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Domination on hyperbolic graphs

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Abstract

If $k \geq 1$ and $G = (V, E)$ is a finite connected graph, $S \subseteq V$ is said a *distance k-dominating set* if every vertex $v \in V$ is within distance k from some vertex of S. The *distance k-domination number* $\gamma_w^k(G)$ is the minimum cardinality among all distance k-dominating sets of G. A set $S \subseteq V$ is a *total dominating set* if every vertex $v \in V$ satisfies $\delta_S(v) \geq 1$ and the *total domination number*, denoted by $\gamma_t(G)$, is the minimum cardinality among all total dominating sets of G. The study of hyperbolic graphs is an interesting topic since the hyperbolicity of any geodesic metric space is equivalent to the hyperbolicity of a graph related to it. In this paper we obtain relationships between the hyperbolicity constant $\delta(G)$ and some domination parameters of a graph G. The results in this work are inequalities, such as $\gamma_w^k(G) \geq 2\delta(G)/(2k+1)$ and $\delta(G) \leq \gamma_t(G)/2 + 3$.

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1. Introduction

The idea of domination in graphs was mathematically formalized by Berge [2] and Ore [24] in 1962. Currently, this topic has been detailed in the two, well-known, books by Haynes, Hedetniemi, and Slater. The theory of domination in graphs is an area of increasing interest in discrete mathematics and combinatorial computing. Besides of the mathematical and combinatorial importance of the theory, it has been applied successfully in different practical problems such as: analysis of social networks [19], efficient identification of web communities [10], bioinformatics [15], foodwebs [20]. Another application of the concept of domination is the study of the transmission of information in the network associated with defense systems [25].

In [6], Cockayne, Gamble and Shepherd defined a generalization of domination in graphs as follows: given a graph $G = (V, E)$, a set $S \subseteq V$ is a *k-dominating set* if every vertex $v \in V \setminus S$ satisfies $\delta_S(v) \geq k$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among all k-dominating sets. A set $S \subseteq V(G)$ is a *total k-dominating set* if every vertex $v \in V(G)$ satisfies $\delta_S(v) \geq k$. The *total k-domination number* $\gamma_{kt}(G)$ is the minimum cardinality among all total k-dominating sets (see [9, 14, 16]), and the *total domination number*, denoted by $\gamma_t(G)$, is the minimum cardinality among all total dominating sets, that is, $\gamma_t(G) = \gamma_{1t}(G)$ (see [5, 9]).

Hyperbolic spaces, defined by Gromov in [12], play an important role in geometric group theory and in the geometry of negatively curved spaces (see [1, 4, 11, 12]). The concept of Gromov hyperbolicity grasps

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the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1, 4, 11, 12]). As observed in [4, Section 1.3], the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it.

We say that a curve $\gamma : [a, b] \to X$ in a metric space X is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t-s|$ for every $s, t \in [a, b]$, where L and d denote length and distance, respectively, and $\gamma|_{[t,s]}$ is the restriction of the curve γ to the interval $[t, s]$ (then γ is equipped with an arc-length parametrization). The metric space X is said *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by uv.

In order to consider a graph G as a geodesic metric space, we identify (by an isometry) any edge $[u, v] \in E(G)$ with the interval $[0, 1]$ in the real line; then the edge $[u, v]$ (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G. In this way, any connected graph G has a natural distance defined on its points, induced by taking shortest paths in G, and we can see G as a metric graph. Throughout this paper, $G = (V, E) = (V(G), E(G))$ denotes a connected finite graph such that every edge has length 1 and $V \neq \emptyset$. These properties guarantee that G is a geodesic metric space. Note that the connectedness of the graph is not an important restriction. Since any domination parameter of a non-connected graph G is the sum of the values of this domination parameter of the connected components of G, and any hyperbolicity constant of a non-connected graph G is the supremum of the values of this hyperbolicity constant of the connected components of G, the results of this paper can be trivially extended to non-connected graphs.

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ is a *geodesic triangle* that will be denoted by $T = \{x_1, x_2, x_3\}$ and we will say that x_1, x_2 and x_3 are the vertices of T; it is usual to write also $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$. We say that T is δ-*thin* if any side of T is contained in the δ-neighborhood of the union of the two other sides. We denote by $\delta(T)$ the sharp thin constant of T, i.e., $\delta(T) := \inf\{\delta \geq 0 : T \text{ is } \delta\text{-thin}\}$. The space X is δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$; then X is hyperbolic if and only if $\delta(X) < \infty$. If we have a triangle with two identical vertices, we call it a "bigon". Obviously, every bigon in a δ-hyperbolic space is δ-thin.

In this paper we obtain relationships between the hyperbolicity constant $\delta(G)$ and some domination parameters of a graph G. The results in this work are inequalities, such as $\gamma_w^k(G) \geq 2\delta(G)/(2k+1)$ (Theorem 2.7) and $\delta(G) \leq \gamma_t(G)/2 + 3$ (Theorem 2.11).

2. Domination and hyperbolicity

In the classical references on this subject (see, e.g., [4, 11]) appear several different definitions of Gromov hyperbolicity, which are equivalent in the sense that if X is δ-hyperbolic with respect to one definition, then it is δ'-hyperbolic with respect to another definition (for some δ' related to δ). We have chosen the Rips definition by its deep geometric meaning [11].

Let us define the *Gromov product* of $x, y \in G$ with base point $w \in G$ by

$$(x, y)_w := \frac{1}{2}\big(d_G(x, w) + d_G(y, w) - d_G(x, y)\big).$$

If G is a Gromov hyperbolic graph, it holds

$$(x, z)_w \geq \min\big\{(x, y)_w, (y, z)_w\big\} - \delta \tag{2.1}$$

for every $x, y, z, w \in G$ and some constant $\delta \ge 0$ (see e.g. [1, 11]). Let us denote by $\delta^*(G)$ the sharp constant for this inequality, *i.e.*,

$$\delta^*(G) := \sup \big\{ \min \big\{(x, y)_w, (y, z)_w \big\} - (x, z)_w : x, y, z, w \in G \big\}.$$

It is well-known that (2.1) is, in fact, equivalent to our definition of Gromov hyperbolicity; furthermore, we have $\delta^*(G) \le 4\,\delta(G)$ and $\delta(G) \le 3\,\delta^*(G)$ (see e.g. [1, 11]). In [28, Proposition II.20] we found the following improvement of the previous inequality $\delta^*(G) \le 2\,\delta(G)$.

A subgraph Γ of G is said *isometric* if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$ (in particular, every isometric graph is connected).

The following result is elementary.

Lemma 2.1. *If Γ is an isometric subgraph of G, then $\delta(\Gamma) \le \delta(G)$ and $\delta^*(\Gamma) \le \delta^*(G)$.*

In [18] is introduced the concept of distance domination (see also [17], [13], [22]). Given a graph G and $k \ge 1$, we say that a subset of vertices $S \subset V(G)$ is *distance k-dominating* if for any vertex $v \in V(G)$ there is $w \in S$ with $d_G(v, w) \le k$. Since $d_G(w, w) = 0 \le k$, we can replace the condition "$d_G(v, w) \le k$ for any $v \in V(G)$" by "$d_G(v, w) \le k$ for any $v \in V(G) \setminus S$".

We say that a subgraph Γ of G is *distance k-dominating* if $V(\Gamma)$ is distance k-dominating.

Theorem 2.2. *Let G be a graph, $k \ge 1$ and Γ an isometric distance k-dominating subgraph of G. Then*

$$\delta^*(\Gamma) \le \delta^*(G) \le \delta^*(\Gamma) + 6k + 3.$$

Proof. Lemma 2.1 gives the first inequality.

Let f be a projection map $f : G \to \Gamma$, i.e., a map such that $d_G(x, f(x)) = d_G(x, \Gamma)$ for every $x \in G$ (in particular, $f|_\Gamma$ is the identity map). Since Γ an isometric distance k-dominating subgraph, we have $d_G(x, f(x)) \le k + 1/2$ and

$$\begin{aligned}
(f(x), f(y))_{f(w)} &= \frac{1}{2}\Big(d_\Gamma(f(x), f(w)) + d_\Gamma(f(y), f(w)) - d_\Gamma(f(x), f(y))\Big) \\
&= \frac{1}{2}\Big(d_G(f(x), f(w)) + d_G(f(y), f(w)) - d_G(f(x), f(y))\Big) \\
&\le \frac{1}{2}\Big(d_G(x, w) + 2k + 1 + d_G(y, w) + 2k + 1 - d_G(x, y) + 2k + 1\Big) \\
&= (x, y)_w + 3k + \frac{3}{2}\,.
\end{aligned}$$

We obtain in a similar way

$$(f(x), f(y))_{f(w)} \ge (x, y)_w - 3k - \frac{3}{2}\,,$$

and thus

$$\begin{aligned}
(x, z)_w &\ge (f(x), f(z))_{f(w)} - 3k - \frac{3}{2} \\
&\ge \min \big\{(f(x), f(y))_{f(w)}, (f(y), f(z))_{f(w)}\big\} - \delta^*(\Gamma) - 3k - \frac{3}{2} \\
&\ge \min \Big\{(x, y)_w - 3k - \frac{3}{2}\,, (y, z)_w - 3k - \frac{3}{2}\Big\} - \delta^*(\Gamma) - 3k - \frac{3}{2} \\
&= \min \big\{(x, y)_w, (y, z)_w\big\} - \delta^*(\Gamma) - 6k - 3.
\end{aligned}$$

Hence, we conclude

$$\delta^*(G) \le \delta^*(\Gamma) + 6k + 3.$$

□

Theorem 2.2 has the following consequence.

Theorem 2.3. *Let G be a graph, $k \geq 1$ and Γ an isometric distance k-dominating subgraph of G. Then*

$$\delta(\Gamma) \leq \delta(G) \leq 6\,\delta(\Gamma) + 18k + 9.$$

Proof. Lemma 2.1 gives the first inequality.

Using the inequalities relating $\delta^*(G)$ and $\delta(G)$ and Theorem 2.2, we conclude

$$\delta(G) \leq 3\,\delta^*(G) \leq 3\big(\delta^*(\Gamma) + 6k + 3\big) \leq 6\,\delta(G) + 18k + 9.$$

□

The following example shows that it is not possible to have the inequality

$$\delta(G) \leq \Psi\big(\delta(\Gamma)\big),$$

for every graph G and distance k-dominating subgraph Γ (not necessarily isometric) and some function Ψ. For each integer $n > 2k$ consider the cycle graph C_n with vertices $V(C_n) = \{v_1, v_2, \dots, v_{n-1}, v_n\}$ and edges $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$ and the subgraph Γ_n induced by $\{v_1, \dots, v_{n-2k}\}$. It is clear that $V(\Gamma_n)$ is a distance k-dominating set. Since Γ_n is a tree, $\delta(\Gamma_n) = 0$. However, $\delta(C_n) = n/4$.

For any graph G, we define, as usual,

$$\operatorname{diam} V(G) := \sup\big\{d_G(v, w) \mid v, w \in V(G)\big\},$$
$$\operatorname{diam} G := \sup\big\{d_G(x, y) \mid x, y \in G\big\},$$

i.e, $\operatorname{diam} V(G)$ is the diameter of the set of vertices of G, and $\operatorname{diam} G$ is the diameter of the whole graph G (recall that in order to have a geodesic metric space, G must contain both the vertices and the points in the interior of any edge of G).

The following result is well-known (see, e.g., [27, Theorem 8] for a proof).

Lemma 2.4. *In any graph G the inequality*

$$\delta(G) \leq \frac{1}{2}\operatorname{diam} G \leq \frac{1}{2}\big(\operatorname{diam} V(G) + 1\big)$$

holds.

Given a graph G, we say that a subset of vertices $S \subset V(G)$ is *dominating* if every vertex $v \in V(G) \setminus S$ has a neighbor in S. We define the *domination number* of G as

$$\gamma(G) := \min\big\{\, |S| \,:\, S \text{ is a dominating set of } G \,\big\}.$$

Given a graph G, we say that a subset of vertices $S \subset V(G)$ is *total-dominating* if every vertex $v \in V(G)$ has a neighbor in S. We define the *total-domination number* of G as

$$\gamma_t(G) := \min\big\{\, |S| \,:\, S \text{ is a total-dominating set of } G \,\big\}.$$

Given a graph G, we define the *distance k-domination number* of G as

$$\gamma^k(G) := \min\big\{\, |S| \,:\, S \text{ is a distance } k\text{-dominating set of } G \,\big\}.$$

It is well-known (see [8, Theorem 4]) that

$$\gamma_t(G) \geq \frac{\operatorname{diam} V(G) + 1}{2}.$$

Thus, Lemma 2.4 gives the following result.

Proposition 2.5. *If G is a graph, then*

$$\delta(G) \le \gamma_t(G)\,.$$

Proposition 2.5 can be improved for graphs with small maximum degree.

Theorem 2.6. *If G is a graph with maximum degree Δ, then*

$$\delta(G) \le \frac{\Delta}{4}\,\gamma_t(G)\,.$$

Proof. Let $S \subseteq V(G)$ be a total dominating set with $|S| = \gamma_t(G)$, and $n := |V(G)|$. Denote by $\overline{S}$ the complement $\overline{S} := V(G) \setminus S$ of the set S, and by $E_{S,\overline{S}}$ the set of edges joining a vertex in S with a vertex in $\overline{S}$. Since S is a dominating set, $\left|\overline{S}\right| \le \left|E_{S,\overline{S}}\right|$. Since S is a total dominating set, $\left|E_{S,\overline{S}}\right| \le (\Delta - 1)|S|$, and we conclude

$$n - |S| = \left|\overline{S}\right| \le \left|E_{S,\overline{S}}\right| \le (\Delta - 1)|S|, \qquad n \le \Delta\gamma_t(G).$$

The inequality $\delta(G) \le n/4$ (see [23, Theorem 30]) gives $\delta(G) \le \Delta\gamma_t(G)/4$. □

We have similar results for $\gamma^k(G)$.

Theorem 2.7. *Let G be a graph and $k \ge 1$. Then*

$$\gamma^k(G) \ge \frac{\operatorname{diam} V(G) + 1}{2k + 1}\,, \qquad \gamma^k(G) \ge \frac{2\delta(G)}{2k + 1}\,.$$

Proof. Let S be a distance k-dominating set of G with $|S| = \gamma^k(G)$, and $\sigma = [uv]$ a geodesic in G with $u, v \in V(G)$ and $d_G(u, v) = \operatorname{diam} V(G)$. Since S is distance k-dominating, there exists $s_1 \in S$ with $d_G(u, s_1) \le k$.

Let $\{u_1, u_2, \dots, u_r, u_{r+1}\} = V(G) \cap \sigma$ with $u_1 = u$, $u_{r+1} = v$, $r = \operatorname{diam} V(G)$ and $u_i u_{i+1} \in E(G)$ for $1 \le i \le r$. Define

$$t_1 := \max\left\{\, 1 \le t \le r + 1 \;:\; d_G(u_t, s_1) \le k \,\right\}.$$

Since σ is a geodesic and the diameter of the closed ball $\overline{B_G}(u_1, k)$ is at most $2k$, we have $t_1 \le 2k + 1$.

If $r + 1 > 2k + 1$, then there exists $s_2 \in S$ with $d_G(u_{t_1+1}, s_2) \le k$. Define

$$t_2 := \max\left\{\, t_1 + 1 \le t \le r + 1 \;:\; d_G(u_t, s_2) \le k \,\right\}.$$

Thus, $t_2 \le 4k + 2$.

If $r + 1 > 4k + 2$, then we can repeat this process obtaining two finite sequences $\{s_1, \dots, s_j\} \subseteq S$ and $1 \le t_1 < t_2 < \cdots < t_j \le r + 1$ with $r + 1 \le (2k + 1)j$. Hence, we obtain

$$\frac{\operatorname{diam} V(G) + 1}{2k + 1} = \frac{r + 1}{2k + 1} \le j = \left|\{s_1, \dots, s_j\}\right| \le |S| = \gamma^k(G),$$

and Lemma 2.4 gives the second inequality. □

Given a graph G, we say that a subset of vertices $S \subset V(G)$ is *k-total-dominating* ($k \ge 1$) if every vertex $v \in V(G)$ has k neighbors in S. Denote by $\langle S \rangle$ the subgraph of G induced by S. We say that S is *k-total-connected-dominating* if it is k-total-dominating and $\langle S \rangle$ is connected. We define the *k-total-connected-domination number* of G as

$$\gamma^k_{tc}(G) := \min\left\{\, |S| : S \text{ is a } k\text{-total-connected-dominating set of } G \,\right\}.$$

As usual, we denote by $\lfloor t \rfloor$ the lower integer part of t, i.e., the largest integer least than or equal to t.

Theorem 2.8. *If G is a graph and $k \geq 2$, then*

$$\delta(G) \leq \frac{1}{2} \max\left\{5, \left\lfloor \frac{3\,\gamma_{tc}^k(G) - 2}{k+1} \right\rfloor + 1\right\}.$$

Proof. Given a graph G, fix a k-total-connected-dominating set S with $|S| = \gamma_{tc}^k(G)$. Define $s := \operatorname{diam}_{\langle S\rangle} S$ and choose $u, v \in V(S)$ with $d_S(u, v) = s$. For each $0 \leq j \leq s$, let $n_j := |S_j|$ with $S_j := \{w \in S \,:\, d_S(w, u) = j\}$. Note that a vertex of S_j and a vertex of $S_0 \cup S_1 \cup \cdots \cup S_{j-2}$ can not be neighbors for $2 \leq j \leq s$. Since $\langle S\rangle$ is connected, we have $\sum_{j=0}^{s} n_j = |S| = \gamma_{tc}^k(G)$, $n_0 = 1$, $n_1 \geq k$ and $n_j \geq 1$ for each $2 \leq j \leq s$.

Since S is a k-total-connected-dominating set S, if $s < 3$, then $\operatorname{diam}_G V(G) \leq s + 2 \leq 4$ and Lemma 2.4 gives $\delta(G) \leq 5/2$. Hence, we can assume that $s \geq 3$.

Define $n_{s+1} := 0$ and

$$a_s := \sum_{j=3}^{s} \big(n_{j-1} + n_j + n_{j+1}\big) = \begin{cases} n_2 + 2n_3 + 3\sum_{j=4}^{s-1} n_j + 2n_s, & \text{if } s > 4, \\ n_2 + 2n_3 + 2n_4, & \text{if } s = 4, \\ n_2 + n_3, & \text{if } s = 3. \end{cases}$$

Note that for any $3 \leq j \leq s$, we have $n_{j-1} + n_j + n_{j+1} \geq k + 1$ and so, $a_s \geq (s-2)(k+1)$. Thus,

$$\begin{aligned} 3|S| &= 3\sum_{j=0}^{s} n_j = 3 + 3n_1 + 2n_2 + n_3 + n_s + a_s \\ &\geq 3 + 3k + 2 + 1 + 1 + (s-2)(k+1) = (s+1)(k+1) + 4, \\ \frac{3|S| - 4}{k+1} &\geq s + 1, \\ \operatorname{diam}_{\langle S\rangle} S &\leq \left\lfloor \frac{3|S| - 4}{k+1} \right\rfloor - 1 = \left\lfloor \frac{3\,\gamma_{tc}^k(G) - 4}{k+1} \right\rfloor - 1. \end{aligned}$$

Since S is a k-total-connected-dominating set, we have that $\operatorname{diam}_G V(G) \leq \operatorname{diam}_{\langle S\rangle} S + 2$. Let us assume that $\operatorname{diam}_G V(G) = \operatorname{diam}_{\langle S\rangle} S + 2$. Hence, there exist $u', v' \in V(G) \setminus S$ and $u, v \in S$ with $uu', vv' \in E(G)$ and $\operatorname{diam}_G V(G) = d_G(u', v') = d_S(u, v) + 2$.

For each $-1 \leq j \leq s + 1$, let $n_j := |S_{j+1}|$ with $S_j := \{w \in S \,:\, d_S(w, u') = j\}$. Using the previous argument, since S is a k-total-connected-dominating set, we have in this case $n_0, n_s \geq k$ and $n_1, n_2, n_3 \geq 1$. Therefore, we deduce

$$\begin{aligned} 3|S| &= 3n_0 + 3n_1 + 2n_2 + n_3 + n_s + a_s \\ &\geq 3k + 3 + 2 + 1 + k + (s-2)(k+1) = (s+2)(k+1) + 2, \\ \operatorname{diam}_{\langle S\rangle} S &\leq \left\lfloor \frac{3\,\gamma_{tc}^k(G) - 2}{k+1} \right\rfloor - 2, \\ \operatorname{diam}_G V(G) &\leq \left\lfloor \frac{3\,\gamma_{tc}^k(G) - 2}{k+1} \right\rfloor. \end{aligned}$$

If $\operatorname{diam}_G V(G) < \operatorname{diam}_{\langle S\rangle} S + 2$, then

$$\operatorname{diam}_G V(G) \leq \operatorname{diam}_{\langle S\rangle} S + 1 \leq \left\lfloor \frac{3\,\gamma_{tc}^k(G) - 4}{k+1} \right\rfloor \leq \left\lfloor \frac{3\,\gamma_{tc}^k(G) - 2}{k+1} \right\rfloor.$$

Hence, we have

$$\operatorname{diam}_G V(G) \leq \max\left\{4, \left\lfloor \frac{3\,\gamma_{tc}^k(G) - 2}{k+1} \right\rfloor\right\},$$

and Lemma 2.4 gives

$$\delta(G) \le \frac{1}{2} \max\left\{5, \left\lfloor \frac{3\,\gamma_{tc}^k(G) - 2}{k+1} \right\rfloor + 1\right\}.$$

□

As usual, by *cycle* in a graph we mean a simple closed curve, i.e., a path with different vertices, except for the last one, which is equal to the first vertex.

Let us denote by $J(G)$ the union of the set $V(G)$ and the midpoints of the edges of G. Consider the set $\mathbb{T}_1$ of geodesic triangles T in G that are cycles and such that the three vertices of the triangle T belong to $J(G)$, and denote by $\delta_1(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_1$ is λ-thin.

The following results, which appear in [3, Theorems 2.7 and 2.6], will be used throughout the paper.

Lemma 2.9. *For any hyperbolic graph G, there exists a geodesic triangle $T \in \mathbb{T}_1$ such that $\delta(T) = \delta(G)$.*

The next result will narrow the possible values for the hyperbolicity constant δ.

Lemma 2.10. *If G is a graph, then $\delta(G)$ is a multiple of* $1/4$.

The two following results improve Proposition 2.5.

Given $s \in \mathbb{R}$, denote by $\lceil s \rceil$ the upper integer part of s, i.e., the smallest integer greater than or equal to s.

Theorem 2.11. *If G is a graph, then*

$$\delta(G) \le \begin{cases} \frac{1}{2}\,\gamma_t(G) + 1, & \text{if } \gamma_t(G) \le 3, \\ \frac{1}{2}\,\gamma_t(G) + 3, & \text{if } \gamma_t(G) \ge 4. \end{cases}$$

Proof. Fix a total dominating set $S \subset V(G)$ with $|S| = \gamma_t(G)$.

Assume first that $\gamma_t(G) \le 3$. Thus, S is a connected set, and we deduce $\operatorname{diam}_G S \le \gamma_t(G) - 1$ and $\operatorname{diam}_G V(G) \le \gamma_t(G) + 1$. Thus, Lemma 2.4 gives $\delta(G) \le \gamma_t(G)/2 + 1$.

Assume now that $\gamma_t(G) \ge 4$.

By Lemma 2.9, there exist a triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $p \in [xy]$ such that $d_G(p, [xz] \cup [zy]) = \delta(G)$. Let $V(G) \cap [xy] = \{a_1, a_2, \dots, a_r\}$ with $a_j a_{j+1} \in E(G) \cap [xy]$ for $1 \le j < r$, $d_G(a_1, x) \le 1/2$ and $d_G(a_r, y) \le 1/2$. Let $V(G) \cap ([xz] \cup [zy]) = \{b_1, b_2, \dots, b_\beta\}$ with $b_j b_{j+1} \in E(G) \cap ([xz] \cup [zy])$ for $1 \le j < \beta$, $d_G(b_1, x) \le 1/2$ and $d_G(b_\beta, y) \le 1/2$ (note that $r \le \beta$, since $[xy]$ is a geodesic and $x, y \in J(G)$). Let $1 \le \alpha \le \alpha' \le \beta$ be such that $V(G) \cap [xz] = \{b_1, b_2, \dots, b_\alpha\}$ and $V(G) \cap [zy] = \{b_{\alpha'}, b_{\alpha'+1}, \dots, b_\beta\}$ (note that $\alpha = \alpha'$ if and only if $z \in V(G)$; otherwise, $\alpha' = \alpha + 1$).

If $a_j \in S$, then we define $s_j := a_j$; since S is a total dominating set, if $a_j \notin S$, then there exists $s_j \in N(a_j) \cap S$. If $b_j \in S$, then we define $\overline{s}_j := b_j$; since S is a total dominating set, if $b_j \notin S$, then there exists $\overline{s}_j \in N(b_j) \cap S$.

We are going to define subsets $S_1, S_2 \subset S$ associated to $[xy]$ and $[xz] \cup [zy]$, respectively.

Since $[xy]$ is a geodesic, if $s_i = s_j$, then $|i - j| \le 2$. Let $\mathfrak{I}$ be the set

$$\mathfrak{I} := \big\{ 1 \le i \le r - 2 : \, s_i = s_{i+1} = s_{i+2} \big\}.$$

If $\mathfrak{I} = \emptyset$, then

$$\big|\{s_1, s_2, \dots, s_r\}\big| \ge \left\lceil \frac{r}{2} \right\rceil.$$

Hence, the set $S_1 := \{s_1, s_2, \dots, s_r\}$ satisfies $|S_1| \ge \lceil r/2 \rceil$.

Since S is a total dominating set, if $\mathfrak{I} \ne \emptyset$ and $i \in \mathfrak{I}$, then there exists $s_i' \in N(s_i) \cap S$. Assume that $i, j \in \mathfrak{I}$ with $i \ne j$ (without loss of generality we can assume that $i < j$, and thus $i + 3 \le j$); then $s_i' \ne s_j'$, since otherwise $5 = i + 3 + 2 - i \le j + 2 - i = d_G(a_i, a_{j+2}) \le d_G(a_i, s_i) + d_G(s_i, s_i') +$

$d_G(s'_j, s_{j+2}) + d_G(s_{j+2}, a_{j+2}) \le 4$, a contradiction. Note that $s'_i \notin \{a_i, a_{i+1}, a_{i+2}\}$; also, $s'_i \notin \{a_1, a_2, \dots, a_r\}$, since otherwise $s_i \in N(a_i) \cap N(a_{i+1}) \cap N(a_{i+2}) \cap N(s'_i)$, a contradiction. Besides, $s'_i \neq s_j$ if $s_j = s_{j+1}$ and $\{i, i+1, i+2\} \cap \{j, j+1\} = \emptyset$. Furthermore, there exists at most one j with $s'_i = s_j$, $j \notin \{i, i+1, i+2\}$ and $s_{j-1} \neq s_j \neq s_{j+1}$. Thus,

$$\big| \cup_{i \in \mathfrak{I}} \{s'_i\} \cup \{s_1, s_2, \dots, s_r\} \big| \ge \left\lceil \frac{r}{2} \right\rceil.$$

Therefore, the set $S_1 := \cup_{i \in \mathfrak{I}} \{s'_i\} \cup \{s_1, s_2, \dots, s_r\}$ satisfies $|S_1| \ge \lceil r/2 \rceil$ in both cases.

Next, we define a similar set associated to $[xz] \cup [zy]$.

Given $v_1, v_2, \dots, v_k \in V(G)$ such that for each $1 \le j < k$ we have either $v_j v_{j+1} \in E(G)$ or $v_j = v_{j+1}$, we denote by $v_1 v_2 \cdots v_k$ the path containing the edges (or vertices) $v_j v_{j+1}$ for $1 \le j < k$.

Let us consider the sets

$$\begin{aligned}
\Gamma_0 &:= \big\{ \gamma \text{ path } \subset G : \gamma = b_1 b_2 \cdots b_\beta \big\}, \\
\Gamma_1 &:= \big\{ \gamma \text{ path } \subset G : \gamma = b_1 b_2 \cdots b_i \overline{s}_i b_j \cdots b_\beta \text{ if } \overline{s}_i = \overline{s}_j \big\}, \\
\Gamma_2 &:= \big\{ \gamma \text{ path } \subset G : \gamma = b_1 b_2 \cdots b_i \overline{s}_i \overline{s}_j b_j \cdots b_\beta \text{ if } \overline{s}_i \overline{s}_j \in E(G) \big\}, \\
\Gamma_3 &:= \big\{ \gamma \text{ path } \subset G : \gamma = b_1 b_2 \cdots b_i \overline{s}_i \overline{s}_0 \overline{s}_j b_j \cdots b_\beta \text{ if } \exists\, \overline{s}_0 \in S \cap N(\overline{s}_i) \cap N(\overline{s}_j) \big\}, \\
\Gamma &:= \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.
\end{aligned}$$

Let us choose $\sigma \in \Gamma$ with

$$L(\sigma) = \min \big\{ L(\gamma) : \gamma \in \Gamma \big\}.$$

Since σ joins b_1 and b_β, we have that $|\sigma \cap V(G)| \ge r$. Let i_0, j_0 be the integers such that $1 \le i_0 \le \alpha \le \alpha' \le j_0 \le \beta$, $b_1, \dots, b_{i_0}, b_{j_0}, \dots, b_\beta \in \sigma$, $b_{i_0+1} \notin \sigma \cap [xz]$ and $b_{j_0-1} \notin \sigma \cap [zy]$.

Let us define the set

$$\overline{\mathfrak{I}} := \big\{ 1 \le i \le i_0 - 2 : \overline{s}_i = \overline{s}_{i+1} = \overline{s}_{i+2} \big\} \cup \big\{ j_0 \le i \le \beta - 2 : \overline{s}_i = \overline{s}_{i+1} = \overline{s}_{i+2} \big\}. \tag{2.2}$$

Since S is a total dominating set, if $i \in \overline{\mathfrak{I}}$, then there exists $\overline{s}'_i \in N(\overline{s}_i)$.

Case A. Assume that $\sigma \notin \Gamma_0$.

Since $\sigma \notin \Gamma_0$, the minimality of σ gives $\overline{s}_{i_0} \neq \overline{s}_i$ for every $1 \le i < i_0$ and $\overline{s}_{j_0} \neq \overline{s}_j$ for every $j_0 < j \le \beta$; in particular, this gives $i_0 - 2, j_0 \notin \overline{\mathfrak{I}}$, and we can write

$$\overline{\mathfrak{I}} = \big\{ 1 \le i < i_0 - 2 : \overline{s}_i = \overline{s}_{i+1} = \overline{s}_{i+2} \big\} \cup \big\{ j_0 < i \le \beta - 2 : \overline{s}_i = \overline{s}_{i+1} = \overline{s}_{i+2} \big\}. \tag{2.3}$$

If $i, j \in \overline{\mathfrak{I}}$ with $i \neq j$ and either $1 \le i, j < i_0 - 2$ or $j_0 < i, j \le \beta - 2$, then the argument in the case of S_1 gives $\overline{s}'_i \neq \overline{s}'_j$. If $1 \le i < i_0 - 2$ and $j_0 < j \le \beta - 2$, then the minimality of σ gives $\overline{s}'_i \neq \overline{s}'_j$.

Also, the minimality of σ gives $\overline{s}'_i \neq \overline{s}_j$ if $i \in \overline{\mathfrak{I}}$ with $1 \le i \le i_0 - 2$ and $j_0 \le j \le \beta$, and $\overline{s}_i \neq \overline{s}'_j$ if $j \in \overline{\mathfrak{I}}$ with $1 \le i \le i_0$ and $j_0 \le j \le \beta - 2$.

If $1 \le i < i_0$ and $j_0 < j \le \beta$, then the minimality of σ also gives $\overline{s}_i \neq \overline{s}_j$, $\overline{s}_{i_0} \neq \overline{s}_j$ and $\overline{s}_i \neq \overline{s}_{j_0}$. Note that in the paths $b_i \overline{s}_i b_j$ (if $\sigma \in \Gamma_1$ and $\overline{s}_i = \overline{s}_j$), $b_i \overline{s}_i \overline{s}_j b_j$ (if $\sigma \in \Gamma_2$ and $\overline{s}_i \overline{s}_j \in E(G)$), and $b_i \overline{s}_i \overline{s}_0 \overline{s}_j b_j$ (if $\sigma \in \Gamma_3$ and there exists $\overline{s}_0 \in S \cap N(\overline{s}_i) \cap N(\overline{s}_j)$), the cardinal of the vertices in S plus 1 is greater than or equal to the cardinal of the points in $V(G) \setminus S$. Thus, the set $S_2 := \cup_{i \in \overline{\mathfrak{I}}} \{\overline{s}'_i\} \cup \{\overline{s}_1, \overline{s}_2, \dots, \overline{s}_r\}$ satisfies $|S_2| \ge \lceil (r-1)/2 \rceil$.

Case B. Assume that $\sigma \in \Gamma_0$.

Note that $\overline{\mathfrak{I}}$ is defined by (2.2); since $\sigma \in \Gamma_0$, (2.3) can be false.

As in Case A, let us define $S_2 := \cup_{i \in \overline{\mathfrak{I}}} \{\overline{s}'_i\} \cup \{\overline{s}_1, \overline{s}_2, \dots, \overline{s}_r\}$.

Assume that $z \notin V(G)$, since the argument when $z \in V(G)$ is analogous. Thus, $i_0 = \alpha$ and $j_0 = \alpha' = \alpha + 1$.

The minimality of σ gives the following six facts:

$\overline{s}_i \neq \overline{s}_j$ for every $1 \le i < i_0$ and $j_0 < j \le \beta$.

$\overline{s}_{i_0} \neq \overline{s}_j$ for every $j_0 + 2 \leq j \leq \beta$ and $\overline{s}_{j_0} \neq \overline{s}_i$ for every $1 \leq i \leq i_0 - 2$.
$\overline{s}'_i \neq \overline{s}'_j$ if $i, j \in \overline{\mathfrak{I}}$ with $i \neq j$ and either $1 \leq i, j \leq i_0 - 2$ or $j_0 \leq i, j \leq \beta - 2$.
$\overline{s}'_i \neq \overline{s}'_j$ if $i, j \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0 - 2$ and $j_0 \leq j \leq \beta - 2$.
$\overline{s}'_i \neq \overline{s}_j$ if $i \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0 - 2$ and $j_0 < j \leq \beta$.
$\overline{s}_i \neq \overline{s}'_j$ if $j \in \overline{\mathfrak{I}}$ with $1 \leq i < i_0$ and $j_0 \leq j \leq \beta - 2$.

Case B.1. If $\overline{s}_i \neq \overline{s}_j$ for every $1 \leq i \leq i_0$ and $j_0 \leq j \leq \beta$, $\overline{s}'_i \neq \overline{s}_j$ for every $i \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0 - 2$ and $j_0 \leq j \leq \beta$, and $\overline{s}_i \neq \overline{s}'_j$ for every $j \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0$ and $j_0 \leq j \leq \beta - 2$, then $|S_2| \geq \lceil r/2 \rceil$.

Case B.2. Assume that we are not in Case B.1. We have five different cases:

Case B.2.1. $i_0 - 2 \in \overline{\mathfrak{I}}$ and $\overline{s}'_{i_0} = \overline{s}_{j_0}$. The minimality of σ gives $\overline{s}_i \neq \overline{s}_j$ for every $1 \leq i \leq i_0$ and $j_0 \leq j \leq \beta$, and $\overline{s}_i \neq \overline{s}'_j$ for every $j \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0$ and $j_0 \leq j \leq \beta - 2$. Besides, the two vertices $\overline{s}_{i_0}$ and $\overline{s}'_{i_0} = \overline{s}_{j_0}$ in S_2 are associated to the four vertices $b_{i_0-2}, b_{i_0-1}, b_{i_0}, b_{j_0}$. Hence, we also conclude that $|S_2| \geq \lceil r/2 \rceil$.

Case B.2.2. $j_0 \in \overline{\mathfrak{I}}$ and $\overline{s}'_{j_0} = \overline{s}_{i_0}$. A symmetric argument to the one in the previous case also gives $|S_2| \geq \lceil r/2 \rceil$.

Case B.2.3. $\overline{s}_{i_0} = \overline{s}_{j_0+1}$ and $\overline{s}_{i_0-1} \neq \overline{s}_{j_0}$. The minimality of σ gives $\overline{s}_i \neq \overline{s}_{i_0}$ for every $1 \leq i < i_0$ and $\overline{s}_{j_0+1} \neq \overline{s}_j$ for every $j_0 + 2 \leq j \leq \beta$. Thus, $i_0 - 2, j_0 \notin \overline{\mathfrak{I}}$ and we conclude $\overline{s}'_i \neq \overline{s}_j$ if $i \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0 - 2$ and $j_0 \leq j \leq \beta$, and $\overline{s}_i \neq \overline{s}'_j$ if $j \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0$ and $j_0 \leq j \leq \beta - 2$. The vertex $\overline{s}_{i_0} = \overline{s}_{j_0+1} \in S_2$ is associated to the three vertices $b_{i_0}, b_{j_0}, b_{j_0+1}$, and so, twice the cardinal of the vertices in S_2 plus 1 is greater than or equal to the cardinal of the points in $\{b_1, b_2, \dots, b_\beta\}$. Hence, $|S_2| \geq \lceil (r-1)/2 \rceil$.

Case B.2.4. $\overline{s}_{i_0-1} = \overline{s}_{j_0}$ and $\overline{s}_{i_0} \neq \overline{s}_{j_0+1}$. A symmetric argument to the one in the previous case also gives $|S_2| \geq \lceil (r-1)/2 \rceil$.

Case B.2.5. $\overline{s}_{i_0-1} = \overline{s}_{j_0}$ and $\overline{s}_{i_0} = \overline{s}_{j_0+1}$. The minimality of σ gives $\overline{s}_{i_0-1} \neq \overline{s}_{j_0+1}$. A similar argument to the one in Case B.2.3 (now, with the two vertices $\overline{s}_{i_0-1} = \overline{s}_{j_0}$, $\overline{s}_{i_0} = \overline{s}_{j_0+1} \in S_2$ associated to the four vertices $b_{i_0-1}, b_{i_0}, b_{j_0}, b_{j_0+1}$) gives $|S_2| \geq \lceil r/2 \rceil$.

Hence, we have in every case $|S_2| \geq \lceil (r-1)/2 \rceil$.

We consider several cases.

(1) Assume first that $S_1 \cap S_2 = \emptyset$. Thus,

$$\gamma_t(G) = |S| \geq |S_1| + |S_2| \geq \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{r-1}{2} \right\rceil = r.$$

Since $x, y \in J(G)$ and $|\,[xy] \cap V(G)| = r$, we conclude $L([xy]) \leq r \leq \gamma_t(G)$, and

$$\delta(G) = d_G(p, [xz] \cup [zy]) \leq d_G(p, \{x, y\}) \leq \frac{1}{2}\, L([xy]) \leq \frac{1}{2}\, \gamma_t(G)\,.$$

(2) Assume now that $S_1 \cap S_2 \neq \emptyset$.

(2.1) Assume that $d_G(p, [xz] \cup [zy]) \leq 5$. Thus,

$$\delta(G) = d_G(p, [xz] \cup [zy]) \leq \frac{4}{2} + 3 \leq \frac{1}{2}\, \gamma_t(G) + 3,$$

since $\gamma_t(G) \geq 4$.

(2.2) Assume that $d_G(p, [xz] \cup [zy]) > 5$. If $p = a_l \in V(G)$, then S_2 does not intersect the subset of S_1 associated to $\{a_l\}$ (i.e., s_l and perhaps s'_l); and if $p \notin V(G)$, then $p \in a_l a_{l+1} \in E(G)$ and S_2 does not intersect the subset of S_1 associated to $\{a_l, a_{l+1}\}$ (i.e., s_l, s_{l+1} and perhaps s'_l and/or s'_{l+1}). Thus, there exists a maximal connected subset $\mathcal{A} := \{a_{i_1}, a_{i_1+1}, \dots, a_{i_2-1}, a_{i_2}\}$ of $[xy] \cap V(G)$ (with respect to the inclusion) such that $p \in [a_{i_1} a_{i_2}]$ and $S_1(\mathcal{A}) \cap S_2 = \emptyset$, where $S_1(\mathcal{A})$ is the subset of S_1 associated to $\mathcal{A}$.

Fix a positive integer u.

(2.2.1) If $i_1 \ge u+1$ and $i_2 \le r-u$, then $|\sigma \cap V(G)| \ge r \ge |\mathcal{A}| + 2u$ and

$$\begin{aligned}\gamma_t(G) = |S| \ge |S_1(\mathcal{A})| + |S_2| &\ge \left\lceil \frac{1}{2}\,|\mathcal{A}| \right\rceil + \left\lceil \frac{1}{2}\left(|\mathcal{A}| + 2u - 1\right) \right\rceil \\ &\ge \left\lceil \frac{1}{2}\,|\mathcal{A}| \right\rceil + \left\lceil \frac{1}{2}\left(|\mathcal{A}| - 1\right) \right\rceil + u = |\mathcal{A}| + u.\end{aligned}$$

The maximality of $\mathcal{A}$ gives $d_G(a_{i_1-1}, [xz] \cup [zy]) \le 4$ and $d_G(a_{i_2+1}, [xz] \cup [zy]) \le 4$. Let g_1 (respectively, g_2) be a geodesic in G joining a_{i_1-1} (respectively, a_{i_2+1}) and $[xz] \cup [zy]$, and ρ the curve

$$\rho := g_1 \cup a_{i_1-1}a_{i_1} \cdots a_{i_2}a_{i_2+1} \cup g_2.$$

Since ρ joins two points in $[xz] \cup [zy]$, $p \in \rho$ and $L(\rho) \le 4 + |\mathcal{A}| + 1 + 4$, we have

$$\delta(G) = d_G(p, [xz] \cup [zy]) \le \frac{1}{2}\,L(\rho) \le \frac{1}{2}\,|\mathcal{A}| + \frac{9}{2} \le \frac{1}{2}\,\gamma_t(G) + \frac{9-u}{2}\,.$$

(2.2.2) If $i_1 \le u$ and $i_2 \ge r-u+1$, then $|\sigma \cap V(G)| \ge r \ge |\mathcal{A}| + 1$ (since $S_1 \cap S_2 \ne \emptyset$) and

$$\gamma_t(G) = |S| \ge |S_1(\mathcal{A})| + |S_2| \ge \left\lceil \frac{1}{2}\,|\mathcal{A}| \right\rceil + \left\lceil \frac{1}{2}\,|\mathcal{A}| \right\rceil \ge |\mathcal{A}|.$$

We also have

$$\begin{aligned}d_G(a_{i_1}, x) &\le d_G(a_{i_1}, a_1) + d_G(a_1, x) \le u - 1 + \frac{1}{2}\,,\\ d_G(a_{i_2}, y) &\le d_G(a_{i_2}, a_r) + d_G(a_r, y) \le u - 1 + \frac{1}{2}\,,\\ L([xy]) &= d_G(x, a_{i_1}) + |\mathcal{A}| - 1 + d_G(a_{i_2}, y) \le \gamma_t(G) + 2u - 2\,,\\ \delta(G) = d_G(p, [xz] \cup [zy]) &\le d_G(p, \{x, y\}) \le \frac{1}{2}\,L([xy]) \le \frac{1}{2}\,\gamma_t(G) + u - 1.\end{aligned}$$

(2.2.3) If $i_1 \le u$ and $i_2 \le r-u$, then $|\sigma \cap V(G)| \ge r \ge |\mathcal{A}| + u$ and

$$\begin{aligned}\gamma_t(G) = |S| \ge |S_1(\mathcal{A})| + |S_2| &\ge \left\lceil \frac{1}{2}\,|\mathcal{A}| \right\rceil + \left\lceil \frac{1}{2}\left(|\mathcal{A}| + u - 1\right) \right\rceil \\ &\ge \left\lceil \frac{1}{2}\,|\mathcal{A}| \right\rceil + \left\lceil \frac{1}{2}\left(|\mathcal{A}| - 1\right) \right\rceil + \left\lfloor \frac{u}{2} \right\rfloor = |\mathcal{A}| + \left\lfloor \frac{u}{2} \right\rfloor.\end{aligned}$$

The maximality of $\mathcal{A}$ gives $d_G(a_{i_2+1}, [xz] \cup [zy]) \le 4$. Let g be a geodesic in G joining a_{i_2+1} and $[xz] \cup [zy]$, and ρ the curve

$$\rho := [xa_{i_1}] \cup a_{i_1} \cdots a_{i_2}a_{i_2+1} \cup g.$$

Thus,

$$\begin{aligned}d_G(a_{i_1}, x) &\le d_G(a_{i_1}, a_1) + d_G(a_1, x) \le u - 1 + \frac{1}{2}\,,\\ L(\rho) &\le u - 1 + \frac{1}{2} + |\mathcal{A}| + 4 = u + \frac{7}{2} + |\mathcal{A}|.\end{aligned}$$

Since ρ joins two points in $[xz] \cup [zy]$ and $p \in \rho$, we have

$$\delta(G) = d_G(p, [xz] \cup [zy]) \le \frac{1}{2}\,L(\rho) \le \frac{1}{2}\Big(u + \frac{7}{2} + |\mathcal{A}|\Big) \le \frac{1}{2}\,\gamma_t(G) + \frac{1}{2}\Big(\frac{7}{2} + u - \left\lfloor \frac{u}{2} \right\rfloor\Big).$$

(2.2.4) If $i_1 \ge u+1$ and $i_2 \ge r-u+1$, then a similar argument to the previous one in (2.2.3) gives the same inequality for $\delta(G)$.

Since the function

$$F(u) := \max\Big\{ \frac{9-u}{2}\,,\, u-1,\, \frac{1}{2}\Big(\frac{7}{2} + u - \Big\lfloor \frac{u}{2} \Big\rfloor \Big) \Big\},$$

with $u \in \mathbb{Z}^+$, attains its minimum value 3 for $u = 3$ and $u = 4$, we have

$$\delta(G) \le \frac{1}{2}\,\gamma_t(G) + 3.$$

□

The following example shows that Theorem 2.11 is asymptotically sharp.

For each integer $k \ge 1$ consider the cycle graph C_{4k} with vertices $V(C_{4k}) = \{v_1, v_2, \dots, v_{4k-1}, v_{4k}\}$ and edges $E(C_{4k}) = \{v_1v_2, v_2v_3, \dots, v_{4k-1}v_{4k}, v_{4k}v_1\}$. Given points $x, y \notin V(C_{4k})$, let G_k be the graph with

$$V(G_k) = \{x, y\} \cup V(C_{4k}),$$
$$E(G_k) = \{xv_1, xv_{4k}, yv_{2k}, yv_{2k+1}\} \cup E(C_{4k}).$$

Consider the geodesics g_1, g_2 in G_k joining x and y with $g_1 \cap g_2 = \{x, y\}$. If p is the midpoint of g_1, then Lemma 2.4 gives

$$\frac{1}{2}\operatorname{diam} G_k \ge \delta(G_k) \ge d_{G_k}(p, g_2) = d_{G_k}(p, \{x, y\}) = \frac{1}{2}\,L(g_1) = k + \frac{1}{2} = \frac{1}{2}\operatorname{diam} G_k,$$

and we conclude $\delta(G_k) = k + 1/2$. [21] gives $\gamma_t(C_{4k}) = 2k$, and one can check that $\gamma_t(G_k) = \gamma_t(C_{4k}) = 2k$. Hence, $\delta(G_k) = k + 1/2 = \gamma_t(G_k)/2 + 1/2$.

One can think that perhaps it is possible to obtain an upper bound of $\gamma_t(G)$ in terms of $\delta(G)$, i.e., the inequality

$$\gamma_t(G) \le \Psi\big(\delta(G)\big), \tag{2.4}$$

for every graph G and some function Ψ. However, this is not possible, as the following example shows. For each integer $n \ge 2$ consider the path graph P_n. Since P_n is a tree, $\delta(P_n) = 0$, but $\lim_{n\to\infty} \gamma_t(P_n) = \infty$.

However, we can obtain (2.4) for a kind of graphs.

Theorem 2.12. *If G is a graph with an isometric dominating cycle C, then*

$$\gamma_t(G) \le 4\delta(G).$$

Proof. Since C is a dominating cycle, $C \cap V(G)$ is a total dominating set and $\gamma_t(G) \le |\,C \cap V(G)| = L(C) = 4\delta(C)$. Since C is an isometric subgraph of G, Lemma 2.1 gives the inequality. □

Theorem 2.13. *If G is a graph with a dominating cycle C, then*

$$\delta(G) \le \frac{1}{2}\Big\lfloor \frac{L(C)}{2} \Big\rfloor + \frac{3}{2}\,,$$

and the inequality is sharp.

Proof. Since C is a dominating cycle, we have

$$\operatorname{diam} V(G) \le \operatorname{diam} V(C) + 2 = \Big\lfloor \frac{L(C)}{2} \Big\rfloor + 2,$$

and Lemma 2.4 gives the inequality. [26, Theorem 3.1] gives that the inequality is sharp. □

Proposition 2.14. *If G is a graph with no induced C_4 or P_4, then*

$$\delta(G) \le \frac{3}{2}\,.$$

Proof. Since G is a graph with no induced C_4 or P_4, [29] (see also [7, Theorem 1]) gives that G has a dominating vertex. Thus, $\operatorname{diam} V(G) \le 2$ and Lemma 2.4 gives the inequality. □

This result can be improved as follows.

Theorem 2.15. *If G is a graph with no induced P_4, then*

$$\delta(G) \le \frac{5}{4}\,,$$

and the inequality is sharp.

Proof. Seeking for a contradiction assume that $\operatorname{diam} V(G) > 2$. Thus, there exist $u, v \in V(G)$ with $d_G(u, v) = 3$. Let $u', v' \in V(G)$ with $uu', u'v', v'v \in E(G)$. Since $uu'v'v$ is a P_4 on G, it is not induced and so, $d_G(u, v) < 3$, a contradiction. Hence, $\operatorname{diam} V(G) \le 2$, $\operatorname{diam} G \le 3$ and Lemma 2.4 gives $\delta(G) \le 3/2$.

Seeking for a contradiction assume that $\delta(G) > 5/4$. Thus, Lemma 2.10 gives $\delta(G) = 3/2$. By Lemma 2.9, there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $\delta(T) = 3/2 = d_G(p, [yz] \cup [zx])$ for some $p \in [xy]$. Then $d_G(p, \{x, y\}) \ge d_G(p, [yz] \cup [zx]) = 3/2$ and $d_G(x, y) \ge 3$. Therefore, $\operatorname{diam} G = 3$, $\operatorname{diam} V(G) = 2$, $x, y \in J(G) \setminus V(G)$ and $p \in V(G)$. Thus, $x \in u_x v_x \in E(G)$ and $y \in u_y v_y \in E(G)$, with $u_x, u_y \in [xy]$ and $d_G(u_y, \{u_x, v_x\}) = 2$, and so, $u_y u_x, u_y v_x \notin E(G)$. Since $v_x u_x p\, u_y$ is a P_4 on G, it is not induced and so, $v_x p \in E(G)$ (recall that $u_y u_x, u_y v_x \notin E(G)$); thus, $3/2 = d_G(p, [yz] \cup [zx]) \le d_G(p, v_x) = 1$, a contradiction. Hence, $\delta(G) \le 5/4$.

Let K_4 be a complete graph with vertices $\{v_1, v_2, v_3, v_4\}$. We denote by G the graph obtained from K_4 by adding a new vertex v_5 and two edges v_5v_1, v_5v_2. Denote by y the midpoint of v_3v_4. Let us consider the geodesic bigon $\{v_5, y\}$ which is the union of the geodesics $\gamma_1 = v_5v_1 \cup v_1v_4 \cup [v_4y]$ and $\gamma_2 = v_5v_2 \cup v_2v_3 \cup [v_3y]$. If p is the midpoint of γ_1, then we have $\delta(G) \ge d_G(p, \gamma_2) = 5/4$. Since $\operatorname{diam} G = 5/2$, Lemma 2.4 gives $\delta(G) \le 5/4$, and we conclude $\delta(G) = 5/4$. □

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