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# Detecting outliers in multivariate volatility models: A wavelet procedure*

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**Abstract**

It is well known that outliers can affect both the estimation of parameters and volatilities when fitting a univariate GARCH-type model. Similar biases and impacts are expected to be found on correlation dynamics in the context of multivariate time series. We study the impact of outliers on the estimation of correlations when fitting multivariate GARCH models and propose a general detection algorithm based on wavelets, that can be applied to a large class of multivariate volatility models. Its effectiveness is evaluated through a Monte Carlo study before it is applied to real data. The method is both effective and reliable, since it detects very few false outliers.

*MSC:* 62M10, 91B84, 62G35, 62H10, 65C05.

*Keywords:* Correlations, multivariate GARCH models, outliers, wavelets.

## 1 Introduction

The correlation structure of security returns is the keystone of both portfolio allocation and risk management decisions. In the literature, there are several models to estimate correlations, the multivariate GARCH being the most popular class of models. Oil and financial series of returns often exhibit excess of kurtosis that can be caused by large unexpected observations. In the univariate context, some authors tried to capture this excess of kurtosis by estimating volatility models with fat tail distributed errors. However, it was observed that the estimated residuals of these models still registered excess kurtosis (see Baillie and Bollerslev, 1989, Teräsvirta, 1996). Furthermore, it is well known that these observations can affect the estimation of the GARCH parameters

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(Fox, 1972, Van Dijk, Franses and Lucas, 1999, Verhoeven and McAleer, 2000, Bali and Guirguis, 2007, Charles and Darné, 2014), the tests of conditional homoscedasticity (Carnero, Peña and Ruiz, 2007, Grossi and Laurini, 2009), the out-of-sample volatility forecasts (Ledolter, 1989, Chen and Liu, 1993, Franses and Ghijsels, 1999, Grané and Veiga, 2010, Boudt, Daníelsson and Laurent, 2013), the volatility estimates (Carnero, Peña and Ruiz, 2012, Behmiri and Manera, 2015) and the risk measures (Grané and Veiga, 2014). For a recent survey on the effects of the outliers on the specification, on the parameter estimation, on the volatility estimation and prediction see Hotta and Trucíos (2018). Analogously, when volatilities and correlations are estimated using multivariate GARCH models similar effects might be expected (see, for instance, Boudt and Croux, 2010).

Portfolios are often composed of commodities that can exhibit negative correlations with stock price returns as, for example, oil returns. Oil is a strategic commodity used as an input in all economic activities, therefore, turmoils in the oil market can propagate to stock markets and affect correlations making them quite negative (see Ramos, Martín-Barragán and Veiga, 2015). Therefore, the first objective of this paper is to study the effect of outliers, in particular, isolated level outliers, patches of level outliers and volatility outliers, on the estimated correlations of multivariate GARCH models (see Hotta and Tsay, 2012, Galeano and Peña, 2013, for a resume on the different types of outliers). We focus on the diagonal Baba-Engle-Kraft-Kroner (D-BEKK) by Engle and Kroner (1995), the conditional constant correlation (CCC) model by Bollerslev (1990) and the dynamic conditional correlation (DCC) model by Engle (2002). We have chosen these models because they are often used in empirical works (see Bauwens, Laurent and Rombout, 2006, Silvennoinen and Teräsvirta, 2009, for excelent surveys on these models). Moreover, as a multivariate GARCH model, the D-BEKK is easier to estimate than the full BEKK because it involves less parameters. Yet, the conclusions and the procedures of this work can be extended to other more sophisticated multivariate volatility models. The second aim is to propose a procedure able to detect outliers in multivariate volatility models that is based on the residuals. The detection of outliers may warn the researcher to use more sophisticate models, filter for outliers or to use robust methods as those proposed, for instance, in Muler and Yohai (2008) for univariate GARCH models and Boudt and Croux (2010) for multivariate GARCH models. Robust estimators work well in the case of additive outliers but lose their efficiency when there is an autoregressive high order dependence. Furthermore and regarding the M-estimators, their asymptotic theory is often based on the hypothesis that the errors are homoscedastic, which is not the case when dealing, for instance, with the DCC model. On the other hand, there are M-estimators that do not depend on homoscedastic errors but they are not so efficient. Therefore, our recommendation is to start by applying a detection procedure and in case outliers are detected, the researcher can either filter them, go for a robust estimation method or use a more sophisticated model that can accommodate outliers.

The Monte Carlo study leads us to conclude that outliers affect the estimated correlations and the effect is stronger for the conditional correlation models (CCC and DCC). Second, our detection procedure is very reliable, not only because the percentage of correct detections is quite high, but also because it detects very few false outliers. This property ensures that when one observation is detected as a possible outlier, it is indeed an outlier.

The advantages of our method are several: first, it can be applied to any multivariate volatility model given that the errors follow a known distribution, second it is well suited for detecting several types of outliers, such as isolated single/multiple outliers and patches of outliers; third, the method is easy and quick to apply, which makes it an attractive tool for academic communities and/or practitioners; fourth, it can be applied to a high number of series, and finally, it is reliable since it detects very few false outliers.

The organization of this paper is as follows. In Section 2 we present the volatility models used in the paper and review two particular types of additive outliers. In Section 3 we study the effect of outliers on the estimated correlations via several simulation studies. In Section 4 we present and evaluate the performance of the detection algorithm and we apply it to several daily series of returns in Section 5. Finally, we conclude in Section 6. Additional Monte Carlo experiments are reported in the Appendix.

## 2 Outliers in multivariate volatility models

Multivariate financial time series of returns exhibit similar patterns to those of univariate series, such as persistent time-varying volatilities. Additionally, they display time-varying correlations that are often modeled by multivariate GARCH models. One advantage of these models is that they are flexible enough to represent the dynamics of the volatilities and correlations. We start this section describing the models we are going to evaluate. Next we present the type of outliers we are going to consider.

### 2.1 Models under evaluation

The models that we consider have been pioneer in the financial econometrics literature and are often applied empirically to many fields such as volatility spillover transmission, contagion, portfolio management, asset allocation, etc. However, the methodology developed in this paper is not restricted to them.

In particular, the models under evaluation are the diagonal Baba-Engle-Kraft-Kroner (D-BEKK) model defined in Engle and Kroner (1995), the constant conditional correlation (CCC) model by Bollerslev (1990), and the dynamic conditional correlation (DCC) model by Engle (2002).

Let $\{\mathbf{y}_t\}$ be a vector stochastic process with dimension $N \times 1$ such that $E(\mathbf{y}_t) = \mathbf{0}$ and $\mathscr{F}_{t-1}$ is the information set till time $t-1$. We consider that $\mathbf{y}_t = \varepsilon_t$ and $\varepsilon_t = \mathbf{H}_t^{1/2}\boldsymbol{\eta}_t$, where $\mathbf{H}_t$ is the conditional covariance matrix of $\mathbf{y}_t$ and $\boldsymbol{\eta}_t$ is an iid vector error process such that $E(\boldsymbol{\eta}_t\boldsymbol{\eta}_t') = \mathbf{I}$, the identity matrix of order $N$. We assume that there is no linear dependence in $\mathbf{y}_t$.

Alternative approaches in the literature propose different models for the dependence of $\mathbf{H}_t$ on past information $\mathscr{F}_{t-1}$. In the D-BEKK, the dependence of $\mathbf{H}_t$ on past information is modeled directly. In contrast, in the CCC and DCC models, first the conditional variances are modeled using univariate specifications and then $\mathbf{H}_t$ is obtained by using these conditional standard deviations together with some specifications of the correlations (constant for CCC and time-varying for DCC).

We now proceed to a more detailed description of these three models.

**DIAGONAL BEKK MODEL** The D-BEKK of first order is a restricted version of the model defined in Engle and Kroner (1995), where the dependence of $\mathbf{H}_t$ on past information is modeled as follows:

$$\mathbf{H}_t = \mathbf{C}\mathbf{C}^{\mathsf{T}} + \mathbf{A}^{\mathsf{T}}\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}^{\mathsf{T}}\mathbf{A} + \mathbf{B}^{\mathsf{T}}\mathbf{H}_{t-1}\mathbf{B}, \tag{1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ diagonal matrices and $\mathbf{C}$ is a $N \times N$ lower triangular matrix. The D-BEKK is covariance stationary if and only if $a_{ii}^2 + b_{ii}^2 < 1$ for all $i$, where $a_{ii}$ and $b_{ii}$ are, respectively, the diagonal elements of $\mathbf{A}$ and $\mathbf{B}$. The conditional covariance matrix is positive definite by construction.

**CONDITIONAL CORRELATION MODELS** The CCC model is given by

$$\mathbf{H}_t = \mathbf{D}_t\mathbf{R}\mathbf{D}_t = \left(\rho_{ij}\sqrt{h_{ii,t}\,h_{jj,t}}\right)_{ij,t},$$

where $\mathbf{D}_t = \mathrm{diag}(h_{11,t}^{1/2}, \ldots h_{NN,t}^{1/2})$. Here $h_{ii,t}$ is defined in a univariate GARCH–type context such as $h_{ii,t} = \alpha_{0i} + \alpha_{1i}\varepsilon_{i,t-1}^2 + \beta_{1i}h_{ii,t-1}$ and $\mathbf{R} = (\rho_{ij})_{1\le i,j\le N}$ is a correlation matrix, that is symmetric and positive definite, with $\rho_{ii} = 1$, $-1 \le \rho_{ij} \le 1$, $\rho_{ij} = \rho_{ji}$ for $i, j = 1, \ldots, N$. If the $N$ conditional variances are positive, since $\mathbf{R}$ is a positive definite matrix, then $\mathbf{H}_t$ is positive definite. The number of parameters to be estimated are $N(N+5)/2$. Furthermore, univariate GARCH models require that $\alpha_{0i} > 0$, $\alpha_{1i} \ge 0$ and $\beta_{1i} \ge 0$ to guarantee positive conditional variances and $\alpha_{1i} + \beta_{1i} < 1$ to enforce stationary (see Duan et al., 2006).

On the other hand, the dynamic conditional correlation model, DCC, by Engle (2002) is defined as

$$\mathbf{H}_t = \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t, \tag{2}$$

with $\mathbf{D}_t$ defined as before and $\mathbf{R}_t = \left(q_{ij,t}/\sqrt{q_{ii,t}\,q_{jj,t}}\right)_{ij,t}$, where $\mathbf{Q}_t = (q_{ij,t})$ is a $N \times N$ symmetric positive definite matrix given by:

$$\mathbf{Q}_t = (1-\alpha-\beta)\,\bar{\mathbf{Q}} + \alpha\,\mathbf{u}_{t-1}\mathbf{u}_{t-1}^{\mathsf{T}} + \beta\,\mathbf{Q}_{t-1}, \tag{3}$$

where $\mathbf{u}_t = (u_{1,t}, \ldots, u_{N,t})^\intercal$ with $u_{i,t} = \varepsilon_{i,t}/\sqrt{h_{ii,t}}$, $\bar{\mathbf{Q}}$ is the unconditional variance matrix of $\mathbf{u}_t$ and $\alpha$ and $\beta$ are non-negative scalar parameters that satisfy $\alpha + \beta < 1$ (see Bauwens et al., 2006).

We now proceed to define the type of outliers we are going to study. Following Hotta and Tsay (2012), we distinguish two type of additive outliers, level and volatility, and propose a simple extension to the multivariate case.

### 2.2 Additive level outliers

Additive level outliers (ALOs) can be caused by institutional changes or market corrections that do not affect volatility. In this case, the conditional mean equation of the multivariate volatility model becomes:

$$\begin{aligned} \mathbf{y}_t &= \boldsymbol{\omega} \cdot I_T(t) + \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_t &= \mathbf{H}_t^{1/2} \boldsymbol{\eta}_t, \end{aligned} \tag{4}$$

where $\boldsymbol{\eta}_t$ is as before, that is, an iid vector error process such that $E(\boldsymbol{\eta}_t \boldsymbol{\eta}_t^\intercal) = \mathbf{I}$, $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N)^\intercal$ is a vector containing the ALOs' sizes and $I_T(t) = 1$ for $t \in T$ and 0 otherwise, representing the presence of ALOs at a given set of times $T$. ALOs can occur simultaneously at the same instant $t$ or not and their sizes can coincide or not.

Note that the conditional covariance matrix $\mathbf{H}_{t+1}$ depends only on the past information through $\boldsymbol{\varepsilon}_t$ and $\mathbf{H}_t$. Since the effect of the outlier is only in $\mathbf{y}_t$, the conditional covariance matrix will not be affected by this type of outliers. Indeed ALOs only affect the level of the series. This is true for all multivariate GARCH models.

### 2.3 Additive volatility outliers

Additive volatility outliers (AVOs) affect both the level of the returns and their volatilities (see Carnero et al., 2007, Grané and Veiga, 2010, Hotta and Tsay, 2012, Hotta and Trucíos, 2018). In this context, the conditional mean equation of the multivariate GARCH model becomes:

$$\begin{aligned} \mathbf{y}_t &= \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_t &= \boldsymbol{\omega} \cdot I_T(t) + \mathbf{H}_t^{1/2} \boldsymbol{\eta}_t, \end{aligned} \tag{5}$$

where $\boldsymbol{\eta}_t, \boldsymbol{\omega}$ and $I_T(t) = 1$ are defined as in section 2.2.

In contrast to ALOs, the effect of AVOs in $\mathbf{y}_t$ is through the term $\boldsymbol{\varepsilon}_t$ which indeed affects the conditional covariance matrix $\mathbf{H}_{t+1}$. This means that the values of the returns following the outlier occurrence will also be affected, since the conditional covariance matrix has been modified by the outlier. In order to highlight this behavior we are going to focus in the D-BEKK model.

Let $\{\mathbf{y}_t\}$ be a vector stochastic process following the D-BEKK model described by equation (1) and $\{\mathbf{y}_t^*\}$ a vector stochastic process following the D-BEKK model contaminated with an AVO at time $s$. Let $\mathbf{H}_t^*$ and $\boldsymbol{\varepsilon}_t^*$ denote the conditional covariance matrix and the vector of errors for the contaminated process $\{\mathbf{y}_t^*\}$, respectively. Using this notation, equation (1) for the process $\{\mathbf{y}_t^*\}$ is:

$$\mathbf{H}_t^* = \mathbf{C}\mathbf{C}^{\mathsf{T}} + \mathbf{A}^{\mathsf{T}}\boldsymbol{\varepsilon}_{t-1}^*\boldsymbol{\varepsilon}_{t-1}^{*,\mathsf{T}}\mathbf{A} + \mathbf{B}^{\mathsf{T}}\mathbf{H}_{t-1}^*\mathbf{B}.$$

At time $s$, when the outlier occurs, $\boldsymbol{\varepsilon}_s^* = \boldsymbol{\omega} + \boldsymbol{\varepsilon}_s$. Hence, at time $s+1$, the conditional covariance matrix of the process $\{\mathbf{y}_t^*\}$ will get contaminated. That is:

$$\mathbf{H}_{s+1}^* = \mathbf{H}_{s+1} + \mathbf{A}^{\mathsf{T}}(\boldsymbol{\omega}\boldsymbol{\omega}^{\mathsf{T}} + \boldsymbol{\omega}\boldsymbol{\varepsilon}_s^{\mathsf{T}} + \boldsymbol{\varepsilon}_s\boldsymbol{\omega}^{\mathsf{T}})\mathbf{A}.$$

Note that, after time $s+1$, the conditional covariance matrix of the process $\{\mathbf{y}_t^*\}$ is different than that of the non–contaminated process $\{\mathbf{y}_t\}$, since it is affected by both the second and the third terms of equation (1). It is easy to see that the third term is affected by the outlier since it ultimately depends on $\mathbf{H}_{t-1}^*$. The second term depends on $\boldsymbol{\varepsilon}_t^*$, whose covariance is actually $\mathbf{H}_t^*$, which is hence different from the non-contaminated vector of errors $\boldsymbol{\varepsilon}_t$.

Regarding, the CCC and DCC models the conditional covariance matrix $\mathbf{H}_t$ is also affected by the AVO, since it depends on the conditional variances obtained with the univariate GARCH models, that are as well affected by the AVO (see Carnero et al., 2007, Grané and Veiga, 2010, Carnero et al., 2012).

## 3 Effects of outliers on the correlations: Simulation studies

In the univariate literature it is well known that outliers can affect the estimation of parameters and volatility in the context of GARCH models. However, there are still few studies devoted to analyse the effects of these observations on the estimated correlations using multivariate GARCH models. In this section we contribute in this line by implementing simulation studies for Gaussian and Student-$t_7$ distributed errors.

**Experimental conditions** All simulation studies involve single, multiple and patches of additive level outliers and additive volatility outliers included in the models described in section 2.3.

The frequency of the simulations is daily and the number of simulated series is $N = 2$. Outliers are placed randomly across the simulated series, but in the same position for each pair of series. We consider that the outlier affects each pair of series at the same instant of time. Each scenario involves 1000 replications and series are simulated from CCC, DCC and D-BEKK(1,1,1) models with either normal or Student-$t_7$ distributed errors (see the Appendix). The number of replications is selected to provide robust results.

Given a model, we analysed 24 scenarios, that are defined from the type and number of outliers (one isolated ALO, multiple ALOs, patches of three ALOs, one isolated AVO), the size of the outlier ($\boldsymbol{\omega} = 5\boldsymbol{\sigma}_{\mathbf{y}}, 10\boldsymbol{\sigma}_{\mathbf{y}}$ for ALOs and $\boldsymbol{\omega} = 25\boldsymbol{\sigma}_{\mathbf{y}}, 50\boldsymbol{\sigma}_{\mathbf{y}}$ for AVOs) and the sample size of the simulated series ($n = 1000, 3000, 5000$).

GAUSSIAN ERRORS Parameter values were chosen by fitting the models to real time series of financial returns including commodities such as oil. In particular, for the D-BEKK model: $\{\text{vec}(\mathbf{C}) = (0.053, 0.042, 0, 0.020)^\intercal, \text{diag}(\mathbf{A}) = (0.161, 0.164)^\intercal, \text{diag}(\mathbf{B}) = (0.983, 0.981)^\intercal\}$; for the CCC: $\{\boldsymbol{\alpha}_0 = (0.010, 0.013), \boldsymbol{\alpha}_1 = (0.049, 0.067), \boldsymbol{\beta}_1 = (0.940, 0.926), \rho_{12} = -0.606)\}$; and for the DCC model: $\{\boldsymbol{\alpha}_0 = (0.010, 0.013), \boldsymbol{\alpha}_1 = (0.049, 0.067), \boldsymbol{\beta}_1 = (0.940, 0.926), \alpha = 0.015, \beta = 0.981\}$. Symbols $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_1$ stand for vectors of parameters of the univariate GARCH(1,1) models (see Grané and Veiga, 2010, for the details).

Simulation results are robust to the choice of parameter values and $N$. Therefore, we use $N = 2$ since it allows presenting results graphically without losing generality.

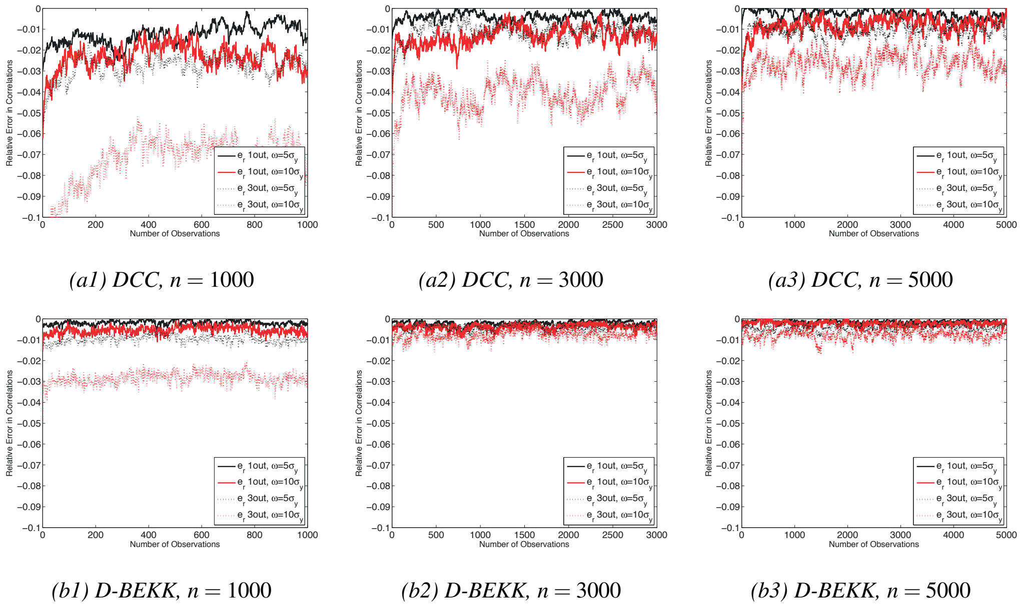
The results of this simulation study are reported in Table 1 (16 top rows) and Figures 1–3.[1]

***Table 1:*** *Relative bias in the estimated correlations obtained from a CCC model from 1000 simulated series of size n that include outliers of different magnitudes.*

CCC Model with Gaussian errors

| | $n$ | Estimated Correlation | Relative Bias | | $n$ | Estimated Correlation | Relative Bias |
|---|---|---|---|---|---|---|---|
| 1 ALO | 1000 | -0.5987 | -0.013 | 3 ALOSs | 1000 | -0.5892 | -0.028 |
| $\boldsymbol{\omega} = 5\boldsymbol{\sigma}_{\mathbf{y}}$ | 3000 | -0.6042 | -0.004 | $\boldsymbol{\omega} = 5\boldsymbol{\sigma}_{\mathbf{y}}$ | 3000 | -0.6007 | -0.010 |
| | 5000 | -0.6051 | -0.002 | | 5000 | -0.6017 | -0.008 |
| 1 ALO | 1000 | -0.5872 | -0.032 | 3 ALOs | 1000 | -0.5545 | -0.086 |
| $\boldsymbol{\omega} = 10\boldsymbol{\sigma}_{\mathbf{y}}$ | 3000 | -0.5970 | -0.016 | $\boldsymbol{\omega} = 10\boldsymbol{\sigma}_{\mathbf{y}}$ | 3000 | -0.5810 | -0.042 |
| | 5000 | -0.6012 | -0.009 | | 5000 | -0.5902 | -0.027 |
| Patch of | 1000 | -0.5972 | -0.015 | 1 AVO | 1000 | -0.5614 | -0.074 |
| 3 ALOs | 3000 | -0.6031 | -0.006 | $\boldsymbol{\omega} = 25\boldsymbol{\sigma}_{\mathbf{y}}$ | 3000 | -0.5805 | -0.043 |
| $\boldsymbol{\omega} = 5\boldsymbol{\sigma}_{\mathbf{y}}$ | 5000 | -0.6041 | -0.004 | | 5000 | -0.5847 | -0.036 |
| Patch of | 1000 | -0.5839 | -0.037 | 1 AVO | 1000 | -0.5318 | -0.123 |
| 3 ALOs | 3000 | -0.5959 | -0.017 | $\boldsymbol{\omega} = 50\boldsymbol{\sigma}_{\mathbf{y}}$ | 3000 | -0.5627 | -0.072 |
| $\boldsymbol{\omega} = 10\boldsymbol{\sigma}_{\mathbf{y}}$ | 5000 | -0.5999 | -0.011 | | 5000 | -0.5642 | -0.070 |
| No outliers | 1000 | -0.6064 | | | | | |
| | 3000 | -0.6065 | | | | | |
| | 5000 | -0.6064 | | | | | |

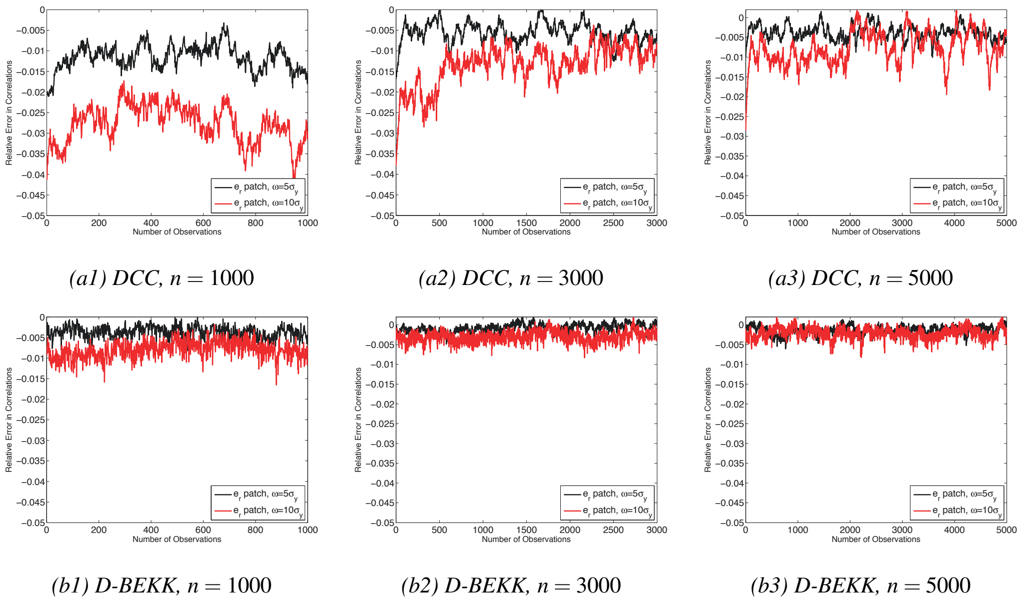
1. Preliminary results concerning isolated ALOs were presented in an invited conference in the 7th International Workshop on Statistical Simulation (Rimini, 2012) and published in the Proceedings of the workshop (see Grané, Veiga and Martín-Barragán, 2014).




*(a1) DCC, n = 1000* *(a2) DCC, n = 3000* *(a3) DCC, n = 5000*

*(b1) D-BEKK, n = 1000* *(b2) D-BEKK, n = 3000* *(b3) D-BEKK, n = 5000*

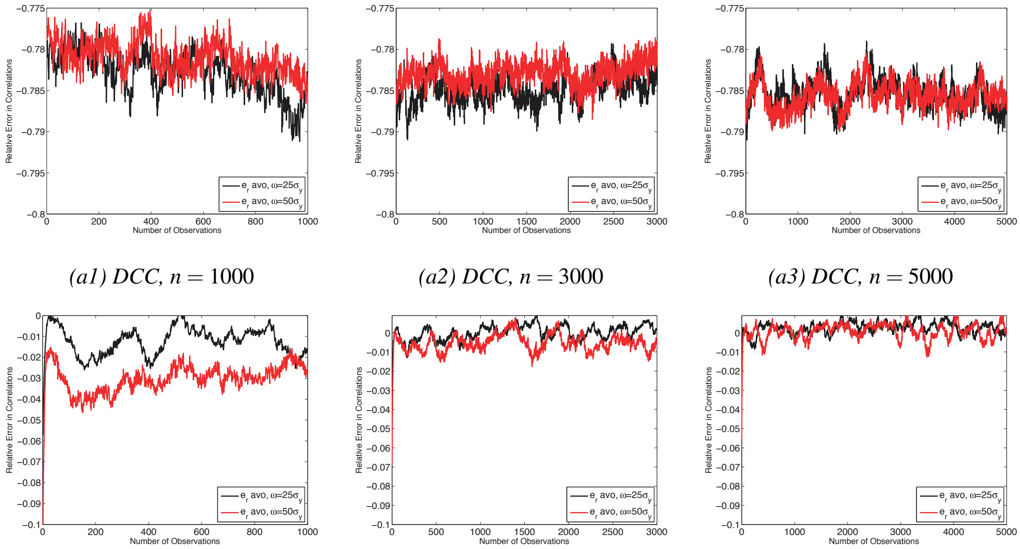
***Figure 1:*** *Relative bias in the estimated correlations obtained from a (a) DCC model and a (b) D-BEKK model with errors following normal distributions from 1000 simulated series of size n that include ALOs of different magnitudes.*




*(a1) DCC, n = 1000* *(a2) DCC, n = 3000* *(a3) DCC, n = 5000*

*(b1) D-BEKK, n = 1000* *(b2) D-BEKK, n = 3000* *(b3) D-BEKK, n = 5000*

***Figure 2:*** *Relative bias in the estimated correlations obtained from a (a) DCC model and a (b) D-BEKK model with errors following normal distributions from 1000 simulated series of size n that include patches of different magnitudes.*




*(a1) DCC,* $n = 1000$ *(a2) DCC,* $n = 3000$ *(a3) DCC,* $n = 5000$

*(b1) D-BEKK,* $n = 1000$ *(b2) D-BEKK,* $n = 3000$ *(b3) D-BEKK,* $n = 5000$

***Figure 3:*** *Relative bias in the estimated correlations obtained from a (a) DCC model and a (b) D-BEKK model with errors following normal distributions from 1000 simulated series of size n that include 1 AVO of different magnitudes.*

Table 1 shows the estimated correlations (each reported value is the sample mean computed on 1000 values) for the CCC model and the relative biases (or relative errors) with respect to the estimated correlations in the absence of outliers. Lines 13-15 correspond to the estimated correlations in the absence of outliers. Figures 1–3 contain the relative errors of DCC and D-BEKK models for different sample sizes. In particular, in Figure 1 we plot the relative bias obtained in the estimation of the correlations using DCC and D-BEKK models for the case of isolated ALOs, whereas Figures 2–3 correspond, respectively, to patches of ALOs and 1 isolated AVO. For each time $t$ (going from 1 to $n$), the plotted value is the sample mean computed from 1000 replications.

From Table 1 and Figures 1–3 we can observe that the estimated correlations are affected by the presence of outliers and the relative errors are higher the higher is the magnitude of the outlier, the larger the number of outliers included in the simulated series and the smaller the sample sizes of the simulated time series. Moreover, the biases in the correlations are higher for the DCC model in comparison to the CCC and D-BEKK models. In particular, the latter seems to be more robust to the presence of outliers since the correlations present small relative errors over the sample size. Finally, another conclusion is that additive outliers (level or volatility) bias the estimated correlations towards zero for the three considered models when the errors are Gaussian.

**STUDENT-*t* ERRORS** Next we perform simulations assuming that the errors follow a Student-*t* distribution (see Appendix A.1). From the results of Table A we can conclude that, for the CCC model and $N = 2$, the biases are of less magnitude (in absolute value) when considering a Student-$t$ distribution. This was expected since the Student-$t$ distribution is more robust to outliers. With respect to DCC and D-BEKK, we also observe that the biases are slightly smaller in absolute value when considering level outliers, isolated or patches. However, when we consider volatility outliers we observe that the impact on the correlations is larger in absolute value.

**MORE SERIES** Finally, we conduct a third simulation study considering $N = 4$ series (see Appendix A.2). The main conclusions are: Firstly, the impact of the outliers on the correlations tend to decrease for all level outliers; And secondly, regarding the volatility outliers, passing from two to four series leads to a decrease of the impact of outliers for the D-BEKK model, whereas for the DCC model the impact of volatility outliers on the correlations remains almost the same.

## 4 Wavelet-based detection procedure

Grané and Veiga (2010) proposed a general outlier detection method based on wavelets for the univariate case. This procedure was evaluated through an intensive Monte Carlo study and compared to other existing competitors. The method was proven to be very effective in detecting isolated level outliers, patches of level outliers and volatility outliers in large univariate financial time series. Additionally, the results showed the reliability of the method (in front of other competitors), since it detected a significantly small number of false outliers. More recently, Kamranfar, Chinipardaz and Mansouri (2017) has extended the procedure of Franses and Ghijsels (1999) to allow for level change and temporary change outliers.

The purpose of this work is to extend the method by Grané and Veiga (2010) to a multivariate setting, preserving as much as possible the good properties already proven in the univariate case, in particular, effectiveness and reliability, and also a feasible implementation in large data sets. This will be achieved by applying the random projection method (Cuesta-Albertos, Fraiman and Ransford, 2006, Cuesta-Albertos et al., 2007), that allows to translate a multivariate problem into a univariate context. These authors intuitively describe the random projection method in the following way. Imagine we have to deal with a problem related to $d$-dimensional objects. The random projection method consists of choosing, at random, a subspace of dimension $k$ (where $k$ is low compared to $d$), solve the problem in the $k$-dimensional subspace and translate the solution to the original ($d$-dimensional) space. In practice, $k = 1, 2$, which is exactly contrary to the Projection Pursuit paradigm, avoiding implementation problems due to high-dimensionality. Random projections have been successfully applied as a simple method for dimensionality reduction in high-dimensional problems, in fields such as computer science, data mining, image processing, etc. (see, for instance, Bingham and Mannila, 2001, Vempala, 2004).

Our procedure is to be applied to the residuals of multivariate GARCH models or other multivariate volatility models such as the stochastic volatility models, although in this paper we focus our attention on the former class. To our knowledge, our procedure is the first to detect outliers in multivariate volatility models.

Next in Section 4.1 we describe the proposed method to detect outliers and evaluate its performance in Section 4.2.

### 4.1 The procedure

The method we propose is an extension of the procedure by Grané and Veiga (2010), that was based on the detail wavelet coefficients resulting from the discrete wavelet transform (DWT) of a univariate series of (standardized) residuals.

The method requires a preliminary step that consists in fitting a multivariate GARCH model and obtaining the series of multivariate residuals. Note that our proposal is model-dependent, but general enough to cope with a wide variety of models.

In the first step, the multivariate series of residuals is transformed into a univariate series to which DWT will be applied. At this point two possibilities are considered whether the conditional covariance matrix of the fitted model fulfills the decomposition property. In case this property is satisfied, it is enough to consider only the univariate marginals. This will be the case for conditional correlation models, such as CCC and DCC. On the other hand, if the decomposition property is not fulfilled, like happens in the D-BEKK model, in addition to the marginals, we consider one randomly chosen projection (see Cuesta-Albertos et al., 2006). Therefore, we end the first step with a vector containing the univariate marginals or either the univariate marginals plus an extra series containing the result of the random projection. Note that only one projection is needed regardless the number of series considered. In Section 4.3 we show that for $N = 10$ series the procedure is still effective and no advantage is achieved by increasing the number of projections.

In the second step we apply the DWT to each of the univariate series under consideration and in further steps the procedure proceeds by identifying outliers as those observations in the original series whose detail coefficients are greater (in absolute value) than a certain threshold (see more details below). This threshold is a percentile of the distribution of a certain test statistic. The underlying idea for the threshold relies in the fact that in the context of financial return time series it is common to assume an underlying model for the data. Therefore, if the fitted model has captured the structure of the data, the residuals are supposed to be independent and identically distributed random variables following a specified distribution. In particular, the threshold is associated to the following test statistic: the maximum of the detail wavelet coefficients (in absolute value) resulting from the DTW of a univariate series of (standardized) residuals. When the univariate series under consideration is a marginal of the multivariate one, the thresholds given in Grané and Veiga (2010) for the univariate case are still valid. In case the univariate series is the result of a random projection, the distribution of the test statistic is obtained via Monte Carlo and the threshold is derived analogously.

***Table 2:*** *Threshold values: Percentiles of the distribution of the test statistics (with Bonferroni correction).*

| | | Gaussian distributed errors | | | | Student's $t$ distributed errors | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | only marginals | | marginals and random projection | | only marginals | | marginals and random projection | |
| | n | 1st level | 2nd level | 1st level | 2nd level | 1st level | 2nd level | 1st level | 2nd level |
| $\alpha = 0.05$ | 1000 | 4.0595 | 3.8827 | 4.1386 | 3.9731 | 5.2583 | 4.6062 | 5.2390 | 4.6399 |
| | 3000 | 4.2995 | 4.1437 | 4.3885 | 4.2383 | 5.7469 | 5.0101 | 5.7214 | 5.0092 |
| | 5000 | 4.4062 | 4.2664 | 4.5027 | 4.3503 | 6.0131 | 5.1269 | 5.9162 | 5.1953 |
| $\alpha = 0.10$ | 1000 | 3.7216 | 3.5280 | 3.9944 | 3.8207 | 4.9470 | 4.3384 | 4.9215 | 4.4039 |
| | 3000 | 3.8965 | 3.7114 | 4.2319 | 4.0873 | 5.4087 | 4.7086 | 5.3850 | 4.7845 |
| | 5000 | 4.2620 | 4.0992 | 4.3607 | 4.2012 | 5.6332 | 4.9015 | 5.6013 | 4.9383 |

In practice, we find that in order to detect isolated ALOs it suffices to work with the first level detail wavelet coefficients. However, if there are patches of ALOs or isolated AVOs, it is necessary to use both first level and second level detail wavelet coefficients. From the simulation study (see section 4.2) we believe that a reasonable threshold to use in the detection of isolated ALOs is the 95-th percentile, whereas for the detection of patches of ALOs and isolated AVOs the 90-th percentile is more useful. An analogous situation occurred in the univariate case.

Since in the multivariate case we are considering more than one series, the thresholds proposed in Grané and Veiga (2010) for the univariate case are not directly applicable and the union-intersection principle (Roy, 1953) with Bonferroni correction is applied. As a reference, Table 2 contains the values of the thresholds after applying the Bonferroni correction for bivariate series. The third, fourth, seventh and eighth columns correspond to the thresholds for the case in which only the marginals are considered and the fifth, sixth, ninth and tenth columns, to the case in which both the marginals and a random projection are considered. The thresholds are shown for two different significance levels $\alpha = 0.05$ and $0.10$, three different sample sizes $n = 1000, 3000$, and $5000$ and two different error distributions.

### A brief description of the algorithm

Next we give a brief description of the procedure for detecting ALOs. Let $\mathbf{X} = (X_1, \ldots, X_N)$ be the multivariate series of residuals of size $n$ obtained after fitting a CCC, DCC or a D-BEKK(1,1,1) model with normal distributed errors.

**Step 1** In case the fitted model is D-BEKK(1,1,1) (or any other model in which the decomposition property does not apply) consider a random vector $h = (h_1, \ldots, h_N)^\top$ such that $\|h\| = 1$ and obtain the random projection of the multivariate series of residuals on that direction, that is, $X_{N+1} = h \cdot \mathbf{X}$. Let $\mathbf{X}^* = (X_1, \ldots, X_N, X_{N+1})$.

**Step 2** Apply the DWT to each marginal of $\mathbf{X}$ (or alternatively $\mathbf{X}^*$) to obtain the first level wavelet detail coefficients $D_j = (d_{i,j})$, $i = 1,\dots,n/2$, $j = 1,\dots,N$ (or alternatively $j = 1,\dots,N+1$).

**Step 3** Set the threshold $k^\alpha$ equal to some percentile of the distribution of the maximum of the first level wavelet detail coefficients (in absolute value) resulting from the DWT of $n$ iid random variables following a standard normal distribution, considering the Bonferroni correction. See Table 2 for some examples and other probability distributions.

**Step 4** Find $S_j = \{i : |d_{i,j}| > k^\alpha\}$, for $j = 1,\dots,N$ (or alternatively $j = 1,\dots,N+1$) and consider $S = \cup_{j\geq 1} S_j$ the set formed by the union of all the elements in the $S_j$'s.

**Step 5** Use $S$ to locate the exact positions of the ALOs in any of the $X_j$'s. Let $s$ be a generic element in $S$. Let $\overline{x}_{n-2}$ be the sample mean of $X_j$ without observations at locations $2s$ and $2s-1$. Then, set the position of the ALO equal to $2s$ if $|X_{2s,j} - \overline{x}_{n-2}| > |X_{2s-1,j} - \overline{x}_{n-2}|$, or equal to $2s-1$, otherwise.

The algorithms that respectively search for patches of ALOs and for AVOs differ from the previous one in the sense that two level wavelet coefficients are computed, and consequently there are two thresholds, one for each set of detail wavelet coefficients $\mathbf{D}^{(1)} = \cup_{j\geq 1} D_j^{(1)}$ and $\mathbf{D}^{(2)} = \cup_{j\geq 1} D_j^{(2)}$, for $j = 1,\dots N$ (or alternatively $j = 1,\dots N+1$). However, the main idea remains unchanged. These algorithms have been implemented in Matlab and are available from the authors upon request.

### *4.2 Performance of the procedure*

In this section we present the results of an intensive simulation study to assess the performance of our detection proposal. In this study, we simulate the contaminated and no–contaminated multivariate series following the experimental conditions described in Section 3. We also consider D-BEKK, CCC and DCC models with Student-*t* distributed errors.[2]

We apply the detection method described in Section 4.1 where the assumed model is the true model used to generate the series. The results are shown in Table 3.[3] The measures used in the performance study are the percentage of times that the localization of the outliers is correctly detected and the percentage of false outliers. The threshold values used in the study are contained in Table 2.

The detection rate is larger for models with Gaussian errors. From Table 3 we can see that when the magnitude of the outlier is $\boldsymbol{\omega} = 10\boldsymbol{\sigma}_{\mathbf{y}}$, the procedure detects more than

2. Parameter values used for models with Student-$t_7$ distributed errors: $\{\text{vec}(\mathbf{C}) = (0.106, 0.110, 0, 0.0371)', \text{diag}(\mathbf{A}) = (0.0571, 0.050)', \text{diag}(\mathbf{B}) = (0.983, 0.985)'\}$ for the D-BEKK model, $\{\boldsymbol{\alpha}_0 = (0.010, 0.013), \boldsymbol{\alpha}_1 = (0.049, 0.067), \boldsymbol{\beta}_1 = (0.740, 0.759), \rho_{12} = 0.506\}$ for the CCC model and $\{\boldsymbol{\alpha}_0 = (0.106, 0.110, 0.0371), \boldsymbol{\alpha}_1 = (0.0571, 0.050), \boldsymbol{\beta}_1 = (0.740, 0.759), \alpha = 0.015, \beta = 0.781\}$ for the DCC model.

3. The Matlab codes are available from the authors upon request.

96% of the isolated outliers, reaching the 100% in two cases. When the magnitude of the outlier is relatively small, $\boldsymbol{\omega} = 5\boldsymbol{\sigma}_{\mathbf{y}}$, the detection rate goes from 36% to 43% for the D-BEKK model and from 68% and 77% for the CCC and DCC models. Regarding patches and volatility outliers, the detection rate also increases with the size of the outlier and it ranges from 24.1% (AVO and D-BEKK) to 99.8% (AVO and DCC). Finally, the percentage of false positives is at most 0.001% in 80% of the cases and under 0.007% in the rest (the only exception is the DCC model for $\boldsymbol{\omega} = 10\boldsymbol{\sigma}_{\mathbf{y}}, n = 1000$). Concerning models with Student-$t$ distributed errors, we observe that, for example, when the magnitude of the outlier is $\boldsymbol{\omega} = 10\boldsymbol{\sigma}_{\mathbf{y}}$, the procedure detects from 70.9% to 99.2% of isolated ALOs. As expected, the detection rate is low when $\boldsymbol{\omega} = 5\boldsymbol{\sigma}_{\mathbf{y}}$, since it is difficult to distinguish small size outliers from the thick tail of Student-$t$ distribution. The percentage of false positives is still very small, being at most 0.006% in more than 77% of the cases. These results lead us to conclude that the method is very reliable.

***Table 3:*** *Percentage of correct detection of outliers and percentage of false outliers in 1000 replications of size n for multivariate GARCH models with either normal or Student-$t_7$ distributed errors.*

| | | Gaussian errors | | | | | | Student-$t_7$ distributed errors | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | % of correct detections | | | % of false outliers | | | % of correct detections | | | % of false outliers | | |
| | $n$ | D-BEKK | CCC | DCC | D-BEKK | CCC | DCC | D-BEKK | CCC | DCC | D-BEKK | CCC | DCC |
| 1 ALO | 1000 | 43.8 | 77.1 | 77.2 | 0.004 | 0.005 | 0.005 | 12.5 | 7.7 | 7.8 | 0.0130 | 0.0082 | 0.0083 |
| $\omega = 5\sigma_y$ | 3000 | 38.7 | 76.2 | 75.3 | 0.001 | 0.001 | 0.001 | 11.5 | 3.2 | 3.2 | 0.0082 | 0.0055 | 0.0056 |
| | 5000 | 36.1 | 69.8 | 70.8 | 0.001 | 0.001 | 0.001 | 11.2 | 2.1 | 2.1 | 0.0067 | 0.0050 | 0.0049 |
| 1 ALO | 1000 | 99.1 | 100.0 | 100.0 | 0.004 | 0.004 | 0.048 | 92.2 | 98.8 | 99.0 | 0.0127 | 0.1080 | 0.0067 |
| $\omega = 10\sigma_y$ | 3000 | 99.3 | 99.9 | 99.9 | 0.001 | 0.001 | 0.001 | 81.7 | 98.4 | 98.4 | 0.0081 | 0.0050 | 0.0050 |
| | 5000 | 99.3 | 99.9 | 99.8 | 0.001 | 0.001 | 0.001 | 73.3 | 99.2 | 99.1 | 0.0068 | 0.0047 | 0.0047 |
| 3 ALOs | 1000 | 36.7 | 69.6 | 68.9 | 0.003 | 0.004 | 0.004 | 12.3 | 5.6 | 5.6 | 0.0113 | 0.0568 | 0.0566 |
| $\omega = 5\sigma_y$ | 3000 | 36.5 | 71.2 | 71.1 | 0.001 | 0.001 | 0.001 | 10.3 | 2.5 | 2.5 | 0.0077 | 0.0050 | 0.0050 |
| | 5000 | 36.1 | 71.5 | 71.4 | 0.001 | 0.001 | 0.001 | 11.4 | 2.2 | 2.2 | 0.0066 | 0.0047 | 0.0048 |
| 3 ALOs | 1000 | 96.5 | 97.8 | 97.8 | 0.002 | 0.005 | 0.005 | 88.2 | 93.0 | 93.3 | 0.0103 | 0.0062 | 0.0059 |
| $\omega = 10\sigma_y$ | 3000 | 97.8 | 98.9 | 98.8 | 0.001 | 0.001 | 0.001 | 78.8 | 97.9 | 97.8 | 0.0075 | 0.0542 | 0.0042 |
| | 5000 | 97.8 | 99.1 | 98.9 | 0.001 | 0.001 | 0.001 | 70.9 | 98.1 | 98.2 | 0.0065 | 0.0043 | 0.0043 |
| Patch of | 1000 | 26.4 | 20.5 | 20.4 | 0.0001 | 0 | 0 | 24.0 | 1.7 | 1.9 | 0 | 0 | 0.0001 |
| 3 ALOs | 3000 | 30.5 | 18.8 | 18.8 | 0 | 0 | 0 | 26.9 | 0.7 | 0.7 | 0.0001 | 0 | 0 |
| $\omega = 5\sigma_y$ | 5000 | 33.1 | 17.7 | 18.9 | 0.00002 | 0 | 0 | 26.6 | 0.1 | 0.1 | 0.0001 | 0.00004 | 0.00002 |
| Patch of | 1000 | 73.2 | 89.2 | 88.5 | 0 | 0 | 0 | 52.3 | 77.8 | 77.8 | 0 | 0 | 0 |
| 3 ALOs | 3000 | 70.4 | 88.1 | 87.6 | 0 | 0 | 0 | 44.1 | 71.1 | 71.1 | 0.0001 | 0 | 0 |
| $\omega = 10\sigma_y$ | 5000 | 70.5 | 86.0 | 85.3 | 0.00002 | 0 | 0 | 41.1 | 63.2 | 63.3 | 0.0001 | 0.00002 | 0.00002 |
| 1 AVO | 1000 | 24.2 | 66.4 | 95.6 | 0.001 | 0.0001 | 0 | 3.2 | 46.3 | 70.1 | 0.0001 | 0 | 0 |
| $\omega = 25\sigma_y$ | 3000 | 24.1 | 66.8 | 94.8 | 0.0003 | 0 | 0 | 3.3 | 47.1 | 66.6 | 0.0001 | 0.0001 | 0 |
| | 5000 | 24.4 | 66.5 | 95.6 | 0.0002 | 0 | 0 | 2.8 | 44.4 | 66.3 | 0.0002 | 0 | 0 |
| 1 AVO | 1000 | 52.2 | 87.0 | 99.6 | 0.004 | 0 | 0 | 15.0 | 75.1 | 94.5 | 0.0002 | 0 | 0 |
| $\omega = 50\sigma_y$ | 3000 | 55.6 | 88.0 | 99.3 | 0.002 | 0 | 0 | 17.7 | 75.5 | 94.0 | 0.0002 | 0 | 0 |
| | 5000 | 53.9 | 88.8 | 99.8 | 0.001 | 0 | 0 | 17.3 | 75.0 | 94.2 | 0.0002 | 0 | 0 |
| No | 1000 | | | | 0.004 | 0.006 | 0.005 | | | | 0.0142 | 0.0092 | 0.0094 |
| outliers | 3000 | | | | 0.001 | 0.001 | 0.001 | | | | 0.0082 | 0.0057 | 0.0058 |
| | 5000 | | | | 0.001 | 0.001 | 0.001 | | | | 0.0068 | 0.0050 | 0.0050 |

In general, the percentage of correctly detected outliers is smaller for the D-BEKK model than for CCC and DCC and this is confirmed by the results presented in Section 3, where we observed that the effect of outliers in the estimation of the correlations was lower for D-BEKK model than for the CCC or DCC models.

These results show the reliability of the detection method for bivariate series. A natural question arises for higher-dimensional cases: is one random projection enough for detecting outliers or should the number of random projections be increased with $N$? As shown in what follows, our results suggest that there is no need to increase the number of random projections considered in the algorithm.

### 4.3 One or more random projections?

We focus now on analysing the performance of our procedure for $N = 10$ series and the D-BEKK model.[4] In particular, we are interested in studying whether increasing the number of random projections may increase the percentage of correctly detected outliers. In this case, threshold values are computed as suggested by Benjamini and Yekutieli (2001), instead of Bonferroni correction, which is too conservative. Results are contained in Table 4. As before, the measures used are the percentage of times that the localization of the outliers is correctly detected and the percentage of false outliers. The number of random projections is shown in the first column. We analyse here the case of 1 ALO of sizes $\boldsymbol{\omega} = 5\boldsymbol{\sigma}_\mathbf{y}, 10\boldsymbol{\sigma}_\mathbf{y}$. For $\boldsymbol{\omega} = 10\boldsymbol{\sigma}_\mathbf{y}$ the proportion of correct detections stays constant when the number of random projections is increased, whereas for $\boldsymbol{\omega} = 5\boldsymbol{\sigma}_\mathbf{y}$ the increase is very low (0.2 percentage points). In contrast, the percentage of false outliers and the computational burden increase with a raise of the number of random projections, suggesting that it is not worth to use more than one random projection for large values of $N$; Similar conclusions where found by Cuesta-Albertos et al. (2006).

***Table 4:*** *Percentage of correct detection of outliers and percentage of false outliers in 1000 replications of size $n = 1000$ for a D-BEKK model with Gaussian distributed errors. Series contaminated with one ALO of two different sizes.*

| | 1 ALO $\boldsymbol{\omega} = 5\boldsymbol{\sigma}_\mathbf{y}$ | | 1 ALO $\boldsymbol{\omega} = 10\boldsymbol{\sigma}_\mathbf{y}$ | |
|---|---|---|---|---|
| num. of random projections | % of correct detections | % of false outliers | % of correct detections | % of false outliers |
| 1 | 21.4 | 0.0065 | 97.7 | 0.0241 |
| 2 | 21.4 | 0.0064 | 97.7 | 0.0178 |
| 5 | 21.4 | 0.0079 | 97.7 | 0.0212 |
| 10 | 21.6 | 0.0085 | 97.7 | 0.0280 |
| 20 | 21.6 | 0.0115 | 97.7 | 0.0377 |
| 50 | 21.6 | 0.0182 | 97.7 | 0.0672 |

4. Regarding the computational burden of this simulation study, we want to remark that estimating 1000 times the D-BEKK model for 10 series took approximately one week in an ordinary computer.

## 5 Empirical application

In this section, we analyse nine time series of returns to illustrate the performance of our method on real data. The data set is composed of the most important indices of the U.S. stock market, such as Nasdaq and S&P500 and company stocks such as Marathon Oil Corporation (MRO), International Business Machines Corporation (IBM), Coca-Cola Corporation (Cola), Colgate-Palmolive Corporation (Colgate), British Petroleum (BP), Microsoft Corporation (Micro) and American Express Company (AE). The data was collected from Yahoo Finance website (http://finance.yahoo.com) and spans the period from January 2, 1990 to January 30, 2015.

Figure 4 depicts the nine return series, $y_t = (\log p_t - \log p_{t-1}) \cdot 100$, where $p_t$ is the value at time $t$ of the corresponding index and Table 5 reports some summary statistics and the results of the Kiefer and Salmon test for normality in the context of conditional heteroscedastic series.

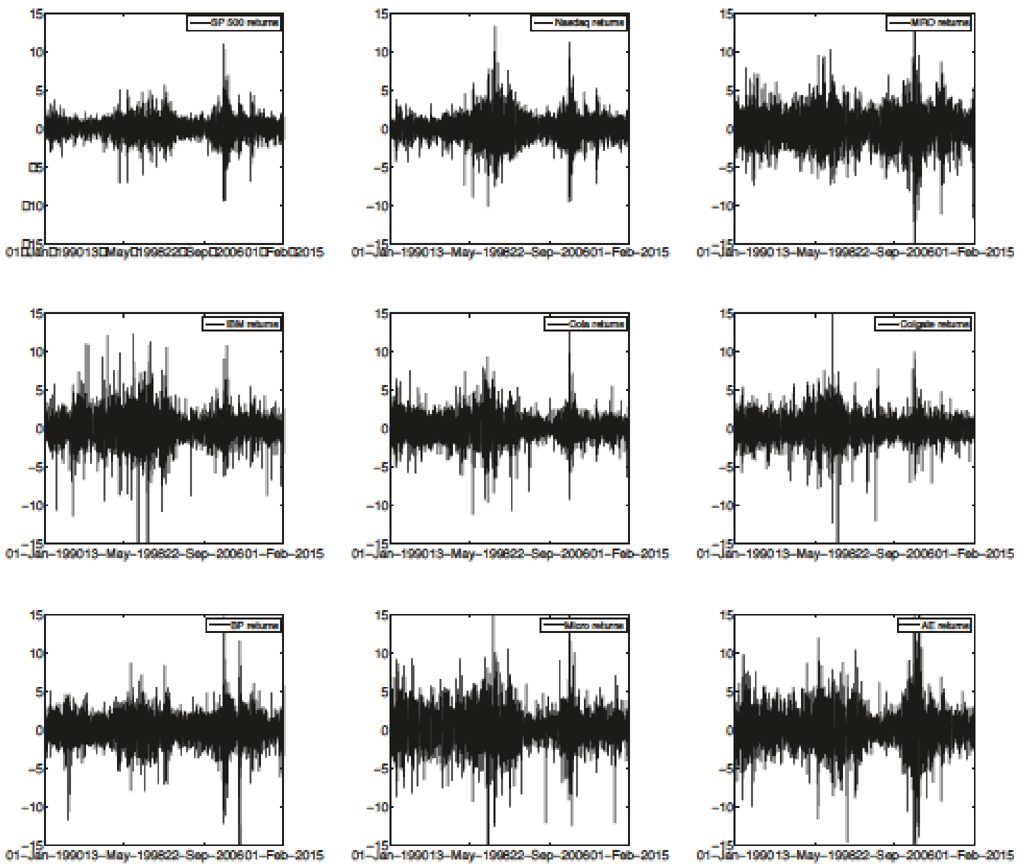



***Figure 4:*** *Returns in percentage for the nine time series considered.*

***Table 5:*** *Descriptive statistics for daily returns.*

Summary statistics of daily returns

| Returns | Nasdaq | S&P 500 | MRO | IBM | Cola | Colgate | BP | Micro | AE |
|---|---|---|---|---|---|---|---|---|---|
| Mean | 0.0366 | 0.0271 | 0.0296 | 0.0360 | 0.0418 | 0.0531 | 0.0297 | 0.0715 | 0.0419 |
| Variance | 2.2333 | 1.3031 | 4.4819 | 3.1945 | 2.1428 | 2.2649 | 2.8337 | 4.3696 | 5.1985 |
| Skewness | -0.0858 | -0.2392 | -0.2547 | -0.0210 | 0.0353 | -0.0230 | -0.3780 | -0.0101 | 0.0294 |
| Kurtosis | 9.1796 | 11.7389 | 10.9489 | 10.3444 | 8.6336 | 13.1178 | 12.3986 | 8.5061 | 10.5041 |

Results of the Kiefer and Salmon test

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $KS_S$ | -2.7843* | -7.7615* | -8.2652* | -0.6809 | 1.1449 | -0.7462 | -12.2656* | -0.3285 | 0.9542 |
| $KS_K$ | 100.2478* | 141.7666* | 128.9508* | 119.1434* | 91.3907* | 164.1346* | 152.4680* | 89.3229* | 121.7340* |

$*$ Means that we reject at 1%, 5% and 10% the nulls of skewness and kurtosis similar to those of a variable that follows a normal distribution.

From Table 5, we observe that the nine return series are in majority negatively skewed (except Cola and AE returns) and have significant kurtosis, ranging from 8.5061 for Micro to 13.1178 for Colgate, which suggests the existence of some outliers. It is known that this type of observations in time series leads to fat tail distributions, and some outlier detection methods, specially in the multivariate context, are based on this information (see for example Peña and Prieto, 2001, Galeano, Peña and Tsay, 2006). Table 5 also contains the results of the Kiefer and Salmon (1983) test (*KS*), which is a formal test of normality in the context of conditional heteroscedastic series.[5] The results of the test confirm the non Gaussianity of the nine return series.

Next, we estimate the three multivariate GARCH models considered in this work: the D-BEKK, the CCC and the DCC with Gaussian and Student-*t* distributed errors, and we proceed by applying our method to detect outliers. The degrees of freedom of the Student-*t* distributions are considered endogenous (they are included in the likelihood) and therefore estimated. Results are shown in Table 6.

Some dates are often detected as outliers or belong to a patch of outliers. Note that, the dates identified as outliers in the conditional correlation models are also identified as outliers in the D-BEKK model. Most of the outliers can be related to specific events. In particular, 19-Mar-91, 20/21-Oct-99, 21-Sep-04, 27/28-Apr-06, 19-Apr-13 and 18-Jul-13. On March 19, 1991 IBM announced that its returns were expected to decreased by half which led to an immediate plunge of its shares. On October 21, 1999 IBM stocks tumbled pulling the rest of the market with it. On April 28, 2006 Microsoft announces lower-than-expected earnings due to research expenses that would hurt future results. On September 21, 2004 CNN announced that the optimism about technologic stocks lifted the U.S. stock market at the open on this day. On April 19, 2013 it was announced that IBM shares posted their biggest one-day percentage drop in eight years. Finally, on

5. The Kiefer and Salmon (1983) test is given by $KS_N = (KS_S)^2 + (KS_K)^2$ (test of normality), where $KS_S = \sqrt{\frac{T}{6}}\left[\frac{1}{T}\sum_{t=1}^{T} y_t^{*3} - \frac{3}{T}\sum_{t=1}^{T} y_t^{*}\right]$ (test of skewness),$KS_K = \sqrt{\frac{T}{24}}\left[\frac{1}{T}\sum_{t=1}^{T} y_t^{*4} - \frac{6}{T}\sum_{t=1}^{T} y_t^{*2} + 3\right]$ (test of kurtosis) and $y_t^*$ are the standardized returns. If the distribution of $y_t^*$ is conditional $N(0,1)$, then $KS_S$ and $KS_K$ are asymptotically $N(0,1)$ and $KS_N$ is asymptotically $\chi^2(2)$.

***Table 6:*** *Observations identified as possible outliers in nine time series of returns.*

| D-BEKK | | | | CCC | | | | DCC | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Gaussian | | Student's *t* | | Gaussian | | Student's *t* | | Gaussian | | Student's *t* | |
| ALOs | Patches | ALOs | Patches | ALOs | Patches | ALOs | Patches | ALOs | Patches | ALOs | Patches |
| 18-Mar-91 | 21-Oct-99 | 19-Mar-91 | 21-Sep-04 | 19-Mar-91 | 21-Sep-04 | 21-Sep-04 | 21-Sep-04 | 19-Mar-91 | 21-Sep-04 | 21-Sep-04 | 21-Sep-04 |
| 25-Jun-92 | 15-Mar-00 | 8-Apr-02 | 28-Apr-06 | 15-Nov-91 | 27-Apr-06 | 27-Apr-06 | | 15-Nov-91 | 27-Apr-06 | 27-Apr-06 | |
| 15-Dec-92 | 21-Sep-04 | 23-Oct-03 | 19-Apr-13 | 25-Jun-92 | 18-Jul-13 | | | 25-Jun-92 | 18-Jul-13 | | |
| 22-Jul-94 | 28-Apr-06 | 21-Sep-04 | 18-Jul-13 | 20-Oct-99 | | | | 20-Oct-99 | | | |
| 27-Oct-97 | 19-Apr-13 | 15-Apr-05 | | 15-Sep-00 | | | | 15-Sep-00 | | | |
| 21-Oct-99 | 18-Jul-13 | 28-Apr-06 | | 17-Sep-01 | | | | 17-Sep-01 | | | |
| 15-Mar-00 | 21-Oct-14 | 29-Oct-07 | | 22-Oct-03 | | | | 22-Oct-03 | | | |
| 14-Sep-00 | | 14-Oct-08 | | 23-Jul-04 | | | | 23-Jul-04 | | | |
| 19-Oct-00 | | 29-Apr-10 | | 21-Sep-04 | | | | 21-Sep-04 | | | |
| 17-Sep-01 | | 1-Jun-10 | | 15-Apr-05 | | | | 15-Apr-05 | | | |
| 8-Apr-02 | | 9-Jun-10 | | 27-Apr-06 | | | | 27-Apr-06 | | | |
| 23-Oct-03 | | 19-Apr-13 | | 29-Jul-08 | | | | 29-Jul-08 | | | |
| 23-Jul-04 | | 18-Jul-13 | | 8-Aug-11 | | | | 8-Aug-11 | | | |
| 21-Sep-04 | | 20-Oct-14 | | 15-Apr-13 | | | | 15-Apr-13 | | | |
| 18-Apr-05 | | 21-Oct-14 | | 19-Apr-13 | | | | 19-Apr-13 | | | |
| 28-Apr-06 | | | | 18-Jul-13 | | | | 18-Jul-13 | | | |
| 29-Oct-07 | | | | 17-Oct-14 | | | | 17-Oct-14 | | | |
| 29-Apr-08 | | | | | | | | | | | |
| 14-Oct-08 | | | | | | | | | | | |
| 29-Apr-10 | | | | | | | | | | | |
| 1-Jun-10 | | | | | | | | | | | |
| 9-Jun-10 | | | | | | | | | | | |
| 29-Jul-10 | | | | | | | | | | | |
| 8-Aug-11 | | | | | | | | | | | |
| 15-Apr-13 | | | | | | | | | | | |
| 19-Apr-13 | | | | | | | | | | | |
| 18-Jul-13 | | | | | | | | | | | |
| 4-Sep-14 | | | | | | | | | | | |
| 20-Oct-14 | | | | | | | | | | | |
| 21-Oct-14 | | | | | | | | | | | |
| 30-Jan-15 | | | | | | | | | | | |

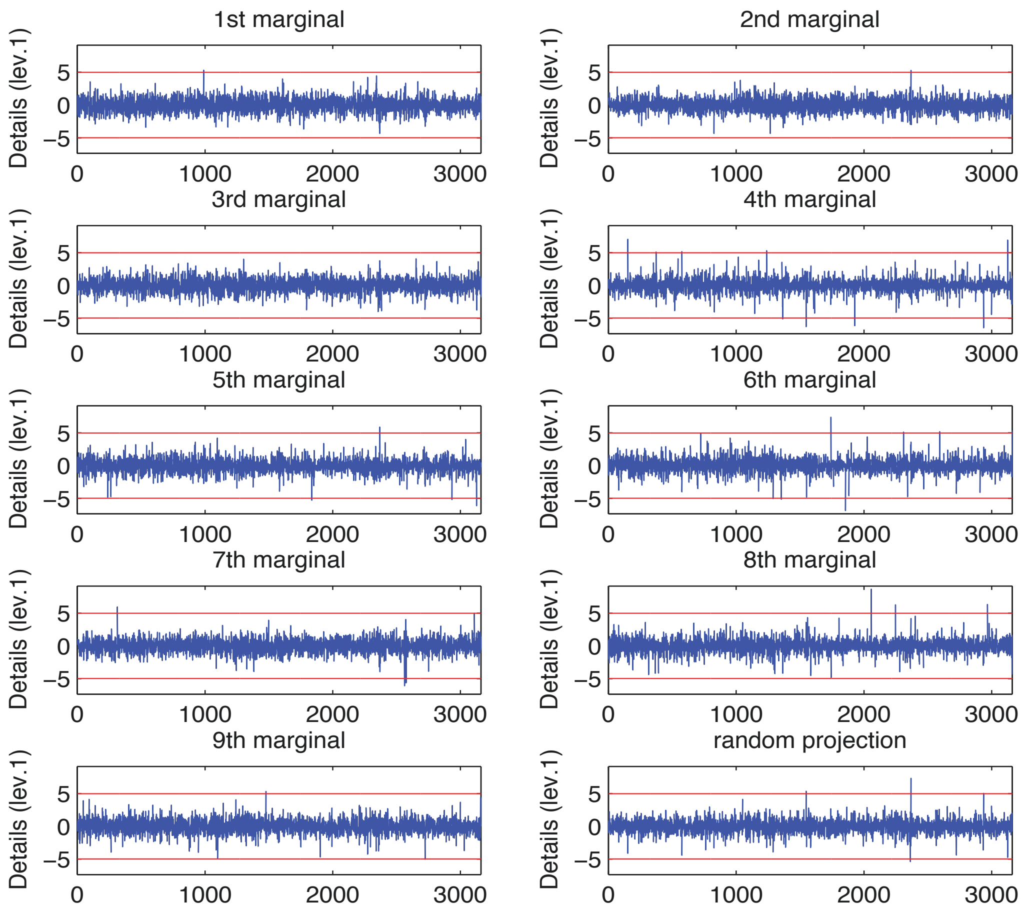



***Figure 5:*** *Graphical output of the wavelet-based procedure for the returns of nine time series estimated with a D-BEKK model with Gaussian errors.*

July 18, 2013 news about the European Union plans to limit fees for using credit and debit cards pushed down the American Express company's shares. Furthermore, other dates like 15-Nov-91, 15-Dec-92, 22-Jul-94 and 29-Jul-08 can be related to oil shocks; In 1990 Irak invades Kuwait and Kuwait cut crude exports until 1994; Oil prices dropped from historic highs of \$144.29 in July 2008, to \$33.87 five months later.

Figure 5 shows a graphical output of the Matlab program, which corresponds to the analysis of the multivariate residuals obtained after fitting a D-BEKK model with Gaussian errors to nine series of returns.

All these observations correspond to important financial crashes or oil shocks that our procedure detected successfully.

## 6 Conclusion

The main contribution of this paper is the proposal of a general detection algorithm based on wavelets that can be applied to a large class of multivariate volatility models. The effectiveness of our method is evaluated both with simulated and real data. The simulations report evidence that our proposal is both effective and reliable since it detects very few false outlier.

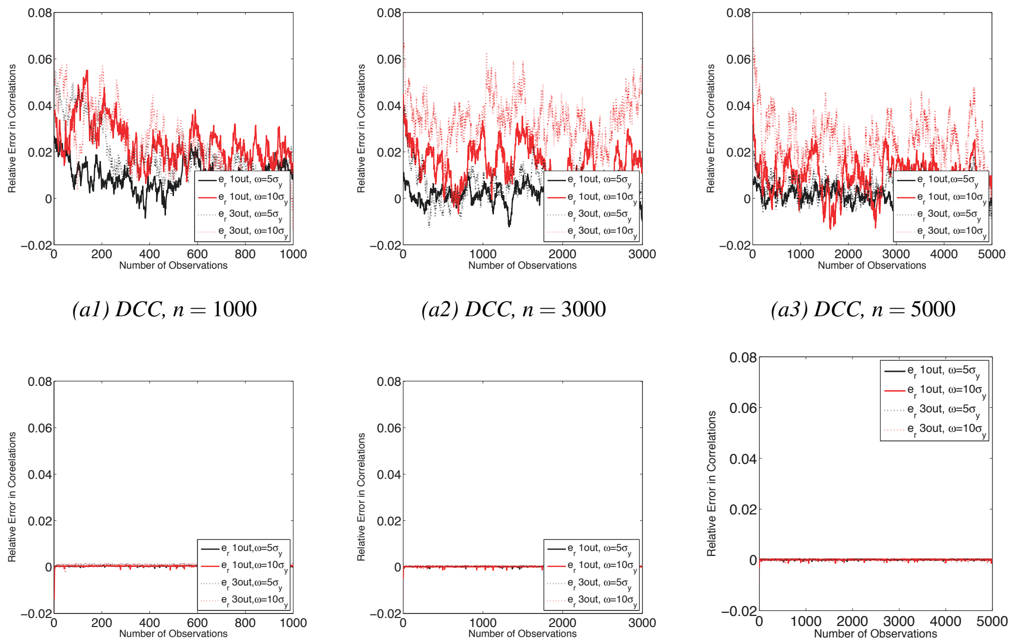
We also study the impact of outliers (isolated level outliers, patches of level outliers and volatility outliers) on the estimation of correlations when fitting well known multivariate GARCH models via several simulation studies. The results of the Monte Carlo experiments show that correlations are considerably affected by the presence of outliers. The impact on the correlations is stronger the higher is the magnitude of the outlier, the larger the number of outliers included in the simulated series and the smaller the sample sizes of the simulated time series. In the simulation, we consider scenarios that try to mimic portfolios that include asset returns and commodity returns such as oil, where the correlation is negative and quite negative when turmoils in the oil market propagate to stock markets. Therefore, the implications of these results are important for investments in oil commodities, as we identify several sources of impacts that are useful for controlling international risks of investments.

## Appendix

### A.1 Student-t distributed errors

The simulations are conducted following the experimental conditions explained in the beginning of section 3. In this case, the parameter values used are: for the D-BEKK model, $\{\text{vec}(\mathbf{C}) = (0.053, 0.042, 0, 0.020)^{\intercal}, \text{diag}(\mathbf{A}) = (0.161, 0.164)^{\intercal}, \text{diag}(\mathbf{B}) = (0.983, 0.981)^{\intercal}\}$; $\{\boldsymbol{\alpha}_0 = (0.010, 0.013), \boldsymbol{\alpha}_1 = (0.019, 0.027), \boldsymbol{\beta}_1 = (0.940, 0.826), \rho_{12} = -0.306)\}$ for the CCC model and $\{\boldsymbol{\alpha}_0 = (0.010, 0.013), \boldsymbol{\alpha}_1 = (0.019, 0.027), \boldsymbol{\beta}_1 = (0.940, 0.826), \alpha = 0.015, \beta = 0.981\}$ for the DCC model.

The idea is similar to the experiments showed in Section 3. We simulate with Student-$t_7$ errors and estimate the CCC, DCC and D-BEKK considering the degree of freedom of the Student-*t* distribution endogenous. Table A and Figures A–C contain the results of this second simulation study.




(a1) DCC, $n = 1000$ (a2) DCC, $n = 3000$ (a3) DCC, $n = 5000$

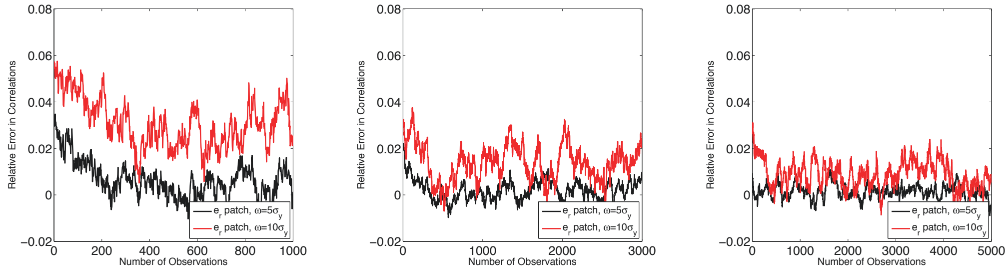
(b1) D-BEKK, $n = 1000$ (b2) D-BEKK, $n = 3000$ (b3) D-BEKK, $n = 5000$

***Figure A:*** *Relative bias in the estimated correlations obtained from a (a) DCC model and a (b) D-BEKK model with* Student-$t_7$ *distributed errors from 1000 simulated series of size n that include ALOs of different magnitudes.*

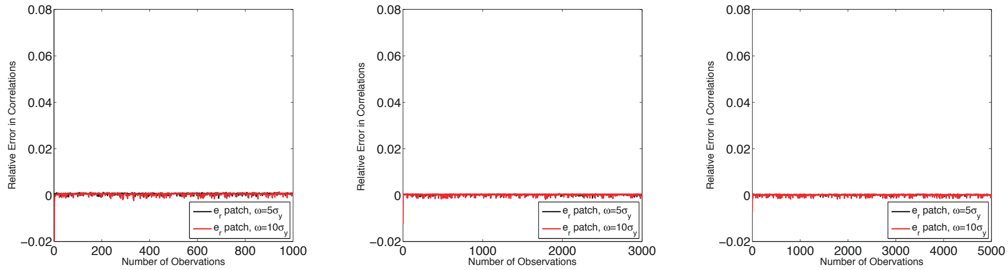
***Table A:*** *Relative bias in the estimated correlations obtained from a CCC model from 1000 simulated series of size n that include outliers of different magnitudes.*

CCC Model with Student-$t_7$ distributed errors

| | $n$ | Estimated Correlation | Relative Bias | | $n$ | Estimated Correlation | Relative Bias |
|---|---|---|---|---|---|---|---|
| 1 ALO | 1000 | -0.3073 | 0.0062 | 3 ALOSs | 1000 | -0.3106 | 0.0170 |
| $\omega = 5\sigma_y$ | 3000 | -0.3071 | 0.0029 | $\omega = 5\sigma_y$ | 3000 | -0.3086 | 0.0078 |
| | 5000 | -0.3070 | 0.0020 | | 5000 | -0.3079 | 0.0049 |
| 1 ALO | 1000 | -0.3093 | 0.0128 | 3 ALOs | 1000 | -0.3141 | 0.0285 |
| $\omega = 10\sigma_y$ | 3000 | -0.3100 | 0.0124 | $\omega = 10\sigma_y$ | 3000 | -0.3145 | 0.0271 |
| | 5000 | -0.3089 | 0.0082 | | 5000 | -0.3128 | 0.0209 |
| Patch of | 1000 | -0.3075 | 0.0069 | 1 AVO | 1000 | -0.2918 | -0.0445 |
| 3 ALOs | 3000 | -0.3072 | 0.0033 | $\omega = 25\sigma_y$ | 3000 | -0.3006 | -0.0183 |
| $\omega = 5\sigma_y$ | 5000 | -0.3072 | 0.0026 | | 5000 | -0.3025 | -0.0127 |
| Patch of | 1000 | -0.3137 | 0.0272 | 1 AVO | 1000 | -0.2745 | -0.1012 |
| 3 ALOs | 3000 | -0.3099 | 0.0121 | $\omega = 50\sigma_y$ | 3000 | -0.2906 | -0.0509 |
| $\omega = 10\sigma_y$ | 5000 | -0.3089 | 0.0082 | | 5000 | -0.2955 | -0.0356 |
| No outliers | 1000 | -0.3054 | | | | | |
| | 3000 | -0.3062 | | | | | |
| | 5000 | -0.3064 | | | | | |

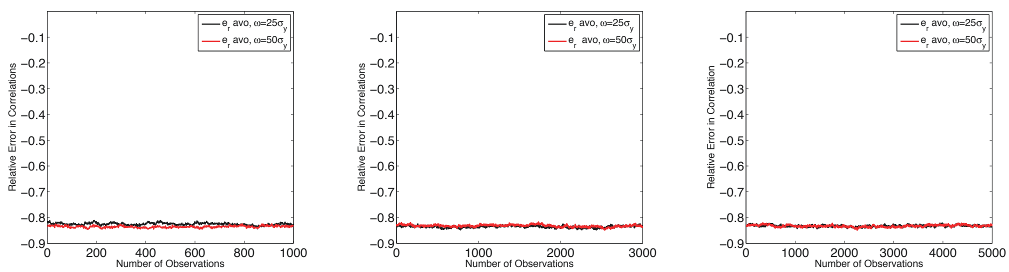



*(a1) DCC, $n = 1000$* *(a2) DCC, $n = 3000$* *(a3) DCC, $n = 5000$*

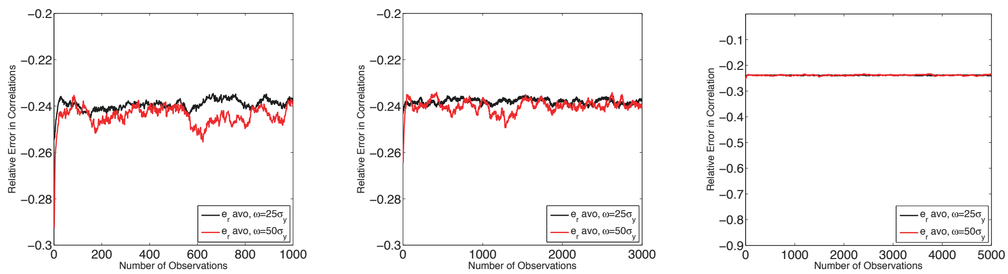



*(b1) D-BEKK, $n = 1000$* *(b2) D-BEKK, $n = 3000$* *(b3) D-BEKK, $n = 5000$*

***Figure B:*** *Relative bias in the estimated correlations obtained from a (a) DCC model and a (b) D-BEKK model with* Student-$t_7$ *distributed errors from 1000 simulated series of size n that include patches of different magnitudes.*




*(a1) DCC, $n = 1000$* *(a2) DCC, $n = 3000$* *(a3) DCC, $n = 5000$*




*(b1) D-BEKK, $n = 1000$* *(b2) D-BEKK, $n = 3000$* *(b3) D-BEKK, $n = 5000$*

***Figure C:*** *Relative bias in the estimated correlations obtained from a (a) DCC model and a (b) D-BEKK model with* Student-$t_7$ *distributed errors from 1000 simulated series of size n that include 1 AVO of different magnitudes.*

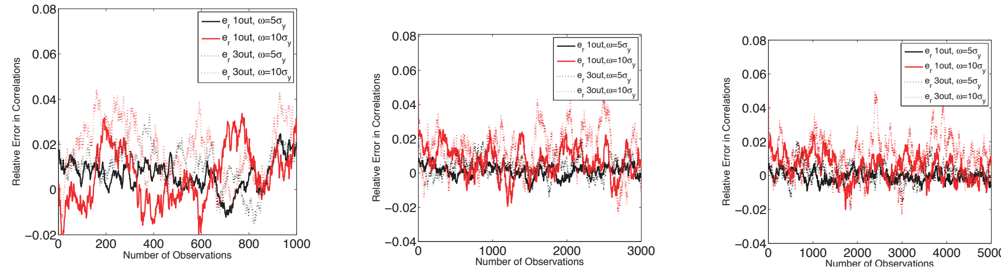
### *A.2* $N = 4$ *and* Student-$t_7$ *distributed errors*

The simulations are conducted following the experimental conditions explained in section 3, with $N = 4$. For better comparison with results of Appendix A.1, we choose the same parameter values for the first two series.

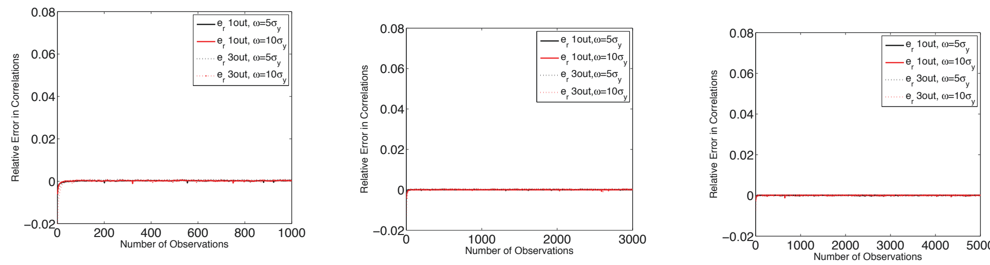
Once again, the outliers are placed randomly across the simulated series, but in the same position for each pair of series. We consider that the outlier affects each pair of series at the same instant. Each scenario involves 1000 replications and series are simulated from CCC, DCC and D-BEKK(1,1,1) models with Student-$t_7$ errors. The number of replications is selected to provide robust results. Given a model, we analysed 24 scenarios, that are defined from the type and number of outliers (one isolated ALO, multiple ALOs, patches of three ALOs, one isolated AVO), the size of the outlier ($\omega = 5\sigma_y, 10\sigma_y$ for ALOs and $\omega = 25\sigma_y, 50\sigma_y$ for AVOs) and the sample size of the simulated series ($n = 1000, 3000, 5000$). Table B and Figures D–F contain the results of this simulation study. In order to simplify the presentation, we only report the results for the correlation between the first two simulated series.

***Table B:*** *Relative bias in the estimated correlations (first and second series) obtained from a CCC model with errors following* Student-$t_7$ *distributions from 1000 simulated series of size n that include outliers of different magnitudes.*

| | $n$ | Estimated Correlation | Relative Bias | | $n$ | Estimated Correlation | Relative Bias |
|---|---|---|---|---|---|---|---|
| 1 ALO | 1000 | -0.3076 | 0.0052 | 3 ALOSs | 1000 | -0.3105 | 0.0147 |
| $\omega = 5\sigma_y$ | 3000 | -0.3069 | 0.0029 | $\omega = 5\sigma_y$ | 3000 | -0.3082 | 0.0072 |
| | 5000 | -0.3061 | 0.0013 | | 5000 | -0.3069 | 0.0039 |
| 1 ALO | 1000 | -0.3093 | 0.0108 | 3 ALOs | 1000 | -0.3123 | 0.0206 |
| $\omega = 10\sigma_y$ | 3000 | -0.3066 | 0.0020 | $\omega = 10\sigma_y$ | 3000 | -0.3153 | 0.0304 |
| | 5000 | -0.3078 | 0.0069 | | 5000 | -0.3121 | 0.0210 |
| Patch of | 1000 | -0.3078 | 0.0058 | 1 AVO | 1000 | -0.2934 | -0.0411 |
| 3 ALOs | 3000 | -0.3065 | 0.0018 | $\omega = 25\sigma_y$ | 3000 | -0.3003 | -0.0186 |
| $\omega = 5\sigma_y$ | 5000 | -0.3061 | 0.0012 | | 5000 | -0.3013 | -0.0144 |
| Patch of | 1000 | -0.3156 | 0.0314 | 1 AVO | 1000 | -0.2713 | -0.1134 |
| 3 ALOs | 3000 | -0.3096 | 0.0117 | $\omega = 50\sigma_y$ | 3000 | -0.2922 | -0.0451 |
| $\omega = 10\sigma_y$ | 5000 | -0.3079 | 0.0071 | | 5000 | -0.2962 | -0.0311 |
| No outliers | 1000 | -0.3060 | | | | | |
| | 3000 | -0.3060 | | | | | |
| | 5000 | -0.3057 | | | | | |

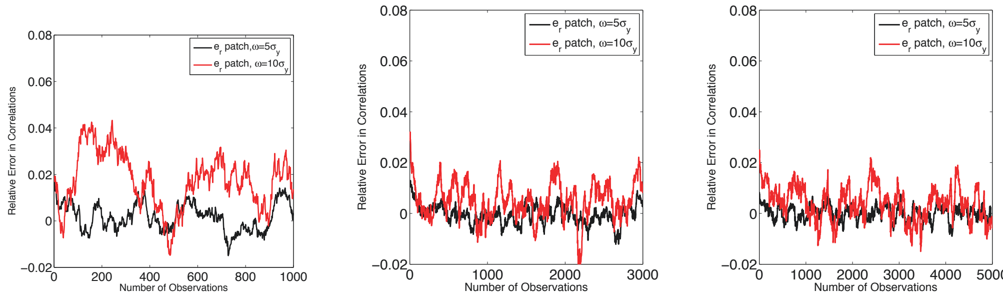



*(a1) DCC,* $n = 1000$ *(a2) DCC,* $n = 3000$ *(a3) DCC,* $n = 5000$

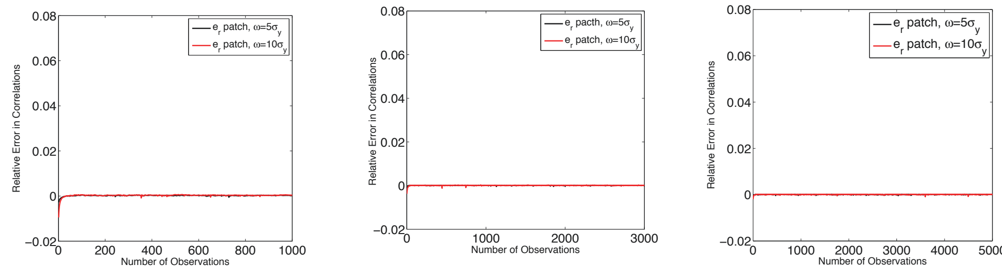



*(b1) D-BEKK,* $n = 1000$ *(b2) D-BEKK,* $n = 3000$ *(b3) D-BEKK,* $n = 5000$

***Figure D:*** *Relative bias in the estimated correlations obtained from a (a) DCC model and a (b) D-BEKK model with errors following* Student-$t_7$ *distributions from 1000 simulated series of size n that include ALOs of different magnitudes.*

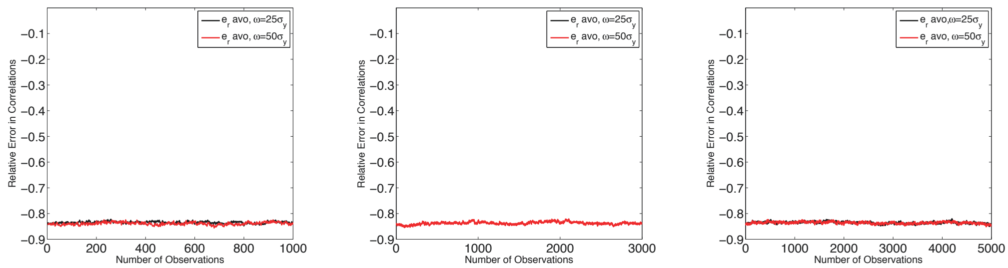



*(a1) DCC,* $n = 1000$ *(a2) DCC,* $n = 3000$ *(a3) DCC,* $n = 5000$

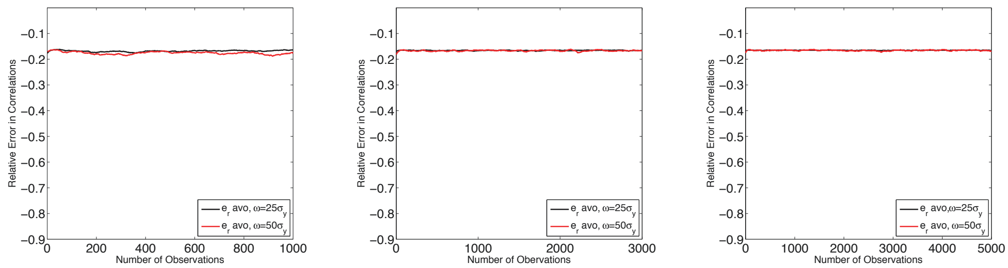



*(b1) D-BEKK,* $n = 1000$ *(b2) D-BEKK,* $n = 3000$ *(b3) D-BEKK,* $n = 5000$

***Figure E:*** *Relative bias in the estimated correlations obtained from a (a) DCC model and a (b) D-BEKK model with errors following* Student-$t_7$ *distributions from 1000 simulated series of size n that include patches of different magnitudes.*




*(a1) DCC,* $n = 1000$ *(a2) DCC,* $n = 3000$ *(a3) DCC,* $n = 5000$




*(b1) D-BEKK,* $n = 1000$ *(b2) D-BEKK,* $n = 3000$ *(b3) D-BEKK,* $n = 5000$

***Figure F:*** *Relative bias in the estimated correlations obtained from a (a) DCC model and a (b) D-BEKK model with errors following* Student-$t_7$ *distributions from 1000 simulated series of size n that include 1 AVO of different magnitudes.*

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