyhedra". In: *Bull. Amer. Math. Soc.* 54 (1948), pp. 431–438.

[Gie+19] J. Giesen et al. "Using Benson's algorithm for regularization parameter tracking". In: *The Thirty-Third AAAI Conference on Artificial Intelligence*. AAAI Press, 2019, pp. 3689–3696.

[GM12] B. Gärtner and J. Matoušek. *Approximation algorithms and semidefinite programming*. Springer, Heidelberg, 2012, pp. xii+251.

[Gob+13] M. A. Goberna et al. "Motzkin decomposition of closed convex sets via truncation". In: *J. Math. Anal. Appl.* 400.1 (2013), pp. 35–47.

[Gom58] R. E. Gomory. "Outline of an algorithm for integer solutions to linear programs". In: *Bull. Amer. Math. Soc.* 64 (1958), pp. 275–278.

[GR95] A. J. Goldman and M. Ramana. "Some geometric results in semidefinite programming". In: *J. Global Optim.* 7.1 (1995), pp. 33–50.

[Gru07] P. M. Gruber. *Convex and discrete geometry*. Vol. 336. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin, 2007, pp. xiv+578.

[GT56] A. J. Goldman and A. W. Tucker. "Polyhedral convex cones". In: *Linear inequalities and related systems*. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N.J., 1956, pp. 19–40.

[GWZ15] F. Guo, C. Wang, and L. Zhi. "Semidefinite representations of noncompact convex sets". In: *SIAM J. Optim.* 25.1 (2015), pp. 377–395.

[Hea97] T. L. Heath. *The works of Archimedes*. Cambridge University Press, 1897, pp. clxxxvi+326.

[Hel00] C. Helmberg. "Semidefinite programming for combinatorial optimization". Habilitation thesis. TU Berlin, 2000.

[HJ13] R. A. Horn and C. R. Johnson. *Matrix analysis*. Second. Cambridge University Press, Cambridge, 2013, pp. xviii+643.

[HK71] K. Hoffman and R. Kunze. *Linear algebra*. Second. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1971, pp. viii+407.

[HL01] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of convex analysis*. Grundlehren Text Editions. Abridged version of *Convex analysis and minimization algorithms. I* [Springer, Berlin, 1993] and *II* [ibid.]. Springer-Verlag, Berlin, 2001, pp. x+259.

[HLR14] A. H. Hamel, A. Löhne, and B. Rudloff. "Benson type algorithms for linear vector optimization and applications". In: *J. Global Optim.* 59.4 (2014), pp. 811–836.

[HN09] J. W. Helton and J. Nie. "Sufficient and necessary conditions for semidefinite representability of convex hulls and sets". In: *SIAM J. Optim.* 20.2 (2009), pp. 759–791.

[HN10] J. W. Helton and J. Nie. "Semidefinite representation of convex sets". In: *Math. Program.* 122.1, Ser. A (2010), pp. 21–64.

[HN12] J. W. Helton and J. Nie. "Semidefinite representation of convex sets and convex hulls". In: *Handbook on semidefinite, conic and polynomial optimization*. Vol. 166. Internat. Ser. Oper. Res. Management Sci. Springer, New York, 2012, pp. 77–112.

[Hof81] K. L. Hoffman. "A method for globally minimizing concave functions over convex sets". In: *Math. Programming* 20.1 (1981), pp. 22–32.

[Hol75] R. B. Holmes. *Geometric functional analysis and its applications*. Graduate Texts in Mathematics, No. 24. Springer-Verlag, New York-Heidelberg, 1975, pp. x+246.

[Hor+91] R. Horst et al. "On solving a d.c. programming problem by a sequence of linear programs". In: *J. Global Optim.* 1.2 (1991), pp. 183–203.

[HV07] J. W. Helton and V. Vinnikov. "Linear matrix inequality representation of sets". In: *Comm. Pure Appl. Math.* 60.5 (2007), pp. 654–674.

[IS10] A. Iusem and A. Seeger. "Distances between closed convex cones: old and new results". In: *J. Convex Anal.* 17.3-4 (2010), pp. 1033–1055.

[Kam02] G. K. Kamenev. "Conjugate adaptive algorithms for the polyhedral approximation of convex bodies". In: *Zh. Vychisl. Mat. Mat. Fiz.* 42.9 (2002), pp. 1351–1367.

[Kam08] G. K. Kamenev. "Duality theory of optimal adaptive methods for the polyhedral approximation of convex bodies". In: *Zh. Vychisl. Mat. Mat. Fiz.* 48.3 (2008), pp. 397–417.

[Kam92] G. K. Kamenev. "A class of adaptive algorithms for the approximation of convex bodies by polyhedra". In: *Zh. Vychisl. Mat. i Mat. Fiz.* 32.1 (1992), pp. 136–152.

[Kam93] G. K. Kamenev. "Efficiency of Hausdorff algorithms for polyhedral approximation of convex bodies". In: *Zh. Vychisl. Mat. i Mat. Fiz.* 33.5 (1993), pp. 796–805.

[Kam94] G. K. Kamenev. "Analysis of an algorithm for approximating convex bodies". In: *Zh. Vychisl. Mat. i Mat. Fiz.* 34.4 (1994), pp. 608–616.

[Kel60] J. E. Kelley Jr. "The cutting-plane method for solving convex programs". In: *J. Soc. Indust. Appl. Math.* 8 (1960), pp. 703–712.

[Kle57] V. L. Klee Jr. “Extremal structure of convex sets”. In: *Arch. Math. (Basel)* 8 (1957), pp. 234–240.

[KLW16] J. Kronqvist, A. Lundell, and T. Westerlund. “The extended supporting hyperplane algorithm for convex mixed-integer nonlinear programming”. In: *J. Global Optim.* 64.2 (2016), pp. 249–272.

[Kno86] W. R. Knorr. *The ancient tradition of geometric problems*. Birkhäuser Boston, Inc., Boston, MA, 1986, pp. x+411.

[Kum16] M. Kummer. “Two results on the size of spectrahedral descriptions”. In: *SIAM J. Optim.* 26.1 (2016), pp. 589–601.

[Las09] J. B. Lasserre. “Convex sets with semidefinite representation”. In: *Math. Program.* 120.2, Ser. A (2009), pp. 457–477.

[LL92] C. Lassez and J.-L. Lassez. “Quantifier elimination for conjunctions of linear constraints via a convex hull algorithm”. In: *Symbolic and Numerical Computation for Artificial Intelligence*. Ed. by B. R. Donald, D. Kapur, and J. L. Mundy. Academic Press, 1992.

[Löh+19] A. Löhne et al. “Solving bilevel problems with polyhedral constraint set”. In: *J. Appl. Numer. Optim.* 1.3 (2019), pp. 243–251.

[Löh11] A. Löhne. *Vector optimization with infimum and supremum*. Vector Optimization. Springer, Heidelberg, 2011, pp. x+206.

[Löh20] A. Löhne. *Approximate vertex enumeration*. 2020. URL: https://arxiv.org/abs/2007.06325.

[LRU14] A. Löhne, B. Rudloff, and F. Ulus. “Primal and dual approximation algorithms for convex vector optimization problems”. In: *J. Global Optim.* 60.4 (2014), pp. 713–736.

[LV12] M. Laurent and F. Vallentin. *Semidefinite optimization*. Lecture notes. 2012. URL: http://www.mi.uni-koeln.de/opt/wp-content/uploads/2015/06/laurent_vallentin_sdo_2012_05.pdf.

[LW16] A. Löhne and B. Weißing. “Equivalence between polyhedral projection, multiple objective linear programming and vector linear programming”. In: *Math. Methods Oper. Res.* 84.2 (2016), pp. 411–426.

[Min03] H. Minkowski. “Volumen und Oberfläche”. In: *Math. Ann.* 57.4 (1903), pp. 447–495.

[Mon10] D. Monniaux. “Quantifier elimination by lazy model enumeration”. In: *Computer aided verification*. Vol. 6174. Lecture Notes in Comput. Sci. Springer, Berlin, 2010, pp. 585–599.

[Mor62] J.-J. Moreau. “Décomposition orthogonale d’un espace hilbertien selon deux cônes mutuellement polaires”. In: *C. R. Acad. Sci. Paris* 255 (1962), pp. 238–240.

[Mos69] U. Mosco. “Convergence of convex sets and of solutions of variational inequalities”. In: *Advances in Math.* 3 (1969), pp. 510–585.

[Mot36] T. S. Motzkin. “Beiträge zur Theorie der linearen Ungleichungen”. PhD thesis. University of Basel, 1936.

[Nem07] A. Nemirovski. “Advances in convex optimization: conic programming”. In: *International Congress of Mathematicians. Vol. I*. Eur. Math. Soc., Zürich, 2007, pp. 413–444.

[Net11] T. Netzer. "Spectrahedra and their Shadows". Habilitation thesis. University of Leipzig, 2011.

[NN94] Y. Nesterov and A. Nemirovskii. *Interior-point polynomial algorithms in convex programming*. Vol. 13. SIAM Studies in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994, pp. x+405.

[NR95] P. E. Ney and S. M. Robinson. "Polyhedral approximation of convex sets with an application to large deviation probability theory". In: *J. Convex Anal.* 2.1-2 (1995), pp. 229–240.

[NS09] T. Netzer and R. Sinn. *A note on the convex hull of finitely many projections of spectrahedra*. 2009. URL: https://arxiv.org/abs/0908.3386.

[PB14] N. Parikh and S. Boyd. "Block splitting for distributed optimization". In: *Math. Program. Comput.* 6.1 (2014), pp. 77–102.

[PT20] P. A. Parrilo and R. R. Thomas, eds. *Sum of squares: theory and applications*. Vol. 77. Proceedings of Symposia in Applied Mathematics. AMS Short Course, Sum of Squares: Theory and Applications, January 14–15, 2019, Baltimore, MD. 2020, pp. vii+142.

[Ram97] Motakuri V. Ramana. "An exact duality theory for semidefinite programming and its complexity implications". In: *Math. Programming* 77.2, Ser. B (1997), pp. 129–162.

[Roc70] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970, pp. xviii+451.

[Rom05] S. Roman. *Advanced linear algebra*. Second. Vol. 135. Graduate Texts in Mathematics. Springer, New York, 2005, pp. xvi+482.

[RTW97] M. V. Ramana, L. Tunçel, and H. Wolkowicz. "Strong duality for semidefinite programming". In: *SIAM J. Optim.* 7.3 (1997), pp. 641–662.

[RW05] S. Ruzika and M. M. Wiecek. "Approximation methods in multiobjective programming". In: *J. Optim. Theory Appl.* 126.3 (2005), pp. 473–501.

[RW98] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis*. Vol. 317. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1998, pp. xiv+733.

[Sch18a] C. Scheiderer. "Semidefinite representation for convex hulls of real algebraic curves". In: *SIAM J. Appl. Algebra Geom.* 2.1 (2018), pp. 1–25.

[Sch18b] C. Scheiderer. "Spectrahedral shadows". In: *SIAM J. Appl. Algebra Geom.* 2.1 (2018), pp. 26–44.

[Sch81] R. Schneider. "Zur optimalen Approximation konvexer Hyperflächen durch Polyeder". In: *Math. Ann.* 256.3 (1981), pp. 289–301.

[SE08a] L. Shao and M. Ehrgott. "Approximately solving multiobjective linear programmes in objective space and an application in radiotherapy treatment planning". In: *Math. Methods Oper. Res.* 68.2 (2008), pp. 257–276.

[SE08b] L. Shao and M. Ehrgott. "Approximating the nondominated set of an MOLP by approximately solving its dual problem". In: *Math. Methods Oper. Res.* 68.3 (2008), pp. 469–492.

[SW79] G. Salinetti and R. J.-B. Wets. "On the convergence of sequences of convex sets in finite dimensions". In: *SIAM Rev.* 21.1 (1979), pp. 18–33.

[TTB83] T. V. Thiệu, B. T. Tâm, and V. T. Ba'n. "An outer approximation method for globally minimizing a concave function over a compact convex set". In: *Acta Math. Vietnam.* 8.1 (1983), 21–40 (1984).

[Ulu18] F. Ulus. "Tractability of convex vector optimization problems in the sense of polyhedral approximations". In: *J. Global Optim.* 72.4 (2018), pp. 731–742.

[VB96] L. Vandenberghe and S. Boyd. "Semidefinite programming". In: *SIAM Rev.* 38.1 (1996), pp. 49–95.

[VB99] L. Vandenberghe and S. Boyd. "Applications of semidefinite programming". In: *Proceedings of the Stieltjes Workshop on High Performance Optimization Techniques (HPOPT '96) (Delft)*. Vol. 29. 3. 1999, pp. 283–299.

[Vei67] A. F. Veinott Jr. "The supporting hyperplane method for unimodal programming". In: *Operations Res.* 15 (1967), pp. 147–152.

[Wag+22] A. Wagner et al. *Algorithms to solve unbounded convex vector optimization problems*. 2022. URL: https://arxiv.org/abs/2207.03200.

[Wei17] B. Weißing. "The polyhedral projection problem". PhD thesis. Friedrich Schiller University Jena, 2017.

[WP95] T. Westerlund and F. Pettersson. "An extended cutting plane method for solving convex MINLP problems". In: *Computers & Chemical Engineering* 19 (1995), pp. 131–136.

[WW67] D. W. Walkup and R. J.-B. Wets. "Continuity of some convex-cone-valued mappings". In: *Proc. Amer. Math. Soc.* 18 (1967), pp. 229–235.

[YBR08] H. Yu, D. P. Bertsekas, and J. Rousu. *An efficient discriminative training method for generative models*. 6th International Workshop on Mining and Learning with Graphs. 2008. URL: https://www.mit.edu/~dimitrib/Yu_Bertsekas_Rousu_Extended.pdf (visited on 07/19/2022).

[Zha05] F. Zhang, ed. *The Schur complement and its applications*. Vol. 4. Numerical Methods and Algorithms. Springer-Verlag, New York, 2005, pp. xvi+295.

[Zie95] G. M. Ziegler. *Lectures on polytopes*. Vol. 152. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. x+370.

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