Optimal Stopping
of
Gauss–Markov processes

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To Aria,
who will never get to read these pages.

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Published and submitted content

The following papers, co-authored by the author of this dissertation and his advisors, have been published or submitted for publication. For the papers declared to be fully included, some minor changes were introduced to correct typos, make use of a better notation, and homogenize the bibliography, although the scientific contributions and proofs remain unaltered. The inclusion of the papers is not indicated by typographical means or references, but rather individually commented below.

Paper A Azze, A., D'Auria, B., and García-Portugués, E. Optimal exercise of American options under time-dependent Ornstein–Uhlenbeck processes. arXiv e-print 2022, https://arxiv.org/abs/2211.04095

- Submitted for publication.
- Fully included in Chapter 2.

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- Published.
- Partially included in Chapter 3.
- Propositions 3.3–3.4 and Lemma 3.1 were added to the thesis to correct the last formula on page 18 of the original paper and its implications in the properties of the optimal stopping boundary.

Paper C Azze, A., D'Auria, B., and García-Portugués, E. Optimal stopping of an Ornstein–Uhlenbeck bridge. arXiv e-print 2021, https://arxiv.org/abs/2110.13056

- Submitted for publication.
- Fully included in Chapter 4.

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Contents

Acknowledgements **v**

Contents **x**

List of Figures **xiv**

1 Introduction **1**
1.1 Motivation 1
1.1.1 (Non-degenerated) Gauss–Markov processes 2
1.1.2 Gauss–Markov bridges 3
1.2 Optimal stopping theory 4
1.2.1 Foundations and early developments 4
1.2.2 The dimension of the OSP 6
1.3 State of the art 8
1.3.1 Optimal stopping and (non-degenerated) GM processes 8
1.3.2 Optimal stopping and GMBs 9
1.3.3 American options and GM processes 10
1.4 Contributions 11
1.5 Structure 13
References 13

2 Optimal exercise of American options under time-dependent Ornstein–Uhlenbeck processes **27**
2.1 Introduction 28
2.2 Formulation of the problem 30
2.3 Regularities of the boundary and the value function 32
2.4 The value formula and the free-boundary equation 44
2.5 Numerical experiments 46
2.6 Concluding remarks 49
References 50

3 Discounted optimal stopping of a Brownian bridge, with application to American options under pinning **55**
3.1 Introduction 56
3.2 Problem setting 58
3.3 Optimally exercising American put options for a Brownian bridge 59
3.4 Boundary computation and inference 62
3.4.1 Solving the free-boundary equation 62

3.4.2 Estimating the volatility . . . 63
3.4.3 Confidence intervals for the boundary . . . 65
3.4.4 Simulations . . . 66
3.5 Pinning-at-the-strike and real-data study . . . 68
3.6 Non-monotonicity of the OSB . . . 72
3.7 Conclusions . . . 73
3.A Main proofs . . . 74
3.B Auxiliary lemmas . . . 82
References . . . 85

4 Optimal stopping of an Ornstein–Uhlenbeck bridge **89**
4.1 Introduction . . . 90
4.2 Formulation of the problem . . . 91
4.3 Reformulation of the problem . . . 91
4.4 Solution of the reformulated problem: a direct approach . . . 93
4.5 Solution of the original problem and some extensions . . . 104
4.6 Numerical results . . . 106
4.7 Conclusions . . . 106
References . . . 108

5 Optimal stopping of Gauss–Markov bridges **111**
5.1 Introduction . . . 111
5.2 Gauss–Markov bridges . . . 114
5.3 Two equivalent formulations of the OSP . . . 119
5.4 Solution of the infinite-horizon OSP . . . 120
5.5 Solution of the original OSP . . . 132
5.6 Numerical results . . . 134
5.7 Concluding remarks . . . 137
References . . . 137

6 Final thoughts and extensions **143**
6.1 Conclusions . . . 143
6.2 Future work . . . 144

List of Figures

1.1 Visual sketch of the reward associated with stopping the process according to $b(t_n) \propto \sqrt{1-t_n}$, and different values of N_p and N_m. The red line is the stepwise function $b(t)$ and the black one represents the evolution of X_{t_n}. We used $\delta = \sqrt{1/(N_p+N_m)}$, $t_n = n\delta^2$, $n = 1, \dots, N_p + N_m$, and $X_0 = 0$. 1

2.1 For the images on top, the solid colored lines represent the OSBs for the different choices of the α, θ, and σ functions shown in the legend. The dashed lines represent the pulling level α and the dotted line is placed at the strike price level A. Φ and ϕ represent the distribution and density functions of a standard normal. We set $T = 1$, $\lambda = 1$, and $A = 0$. The smaller images below provide the errors $\log_{10}(d_k)$ between consecutive boundaries for each iteration $k = 1, 2, \dots$ of (2.55). 48

2.2 For the images on top, the solid colored lines represent the OSBs for different n and ε (see main text). The dotted line is placed at the strike price level A, while the dashed line stands for the pulling level α, and the dotted-dashed lines are the OSBs of the BB (a) and the OUB (b). We set $A = 0$, $\lambda = 0$, and $T = 1$. The smaller images below are analogous to those in Figure 2.1. 49

2.3 For images on top, the solid colored lines represent the OSBs for different values of the discounting rate specified in the legend, while the dashed line is placed at $A = 0$. We set $\theta(t) = 1$, $\alpha(t) = 0$, $\sigma(t) = 1$, $A = 0$, and $T = 1$. The smaller images below are analogous to those in Figure 2.1. 50

3.1 Boundary estimation via Algorithm 3.1 for different partition sizes and for the parameters $S = 10$, $T = 1$, $\lambda = 0$, and $\sigma = 1$. 64

3.2 Inferring the boundary using one third ($n = 66$, $N = 200$) of the Brownian bridge path, for $T = 1$, $S = 10$, $X_0 = 10$, $\lambda = 0$, and $\sigma = 1$. The solid curves represent the true boundary b_σ (red curve), the estimated boundary $\tilde{b}_{\widehat{\sigma}_n}$ (blue curve), the upper confidence curve $\tilde{c}_{1,\widehat{\sigma}_n}$ (orange curve), and the lower confidence curve $\tilde{c}_{2,\widehat{\sigma}_n}$ (green curve). 65

3.3 Pointwise proportion of trials, out of $M = 1000$, in which the true boundary does not belong to the interval delimited by the confidence curves. We use $S = 10$, $X_0 = 10$, $T = 1$, $\lambda = 0$, $\sigma = 1$, and a significance level $\alpha = 0.05$, and a number $M = 1000$ of sample paths. For each path, one third (a) or two thirds (b) of the observations were used to compute σ and then to estimate the confidence curves by (3.20). The continuous line represents the proportion of non-inclusions, the dashed line stands for α, and the dotted lines are placed at $\alpha \pm z_{0.025}\sqrt{\alpha(1-\alpha)/M}$. The spikes at $T = 1$ are numerical artifacts due to the null variance of $\tilde{b}_{\widehat{\sigma}_n}(T)$. . . 66

3.4 $X_t^{(q)}$ for $q = 0.2, 0.4, 0.6, 0.8$, where $X_t^{(q)}$ is the q-quantile of a $\mathcal{N}(0, t(1-t))$, the marginal distribution at time t of a Brownian bridge with unit volatility and $X_0 = X_1 = 10$. The green and orange lines refer to the paths of $(X_t \mid X_{0.2} = X_{0.2}^{(0.2)})_{t=0}^1$ and $(X_t \mid X_{0.8} = X_{0.8}^{(0.8)})_{t=0}^1$ respectively, with $X_{0.2}^{0.2} \approx 9.6649$ and $X_{0.8}^{0.8} \approx 10.3382$. 67

3.5 Mean of the payoff associated with: the true boundary b_σ (red curve), the estimated boundary $\tilde{b}_{\widehat{\sigma}_n}$ (blue curve), the upper confidence curve $\tilde{c}_{1,\widehat{\sigma}_n}$ (orange curve), and the lower confidence curve $\tilde{c}_{2,\widehat{\sigma}_n}$ (green curve). The left column shows the low-frequency scenario ($r = 1$), while the right one stands for the high-frequency scenario ($r = 25$). We use $\sigma = 1$, $T = 1$, $S = 10$, $X_0 = 10$, and $\lambda = 0$. 69

3.6 Variances of the payoff associated with: the true boundary b_σ (red curve), the estimated boundary $\tilde{b}_{\widehat{\sigma}_n}$ (blue curve), the upper confidence curve $\tilde{c}_{1,\widehat{\sigma}_n}$ (orange curve), and lower confidence curve $\tilde{c}_{2,\widehat{\sigma}_n}$ (green curve). The left column shows the low-frequency scenario ($r = 1$), while the right one stands for the high-frequency scenario ($r = 25$). We use $\sigma = 1$, $T = 1$, $S = 10$, $X_0 = 10$, and $\lambda = 0$. 70

3.7 Results of the real data application. The black curve is the relative mean profit $(\mathrm{BB}(p) - \mathrm{GBM}(p))\,/\mathrm{GBM}(p)$ for a pinning deviance p, while the blue dashed curve represents the kernel density estimation of the pinning deviances. 72

3.8 The image on the left shows four boundaries (continuous lines) for different values of λ. The dotted curve represents the (unique) pair $(t, b_\lambda(t))$ where each boundary b_λ changes its monotonicity, computed for a mesh of equally-spaced 5000 points going from $\lambda = 0$ to $\lambda = 2500$. The smallest value of λ where a change of monotonicity was observed was $\lambda = 37.5$. The dashed line is placed at the level of the pinning point ($S = 0$). We set $\sigma = 1$ for the volatility and $N = 5000$ for the logarithmically-spaced grid. The smaller images on the right zoom in the four boundaries for a better appreciation of their change of monotonicity. 73

4.1 Optimal stopping boundary estimation for different values of α. The boundary is pulled towards 0 with a strength that increases as both $|\alpha|$ (values of α with equal absolute values yield the same boundary) and the residual time to the horizon $1-t$ increases. As $\alpha \to 0$, the boundary estimation is shown to converge towards the OSB of a BB (dashed line), which is known to be $z + L\sqrt{1-t}$, for $L \approx 0.8399$. 107

4.2 Optimal stopping boundary estimation for different values of γ. The boundary exhibits an increasing proportional relationship with respect to γ. 107

4.3 Optimal stopping boundary estimation for different values of z and N. We display $t \mapsto \beta(t) - z$ to allow a clearer comparison across the different values of z. As N increases the boundary estimation is seen to converge. 108

5.1 For the images on top, the solid colored lines represent the computed OSBs for the different choices of the volatility coefficient $\widetilde{\nu}$ (image (a)), the partition length N (image (b)), and the type of partition considered (image (c)). Black dashed, dotted, and dashed-dotted lines stand for the OSB of a BB associated with the different values of $\widetilde{\nu}$. Specifications are shown in the legend and caption of each image. Image (c) accounts for a subplot that shows, as a function of the partition size N (x axis), the evolution of the relative L_2 error between the different computed boundaries and the true one (y axis). The smaller images below display the log-errors $\log_{10}(d_k)$ between consecutive boundaries for each iteration $k = 1, 2, \ldots$ of the Picard algorithm. 136

5.2 The first row of three plots shows $1/\widetilde{\theta}$ (continuous line) versus $1/\theta$ (dashed line) for the different choices of the slope $\widetilde{\theta}$ (image (a)), the mean-reverting level $\widetilde{\kappa}$ (image (b)), and the volatility $\widetilde{\nu}$ (image (c)) functions. Specifications of the functions are given in the legend and caption of each image. The second row does the same for $\widetilde{\kappa}$ and κ. The main plot, in the third row, shows in solid colored lines the computed OSBs. The smaller images at the bottom display the log-errors $\log_{10}(d_k)$ between consecutive boundaries for each iteration $k = 1, 2, \ldots$ of the Picard algorithm. 136

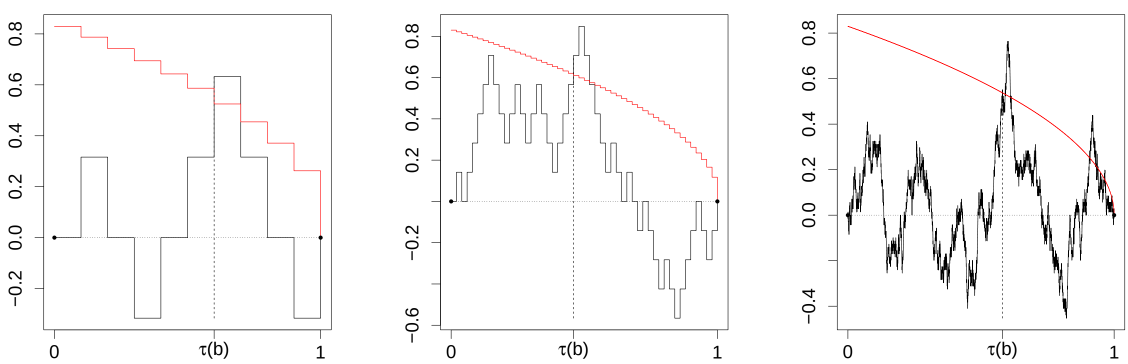
Chapter 1

Introduction

1.1 Motivation

Withdraw balls sequentially, without replacement, from an urn that contains N_p "plus" balls and N_m "minus" balls. Each plus ball gives you δ dollars. Each minus ball takes δ dollars from you. At time t_n, when you have just made the n-th withdrawal, you can decide whether to stop the game and claim the current capital X_{t_n}, or to withdraw another ball. What is the "best" "stopping strategy"? This problem dates back to Shepp (1969). A version of it, formulated with cards instead of balls, is also featured in Crack (2014, Question 1.46).

It seems reasonable to stop the game at time t_n if the reward X_{t_n} exceeds a threshold $b(t_n)$, which must depend on the number of plus (or minus) balls you have already withdrawn, and the number of trials $N_p + N_m - n$ left until you run out of balls. Reasoning like that, a stopping strategy boils down to setting a sequence $b = \{b(t_n)\}_{n=0}^{N_p+N_m}$ of real numbers, and $X_{\tau(b)}$ denotes the associated (random) profit, where $\tau(b) := \min\{t_n : X_{t_n} \geq b(t_n)\}$. Figure 1.1 illustrates the mechanics of the game for different values of N_p and N_m.



(a) $N_p = N_m = 5$ (b) $N_p = N_m = 25$ (c) $N_p = N_m = 2500$

Figure 1.1: Visual sketch of the reward associated with stopping the process according to $b(t_n) \propto \sqrt{1 - t_n}$, and different values of N_p and N_m. The red line is the stepwise function $b(t)$ and the black one represents the evolution of X_{t_n}. We used $\delta = \sqrt{1/(N_p + N_m)}$, $t_n = n\delta^2$, $n = 1, \ldots, N_p + N_m$, and $X_0 = 0$.

The evolution of the capital $\{X_{t_n}\}_{n=0}^{N_p+N_m}$, appropriately rescaled like in Figure 1.1, converges to a Brownian Bridge (BB) as N_p and N_m increase. In such a case, it is more convenient to work out the BB asymptotic approximating solution, rather than the discrete-time original one, to ease the computational burden and rounding errors.

Besides Shepp's illustrative urn game, the question of optimally stopping a BB finds many applications in finance. For instance, Boyce (1970) considered the problem of optimally selling a bond before maturity. Ekström and Wanntorp (2009) motivated the model for optimally exercising an American option in the presence of the so-called *stock-pinning* effect. Baurdoux et al. (2015) generalized Boyce's idea by interpreting bonds as perishable commodities and maturity dates as deadlines after which the item becomes useless. In many of these applications, however, the decision of using a BB model is rooted in its simplicity rather than in financial arguments, and other models, which are also deterministically anchored to a terminal point, are arguably better fits.

A good approach to add flexibility without compromising tractability might be to keep the fundamental features behind the BB simplicity, that is:

Continuity. The value of the process evolves continuously over time, meaning that no "jumps" are allowed.

Markovianity. Given historical data of the process until time t, its future evolution depends only on the value at t and not on previous ones.

Gaussianity. The process's values at every finite collection of times follow a multivariate normal distribution.

Processes that meet these conditions are called (continuous-time) Gauss–Markov (GM) processes and, when they degenerate at a specific value in a terminal time, as the BB does, they are referred to as Gauss–Markov Bridges (GMBs).

This thesis studies when it is advantageous to stop a GM process and a GMB to claim an optimal reward. Before exploring the tools that allow doing so, we motivate the use of these types of processes by giving a glimpse of their reach and applications from different perspectives.

1.1.1 (Non-degenerated) Gauss–Markov processes

GM processes include the class of BMs with time-dependent drifts and volatilities (Buonocore et al. (2013)). Attempting to comprehensively list the applications of these processes would be a futile endeavor, since they are ubiquitous in many applied and theoretical fields. Nevertheless, we highlight the works of Revuz and Yor (1999), Rogers and Williams (2000a), Rogers and Williams (2000b), and Burdzy (2013). The handbook by Gardiner (2004) is also worth mentioning, as it gathers formulae and facts for a convenient manipulation of BMs and related processes.

The Ornstein–Uhlenbeck (OU) process also falls into the class of GM processes. It was first formulated by Langevin in 1908 (see Lemons and Gythiel (1997)) and later formalized by Uhlenbeck and Ornstein (1930) to model motion under friction. Ever since, the classic time-homogeneous OU process has found a plethora of applications in many different fields. In finance, it is used under the name of *Vasicek model* for stochastic interest rates (see, e.g., Vasicek (1977), Korn and Kraft (2002), and Mamon (2004)), and to represent asset's price exhibiting a mean-reverting behavior, such as in pair trading (see Leung and Li (2015a) for the theory and empirical evidence, and Gatev et al. (2006) for a practitioner perspective). Other applications include modeling neuronal activity (see, e.g., Ricciardi and Sacerdote (1979) and Lánský and

Rospars (1995)), phylogenetic processes (see Felsenstein (1988) and Garland et al. (1993)), the dynamics of human microbiome (Kenney et al., 2020), and humans' running endurance (Billat et al., 2018).

Despite its diverse applications, the classic OU process does not capture the effect of time, which makes it a poor model for some real-world dynamics. Noticing this, many authors opted for time-dependent versions. The Vasicek model evolved into the Vasicek–Hull–White (time-dependent OU) model (see Hull and White (1990) and Hunt and Kennedy (2004)), which is far more used by practitioners. Option pricing departed from the classic Black–Scholes model to adopt time-dependent OU processes (see, e.g., Goldenberg (1991), Wildman (2016), Skorupa (2018), and Carr and Itkin (2021)). Similarly, studies in neuronal activity, phylogenetics, environmental research, deteriorating systems, anomalous diffusions, and many others, started using this model to incorporate the influence of time (see Burkitt (2006), Buonocore et al. (2014), Albano and Giorno (2020), Hansen (1997), Gutiérrez et al. (2012), Deng et al. (2016), and Palamarchuk (2018)). An analytic study of time-dependent OU processes can be found in Gardiner (2004), whereas a discussion on efficient parameter estimation is given by Albano and Giorno (2020).

Beyond considering time-inhomogeneity, the next natural extension of OU processes is to keep the main traits that make them tractable and flexible: continuity, Gaussianity, and Markovianity. This leads to the class of GM processes. Although more general, these processes do not significantly widen the class of time-dependent OU processes. They are time-inhomogeneous linear-drifted diffusions. That is, strong solutions of the Stochastic Differential Equation (SDE)

$$\mathrm{d}X_t = (a(t)X_t + b(t))\mathrm{d}t + c(t)\mathrm{d}B_t,$$

for suitable functions $a : \mathbb{R}_+ \to \mathbb{R}$, $b : \mathbb{R}_+ \to \mathbb{R}$, and $c : \mathbb{R}_+ \to \mathbb{R}_+$, and a standard Brownian Motion (BM) $\{B_t\}_{t\in\mathbb{R}_+}$. Mehr and McFadden (1965), Borisov (1983), and Buonocore et al. (2013) provide further characterizations and theoretical results about GM processes.

On top of the already mentioned applications of time-dependent BMs and OU processes, which are inherited by GM processes, we highlight their applicability in the context of first-passage-time problems with time-dependent boundaries (see Mehr and McFadden (1965), Nobile et al. (2008), Molini et al. (2011), Buonocore et al. (2013), Buonocore et al. (2014), and Giorno and Nobile (2019)). This is relevant in biology to model neurons' firing and study population's dynamics (see, e.g., Ricciardi et al. (1999) and Buonocore et al. (2013)), and in finance, for designing optimal strategies of buying/selling assets and pricing derivatives (Patie, 2004). Other applications are listed in Suwansantisuk et al. (2012), Pavliotis (2014), and Redner (2001).

1.1.2 Gauss–Markov bridges

We regard GMBs as processes that result after anchoring non-degenerated GM processes to hit deterministic values at a terminal time. GMBs are appealing models whenever GM processes arise and future information is disclosed.

In finance, BB-like models are often used to capture the viewpoint of insider traders: in trading perishable commodities like flight tickets or contingent claims, where there is a deadline after which the item becomes useless; to represent the convergence of futures' price to spots' price that should theoretically take place at maturity in non-arbitrage markets; in modeling predictions of experts and incorporating algorithms' forecasts; when trading mispriced assets that could rapidly return to their fair price, like it may happen when companies release financial reports or some public market information is disclosed; in the presence of the stock-pinning effect that some heavily traded options tend to exhibit (see Krishnan and Nelken (2001); Ni et al. (2005), Golez and Jackwerth (2012), and Ni et al. (2021) for empirical evidence). Works on these and other insider trading situations include Boyce (1970), Kyle (1985), Brennan and Schwartz (1990), Back (1992), Campi and Çetin (2007), Campi et al. (2011), Campi et al. (2013), Cetin and Xing (2013), Sottinen and Yazigi (2014), Cartea et al. (2016), Angoshtari and Leung (2019), and Chen et al. (2021).

Non-financial applications of a BB include, among many others, the modeling of human and other animals' movement (see Horne et al. (2007), Venek et al. (2016), Kranstauber (2019) and Krumm (2021)). Its connection with the Kolmogorov–Smirnov test (Chow, 2009) makes it particularly important in the theory of empirical processes and goodness-of-fit tests (Vaart and Wellner, 1996). Also, a BB comes up as the limit of sequentially drawing elements without replacement from a large population (Rosén, 1965), which makes it the standard model in many versions of Sheep's urn problem commented in Section 1.1 (see Chen et al. (2015) and Andersson (2012)).

Ornstein–Uhlenbeck Bridges (OUBs) are also canonical examples of GMBs, and they feature in a considerable volume of papers. We remark their financial usage to model arbitrage situations (see Hilliard and Hilliard (2015), Hilliard and Hilliard (2017), and Hilliard et al. (2021)), and their application in biology to capture the dynamics of animal movement when there is interaction within groups (Niu et al., 2016).

Works that linger on the theoretical aspects of GMBs in general, and comment on their applications, are due to Buonocore et al. (2013), Abrahams and Thomas (1981), Barczy and Kern (2013), Hildebrandt and Rœlly (2020), Chen and Georgiou (2016), Barczy and Kern (2011), and Gasbarra et al. (2007).

1.2 Optimal stopping theory

1.2.1 Foundations and early developments

When to take an action to maximize a reward? Answering this question is, in a nutshell, the purpose of Optimal Stopping Theory (OSTh). The only restriction is that, at any time, either you execute the action or lose the opportunity in the hope of a better future one. Problems like these permeate our daily life: buy an item at today's price, or wait for potentially lower upcoming prices (Baumann et al., 2018); accept that job offer or keep on the search; make another bet or quit gambling and leave with your current gains (Hill, 2009); take that parking slot or go hunting for a better one (Tamaki (1982) and Bogoslavskyi et al. (2015)); settle down with a spouse or keep dating until a better match (Lee and Courey, 2021).

The first rigorous mathematical treatment of timing problems like these came in the midst of World War II, fostered by the necessity of more efficient ordnance tests. Abraham Wald conceived a new statistical hypothesis-test method in which the sample size is not given in advance, but observations come over time and one has to determine, at any given moment, whether more observations are needed or there is already enough evidence to render a decision. The technique was proved so effective that it was classified by the USA army until 6 months after the war ended, when the army itself alluded that its dissemination throughout the industry was "imminent and necessary" (Wallis, 1980).

Wald's technique gave birth to the newborn theory of sequential analysis in Wald (1947), which was later leveraged by Snell (1952). He formulated Wald's sequential testing method as a problem of finding the best *stopping random variable* for a discrete-time stochastic process.

A stopping random variable formalizes the notion of a valid stopping strategy by meeting two principles: the decision to stop should be based only on the present and past knowledge; the game cannot be indefinitely prolonged in time. Snell also gave the celebrated characterization of the value function as the smallest super-martingale greater than the gain process, known as the Snell envelope. Methods that follow Snell's approach are referred to as "martingale method". They are extensively developed in Chow et al. (1971) and Shiryaev (2008) in a discrete-time setting, while Fakeev (1970) delves into the continuous-time version.

The three-page report of Dynkin (1963) marked the next milestone in OSTh. He studied Snell's stopping problem when the discrete-time stochastic process is Markovian, and commented on the continuous-time extension. We also owe him the first version of the modern formulation of an Optimal Stopping Problem (OSP):

> For a Markov process $X = \{X_t\}_{t\geq 0}$ with state space $\mathcal{E}$ and defined on the filtered probability space $(\mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P}_x)$, where $\mathbb{P}_x(X_0 = x) = 1$, find a stopping time τ^* (a random variable $\tau^* \geq 0$ such that $\{\tau \leq t\} \in \mathcal{F}_t$, for all $t \geq 0$) that maximizes the mean gain. That is, provide a tractable characterization of the (value) function
>
> $$V(x) = \sup_{\tau} \mathbb{E}_x \left[G(X_\tau) \right], \tag{1.1}$$
>
> where the supremum is taken among all stopping times of X, $G : \mathcal{E} \to \mathbb{R}$ is the (gain) function that outputs the reward for different values of the process, and $\mathbb{E}_x$ is the mean operator with respect to $\mathbb{P}_x$, for $x \in \mathcal{E}$.

Dynkin proved, within the Markovian setting, that V is the smallest superharmonic function (twice continuously differentiable with negative Laplacian) that dominates G, and that the first (random) time τ^* the process enters the (stopping) region

$$\mathcal{D} = \{x \in \mathcal{E} : V(x) = G(x)\}$$

is an Optimal Stopping Time (OST), meaning that $V(x) = \mathbb{E}_x \left[G(X_{\tau^*}) \right]$. Dynkin's approach is known as "Markovian method", and it is thoroughly explored in Shiryaev (2008) and Peskir and Shiryaev (2006).

Dynkin's work unlocked a powerful connection between OSTh and Partial Differential Equation (PDE) theory. Indeed, in the 60s, several authors started to connect Markovian OSPs to Free-Boundary Problems (FBPs). See Chernoff (1961), Lindley (1961), Bather and Walker (1962), Whittle (1964), and Samuelson (1965). However, most of these early works ran into problems when trying to prove the extra boundary condition required for the FBP to have a unique solution, and they either derived it from heuristic principles or conveniently assumed it. The first rigorous proof of its validity came by McKean (1965), for a linear gain function and a geometric BM underneath. In a more general setting, Grigelionis and Shiryaev (1966) (extended by Grigelionis (1967) to time-discontinuous Markovian processes) proved that, for $\mathcal{E} = \mathbb{R}^n$ with $n \geq 1$, V and $\mathcal{D}$ solve the FBP

$$\begin{aligned}
\mathbb{L}V &= 0 && \text{in } \mathcal{E} - \mathcal{D}, \\
\mathbb{L}V &< 0 && \text{in } \mathcal{D}, \\
V &\equiv G && \text{in } \mathcal{D}, \\
\partial_{x_i} V &= \partial_{x_i} G && \text{in } \partial\mathcal{D},\ i = 1, \dots, n.
\end{aligned}$$

Above, $\mathbb{L}$ stands for the infinitesimal generator of the underlying Markov process, and ∂_{x_i} refers to the partial derivative with respect to the ith coordinate of $x \in \mathcal{E}$. $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$, known as the Optimal Stopping Boundary (OSB). The first two conditions rely on Dynkin's superharmonic characterization and the strong Markov property, the third one reflects the definition of $\mathcal{D}$, and the last one is the extra boundary condition, known as "smooth pasting", "high contact", or "smooth fit".

The smooth-fit condition has its own history and specific importance in OSTh's evolution. It allows treating OSPs as FBPs, which unlocks the variational inequality viewpoint of OSPs. Besides obvious theoretical advantages, the variational inequality approach brought up powerful computational methods into OSTh. This link was first noticed by Fleming (1969), Tobias (1973), and Bensoussan and Lions (1973), and it was established in solid grounds by Friedman (1976) and Bensoussan and Lions (1978). The work of Brekke and Øksendal (1990) is also a reference worth mentioning.

OSTh can also be embedded into the more general theory of controlled processes, for which the book of Krylov and Aries (1980) is a canonical reference.

The number of works on OSPs exponentially exploded from the late 60s on as the theory grew solid. For a deep dive into its history and technical details, we recommend the books of Chow et al. (1971), Shiryaev (2008), Peskir and Shiryaev (2006), and Karatzas and Shreve (1998, Appendix D).

1.2.2 The dimension of the OSP

Dynkin's superharmonic characterization of the value function is a powerful tool to verify the validity of a candidate solution of the OSP. Actually, a common recipe for solving OSPs goes as follows:

i. Prove the connection between the OSP and the FBP (sometimes, this step involves guessing the form of the stopping set and/or imposing a regularity pasting condition at the boundary, like the smooth-fit condition).

ii. Solve the FBP.

iii. Verify that the solution of the FBP also solves the OSP (the usual methods involve the use of extended Itô's formulae and the stopping sampling theorem to check the superharmonic characterization).

This guess-and-verify method has been used, among others, by Grigelionis and Shiryaev (1966), Shepp (1969), Salminen (1985), Pederson and Peskir (2000), Dayanik and Karatzas (2003), and Ekström and Wanntorp (2009). However, the methodology involves solving an FBP, which is mathematically challenging.

The dimension of an FBP largely determines its degree of complexity. The FBP associated with the OSP inherits its dimension from that of the underlying process. Introducing time dependencies in the OSP typically adds an extra dimension to the FBP. Some of these dependencies include considering discounting factors, time-dependent gain functions, time-inhomogeneous processes, and finite horizons. We discuss next some of these time dependencies and how they affect the dimension of the FBP.

Exponential discount

In finance, discounting factors reflect the idea that tomorrow's money is worth less than today's. An exponential discount was already used in the work of McKean (1965), where he priced an American option framed as an OSP. Other early implementations of discounts include those of Taylor (1968), Fakeev (1971), Mucci (1978), Salminen (1985), and Beibel and Lerche (1997). More recent contributions, like those of Beibel and Lerche (2001) and Dayanik (2008), explored stochastic discounting rates.

Adding a deterministic exponential discount fundamentally changes the differential operator in the FBP, rather than just increasing its dimension. Indeed, it is possible to bring the OSP back to the non-discounted case by considering the (exponentially) killed version of the underlying process (see Peskir and Shiryaev (2006) and Øksendal (2010)), which adds an extra zero-degree term in its infinitesimal generator.

Time-dependent gain function

Oftentimes the gain function is time-dependent to reflect the profit variation over time. However, one can consider the time-space process (t, X_t) to treat the time coordinate as a spatial one, which brings the OSP back to the canonical time-homogeneous formulation (1.1). This technique, which is formally elaborated in Krylov and Aries (1980) and Babilua et al. (2009), increases the dimension of the FBP.

Some time dependencies can be brought back to the original dimension. For instance, we already discussed the case of adding an exponential discount. Also, convenient time transformations can get rid of some types of time dependencies. Pederson and Peskir (2000) fairly describe this method and offer several examples. More complex time dependencies, however, cannot be managed in a straightforward way. The work of Peskir and Uys (2003) in pricing an Asian-type option is an example of this. Studies that deal with time-dependent gain functions include the works of Shepp (1969), Mucci (1978), Jacka and Lynn (1992), Pederson and Peskir (2000), and Babilua et al. (2009).

Finite horizon

When stopping must happen before a deadline, it is said that the OSP has a finite horizon. In this case, even when working with a time-homogeneous process and a time-independent gain function, the OSB becomes time-dependent and the associated FBP increases its dimension. Typical OSPs dealing with finite horizons include sequential testing (Gapeev and Peskir, 2006) and quickest detection problems (Gapeev and Peskir, 2004), as well as the valuation of different types of options (Peskir and Uys (2003), Peskir (2005a), and Peskir (2005b)). In some finite-horizon cases, however, the burden of a higher dimension can be avoided. Take the example of Shepp (1969), who transformed the OSP of a BB into that of BM and used a time-scaling argument to solve the latter. The same technique was applied by Ekström and Wanntorp (2009) and D'Auria and Ferriero (2020) to handle a wider class of bridge processes. This trick, however, does not necessarily add simplicity, and more often than not the degree of complexity remains the same, now reallocated into the time dependency of the new transformed gain function and/or the underlying process.

Time inhomogeneity

By using the time-space process (t, X_t) and extending the domain of the gain function to include the time coordinate, one can transform a time-inhomogeneous OSP into a homogeneous one. Dochviri (1995) formalizes this transformation, although the idea was around way before, in early works like the one of Taylor (1968) and the 1976 edition of the book by Shiryaev (2008). This homogenization technique increases the dimension of the underlying process and, consequently, that of the FBP.

There are exceptions. We already mentioned how adding exponential discounts and dealing with a time-dependent gain function does not always increase the dimension. Authors like Shepp (1969), Ekström and Wanntorp (2009), Ernst and Shepp (2015), and D'Auria and Ferriero (2020) have succeeded in reducing the dimension by linking time-inhomogeneous OSPs to time-homogeneous ones.

However, there is no trivial way to deal with time-inhomogeneous OSPs in a general setting. Indeed, one-dimensional time-homogeneous OSPs had already been solved by the early works of Dynkin and Yushkevich (1969) and Salminen (1985), and with greater generality in more recent years by Dayanik and Karatzas (2003) and Lamberton and Zervos (2013). In contrast, most of the available results in time-inhomogeneous schemes are either partial or address specific cases of simplified time-dependencies (see Krylov and Aries (1980), Oshima (2006), Yang (2014), Friedman (1975b), Jacka and Lynn (1992), Friedman (1975a), and Peskir (2019)).

Among the few, full results in time-inhomogeneous settings, we mention the work of Gapeev and Stoev (2017), who solved the quickest detection problem for a BM with a smooth time-dependent drift and constant volatility, and Carr and Itkin (2021), who optimally exercised an American option under a time-dependent OU process.

1.3 State of the art

To properly describe the contributions of this thesis, we offer here a clear picture of the current level of developments in those areas of OSTh that are relevant to our settings. In contrast to the previous section, where we discussed the landscape of OSTh in terms of different time dependencies of the OSP, this section surveys the connection between OSPs and GM processes, and its applicability to optimally exercising American-type options. More details, tailored for the specifics of each model, are provided in each chapter's introduction.

1.3.1 Optimal stopping and (non-degenerated) GM processes

Every application of GM processes is a potential niche for OSPs whenever the variable of interest is the time to execute an action. However, besides the classic BM and OU processes, we are not aware of many solutions of OSPs with GM processes in a general setting. Nor have we found many applications outside the financial realm. Since a BM can be considered a special case of an OU process (with null drift), and due to the large volume of papers addressing it, we exclude the BM case from the following survey.

To the best of our knowledge, the first treatment of OU processes related to OSTh was due to Taylor (1968). He proved that, with the identity as the gain function, an exponential discount, and an infinite horizon, it is optimal to stop the OU process as soon as it hits the interval $[b, \infty)$, for $b \approx 0.8399$. Mucci (1978) and Salminen (1984) revisited the same problem from different perspectives, and Pederson and Peskir (2000) extended it to different functionals of an OU process. Other works are motivated by the phenomenon of mean-reversion in finance.

Levendorskii (2005) found the optimal time to exercise a perpetual (infinite-horizon) American option when the underlying stock price behaves like the exponential of an OU process, and compared the result with that of a BM. Ekström et al. (2011) found the best time to liquidate the spread in pair-trading in the presence of a stop-loss level. In the same context, Leung and Li (2015b) used a discounted double OSP to compute the optimal buy-low-sell-high strategy in a perpetual frame. Kitapbayev and Leung (2017) extended Leung's work to finite-horizon cases. Recently, Carr and Itkin (2021) found the optimal exercise strategy of an American option when a time-dependent OU process models the asset price.

In the more general framework of GM processes, we highlight the work of Babilua et al. (2011) and Babilua et al. (2018) on OSPs under limited information.

1.3.2 Optimal stopping and GMBs

GMBs account for two fundamental layers of complexity when it comes to OSPs. First, they are time-inhomogeneous processes, even when the unconditioned GM process is not. Moreover, their drifts explode as the time approaches the horizon, hence they fail to meet the common assumption of Lipschitz continuity (see Krylov and Aries (1980, Chapter 3) and Jacka and Lynn (1992)).

Shepp (1969) pioneered the study of OSPs with GMBs by solving the OSP of a BB. He used a guess-and-verify solution method, but left unaddressed the verification part and added comments on how to solve it. A verification theorem was later provided by Ekström and Wanntorp (2009) and Ernst and Shepp (2015), who revisited Shepp's problem also in a guess-and-verify fashion, reducing the related FBP to an ordinary differential equation. The latter authors developed an argument based on the Taylor expansion of the value function to heuristically back up the guessing part, while the former expanded the methodology to treat different functionals of a BB, including odd powers, reflections, and integrals. De Angelis and Peskir (2020) contributed to widen the class of gain functions by solving the OSP of the exponential of a BB. Instead of proposing a candidate solution and verifying its validity, they used a direct approach in which the free-boundary equation is derived and proved to have the OSB as its unique solution. Chen et al. (2015) added an element of risk aversion to Shepp's problem by considering an absorbing lower boundary for the BB, which is interpreted as a stop-loss level for a more conservative trader. Glover (2019) and Glover (2020) introduced uncertainty in the horizon and used a Bayesian framework to address it. The former considered different prior distributions, including gamma, beta, and Bernoulli ones, and provided closed-form solutions for some of these cases. The latter worked with the randomness coming from allowing a chance of replacing balls in Shepp's urn game. Föllmer (1972), Leung et al. (2018), and Ekström and Vaicenavicius (2020), on the other hand, considered the randomization of the BB pinning point. Motivated by empirical evidence in the work of Boyce (1970), Föllmer (1972) consider a Gaussian prior for the pinning-point distribution. Leung et al. (2018) adopted the perspective of a trader who wants to liquidate a European option before maturity, and numerically solved the OSP via variational inequality techniques. By using a Bayesian methodology, Ekström and Vaicenavicius (2020) derived smoothing properties of the value function when the prior distribution of the pinning point is fairly general, and obtained richer results for Gaussian and Bernoulli priors.

Apart from financial applications, Lisovskii (2019) found the Bayes sequential testing of two simple hypotheses about the mean of a BB. In a more general setting, D'Auria and Ferriero (2020) solved the OSPs related to a class of GMBs, which includes, but is not limited to, BBs and α-Wiener bridges.

1.3.3 American options and GM processes

American options are among the most traded financial derivatives. They give their holders the right (but not the obligation) to exercise the option at any time before the expiration date. This extra flexibility makes their valuation particularly challenging.

The arbitrage-free price of an American contingent claim under a complete market was first proved to be the solution of an OSP in the mid-1980s (see Bensoussan (1984, Lemma 3.1) and Karatzas (1988, Theorem 5.4)). However, up to 20 years earlier, American-type options were already connected to OSPs in the work of Samuelson (1965), where the underlying stock is modeled by a geometric BM. As an addendum to the same paper, a solution method was proposed by McKean (1965), who also pioneered the linkage of OSPs and FBPs. However, even with the simple geometric BM model, it took almost 40 years to reach a complete and rigorous derivation of its solution. This was given in Peskir (2005a), where the optimal exercise strategy was characterized as the unique solution of a type-two Volterra integral equation. A good historical survey on the topic can be found in Myneni (1992) and Barone-Adesi (2005).

Black–Scholes's time-homogeneity has been long criticized (see, e.g., Dupire (1994), Derman and Kani (1994), Fortune (1996), and Dumas et al. (1998)). However, working with time-inhomogeneity often hindrances subsequent derivations of analytical results, and leads to tackling only the pragmatical aspect of option pricing through numerical methods, such as binomial trees, Monte Carlo simulations, finite differences methods, and neural network approaches (see Zhao (2018) and Ruf and Wang (2020)). Among the few analytical results, it is worth mentioning that of Ekström (2004) in a time-dependent volatility scheme, later extended by Rehman and Shashiashvili (2009), who additionally considered time-dependent interest rates. Jaillet et al. (1990) and Blanchet et al. (2006) addressed time-dependent diffusions from the variational inequality and the partial differential equation perspectives. The recent paper of Carr and Itkin (2021) also works within the time-inhomogeneous scenario. This paper distances itself from the geometric BM setup by solving the American pricing problem when the underlying asset behaves like a time-dependent OU process. This is one of the scanty incursions into the analytical valuation of American options with time-dependent GM processes. Actually, to the extent of our knowledge, it is the only one that considers such processes to model assets' prices instead of interest rates, as it is customary (see, e.g., Galluccio (1999) and Cai et al. (2022)).

Many authors have attempted more model-free approaches, but often at the cost of reducing the completeness of results. For example, Detemple and Tian (2002) treated general diffusions and proved that the optimal strategy satisfies the free-boundary equation, but it left the uniqueness of the solution unaddressed. Also in great generality, Zhao and Wong (2012) expressed the optimal stopping strategy in terms of Maclaurin series, which has great theoretical value but lacks computational power. As Detemple and Tian (2002), they required a boundedness assumption on the process's drift that excludes the class of GMBs.

In finance, GMBs have attracted attention to model situations where future information is disclosed. See, e.g., Pikovsky and Karatzas (1996), Schweizer et al. (2003), Biagini and Øksendal (2005), and D'Auria and Salmerón (2020). However, these results are more focused on quantifying the value of the insider information and do not deal with its effect on the option execution strategy. The work of Hilliard and Hilliard (2015) is a recent exception, which used an OUB to model the interest rate in the presence of short-lived arbitrage, although it only provides numerical results based on a binomial-tree algorithm. As far as we are aware, besides the work of Hilliard and Hilliard (2015), the valuation of American options with GMBs has been limited to the study of a BB (Shepp, 1969) and some modifications of it (see Ekström and Wanntorp (2009) and D'Auria and Ferriero (2020)).

1.4 Contributions

This section portrays the thesis contributions within the OSTh landscape in terms of the four types of time dependencies discussed in Section 1.2.2. It also provides a summary of the most relevant OSBs properties obtained and the techniques employed to do so. An additional third perspective is given in financial terms.

The two classes of processes we work with, GM processes and bridges derived from them, are non-homogeneous in time. More than that, the latter does not account for the ubiquitous Lipschitz-continuity assumption on the diffusion coefficients. We work in (exponentially) discounted and non-discounted scenarios, and infinite and finite-horizon ones. Our settings yield non-monotonic OSBs, which makes it particularly difficult to prove the smooth-fit principle.

With an approach that is mostly probabilistic and is inspired by the work of Peskir (2005a), we find enough smoothness on the value function and the OSB to apply a relaxed Itô lemma and come up with the free-boundary equation, given in terms of a second-type, non-linear Volterra integral equation with a unique solution. Among these smoothing conditions, the Lipschitz continuity (and, thus, differentiability almost anywhere) of the OSB away from the horizon stands out as a remarkable property both in theoretical and practical terms. The method to obtain such a degree of smoothness, which is an adaptation of the work of De Angelis and Stabile (2019) to meet our settings in each different chapter, consists in proving that the OSB is the uniform limit of Lipschitz continuous functions defined on closed intervals excluding the horizon. The smooth-fit principle, upon which the uniqueness of the solution of the OSP relies, is then derived from the local Lipschitz continuity of the OSBs.

The numerical aspect of solving the free-boundary equation, being fundamentally a fixed-point problem, is discussed and addressed by implementing different algorithms, which are then used to shed light on the OSBs' shape for a varying set of values of the parametrizations of the processes. In particular, the non-monotonicity of the OSBs is revealed with numerical experiments. GitHub repositories are provided with all the R code necessary to reproduce the numerical studies.

The particular contributions of each chapter are reported as follows and, in a deeper view, in each chapter's introduction.

Chapter 2

In this chapter we offer the solution of the exponentially-discounted finite-horizon OSP when the underlying process behaves like a time-dependent OU, and the gain function is that of an American put option. The time-inhomogeneity of the process adds the challenge of dealing with a non-monotonic OSB. We develop a comparison argument to obtain the boundary lower-boundedness from the solution of a discounted infinite-horizon OSP with a BM underneath. A put-call parity formula extends all previous results to the call option case.

We show numerical evidence on the OSB continuity with respect to the process coefficients. This is done by showing the convergence of the OSBs of non-degenerated GMBs to those of the BB and the OUB.

Chapter 3

In this chapter we obtain the optimal time to exercise an American put option written on top of a BB. Framed as an OSP, we depart from Shepp (1969) and Ekström and Wanntorp (2009) by including an exponential discount, which is more financially realistic, especially for options with long maturity periods. The discount disables Shepp's scaling argument to reduce the OSP dimension, producing a two-dimensional FBP. It also challenges the guess-and-verify method used in Ekström and Wanntorp (2009). Furthermore, by adding an exponential discount we step into the realm of non-monotonic OSBs.

This chapter accounts for the most comprehensive numerical study, in both simulated and real-data situations. We estimate by maximum likelihood the unknown volatility of the BB, asymptotically quantify the estimation error, and extend the inference to the OSB via the delta method by creating (point-wise) confidence curves. Investors with access to discrete data can benefit from this inference method. It also allows incorporating a risk-preference element by stopping nearest the lower (risk-lover) or upper (risk-averse) confidence curves.

Tested on a real data set comprised of Apple's and IBM's financial options, we show that our model is competitive when compared with the Black–Scholes one. As intuition dictates, the best performances are obtained when the stock price exhibits a strong pinning-at-the-strike.

Finally, we comment on how continuity and piecewise monotonicity of the OSB suffice for the smooth-fit condition to hold true, hence providing an alternative potential path to obtain this property rather than relying on the local Lipschitz continuity of the OSB. We numerically show that some values of the discounting rate produce a non-monotonic OSB, where the change of monotonicity occurs just once. Specifically, in those cases, the OSB evolves decreasingly, reaches a minimum, and increases afterward until hitting the horizon with an infinite slope.

Chapter 4

In this chapter we solve the non-discounted, finite-horizon OSP of an OUB. Similarly to Shepp (1969), our methodology relies on a time-space change that casts the original problem into an infinite-horizon OSP with a BM underneath. Differently, the dimension of the new problem cannot be reduced and, thus, it is required to deal with a two-dimensional FBP. The solution, given in terms of the free-boundary (Volterra integral) equation, is provided in both the original and transformed coordinates. This allows choosing the most convenient representation for numerical computations and theoretical analysis.

We bypass the added challenge of not having a monotonic OSB by proving its Lipschitz continuity and then obtaining the smooth-fit condition. Furthermore, we develop a comparison argument that obtains an explicit upper bound for our OSB from the exact solution of the OSB for the BB case, featured in Shepp (1969). It is worth mentioning that our framework includes that of the BB as a limit case.

Chapter 5

This chapter is the natural extension of Chapters 3 and 4, and the perfect complement of Chapter 2. Here, we work out the solution of OSPs with GMBs. We first provide a throughout characterization of these types of processes from different perspectives. One of these viewpoints generalizes Shepp's change-of-variable technique (Shepp, 1969), and allows studying an easier infinite-horizon OSP with a BM underneath. An equivalence result allows switching between the solution in original (GMB) terms or transformed (BM) ones.

The comparison argument developed in Chapter 4 to prove the OSB (upper) boundedness is extended to fit GMBs in general, as it was the methodology used to prove the boundary's Lipschitz continuity.

1.5 Structure

To conclude, we outline next the content of the thesis and its structure. Each chapter is self-contained and presents the same structure and content as the article on which it is based.

Chapter 2 solves the problem of optimally executing an American put option when a time-dependent OU process runs underneath. We introduce the notation and model definition required for a proper formulation of the problem in Section 2.2. Section 2.3 contains all the technical work, including the regularities of the value function and the stopping boundary that, later in Section 2.4, lead to the free-boundary equation and the pricing formula. Numerical results showing the shape and changes of the OSB for different sets of drifts and volatilities are given in Section 2.5.

Chapter 3 offers the solution of the finite-horizon discounted OSP with a BB as the underlying process and the gain function related to an American put option. We set the problem in Section 3.2, along with convenient notation and necessary definitions. Section 3.3 accounts for the theoretical results and the derivation of the free-boundary equation. Section 3.4 provides the mechanism to construct (point-wise) confidence curves for the OSB, when the stock volatility is unknown and estimated via maximum likelihood. Our model is compared in Section 3.5 against the geometric BM by using real data that exhibits pinning-at-the-strike with different degrees of intensity. We study the nature of the non-monotonicity of the OSB and its implications in Section 3.6. We relegate to Appendices 3.A and 3.B all the proofs and auxiliary lemmas required for Section 3.3.

Chapter 4 addresses the problem of optimally stopping an OUB. Section 4.2 presents the OSP and introduces some useful notation. The auxiliary time-space transformed OSP is derived in Section 4.3, where its equivalence to the original one is proved. Section 4.4 gathers the heaviest technical part of the chapter. There, we derive the solution of the reformulated OSP. Section 4.5 expresses the solution back in terms of the original OSP, and remarks that a BB, and an OUB with a general slope coefficient and terminal time, are extensions of our settings. Numerical insights into the OSB's shape are given in Section 4.6 by exploring illustrative cases for different values of the OUB's parameters.

Chapter 5 takes up Chapters 3 and 4 by embedding them into a unifying framework. It solves the OSP of a GMB in great generality. Four different characterizations of these types of processes are offered in Section 5.2. One of these representations allows establishing, in Section 5.3, the equivalence between the original OSP and one in terms of a time-space transformed BM. The latter is solved in Section 5.4, and the solution is then expressed in original coordinates in Section 5.5. A (fixed-point) numerical approach to compute the OSB is explored in Section 5.6.

Global concluding remarks, discussions on future work, and possible extensions are relegated to Chapter 6.

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Chapter 2

Optimal exercise of American options under time-dependent Ornstein–Uhlenbeck processes

Abstract

We study the barrier that gives the optimal time to exercise an American option written on a time-dependent Ornstein–Uhlenbeck process, a diffusion often adopted by practitioners to model commodity prices and interest rates. By framing the optimal exercise of the American option as a problem of optimal stopping and relying on probabilistic arguments, we provide a non-linear Volterra-type integral equation characterizing the exercise boundary, develop a novel comparison argument to derive upper and lower bounds for such a boundary, and prove its Lipschitz continuity in any closed interval that excludes the expiration date and, thus, its differentiability almost everywhere. We implement a Picard iteration algorithm to solve the Volterra integral equation and show illustrative examples that shed light on the boundary's dependence on the process's drift and volatility.

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Contents

2.1 Introduction 28
2.2 Formulation of the problem 30
2.3 Regularities of the boundary and the value function 32
2.4 The value formula and the free-boundary equation 44
2.5 Numerical experiments 46
2.6 Concluding remarks 49
References 50

2.1 Introduction

The extra flexibility of being able to exercise at any time before maturity makes American options a popular financial instrument among traders, but it makes these types of vanilla options significantly harder to price when compared to their European counterparts. Indeed, while the arbitrage-free price of the European option written on a geometric Brownian motion was settled down in the notorious works of Merton (1973) and Black and Scholes (1973), the valuation of the American option had to go through a much longer process that took 40 years to complete, starting from the early works of Samuelson (1965) and McKean (1965), reaching important milestones in the works of Kim (1990), Jacka (1991), and Carr et al. (1992), and concluding in Peskir (2005b). This pricing journey is registered in the surveys of Myneni (1992) and Barone-Adesi (2005).

The classic geometric Brownian motion is not a suitable model in many financial contexts. Examples are the so-called mean-reverting strategies that arise when trading assets that tend to revert to an average price. *Pair-trading* strategies are the most popular of these scenarios. They take place when the spread between correlated assets is pulled towards a baseline level. In such a case, it is common to model the spread by means of an Ornstein–Uhlenbeck (OU) process to capture the mean-reverting behavior. See Leung and Li (2015) for theory and empirical evidence, Gatev et al. (2006) for a practitioner perspective, and the handbook of Ehrman (2006) for a quick reference guide. Mean-reverting strategies also find room when trading options. Indeed, imagine that a trader holds two American options, one put and one call, written on two different stocks whose price at time t is $X_t^{(1)}$ and $X_t^{(2)}$, with the same maturity date T and strike prices given by A_1 and A_2. At any time t before maturity, the trader can exercise both options, thus making the profit $(\alpha_1 X_t^{(1)} - A_1) + (A_2 - \alpha_2 X_t^{(2)}) = (A_2 - A_1) - (\alpha_2 X_t^{(2)} - \alpha_1 X_t^{(1)})$, where α_1 and α_2 represent the shares of each asset held by the trader in each American option position. The trader can then carve out a new American put option written on the spread $X_t = X_t^{(2)} - X_t^{(1)}$ and with strike price given by $A = A_2 - A_1$, and the spread could be modeled as an OU process by conveniently choosing α_1 and α_2 (see Chapter 2 in Leung and Li (2015)).

An OU process has also been shown convenient for modeling certain commodity prices (see, e.g., Chaiyapo and Phewchean (2017) and Ogbogbo (2018)). However, it is far more common to use it to model log-prices (see, e.g., Schwartz (1997), Boyarchenko and Levendorskii (2007), Zhang et al. (2012), and Mejía Vega (2018)), which removes the drawback of allowing negative prices. Further flexibility can be added to the OU model if its parameters are time-dependent. Some authors, for instance, have relied on a time-dependent OU process to capture the effect of seasonality on commodity prices (see, e.g., Zapranis and Alexandridis (2008), Back et al. (2013), and Tong and Liu (2021)) and to cope with predictable assets (Lo and Wang, 1995; Carmona et al., 2012). A time-dependent OU process is also a go-to for practitioners when modeling interest rates, as it accommodates the popular Hull–White model (Hull and White, 1990), an iteration of the Vasicek model (i.e., an OU process with constant parameters).

The analytical valuation of American options is fundamentally an Optimal Stopping Problem (OSP), as McKean (1965) established, and Bensoussan (1984) and Karatzas (1988) later proved for the geometric Brownian motion case. The solution of the OSP is both the option's price, often called the value function, and the Optimal Stopping Boundary (OSB), that is, the optimal exercise barrier that sets the rule for deciding, at any time before maturity, whether it is optimal or not to exercise. Due to the connection between OSPs and free-boundary problems that started to be noticed in the 1960s by works like McKean (1965), Grigelionis and Shiryaev (1966), and Grigelionis (1967), and later detailed, among others, in the book by Peskir and

Shiryaev (2006), the valuation of an American option can also be seen through the lens of partial differential equations and, consequently, embedded into the well-consolidated theory of variational inequalities. The works by Jaillet et al. (1990) and Blanchet et al. (2006) go in that direction. The downside of variational inequality methods, compared to probabilistic ones, is that they tend to lead to less explicit solutions of both the value function and the OSB.

To bypass the complexity of pricing American options analytically, a plethora of numerical methods based on Monte Carlo simulation, finite differences, and binomial/trinomial trees have been developed. Zhao (2018) performs a comparison between eight of the most popular of these methods. Nevertheless, if available, analytical approaches are typically preferred over (pure) numerical ones, as the former are generally more precise and computationally expedient. Working out analytical solutions is not trivial, even if the American option is written on top of relatively simple models like the plain geometric Brownian motion. It becomes even harder when the coefficients of the underlying process are time-dependent. In this case, the OSB cannot be guaranteed to be monotonic or convex, and proving the so-called *smooth-fit condition* (the extra boundary condition that is required to offset not knowing the boundary in advance) becomes mathematically challenging. Under a geometric Brownian motion scheme, Ekström (2004) worked with a time-dependent volatility and proved smoothing properties of the price function. Rehman and Shashiashvili (2009) took Ekström's work further by also allowing a time-dependent interest rate and providing the fair price of the American option in terms of the early exercise premium characterization, that is, the price of the European option plus the early exercise premium associated with the possibility of exercising before maturity. They did not prove the smooth-fit condition, which led to a somewhat difficult proof of the vanishing of the local-time term that the Itô–Tanaka formula yields for the pricing formula. Escaping the framework of a geometric Brownian motion, Detemple (2005, Section 4.6) provided the early exercise premium characterization of the American option's price written on a diffusion with time- and space-dependent coefficients, these having somehow restrictive conditions that lead to a monotonic OSB. Using variational inequalities and partial differential equation techniques, Jaillet et al. (1990) and Blanchet et al. (2006) also worked with time-inhomogeneous diffusions.

In pricing American options, OU processes have been mostly used as a model for interest rates and discounting factors, rather than to model the underlying asset price. To consolidate their use in the latter case, we offer in this paper the solution to the problem of optimally exercising an American put option written on top of a time-inhomogeneous OU process with continuously differentiable coefficients. The associated OSB is given in terms of a non-linear Volterra-type integral equation that holds an explicit and relatively simple dependency on the OU parameters, which allows deriving a clean intuition on how those affect the OSB shape. We comment on two algorithms and implement one of them to work out the solution of the integral equation. The solution method we used, which is primarily probabilistic, yields important properties about the OSB, such as its local Lipschitz continuity and, thus, differentiability almost everywhere. This order of smoothness of the OSB, which goes beyond its typical differentiability almost everywhere, is usually difficult to prove, especially for non-convex OSBs and non-differentiable gain functions. We also derive explicit upper and lower bounds for the OSB by relying on a comparison method that, to the best of our knowledge, has not been used within the framework of optimal stopping and could potentially be used in generic processes with drifts that are monotone in the space component. The value function is characterized by means of the early-exercise premium representation, which has a clear economic interpretation. The shape of the OSB is explored numerically for different sets of values of the drift and volatility of the OU process. We also cover the American call option scenario by proving that its OSB is the reflection of the OSB of its put counterpart with respect to the strike price.

Recently, by using a different approach, Carr and Itkin (2021) also tackled the problem of optimally exercising an American option with an OU process with differentiable time-dependent coefficients underneath. They performed time-space transformations to cast the partial differential operator of the associated free-boundary problem into that of the heat equation, which they solved by means of the heat potential method; the recent paper of Lipton and Kaushansky (2020) explains this method within the context of the American put option for a geometric Brownian motion. As a result, the OSB is proved to satisfy an integral equation of Fredholm type, although the uniqueness of the solution was not addressed, nor numerical methods for finding approximate solutions were discussed. In addition, the dependence of the integral equation with respect to the process's coefficients does not seem to be evident, as neither is the interpretation of the value formula in financial terms. Remarkably, Carr and Itkin (2021) also allow for a time-dependent discount rate and repulsion drifts, which we do not account for in our setting.

The rest of the paper is structured as follows. Section 2.2 sets the problem and introduces the basic notation. Section 2.3 gathers all the properties required to come up, in Section 2.4, with the value formula and the free-boundary equation. Numerical experiments are displayed in Section 2.5, while concluding remarks are relegated to Section 2.6.

2.2 Formulation of the problem

Define $\mu : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that $\mu(s, x) := \theta(s)(\alpha(s) - x)$, where $\theta, \alpha : [0, T] \to \mathbb{R}$ are continuously differentiable functions, and $\theta(s) > 0$ for all $s \in [0, T]$. Let $X = (X_s)_{s\in[0,T]}$ be the unique strong solution (see, e.g., Daniel and Marc (2010, Theorem 2.1, Chapter IX)) of the stochastic differential equation

$$\mathrm{d}X_s = \mu(s, X_s)\,\mathrm{d}s + \mathrm{d}W_s, \quad 0 \le s \le T, \tag{2.1}$$

in the filtered space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_s)_{s\in[0,T]})$, where $(\mathcal{F}_s)_{s\in[0,T]}$ is the natural filtration of the underlying standard Brownian motion $(W_s)_{s\in[0,T]}$. We refer to θ and α as the slope and pulling level (functions), respectively.

From (2.1), we get that the infinitesimal generator $\mathbb{L}$ of the process $((s, X_s))_{s\in[0,T]}$ is given by

$$(\mathbb{L}f)(s, x) = \partial_t f(s, x) + \theta(s)(\alpha(s) - x)\partial_x f(s, x) + \frac{1}{2}\partial_{xx} f(s, x). \tag{2.2}$$

Above and henceforth, we use ∂_t and ∂_x to denote, respectively, the usual partial derivatives with respect to time and space of a function with arguments (t, x), and ∂_{xx} is used as a shorthand for $\partial_x\partial_x$.

Consider the finite-horizon, discounted Optimal Stopping Problem (OSP)

$$V(t, x) = \sup_{0\le\tau\le T-t} \mathbb{E}_{t,x}\left[e^{-\lambda\tau}G(X_{t+\tau})\right], \tag{2.3}$$

where V is the value function,

$$G(x) := (A - x)^+ \tag{2.4}$$

is the gain function for some $A \in \mathbb{R}$ that represents the strike price of the option, and $\lambda \ge 0$ is the discounting rate. The supremum above is taken over all random times τ such that $t + \tau$ is a stopping time in $(\mathcal{F}_s)_{s\in[0,T]}$ and $\mathbb{E}_{t,x}$ represents the expectation under the probability measure

$\mathbb{P}_{t,x}$ defined as $\mathbb{P}_{t,x}(\cdot) := \mathbb{P}(\cdot \mid X_t = x)$. For simplicity and in an abuse of notation, in the sequel, we will refer to τ as a stopping time while keeping in mind that $t+\tau$ is the actual stopping time in the filtration $(\mathcal{F}_s)_{s\in[0,T]}$. The particular form of G implies that

$$-(y-x)^- \le G(x) - G(y) \le (y-x)^+ \tag{2.5}$$

for all $x, y \in \mathbb{R}$, inequalities which will be recurrently used throughout the paper.

It is useful to keep track of the condition $X_t = x$ in a way that does not change the probability measure whenever t or x changes. To do so, we denote the process $X^{t,x} = (X_s^{t,x})_{s\in[0,T-t]}$ in the filtered space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_s)_{s\in[0,T-t]})$ as the unique solution of

$$\begin{cases} \mathrm{d}X_s^{t,x} = \mu(t+s, X_s^{t,x})\,\mathrm{d}s + \mathrm{d}W_s, \quad 0 \le s \le T-t, \\ \;X_0^{t,x} = x, \end{cases}$$

where the underlying Brownian motion $(W_s)_{s\in\mathbb{R}_+}$ remains the same for all $t \in [0,T]$. Theorem 38 from Chapter V in Protter (2003) guarantees that $X^{t,x}$ is well defined and continuous with respect to x.

Theorem 39 from Chapter V in Protter (2003) ensures that $X^{t,x}$ is continuously differentiable with respect to x and t, and combined with (2.1), it states that the processes $\partial_t X^{t,x} = (\partial_t X_s^{t,x})_{s\in[0,T-t]}$ and $\partial_x X^{t,x} = (\partial_x X_s^{t,x})_{s\in[0,T-t]}$, defined as the following $\mathbb{P}$-a.s. limits

$$\partial_t X_s^{t,x} := \lim_{\varepsilon\to 0} \left(X_s^{t+\varepsilon,x} - X_s^{t,x} \right) \varepsilon^{-1} \quad \text{and} \quad \partial_x X_s^{t,x} := \lim_{\varepsilon\to 0} \left(X_s^{t,x+\varepsilon} - X_s^{t,x} \right) \varepsilon^{-1},$$

take the form

$$\begin{aligned} \partial_t X_s^{t,x} &= \int_0^s \left(\partial_t \mu(t+u, X_u^{t,x}) + \partial_x \mu(t+u, X_u^{t,x}) \partial_t X_u^{t,x} \right) \mathrm{d}u \\ &= \int_0^s \left(\theta'(t+u)(\alpha(t+u) - X_u^{t,x}) + \theta(t+u)(\alpha'(t+u) - \partial_t X_u^{t,x}) \right) \mathrm{d}u \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \partial_x X_s^{t,x} &= 1 + \int_0^s \partial_x \mu(t+u, X_u^{t,x}) \partial_x X_u^{t,x}\,\mathrm{d}u \\ &= 1 - \int_0^s \theta(t+u) \partial_x X_u^{t,x}\,\mathrm{d}u. \end{aligned} \tag{2.7}$$

Mind the difference between the differential operators ∂_t and ∂_x and the processes $\partial_t X_s^{t,x}$ and $\partial_x X_s^{t,x}$. From (2.7), it follows that the process $\partial_x X_s^{t,x}$ is deterministic, i.e.

$$\partial_x X_s^{t,x} = \exp\left\{ -\int_0^s \theta(t+u)\,\mathrm{d}u \right\}, \tag{2.8}$$

and that $\partial_x X_s^{t,x} \in (0,1]$ for all $s \in [0, T-t]$.

The results in Section 2.2 from Peskir and Shiryaev (2006), along with the fact that X is a Feller process (see, e.g., Daniel and Marc (2010, Theorem 2.5, Chapter IX)), give us the existence and characterization of a stopping time that is optimal in (2.3). Specifically, if we denote by $\mathcal{D} := \{V = G\}$ and $\mathcal{C} := \mathcal{D}^c = \{V > G\}$ to the so-called stopping set and continuation set, respectively, then, if

$$\mathbb{E}_{t,x}\left[\sup_{0\le s\le T-t} e^{-\lambda s} G(X_{t+s}) \right] < \infty \tag{2.9}$$

for all $(t, x) \in [0, T) \times \mathbb{R}$, we can guarantee that, under $\mathbb{P}_{t,x}$, the first hitting time of $(X_{t+s})_{s\in[0,T-t]}$ into $\mathcal{D}$, denoted by $\tau^* = \tau^*(t, x)$, satisfies

$$V(t, x) = \mathbb{E}_{t,x}\left[e^{-\lambda\tau^*}G(X_{t+\tau^*})\right]. \tag{2.10}$$

If there is another Optimal Stopping Time (OST) τ, then $\tau^* \leq \tau$ $\mathbb{P}_{t,x}$-a.s. (see Equation 2.2.1 from Peskir and Shiryaev (2006)). Solving the OSP (2.3) means to provide tractable expressions for both the value function V and the OST τ^*.

The boundary of $\mathcal{D}$ (or $\mathcal{C}$), denoted by $\partial\mathcal{D}$ (or $\partial\mathcal{C}$), is called the Optimal Stopping Boundary (OSB). It turns out that both the value function V and the OSB $\partial\mathcal{D}$ are the solution of a Stefan problem with the infinitesimal generator $\mathbb{L}$ being the differential operator acting on V in $\mathcal{C}$. For this reason, the OSB is also referred to as the free boundary; see Peskir and Shiryaev (2006) for more information on the relation between OSPs and free-boundary problems. If the OSB can be depicted by the graph of a function $b : [0, T] \to \mathbb{R}$, i.e., if $\partial\mathcal{D} = \{(t, b(t)) : t \in [0, T]\}$, then b is referred to as the OSB too.

Finally, it is convenient to recall the martingale and supermartingale properties of V:

$$\mathbb{E}_{t,x}\left[V(t + \tau^* \wedge s, X_{t+\tau^*\wedge s})\right] = V(t, x), \tag{2.11}$$

$$\mathbb{E}_{t,x}\left[V(t + s, X_{t+s})\right] \leq V(t, x), \tag{2.12}$$

for all $0 \leq s < T - t$, as they will be used to prove results in Section 2.3.

2.3 Regularities of the boundary and the value function

In this section we derive regularity conditions about the OSB and the value function. These conditions allow obtaining a solution for the OSP (2.3), later addressed in Section 2.4, by using an extension of the Itô formula to derive a characterization of the OSB via a Volterra integral equation.

We first point out that (2.9) holds true, as it is a consequence of Lemma 2.1 below. This allows us to prove the existence of the OST as well as its characterization in terms of the stopping set claimed in Section 2.2. We then shed light on the geometry of the stopping and continuation regions in the next proposition, which entails that the stopping set lies below the continuation set and that the boundary between them can be seen as the graph of a bounded function. We highlight the comparison argument used to derive the lower bound of the OSB. Besides its simplicity, it relies only on the monotonicity of the drift with respect to the space component, which makes it usable beyond the specifics of the OU process.

Proposition 2.1 (Boundary existence and shape of the stopping set)**.**
There exists a function $b : [0, T] \to \mathbb{R}$ such that $-\infty < b(t) < A$ for all $t \in [0, T)$ and $\mathcal{D} = \{(t, x) \in [0, T) \times \mathbb{R} : x \leq b(t)\} \cup (\{T\} \times \mathbb{R})$, where A is as in (2.4). Moreover, $b(t) \leq \gamma(t)$ for all $t \in [0, T)$, where

$$\gamma(t) := (\theta(t) + \lambda)^{-1}(\theta(t)\alpha(t) + \lambda A) \tag{2.13}$$

and $b(T) := \lim_{t\to T} b(t) = \min\{A, \gamma(T)\}$.

Proof. Define $b : [0, T] \to \mathbb{R}$ such that $b(t) := \sup\{x \in \mathbb{R} : (t, x) \in \mathcal{D}\}$ for all $t \in [0, T)$ and $b(T) = \lim_{t\to T} b(t)$. We first prove that $b(t) < A$ for all $t \in [0, T)$. Let $(t, x) \in \mathcal{A} := [0, T) \times [A, \infty)$ and define the stopping time $\tau_\delta := \inf\{s \in [0, T-t] : X_{t+s} \leq A - \delta\}$ for some $\delta > 0$, assuming that $\inf\{\emptyset\} = T - t$. Since $\tau_\delta \leq T - t$ and $\mathbb{P}(\tau_\delta < T - t) > 0$,

$$V(t,x) \geq \mathbb{E}_{t,x}\left[e^{-\lambda\tau_\delta} G(X_{t+\tau_\delta})\right] \geq e^{-\lambda(T-t)}\mathbb{P}(\tau_\delta < T-t)\,\delta > 0 = G(x),$$

that is, $(t,x) \in \mathcal{C}$. Therefore, $\mathcal{A} \subset \mathcal{C}$.

We now prove that $\mathcal{D} = \{(t,x) \in [0,T]\times\mathbb{R} : x \leq b(t)\} \cup (\{T\} \times \mathbb{R})$. The fact that $\{T\}\times\mathbb{R} \subset \mathcal{D}$ is a straightforward consequence of the definition $V(T,x) = G(x)$ for all $x \in \mathbb{R}$. Moreover, if $(t, x_1) \in \mathcal{D}$, then, for any $x_2 < x_1$ we have that, for $\tau^* = \tau^*(t, x_2)$,

$$\begin{aligned} V(t,x_2) - V(t,x_1) &\leq \mathbb{E}\left[e^{-\lambda\tau^*}\left(G\left(X_{\tau^*}^{t,x_2}\right) - G\left(X_{\tau^*}^{t,x_1}\right)\right)\right] \\ &\leq \mathbb{E}\left[X_{\tau^*}^{t,x_1} - X_{\tau^*}^{t,x_2}\right] = (x_1 - x_2)\mathbb{E}\left[\partial_x X_{\tau^*}^{t,x}\right] \\ &\leq x_1 - x_2 = G(x_2) - G(x_1), \end{aligned} \tag{2.14}$$

where in the second inequality we used (2.5) and the fact that $x \mapsto X_s^{t,x}$ is an increasing function for all $s \in [0, T-t)$ ($\partial_x X_s^{t,x} \geq 0$ for all s according to (2.8)). We also used the mean value theorem alongside (2.8) to come up with the equality. Since $V(t,x_1) = G(x_1)$, it follows from (2.14) that (t, x_2) belongs to $\mathcal{D}$. Finally, being $\mathcal{D}$ closed, $(t, b(t)) \in \mathcal{D}$ for all $t \in [0,T)$, which guarantees that $\mathcal{D}$ has the claimed shape.

We now use the Itô–Tanaka formula to get that, for all $(t,x) \in [0,T) \times \mathbb{R}$,

$$\begin{aligned} e^{-\lambda s}G(X_s^{t,x}) = G(x) &+ \int_0^s F_t\left(u, X_u^{t,x}\right)\mathbb{1}(X_u^{t,x} < A)\,\mathrm{d}u \\ &- \int_0^s e^{-\lambda u}\mathbb{1}(X_u^{t,x} < A)\,\mathrm{d}B_u + \frac{1}{2}\int_0^s e^{-\lambda u}\mathbb{1}(X_u^{t,x} = A)\,\mathrm{d}l_u^A(X^{t,x}), \end{aligned} \tag{2.15}$$

where $F_t(u,x) = -e^{-\lambda u}(\lambda G(x) + \mu(t+u,x))$, and $l_s^A(X)$ is the local time of the process X at A and up to time s, that is,

$$l_s^A(X) = \lim_{h\downarrow 0}\frac{1}{2h}\int_0^s \mathbb{1}(A - h \leq X_u \leq A + h)\,\mathrm{d}\langle X, X\rangle_u.$$

We have already proved that $b(t) \leq \gamma(t)$ whenever $\gamma(t) \geq A$, so we will assume that $\gamma(t) < A$. Fix (t,x) such that $\gamma(t) < x < A$ and notice that, in such a case,

$$\begin{aligned} F_t(u,x) > 0 \;\text{ if }\; x > \gamma(t+u), \\ F_t(u,x) < 0 \;\text{ if }\; x < \gamma(t+u). \end{aligned}$$

Take a ball $\mathcal{B} \subset [0,T] \times (-\infty, A)$ centered at (t,x) and sufficiently small such that $F_t(u,y) > 0$ for all $(u,y) \in \mathcal{B}$. In (2.15), replacing s by the first exit time from $\mathcal{B}$ (denoted by $\tau_{\mathcal{B}^c}$), taking $\mathbb{P}_{t,x}$-expectation (cancels the martingale term), and noticing that X_{t+u} never touches A for $u \leq \tau_{\mathcal{B}^c}$ under $\mathbb{P}_{t,x}$ (i.e., the local-time term is null), we get

$$V(t,x) \geq \mathbb{E}_{t,x}\left[e^{-\lambda\tau_{\mathcal{B}^c}}G(X_{t+\tau_{\mathcal{B}^c}})\right] = G(x) + \mathbb{E}_{t,x}\left[\int_t^{t+\tau_{\mathcal{B}^c}} F_t\left(u, X_u^{t,x}\right)\,\mathrm{d}u\right] > G(x),$$

meaning that $(t,x) \in \mathcal{C}$ and, therefore, that $b(t) \leq \gamma(t)$ for all $t \in [0,T)$.

We now prove that $(t,x) \in \mathcal{D}$ for all $x < \min\{A, \gamma(T)\}$ and all t sufficiently close to T. This, along with the fact that $b(t) \leq \min\{A, \gamma(t)\}$ that we have already proved, shows that $\lim_{t\to T} b(t)$ exists and takes the claimed value. Notice that

$$\mathbb{E}_{t,x}\left[\int_0^s F_t\left(u, X_u^{t,x}\right)\mathbb{1}(X_u^{t,x} < A)\,\mathrm{d}u\right] \leq s\mathbb{E}_{t,x}\left[\sup_{0\leq u\leq s} F_t\left(u, X_u^{t,x}\right)\mathbb{1}(X_u^{t,x} < A)\right] \\ \leq s\left(K_1 p(t,x,s,\varepsilon) + K_2(1 - p(t,x,s,\varepsilon))\right) \tag{2.16}$$

and

$$\mathbb{E}_{t,x}\left[\int_0^s e^{-\lambda u}\mathbb{1}(X_u^{t,x} = A)\,\mathrm{d}l_u^A(X^{t,x})\right] \leq s(1 - p(t,x,s,\varepsilon)), \tag{2.17}$$

with

$$p(t,x,s,\varepsilon) := \mathbb{P}\left(\sup_{0\leq u\leq s} X_u^{t,x} < x + \varepsilon\right)$$

and

$$K_1 := \sup_{0\leq u\leq s} F_t\left(u, x+\varepsilon\right) < 0, \quad K_2 := \left(\sup_{0\leq u\leq s} F_t\left(u, A\right)\right)^+,$$

for $\varepsilon > 0$ and $t \in [0,T)$ such that $x + \varepsilon \leq \min\left\{A, \min_{u\in[0,T-t]}\gamma(t+u)\right\}$. This selection of ε and t is possible, as the continuity of γ, inherited from the same level of smoothness of θ and α, guarantees that $\gamma(T)$ gets arbitrarily close to $\gamma(T)$ as t converges to T. Likewise, t close enough to T yields $p(t,x,s,\varepsilon) \geq (K_2+1)/(K_2+1-K_1)$. Indeed,

$$\begin{aligned}\lim_{t\to T}\mathbb{P}\left(\sup_{0\leq u\leq s} X_u^{t,x} < x+\varepsilon\right) &\geq \lim_{t\to T}\mathbb{P}\left(\sup_{0\leq u\leq T-t} X_u^{t,x} - x - \varepsilon < 0\right) \\ &= \lim_{t\to T}\mathbb{P}\left(\sup_{0\leq u\leq T-t}\frac{X_u^{t,x} - x - \varepsilon}{2u\ln(\ln(1/u))} < 0\right) \\ &= \mathbb{P}\left(\lim_{t\to T}\sup_{0\leq u\leq T-t}\frac{X_u^{t,x} - x - \varepsilon}{2u\ln(\ln(1/u))} < 0\right) \\ &= \mathbb{P}\left(\limsup_{u\to 0}\frac{X_u^{t,x} - x - \varepsilon}{2u\ln(\ln(1/u))} < 0\right) = 1,\end{aligned}$$

where the last inequality comes after (2.44) below. Therefore, after using (2.15) along with (2.16) and (2.17), we get that

$$V(t,x) - G(x) \leq \sup_{s\in[0,T-t]} s\left(K_1 p(t,x,s,\varepsilon) + (K_2+1)(1 - p(t,x,s,\varepsilon))\right) \leq 0,$$

meaning that $(t,x) \in \mathcal{D}$. Hence, we conclude that $b(T) := \lim_{t\to T} b(t) = \min\{\gamma(T), A\}$.

The lower boundedness of b is now addressed. Define $m := \inf_{t\in[0,T]}\alpha(t)$. The following inequalities hold $\mathbb{P}$-a.s.:

$$X_s^{t,x} \geq Z_s^{t,x} \geq \alpha(t+s) - |\alpha(t+s) - B_s^x| \geq m - |m - B_s^x|, \tag{2.18}$$

where $Z_s^{t,x} := \alpha(t+s) - |\alpha(t+s) - X_s^{t,x}|$, $B_t^x = x + W_t$ and $(W_t)_{t\geq 0}$ is the underlying standard Brownian motion in (2.1). The second inequality holds since the drift of $Z^{t,x} := (Z_s^{t,x})_{s\in[0,T-t]}$ is greater than the drift of the reflection of B^x with respect to α for all $(t,x) \in [0,T) \times \mathbb{R}$, and therefore we can assume that the first process is greater than the last one pathwise $\mathbb{P}$-a.s. (see Peng and Zhu, 2006, Corollary 3.1). Indeed, for $\varepsilon > 0$, define the function

$$g_\varepsilon(x,\alpha) := \begin{cases} |x-\alpha|, & \text{if } |x-\alpha| \geq \varepsilon, \\ \frac{1}{2}\left(\varepsilon + \varepsilon^{-1}(x-\alpha)^2\right), & \text{if } |x-\alpha| < \varepsilon \end{cases}$$

and the processes

$$Y_t^{(1),\varepsilon} := \alpha(t) - g_\varepsilon\left(X_t, \alpha(t)\right), \quad Y_t^{(2),\varepsilon} := \alpha(t) - g_\varepsilon\left(W_t, \alpha(t)\right).$$

Denote by $\mu^{(i),\varepsilon}$ the drift of $Y^{(i),\varepsilon}$, $i = 1,2$. By using the Itô formula, we obtain that

$$(\mu^{(1),\varepsilon} - \mu^{(2),\varepsilon})(t,x) = \begin{cases} \theta(t)|\alpha(t) - x|, & \text{if } |\alpha(t) - x| \geq \varepsilon, \\ \varepsilon^{-1}\theta(t)(\alpha(t) - x)^2, & \text{if } |\alpha(t) - x| < \varepsilon. \end{cases}$$

Therefore, $\mu^{(1),\varepsilon} \geq \mu^{(2),\varepsilon}$, which allows us to use Corollary 3.1 from Peng and Zhu (2006) to state that $Y_t^{(1),\varepsilon} \geq Y_t^{(2),\varepsilon}$ for all $t \in [0,T]$ $\mathbb{P}$-a.s. and obtain the claimed result after taking $\varepsilon \to 0$ and realizing that, in such a case, $g_\varepsilon(c,x) \downarrow |c-x|$. Indeed,

$$\begin{aligned} &\mathbb{P}\left(\sup_{s\in[0,T-t]} \left\{Z_s^{t,x} - (\alpha(t+s) - |\alpha(t+s) - B_s^x|)\right\} < 0\right) \\ &\quad = \mathbb{P}_{t,x}\left(\lim_{n\to\infty} \sup_{s\in[0,T-t]} \left\{Y_{t+s}^{(1),1/n} - Y_{t+s}^{(2),1/n}\right\} < 0\right) \\ &\quad = \mathbb{P}_{t,x}\left(\bigcup_{N\geq 0} \bigcap_{n\geq N} \left\{\sup_{s\in[0,T-t]} \left\{Y_{t+s}^{(1),1/n} - Y_{t+s}^{(2),1/n}\right\} < 0\right\}\right) = 0. \end{aligned}$$

In the first equality, the interchange of the limit and supremum operators is justified since the convergence of g_ε is uniform, while the last equality comes since we are considering the probability of a numerable union and intersection of $\mathbb{P}_{t,x}$-null sets.

The inequalities in (2.18), and the fact that $G(x)$ is a decreasing function, guarantee that the value function (2.3) is lower than the value function associated with the infinite-horizon, discounted OSP of a reflected (with respect to m) Brownian motion. Thus, the respective OSBs hold the reverse inequality, that is, the stopping region of the latter process is contained in the stopping region of the former. It is easy to show that the free boundary for the infinite-horizon, discounted OSP with the gain function G and a m-reflected Brownian motion, is finite. Actually, one can explicitly obtain the (constant) OSB by directly solving the associated free-boundary problem. Therefore b is bounded from below. □

The value function satisfies the regularity properties listed in the next proposition. The method used to obtain the Lipschitz continuity of V, based on properties (2.11) and (2.12), is a powerful technique to solve OSPs framed within Markovian processes. For example, the work of De Angelis and Stabile (2019) provides Lipschitz continuity of the value function in a similar fashion, but for differentiable gain functions.

Proposition 2.2 (Regularity of V)**.**
The value function V satisfies the following properties:

(i) V is locally Lipschitz continuous.

(ii) V is $C^{1,2}$ on $\mathcal{C}$ and on $\mathcal{D}$, and $\mathbb{L}V = \lambda V$ on $\mathcal{C}$.

(iii) $x \mapsto V(t,x)$ is decreasing for all $t \in [0,T)$. Moreover, for $x \neq b(t)$,

$$0 \geq \partial_x V(t,x) \geq -\mathbb{E}\left[e^{-\lambda \tau^*} \partial_x X_{\tau^*}^{t,x}\right], \tag{2.19}$$

where $\tau^* = \tau^*(t,x)$. Additionally,

$$\partial_t V(t,x) \leq \mathbb{E}\left[\left|\partial_t X_{\tau^*}^{t,x}\right|\right], \tag{2.20}$$

$$\partial_t V(t,x) \geq -\mathbb{E}\left[\left|\partial_t X_{\tau^*}^{t,x}\right|\right] - L\mathbb{P}\left(\tau^* = T-t\right), \tag{2.21}$$

for some positive constant L.

Proof. *(i)* Fix $(t,x) \in [0,T] \times \mathbb{R}$. Since $\tau^* = \tau^*(t,x)$ is optimal for the OSP with initial conditions (t,x) and suboptimal for any different initial condition, and due to inequality (2.5), it follows that, for $\delta > 0$ small enough,

$$\begin{aligned} V(t,x) - V(t-\delta,x) &\leq \mathbb{E}\left[G\left(X_{\tau^*}^{t,x}\right) - G\left(X_{\tau^*}^{t-\delta,x}\right)\right] \leq \mathbb{E}\left[\left(X_{\tau^*}^{t-\delta,x} - X_{\tau^*}^{t,x}\right)^+\right] \\ &= \delta \mathbb{E}\left[\left(-\partial_t X_{\tau^*}^{t_\delta,x}\right)^+\right] \leq \delta L_x, \end{aligned} \tag{2.22}$$

where the last equality holds for a certain (random) time $t_\delta \in (t-\delta, t)$ due to the mean value theorem. In (2.22) and thereafter we use the notation

$$L_x := \sup_{t\in[0,T]} \mathbb{E}\bigg[\sup_{s\leq T-t} \left|\partial_t X_s^{t,x}\right|\bigg].$$

By (2.43) below, the value of L_x is bounded. Applying similar arguments as those used in (2.22), relying on the martingale and supermartingale properties of the value function, and noticing that $\tau^* \wedge (T-t-\delta)$ is admissible for $V(t-\delta,x)$, we have

$$\begin{aligned} &V(t+\delta,x) - V(t,x) \\ &\quad \geq \mathbb{E}\left[V\left(t+\delta+\tau^* \wedge (T-t-\delta), X_{\tau^*\wedge(T-t-\delta)}^{t+\delta,x}\right)\right] - \mathbb{E}\left[V\left(t+\tau^*, X_{\tau^*}^{t,x}\right)\right] \\ &\quad = \mathbb{E}\left[\mathbb{1}(\tau^* \leq T-t-\delta)\left(V\left(t+\delta+\tau^*, X_{\tau^*}^{t+\delta,x}\right) - V\left(t+\tau^*, X_{\tau^*}^{t,x}\right)\right)\right] \\ &\qquad + \mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(V\left(T, X_{T-t-\delta}^{t+\delta,x}\right) - V\left(t+\tau^*, X_{\tau^*}^{t,x}\right)\right)\right] \\ &\quad \geq \mathbb{E}\left[\mathbb{1}(\tau^* \leq T-t-\delta)\left(G\left(X_{\tau^*}^{t+\delta,x}\right) - G\left(X_{\tau^*}^{t,x}\right)\right)\right] \\ &\qquad + \mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(G(X_{T-t-\delta}^{t+\delta,x}) - G(X_{T-t-\delta}^{t,x})\right)\right] \\ &\qquad + \mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(G(X_{T-t-\delta}^{t,x}) - V(t+\tau^*, X_{\tau^*}^{t,x})\right)\right] \\ &\quad \geq -\mathbb{E}\left[\mathbb{1}(\tau^* \leq T-t-\delta)\left(X_{\tau^*}^{t,x} - X_{\tau^*}^{t+\delta,x}\right)^-\right] \end{aligned}$$

$$
\begin{aligned}
&- \mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(X^{t,x}_{T-t-\delta} - X^{t+\delta,x}_{T-t-\delta}\right)^-\right] \\
&+ \mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(G(X^{t,x}_{T-t-\delta}) - \mathbb{E}\left[V\left(T-\delta+\rho^*, X^{T-\delta, X^{t,x}_{T-t-\delta}}_{\rho^*}\right)\right]\right)\right] \\
\geq &- \delta L_x + \mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(G(X^{t,x}_{T-t-\delta}) - V(T-\delta, X^{t,x}_{T-t-\delta})\right)\right], \qquad (2.23)
\end{aligned}
$$

where $\rho^* := \tau^*(T-\delta, X^{t,x}_{T-t-\delta})$ and, in the second inequality, we added and subtracted the term $G(X^{t,x}_{T-t-\delta})$. For the third inequality, we used the tower property of the expectation along with the Markovian nature of X, to get

$$
\mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(V(t+\tau^*, X^{t,x}_{\tau^*}) - \mathbb{E}\left[V\left(T-\delta+\rho^*, X^{T-\delta, X^{t,x}_{T-t-\delta}}_{\rho^*}\right)\right]\right)\right] = 0.
$$

We now analyze the last term in (2.23) in detail. By the definition of V in (2.3), using the Itô–Tanaka formula, and acknowledging that $(T-\delta, X^{t,x}_{T-t-\delta}) \in \mathcal{C}$ in the set $\{\tau^* > T-t-\delta\}$, we derive the following inequality for ρ^* in $\{\tau^* > T-t-\delta\}$ and $Y_u = X_u^{T-\delta, X^{t,x}_{T-t-\delta}}$:

$$
\begin{aligned}
V(T-\delta,\, &X^{t,x}_{T-t-\delta}) - G(X^{t,x}_{T-t-\delta}) \\
&= \mathbb{E}\left[e^{-\lambda\rho^*} G\left(Y_{\rho^*}\right)\right] - G(X^{t,x}_{T-t-\delta}) \\
&= \mathbb{E}\Bigg[\frac{1}{2}\int_0^{\rho^*} e^{-\lambda u}\mathbb{1}\left(Y_u = A\right)\, \mathrm{d}l^A_u\left(Y\right) \\
&\qquad - \int_0^{\rho^*} e^{-\lambda u}\mathbb{1}\left(Y_u < A\right)\left(\lambda\left(A - Y_u\right) + \mu\left(T-\delta+u, Y_u\right)\right)\mathrm{d}u\Bigg] \\
&\leq \mathbb{E}\left[\frac{1}{2}\int_0^{\rho^*} \mathbb{1}\left(Y_u = A\right)\, \mathrm{d}l^A_u\left(Y\right) - \int_0^{\rho^*} e^{-\lambda u}\mathbb{1}\left(Y_u < A\right)\mu\left(T-\delta+u, Y_u\right)\mathrm{d}u\right] \\
&\leq \delta L, \qquad (2.24)
\end{aligned}
$$

where $L = \frac{1}{2} + \max\{|\mu(t,x)| : 0 \leq t \leq T,\ b(t) \leq x \leq A\} < \infty$.

Plugging (2.24) into (2.23), we obtain

$$
V(t+\delta, x) - V(t,x) \geq -\delta\left(L_x + L\mathbb{P}\left(\tau^* > T-t-\delta\right)\right). \qquad (2.25)
$$

Now consider $\tau_\delta = \tau^*(t-\delta, x)$ for $0 \leq \delta \leq t \leq T$, and notice that we get the following from (2.25):

$$
V(t,x) - V(t-\delta, x) \geq -\delta\left(L_x + L\right).
$$

For $0 \leq \delta \leq T-t$ and $\tau^\delta = \tau^*(t+\delta, x)$ one gets the following by proceeding as in (2.22):

$$
V(t+\delta, x) - V(t,x) \leq \delta L_x.
$$

So far, since L_x is finite (see Lemma 2.1), we have proved that, for any $x \in \mathbb{R}$, $t \mapsto V(t,x)$ is Lipschitz continuous on $[0,T]$. We will now prove that V is also Lipschitz continuous with respect to $x \in \mathbb{R}$ for all $t \in [0,T]$, which will complete the proof of *(i)*.

Since G is decreasing and $x \mapsto X_s^{t,x}$ is increasing for all $s \in [0, T-t)$ and $t \in [0,T)$ ($\partial_x X_s^{t,x} \geq 0$), then $x \mapsto V(t,x)$ is decreasing for all $t \in [0,T)$. Fix $(t,x) \in [0,T) \times \mathbb{R}$ and $\delta > 0$. Consider $\tau^* = \tau^*(t,x)$, recall (2.8), and argue as in (2.14) to get

$$0 \geq V(t, x+\delta) - V(t,x) \geq -\delta \mathbb{E}\left[e^{-\lambda\tau^*} \exp\left\{-\int_0^{\tau^*} \theta(t+u)\,\mathrm{d}u\right\}\right] \geq -\delta. \tag{2.26}$$

Then, $x \mapsto V(t,x)$ is Lipschitz continuous for all $t \in [0,T]$, which, alongside the Lipschitz continuity of $t \mapsto V(t,x)$ for all $x \in \mathbb{R}$, allows us to conclude that V is Lipschitz continuous on $[0,T] \times \mathbb{R}$.

(ii) Since V is continuous on $\mathcal{C}$ (see *(i)*), μ in (2.1) is Lipschitz continuous (actually, it suffices to require local α-Hölder continuity) in $[0,T] \times \mathbb{R}$, and the diffusion coefficient is constant, one can borrow a classic result from the theory of parabolic partial differential equations (see Friedman, 1964, Section 3, Theorem 9) to guarantee that, for an open rectangle $\mathcal{R} \subset \mathcal{C}$, the first initial-boundary value problem

$$\mathbb{L}f - \lambda f = 0 \qquad \text{in } \mathcal{R}, \tag{2.27}$$

$$f = V \qquad \text{on } \partial\mathcal{R}, \tag{2.28}$$

has a unique solution $f \in C^{1,2}(\mathcal{R})$. Fix $(t,x) \in \mathcal{R}$ and let $\tau_{\mathcal{R}^c}$ be the first time $X_s^{t,x}$ exits $\mathcal{R}$. Therefore, we can use the Itô formula on $f(X_s^{t,x})$ at $s = \tau_{\mathcal{R}^c}$, together with (2.27) and (2.28), to get the equality $\mathbb{E}_{t,x}[V(X_{t+\tau_{\mathcal{R}^c}})] = f(t,x)$. Finally, notice that, due to the strong Markov property, $\mathbb{E}_{t,x}[V(X_{t+\tau_{\mathcal{R}^c}})] = V(t,x)$, meaning that $f = V$ (see Peskir and Shiryaev, 2006, Section 7.1, for further details). The fact that V is $C^{1,2}$ on $\mathcal{D}$ follows straightforwardly from $V = G$ on $\mathcal{D}$.

(iii) We already mentioned that $x \mapsto V(t,x)$ is decreasing since G is decreasing as well and $x \mapsto X_s^{t,x}$ is increasing. For $(t,x) \in [0,T) \times \mathbb{R}$ such that $x \neq b(t)$, (2.19) follows after recalling that V is differentiable with respect to x in $\mathcal{C}$ and $\mathcal{D}$, dividing by δ in (2.26), and taking $\delta \to 0$, while using the dominated convergence theorem. The same procedure used in (2.22) and (2.25) yields (2.20) and (2.21). □

Next, we look for smoothness of the free boundary in order to prove the smooth-fit condition on which relies the uniqueness of the solution of the free-boundary problem associated to the OSP (2.3). The works of De Angelis (2015), Peskir (2019), and De Angelis and Stabile (2019) are a good compendium on the smoothness of the OSB. For time-homogeneous processes and smooth gain functions, De Angelis (2015) provides the continuity of the free boundary for one-dimensional processes with locally Lipschitz-continuous drift and volatility. The two-dimensional case (including time-space diffusions) is addressed by Peskir (2019) in fairly general settings, which proves the impossibility of first-type discontinuities of the OSB for Mayer–Lagrange OSPs. Unfortunately, continuity of the OSB is not enough to prove the smooth-fit condition. Local Lipschitz continuity, however, suffices to derive the smooth-fit condition in our settings, as we prove later in Proposition 2.4.

In De Angelis and Stabile (2019), the local Lipschitz continuity of the free boundary in a high-dimensional framework is obtained, also for Mayer–Lagrange OSPs. However, some restrictive conditions are imposed that are not met in our setting, like C^3 smoothness of the gain function and a particular relation between the partial derivatives of $\mathbb{L}G$ (see conditions (D), (F) and (G) in De Angelis and Stabile (2019)).

The next proposition proves the local Lipschitz continuity of the OSB by adapting the methodology used by De Angelis and Stabile (2019) to deal with our settings.

Proposition 2.3 (Lipschitz continuity of b)**.**
The OSB $b : [0, T] \to \mathbb{R}$ *is Lipschitz continuous on any closed interval in* $[0, T)$.

Proof. Consider the function $W(t, x) = V(t, x) - G(x)$ and the closed interval $I = [\underline{t}, \bar{t}] \subset [0, T)$. Proposition 2.1 guarantees that b is bounded from above and, hence, we can choose $r > \sup\{b(t) : t \in I\}$. Hence, $I \times \{r\} \subset \mathcal{C}$, and then $W(t, r) > 0$ for all $t \in I$. Since W is continuous on $\overline{\mathcal{C}}$ (see Proposition 2.2), there exists a constant $a > 0$ such that $W(t, r) \geq a$ for all $t \in I$. Therefore, for all δ such that $0 < \delta \leq a$, the equation $W(t, x) = \delta$ has a solution in $\mathcal{C}$ for all $t \in I$. This solution is unique for each t, as $\partial_x W > 0$ in $\mathcal{C}$ (see (2.19)). Hence we can denote it by $b_\delta(t)$, where $b_\delta : I \to (b(t), r]$. Away from the boundary, W is regular enough to apply the implicit function theorem, which guarantees that b_δ is differentiable and

$$b_\delta'(t) = -\partial_t W(t, b_\delta(t))/\partial_x W(t, b_\delta(t)). \tag{2.29}$$

Notice that b_δ is increasing in δ and therefore it converges pointwise to some limit function b_0, which satisfies $b_0 \geq b$ in I as $b_\delta > b$ for all δ. Since $W(t, b_\delta(t)) = \delta$ and W is continuous, it follows that $W(t, b_0(t)) = 0$ after taking $\delta \downarrow 0$, which means that $b_0 \leq b$ in I and hence $b_0 = b$ in I.

For $(t, x) \in \mathcal{C}$ such that $t \in I$ and $x < r$, consider the stopping times $\tau^* = \tau^*(t, x)$ and

$$\tau_r = \tau_r(t, x) := \inf\{s \geq 0 : X_s^{t,x} \notin I \times (-\infty, r)\}.$$

Using (2.42) from Lemma 2.1, alongside the fact that $\tau^* \leq T - \underline{t}$, it readily follows after (2.20) that

$$\partial_t W(t, x) \leq \mathbb{E}\left[\left|\partial_t X_{\tau^*}^{t,x}\right|\right] \leq L\left(1 + (T - \underline{t})e^{L(T-\underline{t})}\right) m(t, x), \tag{2.30}$$

where L is a positive constant coming from the Lipschitz continuity of $\partial_t \mu$ and m is the function defined as

$$m(t, x) := \mathbb{E}\left[\int_0^{\tau^*} \left(\left|X_u^{t,x}\right| + 1\right) \mathrm{d}u\right].$$

Due to the tower property of conditional expectation and the strong Markov property, we have that

$$m(t, x) = \mathbb{E}\left[\int_0^{\tau^* \wedge \tau_r} \left(\left|X_u^{t,x}\right| + 1\right) \mathrm{d}u + \mathbb{1}(\tau_r \leq \tau^*) m\left(t + \tau_r, X_{t+\tau_r}\right)\right]. \tag{2.31}$$

Notice that, for $b(t) < x < r$, $(t + \tau_r, X_{t+\tau_r}^{t,x}) \in \Gamma_t := ((t, \bar{t}) \times \{r\}) \cup (\{\bar{t}\} \times (b(\bar{t}), r])$ on the set $\{\tau_r < \tau^*\}$. Then, we get the following upper bound on $\{\tau_r < \tau^*\}$:

$$\begin{aligned}
m\left(t + \tau_r, X_{t+\tau_r}\right) \leq \sup_{(s,y)\in\Gamma_t} m\left(s, y\right) &= \sup_{(s,y)\in\Gamma_t} \mathbb{E}\left[\int_0^{\tau^*(s,y)} \left(|X_u^{s,y}| + 1\right) \mathrm{d}u\right] \\
&\leq T \sup_{(s,y)\in\Gamma_t} \sqrt{\mathbb{E}\left[\sup_{u\leq T-s} \left(|X_u^{s,y}| + 1\right)^2\right]} \\
&\leq T \sup_{(s,y)\in\Gamma_t} \sqrt{M_{T-s}^{(1)} \left(|y|^2 + 1\right)} \\
&\leq T\sqrt{M_T^{(1)} \left(\max\left\{|b(\bar{t})|^2, |r|^2\right\} + 1\right)},
\end{aligned} \tag{2.32}$$

where we used the Cauchy–Schwarz inequality and (2.40) from Lemma 2.1. Plugging (2.32) into (2.31), recalling (2.30), and noticing that, for $u \leq \tau^* \wedge \tau_r$, $|X_u^{t,x}| \leq \max\{|\underline{b}|, r\}$ with $\underline{b} := \inf\{b(t) : t \in [0,T]\}$, we get that

$$\partial_t W(t,x) \leq K_I^{(1)} \mathbb{E}\left[\tau_r \wedge \tau^* + \mathbb{1}(\tau_r \leq \tau^*)\right] \tag{2.33}$$

for some positive constant $K_I^{(1)}$.

Using (2.19) and (2.41) from Lemma 2.1, and proceeding in the same way as for (2.31), we get the following lower bound for $\partial_x W(t,x)$:

$$\partial_x W(t,x) \geq M_{T-\underline{t}}^{(2)} \mathbb{E}\left[\tau^*\right] = M_{T-\underline{t}}^{(2)} \mathbb{E}\left[(\tau_r \wedge \tau^*) + \mathbb{1}(\tau_r \leq \tau^*) \mathbb{E}\left[\tau^*\left(t + \tau_r, X_{\tau_r}^{t,x}\right)\right]\right]. \tag{2.34}$$

Now, by combining equations (2.29), (2.33), and (2.34), we obtain the following inequalities for a constant $K_I^{(2)} > 0$, $x_\delta = b_\delta(t)$, $\tau_\delta = \tau^*(t, x_\delta)$, and $\tau_r = \tau_r(t, x_\delta)$:

$$\begin{aligned}
b_\delta'(t) \geq& - K_I^{(2)} \frac{\mathbb{E}\left[\tau_\delta \wedge \tau_r + \mathbb{1}(\tau_r \leq \tau_\delta)\right]}{\mathbb{E}\left[\tau_\delta \wedge \tau_r + \mathbb{1}(\tau_r \leq \tau_\delta) \mathbb{E}\left[\tau^*(t + \tau_r, X_{\tau_r}^{t,x_\delta})\right]\right]} \\
\geq& - K_I^{(2)} \left(1 + \frac{\mathbb{P}\left(\tau_r \leq \tau_\delta\right)}{\mathbb{E}\left[\tau_\delta \wedge \tau_r + \mathbb{1}(\tau_r \leq \tau_\delta) \mathbb{E}\left[\tau^*(t + \tau_r, X_{\tau_r}^{t,x_\delta})\right]\right]}\right) \\
\geq& - K_I^{(2)} \left(1 + \frac{\mathbb{P}\left(\tau_r \leq \tau_\delta, \tau_r = \bar{t} - t\right)}{\mathbb{E}\left[\tau_\delta \wedge \tau_r\right]} + \frac{\mathbb{P}\left(\tau_r \leq \tau_\delta, \tau_r < \bar{t} - t\right)}{\mathbb{E}\left[\mathbb{1}(\tau_r \leq \tau_\delta) \mathbb{E}\left[\tau^*(t + \tau_r, X_{\tau_r}^{t,x_\delta})\right]\right]}\right) \\
\geq& - K_I^{(2)} \left(1 + \frac{\mathbb{P}\left(\tau_r \leq \tau_\delta, \tau_r = \bar{t} - t\right)}{\mathbb{E}\left[\mathbb{1}(\tau_r \leq \tau_\delta, \tau_r = \bar{t} - t)(\tau_\delta \wedge \tau_r)\right]}\right. \\
&\left. + \frac{\mathbb{P}\left(\tau_r \leq \tau_\delta, \tau_r < \bar{t} - t\right)}{\mathbb{E}\left[\mathbb{1}(\tau_r \leq \tau_\delta, \tau_r < \bar{t} - t) \mathbb{E}\left[\tau^*(t + \tau_r, X_{\tau_r}^{t,x_\delta})\right]\right]}\right) \\
\geq& - K_I^{(2)} \left(1 + \frac{1}{\bar{t} - t} + \frac{1}{\inf_{t \in I} \mathbb{E}\left[\tau^*(t, r)\right]}\right),
\end{aligned} \tag{2.35}$$

where in the last inequality we used that $X_{\tau_r}^{t,x_\delta} = r$ in the set $\{\tau_r \leq \tau_\delta, \tau_r < \bar{t} - t\}$.

Since $I \times \{r\} \subset \mathcal{C}$, there exists $\varepsilon > 0$ such that $\mathcal{R}_\varepsilon := [\underline{t}, \bar{t} + \varepsilon] \times (r - \varepsilon, r + \varepsilon) \subset \mathcal{C}$. Notice that $\tau^*(t,r) > \tau_\varepsilon(t,r)$ for all $t \in I$, with $\tau_\varepsilon(t,r) = \inf\left\{u \geq 0 : (t + u, X_u^{t,r}) \notin \mathcal{R}_\varepsilon\right\}$. Hence,

$$\begin{aligned}
\inf_{t \in I} \mathbb{E}\left[\tau^*(t,r)\right] &\geq \inf_{t \in I} \mathbb{E}\left[\tau_\varepsilon(t,r)\right] \\
&\geq \inf_{t \in I} \left\{(\bar{t} + \varepsilon - t) \mathbb{P}\left(\sup_{u \leq \bar{t} + \varepsilon - t} \left|X_u^{t,r} - r\right| < \varepsilon\right)\right\} \\
&\geq \varepsilon \inf_{t \in I} \mathbb{P}\left(\sup_{u \leq \bar{t} + \varepsilon - t} \left|X_u^{t,r} - r\right| < \varepsilon\right) =: K_{I,\varepsilon} > 0.
\end{aligned}$$

Therefore, going back to (2.35), we have that, for all $t \in I' = [\underline{t}, \bar{t} - \varepsilon]$ and $\varepsilon > 0$ small enough,

$$b_\delta'(t) \geq -K_I^{(2)} \left(1 + \varepsilon^{-1} + K_{I,\varepsilon}\right). \tag{2.36}$$

To find a bound in the opposite direction that is also uniform with respect to δ and for all $t \in I'$, we consider (2.21) and (2.30) and use the Markov inequality to get

$$\partial_t W(t, x_\delta) \geq - L\left(1 + (T - \underline{t}) e^{L(T - \underline{t})}\right) \mathbb{E}\left[\int_0^{\tau_\delta} \left(\left|X_u^{t,x_\delta}\right| + 1\right) \mathrm{d}u\right]$$

$$- \widetilde{L}(T-t)^{-1}\mathbb{E}\left[\tau_\delta\right] \tag{2.37}$$

for some positive constant $\widetilde{L}$. Hence, relying on the same arguments as those used to get (2.36), but with (2.37) instead of (2.33), we obtain

$$b_\delta'(t) \leq K\left(1+\varepsilon^{-1}+K_{I,\varepsilon}\right)+\widetilde{L}(T-\bar{t})^{-1}. \tag{2.38}$$

We have proved that $|b_\delta'(t)|$ is bounded by a constant, uniformly in δ and for all $t\in I'$. Then the Arzelà–Ascoly's theorem gives that b_δ converges uniformly to b in I' as $\delta\to 0$, which implies that b is Lipschitz continuous on I'. □

The next proposition gives the smooth-fit condition for our OSP. Its proof relies on Lemma 2.2 and the fact that b is locally Lipschitz continuous. These two combined results imply that the process $X^{t,x}$ enters the interior of $\mathcal{D}$ immediately for $x=b(t)$ (see Remark 4.5 in De Angelis and Stabile (2019)). The work in De Angelis and Peskir (2020), specifically Corollary 6, is then used to obtain $\tau^*(t,b(t)+\varepsilon)\to 0$ $\mathbb{P}$-a.s. when $\varepsilon\downarrow 0$, from which the smooth-fit condition follows.

Proposition 2.4 (Smooth-fit condition)**.**
The smooth-fit condition holds, i.e., $\partial_x V(t,b(t))=-1$ *for all* $t\in[0,T)$.

Proof. Take a point $(t,b(t))$ for $t\in[0,T)$ and consider $\delta>0$. Since $(t,b(t))\in\mathcal{D}$ and $(t,b(t)+\delta)\in\mathcal{C}$, we get that $\delta^{-1}(V(t,b(t)+\delta)-V(t,b(t)))\geq\delta^{-1}(G(b(t)+\delta)-G(b(t)))=-1$. Therefore, $\partial_x^+V(t,b(t))\geq -1$.

Besides, reasoning as in (2.26), we get that

$$\begin{aligned}\delta^{-1}(V(t,b(t)+\delta)-V(t,b(t))) &\leq -\mathbb{E}\left[e^{-\lambda\tau_\delta}\partial_x X_{\tau_\delta}^{t,x_\delta}\right]\\ &\leq \sup_{x\in(b(t),A)}\mathbb{E}\Big[\sup_{s\leq T-t}\partial_x X_s^{t,x}\Big]<\infty,\end{aligned}$$

where $\tau_\delta=\tau^*(t,b(t)+\delta)$ and $x_\delta\in(b(t),b(t)+\delta)$. From (2.8), it is easy to see that the supremum is finite (actually, it is lower than 1) since $\partial_x\mu<0$. Then, we can apply the dominated convergence theorem to obtain

$$\partial_x^+V(t,b(t))\leq -\mathbb{E}\left[e^{-\lambda\tau_0}\partial_x X_{\tau_0}^{t,b(t)}\right] \tag{2.39}$$

with $\tau_0:=\lim_{\delta\to 0}\tau_\delta$ (τ_0 is well-defined since the sequence τ_δ decreases with respect to δ $\mathbb{P}$-a.s.).

We now prove that $\mathbb{P}_{t,b(t)}\left(\tau_0=0\right)=1$. We do so by summoning Corollary 6 from De Angelis and Peskir (2020) and proving that $(t,b(t))$ is probabilistically regular for $\mathcal{D}^{\mathrm{o}}$, that is, $\mathbb{P}_{t,b(t)}\left(\tau_{\mathcal{D}^{\mathrm{o}}}=0\right)=1$, where

$$\tau_{\mathcal{D}^{\mathrm{o}}}:=\inf\left\{u\geq 0:(t+u,X_{t+u})\in\mathcal{D}^{\mathrm{o}}\right\}.$$

Indeed,

$$\begin{aligned}\mathbb{P}_{t,b(t)}\left(\tau_{\mathcal{D}^{\mathrm{o}}}=0\right) &= \lim_{\varepsilon\downarrow 0}\mathbb{P}_{t,b(t)}\left(\tau_{\mathcal{D}^{\mathrm{o}}}<\varepsilon\right)\\ &= \lim_{\varepsilon\downarrow 0}\mathbb{P}_{t,b(t)}\left(\inf_{u\in(0,\varepsilon)}\left(X_{t+u}-b(t+u)\right)<0\right)\\ &= \lim_{\varepsilon\downarrow 0}\mathbb{P}_{t,b(t)}\left(\inf_{u\in(0,\varepsilon)}\frac{X_{t+u}-b(t+u)}{\sqrt{2u\ln(\ln(1/u))}}<0\right)\end{aligned}$$

$$\geq \lim_{\varepsilon \downarrow 0} \mathbb{P}_{t,b(t)} \left(\inf_{u \in (0,\varepsilon)} \frac{X_{t+u} - b(t) + Lu}{\sqrt{2u \ln(\ln(1/u))}} < 0 \right)$$
$$= \mathbb{P}_{t,b(t)} \left(\liminf_{u \downarrow 0} \frac{X_{t+u} - b(t) + Lu}{\sqrt{2u \ln(\ln(1/u))}} < 0 \right) = 1,$$

where, in the first inequality, L is a positive constant that comes from the local Lipschitz continuity of b (see Proposition 2.3), and the last equality holds due to (2.45). Hence, (2.39) yields $\partial_x^+ V(t, b(t)) \leq -1$ and, consequently, $\partial_x^+ V(t, b(t)) = -1$.

The smooth-fit condition arises by recalling that $\partial_x^- V(t, b(t)) = -1$ since $V = G$ on $\mathcal{D}$. $\square$

We conclude this section with two technical lemmas. One on the boundedness of several functionals of the underlying process X that have been used throughout the previous proofs, and another on a law of the iterated logarithm for X.

Lemma 2.1 (Some bounds on functionals of X).
The following inequalities hold for positive constants L, $M_s^{(i)}$, $i = 1, 2, 3$, and $(t, x) \in [0, T) \times \mathbb{R}$ and $s \in (0, T - t]$:

$$\mathbb{E}\left[\sup_{u \leq s} \left(\left|X_u^{t,x}\right| + 1\right)^2\right] \leq M_s^{(1)} \left(|x|^2 + 1\right), \tag{2.40}$$
$$\partial_x X_s^{t,x} \leq 1 - M_s^{(2)} s, \tag{2.41}$$
$$\mathbb{E}\left[\sup_{u \leq s} \left|\partial_t X_u^{t,x}\right|\right] \leq L\left(1 + s e^{Ls}\right) \mathbb{E}\left[\int_0^s \left(\left|X_u^{t,x}\right| + 1\right) \mathrm{d}u\right] \tag{2.42}$$
$$\leq M_s^{(3)} \left(|x| + 1\right). \tag{2.43}$$

Moreover, $s \mapsto M_s^{(i)}$ is an increasing function for $i = 1, 3$, and is decreasing for $i = 2$.

Proof. Since $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ and due to the Lipschitz continuity of $x \mapsto \mu(t, x)$,

$$\left(\left|X_s^{t,x}\right| + 1\right)^2 \leq 3\left(|x| + 1\right)^2 + 3\int_0^s \left(\mu(t + r, X_r^{t,x})\right)^2 \mathrm{d}r + 3\left(W_s\right)^2$$
$$\leq 3\left(|x| + 1\right)^2 + 3L\int_0^s \left(\left|X_r^{t,x}\right| + 1\right)^2 \mathrm{d}r + 3\left(W_s\right)^2$$

for some positive constant L. Hence, using the maximal inequalities in Theorem 14.13 (d) from Schilling et al. (2012), it follows that

$$\mathbb{E}\left[\sup_{u \leq s} \left(\left|X_u^{t,x}\right| + 1\right)^2\right] \leq 3\left(|x| + 1\right)^2 + 3L\int_0^s \mathbb{E}\left[\left(\left|X_r^{t,x}\right| + 1\right)^2\right] \mathrm{d}r + 3\mathbb{E}\left[\sup_{u \leq s} \left(W_u\right)^2\right]$$
$$\leq 3\left(|x| + 1\right)^2 + 12s + 3L\int_0^s \mathbb{E}\left[\sup_{u \leq r} \left(\left|X_u^{t,x}\right| + 1\right)^2\right] \mathrm{d}r.$$

Therefore, Gronwall's inequality (Schilling et al., 2012, Theorem A.43) guarantees that

$$\mathbb{E}\left[\sup_{u \leq s} \left(\left|X_u^{t,x}\right| + 1\right)^2\right] \leq 3\left(|x| + 1\right)^2 + 12s + 3L\int_0^s \left(3\left(|x| + 1\right)^2 + 12u\right) e^{3L(s-u)} \mathrm{d}u$$
$$\leq 3\left(|x| + 1\right)^2 + 12s + 3Le^{3Ls}\left(3\left(|x| + 1\right)^2 s + 6s^2\right),$$

from which (2.40) follows.

To obtain (2.41), we use that μ is Lipschitz continuous and $\partial_x \mu < 0$, together with representation (2.8) and (2.19), leading to

$$\begin{aligned} 1 - \partial_x X_s^{t,x} &= \int_0^s -\partial_x \mu(t+u, X_u^{t,x}) \partial_x X_u^{t,x} \, \mathrm{d}u \\ &= \int_0^s -\partial_x \mu(t+u, X_u^{t,x}) \exp\left\{ \int_0^u \partial_x \mu(t+r, X_r^{t,x}) \, \mathrm{d}r \right\} \mathrm{d}u \\ &\geq \exp\left\{ \int_0^s \partial_x \mu(t+u, X_u^{t,x}) \, \mathrm{d}u \right\} \int_0^s -\partial_x \mu(t+u, X_u^{t,x}) \, \mathrm{d}u \\ &\geq e^{-Ls} \int_0^s -\partial_x \mu(t+u, X_u^{t,x}) \, \mathrm{d}u \\ &\geq e^{-Ls} \min_{u \in [0,T]} \{-\bar{\mu}(u)\} \, s, \end{aligned}$$

where L is the Lipschitz constant.

Let us prove now (2.43). To do so, notice first that (2.6) implies

$$\left| \partial_t X_s^{t,x} \right| \leq a_1(s,t,x) + \int_0^s a_2(u,t,x) \left| \partial_t X_u^{t,x} \right| \, \mathrm{d}u,$$

with

$$a_1(s,t,x) := \int_0^s \left| \partial_t \mu(t+u, X_u^{t,x}) \right| \mathrm{d}u, \quad a_2(u,t,x) := \left| \partial_x \mu(t+u, X_u^{t,x}) \right|.$$

Therefore, an application of Gronwall's inequality yields

$$\left| \partial_t X_s^{t,x} \right| \leq a_1(s,t,x) + \int_0^s a_1(u,t,x) a_2(u,t,x) \exp\left\{ \int_u^s a_2(r,t,x) \, \mathrm{d}r \right\} \mathrm{d}u.$$

Hence, due to the Lipschitz continuity of μ and $\partial_t \mu$, and using (2.40) alongside the Cauchy–Schwarz inequality, we have that

$$\begin{aligned} \mathbb{E}\left[\sup_{u \leq s} \left| \partial_t X_u^{t,x} \right| \right] &\leq \mathbb{E}\left[a_1(s,t,x) \left(1 + \int_0^s a_2(u,t,x) \, \mathrm{d}u \exp\left\{ \int_0^s a_2(u,t,x) \, \mathrm{d}u \right\} \right) \right] \\ &\leq L \left(1 + s e^{Ls} \right) \mathbb{E}\left[\int_0^s \left(\left| X_u^{t,x} \right| + 1 \right) \mathrm{d}u \right] \\ &\leq L \left(1 + s e^{Ls} \right) s \sqrt{ \mathbb{E}\left[\sup_{u \leq s} \left(\left| X_u^{t,x} \right| + 1 \right)^2 \right] } \\ &\leq L \left(1 + s e^{Ls} \right) s \sqrt{ M_s^{(1)} \left(|x|^2 + 1 \right) }, \end{aligned}$$

which entails (2.42) and (2.43). □

Lemma 2.2 (Law of the iterated logarithm for X)**.** *The following relations hold for all $(t,x) \in [0,T] \times \mathbb{R}$:*

$$\mathbb{P}_{t,x}\left(\limsup_{u \downarrow 0} \frac{X_{t+u} - x}{\sqrt{2u \ln(\ln(1/u))}} = 1 \right) = 1, \tag{2.44}$$

and

$$\mathbb{P}_{t,x}\left(\liminf_{u \downarrow 0} \frac{X_{t+u} - x}{\sqrt{2u \ln(\ln(1/u))}} = -1 \right) = 1. \tag{2.45}$$

Proof. Let $m_{t,x}(u) := \mathbb{E}_{t,x}\left[X_{t+u}\right]$, $f_t(u) = \exp\left\{-\int_t^{t+u}\theta(r)\,\mathrm{d}r\right\}$, and $h_t(u) = \int_t^{t+u} f_t^{-2}(u)\,\mathrm{d}u$. It is known (see, e.g., Buonocore et al. (2013)) that, under $\mathbb{P}_{t,x}$, X_{t+u} admits the representation $X_{t+u} = m_{t,x}(u) + f_t(u)B_{h_t(u)}$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Hence, the law of the iterated logarithm for the Brownian motion establishes that

$$\mathbb{P}_{t,x}\left(\limsup_{u\to 0}\frac{(X_{t+u}-m_{t,x}(u))/f_t(u)}{\sqrt{2h_t(u)\ln(\ln(1/h_t(u)))}} = 1\right) = 1.$$

Then, (2.44) is proved after realizing that $m_{t,x}(u)\to x$, $f_t(u)\to 1$, and $h_t(u)/u\to 1$, as $u\downarrow 0$. Relation (2.45) follows identical steps. $\square$

2.4 The value formula and the free-boundary equation

Our main result is collected in the following theorem. It gives a characterization of the OSB as the unique solution, up to regularity conditions, of a type-two Volterra integral equation. It also provides a formula for the value function, which can be regarded as the fair price of the American option under the natural measure of the OU process. This contract might find application from the point of view of small investors who, due to their limited capital and impossibility to trade in continuous time (also due to transaction costs), cannot perform hedging strategies to reproduce the conditions of a risk-free measure settings under which the valuation of options is typically done to keep the market free of arbitrage opportunities.

The value formula is given in terms of the early-exercise premium representation. That is, the sum of the price of the European put option written on the same asset and expiring on the same date, and the early-exercise premium, i.e., the cost of being able to exercise the option before maturity.

Theorem 2.1 (Free-boundary equation and value formula).
The OSB related to the OSP (2.3) *satisfies the integral equation*

$$\begin{aligned} b(t) &= A - K_\lambda(A,1,t,b(t),T,A) \\ &\quad - \int_t^T K_\lambda(\lambda A+\theta(u)\alpha(u), \lambda+\theta(u), t, b(t), u, b(u))\,\mathrm{d}u, \end{aligned} \tag{2.46}$$

where, for $c_1, c_2, x_1, x_2 \in \mathbb{R}$ *and* $t_1, t_2 \in [0,T]$ *such that* $t_2 \geq t_1$,

$$\begin{aligned} K_\lambda(c_1,c_2,t_1,x_1,t_2,x_2) &:= e^{-\lambda(t_2-t_1)}\mathbb{E}_{t_1,x_1}\left[(c_1-c_2X_{t_2})\,\mathbb{1}(X_{t_2}\leq x_2)\right] \\ &= e^{-\lambda(t_2-t_1)}\bigg\{(c_1-c_2\nu(t_1,x_1,t_2))\Phi\left(\frac{x_2-\nu(t_1,x_1,t_2)}{\gamma(t_1,t_2)}\right) \\ &\quad + c_2\gamma(t_1,t_2)\phi\left(\frac{x_2-\nu(t_1,x_1,t_2)}{\gamma(t_1,t_2)}\right)\bigg\}, \end{aligned} \tag{2.47}$$

where ϕ and Φ are the density and distribution functions of a standard normal, and ν and γ depend on the time-dependent OU parameters:

$$\nu(t_1,x,t_2) = \exp\left\{-\int_{t_1}^{t_2}\theta(r)\,\mathrm{d}r\right\}x + \int_{t_1}^{t_2}\exp\left\{-\int_r^{t_2}\theta(s)\,\mathrm{d}s\right\}\theta(r)\alpha(r)\,\mathrm{d}r, \tag{2.48}$$

$$\gamma^2(t_1,t_2) = \int_{t_1}^{t_2}\exp\left\{-2\int_r^{t_2}\theta(s)\,\mathrm{d}s\right\}\mathrm{d}r. \tag{2.49}$$

Moreover, (2.46) *has a unique solution among the class of continuous functions of bounded variation* $c : [0, T] \to \mathbb{R}$ *that satisfy* $c(t) < A$ *for all* $t \in [0, T)$.

Also, the associated value function is given by

$$\begin{aligned} V(t, x) &= K_\lambda(A, 1, t, x, T, A) \\ &\quad + \int_t^T K_\lambda(\lambda A + \theta(u)\alpha(u), \lambda + \theta(u), t, x, u, b(u))\, \mathrm{d}u. \end{aligned} \tag{2.50}$$

Proof. Propositions 2.1–2.4 allow applying an extension of the Itô formula to $e^{-\lambda s}V(t+s, X_{t+s})$. This extension is derived in Peskir (2005a) and is reformulated in Lemma A2 from D'Auria et al. (2020) in a way that fits our settings more straightforwardly. After setting $s = T - t$, taking $\mathbb{P}_{t,x}$-expectation, and shifting the integrating variable t units, it follows that

$$V(t, x) = e^{-\lambda(T-t)}\mathbb{E}_{t,x}\left[G(X_T)\right] - \int_t^T \mathbb{E}_{t,x}\left[e^{-\lambda(u-t)}(\mathbb{L}V - \lambda V)(u, X_u)\right] \mathrm{d}u, \tag{2.51}$$

where the martingale term vanishes after taking $\mathbb{P}_{t,x}$-expectation, and the local time term does not appear due to the smooth-fit condition. Recall that $\mathbb{L}V = \lambda V$ on $\mathcal{C}$ and $V = G$ on $\mathcal{D}$. Therefore, (2.51) turns into the value formula

$$\begin{aligned} V(t, x) &= e^{-\lambda(T-t)}\mathbb{E}_{t,x}\left[(A - X_T)\,\mathbb{1}\,(X_T \le A)\right] \\ &\quad + \int_t^T \mathbb{E}_{t,x}\left[e^{-\lambda(u-t)}\left(\lambda\left(A - X_u\right) + \mu(u, X_u)\right)\mathbb{1}\left(X_u \le b(u)\right)\right] \mathrm{d}u. \end{aligned} \tag{2.52}$$

By taking $x \uparrow b(t)$ in (2.52) we get the free-boundary equation

$$\begin{aligned} b(t) &= A - e^{-\lambda(T-t)}\mathbb{E}_{t,b(t)}\left[(A - X_T)\,\mathbb{1}\,(X_T \le A)\right] \\ &\quad - \int_t^T \mathbb{E}_{t,b(t)}\left[e^{-\lambda(u-t)}\left(\lambda\left(A - X_u\right) + \mu(u, X_u)\right)\mathbb{1}\left(X_u \le b(u)\right)\right] \mathrm{d}u. \end{aligned} \tag{2.53}$$

The uniqueness of the solution of (2.53), up to the conditions stated in the theorem, follows well-known arguments derived by Peskir (2005b, Theorem 3.1).

We can provide a more tractable expression for both the value formula (2.52) and the free-boundary equation (2.53) by taking advantage of the Gaussianity of X. Indeed, it is easy to get (see, e.g., Gardiner, 2004, Section 4.4.9) that, under $\mathbb{P}_{t,x}$, X_u has a normal distribution with mean and variance given by (2.48) and (2.49) for all $u \in [t, T]$. This leads, after some algebraic manipulation, to equations (2.46) and (2.50). □

Further flexibility of the underlying process can be achieved by allowing time-dependent volatility, like in the popular Hull–White model. We show in the next remark that such an extension requires no extra analysis, as it can be reduced to the constant-volatility model (2.1) by means of a time change.

Remark 2.1 (Time-dependent deterministic volatility).
Let $X_t(\theta, \alpha, \sigma, T) = (X_t)_{t\in[0,T]}$ *be a stochastic process satisfying the stochastic differential equation*

$$\mathrm{d}X_t(\theta, \alpha, \sigma, T) = \theta(t)(\alpha(t) - X_t(\theta, \alpha, \sigma, T))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W_t, \quad 0 \le t \le T, \tag{2.54}$$

where $\theta, \alpha, \sigma : [0, T] \to \mathbb{R}$ *are continuously differentiable functions such that* $\theta(t) > 0$ *and* $\sigma(t) > 0$ *for all* $t \in [0, T]$*, and* $\int_0^T \sigma^2(t)\,\mathrm{d}t < \infty$*. Define the time change* $\tilde{t} = \tilde{h}(t) = \int_0^t \sigma^2(u)\,\mathrm{d}u$

and the inverse transformation $h = \tilde{h}^{-1}$. *Consider the time-changed process* $\widetilde{X} = (\widetilde{X}_{\tilde{t}})_{\tilde{t}\in[0,\widetilde{T}]}$ *defined by* $\widetilde{X}_{\tilde{t}} = X_{h(\tilde{t})}(\theta, \alpha, \sigma, T)$ *and* $\widetilde{T} = \tilde{h}(T)$. *Then,*

$$\widetilde{X}_{\tilde{t}} = X_0(\theta, \alpha, \sigma, T) + \int_0^{h(\tilde{t})} \theta(u)(\alpha(u) - X_u(\theta, \alpha, \sigma, T))\,\mathrm{d}u + \int_0^{h(\tilde{t})} \sigma(u)\,\mathrm{d}W_u.$$

The process $\widetilde{W}_{\tilde{t}} = \int_0^{h(\tilde{t})} \sigma(u)\,\mathrm{d}W_u$ *is a standard Brownian motion. Hence,* $\widetilde{X}$ *solves*

$$\begin{aligned}\mathrm{d}\widetilde{X}_{\tilde{t}} &= h'(\tilde{t})\theta(h(\tilde{t}))\big(\alpha(h(\tilde{t})) - X_{h(\tilde{t})}(\theta, \alpha, \sigma, T)\big)\,\mathrm{d}\tilde{t} + \mathrm{d}\widetilde{W}_{\tilde{t}} \\ &= \widetilde{\theta}(\tilde{t})\big(\widetilde{\alpha}(\tilde{t}) - \widetilde{X}_{\tilde{t}}\big)\,\mathrm{d}\tilde{t} + \mathrm{d}\widetilde{W}_{\tilde{t}},\end{aligned}$$

where $\widetilde{\theta}(\tilde{t}) = \sigma^{-2}(h(\tilde{t}))\theta(h(\tilde{t}))$ *and* $\widetilde{\alpha}(\tilde{t}) = \alpha(h(\tilde{t}))$*, that is,* $\widetilde{X}$ *is a time-dependent OU process with constant volatility. Hence, we can rely on equations* (2.50) *and* (2.46) *to compute the value function and OSB associated with* $\widetilde{X}$, $\widetilde{V}$, *and* $\tilde{b}$, *from where we can simply obtain the value function and OSB related to* $X(\theta, \alpha, \sigma, T)$ *as* $V(t, x) = \widetilde{V}(h(\tilde{t}), x)$ *and* $b(t) = \tilde{b}(h(\tilde{t}))$.

The optimal exercise of the put and the call American options is fundamentally the same problem due to their symmetry with respect to the strike price, as shown in the next remark.

Remark 2.2 (Put-call parity)**.**
Let V_p *be the value function associated with an American put option written on the time-dependent OU processes* $X^{(\mathrm{p})} = (X_t^{(\mathrm{p})})_{t\in[0,T]}$ *and with strike price* A. *Then,*

$$\begin{aligned}V_\mathrm{p}(t, x) &:= \sup_{\tau\le T-t} \mathbb{E}_{t,x}\left[e^{-\lambda\tau}\left(A - X_{t+\tau}^{(\mathrm{p})}\right)^+\right] \\ &= \sup_{\tau\le T-t} \mathbb{E}_{t,x}\left[e^{-\lambda\tau}\left(\left(2A - X_{t+\tau}^{(\mathrm{p})}\right) - A\right)^+\right] \\ &= \sup_{\tau\le T-t} \mathbb{E}_{t,2A-x}\left[e^{-\lambda\tau}\left(X_{t+\tau}^{(\mathrm{c})} - A\right)^+\right] =: V_\mathrm{c}(t, 2A - x),\end{aligned}$$

where $X^{(\mathrm{c})} = (X_t^{(\mathrm{c})})_{t\in[0,T]}$ *is the time-dependent OU process such that* $X^{(\mathrm{c})} = 2A - X^{(\mathrm{p})}$*, and* V_c *is the value function of the American call option with strike price* A *and written on* $X^{(\mathrm{c})}$. *Similarly, the optimal exercise strategy for the call option is to stop the process* $X^{(\mathrm{c})}$ *the first time it is below the function* $b_\mathrm{c} : [0, T] \to \mathbb{R}$ *such that* $b_\mathrm{c}(t) = 2A - b_\mathrm{p}(t)$*, where* b_p *is the OSB associated to* V_p.

2.5 Numerical experiments

Non-linear Volterra-type integral equations are hard to solve, and the free-boundary equation (2.46) is not an exception. Mainly, there are two methods to solve these types of integral equations arising from OSPs, both relying on the contraction principle: (1) a backward induction approach that recursively builds the boundary by leveraging that $b(t)$ depends only on $\{b(s)\}_{s\in[t,T]}$; and (2) the well-known method of Picard iterations that, given an initial candidate boundary b^0, consequently computes subsequent boundaries until a convergence criterion is fulfilled. See Detemple (2005, Chapter 8) for more details on the method of backward induction and Pedersen and Peskir (2002) for implementations. The Picard iteration scheme has been used by, e.g., Detemple and Kitapbayev (2020) and De Angelis and Milazzo (2020). As far as we know, the convergence of both approaches has not been formally addressed in general settings, and authors tend to provide numerical evidence of the error decreasing as the number of iterations grows. This happens because the kernel of the integral equation is typically highly non-linear and, hence, it becomes challenging to prove the contracting mapping principle for the integral operator that characterizes the OSB, which is the most obvious and immediate strategy to follow to get the convergence of Picard's iteration algorithm. We adopted the Picard method to display visual insight into the OSB's shape, as our numerical experiments suggested similar accuracy and faster computations compared to the backward induction approach (within the set of explored settings).

Take the partition $0 = t_0 < t_1 < \cdots < t_N = T$ for $N \in \mathbb{N}$. We initialize the Picard scheme by starting with the constant boundary $b^{(0)}(t) = b(T)$ for all $t \in [0, T]$. Recall from Proposition 2.1 that $b(T) = \min\{A, \gamma(T)\}$ is known, with $\gamma(T)$ as in (2.13). Each iteration of the algorithm updates the boundary according to the following formula, which is a right Riemann sum version of the integral in (2.46):

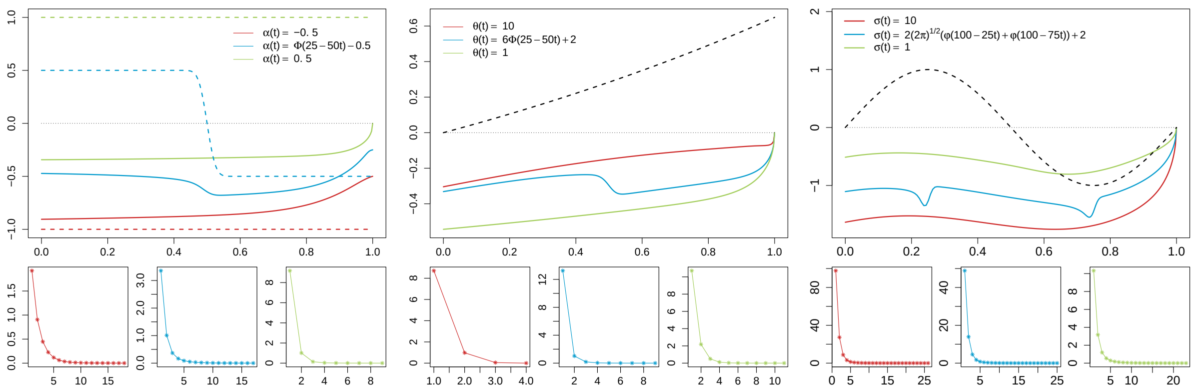
$$\begin{aligned} b^{(k)}(t_i) &= A - K_\lambda(A, 1, t_i, b^{(k-1)}(t_i), T, A) \\ &\quad - \sum_{j=i+1}^{N} K_\lambda(\lambda A + \theta(t_j)\alpha(t_j), \lambda + \theta(t_j), t_i, b^{(k-1)}(t_i), t_j, b^{(k-1)}(t_j)). \end{aligned} \tag{2.55}$$

We stop the iterations of the fixed-point algorithm (2.55) at the first $k = 1, 2, \ldots$ such that the (squared) distance $d_k := \sum_{i=0}^{N} \left(b^{(k)}(t_i) - b^{(k-1)}(t_i)\right)^2$ is less than $\delta = 10^{-3}$. Computational experiments suggested that a mesh that is denser at the terminal time T performs better and that it is preferable that the distances between consecutive nodes narrow smoothly. For that reason, we used the logarithmically-spaced partition $t_i = \ln(1 + i(e-1)/N)$ with $N = 200$ (unless otherwise specified).

For Figures 2.1–2.3, the images on top show computer drawings of the obtained OSB from the Picard iteration (2.55). The images below represent the sequences of errors of the algorithm, that is, the x-axis accounts for the iteration number $k = 1, 2, \ldots$ and the y-axis for d_k. The decrease in these error sequences empirically corroborates the convergence of the algorithm.

Figure 2.1 provides insights into the shape of the OSB for different sets of slopes and pulling levels. We also allow the process to have non-constant volatility, as discussed in Remark 2.1. In each image, two of the three functions that define the dynamics in (2.54) are fixed, and the remaining one varies to make clear the marginal effect it has on the OSB's shape. We use the functions Φ and ϕ to model abrupt changes in a smooth fashion. Figure 2.1(a) models a regime change of the stock price, where the switching happens smoothly, deterministically, and depends only on the time variable. It also reflects the attracting behavior of the OSB towards α. Figure 2.1(b) exemplifies how this attraction strengthens as θ increases, and also depicts a sudden increase of the pulling strength, which could represent a sharp increase of confidence in a belief of the price evolution. Figure 2.1(c) introduces periods where the volatility spikes to unusual levels before returning to a baseline. It also shows how the OSB is repelled from the pulling level as the volatility increases, which, in agreement with Remark 2.1, coincides with the effect of reducing the slope.

Recent interest in OSPs related to diffusion bridges whose drift is linear in the space component (see, e.g., D'Auria et al. (2020) and D'Auria et al. (2021)) led us to investigate the applicability of our OSB within this context. The difficulty of dealing with these processes is that their slopes θ explode as the time approaches the horizon, hence they do not fit our assumption of a bounded slope. However, we suggest in Figure 2.2 that this inconvenience could be



(a) $\theta(t) = 1$, $\sigma(t) = 1$. (b) $\sigma(t) = 1$, $\alpha(t) = e^{0.5t}$. (c) $\theta(t) = 1$, $\alpha(t) = \sin(2\pi t)$.

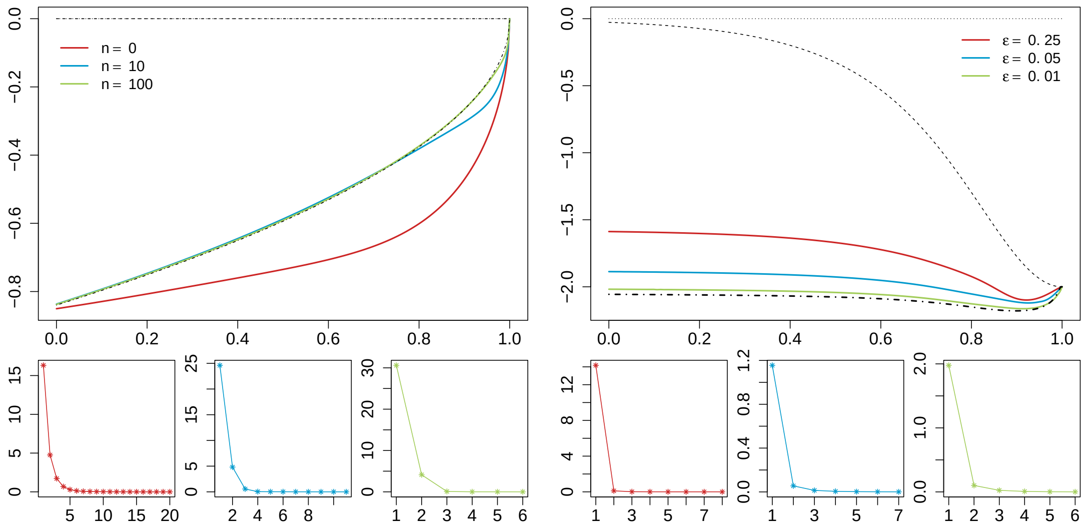
Figure 2.1: For the images on top, the solid colored lines represent the OSBs for the different choices of the α, θ, and σ functions shown in the legend. The dashed lines represent the pulling level α and the dotted line is placed at the strike price level A. Φ and ϕ represent the distribution and density functions of a standard normal. We set $T = 1$, $\lambda = 1$, and $A = 0$. The smaller images below provide the errors $\log_{10}(d_k)$ between consecutive boundaries for each iteration $k = 1, 2, \dots$ of (2.55).

circumvented in practice by arbitrarily approximating the explosion with non-exploding smooth drifts of time-dependent OU processes. We do so by relying on the OSB of a Brownian Bridge (BB) and an Ornstein–Uhlenbeck Bridge (OUB) in the non-discounted scenario, whose solutions are available for some particular settings.

The OSB for the BB with pinning point A and gain function $x \mapsto (A - x)^+$ takes the form $A - B\sigma\sqrt{T - t}$, for $B \approx 0.8399$, a result that goes back to the work of Shepp (1969) and its connection to the American option by D'Auria et al. (2020). We use, in Figure 2.2(a), the Taylor-expansion approximation (at $t = 0$) of the BB slope $(1 - t)^{-1}$, namely, $\sum_{i=0}^{n} t^i$, for different values of the order n indicated in the legend. The results in Figure 2.2(a) show that the application of the Picard iteration (2.55) on the time-dependent OU that order-n approximates a BB converges to the expected, known, OSB.

The OSB of the OUB is not known in closed-form, but rather as a characterization via the free-boundary equation provided in D'Auria et al. (2021), which can be also solved by means of a Picard algorithm. In that paper the authors treated the OUB case with the identity as the gain function. However, a simple translation with respect to A provides the OSB for $x \mapsto (A - x)^+$ as the gain function whenever the pinning point is lower than A. This is easy to note as both gain functions, $x \mapsto A - x$ and $x \mapsto (A - x)^+$ coincide for all $x \leq A$. Then, there is no fundamental difference in both OSPs (for $\lambda = 0$) whenever the boundary associated with the OUB lies below A. In Figure 2.2(b), to avoid the explosion of the OUB slope, $t \mapsto a\coth(a(1-t))$, we use instead the smooth function $t \mapsto \mathbb{1}(t \leq 1-\varepsilon)\{a\coth(a(1-t))\} + \mathbb{1}(t > 1-\varepsilon)\{1 - \exp\{-a^2(t-1+\varepsilon)/\sinh^2(a\varepsilon)\} + a\coth(a\varepsilon)\}$, for $a = 5$, and different values of ε specified in the legend. Figure 2.2(b) shows that the application of (2.55) on the time-dependent OU that approximates an OUB with pinning point $z = 2$ converges to the (numerically-computed) OSB of the OUB.

Finally, Figure 2.3 illustrates how the algorithm's output seems to converge as the partition size increases. Observe the relatively good performance with just a few points in the partition. Additionally, we show how changing the discounting rate affects the OSB's shape: larger



(a) $\alpha(t) = 0$, $\sigma(t) = 1$. (b) $\alpha(t) = 2\,\mathrm{sech}(5(1-t))$, $\sigma(t) = 1$.

Figure 2.2: For the images on top, the solid colored lines represent the OSBs for different n and ε (see main text). The dotted line is placed at the strike price level A, while the dashed line stands for the pulling level α, and the dotted-dashed lines are the OSBs of the BB (a) and the OUB (b). We set $A = 0$, $\lambda = 0$, and $T = 1$. The smaller images below are analogous to those in Figure 2.1.

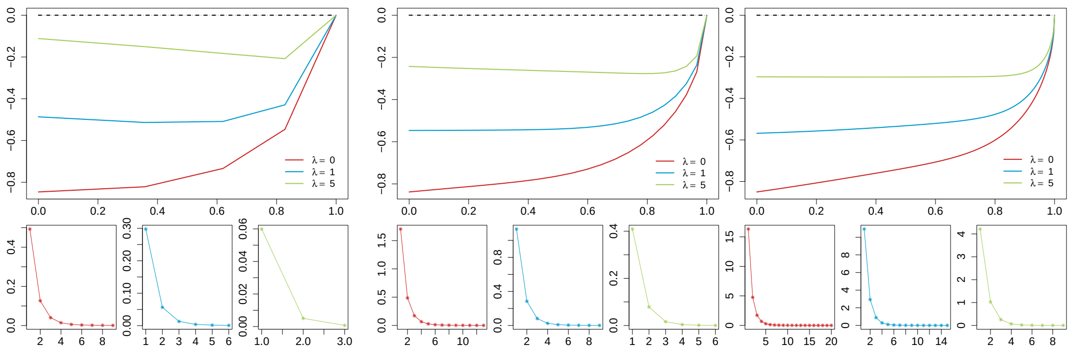
discounts decrease the separation of the OSB with respect to the strike price.

The repository `https://github.com/aguazz/AmOpTDOU` contains all the required R scripts to reproduce the numerical experiments.

2.6 Concluding remarks

If a time-dependent Ornstein–Uhlenbeck model fits well the price of the underlying asset, the strategy that maximizes the mean gain of a holder of an American put option on that asset is that of exercising once the price lies below the solution of the integral equation in Theorem 2.1. This integral equation can be solved numerically using the algorithm from Section 2.5.

We rely on a probabilistic methodology similar to that used in Peskir (2005b). That is, we obtained enough regularity conditions on the value function and on the optimal stopping boundary to apply an extension of the Itô formula and come up with the free-boundary (Volterra integral) equation. Contrary to Peskir (2005b) and many papers that followed the same approach (see, e.g., Peskir and Uys (2003); Peskir (2005c); Glover et al. (2010, 2011); Kitapbayev (2014); De Angelis and Milazzo (2020)), our optimal stopping boundary is not necessarily monotonic (see Figure 2.1), which makes more complicated the derivation of certain properties, especially the smooth-fit condition. To overcome this handicap, we obtained the Lipschitz continuity of the boundary by adapting the work of De Angelis and Stabile (2019) to fit our non-differentiable gain function and relax other restrictive assumptions. We then proved that Lipschitz continuity suffices to obtain that the underlying process enters the stopping set immediately after starting on the optimal stopping boundary and, then, by relying on the work of De Angelis and Peskir (2020), we derived the smooth-fit condition. It is worth highlighting the comparison method used to get the lower bound of the optimal boundary in Proposition 2.1, as it seems to be extensible to other settings.



(a) $N = 5$. (b) $N = 20$. (c) $N = 200$.

Figure 2.3: For images on top, the solid colored lines represent the OSBs for different values of the discounting rate specified in the legend, while the dashed line is placed at $A = 0$. We set $\theta(t) = 1$, $\alpha(t) = 0$, $\sigma(t) = 1$, $A = 0$, and $T = 1$. The smaller images below are analogous to those in Figure 2.1.

Numerical experiments performed in Section 2.5 revealed a wide flexibility in the shape of the optimal boundary generated by changing the form of the time-dependent parameters. It also showed that the errors produced by the Picard iteration algorithm decrease, suggesting that the fixed-point operator in the free-boundary equation (2.46) could be a contracting map. Figure 2.2 suggests that the limiting boundedness assumption on the process's coefficients (they are continuous on $[0, T]$) can be escaped to approximate the optimal boundary of a Brownian bridge and an Ornstein–Uhlenbeck bridge, and, potentially, other diffusion bridges whose drift is linear in the space component.

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Chapter 3

Discounted optimal stopping of a Brownian bridge, with application to American options under pinning

Abstract

Mathematically, the execution of an American-style financial derivative is commonly reduced to solving an optimal stopping problem. Breaking the general assumption that the knowledge of the holder is restricted to the price history of the underlying asset, we allow for the disclosure of future information about the terminal price of the asset by modeling it as a Brownian bridge. This model may be used under special market conditions. In particular, we focus on what in the literature is known as the "pinning effect", that is, when the price of the asset approaches the strike price of a highly-traded option close to its expiration date. Our main mathematical contribution is in characterizing the solution to the optimal stopping problem when the gain function includes the discount factor. We show how to numerically compute the solution and we analyze the effect of the volatility estimation on the strategy by computing the confidence curves around the optimal stopping boundary. Finally, we compare our method with the optimal exercise time of a geometric Brownian motion by using real data exhibiting pinning.

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D'Auria, B., García-Portugués, E., and Guada, A. Discounted optimal stopping of a Brownian bridge, with application to American options under pinning. *Mathematics* 2020, 8(7), 1159, https://doi.org/10.3390/math8071159

Contents

3.1 Introduction . . . 56

3.2 Problem setting . . . 58

3.3 Optimally exercising American put options for a Brownian bridge . 59

3.4 Boundary computation and inference . . . 62

3.4.1 Solving the free-boundary equation . . . 62

3.4.2 Estimating the volatility . . . 63

3.4.3 Confidence intervals for the boundary . . . 65

3.4.4 Simulations . . . 66

3.5 Pinning-at-the-strike and real-data study 68
3.6 Non-monotonicity of the OSB 72
3.7 Conclusions 73
3.A Main proofs 74
3.B Auxiliary lemmas 82
References 85

3.1 Introduction

American options are a special type of vanilla option that can be considered among the most basic financial derivatives. By allowing for the possibility to exercise at any time before the day of expiration, they add a new dimension to their valuation that escapes the consolidated hedging and arbitrage-free pricing frameworks. The methodology to value an American option can be traced back to McKean (1965), which suggested transforming this problem into a free-boundary problem. However, even in the simplest case first proposed by Samuelson (1965), where the underlying stock is modeled by a geometric Brownian motion, it took almost 40 years to reach a complete and rigorous derivation of its solution. This was given in Peskir (2005b), where it was finally proved that the free-boundary equation characterizes the optimal stopping boundary. A good historical survey on the topic can be found in Myneni (1992).

The literature on the valuation of American options is considerable, and there have been many attempts to extend the classes of stochastic processes that could model the dynamics of the underlying stock. However, sometimes this has been achieved at the cost of reducing the completeness of the result. For example, Detemple and Tian (2002) treats more general diffusion processes and proves that the optimal strategy satisfies the free-boundary equation, but it leaves open the proof of the uniqueness of the solution. The more recent work Zhao and Wong (2012) provides a closed-form expression of the optimal stopping boundary for a fairly general class of diffusion processes by expressing it in terms of Maclaurin series. However, as in Detemple and Tian (2002), it requires a boundedness assumption on the derivative of the drift term that excludes the class of Gaussian bridges. This class of processes has recently attracted attention to model situations where some future knowledge about the dynamics of the underlying assets is disclosed to the trading agent, see, e.g., Pikovsky and Karatzas (1996), Schweizer et al. (2003), Biagini and Øksendal (2005), and D'Auria and Salmerón (2020). However, these results focus mainly on quantifying the value of the disclosed information and do not address its effect on the execution strategy of a held option.

In the area of option pricing, a first analysis of the Brownian bridge was done in Shepp (1969) by exploiting a time transformation that converts the problem into one about the more tractable Brownian motion. Later, the applicability of this work in the problem of optimally selling a bond was highlighted in Boyce (1970). Successively, in Ekström and Wanntorp (2009), the authors take up the problem in Shepp (1969) by reframing it into the wider context of the free-boundary problem and extend its solution to a wider class of gain functions. In particular, in Ekström and Wanntorp (2009), the Brownian bridge process is presented as a possible model for financial applications under special market conditions, such as the so-called "pinning effect".

The pinning effect refers to the situation in which the price of a given stock approaches the strike price of a highly-traded option close to its expiration date. Evidence for the pinning effect was first reported in Krishnan and Nelken (2001), where the authors employ a bridge process to model stock pinning by tuning a geometric Brownian motion. In Avellaneda and Lipkin (2003), the authors postulate that the pinning behavior is mainly driven by delta-hedging of long option positions and consider, as a model for the stock price, a stochastic differential equation with a drift that pulls the price toward a neighborhood of the strike price. Later, the results in Avellaneda et al. (2012) add real data evidence in support of this model. In Ni et al. (2005), the same assumption is validated and a comprehensive set of evidence for the pinning phenomenon is reported. The model of Avellaneda and Lipkin (2003) is generalized in Jeannin et al. (2008) by adding a diffusion term that shrinks the volatility near the strike price.

Motivated by these findings, we study in this paper the best strategy for executing an American put option in the presence of stock pinning. Similarly to Ekström and Wanntorp (2009), we model the underlying stock by a Brownian bridge. Differently, we include a discount factor that makes the problem more realistic, yet significantly more challenging, as the optimal stopping boundary is no longer monotonic, making it more difficult to obtain the smooth-fit condition. To bypass this added challenge, we obtain the local Lipschitz continuity of the boundary and prove that it suffices for the smooth-fit condition to hold. We solve the optimal stopping problem, in the spirit of Peskir (2005b) and De Angelis and Milazzo (2020), by characterizing the optimal stopping boundary as the unique solution of a Volterra integral equation, up to some regularity conditions.

Besides contributing to the solution to this original problem, we also explore its applicability in real situations. The studied model may be too simple to be applied with real data, but it allows computing exact solutions and easily quantifying the uncertainty about the knowledge of its parameters. For this reason, we describe an algorithm to numerically compute the optimal strategy and inferential method that provide the confidence curves around the optimal stopping boundary when the stock volatility is estimated via maximum likelihood. This inferential method is potentially relevant for an investor that only has access to discrete data. We test our results on a real dataset comprised by financial options on Apple and IBM equities. Our model is competitive when compared with a model based on a geometric Brownian motion and, in accordance with the motivation of our work, the best performance is obtained when the stock price exhibits a pinning-at-the-strike behavior.

Finally, we support with numerical evidence the conjecture that the boundary is not monotone and has at most one change of monotonicity, and compute the time at which this change happens as a function of the discounting rate. This numerical study might be convenient as a starting point to obtain the piecewise monotonicity of the boundary, which, along with its continuity, is known to produce the smooth-fit condition (see Example 7 from De Angelis and Peskir (2020) and Corollary 8 from Cox and Peskir (2015)), hence offering an alternative way to get this property rather than the one based on the Lipschitz continuity of the boundary.

We conclude by mentioning related works using similar models. In Föllmer (1972), the authors tackle the non-discounted problem for a Brownian bridge with a normally-distributed ending point. More recently, Ekström and Vaicenavicius (2020) solves the same problem for small values of the variance, finding bounds for the value function when the pinning point follows a general distribution with a finite first moment. A double-stopping problem, for which the aim is to maximize the mean difference between two stopping times, is analyzed in Baurdoux et al. (2015). The recent paper De Angelis and Milazzo (2020) solves the non-discounted problem using the exponential of a Brownian bridge to model the stock prices. A Brownian bridge with unknown pinning random distribution and a Bayesian approach is advocated by Glover (2020). The analytical results in Ekström and Wanntorp (2009) are extended in D'Auria and Ferriero (2020) by looking at a class of Gaussian bridges that share the same optimal stopping boundary. The discounted problem of a Brownian bridge with a random pinning point is addressed in Leung et al. (2018), under regularity assumptions on the gain function that allow for an application of the standard Itô's formula (something that does not hold in our setting).

The rest of the paper is structured as follows. In Section 3.2, we introduce the model along with notations and definitions. Section 3.3 provides the theoretical results required to obtain the free-boundary equation. Section 3.4 deals with the problem of computing the optimal stopping boundary and quantifying the uncertainty associated with the estimation of the stock volatility. Section 3.5 compares our method with the optimal exercise time based on a geometric Brownian motion by using real data exhibiting various degrees of pinning. Section 3.6 comments on the non-monotonicity of the optimal stopping boundary and, finally, Section 3.7 offers some conclusions.

Proofs and technical lemmas required to back up Section 3.3 are relegated to Appendix 3.A and 3.B, respectively.

3.2 Problem setting

We next introduce the model of the financial asset and we define the optimal stopping problem whose solution constitutes the best strategy to exercise an American put option based on that asset.

We assume that the financial option has a strike price $S > 0$ and a maturity date $T > 0$. To model the pinning effect, we use a Brownian bridge process for the dynamics of the underlying asset. Indeed, this process may be seen as a Brownian motion conditioned to terminate at a known terminal value that in our case is fixed to the strike price S (see Remark 3.3 for a discussion on the relaxation of this assumption). That is, by calling $X^{[t,T]} = (X_{t+s}, 0 \leq s \leq T-t)$ to the asset price process, with $0 \leq t < T$, we assume it satisfies the SDE:

$$X_t = x, \qquad \mathrm{d}X_{t+s} = \frac{S - X_{t+s}}{T-t-s}\,\mathrm{d}s + \sigma\,\mathrm{d}W_s, \tag{3.1}$$

with $0 \leq s \leq T-t$ or, equivalently, it has the explicit expression

$$X_{t+s} = x\frac{T-t-s}{T-t} + S\frac{s}{T-t} + \sigma(T-t-s)\int_0^s \frac{\mathrm{d}W_u}{T-t-u}, \tag{3.2}$$

again with $0 \leq s \leq T-t$ and where, in both equations, $(W_s, 0 \leq s \leq T-t)$ denotes a standard Brownian motion. To emphasize that the process almost surely satisfies the relation $X_t = x$, we will use the notation $\mathbb{P}_{t,x}$ and $\mathbb{E}_{t,x}$ to denote the corresponding probability and mean operators.

Denoting by $G(x) = (S-x)^+$ the gain function of the put option and by $\lambda \geq 0$ the discounting rate, we can finally write the optimal expected reward for exercising the American option as the *Optimal Stopping Problem* (OSP)

$$V(t,x) = \sup_{0\leq\tau\leq\ T-t} \mathbb{E}_{t,x}\left[e^{-\lambda\tau}G(X_{t+\tau})\right]. \tag{3.3}$$

The function V is called the *value function* and the supreme above is taken over all the stopping times τ of $X^{[t,T]}$ with respect to its natural filtration $(\mathcal{F}_s)_{s=0}^T$.

Under mild conditions, namely V being lower semi-continuous and G upper semi-continuous (see Corollary 2.9 by Peskir and Shiryaev (2006)), it is guaranteed that the supremum in (3.3) is achieved. The *Optimal Stopping Time* (OST), $\tau^*(t,x)$, is defined as the smallest stopping time attaining the supremum (3.3) and can be characterized as the hitting time of a closed set D, referred to as the *stopping set*. Since these conditions on V and G are satisfied in our settings (see Remark 2.10 in Peskir and Shiryaev (2006)), we can write

$$\tau^*(t,x) := \inf\{0 \le s \le T-t : X_{t+s} \in D \mid X_t = x\}, \tag{3.4}$$

where D is defined as

$$D := \{(t,x) \in [0,T] \times \mathbb{R} : V(t,x) = G(x)\}. \tag{3.5}$$

We then define the *continuation set* C as the complement of the set D and we denote by ∂C its boundary.

The OST, defined in (3.4), can be interpreted as the best exercise strategy for the American option, and it allows for rewriting the value function V in the simplified form

$$V(t,x) = \mathbb{E}_{t,x}\left[e^{-\lambda\tau^*(t,x)} G\left(X_{t+\tau^*(t,x)}\right)\right]. \tag{3.6}$$

To solve the OSP given in (3.3), we follow the well-known approach of reformulating it as a free-boundary problem for the unknowns V and ∂C. The latter is commonly called the *Optimal Stopping Boundary* (OSB). Our OSP is a finite-horizon problem that involves a time-non-homogeneous process, with its associated free-boundary problem being

$$\partial_t V + \mathbb{L}_X V = \lambda V \qquad \text{on } C, \tag{3.7a}$$

$$V > G \qquad \text{on } C, \tag{3.7b}$$

$$V = G \qquad \text{on } D, \tag{3.7c}$$

$$\partial_x V = \partial_x G \qquad \text{on } \partial C, \tag{3.7d}$$

where $\mathbb{L}_X$ is the infinitesimal generator of the Brownian bridge $X^{[0,T]}$. Given a suitably smooth function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, the application of the operator $\mathbb{L}_X$ to it returns the function

$$\left(\mathbb{L}_X f\right)(t,x) = \frac{S-x}{T-t}\partial_x f(t,x) + \frac{\sigma^2}{2}\partial_{x^2} f(t,x). \tag{3.8}$$

Equations (3.7a), (3.7b), and (3.7c) easily come from the definitions of D, C, and $\tau^*(t,x)$ (see Proposition 3.2 below), whereas (3.7d), generally known as the *smooth-fit condition*, depends on how well-behaved the OSB is for the underlying process. The regularity of the OSB is an important factor in finding and characterizing the solution of the problem itself, and for this reason we will study it in detail in later sections. An in-depth survey on the optimal stopping theory that exploits the free-boundary approach can be found in Peskir and Shiryaev (2006).

3.3 Optimally exercising American put options for a Brownian bridge

We present in this section the main result, consisting of the solution of the problem (3.3). In particular, we solve the free-boundary problem defined in (3.7) by showing that the OSB can be written in terms of a function b such that $\partial C = \{(t, b(t)) : t \in [0,T]\}$, and that this function can be computed as the solution to a Volterra integral equation.

From an application perspective, the function b defines the optimal strategy to follow in order to maximize the profit from the execution of the American put option. It is best to exercise the option the first time the price of the underlying financial asset crosses at time $t \in [0,T]$ the level $b(t)$.

Theorem 3.1. *The optimal stopping time in* (3.4) *can be written as*

$$\tau(t,x) = \inf\{s \in [0,T-t] : X_{t+s} \leq b(t+s)\}, \quad x \in \mathbb{R},\ 0 \leq t \leq T,$$

where the function b is defined as the unique solution, among the class of continuous functions of bounded variation lying below S, of the integral equation

$$b(t) = S - \int_t^T K_{\sigma,\lambda}(t, b(t), u, b(u))\,\mathrm{d}u. \tag{3.9}$$

In addition, the value function V in (3.6) *can be expressed as*

$$V(t,x) = \int_t^T K_{\sigma,\lambda}(t, x, u, b(u))\,\mathrm{d}u. \tag{3.10}$$

The kernel $K_{\sigma,\lambda}$ in (3.9) *and* (3.10) *is defined as*

$$\begin{aligned} K_{\sigma,\lambda}(t,x_1,u,x_2) := e^{-\lambda(u-t)}\frac{1+\lambda(T-u)}{T-u}\Big[&(S-\mu(t,x_1,u))\Phi(z_\sigma(t,x_1,u,x_2)) \\ &+ \nu_\sigma(t,u)\phi(z_\sigma(t,x_1,u,x_2))\Big], \end{aligned} \tag{3.11}$$

where Φ and ϕ are, respectively, the distribution and density functions of a standard normal random variable,

$$\mu(t,x,u) := x\frac{T-u}{T-t} + S\frac{u-t}{T-t}, \tag{3.12}$$

$$\nu_\sigma(t,u) := \sigma\sqrt{\frac{(u-t)(T-u)}{T-t}}, \tag{3.13}$$

and $z_\sigma(t,x_1,u,x_2) := (x_2 - \mu(t,x_1,u))/\nu_\sigma(t,u)$.

The proof of the theorem makes use of some important partial results that we state in the following propositions. All of the proofs are deferred to Appendix 3.A.

The next result sheds some light on the shapes of the sets D and C by showing that their common border can be expressed by means of a function b that satisfies some regularity conditions. In the proof, we focus on regions where it is easy to prove that the value function either exceeds or equals the immediate reward, thus revealing subsets of C and D, respectively. These regions come from the facts that G is null above S and positive below, that the paths of the process decrease with x (for a fixed realization), and that V is non-increasing with respect to x, t, and λ.

Proposition 3.1. *There exists a function $b : [0,T] \to \mathbb{R}$ such that $b(t) < S$ for all $t \in [0,T)$, $b(T) = S$, and $D = \{(t,x) \in [0,T] \times \mathbb{R} : x \leq b(t)\}$.*

The following proposition analyzes some regularity properties of the value function. It exploits the regularity properties of the function b proved in Proposition 3.1. We later use these results in Proposition 3.3 to show the local Lipschitz continuity of b. Part *(i)* comes from standard arguments on parabolic partial differential equations in conjunction with the Markovian property of the Brownian bridge. The rest of the proposition employs different methods, but they all rely on the fact that the OST for a pair (t,x) is sub-optimal under different initial conditions.

Proposition 3.2. *The value function V defined in (3.3) satisfies the following conditions:*

(i) V *is* $\mathcal{C}^{1,2}$ *on* C *and on* D, *and* $\partial_t V + \mathbb{L}_X V = \lambda V$ *on* C.

(ii) $x \mapsto V(t,x)$ *is convex and strictly decreasing for all* $t \in [0,T]$. *Moreover,*

$$\partial_x V(t,x) = -\mathbb{E}\left[e^{-\lambda\tau^*(t,x)}\frac{T-t-\tau^*(t,x)}{T-t}\right]. \tag{3.14}$$

(iii) V *is continuous.*

From the previous proposition we are able to get a stronger result on the smoothness of b, namely its Lipschitz continuity on any interval that excludes the horizon. The proof is an adaptation of Proposition 6 from D'Auria et al. (2021), and it relies on the same Brownian motion representation that was originally used by Shepp (1969) to solve the OSP of a BB without discount. This higher smoothness of b allows obtaining the smooth-fit condition and, consequently, the uniqueness of the solution of the OSP.

Proposition 3.3. *The optimal stopping boundary b for the problem (3.3) is Lipschitz continuous on any closed interval* $I \subset [0,T)$.

Local Lipschitz continuity of the OSB suffices for the smooth-fit condition to hold true, as we show in the following proposition.

Proposition 3.4. *The smooth-fit condition holds, i.e.,* $\partial_x V(t,b(t)) = -1$ *for all* $t \in [0,T]$.

Finally, the next proposition shows that the OSB satisfies the Volterra integral equation (3.9), and that it is the only solution up to some regularity conditions. The proof follows well-known procedures based on probabilistic arguments (seePeskir (2005b)) rather than relying on integral equation theory, which usually uses some variation of the contraction mapping principle.

Proposition 3.5. *The optimal stopping boundary b for the problem (3.3) can be characterized as the* unique *solution of the second-type nonlinear Volterra integral equation (3.9), within the class of continuous functions of bounded variation* $c : [0,T] \to \mathbb{R}$ *such that* $c(t) < S$ *for all* $t \in (0,T)$.

Remark 3.1. *All the results in this section have their own analog when it comes to optimally exercising American* call *options, that is, when the gain function in (3.3) is substituted with* $G(x) = (x-S)^+$. *Indeed, exploiting the symmetry of the Brownian bridge and the gain functions, it is easy to check that the relation* $b_c(t) = 2S - b_p(t)$ *holds, where* b_c *and* b_p *stand for the OSBs for the call and the put option, respectively.*

Remark 3.2. *The OSB for* $\lambda = 0$*, that is, the one that maximizes the mean of a Brownian bridge, can be obtained from the early work of Shepp (1969); see also Ekström and Wanntorp (2009) and Ernst and Shepp (2015). Its explicit expression is* $b_0(t) = S - \sigma B\sqrt{T-t}$*, with* $B \approx 0.8399$.

Remark 3.3. *We have worked under the assumption that the pinning point of the BB equals the strike price of the American option. Relaxing such an assumption, however, does not seem to bring up fundamental extra challenges, and the same methodology could be used to obtain the new free-boundary equation, which, in such a case, should take the form*

$$b(t) = S - e^{-\lambda(T-t)}(S-A)^+ - \int_t^T K_{\sigma,\lambda}(t,b(t),u,b(u))\,\mathrm{d}u, \tag{3.15}$$

where A and S are the pinning point and the strike price, respectively, and with

$$K_{\sigma,\lambda}(t,x_1,u,x_2) := e^{-\lambda(u-t)}\left[\left(\frac{S}{T-u}+\lambda A-\left(\frac{1}{T-u}+\lambda\right)\mu(t,x_1,u)\right)\Phi(z_\sigma(t,x_1,u,x_2)) \right. \\ \left. + \nu_\sigma(t,u)\phi(z_\sigma(t,x_1,u,x_2))\right].$$

3.4 Boundary computation and inference

3.4.1 Solving the free-boundary equation

The lack of an explicit solution for (3.9) requires a numerical approach to compute the OSB. Let $(t_i)_{i=0}^N$ be a grid in the interval $[0,T]$ for some $N \in \mathbb{N}$. The method we consider builds on a proposal by Pedersen and Peskir (2002). They suggested approximating the integral in (3.9) by a right Riemann sum, hence enabling the computation of the value of $b(t_i)$, for $i = 0,\dots,N-1$, by using only the values $b(t_j)$, with $j = i+1,\dots,N$. Therefore, by knowing the value of the boundary at the last point ($b(t_N) = b(T) = S$), one can obtain its value at the second last point $b(t_{N-1})$ and recursively construct the whole OSB evaluated at $(t_i)_{i=0}^N$.

Under our settings, the right Riemann sum is no longer a valid option because we know from (3.11) that, depending on the shape of the boundary b near the expiration date, $K(t_i, b(t_i), u, b(u))$ could explode as $u \to T$, so we cannot evaluate the kernel K at the right point in the last subinterval $(t_{N-1}, T]$. To deal with this issue, we employ a right Riemann sum approximation along all the subintervals except the last one, ending up with the following discrete version of the Volterra integral equation (3.9):

$$b(t_i) \approx S - \sum_{j=i+1}^{N-1}(t_j - t_j)K_{\sigma,\lambda}(t_i, b(t_i), t_j, b(t_j)) - I(t_i, t_{N-1}), \tag{3.16}$$

for $i = 0,1,\dots,N-1$, where $I(t_i,t_{N-1}) := \int_{t_{N-1}}^T K_\sigma(t_i, b(t_i), u, b(u))\,\mathrm{d}u$. It can be shown that $0 \le I(t_i,t_{N-1}) \le H(t_i,t_{N-1})$, where

$$\begin{aligned} H(t_i,t_{N-1}) &:= e^{-\lambda(t_{N-1}-t_i)}\int_{t_{N-1}}^T (1+\lambda(T-u))\left(\frac{S-b(t_i)}{T-t_i}+\sigma\sqrt{\frac{1}{2\pi(T-u)}}\right)\mathrm{d}u \\ &= e^{-\lambda(t_{N-1}-t_i)}\left((S-b(t_i))\frac{T-t_{N-1}}{T-t_i}\left(1+\frac{\lambda}{2}(T-t_{N-1})\right)+\right. \\ &\qquad\left.\sigma\sqrt{\frac{2(T-t_{N-1})}{\pi}}\left(1+\frac{\lambda}{3}(T-t_{N-1})\right)\right), \end{aligned}$$

by using (3.12) and (3.13), the form of the kernel (3.11), and the fact that $\Phi(x) \le 1$ and $\phi(x) \le (2\pi)^{-1/2}$ for all $x \in \mathbb{R}$. Therefore, $I(t_i,t_{N-1}) \approx H(t_i,t_{N-1})/2$ can be seen as a reasonable approximation, admitting an upper bound for the error $\varepsilon(t_i,t_{N-1}) := |H(t_i,t_{N-1})/2 - I(t_i,t_{N-1})|$, namely $\varepsilon(t_i,t_{N-1}) \le H(t_i,t_{N-1})/2$. Moreover, $H(t_i,t_{N-1}) = \mathcal{O}(\sqrt{T-t_{N-1}})$ as $t_{N-1} \to T$. After substituting $I(t_i,t_{N-1})$ for $H(t_i,t_{N-1})/2$ in (3.16), we get

$$\begin{aligned} b(t_{N-1}) \approx &\left(\frac{1}{2}-\frac{\lambda}{4}(T-t_{N-1})\right)^{-1} \\ &\times\left(\frac{S}{2}\left(1-\frac{\lambda}{2}(T-t_{N-1})\right)-\sigma\sqrt{\frac{T-t_{N-1}}{2\pi}}\left(1+\frac{\lambda}{3}(T-t_{N-1})\right)\right), \end{aligned} \tag{3.17}$$

$$\begin{aligned} b(t_i) \approx & \left(1 - \frac{1}{2} e^{-\lambda(t_{N-1} - t_i)} \left(1 + \frac{\lambda}{2}(T - t_{N-1})\right) \frac{T - t_{N-1}}{T - t_i}\right)^{-1} \\ & \times \left(S - \sum_{j=i+1}^{N-1} (t_j - t_j) K_{\sigma,\lambda}(t_i, b(t_i), t_j, b(t_j)) \right. \\ & \quad - \frac{1}{2} e^{-\lambda(t_{N-1} - t_i)} \left(S \frac{T - t_{N-1}}{T - t_i} \left(1 + \frac{\lambda}{2}(T - t_{N-1})\right) \right. \\ & \quad \left.\left. + \sigma \sqrt{\frac{2(T - t_{N-1})}{\pi}} \left(1 + \frac{\lambda}{3}(T - t_{N-1})\right)\right)\right). \end{aligned} \tag{3.18}$$

The procedure for computing the estimated boundary according to the previous approximations is laid down in Algorithm 3.1. From now on, we will use $\tilde{b}$ to denote the cubic-spline interpolating curve that goes through the numerical approximation of the boundary at the points $(t_i)_{i=0}^N$ via Algorithm 3.1.

Algorithm 3.1: Optimal stopping boundary computation

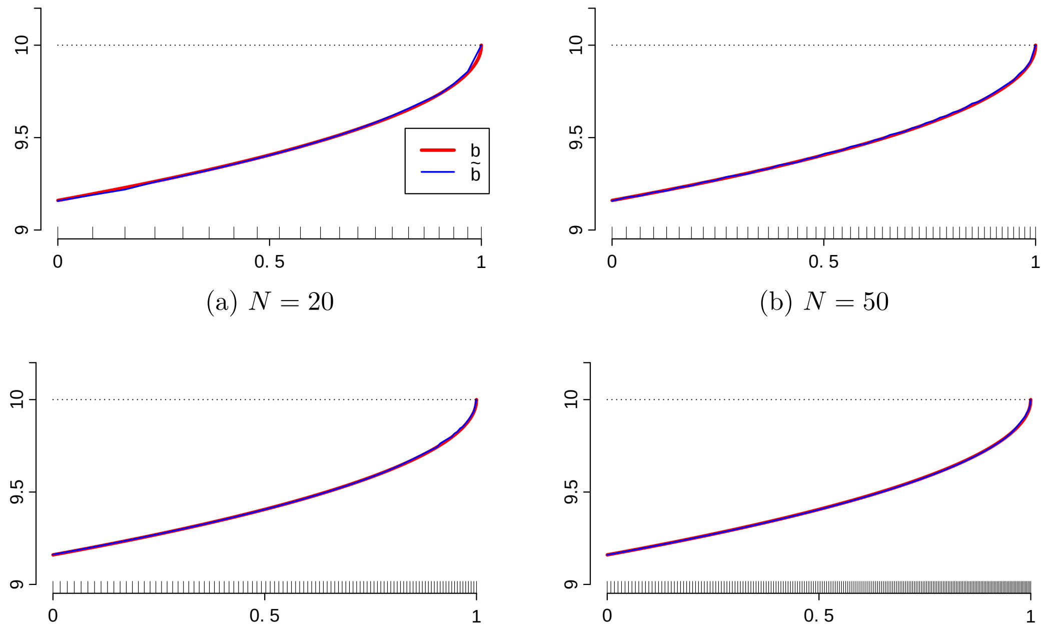
```
Input:  S, λ, (t_i)_{i=0}^N, δ
Output: (b̃(t_i))_{i=0}^N
Code:
b̃(T) ← S
Update b̃(t_{N−1}) according to (3.17)
for i = N − 2 to 0 do
    b̃(t_i) ← b̃(t_{i+1})
    ε ← 1
    while ε > δ do
        b̃_old(t_i) ← b̃(t_i)
        Update b̃(t_i) according to (3.18)
        ε ← |b̃_old(t_i) − b̃(t_i)|/|b̃_old(t_i)|
    end
end
```

Recall from Remark 3.2 that our OSB for $\lambda = 0$ takes the form $b_0(t) = S - B\sigma\sqrt{T - t}$. Having the explicit form of b_0 allows us to validate the accuracy of Algorithm 3.1 and tune its parameters. We empirically determined that $\delta = 10^{-3}$ offers a good trade-off between accuracy and computational time. This value was considered every time Algorithm 3.1 was employed.

We decided to use a logarithmically-spaced grid that is $t_i = \log(1 + \frac{i}{N}(e^T - 1))$, $i = 0, \ldots, N$, with $N = 200$, after systematically observing that uniform partitions tend to misbehave near the expiration date T. In addition, it is preferable that the partition gets thinner close to T in a smooth way. Figure 3.1 shows how precise the Algorithm 3.1 is by comparing the computed boundary $\tilde{b}_0$ versus its explicit form for $S = 10$, $T = 1$, $\lambda = 0$, and $\sigma = 1$.

3.4.2 Estimating the volatility

We assume next that the volatility of the underlying process is unknown, as it may occur in real situations. It is well known that, under model (3.1), one can exactly compute the volatility if the price dynamics are continuously observed. However, investors in real life have to deal with



(a) $N = 20$ (b) $N = 50$

(c) $N = 100$ (d) $N = 200$

Figure 3.1: Boundary estimation via Algorithm 3.1 for different partition sizes and for the parameters $S = 10$, $T = 1$, $\lambda = 0$, and $\sigma = 1$.

discrete-time observations and thus they would have to estimate σ to obtain the OSB.

We start by assuming that we have recorded the price values at the times $t_0 = 0 < t_1 < \cdots < t_{N-1} < t_N = T$, for $N \in \mathbb{N}$, so at t_n, with $n \in \{0, 1, \ldots, N\}$, we have gathered a sample $(X_{t_i})_{i=0}^n$ from the historical path of the Brownian bridge $(X_t)_{t=0}^T$ with $X_T = S$. From (3.2), we have that

$$X_{t_i} \mid X_{t_{i-1}} \sim \mathcal{N}\left(\mu(t_{i-1}, X_{t_{i-1}}, t_i), \nu_\sigma^2(t_{i-1}, t_i)\right), \quad i = 1, \ldots, n,$$

and the log-likelihood function of the volatility takes the form

$$\ell(\sigma \mid (t_i, X_{t_i})_{i=0}^n) = C - n\log(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^n \left(\frac{X_{t_i} - \mu(t_{i-1}, X_{t_{i-1}}, t_i)}{\nu_1(t_{i-1}, t_i)}\right)^2,$$

where C is a constant independent of σ. The maximum likelihood estimator for σ is given by

$$\widehat{\sigma}_n = \sqrt{\frac{1}{n}\sum_{i=1}^n \left(\frac{X_{t_i} - \mu(t_{i-1}, X_{t_{i-1}}, t_i)}{\nu_1(t_{i-1}, t_i)}\right)^2}.$$

Under an equally spaced partition ($t_i = i\frac{T}{N}$, $i = 0, 1, \ldots, N$), standard results on maximum likelihood (see Dacunha-Castelle and Florens-Zmirou (1986)) give that

$$\sqrt{n}(\widehat{\sigma}_n - \sigma) \rightsquigarrow \mathcal{N}\left(0, \frac{\sigma^2}{2}\right),$$

when $n \to \infty$ (hence $N \to \infty$) and $T \to \infty$ such that $t_i - t_{i-1} = T/N$ remains constant, with $i = 1, \ldots, N$.

3.4.3 Confidence intervals for the boundary

We present as follows the uncertainty propagated by the estimation of σ to the computation of the OSB. In order to do so, we assume that the OSB is differentiable with respect to σ, so we are allowed to apply the delta method, under the previous asymptotic conditions. This entails that

$$\sqrt{n}(b_{\widehat{\sigma}_n}(t) - b_\sigma(t)) \rightsquigarrow \mathcal{N}\left(0, \left(\frac{\partial b_\sigma}{\partial \sigma}(\sigma, t)\right)^2 \frac{\sigma^2}{2}\right), \tag{3.19}$$

where b_σ represents the OSB defined at (3.9) associated with a process with volatility σ. Plugging the estimate $\widehat{\sigma}_n$ into (3.19) gives the following asymptotic $100(1-\alpha)\%$ (pointwise) confidence curves for b_σ:

$$\left(c_{1,\widehat{\sigma}_n}(t),\ c_{2,\widehat{\sigma}_n}(t)\right) := \left(b_{\widehat{\sigma}_n}(t) \pm \frac{z_{\alpha/2}}{\widehat{\sigma}_n\sqrt{n/2}} \left|\frac{\partial b_\sigma}{\partial \sigma}(t)\Big|_{\sigma=\widehat{\sigma}_n}\right|\right), \tag{3.20}$$

where $z_{\alpha/2}$ represents the $\alpha/2$-upper quantile of a standard normal distribution. Algorithm 3.1 can be used to compute an approximation of the term $\frac{\partial b_\sigma}{\partial \sigma}(\cdot)$ by means of $(b_{\widehat{\sigma}_n+\varepsilon}(\cdot) - b_{\widehat{\sigma}_n}(\cdot))/\varepsilon$ for some small $\varepsilon > 0$. We denote by $\left(\tilde{c}_{1,\widehat{\sigma}_n}(t),\ \tilde{c}_{2,\widehat{\sigma}_n}(t)\right)$ the approximation of the confidence interval (3.20) coming from this approach at $t \in [0, T]$. Through the paper, we use $\varepsilon = 10^{-2}$, as has been empirically checked to provide, along with $\delta = 10^{-3}$ for Algorithm 3.1, a good compromise between accuracy, stability, and computational speed in calculating the confidence curves. Figure 3.2 illustrates, for one path of a Brownian bridge, how the boundary estimation and its confidence curves work. Figure 3.3 empirically validates the approximation of the confidence curves by marginally computing for each t_n the proportion of trials, out of $M = 1000$, in which the true boundary does not belong to the interval delimited by the confidence curves.

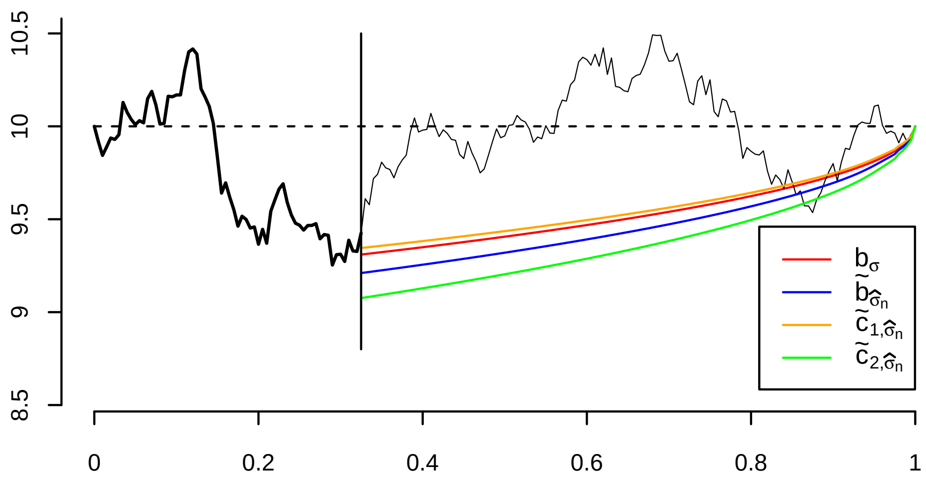
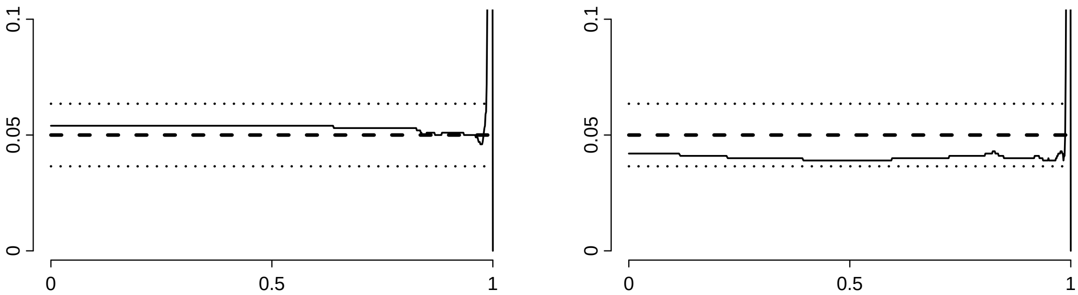


Figure 3.2: Inferring the boundary using one third ($n = 66$, $N = 200$) of the Brownian bridge path, for $T = 1$, $S = 10$, $X_0 = 10$, $\lambda = 0$, and $\sigma = 1$. The solid curves represent the true boundary b_σ (red curve), the estimated boundary $\tilde{b}_{\widehat{\sigma}_n}$ (blue curve), the upper confidence curve $\tilde{c}_{1,\widehat{\sigma}_n}$ (orange curve), and the lower confidence curve $\tilde{c}_{2,\widehat{\sigma}_n}$ (green curve).

The spikes visible in Figure 3.3 near the last point $t_N = T = 1$ indicate that the true boundary rarely lies within the confidence curves at those points. This happens because the confidence curves have zero variance at the maturity date T (actually $\tilde{c}_{1,\widehat{\sigma}_n}(T) = \tilde{c}_{2,\widehat{\sigma}_n}(T) = S$), and the numerical approximation of $b(t_{N-1})$ given at (3.17) is slightly biased. This affects the accuracy of the estimated boundary by frequently leaving the true boundary outside the confidence curves near maturity. This drawback is negligible in practice, since the estimated boundary and the confidence curves are very close to the true boundary in terms of absolute distance.



(a) One third of the paths ($n = 66$). (b) Two thirds of the paths ($n = 133$).

Figure 3.3: Pointwise proportion of trials, out of $M = 1000$, in which the true boundary does not belong to the interval delimited by the confidence curves. We use $S = 10$, $X_0 = 10$, $T = 1$, $\lambda = 0$, $\sigma = 1$, and a significance level $\alpha = 0.05$, and a number $M = 1000$ of sample paths. For each path, one third (a) or two thirds (b) of the observations were used to compute σ and then to estimate the confidence curves by (3.20). The continuous line represents the proportion of non-inclusions, the dashed line stands for α, and the dotted lines are placed at $\alpha \pm z_{0.025}\sqrt{\alpha(1-\alpha)/M}$. The spikes at $T = 1$ are numerical artifacts due to the null variance of $\tilde{b}_{\widehat{\sigma}_n}(T)$.

3.4.4 Simulations

The ability to perform inference for the true OSB rises some natural questions: how much optimality is lost by $\tilde{b}_{\widehat{\sigma}_n}$ when compared to b_σ? How do the stopping strategies associated with the curves $\tilde{c}_{i,\widehat{\sigma}_n}$, $i = 1, 2$, compare with the one for $\tilde{b}_{\widehat{\sigma}_n}$? For example, a risk-averse (or risk-lover) strategy would be to consider the upper (lower) confidence curve $\tilde{c}_{1,\widehat{\sigma}_n}$ ($\tilde{c}_{2,\widehat{\sigma}_n}$) as the stopping rule, this being the most conservative (liberal) option within the uncertainty on estimating b_σ. A balanced strategy would be to consider the estimated boundary $\tilde{b}_{\widehat{\sigma}_n}$.

In the following, we investigate how these stopping strategies behave, assuming $\sigma = 1$, $T = 1$, $S = 10$, $X_0 = 10$, and $\lambda = 0$. We first estimate the payoff associated with each of them, and then we compare these payoffs with the one generated by considering the true boundary in its explicit form (see Remark 3.2). The choice of $\sigma = 1$ is not restrictive, as it is enough to rescale time by $1/\sigma$ and space (i.e., the price values) by $1/\sqrt{\sigma}$.

To perform the comparison, we defined a subset of $[0, T] \times \mathbb{R}$ where the payoffs were computed. We carried out the comparison along the pairs $(t_i, X_{t_i}^{(q)})$, for $i = 1, \ldots, N$, $N = 200$, and $q = 0.2, 0.4, 0.6, 0.8$, where $t_i = i\frac{T}{N}$ and $X_{t_i}^{(q)}$ represents the q-quantile of the marginal distribution of the process at time t_i that is $\mathcal{N}(0, t_i(T - t_i)/T)$ (see Figure 3.4).

For each i and q, we generated $M = 1000$ different paths $(s_j, X_{s_j})_{j=0}^{rN}$ of a Brownian bridge with volatility $\sigma = 1$ going from $(0, 0)$ to $(1, 0)$. Each path was sampled at times $s_j = j\frac{T}{rn}$, for $j = 0, 1, \ldots, rN$, for $r = 1$ and $r = 25$. The idea behind this setting is to tackle both the low-frequency scenario, which regards investors with access to daily prices or less frequent data, and the high-frequency scenario, addressing high volumes of information as it happens to be when recording intraday prices. We forced each path to pass through $(t_i, X_{t_i}^{(q)})$ (see Figure 3.4), and used the past $(s_j, X_{s_j})_{j=0}^{ri}$ of each trajectory to estimate the boundary and the confidence curves. The future $(s_j, X_{s_j})_{j=ri}^{rN}$ was employed to gather M observations of the payoff associated with each stopping rule, whose means and variances are shown below in Figures 3.5 and 3.6, respectively.

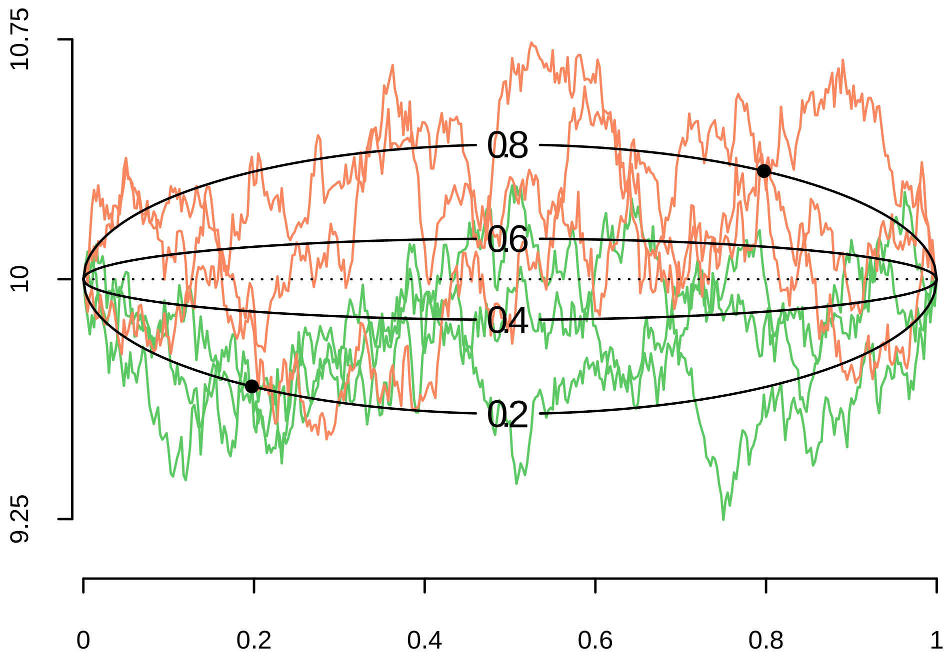


Figure 3.4: $X_t^{(q)}$ for $q = 0.2, 0.4, 0.6, 0.8$, where $X_t^{(q)}$ is the q-quantile of a $\mathcal{N}(0, t(1-t))$, the marginal distribution at time t of a Brownian bridge with unit volatility and $X_0 = X_1 = 10$. The green and orange lines refer to the paths of $(X_t \mid X_{0.2} = X_{0.2}^{(0.2)})_{t=0}^{1}$ and $(X_t \mid X_{0.8} = X_{0.8}^{(0.8)})_{t=0}^{1}$ respectively, with $X_{0.2}^{0.2} \approx 9.6649$ and $X_{0.8}^{0.8} \approx 10.3382$.

Figure 3.5 shows the value functions associated with each stopping rule, the red curve being the one associated with the OSB. An important fact revealed by Figure 3.5 is that in both the low- and high-frequency scenarios the estimate $\tilde{b}_{\widehat{\sigma}_n}$ behaves almost indistinguishably to b_σ in terms of the mean payoff after just a few initial observations.

Despite the variance payoff not being an optimized criterion in (3.3), it is worth knowing how it behaves for the three different stopping strategies, as it represents the risk associated with adopting each stopping rule as an exercise strategy. As expected, for any pair (t, x), a higher stopping boundary implies a smaller payoff variance.

Figure 3.6 not only reflects this behavior by suggesting the upper confidence curve as the best stopping strategy, but also reveals that the variance does exhibit considerable differences for the stopping rules in the low-frequency scenario. These differences increase when the time gets closer to the initial point $t = 0$ and also when the quantile level q decreases. In the high-frequency scenario, this effect is alleviated.

Figures 3.5 and 3.6 also reveal that both the mean and the variance of the payoff associated with the estimated boundary $\tilde{b}_{\widehat{\sigma}}$ converge to the ones associated with the true boundary as more data are taken to estimate σ.

The pragmatic bottom line of the simulation study can be summarized in the following rules-of-thumb: if $15 < n < 1000$, it is advised to adopt the upper confidence curve as the stopping rule because $\tilde{c}_{1,\widehat{\sigma}_n}$ has almost the same mean payoff as all the other stopping rules while having considerable less variance; if $n \geq 1000$, the means and the variances of the payoff of the three stopping rules are quite similar, being the most efficient option to just assume $\tilde{b}_{\widehat{\sigma}_n}$ without computing the confidence curves.

For $n \leq 15$, the best candidate for the execution strategy is not obvious, and it would depend on which criterion is chosen to measure the mean-variance trade-off of the three strategies.

3.5 Pinning-at-the-strike and real-data study

In this section, we compare the performance of the optimal stopping strategy using the Brownian bridge model with a classical approach that uses the geometric Brownian motion (Peskir, 2005b). The latter does not take into account the pinning information of the asset's price at maturity. We do so by a real data study analyzing various scenarios showing different degrees of intensity of pinning-at-the-strike.

The pinning behavior is more likely to take place among heavily traded options, as shown in Ni et al. (2005) and Krishnan and Nelken (2001). This is why we consider the options based on Apple and IBM expiring within the span of January 2011–September 2018, in particular 8905 options for Apple and 4833 for IBM. We denote by M the total number of options of each company and, for the j-th option, we let $\big(X_{t_i}^{(j)}\big)_{i=0}^{N_j}$ be the 5-min tick close price of the underlying stock divided by the strike price S_j. In order to quantify the strength of the pinning effect, we define the *pinning deviance* as $p_j := \big|X_{t_{N_j}}^{(j)} - 1\big|$, $j = 1, \ldots, M$. Therefore, under perfect pinning, we should expect $X_{t_{N_j}}^{(j)} = 1$ and $p_j = 0$.

We perform the following steps in the real data application:

i. We split each path $\big(X_{t_i}^{(j)}\big)_{i=0}^{N_j}$ into two subsets by using a factor $\rho \in \mathcal{P} = \{0.1, 0.2, \ldots, 0.9\}$. We call *historical set* to the first $\rho 100\%$ values of the prices $\big(X_{t_i}^{(j)}\big)_{i=0}^{\lfloor \rho N_j \rfloor}$ and *future set* to the remaining part (including the present value) $\big(X_{t_i}^{(j)}\big)_{i=\lfloor \rho N_j \rfloor}^{N_j}$. Here, $j = 1, \ldots, M$ while $1 - \rho$ represents the proportion of life time of the option.

ii. We use the historical set to estimate the volatility as described in Section 3.4.2.

iii. We compute the risk-free interest rate $\lambda_{j,\rho}$ as the 52-week treasury bill rate (extracted from U.S. Department of the Treasury (2018)) held by the market when the split of $\big(X_{t_i}^{(j)}\big)_{i=0}^{N_j}$ was done.

iv. We set the drift of the geometric Brownian motion to the risk-free interest rate such that the discounted process is a martingale.

v. We compute the OSBs using Algorithm 3.1 for the Brownian bridge model (3.9) and use the method exposed in Pederson and Peskir (2000, p. 12) for the geometric Brownian motion model studied in Peskir (2005b). Both numerical approaches are similar, the

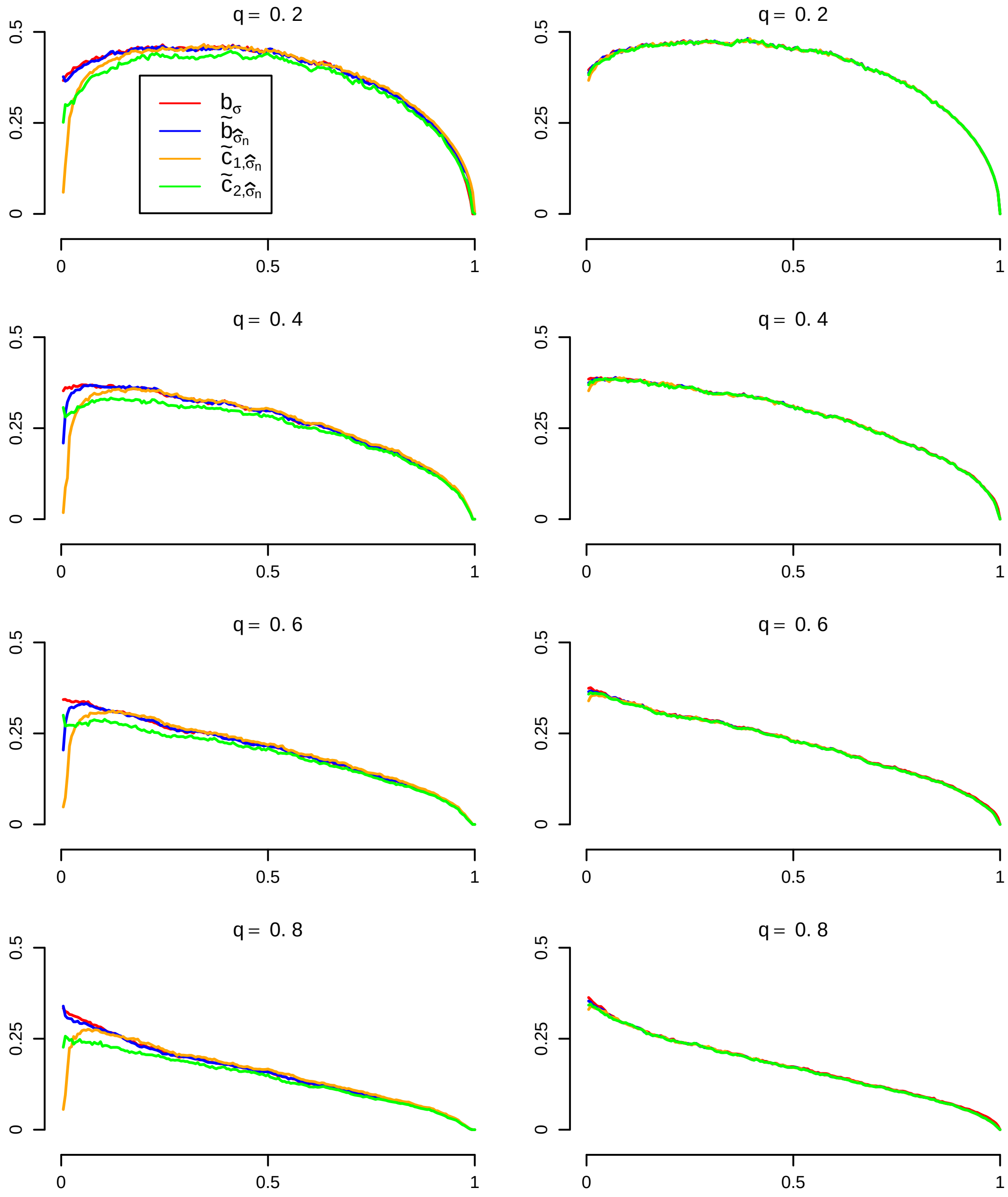


Figure 3.5: Mean of the payoff associated with: the true boundary b_σ (red curve), the estimated boundary $\tilde{b}_{\widehat{\sigma}_n}$ (blue curve), the upper confidence curve $\tilde{c}_{1,\widehat{\sigma}_n}$ (orange curve), and the lower confidence curve $\tilde{c}_{2,\widehat{\sigma}_n}$ (green curve). The left column shows the low-frequency scenario ($r = 1$), while the right one stands for the high-frequency scenario ($r = 25$). We use $\sigma = 1$, $T = 1$, $S = 10$, $X_0 = 10$, and $\lambda = 0$.

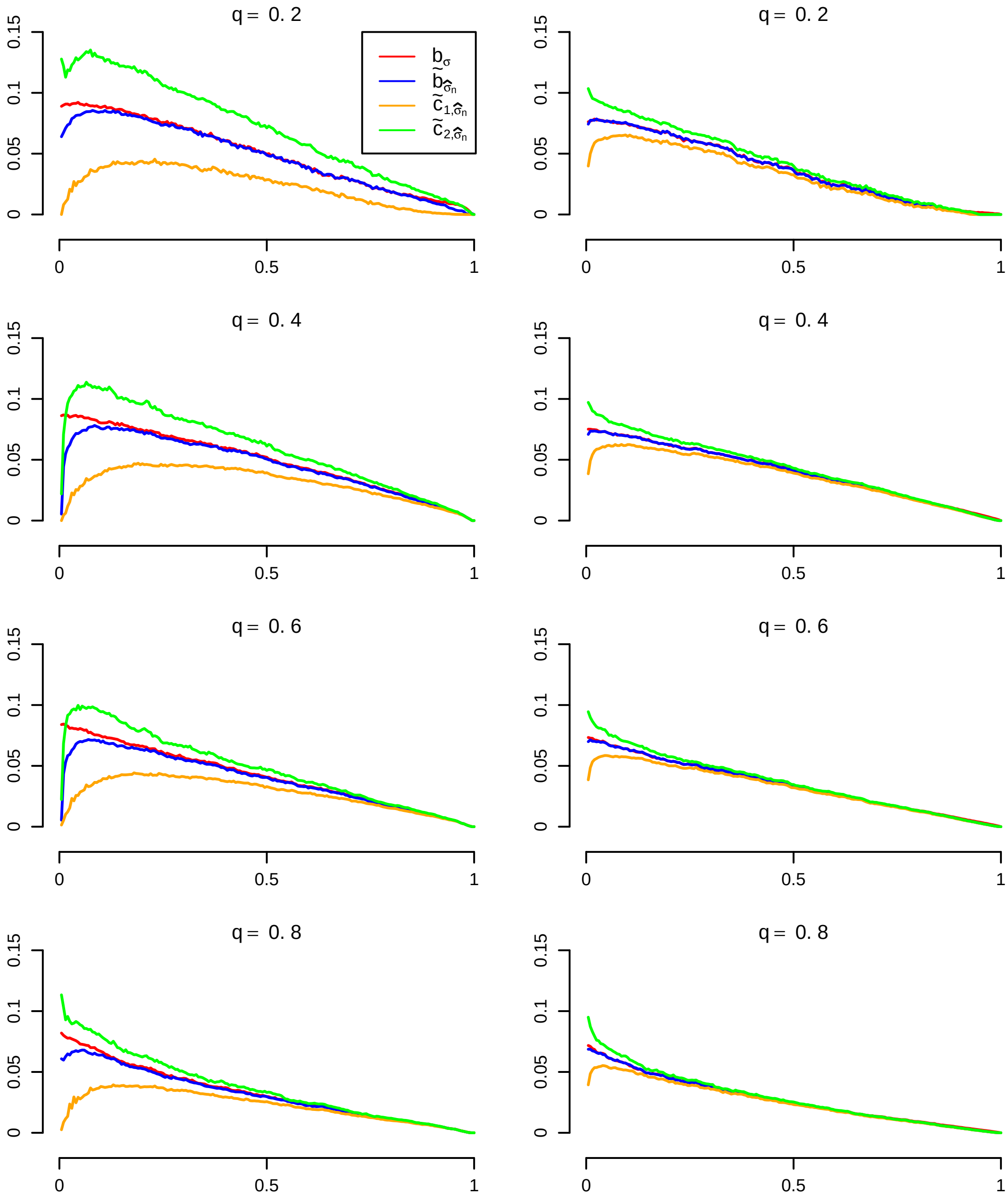


Figure 3.6: Variances of the payoff associated with: the true boundary b_σ (red curve), the estimated boundary $\tilde{b}_{\widehat{\sigma}_n}$ (blue curve), the upper confidence curve $\tilde{c}_{1,\widehat{\sigma}_n}$ (orange curve), and lower confidence curve $\tilde{c}_{2,\widehat{\sigma}_n}$ (green curve). The left column shows the low-frequency scenario ($r = 1$), while the right one stands for the high-frequency scenario ($r = 25$). We use $\sigma = 1$, $T = 1$, $S = 10$, $X_0 = 10$, and $\lambda = 0$.

only subtle difference relies on the Brownian bridge requiring the last part of the integral to be computed as in Algorithm 3.1, while the geometric Brownian motion needs no special treatment. The OSBs are computed with $S = 1$ (the stock prices were previously normalized by using the strike prices), $T = 1$ (all the maturity dates were standardized to 1), and 201 nodes for the time partitions described in Section 3.4.1.

vi. We compute the profit generated by optimally exercising the option within the remaining time by using the future set. This is done by calculating $e^{-\lambda_{j,\rho}\tau_{\mathrm{BB}}^{j,\rho}}\left(1 - X^{(j)}_{t_{\lfloor \rho N_j \rfloor}+\tau_{\mathrm{BB}}^{j,\rho}}\right)$ and $e^{-\lambda_{j,\rho}\tau_{\mathrm{GBM}}^{j,\rho}}\left(1 - X^{(j)}_{t_{\lfloor \rho N_j \rfloor}+\tau_{\mathrm{GBM}}^{j,\rho}}\right)$, where $\tau_{\mathrm{BB}}^{j,\rho}$ and $\tau_{\mathrm{GBM}}^{j,\rho}$ are the OSTs associated, respectively, to the Brownian bridge and geometric Brownian motion strategies under the initial condition $\left(t_{\lfloor \rho N_j \rfloor}, X^{(j)}_{t_{\lfloor \rho N_j \rfloor}}\right)$.

vii. We compute the "ρ-aggregated" cumulative profit, as defined below, to measure the goodness of both models (BB stays for Brownian bridge while GBM stays for geometric Brownian motion):

$$
\begin{aligned}
\mathrm{BB}(p) &= \frac{1}{|\mathcal{P}||\mathcal{J}(p)|}\sum_{j\in\mathcal{J}(p)}\sum_{\rho\in\mathcal{P}} e^{-\lambda_{j,\rho}\tau_{\mathrm{BB}}^{j,\rho}}\left(1 - X^{(j)}_{t_{\lfloor \rho N_j \rfloor}+\tau_{\mathrm{BB}}^{j,\rho}}\right),\\
\mathrm{GBM}(p) &= \frac{1}{|\mathcal{P}||\mathcal{J}(p)|}\sum_{j\in\mathcal{J}(p)}\sum_{\rho\in\mathcal{P}} e^{-\lambda_{j,\rho}\tau_{\mathrm{GBM}}^{j,\rho}}\left(1 - X^{(j)}_{t_{\lfloor \rho N_j \rfloor}+\tau_{\mathrm{GBM}}^{j,\rho}}\right),
\end{aligned}
$$

where $\mathcal{J}(p) := \{j = 1, \ldots, M : p_j < p\}$, and $|\mathcal{P}|$ and $|\mathcal{J}(p)|$ are the number of elements in $\mathcal{P}$ and $\mathcal{J}(p)$, respectively.

viii. We finally compute the relative mean profit $(\mathrm{BB}(p) - \mathrm{GBM}(p))/\mathrm{GBM}(p)$.

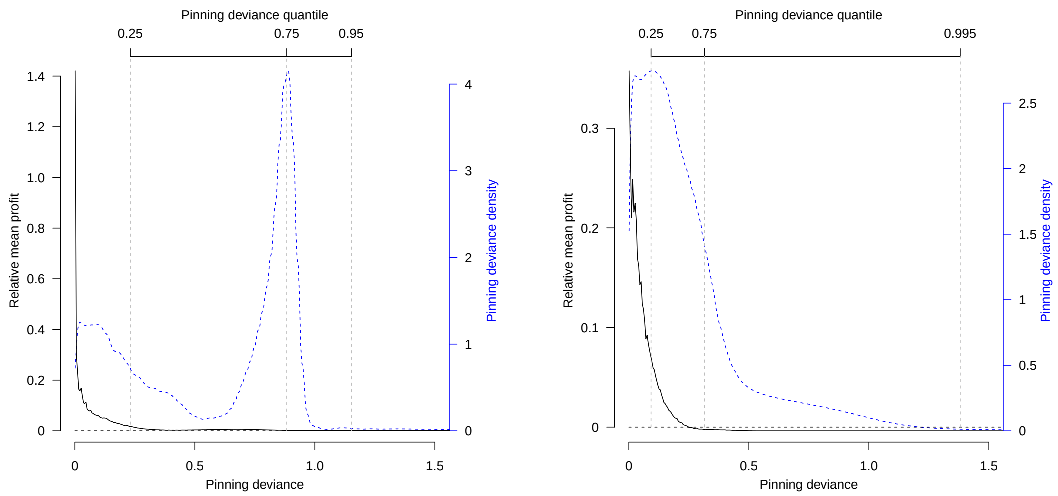
ix. We plot the pinning deviances p versus the relative mean profit (see Figure 3.7).

The Brownian bridge model behaves better than the geometric Brownian motion for options with low pinning deviance. This advantage fades away as we take distance from an ideal pinning-at-the-strike scenario, that is, when the pinning deviance increases. While the Brownian bridge model outperforms the geometric Brownian motion when applied to the Apple options along the whole dataset, when we consider the IBM options, the advantage is only present in 60% of the options with lower pinning deviances.

Remark 3.4. *Besides using the OSB, we also considered in the analysis the confidence curves described in Section 3.4.3. However, since this is a high-frequency sampling scenario, both confidence curves provided almost indistinguishable results and were omitted to avoid redundancy.*

Remark 3.5. *We did not consider the prices to buy the options when computing the profits in Figure 3.7, as we are interested in when it is optimal to exercise the option rather than in whether it is profitable to buy the option held.*

It is clear that an application of the Brownian bridge model is profitable in the presence of pinning-at-the-strike effect. However, it is far from being trivial to know beforehand if a stock will pin or not. Even if pinning forecasting is not the scope of this paper (for a systematic treatment, we refer to Avellaneda and Lipkin (2003), Jeannin et al. (2008), and Avellaneda et al. (2012)), we provide some basic evidence about the possibility to predict the appearance of the pinning effect by means of the trading volume of the options associated with a stock. For that, we study the association between the pinning deviances $(p_j)_{j=1}^{M}$ and the number of open contracts for a given option that we call the *Open Interest* (OI). In particular, we compute the *weighted OI* for options expiring during the year 2017. Its definition is $\mathrm{wOI}_j := \sum_{k=0}^{K_j} w_{j,k} o_{j,k}$, where $o_{j,k}$ is the OI of the j-th option at day k after it was opened, K_j is the total number of days the option remained available, and the weights $w_{j,k} := e^{-(1-k/K_j)} / \sum_{i=0}^{K_j} e^{-(1-i/K_j)}$, $j = 1, \dots, M$, place more importance to OIs closer to the maturity date. We highlight that the wOI is an observable quantity. The Spearman's rank correlation coefficient between the wOI and the pinning deviances scored -0.5932 for Apple and -0.4281 for IBM, thus revealing a significant (p-values $< 10^{-16}$) positive dependence between wOI and the pinning strength.



(a) Apple.

(b) IBM.

Figure 3.7: Results of the real data application. The black curve is the relative mean profit $(\mathrm{BB}(p) - \mathrm{GBM}(p)) / \mathrm{GBM}(p)$ for a pinning deviance p, while the blue dashed curve represents the kernel density estimation of the pinning deviances.

3.6 Non-monotonicity of the OSB

In Proposition 3.3, we proved the local Lipschitz continuity of b off the horizon to later obtain the smooth-fit condition, which leads to the free-boundary equation (3.9). An alternative path to achieve the same result is to obtain the monotonicity and continuity of the OSB. Monotonicity of the OSB is guaranteed in many finite-horizon OSPs, especially for those whose underlying process is time-homogeneous (see, e.g., Peskir and Uys (2003); Peskir (2005b,c); Glover et al. (2010, 2011); Kitapbayev (2014))), but it might be compromised when time-inhomogeneity or a discount factor is introduced. Our setting features a non-monotonic OSB (see Figure 3.8) due to the exponential discount (the non-discounted version does yield a monotonic boundary, as shown in Shepp (1969)). Continuity and piecewise monotonicity of the OSB, however, also suffice for the smooth-fit condition to hold (see Example 7 from De Angelis and Peskir (2020) and Corollary 8 from Cox and Peskir (2015)).

The continuity of the OSB has been extensively studied in a wide variety of frameworks. See, for instance, the work of De Angelis (2015) and Peskir (2019). In particular, Peskir (2019) can be applied straightforwardly to our settings to show that b cannot have first-order discontinuities.

Second-order discontinuities could be ruled out by proving the piecewise monotonicity of the OSB. However, this property resisted all our attempts to prove it, and the related literature addressing it is scant. To our knowledge, the soundest work on piecewise monotonic OSBs dates back to Friedman (1975), but only tackles a non-discounted OSP with time-dependent gain function and time-homogeneous process.

We include here evidence supporting the conjecture that b changes its monotonicity at most once. The OSB seems to be either increasing everywhere, or to start at the origin in a decreasing mode, reaching a global minimum, and going increasing afterward, until it hits the pinning point at the horizon with a positive infinite slope. Figure 3.8 illustrates this behavior.

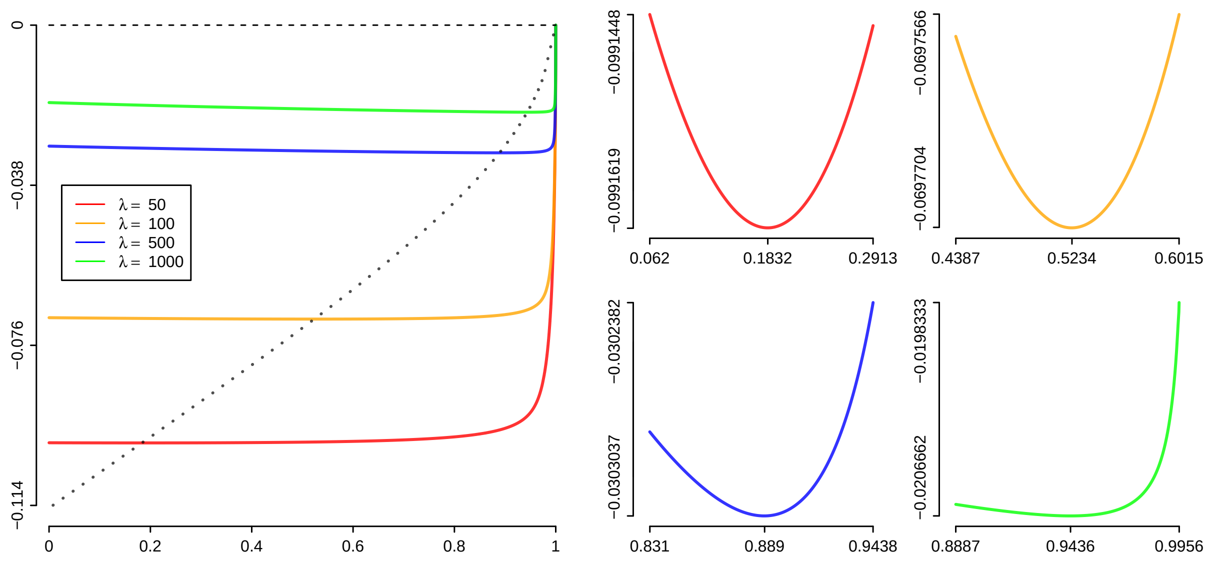


Figure 3.8: The image on the left shows four boundaries (continuous lines) for different values of λ. The dotted curve represents the (unique) pair $(t, b_\lambda(t))$ where each boundary b_λ changes its monotonicity, computed for a mesh of equally-spaced 5000 points going from $\lambda = 0$ to $\lambda = 2500$. The smallest value of λ where a change of monotonicity was observed was $\lambda = 37.5$. The dashed line is placed at the level of the pinning point ($S = 0$). We set $\sigma = 1$ for the volatility and $N = 5000$ for the logarithmically-spaced grid. The smaller images on the right zoom in the four boundaries for a better appreciation of their change of monotonicity.

3.7 Conclusions

In this work we solved the problem of optimally exercising an American put option, in the presence of a discount, by modeling the stock price with a Brownian bridge terminating at the strike price. The OSP was translated into a free-boundary problem shown to have a unique solution within a certain class of functions. We used a recursive fixed-point algorithm to compute the OSB in practice, corroborating its accuracy in the case without discount. Using a maximum likelihood estimation for the volatility, we computed pointwise confident curves around the estimated OSB, and we analyzed some correlated alternative stopping rules. The simulation study done for the non-discounted case showed that the lower confidence curve is the most appealing stopping decision because of the resulting reduced variance of the optimal profit. Finally, we performed a real data study that empirically concluded that the Brownian bridge model behaves considerably better than a classical geometric Brownian motion when the stock prices exhibit the pinned-at-the-strike effect.

Our model not only requires accurate insider information on the final value, but is also limited to the case in which the final value coincides with the strike price (but see Remark 3.3). A natural extension would be to admit randomness at the pinning point. Another interesting extension would be to use a model that excludes negative values of the stock price, like, for instance, the exponential of a Brownian bridge or a geometric Brownian bridge.

Supplementary materials

All the code required to implement Algorithm 3.1 and reproduce the results in Section 3.4 is available at `https://github.com/aguazz/AmOpBB`.

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3.A Main proofs

Proof of Proposition 3.1. Take an admissible pair (t, x) satisfying $x \geq S$ and $t < T$, and consider the stopping time $\tau_\varepsilon := \inf\{0 \leq s \leq T - t : X_{t+s} \leq S - \varepsilon \mid X_t = x\}$ (assume for convenience that $\inf\{\emptyset\} = T - t$), for $\varepsilon > 0$. Notice that $\mathbb{P}_{t,x}[\tau_\varepsilon < T - t] > 0$, which implies that $V(t, x) \geq \mathbb{E}_{t,x}\left[e^{-\lambda\tau_\varepsilon} G(X_{t+\tau_\varepsilon})\right] > 0 = G(x)$, from where it comes that $(t, x) \in C$.

Define $b(t) := \sup\{x \in \mathbb{R} : (t, x) \in D\}$. The above arguments guarantee that $b(t) < S$ for all $t \in [0, T)$, and we get from (3.3) that $b(T) = S$. Furthermore, from (3.3), it can be easily seen that, as λ increases, $V(t, x)$ decreases, and $b(t)$ increases. Hence, since $b(t)$ is known to be finite for all t when $\lambda = 0$ (see Remark 3.2), then we can guarantee that $b(t) > -\infty$ for all values of λ.

Notice that, since D is a closed set, $b(t) \in D$ for all $t \in [0, T]$. To prove that D has the form claimed in Proposition 3.1, let us take $x < b(t)$ and consider the OST $\tau^* = \tau^*(t, x)$. Then, relying on (3.3), (3.2), and (3.6), we get

$$\begin{aligned} V(t, x) - V(t, b(t)) &\leq \mathbb{E}_{t,x}\left[e^{-\lambda\tau^*} G(X_{t+\tau^*})\right] - \mathbb{E}_{t,b(t)}\left[e^{-\lambda\tau^*} G(X_{t+\tau^*})\right] \\ &\leq \mathbb{E}_{t,0}\left[\left(X_{t+\tau^*} + b(t)\frac{T - t - \tau^*}{T - t} - X_{t+\tau^*} - x\frac{T - t - \tau^*}{T - t}\right)^+\right] \\ &= (b(t) - x)\mathbb{E}\left[\frac{T - t - \tau^*}{T - t}\right] \\ &\leq b(t) - x, \end{aligned} \tag{3.21}$$

where we used the relation

$$G(a) - G(b) \leq (b-a)^+, \tag{3.22}$$

for all $a, b \in \mathbb{R}$, for the second inequality. Since $V(t, b(t)) = S - b(t)$, we get from the above relation that $V(t,x) \leq S - x = G(x)$, which means that $(t,x) \in D$ and therefore $\{(t,x) \in [0,T] \times \mathbb{R} : x \leq b(t)\} \subset D$. On the other hand, if $(t,x) \in D$, then $x \geq b(t)$, which proves the reverse inclusion. $\square$

Proof of Proposition 3.2. *(i)* Half of the statement relies on the results obtained by Peskir and Shiryaev (2006, Section 7.1) relative to the Dirichlet problem. It states that $\partial_t V + \mathbb{L}_X V = \lambda V$ on C. In addition, it indicates how to prove that V is $\mathcal{C}^{1,2}$ from a solution of the parabolic Partial Differential Equation (PDE)

$$\begin{cases} \partial_t f + \mathbb{L}_X f - \lambda f = 0 & \text{in } R, \\ f = V & \text{on } \partial R, \end{cases}$$

where $R \in C$ is a sufficiently regular region. If we consider R as an open rectangle, since V is continuous by *(iii)* below, and both μ and σ are locally Hölder continuous, then the above PDE has a unique solution (see Theorem 9, Section 3, Friedman (1983)). Finally, since $V(t,x) = G(x) = S - x$ for all $(t,x) \in D$, V is $\mathcal{C}^{1,2}$ also on D.

(ii) We easily get the convexity of $x \mapsto V(t,x)$ by plugging (3.2) into (3.3). To prove (3.14), let us fix an arbitrary point $(t,x) \in [0,T] \times \mathbb{R}$, and consider $\tau^* = \tau^*(t,x)$ and $\varepsilon > 0$. Arguing similarly to (3.21), we get

$$\varepsilon^{-1}(V(t, x+\varepsilon) - V(t,x)) \geq -\mathbb{E}\left[e^{-\lambda\tau^*}\frac{T-t-\tau^*}{T-t}\right]. \tag{3.23}$$

For $\varepsilon < 0$, the reverse inequality emerges, giving us, after taking $\varepsilon \to 0$, the relation $\partial_x^- V(t,x) \leq -\mathbb{E}\left[e^{-\lambda\tau^*}\frac{T-t-\tau^*}{T-t}\right] \leq \partial_x^+ V(t,x)$, which, due to the continuity of $x \mapsto \partial_x V(t,x)$ on $(-\infty, b(t))$ and on $(b(t), \infty)$ for all $t \in [0,T]$ (V is $\mathcal{C}^{1,2}$ on C and on D), turns into $\partial_x V(t,x) = -\mathbb{E}\left[e^{-\lambda\tau^*}\frac{T-t-\tau^*}{T-t}\right]$ for all (t,x) where $t \in [0,T]$ and $x \neq b(t)$. For $x = b(t)$, Equation (3.14) also holds and becomes the smooth-fit condition, later proved in Proposition 3.4.

Furthermore, since $\mathbb{P}_{t,x}[\tau^* < T - t] > 0$, (3.14) shows that $\partial_x V < 0$ and therefore $x \mapsto V(t,x)$ is strictly decreasing for all $t \in [0,T]$.

(iii) Let $(X_{t_i+s}^{[t_i,T]})_{s\geq 0}^{[0,T-t_i]}$ be a Brownian bridge going from $X_{t_i} = x$ to $X_T = S$ for any $x \in \mathbb{R}$, with $i = 1,2$. Notice that, according to (3.2), the following holds:

$$X_{t_2+s'}^{[t_2,T]} \stackrel{d}{=} r^{1/2} X_{t_1+s}^{[t_1,T]} + (1 - r^{1/2})(S - x)\frac{s}{T - t_1}, \tag{3.24}$$

where $r = \frac{T-t_2}{T-t_1}$, $s \in [0, T-t_1]$, and $s' = sr \in [0, T-t_2]$.

Take $0 \leq t_1 < t_2 < T$, consider $\tau_1 := \tau^*(t_1, x)$, and set $\tau_2 := \tau_1 r$. Since $t \mapsto V(t,x)$ is decreasing for every $x \in \mathbb{R}$, then

$$\begin{aligned} 0 &\leq V(t_1,x) - V(t_2,x) \\ &\leq \mathbb{E}_{t_1,x}\left[e^{-\lambda\tau_1} G\left(X_{t_1+\tau_1}^{[t_1,T]}\right)\right] - \mathbb{E}_{t_2,x}\left[e^{-\lambda\tau_2} G\left(X_{t_2+\tau_2}^{[t_2,T]}\right)\right] \\ &\leq \mathbb{E}\left[e^{-\lambda\tau_2}\left(G\left(X_{t_1+\tau_1}^{[t_1,T]}\right) - G\left(X_{t_2+\tau_2}^{[t_2,T]}\right)\right)\right] \end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\left(X_{t_2+\tau_2}^{[t_2,T]} - X_{t_1+\tau_1}^{[t_1,T]}\right)^+\right] \\
&= \mathbb{E}\left[\left(\left(r^{1/2}-1\right)\left(X_{t_1+\tau_1}^{[t_1,T]} + (S-x)\frac{\tau_1}{T-t_1}\right)\right)^+\right] \\
&\leq \left(\left(r^{1/2}-1\right)(S + \mathbb{1}(x \leq S)(S-x))\right)^+,
\end{aligned}
$$

where the first equality comes after applying (3.24) and the last inequality takes place since $r < 1$ and $X_{t_1+\tau_1}^{[t_1,T]} \leq S$.

Hence, $V(t_1,x) - V(t_2,x) \to 0$ as $t_1 \to t_2$, i.e., $t \mapsto V(t,x)$ is continuous for every $x \in \mathbb{R}$. Thus, to address the continuity of V, it is sufficient to prove that, for a fixed t, $x \mapsto V(t,x)$ is uniformly continuous within a neighborhood of t. The latter comes after the following inequality, which comes right after applying similar arguments to those used in (3.21):

$$
0 \leq V(t,x_1) - V(t,x_2) \leq (x_2 - x_1)\mathbb{E}\left[e^{-\lambda\tau^*}\frac{T-t-\tau^*}{T-t}\right] \leq x_2 - x_1,
$$

where $x_1, x_2 \in \mathbb{R}$ are such that $x_1 \leq x_2$ and $\tau^* = \tau^*(t,x_1)$. □

Proof of Proposition 3.3. We set $S = 0$ and $\sigma = T = 1$ for the proof. These restrictions are merely for notation sobriety. Working out the general case follows identical steps. For the proof, we import the notation and results from Lemma 3.1. Also, note that the Lipschitz continuity of b in closed intervals of $\mathbb{R}$ implies that of b in closed intervals within $[0,T)$. In this proof we tackle the former statement.

Consider the function $H : I \times \mathbb{R} \to \mathbb{R}_+$, for a closed interval $I \subset \mathbb{R}_+$, defined as $H(s,y) = W(s,y) - \mathrm{G}(s,y)$. Take a constant $r \in \mathbb{R}$ such that $0 > r > \sup\{\mathsf{b}(s) : s \in I\}$. Since $I \times \{r\} \subset \mathrm{C}$, H is continuous and $H|_{I\times\{r\}} > 0$. Then, there exists $a > 0$ such that $H(s,r) > a$ for all $s \in I$. Therefore, for all δ such that $0 < \delta \leq a$, the equation $H(s,y) = \delta$ has a solution in C for all $s \in I$. Moreover, this solution is unique for each s since $\partial_x H < 0$ in C, and we denote it by $\mathsf{b}_\delta(s)$, where $\mathsf{b}_\delta : I \to \mathbb{R}$. Away from the boundary, H is regular enough to apply the implicit function theorem that guarantees that b_δ is differentiable and

$$
\mathsf{b}_\delta'(s) = -\partial_t H(s,\mathsf{b}_\delta(s))/\partial_x H(s,\mathsf{b}_\delta(s)). \tag{3.25}
$$

Notice that b_δ is increasing in δ and therefore converges pointwise to some limit function b_0, which satisfies $\mathsf{b}_0 \geq \mathsf{b}$ in I as $\mathsf{b}_\delta > \mathsf{b}$ for all δ. Since $H(s,\mathsf{b}_\delta(s)) = \delta$ and H is continuous, it follows that $H(s,\mathsf{b}_0(s)) = 0$ after taking $\delta \to 0$, which means that $\mathsf{b}_0 \leq \mathsf{b}$ in I and hence $\mathsf{b}_0 = \mathsf{b}$ in I.

Take $(s,y) \in \mathrm{C}$ such that $y < r$. Set $\sigma^* = \sigma^*(s,y)$ and consider

$$
\sigma_r = \sigma_r(s,y) := \inf\left\{v \geq 0 : (s+v, Y_v^{s,y}) \notin I \times (-\infty, r)\right\}.
$$

Recalling (3.47), it is easy to check that there exists a constant $K_I^{(1)} > 0$ such that

$$
|\partial_t H(s,y)| \leq K_I^{(1)} m(s,y) \tag{3.26}
$$

with

$$
m(s,y) := \mathsf{E}_{s,y}\left[\int_s^{s+\sigma^*}\left(1 + \frac{|Y_v|}{(1+v)^2}\right)\mathrm{d}v\right].
$$

Using the tower property of conditional expectation, alongside the strong Markov property, we get that

$$m(s,y) = \mathsf{E}_{s,y}\left[\int_s^{s+\sigma^*\wedge\sigma_r}\left(1+\frac{|Y_v|}{(1+v)^2}\right)\mathrm{d}v + \mathbb{1}\left(\sigma_r \le \sigma^*\right) m(s+\sigma_r, Y_{s+\sigma_r})\right]. \tag{3.27}$$

Notice that, for $r > y > \mathsf{b}(s)$, $(s+\sigma_r, Y_{s+\sigma_r}) \in \Gamma_s$ $\mathsf{P}_{s,y}$-a.s. whenever $\sigma_r \le \sigma^*$, with $\Gamma_s := \{(s,\bar{s})\times\{r\}\}\cup\{\bar{s}\times(\mathsf{b}(\bar{s}),r]\}$ and $\bar{s} := \sup\{s : s\in I\}$. Hence, the following holds true $\mathsf{P}_{s,y}$-a.s. on the set $\{\sigma_r \le \sigma^*\}$:

$$\begin{aligned}
m\left(s+\sigma_r, Y_{s+\sigma_r}\right) &\le \sup_{(t,x)\in\Gamma_s} m\left(t,x\right) \le \sup_{(t,x)\in\Gamma_s} \mathsf{E}_{t,x}\left[\int_t^\infty\left(1+\frac{|Y_v|}{(1+v)^2}\right)\mathrm{d}v\right] \\
&\le \sup_{(t,x)\in\Gamma_s}\int_t^\infty\left(1+\frac{|x|}{(1+v)^2}\right)\mathrm{d}v + \int_t^\infty \frac{\mathsf{E}_{s,0}\left[|Y_v|\right]}{(1+v)^2}\mathrm{d}v \\
&\le \int_0^\infty\left(1+\frac{|\mathsf{b}(\bar{s})|}{(1+v)^2}\right)\mathrm{d}v + \int_0^\infty\sqrt{\frac{2}{\pi}}\frac{\sqrt{v}}{(1+v)^2}\mathrm{d}v < \infty
\end{aligned} \tag{3.28}$$

By plugging (3.28) into (3.27), after observing that $(1+|Y_v|/(1+v)^2) \le 1+|\inf_{s\in I}\mathsf{b}(s)|$ for $v\in(s, s+\sigma^*\wedge\sigma_r)$, and recalling (3.26), we obtain the following for some constant $K_I^{(2)} > 0$:

$$|\partial_t H(s,y)| \le K_I^{(2)}\mathsf{E}_{s,y}\left[\sigma_\delta\wedge\sigma_r + \mathbb{1}\left(\sigma_r\le\sigma_\delta\right)\right]. \tag{3.29}$$

Arguing as in (3.27) and setting $f_s(v) := \frac{e^{-\Lambda(s,s+v)}}{1+(s+v)}$, we get that

$$\begin{aligned}
|\partial_x H(s,y)| &= \mathsf{E}_{s,y}\left[f_s(0)-f_s(\sigma^*)\right] = \mathsf{E}_{s,y}\left[\int_s^{s+\sigma^*} f_s'(v)\,\mathrm{d}v\right] &(3.30)\\
&= \mathsf{E}_{s,y}\left[\int_s^{s+\sigma^*\wedge\sigma_r} f_s'(v)\,\mathrm{d}v + \mathbb{1}\left(\sigma_r\le\sigma^*\right)|\partial_x H(s+\sigma_r, Y_{s+\sigma_r})|\right], &(3.31)
\end{aligned}$$

Take $\varepsilon > 0$ such that $\mathcal{R}_\varepsilon := [\underline{s}, \bar{s}+\varepsilon]\times(r-\varepsilon, r+\varepsilon) \subset \mathrm{C}$, and consider the stopping time $\sigma_\varepsilon = \inf\{v\ge 0 : (s+v, Y_{s+v}) \notin \mathcal{R}_\varepsilon\}$. Observe that $\sigma^* = \sigma^*(s,r) > \sigma_\varepsilon$ for all $s\in I$. Then,

$$\begin{aligned}
|\partial_x H(s+\sigma_r, r)| &\ge \inf_{s\in I}|\partial_x H(s,r)| = \inf_{s\in I}\mathsf{E}_{s,r}\left[f_s(0)-f_s(\sigma^*)\right] \\
&\ge \inf_{s\in I}\mathsf{E}_{s,r}\left[f_s(0)-f_s(\sigma_\varepsilon)\right] \\
&\ge \inf_{s\in I}\left(f_s(0)-f_s(\bar{s}+\varepsilon-s)\right)\mathsf{P}_{s,r}\left(\sigma_\varepsilon = \bar{s}+\varepsilon-s\right) \\
&\ge \left(f_{\bar{s}}(0)-f_{\bar{s}}(\varepsilon)\right)\mathsf{P}_{s,r}\left(\sup_{u\le\bar{s}+\varepsilon-\underline{s}}|Y_{\underline{s}+u}| < \varepsilon\right) > 0,
\end{aligned} \tag{3.32}$$

where we used the fact that $s\mapsto f_s(0)-f_s(v)$ is decreasing for all $v\ge 0$. After noticing that f_s' is positive and decreasing, which means that $f_s'(s+v)\ge f_s'(\bar{s}) > 0$ for all $v\le\sigma_r$, and by plugging (3.32) into (3.31), we obtain, for a constant $K_{I,\varepsilon}^{(3)} > 0$,

$$|\partial_x H(s,y)| \ge K_I^{(3)}\mathsf{E}_{s,y}\left[\sigma^*\wedge\sigma_r + \mathbb{1}\left(\sigma_r\le\sigma^*, \sigma_r < \bar{s}-s\right)\right]. \tag{3.33}$$

Therefore, using (3.29) and (3.33) in (3.25) yields the following bound for some constant $K_I^{(4)} > 0$, $y_\delta = \mathsf{b}_\delta(s)$, and $\sigma_\delta = \sigma^*(s, y_\delta)$:

$$
\begin{aligned}
|\mathsf{b}_\delta'(s)| &\leq K_I^{(4)} \frac{\mathsf{E}_{s,y_\delta}\left[\sigma_\delta \wedge \sigma_r + \mathbb{1}\left(\sigma_r \leq \sigma_\delta\right)\right]}{\mathsf{E}_{s,y_\delta}\left[\sigma_\delta \wedge \sigma_r + \mathbb{1}\left(\sigma_r \leq \sigma_\delta, \sigma_r < \overline{s} - s\right)\right]} \\
&\leq K_I^{(4)}\left(1 + \frac{\mathsf{P}_{s,y_\delta}\left(\sigma_r \leq \sigma_\delta\right)}{\mathsf{E}_{s,y_\delta}\left[\sigma_\delta \wedge \sigma_r + \mathbb{1}\left(\sigma_r \leq \sigma_\delta, \sigma_r < \overline{s} - s\right)\right]}\right) \\
&\leq K_I^{(4)}\left(1 + \frac{\mathsf{P}_{s,y_\delta}\left(\sigma_r \leq \sigma_\delta, \sigma_r = \bar{s} - s\right)}{\mathsf{E}_{s,y_\delta}\left[\sigma_\delta \wedge \sigma_r\right]} + \frac{\mathsf{P}_{s,y_\delta}\left(\sigma_r \leq \sigma_\delta, \sigma_r < \bar{s} - s\right)}{\mathsf{E}_{s,y_\delta}\left[\mathbb{1}\left(\sigma_r \leq \sigma_\delta, \sigma_r < \overline{s} - s\right)\right]}\right) \\
&\leq K_I^{(4)}\left(2 + \frac{\mathsf{P}_{s,y_\delta}\left(\sigma_r \leq \sigma_\delta, \sigma_r = \bar{s} - s\right)}{\mathsf{E}_{s,y_\delta}\left[\mathbb{1}\left(\sigma_r \leq \sigma_\delta, \sigma_r = \bar{s} - s\right)\left(\sigma_\delta \wedge \sigma_r\right)\right]}\right) \\
&\leq K_I^{(4)}\left(2 + \frac{1}{\bar{s} - s}\right).
\end{aligned} \tag{3.34}
$$

If we set $I_\varepsilon = [\underline{s}, \bar{s} - \varepsilon]$ for $\varepsilon > 0$ small enough, then, by relying on (3.34), we obtain the existence of a constant $L_{I_\varepsilon} > 0$, independent from δ, such that $|\mathsf{b}_\delta'(s)| < L_{I_\varepsilon}$ for all $s \in I_\varepsilon$ and $0 < \delta \leq a$. We are thus able to use the Arzelà–Ascoly's theorem to guarantee that b_δ converges to b uniformly with respect to δ in I_ε, meaning that b is Lipschitz continuous on I_ε. □

Proof of Proposition 3.4. Take a pair $(t, x) \in [0, T) \times \mathbb{R}$ lying on the OSB, i.e., $x = b(t)$, and consider $\varepsilon > 0$. Since $(t, x) \in D$ and $(t, x+\varepsilon) \in C$, we have that $V(t, x) = G(x)$ and $V(t, x+\varepsilon) > G(x + \varepsilon)$. Thus, taking into account the inequality (3.22), we get $\varepsilon^{-1}(V(t, x + \varepsilon) - V(t, x)) > \varepsilon^{-1}(G(x + \varepsilon) - G(x)) \geq -1$, which, after taking $\varepsilon \to 0$ turns into $\partial_x^+ V(t, x) \geq -1$. On the other hand, consider the OST $\tau_\varepsilon := \tau^*(t, x + \varepsilon)$ and follow arguments similar to (3.21) to get

$$
\varepsilon^{-1}(V(t, x + \varepsilon) - V(t, x)) \leq -\mathbb{E}\left[e^{-\lambda \tau_\varepsilon} \frac{T - t - \tau_\varepsilon}{T - t}\right]. \tag{3.35}
$$

Since b is locally Lipschitz continuous (see Proposition 3.3), there exists $L_t > 0$ such that

$$
\begin{aligned}
\lim_{\varepsilon \downarrow 0} \mathbb{P}_{t,b(t)}\left(\inf_{u \in (0,\varepsilon)} (X_{t+u} - b(t+u)) < 0\right) &= \lim_{\varepsilon \downarrow 0} \mathbb{P}_{t,b(t)}\left(\inf_{u \in (0,\varepsilon)} \frac{X_{t+u} - b(t+u)}{\sqrt{2u \ln(\ln(1/u))}} < 0\right) \\
&\geq \lim_{\varepsilon \downarrow 0} \mathbb{P}_{t,b(t)}\left(\inf_{u \in (0,\varepsilon)} \frac{X_{t+u} - b(t) + L_t u}{\sqrt{2u \ln(\ln(1/u))}} < 0\right) \\
&= \mathbb{P}_{t,b(t)}\left(\liminf_{u \downarrow 0} \frac{X_{t+u} - b(t) + L_t u}{\sqrt{2u \ln(\ln(1/u))}} < 0\right) = 1,
\end{aligned}
$$

where the last inequality comes after the law of the iterated logarithm and representation (3.48). Hence, $(t, b(t))$ is probabilistically regular for the interior of D, that is, X_{t+u} falls below the boundary immediately $\mathbb{P}_{t,b(t)}$-a.s. and, therefore, Corollary 6 from De Angelis and Peskir (2020) entails that $\tau_\varepsilon \to 0$ a.s., which, along with the dominated convergence theorem and (3.35) give us that $\partial_x^+ V(t, b(t)) \leq -1$. Since $V = G$ in D, it follows straightforwardly that $\partial_x^- V(t, b(t)) = -1$, and hence the smooth-fit condition holds. □

Proof of Proposition 3.5. Assume we have a function $c : [0, T] \to \mathbb{R}$ that solves (3.9) and define

$$
V^c(t, x) := \int_t^T e^{-\lambda(u-t)} \left(\frac{1}{T - u} + \lambda\right) \mathbb{E}_{t,x}\left[(S - X_u)\, \mathbb{1}\left(X_u \leq c(u)\right)\right] \mathrm{d}u \tag{3.36}
$$

$$= \int_t^T K_{\sigma,\lambda}(t, x, u, c(u))\, \mathrm{d}u,$$

where $X = \{X_s\}_{s=0}^T$ is a Brownian bridge with σ volatility that ends at $X_T = S$, and $K_{\sigma,\lambda}$ is defined at (3.11). It turns out that $x \mapsto K_{\sigma,\lambda}(t, x, u, c(u))$ is twice continuously differentiable and therefore differentiating inside the integral symbol at (3.36) yields $\partial_x V^c(t, x)$ and $\partial_{x^2} V^c(t, x)$, and furthermore ensures their continuity on $[0, T) \times \mathbb{R}$.

Let us compute the operator $\partial_t + \mathbb{L}_X$ acting on the function V^c:

$$\partial_t V^c + \mathbb{L}_X V^c(t, x) = \lim_{h \downarrow 0} \frac{\mathbb{E}_{t,x}[V^c(t + h, X_{t+h})] - V^c(t, x)}{h}.$$

Define the function

$$I(t, u, x_1, x_2) := e^{-\lambda(u-t)} \left(\frac{1}{T - u} + \lambda \right) (S - x_1)\, \mathbb{1}\,(x_1 \leq x_2) \tag{3.37}$$

and notice that

$$\begin{aligned}
\mathbb{E}_{t,x}[V^c(t + h, X_{t+h})] &= \mathbb{E}_{t,x} \left[\mathbb{E}_{t+h, X_{t+h}} \left[\int_{t+h}^T I(t + h, u, X_u, c(u))\, \mathrm{d}u \right] \right] \\
&= \mathbb{E}_{t,x} \left[\mathbb{E}_{t,x} \left[\int_{t+h}^T I(t + h, u, X_u, c(u))\, \mathrm{d}u \;\middle|\; \mathcal{F}_{t+h} \right] \right] \\
&= \mathbb{E}_{t,x} \left[\int_{t+h}^T I(t + h, u, X_u, c(u))\, \mathrm{d}u \right],
\end{aligned}$$

where $(\mathcal{F}_s)_{s=0}^T$ is the natural filtration of X. Therefore,

$$\begin{aligned}
&\partial_t V^c + \mathbb{L}_X V^c(t, x) \\
&\quad = \lim_{h \downarrow 0} \frac{\mathbb{E}_{t,x} \left[\int_{t+h}^T I(t + h, u, X_u, c(u))\, \mathrm{d}u \right] - \mathbb{E}_{t,x} \left[\int_t^T I(t, u, X_u, c(u))\, \mathrm{d}u \right]}{h} \\
&\quad = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}_{t,x} \left[\int_{t+h}^T \left(e^{\lambda h} - 1 \right) I(t, u, X_u, c(u))\, \mathrm{d}u \right] - \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}_{t,x} \left[\int_t^{t+h} I(t, u, X_u, c(u))\, \mathrm{d}u \right] \\
&\quad = \lambda V(t, x) - (S - x) \left(\frac{1}{T - t} + \lambda \right) \mathbb{1}(x \leq c(t)).
\end{aligned}$$

From this result, alongside with (3.8) and the fact that V^c, $\partial_x V^c$, and $\partial_{x^2} V^c$ are continuous on $[0, T) \times \mathbb{R}$, we get the continuity of $\partial_t V^c$ on $C_1 \cup C_2$, where

$$C_1 := \{(t, x) \in [0, T) \times \mathbb{R} : x > c(t)\}, \quad C_2 := \{(t, x) \in [0, T) \times \mathbb{R} : x < c(t)\}.$$

Now, define the function $F^{(t)}(s, x) := e^{-\lambda s} V^c(t + s, x)$ with $s \in [0, T - t)$, $x \in \mathbb{R}$, and consider

$$C_1^t := \{(s, x) \in C_1 : t \leq s < T\}, \quad C_2^t := \{(s, x) \in C_2 : t \leq s < T\}.$$

We claim that $F^{(t)}$ satisfies the *(iii-b)* version of the hypothesis of Lemma 3.2 taking $C = C_1^t$ and $D^\circ = C_2^t$. Indeed: $F^{(t)}$, $\partial_x F^{(t)}$, and $\partial_{x^2} F^{(t)}$ are continuous on $[0, T) \times \mathbb{R}$; it has been proved that $F^{(t)}$ is $\mathcal{C}^{1,2}$ on C_1^t and C_2^t; we are assuming that c is a continuous function of bounded variation; and $(\partial_t F^{(t)} + \mathbb{L}_X F^{(t)})(s,x) = -e^{-\lambda s}(S-x)\left(\frac{1}{T-t-s}+\lambda\right)\mathbb{1}(x \leq c(t+s))$ is locally bounded on $C_1^t \cup C_2^t$.

Thereby, we can use the (*iii-b*) version of Lemma 3.2 to obtain the following change-of-variable formula, which is missing the local time term due to the continuity of F_x on $[0,T)\times\mathbb{R}$:

$$\begin{aligned} &e^{-\lambda s} V^c(t+s,\ X_{t+s}) \\ &\quad = V^c(t,x) - \int_t^{t+s} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}+\lambda\right)\mathbb{1}(X_u \leq c(u))\,\mathrm{d}u + M_s^{(1)}, \end{aligned} \tag{3.38}$$

with $M_s^{(1)} = \int_t^{t+s} e^{-\lambda(u-t)}\sigma\partial_x V^c(u,X_u)\,\mathrm{d}B_u$. Note that $(M_s^{(1)})_{s=0}^{T-t}$ is a martingale under $\mathbb{P}_{t,x}$.

In the same way, we can apply the (*iii-b*) version of Lemma 3.2 using the function $F(s,x) = e^{-\lambda s}G(X_{t+s})$, and taking $C = \{(s,x) \in [0,T-t)\times\mathbb{R} : x > S\}$ and $D^\circ = \{(s,x) \in [0,T-t)\times\mathbb{R} : x < S\}$, thereby getting

$$\begin{aligned} e^{-\lambda s}G(X_{t+s}) = G(x) &- \int_t^{t+s} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}+\lambda\right)\mathbb{1}(X_u < S)\,\mathrm{d}u \\ &- M_s^{(2)} + \frac{1}{2}\int_t^{t+s} e^{-\lambda(u-t)}\mathbb{1}(X_u = S)\,\mathrm{d}l_s^S(X), \end{aligned} \tag{3.39}$$

where $M_s^{(2)} = \sigma\int_t^{t+s} e^{-\lambda(u-t)}\mathbb{1}(X_u < S)\,\mathrm{d}B_u$, with $0 \leq s \leq T-t$, is a martingale under $\mathbb{P}_{t,x}$.

Consider the following stopping time for (t,x) such that $x \leq c(t)$:

$$\rho_c := \inf\left\{0 \leq s \leq T-t : X_{t+s} \geq c(t+s) \mid X_t = x\right\}. \tag{3.40}$$

In this way, along with assumption $c(t) < S$ for all $t \in (0,T)$, we can ensure that $\mathbb{1}(X_{t+s} \leq c(t+s)) = \mathbb{1}(X_{t+s} \leq S) = 1$ for all $s \in [0,\rho_c)$, as well as $\int_t^{t+s} e^{-\lambda(u-t)}\mathbb{1}(X_u = S)\,\mathrm{d}l_s^S(X) = 0$. Recall that $V^c(t,c(t)) = G(c(t))$ for all $t \in [c,T)$ since c solves (3.9). Moreover, $V^c(T,S) = 0 = G(S)$. Hence, $V^c(t+\rho_c, X_{t+\rho_c}) = G(X_{t+\rho_c})$. Therefore, we are able now to derive the following relation from equations (3.38) and (3.39):

$$\begin{aligned} V^c(t,x) &= \mathbb{E}_{t,x}[e^{-\lambda\rho_c}V^c(t+\rho_c,\ X_{t+\rho_c})] \\ &\quad + \mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}+\lambda\right)\mathbb{1}(X_u \leq c(u))\,\mathrm{d}u\right] \\ &= \mathbb{E}_{t,x}\left[e^{-\lambda\rho_c}G(X_{t+\rho_c})\right] \\ &\quad + \mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}+\lambda\right)\mathbb{1}(X_u \leq S)\,\mathrm{d}u\right] \\ &= G(x). \end{aligned}$$

The vanishing of the martingales $M_{\rho_c}^{(1)}$ and $M_{\rho_c}^{(2)}$ comes after using the optional stopping theorem (see, e.g., Section 3.2 from Peskir and Shiryaev (2006)). Therefore, we have just proved that $V^c = G$ on C_2.

Now, define the stopping time

$$\tau_c := \inf\{0 \leq u \leq T-t : X_{t+u} \leq c(t+u) \mid X_t = x\}$$

and plugging it into (3.38) to obtain the expression

$$V^c(t,x) = e^{-\lambda\tau_c}V^c(t+\tau_c,\ X_{t+\tau_c})$$

$$+\int_t^{t+\tau_c} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}+\lambda\right)\mathbb{1}(X_u\le c(u))\,\mathrm{d}u - M^{(1)}_{\tau_c}.$$

Notice that, due to the definition of τ_c, $\mathbb{1}(X_{t+u}\le c(t+u))=0$ for all $0\le u<\tau_c$ whenever $\tau_c>0$ (the case $\tau_c=0$ is trivial). In addition, the optional sampling theorem ensures that $\mathbb{E}_{t,x}[M^{(1)}_{\tau_c}]=0$. Therefore, the following formula comes after taking $\mathbb{P}_{t,x}$-expectation in the above equation and considering that $V^c=G$ on C_2:

$$V^c(t,x)=\mathbb{E}_{t,x}[e^{-\lambda\tau_c}V^c(t+\tau_c,X_{t+\tau_c})]=\mathbb{E}_{t,x}\left[e^{-\lambda\tau_c}G(X_{t+\tau_c})\right],$$

for all $(t,x)\in[0,T)\times\mathbb{R}$. Recalling the definition of V from (3.3), the above equality leads to

$$V^c(t,x)\le V(t,x), \tag{3.41}$$

for all $(t,x)\in[0,T)\times\mathbb{R}$.

Take $(t,x)\in C_2$ satisfying $x<\min\{b(t),c(t)\}$, where b is the OSB for (3.3), and consider the stopping time ρ_c defined as

$$\rho_b:=\inf\left\{0\le s\le T-t : X_{t+s}\ge b(t+s)\mid X_t=x\right\}.$$

Since $V=G$ on D, the following equality holds due to (3.42) and from noticing that $\mathbb{1}(X_{t+u}\le b(t+u))=1$ for all $0\le u<\rho_b$:

$$\mathbb{E}_{t,x}[e^{-\lambda\rho_b}V(t+\rho_b,X_{t+\rho_b})]=G(x)-\mathbb{E}_{t,x}\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}+\lambda\right)\mathrm{d}u\right].$$

On the other hand, we get the next equation after substituting s for ρ_b at (3.38) and recalling that $V=G$ on C_2:

$$\mathbb{E}_{t,x}[e^{-\lambda\rho_b}V(t+\rho_b,X_{t+\rho_b})]=G(x)-\mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}+\lambda\right)\mathbb{1}(X_u\le c(u))\,\mathrm{d}u\right].$$

Therefore, we can use (3.41) to merge the two previous equalities into

$$\begin{aligned}\mathbb{E}_{t,x}&\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}-\lambda\right)\mathbb{1}(X_u\le c(u))\,\mathrm{d}u\right]\\&\ge\mathbb{E}_{t,x}\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}(S-X_u)\left(\frac{1}{T-u}-\lambda\right)\mathrm{d}u\right],\end{aligned}$$

meaning that $b(t)\le c(t)$ for all $t\in[0,T]$ since c is continuous.

Suppose there exists a point $t\in(0,T)$ such that $b(t)<c(t)$ and fix $x\in(b(t),c(t))$. Consider the stopping time

$$\tau_b:=\inf\{0\le u\le T-t : X_{t+u}\le b(t+u)\mid X_t=x\}$$

and plugging it both into (3.42) and (3.38) replacing s before taking the $\mathbb{P}_{t,x}$-expectation. We obtain

$$\begin{aligned}&\mathbb{E}_{t,x}[e^{-\lambda\tau_b}V^c(t+\tau_b,X_{t+\tau_b})]\\&\quad=\mathbb{E}_{t,x}[e^{-\lambda\tau_b}G(X_{t+\tau_b})]\end{aligned}$$

$$= V^c(t,x) - \mathbb{E}_{t,x}\left[\int_t^{t+\tau_b} e^{-\lambda(u-t)}(S - X_u)\left(\frac{1}{T-u} + \lambda\right)\mathbb{1}(X_u \leq c(u))\,\mathrm{d}u\right]$$

and

$$\mathbb{E}_{t,x}[e^{-\lambda\tau_b}V(t+\tau_b, X_{t+\tau_b})] = \mathbb{E}_{t,x}[e^{-\lambda\tau_b}G(X_{t+\tau_b})] = V(t,x).$$

Thus, from (3.41), we get

$$\mathbb{E}_{t,x}\left[\int_t^{t+\tau_b} e^{-\lambda(u-t)}(S - X_u)\left(\frac{1}{T-u} + \lambda\right)\mathbb{1}(X_u \leq c(u))\,\mathrm{d}u\right] \leq 0.$$

Using the fact that $x > b(t)$ and the time-continuity of the process X, we can state that $\tau_b > 0$. Therefore, the previous inequality can only happen if $\mathbb{1}(X_s \leq c(s)) = 0$ for all $t \leq s \leq t + \tau_b$, meaning that $b(s) \geq c(s)$ for all $t \leq s \leq t+\tau_b$, which contradicts the assumption $b(t) < c(t)$. □

Proof of Theorem 3.1. Propositions 3.1–3.4 give the required conditions to apply the Itô's formula extension exposed in the supplement to the function $F(s,x) = e^{-\lambda s}V(t+s,x)$, from where we get that

$$e^{-\lambda s}V(t+s, X_{t+s}) = V(t, X_t) + \int_0^s e^{-\lambda u}(\partial_t V + \mathbb{L}_X V - \lambda V)(t+u, X_{t+u})\,\mathrm{d}u \tag{3.42}$$
$$+ \int_0^s \sigma e^{-\lambda u}\partial_x V(t+u, X_{t+u})\,\mathrm{d}B_u.$$

Notice that the above formula is missing the local time term due to the continuity of $x \mapsto \partial_x V(t,x)$ for all $t \in [0,T]$.

Recalling that $\partial_t V + \mathbb{L}_X V = \lambda V$ on C and $(\partial_t V + \mathbb{L}_X V - \lambda V)(t,x) = -(S-x)\left((T-t)^{-1} + \lambda\right)$ for all $(t,x) \in D$, taking $\mathbb{P}_{t,x}$-expectation (causing the vanishing of the martingale term), setting $s = T-t$, and making a simple change of variable in the integral, we get from (3.42) the following pricing formula for the American put option:

$$V(t,x) = \int_t^T e^{-\lambda(u-t)}\left(\frac{1}{T-u} + \lambda\right)\mathbb{E}_{t,x}\left[(S - X_u)\,\mathbb{1}\left(X_u \leq b(u)\right)\right]\,\mathrm{d}u. \tag{3.43}$$

We know from (3.2) that, for $u \in [t,T]$, $X_u^{[t,T]} \sim \mathcal{N}\left(\mu(t,x,u), \nu_\sigma^2(t,u)\right)$ under $\mathbb{P}_{t,x}$, where μ and ν_σ are given in (3.12) and (3.13), respectively.

For any random variable Y, we have that $\mathbb{E}\left[Y\mathbb{1}(Y \leq a)\right] = \mathbb{P}[Y \leq a]\mathbb{E}\left[Y \mid Y \leq a\right]$. In addition, if $Y \sim \mathcal{N}(\mu, \nu^2)$, then $\mathbb{E}\left[Y \mid Y \leq a\right] = \mu - \nu\phi(z)/\Phi(z)$, where $z = (a-\mu)/\nu$, and ϕ and Φ denote, respectively, the density and distribution functions. Then, the more tractable representation (3.10) for V follows.

Since $V(t,x) = S - x$ for all $(t,x) \in D$, we can take $x \uparrow b(t)$ on both sides in (3.10) in order to obtain the type two Volterra nonlinear integral Equation (3.9) for the OSB b.

Finally, due to Proposition 3.5, we obtain that the solution of Equation (3.9) is unique up to the regularity conditions considered in Theorem 3.1. □

3.B Auxiliary lemmas

Lemma 3.1. *Consider the OSP*

$$W(s,y) := \sup_{\sigma \geq 0} \mathsf{E}_{s,y}\left[e^{-\Lambda(s,s+\sigma)}\mathrm{G}(s+\sigma, Y_{s+\sigma})\right], \quad (s,y) \in \mathbb{R}_+ \times \mathbb{R}, \tag{3.44}$$

where Λ *and* G *are the time-dependent discount and gain functions, respectively, given by,*

$$\mathrm{G}(s,y) = \left(\frac{-y}{1+s}\right)^{+}; \quad \Lambda(s,s+v) = \int_s^{s+v} r(\rho)\,\mathrm{d}\rho; \quad r(\rho) := \frac{\lambda}{1+\rho},$$

where the process $\{Y_v\}_{v\in\mathbb{R}_+}$, *defined on the probability space* $(\Omega,\mathcal{F},\mathsf{P})$, *is a BM. For* $(s,y) \in \mathbb{R}_+ \times \mathbb{R}$, *the probability* $\mathsf{P}_{s,y}$ *is such that* $\mathsf{P}_{s,y}(Y_s = y) = 1$, *and* $\mathsf{E}_{s,y}$ *represents the expectation operator with respect to* $\mathsf{P}_{s,y}$.

Then, for $S = 0$ *and* $\sigma = T = 1$, *the OSP* (3.3) *is equivalent to* (3.44). *Specifically,*

$$W(s,y) = V(t,x), \tag{3.45}$$

for $s = t/(1-t)$ *and* $y = x(1+s)$.

Furthermore, denote by C *and* D *to the continuation set and the stopping set of* (3.44), *and by* $\sigma^* = \sigma^*(s,y)$ *and* b *to its OST and OSB, respectively. Then:*

(i) b *is bounded and* $\mathrm{D} = \{(s,y) : y \leq \mathsf{b}(s)\}$.

(ii) W *is continuous.*

(iii) W *is* $\mathcal{C}^{1,2}$ *on* C *and on* D, *and* $\mathrm{L}W = \lambda W$ *on* C, $\mathrm{L} = \partial_t + \frac{1}{2}\partial_{xx}$.

(iv) $y \mapsto W(s,y)$ *is convex and strictly decreasing for all* $s \in \mathbb{R}_+$. *Moreover,*

$$\partial_x W(s,y) = -\mathsf{E}_{s,y}\left[\frac{e^{-\Lambda(s,s+\sigma^*)}}{1+(s+\sigma^*)}\right]. \tag{3.46}$$

(v) For $(s,y) \in \mathrm{C}$,

$$\begin{aligned}\partial_t W(s,y) - \partial_t \mathrm{G}(s,y) = \mathsf{E}_{s,y}&\left[\int_s^{s+\sigma^*} \frac{e^{-\Lambda(s,v)}}{(1+v)^3}\left(2Y_v + \lambda(1+v) - \lambda(1+v)^2\right)\mathbb{1}\left(Y_{s+v} < 0\right)\mathrm{d}v\right] \\ &+ \frac{1}{2}\mathsf{E}_{s,y}\left[\int_s^{s+\sigma^*} \frac{e^{-\Lambda(s,v)}}{1+v}\mathbb{1}\left(Y_v = 0\right)\mathrm{d}\mathrm{l}_v^0\right],\end{aligned} \tag{3.47}$$

where l_v^0 *is the local-time measure of a Brownian motion* $B = \{B_v\}_{v\in\mathbb{R}_+}$ *starting at* y *under* $\mathsf{P}_{s,y}$, *that is,*

$$\mathrm{l}_v^0 = \lim_{\varepsilon\to 0}\int_0^v \mathbb{1}\left(-\varepsilon \leq B_r \leq +\varepsilon\right)\mathrm{d}r.$$

Proof of Lemma 3.1. Relation (3.45) is a straightforward consequence of the following BM representation of the BB (3.2):

$$\mathrm{Law}\left(\{X_{t+u}\}_{u\in[0,1-t]}, \mathbb{P}_{t,x}\right) = \mathrm{Law}\left(\{G(s+v, Y_{s+v})\}_{v\in\mathbb{R}_+}, \mathsf{P}_{s,y}\right), \tag{3.48}$$

for $s + v = (t+u)/(1-(t+u))$.

From (3.2) and the results already obtained for V and b in Propositions 3.1 and 3.2, it readily follows *(i)*, *(ii)*, *(iii)*, and *(iv)*.

To obtain (3.47), take $(s,y) \in \mathrm{C}$ and set $F(s,v,y) := e^{-\Lambda(s,v)}\mathrm{G}(v,y)$. Notice that, for $\varepsilon > 0$ and $\sigma^* = \sigma^*(s,y)$,

$$\varepsilon^{-1}\left(W(s,y) - W(s-\varepsilon,y)\right) \leq \varepsilon^{-1}\mathsf{E}_{s,y}\left[F(s,s+\sigma^*,Y_{s+\sigma^*}) - F(s-\varepsilon,s-\varepsilon+\sigma^*,Y_{s+\sigma^*})\right].$$

Hence, by letting $\varepsilon \to 0$ and setting

$$f(s,v,y) := (\partial_1 + \partial_2)F(s,v,y) = e^{-\Lambda(s,v)}\left(\partial_t \mathrm{G}(v,y) - (\rho(v) - \rho(s))\right),$$

where ∂_i refers to the partial derivative with respect to the i-th coordinate, we get that

$$\begin{aligned} \partial_t W(s,y) &\leq \mathsf{E}_{s,y}\left[f\left(s, s+\sigma^*, Y_{s+\sigma^*}\right)\right] \\ &= \partial_t \mathrm{G}(s,y) + \mathsf{E}_{s,y}\left[\int_s^{s+\sigma^*} \mathrm{L}f(s,v,Y_v)\mathbb{1}\left(Y_{s+v} < 0\right)\,\mathrm{d}v\right] \\ &\quad + \frac{1}{2}\mathsf{E}_{s,y}\left[\int_s^{s+\sigma^*} \frac{e^{-\Lambda(s,v)}}{1+v}\mathbb{1}\left(Y_v = 0\right)\,\mathrm{dl}_v^0\right], \end{aligned} \tag{3.49}$$

In the same fashion, we obtain

$$\varepsilon^{-1}\left(W(s+\varepsilon,y) - W(s,y)\right) \geq \varepsilon^{-1}\mathsf{E}_{s,y}\left[F(s,s+\sigma^*,Y_{s+\sigma^*}) - F(s-\varepsilon, s-\varepsilon+\sigma^*, Y_{s+\sigma^*})\right].$$

Thus, by arguing as in (3.49) we get the reverse inequality and, therefore, (3.47) is proved after computing $\mathrm{L}f(s,v,y)$.

To get (3.46), notice that, for $\varepsilon > 0$ small enough,

$$\begin{aligned} &\varepsilon^{-1}\left(W(s,y) - W(s,y-\varepsilon)\right) \\ &\leq \varepsilon^{-1}\mathsf{E}_{s,y}\left[e^{-\Lambda(s,s+\sigma^*)}\mathrm{G}\left(s+\sigma^*, Y_{s+\sigma^*}\right)\right] - \mathsf{E}_{s,y-\varepsilon}\left[e^{-\Lambda(s,s+\sigma^*)}\mathrm{G}\left(s+\sigma^*, Y_{s+\sigma^*}\right)\right] \\ &= -\mathsf{E}_{s,y}\left[\frac{e^{-\Lambda(s,s+\sigma^*)}}{1+(s+\sigma^*)}\right], \end{aligned}$$

while the same reasoning yields the inequality $\varepsilon^{-1}\left(W(s,y+\varepsilon) - W(s,y)\right) \geq -\mathsf{E}_{s,y}\left[\frac{e^{-\Lambda(s,s+\sigma^*)}}{1+(s+\sigma^*)}\right]$, and then, by letting $\varepsilon \to 0$, we get (3.46). $\square$

For the sake of completeness, we formulate the following change-of-variable result by taking Theorem 3.1 from Peskir (2005a) and changing some of its hypotheses according to Remark 3.2 from Peskir (2005a). Specifically, the *(iii-a)* version of Lemma 3.2 comes after changing, in Peskir (2005a), (3.27) and (3.28) for the joint action of (3.26), (3.35), and (3.36). The *(iii-b)* version relaxes condition (3.35) into (3.37) in Peskir (2005a).

Lemma 3.2. *Let $X = (X_t)_{t=0}^T$ be a diffusion process solving the SDE*

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t, \qquad 0 \leq t \leq T,$$

in Itô's sense. Let $b : [0,T] \to \mathbb{R}$ be a continuous function of bounded variation, and let $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$F \text{ is } \mathcal{C}^{1,2} \text{ on } C,$$
$$F \text{ is } \mathcal{C}^{1,2} \text{ on } D^\circ,$$

where $C = \{(t,x) \in [0,T] \times \mathbb{R} : x > b(t)\}$ and $D^\circ = \{(t,x) \in [0,T] \times \mathbb{R} : x < b(t)\}$.

Assume there exists $t \in [0,T]$ such that the following conditions are satisfied:

(i) $\partial_t F + \mu \partial_x F + (\sigma^2/2)\partial_{x^2} F$ *is locally bounded on $C \cup D^\circ$;*

(ii) *the functions* $s \mapsto \partial_x F(s, b(s)^{\pm}) := \partial_x F(s, \lim_{h \to 0+} b(s) \pm h)$ *are continuous on* $[0, t]$*;*

(iii) *and either*

(iii-a) $x \mapsto F(s, x)$ *is convex on* $[b(s) - \delta, b(s)]$ *and convex on* $[b(s), b(s) + \delta]$ *for each* $s \in [0, t]$*, with some* $\delta > 0$*;*

(iii-b) $\partial_{x^2} F = G_1 + G_2$ *on* $C \cup D^{\circ}$*, where* G_1 *is non-negative (or non-positive) and* G_2 *is continuous on* $\bar{C}$ *and* $\bar{D}^{\circ}$*.*

Then, the following change-of-variable formula holds:

$$
\begin{aligned}
F(t, X_t) = F(0, X_0) &+ \int_0^t (\partial_t F + \mu \partial_x F + (\sigma^2/2) \partial_{x^2} F)(s, X_s) \mathbb{1}(X_s \neq b(s)) \, \mathrm{d}s \\
&+ \int_0^t (\sigma \partial_x F)(s, X_s) \mathbb{1}(X_s \neq b(s)) \, \mathrm{d}B_s \\
&+ \frac{1}{2} \int_0^t (\partial_x F(s, X_s^+) - \partial_x F(s, X_s^-)) \mathbb{1}(X_s = b(s)) \, \mathrm{d}l_s^b(X),
\end{aligned}
$$

where $\mathrm{d}l_s^b(X)$ *is the local time of* X *at the curve* b *up to time* t*, i.e.,*

$$
l_s^b(X) = \lim_{\varepsilon \to 0} \int_0^t \mathbb{1}(b(s) - \varepsilon \leq X_s \leq b(s) + \varepsilon) \, \mathrm{d}\langle X, X \rangle_s, \tag{3.50}
$$

where $\langle X, X \rangle$ *is the predictable quadratic variation of* X*, and the limit above is meant in probability.*

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Chapter 4

Optimal stopping of an Ornstein–Uhlenbeck bridge

Abstract

Markov bridges may be useful models in finance to describe situations in which information on the underlying processes is known in advance. However, within the framework of optimal stopping problems, Markov bridges are inherently challenging processes as they are time-inhomogeneous and account for explosive drifts. Consequently, few results are known in the literature of optimal stopping theory related to Markov bridges, all of them confined to the simplistic Brownian bridge.

In this paper we make a rigorous analysis of the existence and characterization of the free boundary related to the optimal stopping problem that maximizes the mean of an Ornstein–Uhlenbeck bridge. The result includes the Brownian bridge problem as a limit case. The methodology hereby presented relies on a time-space transformation that casts the original problem into a more tractable one with an infinite horizon and a Brownian motion underneath. We conclude by commenting on two different numerical algorithms to compute the free-boundary equation and discuss illustrative cases that shed light on the boundary's shape. In particular, the free boundary does not generally share the monotonicity of the Brownian bridge case.

Reference

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Contents

4.1 Introduction **90**
4.2 Formulation of the problem **91**
4.3 Reformulation of the problem **91**
4.4 Solution of the reformulated problem: a direct approach **93**
4.5 Solution of the original problem and some extensions **104**
4.6 Numerical results **106**
4.7 Conclusions **106**
References **108**

4.1 Introduction

Since their first appearance in the seminal monograph of Wald (1947), Optimal Stopping Problems (OSPs) have become ubiquitous tools in mathematical finance, stochastic analysis, and mathematical statistics, among many other fields. Particularly, OSPs that are non-homogeneous in time are known to be mathematically challenging and, compared to the time-homogeneous counterpart, the literature addressing this topic is scarce, non-comprehensive, and often heavy on smoothness conditions. Markov bridges are not only time-inhomogeneous processes, but they also fail to meet the common assumption of Lipschitz continuity of the underlying drift (see, e.g., Krylov and Aries (1980, Chapter 3), or Jacka and Lynn (1992)), as their drifts explode when time approaches the horizon, thus inherently adding an extra layer of complexity.

The first result in OSPs with Markov bridges was given by Shepp (1969), who circumvented the complexity of dealing with a Brownian Bridge (BB) by using a time-space transformation that allowed reformulating the problem into a more tractable one with a Brownian motion underneath. Since then, more than fifty years ago, the use of Markov bridges in the context of OSPs has been narrowed to extending the result of Shepp (1969): Ekström and Wanntorp (2009) and Ernst and Shepp (2015) studied alternative methods of solutions; Ekström and Wanntorp (2009) and De Angelis and Milazzo (2020) looked at a broader class of gain functions, Glover (2020) randomized the horizon while Föllmer (1972), Leung et al. (2018), and Ekström and Vaicenavicius (2020) analyzed the randomization of the bridge's terminal point.

In finance, the use of a BB in OSPs has been motivated by several applications. Boyce (1970) applied it to the optimal selling of bonds; Baurdoux et al. (2015) suggested the use of a BB to model mispriced assets that could rapidly return to their fair price, or perishable commodities that become useless after a given deadline; and Ekström and Wanntorp (2009) used a BB to model the *stock-pinning* effect, that is, the phenomenon in which the price of a stock tends to be pulled towards the strike price of one of its underlying options with massive trading volumes at the expiration date. While these motivations encourage the investor to rely on a model with added information at the horizon, none of them are exclusive to a BB, its usage being rather driven by tractability issues. Thus, in those same scenarios, other bridge processes could be more appealing than the over-simplistic BB. In particular, we drive our attention to an Ornstein–Uhlenbeck Bridge (OUB) process, since its version without a fixed terminal point, the Ornstein–Uhlenbeck (OU) process, is often the reference model in many financial problems.

Indeed, OU processes are a go-to in finance when it comes to modeling assets with prices that fluctuate around a given level. This mean-reverting phenomenon has been systematically observed in a wide variety of markets. A good reference for either theory, applications, or empirical evidence of mean-reverting problems is Leung and Li (2015a). An example is given by the pair trading strategy, which consists on holding a position in one asset as well as the opposite position in another, both assets known to be correlated in a way that the spread between their prices shows mean reversion. Recently, many authors have tackled pair trading by using an OSP approach with an OU process. Ekström et al. (2011) found the best time to liquidate the spread in the presence of a stop-loss level; Leung and Li (2015b) used a discounted double OSP to compute the optimal buy-low-sell-high strategy in a perpetual frame; and Kitapbayev and Leung (2017) extended that result to a finite horizon and took the viewpoint of investors entering the spread either buying or shorting.

In this paper we solve the finite-horizon OSP featuring the identity as the gain function and an OUB as the underlying process. The solution is provided in terms of a non-linear, Volterra-type integral equation. Similarly to Shepp (1969), our methodology relies on a time-space change that casts the original problem into an infinite-horizon OSP with a Brownian motion as the underlying process. Due to the complexity of our resulting OSP, we use a direct approach to solve it rather than using the common candidate-verification scheme. We then show that one can either apply the inverse transformation to recover the solution of the original OSP or, equivalently, solve the Volterra integral equation reformulated back in terms of OUB. It is worth highlighting that the BB framework is included in our analysis as a limit case.

The rest of the paper is structured as follows. Section 4.2 introduces the main problem and some useful notation. In Section 4.3 we derive the transformed OSP and establish its equivalence to the original one. The most technical part of the paper is relegated to Section 4.4, in which we derive the solution of the reformulated OSP. From it, we use the reverse transformation to get the solution back to the original OSP in Section 4.5, where we also remark that both a BB and an OUB with general pulling level and terminal time are immediate consequences of our results. An algorithm for numerical approximations of the solution is given in Section 4.6, along with a compendium of illustrative cases for different values of the OUB's parameters. Concluding remarks are relegated to Section 4.7.

4.2 Formulation of the problem

Let $X = \{X_t\}_{t\in[0,1]}$ be an OUB with terminal value $X_1 = z$, $z \in \mathbb{R}$, and defined in the filtered space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,1]})$. That is, for an OU process $\widetilde{X} = \{\widetilde{X}_t\}_{t\in[0,1]}$, take X such that $\mathrm{Law}(X, \mathbb{P}) = \mathrm{Law}(\widetilde{X}, \widetilde{\mathbb{P}}_z)$, where $\widetilde{\mathbb{P}}_z := \mathbb{P}(\,\cdot\,|\widetilde{X}_1 = z)$. It is well known (see, e.g., Barczy and Kern (2013)) that X is the unique strong solution of the Stochastic Differential Equation (SDE)

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \gamma \mathrm{d}B_t, \quad 0 \le t \le 1, \tag{4.1}$$

with $\gamma > 0$ and

$$\mu(t, x) = \alpha \frac{z - \cosh(\alpha(1-t))x}{\sinh(\alpha(1-t))}, \quad \alpha \neq 0. \tag{4.2}$$

Note that we can take $\{\mathcal{F}_t\}_{t\in[0,1]}$ as the natural filtration of the underlying standard Brownian motion $\{B_s\}_{t\in[0,1]}$ in (4.1).

Consider the finite-horizon OSP

$$V(t, x) := \sup_{\tau \le 1-t} \mathbb{E}_{t,x}\left[X_{t+\tau}\right], \tag{4.3}$$

where V is the value function and $\mathbb{E}_{t,x}$ represents the expectation under the probability measure $\mathbb{P}_{t,x}$ defined as $\mathbb{P}_{t,x}(\cdot) := \mathbb{P}(\cdot|X_t = x)$. The supremum above is taken under all random times τ in the underlying filtration, such that $t + \tau$ is a stopping time in $\{\mathcal{F}_t\}_{t\in[0,1]}$. Henceforth, we will call τ a stopping time while keeping in mind that $t + \tau$ is the actual stopping time.

4.3 Reformulation of the problem

Barczy and Kern (2013) provide the following space-time transformed representation for X:

$$X_t = a_1(t, X_0, z) + a_2(t)B_{\psi(t)},$$

where the functions a_1 and a_2 take the form

$$a_1(t,x,z) := x\frac{\sinh(\alpha(1-t))}{\sinh(\alpha)} + z\frac{\sinh(\alpha t)}{\sinh(\alpha)}, \quad a_2(t) := \gamma e^{\alpha t}\frac{\kappa(1)-\kappa(t)}{\kappa(1)},$$

and $\psi : [0,1) \to \mathbb{R}_+$ is the time transformation $\psi(t) := \kappa(t)\kappa(1)/(\kappa(1)-\kappa(t))$, with $\kappa(t) := (2\alpha)^{-1}(1-e^{-2\alpha t})$. Notice that $t = \kappa^{-1}\left(\psi(t)\kappa(1)/(\psi(t)+\kappa(1))\right)$, where $\kappa^{-1}(s) = -(2\alpha)^{-1}\ln(1-2\alpha s)$. The following identities can be easily checked:

$$a_1(t,x,z) = \left(x + z\frac{\psi(t)e^{-\alpha}}{\kappa(1)}\right)\frac{1}{f\left(\frac{\psi(t)e^{-\alpha}}{\kappa(1)}\right)}, \quad a_2(t) = \frac{\gamma}{f\left(\frac{\psi(t)e^{-\alpha}}{\kappa(1)}\right)},$$

with

$$f(s) := \sqrt{\left(e^{\alpha}+s\right)\left(e^{-\alpha}+s\right)}. \tag{4.4}$$

Therefore, if we set the time change $s = \psi(t)e^{-\alpha}/\kappa(1)$, we get the space change

$$X_t = \frac{X_0 + zs}{f(s)} + \frac{\gamma}{f(s)}B_{s\kappa(1)e^{\alpha}} = \frac{zs + \gamma\sqrt{\kappa(1)e^{\alpha}}}{f(s)}\left(B_s + \frac{X_0}{\gamma\sqrt{\kappa(1)e^{\alpha}}}\right). \tag{4.5}$$

Let $Y = \{Y_s\}_{s\geq 0}$ be a Brownian motion starting at $Y_0 = y$ under the probability measure $\mathbb{P}_y$, that is, $\mathbb{P}_y\left(Y_0 = y\right) = 1$. Consider the infinite-horizon OSP

$$W_c(s,y) := \sup_{\sigma}\mathbb{E}_y\left[G_c(s+\sigma, Y_\sigma)\right], \tag{4.6}$$

with gain function

$$G_c(s,y) := \frac{cs + y}{f(s)} \tag{4.7}$$

and $c \in \mathbb{R}$. The operator $\mathbb{E}_y$ emphasizes that we are taking the mean with respect to $\mathbb{P}_y$, and the supremum in (4.6) is taken over all the stopping times σ in the natural filtration of $\{Y_s\}_{s\geq 0}$.

Solving an OSP means giving a tractable expression for the value function and finding a stopping time in which the supremum is attained. Thereby, we show in the next proposition the equivalence between (4.3) and (4.6), by providing formulae that relate V to W, and switch from a stopping time that is optimal in the former problem (if it exists) to one optimal in the latter.

Proposition 4.1. *(Time-space equivalence)*
Consider the time change $\upsilon : [0,1] \to \mathbb{R}$ such that $\upsilon(t) = \psi(t)e^{-\alpha}/\kappa(1)$. Take $(t,x) \in [0,1) \times \mathbb{R}$ and set $s = \upsilon(t)$, $c_z := z/(\gamma\sqrt{\kappa(1)e^{\alpha}})$, and $y = c_x$. Then:

(i) The following equation holds:

$$V(t,x) = \frac{z}{c_z}W_{c_z}(s,y). \tag{4.8}$$

(ii) The stopping time $\sigma^*(s,y)$ is optimal in (4.6) under $\mathbb{P}_y$ for $c = c_z$ if and only if

$$\tau^*(t,x) := \upsilon^{-1}\left(\sigma^*(s,y)\right) \tag{4.9}$$

is optimal in (4.3) under $\mathbb{P}_{t,x}$.

Proof. (*i*) We have already proved this part of the proposition. Indeed, (4.8) follows trivially from (4.3) and (4.5)–(4.7).

(*ii*) Suppose that $\sigma^* = \sigma^*(s,y)$ is optimal in (4.6) under $\mathbb{P}_y$ for $c = c_z$. Assume that there exists a stopping time $\tau' = \tau'(t,x)$ that outperforms $\tau^* = \tau^*(t,x)$ defined in (4.9), and set $\sigma' = \sigma'(s,y) := \upsilon^{-1}(\tau')$. Then, by relying on (4.5), we get that

$$\mathbb{E}_y\left[G_{c_z}\left(s+\sigma', Y_{\sigma'}\right)\right] = \mathbb{E}_{t,x}\left[X_{t+\tau'}\right] > \mathbb{E}_{t,x}\left[X_{t+\tau'}\right] = \mathbb{E}_y\left[G_{c_z}\left(s+\sigma^*, Y_{\sigma^*}\right)\right],$$

which contradicts the fact that σ^* is optimal in (4.6). Then, we have proved the *only if* part of the statement. The *if* direction follows by similar arguments. ☐

4.4 Solution of the reformulated problem: a direct approach

In this section we will work out a solution for the OSP (4.6). For the sake of briefness and since there is no risk of confusion, throughout the section we will use the notations $W = W_c$ and $G = G_c$, so that (4.6) can be rewritten as

$$W(s,y) = \sup_{\sigma} \mathbb{E}_y\left[G(s+\sigma, Y_\sigma)\right]. \tag{4.10}$$

Notice that $0 \le s/f(s) \le 1$ and $f(s) \ge \sqrt{1+s^2}$ for all $s \in \mathbb{R}_+$, $f(0) = 1$, and f is increasing. Hence, the following holds for $M := \mathbb{E}\left[\sup_{0\le u\le 1}|B_u|\right]$ and all $(s,y) \in \mathbb{R}_+ \times \mathbb{R}$:

$$\begin{aligned}
\mathbb{E}_y\left[\sup_{u\ge 0}\left|G\left(s+u, Y_u\right)\right|\right] &\le |c| + \mathbb{E}_y\left[\sup_{u\ge 0}\frac{|Y_u|}{f(u)}\right] \le |c| + |y| + \mathbb{E}\left[\sup_{u\ge 0}\frac{|B_u|}{\sqrt{1+u^2}}\right] \\
&\le |c| + |y| + M + \mathbb{E}\left[\sup_{u\ge 1}\frac{|B_u|}{\sqrt{1+u^2}}\right] \\
&= |c| + |y| + M + \mathbb{E}\left[\sup_{u\ge 1}\frac{u}{\sqrt{1+u^2}}\left|B_{1/u}\right|\right] \\
&\le |c| + |y| + M + \mathbb{E}\left[\sup_{u\ge 1}\left|B_{1/u}\right|\right] = |c| + |y| + 2M,
\end{aligned} \tag{4.11}$$

where we used the time-inversion property of a Brownian motion in the first equality. Thereby, since $M < \infty$ and G is continuous, we get that (see, e.g., Corollary 2.9, Remark 2.10, and Equation (2.2.80) in Peskir and Shiryaev (2006)) the first hitting time

$$\sigma^*(s,y) = \inf\left\{u \ge 0 : (s+u, Y_u) \in \mathcal{D}\right\} \tag{4.12}$$

into the stopping set $\mathcal{D} := \{W = G\}$ is optimal for (4.10). That is,

$$W(s,y) = \mathbb{E}_y\left[G\left(s+\sigma^*(s,y), Y_{\sigma^*(s,y)}\right)\right]. \tag{4.13}$$

After applying Itô's lemma to both (4.6) and (4.13) we get the following alternative representations of W:

$$W(s,y) - G(s,y) = \sup_{\sigma} \mathbb{E}_y\left[\int_0^{\sigma} \mathbb{L}G\left(s+u, Y_u\right)\,\mathrm{d}u\right] = \mathbb{E}_y\left[\int_0^{\sigma^*(s,y)} \mathbb{L}G\left(s+u, Y_u\right)\,\mathrm{d}u\right], \tag{4.14}$$

where $\mathbb{L} = \partial_t + \frac{1}{2}\partial_{xx}$ is the infinitesimal generator of $\{(s+u, Y_u)\}_{u\geq 0}$. Here and thereafter, ∂_t and ∂_x will stand, respectively, for the differential operator with respect to time and space, while ∂_{xx} is a shorthand for $\partial_x\partial_x$. Note that $\mathbb{L}G = \partial_t G$. Since many of the proofs rely on the first-order partial derivatives of the gain function, we display them next for quick reference:

$$\partial_t G(s,y) = \frac{c\,(f(s) - sf'(s)) - f'(s)y}{f^2(s)}, \tag{4.15}$$

$$\partial_x G(s,y) = \frac{1}{f(s)}. \tag{4.16}$$

To keep track of the initial condition in a way that does not change the underlying probability measure, we introduce the process $Y^y = \{Y_s^y\}_{s\geq 0}$ such that

$$\mathrm{Law}\left(\{Y_s^y\}_{s\geq 0}, \mathbb{P}\right) = \mathrm{Law}\left(\{Y_s\}_{s\geq 0}, \mathbb{P}_y\right).$$

Notice that the characterization of the Optimal Stopping Time (OST) in (4.12) is too abstract to work with. In the next proposition we characterize $\sigma^*(s,y)$ by means of a function called the Optimal Stopping Boundary (OSB), which is the frontier between $\mathcal{D}$ and its complement $\mathcal{C} := \{W > G\}$. We also derive some properties about the shape of the OSB that shed light on the geometry of $\mathcal{D}$ and $\mathcal{C}$.

Proposition 4.2 (Existence and shape of the optimal stopping boundary)**.**
There exists a function $b : \mathbb{R}_+ \to \mathbb{R}$ such that $\mathcal{D} = \{(s,y) : y \geq b(s)\}$. Moreover, $c(f(s) - sf'(s))/f(s) < b(s) < \infty$ for all $s \in \mathbb{R}_+$.

Proof. The claimed shape for the stopping set, $\mathcal{D} = \{(s,y) : y \geq b(s)\}$, is a straightforward consequence of the fact that $y \mapsto (W - G)(s,y)$ is decreasing for all $s \in \mathbb{R}_+$, which follows after (4.4), (4.14), and (4.15).

We now see that $b(s) > c(f(s) - sf'(s))/f(s)$ for all $s > 0$. Fix a pair (s,y) such that $\partial_t G(s,y) > 0$. Then, the continuity of $\partial_t G$ allows to pick a ball $\mathcal{B}$ such that $(s,y) \in \mathcal{B}$ and $\partial_t G > 0$ in $\mathcal{B}$. After recalling (4.14) and setting $\sigma_{\mathcal{B}}$ as the first exit time of $\{(s+u, Y_u^y)\}_{u\geq 0}$ from $\mathcal{B}$, we get that

$$W(s,y) - G(s,y) \geq \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{B}}} \partial_t G\,(s+u, Y_u)\,\mathrm{d}u\right] > 0.$$

We conclude then that $(s,y) \in \mathcal{C}$. Finally, the claimed lower bound for b comes after using (4.15) to realize that $\partial_t G(s,y) > 0$ if and only if $y < c(f(s) - sf'(s))/f(s)$.

We now prove $b(s) < \infty$ for all $s > 0$. Let $\widetilde{X} = \{\widetilde{X}_t\}_{t\in[0,1]}$ be a BB with pinning point $\widetilde{X}_1 = z$. The drift of $\widetilde{X}$ has the form $\widetilde{\mu}(t,x) = (z-x)/(1-t)$. Define $m_z : [0,1) \to \mathbb{R}$ such that

$$m_z(t) = z\frac{\sinh(\alpha(1-t)) - \alpha(1-t)}{\sinh(\alpha(1-t)) - \alpha(1-t)\cosh(\alpha(1-t))},$$

and notice that $\mu(t,x) \leq \widetilde{\mu}(t,x)$ if and only if $x \geq m_z(t)$. Take $\overline{M}_z := \sup_{t\in[0,1)} m_z(t) < \infty$ and notice the following relation:

$$X_t \leq m_z(t) + |X_t - m_z(t)| \leq m_z(t) + |\widetilde{X}_t - m_z(t)| \leq \overline{M}_z + |\widetilde{X}_t - \overline{M}_z|.$$

The second inequality holds since the drift of the process $t \mapsto m_z(t) + |X_t - m_z(t)|$ is lower than the drift of $t \mapsto m_z(t) + |\widetilde{X}_t - m_z(t)|$, and therefore we can ensure that, pathwise, the first process is lower than the last one $\mathbb{P}$-a.s. (see Corollary 3.1 by Peng and Zhu (2006)). The third inequality is straightforward from the definition of $\overline{M}_z$. Therefore, if we consider the OSP

$$\widetilde{V}_{\overline{M}_z}(t,x) = \sup_{\tau \le 1-t} \mathbb{E}_{t,x}\left[\overline{M}_z + |\widetilde{X}_{t+\tau} - \overline{M}_z|\right],$$

we are allowed to state that $V \le \widetilde{V}_{\overline{M}_z}$. If we take a pair $(t,x) \in [0,1] \times [\overline{M}_z, \infty)$ within the stopping set related to $V_{\overline{M}_z}$, then $V(t,x) \le V_{\underline{M}_z}(t,x) = x$, meaning that (t,x) lies in the stopping set of V. Since it is known that the OSB related to $V_{\underline{M}_z}$ is finite (actually, this is one of the few cases in which the explicit form of the OSP with a finite horizon is available; see, e.g., Theorem 3.2 in Ekström and Wanntorp (2009)), so is the one related to V. Then, using (4.8), we conclude that b is bounded from above. □

We next show that W is Lipschitz continuous on sets of the type $\mathbb{R}_+ \times \mathcal{R}$, where $\mathcal{R}$ stands for a compact set in $\mathbb{R}$.

Proposition 4.3 (Lipschitz continuity of the value function)**.**
For any compact set $\mathcal{R} \subset \mathbb{R}$, there exists a constant $L_{\mathcal{R}} > 0$ such that

$$|W(s_1,y_1) - W(s_2,y_2)| \le L_{\mathcal{R}}\left(|s_1 - s_2| + |y_1 + y_2|\right),$$

for all $(s_1,y_1),(s_2,y_2) \in \mathbb{R}_+ \times \mathcal{R}$.

Proof. Take $(s_1,y_1),(s_2,y_2) \in \mathbb{R}_+ \times \mathcal{R}$ and realize that

$$\begin{aligned} W(s_1,y_1) - W(s_2,y_2) = &\sup_\sigma \mathbb{E}_{y_1}\left[G(s_1+\sigma, Y_\sigma)\right] - \sup_\sigma \mathbb{E}_{y_2}\left[G(s_1+\sigma, Y_\sigma)\right] \\ &+ \sup_\sigma \mathbb{E}_{y_2}\left[G(s_1+\sigma, Y_\sigma)\right] - \sup_\sigma \mathbb{E}_{y_2}\left[G(s_2+\sigma, Y_\sigma)\right]. \end{aligned}$$

Notice from (4.15) that the following relation holds:

$$|\partial_t G(s,y)| \le K\left(1 + \frac{|y|}{f(u)}\right).$$

Then, since $|\sup_\sigma a_\sigma - \sup_\sigma b_\sigma| \le \sup_\sigma |a_\sigma - b_\sigma|$, alongside Jensen's inequality, and (4.15) and (4.16), we get that

$$\begin{aligned} &\left|\sup_\sigma \mathbb{E}_{y_1}\left[G(s_1+\sigma, Y_\sigma)\right] - \sup_\sigma \mathbb{E}_{y_2}\left[G(s_1+\sigma, Y_\sigma)\right]\right| \\ &\qquad\le \sup_\sigma \mathbb{E}\left[|G(s_1+\sigma, Y_\sigma^{y_1}) - G(s_1+\sigma, Y_\sigma^{y_2})|\right] \\ &\qquad= \sup_\sigma \mathbb{E}\left[\frac{|Y_\sigma^{y_1} - Y_\sigma^{y_2}|}{f(s_1+\sigma)}\right] = \frac{|y_1 - y_2|}{f(s_1)} \le |y_1 - y_2|, \end{aligned}$$

and

$$\begin{aligned} &\left|\sup_\sigma \mathbb{E}_{y_2}\left[G(s_1+\sigma, Y_\sigma)\right] - \sup_\sigma \mathbb{E}_{y_2}\left[G(s_2+\sigma, Y_\sigma)\right]\right| \\ &\qquad\le \sup_\sigma \mathbb{E}\left[|G(s_1+\sigma, Y_\sigma^{y_2}) - G(s_2+\sigma, Y_\sigma^{y_2})|\right] \\ &\qquad= |s_1 - s_2| \sup_\sigma \mathbb{E}\left[|\partial_t G(\xi, Y_\sigma^{y_2})|\right] \\ &\qquad\le |s_1 - s_2| K\left(1 + \mathbb{E}\left[\sup_{s\ge 0} \frac{|Y_s^{y_2}|}{f(s)}\right]\right), \end{aligned}$$

where $\xi \in (\min\{s_1, s_2\}, \max\{s_1, s_2\})$ follows from the mean value theorem. Since we already proved in (4.11) that $\mathbb{E}\left[\sup_{s\geq 0}|Y_s^{y_2}|/f(s)\right] < \infty$, the Lipschitz continuity of W in $\mathbb{R}_+ \times \mathcal{R}$ follows. $\square$

Beyond Lipschitz continuity, it turns out that the value function attains a higher smoothness away from the boundary. While this assertion is trivial in the interior of the stopping region, where $W = G$, we prove in the next proposition that it also holds in the continuation set. In addition, we show that $\mathbb{L}W$ vanishes in $\mathcal{C}$, which establishes the equivalence between (4.10) and a free-boundary problem.

Proposition 4.4 (Higher smoothness of the value function and the free-boundary problem). *$W \in C^{1,2}(\mathcal{C})$ and $\mathbb{L}W = 0$ in $\mathcal{C}$.*

Proof. The fact that $\mathbb{L}W = 0$ in $\mathcal{C}$ comes right after the strong Markov property of $\{(s+u, Y_u)\}_{u\geq 0}$; see Peskir and Shiryaev (2006, Section 7.1) for more details.

Since W is continuous on $\mathcal{C}$ (see Proposition 4.3) and the coefficients in the parabolic operator $\mathbb{L}$ are smooth enough (it suffices to require local α-Hölder continuity), then standard theory from parabolic partial differential equations Friedman (1964, Section 3, Theorem 9) guarantees that, for an open rectangle $R \subset \mathcal{C}$, the first initial-boundary value problem

$$\mathbb{L}f = 0 \qquad \text{in } R, \tag{4.17a}$$

$$f = V \qquad \text{on } \partial R \tag{4.17b}$$

has a unique solution $f \in C^{1,2}(R)$. Therefore, we can use Itô's formula on $f(s+u, Y_u)$ at $u = \tau_{R^c}$, that is, the first time $(s+u, Y_u)$ exits R, and then take $\mathbb{P}_y$-expectation with $y \in R$, which guarantees the vanishing of the martingale term and yields, together with (4.17a) and (4.17b), the equality $\mathbb{E}_y[W(s+\tau_{R^c}, Y_{\tau_{R^c}})] = f(t,x)$. Finally, due to the strong Markov property, $\mathbb{E}_y[W(s+\tau_{R^c}, Y_{\tau_{R^c}})] = W(s,y)$. $\square$

Not only the gain function has continuous partial derivatives away from the boundary, but we can provide relatively explicit forms for those derivatives, as shown in the next proposition.

Proposition 4.5 (Partial derivatives of the value function). *Let $\sigma^* = \sigma^*(s,y)$, for $(s,y) \in \mathcal{C}$, and $a := e^{-\alpha} + e^{\alpha}$. Then,*

$$\partial_t W(s,y) = \partial_t G(s,y) + \mathbb{E}\left[\int_s^{s+\sigma^*} \frac{1}{f^3(u)}\left(-c\,(a+3u) + \frac{3\,(a+2u)^2}{4f^2(u)} - Y_{u-s}^y\right)\mathrm{d}u\right] \tag{4.18}$$

and

$$\partial_x W(s,y) = \mathbb{E}\left[\frac{1}{f(s+\sigma^*)}\right]. \tag{4.19}$$

Proof. Take $(s,y) \in \mathcal{C}$ and $\varepsilon > 0$. Due to (4.10) and (4.13), we get the following for $\sigma^* = \sigma^*(s,y)$:

$$\varepsilon^{-1}\left(W(s,y) - W(s-\varepsilon,y)\right) \leq \varepsilon^{-1}\mathbb{E}\left[G(s+\sigma^*, Y_{\sigma^*}^y) - G(s-\varepsilon+\sigma^*, Y_{\sigma^*}^y)\right].$$

Hence, by letting $\varepsilon \to 0$ and recalling that $W \in C^{1,2}(\mathcal{C})$ (see Proposition 4.4), we get that

$$\partial_t W(s,y) \leq \mathbb{E}\left[\partial_t G(s+\sigma^*, Y_{\sigma^*}^y)\right] = \partial_t G(s,y) + \mathbb{E}\left[\int_s^{s+\sigma^*} \mathbb{L}\partial_t G(u, Y_{s-u}^y)\,\mathrm{d}u\right]. \tag{4.20}$$

In the same fashion, we obtain

$$\varepsilon^{-1}\left(W(s+\varepsilon,y)-W(s,y)\right) \geq \varepsilon^{-1}\mathbb{E}\left[G(s+\varepsilon+\sigma^*,Y^y_{\sigma^*})-G(s+\sigma^*,Y^y_{\sigma^*})\right].$$

Thus, by arguing as in (4.20) we get the reverse inequality, and therefore (4.18) is proved after computing $\mathbb{L}\partial_t G(u,Y^y_{s-u}) = \partial_{tt}G(u,Y^y_{s-u})$.

To get the analog result for the space coordinate, notice that

$$\begin{aligned}
\varepsilon^{-1}\left(W(s,y)-W(s,y-\varepsilon)\right) &\leq \varepsilon^{-1}\mathbb{E}\left[W(s+\sigma^*,Y^y_{\sigma^*})-W(s+\sigma^*,Y^{y-\varepsilon}_{\sigma^*})\right] \\
&\leq \varepsilon^{-1}\mathbb{E}\left[G(s+\sigma^*,Y^y_{\sigma^*})-G(s+\sigma^*,Y^{y-\varepsilon}_{\sigma^*})\right] \\
&= \mathbb{E}\left[\frac{1}{f(s+\sigma^*)}\right],
\end{aligned}$$

while the same reasoning yields the inequality $\varepsilon^{-1}\left(W(s,y+\varepsilon)-W(s,y)\right) \geq \mathbb{E}\left[1/f(s+\sigma^*)\right]$, and then, by letting $\varepsilon \to 0$, we get (4.19). □

So far we have proved that solving (4.10) is equivalent to solving the free-boundary problem

$$\begin{aligned}
\mathbb{L}W(s,y) &= 0 && \text{for } y < b(t), && (4.21a)\\
W(s,y) &> G(s,y) && \text{for } y < b(t), && (4.21b)\\
W(s,y) &= G(s,y) && \text{for } y \geq b(t). && (4.21c)
\end{aligned}$$

However, an additional condition for the value function on the free boundary is required to guarantee a unique solution. Roughly speaking, this condition comes in the form of smoothly binding the value and the gain functions with respect to the space coordinate, provided that the optimal boundary is (probabilistically) regular for the interior of D, that is, if after starting at a point $(s,y) \in \partial\mathcal{C}$, the process enters the interior of D immediately $\mathbb{P}_y$-a.s. (see De Angelis and Peskir (2020)). This type of regularity can be obtained for locally Lipschitz continuous OSBs (see Proposition 4.7 ahead).

In the next proposition we show that the boundary is Lipschitz continuous on any bounded interval. The proof is inspired by Theorem 4.3 from De Angelis and Stabile (2019), which states the boundary's local Lipschitz continuity for high-dimensional processes with some regularity conditions. Our settings do not satisfy Assumption (D) in De Angelis and Stabile (2019), which establishes a relation between the partial derivatives of G.

Proposition 4.6 (Lipschitz continuity of the optimal stopping boundary). *For any closed interval $I := [\underline{s},\overline{s}] \subset \mathbb{R}_+$, there exists a constant $L_I > 0$ such that*

$$|b(s_1)-b(s_2)| \leq L_I, \tag{4.22}$$

whenever $s_1, s_2 \in I$.

Proof. Consider the function $H : I\times\mathbb{R} \to \mathbb{R}_+$, for a closed interval $I \subset \mathbb{R}_+$, defined as $H(s,y) = W(s,y)-G(s,y)$. Proposition 4.2 entails that b is bounded from below, and thus we can choose a constant $r \in \mathbb{R}$ such that $r < \inf\{b(s) : s \in I\}$. Since $I\times\{r\} \subset \mathcal{C}$, H is continuous (see Proposition 4.3) and $H|_{I\times\{r\}} > 0$. Then, there exists $a > 0$ such that $H(s,r) > a$ for all $s \in I$. Therefore, for all δ such that $0 < \delta \leq a$, the equation $H(s,y) = \delta$ has a solution in $\mathcal{C}$ for all $s \in I$. Moreover, this solution is unique for each s since $\partial_x H < 0$ in $\mathcal{C}$ (see Proposition 4.5), and we denote it by $b_\delta(s)$, where $b_\delta : I \to \mathbb{R}$. Away from the boundary, H is regular enough to apply the implicit function theorem that guarantees that b_δ is differentiable and

$$b_\delta'(s) = -\partial_t H(s, b_\delta(s))/\partial_x H(s, b_\delta(s)). \tag{4.23}$$

Note that b_δ is decreasing in δ and therefore converges pointwise to some limit function b_0, which satisfies $b_0 \le b$ in I as $b_\delta < b$ for all δ. Since $H(s, b_\delta(s)) = \delta$ and H is continuous, it follows that $H(s, b_0(s)) = 0$ after taking $\delta \to 0$, which means that $b_0 \ge b$ in I and hence $b_0 = b$ in I.

Take $(s, y) \in \mathcal{C}$ such that $y > r$. Set $\sigma^* = \sigma^*(s, y)$ and consider

$$\sigma_r = \sigma_r(s, y) := \inf\{u \ge 0 : (s + u, Y_u^y) \notin I \times (r, \infty)\}.$$

Recalling (4.18), it is easy to check that there exists a constant $K_I^{(1)} > 0$ such that

$$|\partial_t H(s, y)| \le K_I^{(1)} m(s, y) \tag{4.24}$$

with

$$m(s, y) := \mathbb{E}_y\left[\int_0^{\sigma^*}\left(1 + \frac{|Y_u|}{f^2(s + u)}\right)\mathrm{d}u\right].$$

Using the tower property of conditional expectation, alongside the strong Markov property, we get

$$\begin{aligned}
&m(s, y)\\
&= \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r}\left(1 + \frac{|Y_u|}{f^2(s+u)}\right)\mathrm{d}u + \mathbb{1}(\sigma_r \le \sigma^*)\int_{\sigma_r}^{\sigma^*}\left(1 + \frac{|Y_u|}{f^2(s+u)}\right)\mathrm{d}u\right]\\
&= \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r}\left(1 + \frac{|Y_u|}{f^2(s+u)}\right)\mathrm{d}u + \mathbb{1}(\sigma_r \le \sigma^*)\,\mathbb{E}_y\left[\int_{\sigma_r}^{\sigma_r + \sigma^*(\sigma_r, Y_{\sigma_r})}\left(1 + \frac{|Y_u|}{f^2(s+u)}\right)\mathrm{d}u\Big|\mathcal{F}_{\sigma_r}\right]\right]\\
&= \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r}\left(1 + \frac{|Y_u|}{f^2(s+u)}\right)\mathrm{d}u + \mathbb{1}(\sigma_r \le \sigma^*)\,\mathbb{E}_{Y_{\sigma_r}}\left[\int_0^{\sigma^*(\sigma_r, Y_{\sigma_r})}\left(1 + \frac{|Y_u|}{f^2(s+\sigma_r+u)}\right)\mathrm{d}u\right]\right]\\
&= \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r}\left(1 + \frac{|Y_u|}{f^2(s+u)}\right)\mathrm{d}u + \mathbb{1}(\sigma_r \le \sigma^*)\,m(s + \sigma_r, Y_{\sigma_r})\right].
\end{aligned} \tag{4.25}$$

Notice that, for $c < r < y < b(s)$, $(s + \sigma_r, Y_{\sigma_r}^y) \in \Gamma_s$ on the set $\{\sigma_r \le \sigma^*\}$, with $\Gamma_s := \{(s, \bar{s}) \times \{r\}\} \cup \{\bar{s} \times [r, b(\bar{s}))\}$ and $\bar{s} := \sup\{s : s \in I\}$. Hence, the following holds true on the set $\{\sigma_r \le \sigma^*\}$:

$$\begin{aligned}
m\left(s + \sigma_r, Y_{\sigma_r}^y\right) &\le \sup_{(t,x)\in\Gamma_s} m(t, x)\\
&\le \sup_{(t,x)\in\Gamma_s} \mathbb{E}_x\left[\int_0^\infty\left(1 + \frac{|Y_u|}{f^2(t+u)}\right)\mathrm{d}u\right]\\
&\le \sup_{(t,x)\in\Gamma_s}\int_0^\infty\left(1 + \frac{|x|}{f^2(t+u)}\right)\mathrm{d}u + \int_0^\infty \frac{\mathbb{E}[|B_u|]}{f^2(t+u)}\mathrm{d}u\\
&\le \int_0^\infty\left(1 + \frac{|b(\bar{s})|}{f^2(u)}\right)\mathrm{d}u + \int_0^\infty\sqrt{\frac{2}{\pi}}\frac{\sqrt{u}}{f^2(u)}\mathrm{d}u < \infty.
\end{aligned} \tag{4.26}$$

By plugging (4.26) into (4.25), after observing that $(1 + |Y_u|/f^2(s+u)) \le 1 + \max\{|\sup_{s\in I} b(s)|, |r|\}$, and recalling (4.24), we obtain the following for some constant $K_I^{(2)} > 0$:

$$|\partial_t H(s, y)| \le K_I^{(2)}\mathbb{E}_y\left[\sigma_\delta \wedge \sigma_r + \mathbb{1}(\sigma_r \le \sigma_\delta)\right]. \tag{4.27}$$

Arguing as in (4.25) and recalling (4.16) along with (4.19), we get that

$$\begin{aligned}
&|\partial_x H(s,y)| \\
&= \mathbb{E}_y\left[\frac{1}{f(s)} - \frac{1}{f(s+\sigma^*)}\right] = \mathbb{E}_y\left[\int_0^{\sigma^*} -\partial_t(1/f)(s+u)\,\mathrm{d}u\right] \\
&= \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r} -\partial_t(1/f)(s+u)\,\mathrm{d}u + \mathbb{1}\left(\sigma_r \le \sigma^*\right)|\partial_x H(s+\sigma_r, Y_{\sigma_r})|\right] \\
&\ge \mathbb{E}_y\left[\int_0^{\sigma^*\wedge\sigma_r} -\partial_t(1/f)(s+u)\,\mathrm{d}u + \mathbb{1}\left(\sigma_r \le \sigma^*, \sigma_r < \bar{s}-s\right)|\partial_x H(s+\sigma_r, r)|\right].
\end{aligned} \tag{4.28}$$

Take $\varepsilon > 0$ such that $\mathcal{R}_\varepsilon := [\underline{s}, \bar{s}+\varepsilon] \times (r-\varepsilon, r+\varepsilon) \subset \mathcal{C}$, and consider the stopping time $\sigma_\varepsilon = \inf\left[u \ge 0 : Y_u^r \notin \mathcal{R}_\varepsilon\right]$. Observe that $\sigma^*(s,r) > \sigma_\varepsilon$ for all $s \in I$. Then,

$$\begin{aligned}
|\partial_x H(s+\sigma_r, r)| &\ge \inf_{s\in I} |\partial_x H(s,r)| = \inf_{s\in I} \mathbb{E}_r\left[\frac{1}{f(s)} - \frac{1}{f(s+\sigma^*(s,r))}\right] \\
&\ge \inf_{s\in I} \mathbb{E}_r\left[\frac{1}{f(s)} - \frac{1}{f(s+\sigma_\varepsilon)}\right] \\
&\ge \inf_{s\in I}\left(\frac{1}{f(s)} - \frac{1}{f(\bar{s}+\varepsilon)}\right)\mathbb{P}_r\left(\sigma_\varepsilon = \bar{s}+\varepsilon-s\right) \\
&= \left(\frac{1}{f(\bar{s})} - \frac{1}{f(\bar{s}+\varepsilon)}\right)\mathbb{P}\left(\sup_{u\le\bar{s}+\varepsilon-\underline{s}} |B_u| < \varepsilon\right) > 0,
\end{aligned} \tag{4.29}$$

where we used the fact that $s \mapsto 1/f(s) - 1/f(s+u)$ is decreasing for all $u \ge 0$. After noticing that $-\partial_t(1/f)$ is positive and decreasing, which means that $-\partial_t(1/f)(s+u) \ge -\partial_t(1/f)(\bar{s}) > 0$ for all $u \le \sigma_r$, and by plugging (4.29) into (4.28), we obtain, for a constant $K_{I,\varepsilon}^{(3)} > 0$,

$$|\partial_x H(s,y)| \ge K_I^{(3)} \mathbb{E}_y\left[\sigma^* \wedge \sigma_r + \mathbb{1}\left(\sigma_r \le \sigma^*, \sigma_r < \bar{s}-s\right)\right]. \tag{4.30}$$

Therefore, using (4.27) and (4.30) in (4.23) yields the following bound for some constant $K_I^{(4)} > 0$, $y_\delta = b_\delta(s)$, and $\sigma_\delta = \sigma^*(s, y_\delta)$:

$$\begin{aligned}
|b_\delta'(s)| &\le K_I^{(4)} \frac{\mathbb{E}_{y_\delta}\left[\sigma_\delta \wedge \sigma_r + \mathbb{1}\left(\sigma_r \le \sigma_\delta\right)\right]}{\mathbb{E}_{y_\delta}\left[\sigma_\delta \wedge \sigma_r + \mathbb{1}\left(\sigma_r \le \sigma_\delta, \sigma_r < \bar{s}-s\right)\right]} \\
&\le K_I^{(4)}\left(1 + \frac{\mathbb{P}_{y_\delta}\left(\sigma_r \le \sigma_\delta\right)}{\mathbb{E}_{y_\delta}\left[\sigma_\delta \wedge \sigma_r + \mathbb{1}\left(\sigma_r \le \sigma_\delta, \sigma_r < \bar{s}-s\right)\right]}\right) \\
&\le K_I^{(4)}\left(1 + \frac{\mathbb{P}_{y_\delta}\left(\sigma_r \le \sigma_\delta, \sigma_r = \bar{s}-s\right)}{\mathbb{E}_{y_\delta}\left[\sigma_\delta \wedge \sigma_r\right]} + \frac{\mathbb{P}_{y_\delta}\left(\sigma_r \le \sigma_\delta, \sigma_r < \bar{s}-s\right)}{\mathbb{E}_{y_\delta}\left[\mathbb{1}\left(\sigma_r \le \sigma_\delta, \sigma_r < \bar{s}-s\right)\right]}\right) \\
&\le K_I^{(4)}\left(2 + \frac{\mathbb{P}_{y_\delta}\left(\sigma_r \le \sigma_\delta, \sigma_r = \bar{s}-s\right)}{\mathbb{E}_{y_\delta}\left[\mathbb{1}\left(\sigma_r \le \sigma_\delta, \sigma_r = \bar{s}-s\right)\left(\sigma_\delta \wedge \sigma_r\right)\right]}\right) \\
&\le K_I^{(4)}\left(2 + \frac{1}{\bar{s}-s}\right).
\end{aligned} \tag{4.31}$$

If we set $I_\varepsilon = [\underline{s}, \bar{s}-\varepsilon]$ for $\varepsilon > 0$ small enough, then, by relying on (4.31), we obtain the existence of a constant $L_{I_\varepsilon} > 0$, independent from δ, such that $|b_\delta'(s)| < L_{I_\varepsilon}$ for all $s \in I_\varepsilon$ and $0 < \delta \le a$. We are thus able to use the Arzelà–Ascoly's theorem to guarantee that b_δ converges to b uniformly with respect to δ in I_ε. ☐

Once we have the Lipschitz continuity of the boundary on bounded sets, we proceed to illustrate in the following proposition how to obtain the principle of smooth fit, which, as we highlighted before, is required to provide a unique solution to the associated free-boundary problem (4.21a)–(4.21c).

Proposition 4.7 (The smooth-fit condition)**.**
For all $s \geq 0$, $y \mapsto W(s, y)$ is differentiable at $y = b(s)$. Moreover, $\partial_x W(s, b(s)) = \partial_x G(s, b(s))$.

Proof. Recall that we have already obtained in (4.19) an explicit form for $\partial_x W$ away from the boundary, namely,

$$\partial_x W(s, y) = \mathbb{E}\left[\frac{1}{f(s + \sigma^*(s, y))}\right], \quad (s, y) \in \mathcal{C}.$$

The principle of smooth fit is just validation of this formula whenever $y = b(s)$, $s \in \mathbb{R}_+$.

We have that $\partial_x W(s, b(s)^+) = \partial_x G(s, b(s)) = 1/f(s)$, as $\sigma^*(s, y) = 0$ for all $y \geq b(s)$. By relying on the law of the iterated logarithm alongside the local Lipschitz continuity of b, we get that $(s, b(s))$ is probabilistically regular for the interior of $\mathcal{D}$, that is,

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} \mathbb{P}\left(\inf_{u \in (0,\varepsilon)} \left(Y_u^{s,b(s)} - b(s+u)\right) < 0\right) &= \lim_{\varepsilon \downarrow 0} \mathbb{P}\left(\inf_{u \in (0,\varepsilon)} \frac{Y_u^{s,b(s)} - b(s+u)}{\sqrt{2u \ln(\ln(1/u))}} < 0\right) \\
&\geq \lim_{\varepsilon \downarrow 0} \mathbb{P}\left(\inf_{u \in (0,\varepsilon)} \frac{Y_u^{s,b(s)} - b(s) + L_s u}{\sqrt{2u \ln(\ln(1/u))}} < 0\right) \\
&= \mathbb{P}\left(\liminf_{u \downarrow 0} \frac{Y_u^{s,b(s)} - b(s) + L_s u}{\sqrt{2u \ln(\ln(1/u))}} < 0\right) = 1,
\end{aligned}$$

for some $L_s > 0$. Corollary 6 from De Angelis and Peskir (2020) then provides that $\sigma^*(s, b(s)^-) = \sigma^*(s, b(s)) = 0$ $\mathbb{P}$-a.s. and, hence, the dominated convergence theorem entails that $\partial_x W(s, b(s)^-) = 1/f(s) = \partial_x G(s, b(s))$, thus concluding that the smooth-fit condition holds. □

We are now in the position of getting a tractable characterization of both the value function and the OSB. Propositions 4.2–4.7 allow us to use an extension of Itô's lemma on the function $W(s+t, Y_t)$ for $t \geq 0$. This extension was originally derived by Peskir (2005a) and later restated, in a way that applies more directly to our framework, in Lemma A2 from D'Auria et al. (2020). Recalling that $\mathbb{L}W = 0$ on $\mathcal{C}$ and $W = G$ on $\mathcal{D}$, and after taking $\mathbb{P}_y$-expectation (which cancels the martingale term), we get

$$\begin{aligned}
W(s, y) &= \mathbb{E}_y\left[W(s+t, Y_t)\right] - \mathbb{E}_y\left[\int_0^t (\mathbb{L}W)(s+u, Y_u)\,\mathrm{d}u\right] \\
&= \mathbb{E}_y\left[W(s+t, Y_t)\right] - \mathbb{E}_y\left[\int_0^t \partial_t G(s+u, Y_u)\,\mathbb{1}\left(Y_u \geq b(s+u)\right)\mathrm{d}u\right], \qquad (4.32)
\end{aligned}$$

where the local-time term does not appear due to the smooth-fit condition.

Lemma 4.1. *For all $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$,*

$$\lim_{u \to \infty} \mathbb{E}_y\left[W(s+u, Y_u)\right] = c.$$

Proof. The Markov property of Y, together with the fact that both $s \mapsto s/f(s)$ and $s \mapsto f(s)$ are increasing and $s/f(s) \to 1$ as $s \to \infty$, implies that

$$\mathbb{E}_y\left[W(s+u, Y_u)\right] = \mathbb{E}_y\left[\sup_\sigma \mathbb{E}_{Y_u}\left[G(s+u+\sigma, Y_\sigma)\right]\right] \leq \mathbb{E}_y\left[\mathbb{E}_{Y_u}\left[\sup_{r \geq 0} G(s+u+r, Y_r)\right]\right]$$

$$
\begin{aligned}
&= \mathbb{E}_y \left[\mathbb{E}_{Y_u} \left[\sup_{r \geq 0} \left\{ c \frac{s+u+r}{f(s+u+r)} + \frac{Y_r}{f(s+u+r)} \right\} \right] \right] \\
&\leq c \left(\mathbb{1}(c > 0) + \frac{s+u}{f(s+u)} \mathbb{1}(c \leq 0) \right) + \mathbb{E}_y \left[\sup_{r \geq 0} \frac{Y_{u+r}}{f(u+r)} \right], \qquad (4.33)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}_y \left[W(s+u, Y_u) \right] &\geq \mathbb{E}_y \left[\mathbb{E}_{Y_u} \left[\inf_{r \geq 0} G\left(s+u+r, Y_r\right) \right] \right] \\
&\geq c \left(\mathbb{1}(c < 0) + \frac{s+u}{f(s+u)} \mathbb{1}(c \geq 0) \right) + \mathbb{E}_y \left[\inf_{r \geq 0} \frac{Y_{u+r}}{f(s+u+r)} \right]. \qquad (4.34)
\end{aligned}
$$

Notice that

$$
\lim_{u \to \infty} \mathbb{E}_y \left[\sup_{r \geq 0} \frac{Y_{u+r}}{f(u+r)} \right] = \mathbb{E}_y \left[\lim_{u \to \infty} \sup_{r \geq u} \frac{Y_r}{f(r)} \right] = \mathbb{E}_y \left[\limsup_{u \to \infty} \frac{Y_u}{f(u)} \right] = 0,
$$

where in the first equality we applied the monotone convergence theorem and in the third one we used the law of the iterated logarithm as an estimate of the convergence of the process in the numerator. A similar argument yields

$$
\lim_{u \to \infty} \mathbb{E}_y \left[\inf_{r \geq 0} \frac{Y_{u+r}}{f(s+u+r)} \right] = 0.
$$

Thus, we can take $u \to \infty$ in both (4.33) and (4.34) to complete the proof. □

By taking $t \to \infty$ in (4.32) and relying on Proposition 4.1, we get the following pricing formula for the value function:

$$
\begin{aligned}
W(s, y) &= c - \mathbb{E}_y \left[\int_0^\infty \left(\mathbb{L} W \right) \left(s+u, Y_u \right) \, \mathrm{d}u \right] \\
&= c - \mathbb{E}_y \left[\int_0^\infty \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \geq b(s+u)\right) \, \mathrm{d}u \right]. \qquad (4.35)
\end{aligned}
$$

We can obtain a more tractable version of (4.35) by exploiting the linearity of $y \mapsto \partial_t G(s, y)$ (see (4.15)) as well as the Gaussianity of Y_u. Specifically, since $Y_u \sim \mathcal{N}(y, u)$ under $\mathbb{P}_y$, then $\mathbb{E}_y \left[Y_u \mathbb{1}\left(Y_u \geq x\right) \right] = \bar{\Phi}((x-y)/\sqrt{u}) y + \sqrt{u} \phi((x-y)/\sqrt{u})$, where $\bar{\Phi}$ and ϕ denote the survival and the density functions of a standard normal random variable, respectively. By shifting the integrating variable s units to the right, we get that

$$
W(s, y) = c - \int_s^\infty \frac{1}{f(u)} \left(c \bar{\Phi}_{s,y,u,b(u)} - \frac{(a+2u) \left((y+cu) \bar{\Phi}_{s,y,u,b(u)} + \sqrt{u-s} \phi_{s,y,u,b(u)} \right)}{2 f^2(u)} \right) \mathrm{d}u, \qquad (4.36)
$$

where $a = e^{-\alpha} + e^{\alpha}$ and

$$
\bar{\Phi}_{s_1, y_1, s_2, y_2} := \bar{\Phi} \left(\frac{y_2 - y_1}{\sqrt{s_2 - s_1}} \right), \quad \phi_{s_1, y_1, s_2, y_2} := \phi \left(\frac{y_2 - y_1}{\sqrt{s_2 - s_1}} \right), \quad y_1, y_2 \in \mathbb{R}, s_2 \geq s_1 \geq 0.
$$

Take now $y \downarrow b(s)$ in both (4.35) and (4.36) to derive the free-boundary equation

$$
G(s, b(s)) = c - \mathbb{E}_{b(s)} \left[\int_0^\infty \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \geq b(s+u)\right) \, \mathrm{d}u \right], \qquad (4.37)
$$

alongside its more explicit expression

$$G(s,b(s))$$
$$= c - \int_s^\infty \frac{1}{f(u)}\left(c\bar{\Phi}_{s,b(s),u,b(u)} - \frac{(a+2u)\left((b(s)+cu)\bar{\Phi}_{s,b(s),u,b(u)} + \sqrt{u-s}\phi_{s,b(s),u,b(u)}\right)}{2f^2(u)}\right) \mathrm{d}u.$$

It turns out that there exists a unique function b that solves (4.37), as we state in the next theorem. The proof of such an assertion follows from adapting the methodology used in Peskir (2005b, Theorem 3.1), where the uniqueness of the solution of the free-boundary equation is addressed for an American put option with a geometric Brownian motion.

Theorem 4.1. *The integral equation (4.37) admits a unique solution among the class of continuous functions $\beta : \mathbb{R}_+ \to \mathbb{R}$ of bounded variation and such that $\beta(s) > c$ for all $s \in \mathbb{R}_+$.*

Proof. Suppose there exists a function $\beta : \mathbb{R}_+ \to \mathbb{R}$ solving the integral equation (4.37), and define W^β as in (4.35), but with β instead of b. We can conclude from (4.35) that the integrand is twice continuously differentiable with respect to y and, therefore, we can obtain $\partial_x W^\beta$ and $\partial_{xx} W^\beta$ by differentiating inside the integral symbol and ensure they are continuous functions on $\mathbb{R}_+ \times \mathbb{R}$. Besides, the following expression for $\mathbb{L}W^\beta$ can be easily computed from (4.35):

$$\mathbb{L}W^\beta(s,y) = \partial_t G(t,y)\mathbb{1}(y \geq \beta(s)).$$

Define the sets

$$\mathcal{C}_\beta := \{(s,y) \in \mathbb{R}_+ \times \mathbb{R} : y < \beta(s)\}, \quad \mathcal{D}_\beta := \{(s,y) \in \mathbb{R}_+ \times \mathbb{R} : y \geq \beta(s)\}.$$

It turns out that, on both sets, W^β is regular enough to apply the extension of Itô's formula given in D'Auria et al. (2020, Lemma A2), which yields

$$W^\beta(s,y) = \mathbb{E}_y\left[W^\beta(s+t,Y_t)\right] - \mathbb{E}_y\left[\int_0^t \partial_t G(s+u,Y_u)\,\mathbb{1}\left(Y_u \geq \beta(s+u)\right)\mathrm{d}u\right], \tag{4.38}$$

where the martingale term is canceled after taking $\mathbb{P}_y$-expectation and the local time term is missing due to the continuity of $\partial_x W^\beta$ on $\partial\mathcal{C}_\beta$. In addition,

$$G(s,y) = \mathbb{E}_y\left[G(s+t,Y_t)\right] - \mathbb{E}_y\left[\int_0^t \partial_t G(s+u,Y_u)\,\mathrm{d}u\right]. \tag{4.39}$$

Consider the first hitting time $\sigma_{\mathcal{C}_\beta}$ into $\mathcal{C}_\beta$, fix $(s,y) \in \mathcal{D}_\beta$, and notice that $\mathbb{P}_y(Y_u \geq \beta(t+s)) = 1$ for all $0 \leq u \leq \rho_{\mathcal{C}_\beta}$. Recall that $W^\beta(s,\beta(s)) = G(s,\beta(s))$ for all $s \in \mathbb{R}_+$, as β solves (4.37). Due to the law of the iterated logarithm, the dominated convergence theorem, the fact that W^β satisfies (4.35) with β instead of b, and recalling (4.7), we get

$$\lim_{u\to\infty} W^\beta(s+u,Y_u) = \lim_{u\to\infty} G(s+u,Y_u) = c$$

$\mathbb{P}_y$-a.s. for all $y \in \mathbb{R}$. Hence, $W^\beta(s+\sigma_{\mathcal{C}_\beta},Y_{\sigma_{\mathcal{C}_\beta}}) = G(s+\sigma_{\mathcal{C}_\beta},Y_{\sigma_{\mathcal{C}_\beta}})$. From (4.38) and (4.39) it follows that

$$W^\beta(s,y) = \mathbb{E}_y\left[W^\beta(s+\sigma_{\mathcal{C}_\beta},Y_{\sigma_{\mathcal{C}_\beta}})\right] - \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}_\beta}} \partial_t G(s+u,Y_u)\,\mathrm{d}u\right]$$

$$\begin{aligned}
&= \mathbb{E}_y\left[G^\beta(s+\sigma_{\mathcal{C}_\beta}, Y_{\sigma_{\mathcal{C}_\beta}})\right] - \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}_\beta}} \partial_t G\left(s+u, Y_u\right) \mathrm{d}u\right] \\
&= G(s,y),
\end{aligned}$$

which proves that $W^\beta = G$ on $\mathcal{D}_\beta$.

Define now the first hitting time $\sigma_{\mathcal{D}_\beta}$ into $\mathcal{C}_\beta$. Note that either $\sigma_{\mathcal{D}_\beta} = 0$ for $(s,y) \in \mathcal{D}_\beta$, on which $W^\beta = G$, or $Y_u < \beta(s+u)$ for all $0 \le u < \sigma_{\mathcal{D}_\beta}$. We derive from (4.38) that

$$W^\beta(s,y) = \mathbb{E}_y\left[W^\beta\left(s+\sigma_{\mathcal{D}_\beta}, Y_{\sigma_{\mathcal{D}_\beta}}\right)\right] = \mathbb{E}_y\left[G\left(s+\sigma_{\mathcal{D}_\beta}, Y_{\sigma_{\mathcal{D}_\beta}}\right)\right]$$

for all $(s,y) \in \mathbb{R}_+ \times \mathbb{R}$, which, after recalling the definition of W in (4.6), proves that $W^\beta \le W$.

Take $(s,y) \in \mathcal{D}_\beta \cap \mathcal{D}$ and consider the first hitting time $\sigma_{\mathcal{C}}$ into the continuation set $\mathcal{C}$. Since $W = G$ on $\mathcal{D}$ and $W^\beta = G$ on $\mathcal{D}_\beta$, by relying on (4.35), (4.38), and the fact that $\mathbb{P}_y\left(Y_u \ge b(s+u)\right) = 1$ for all $0 \le u < \sigma_{\mathcal{C}}$, we get

$$\begin{aligned}
\mathbb{E}_y\left[W\left(s+\sigma_{\mathcal{C}}, Y_{\sigma_{\mathcal{C}}}\right)\right] &= G(s,y) + \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}}} \partial_t G\left(s+u, Y_u\right) \mathrm{d}u\right], \\
\mathbb{E}_y\left[W^\beta\left(s+\sigma_{\mathcal{C}}, Y_{\sigma_{\mathcal{C}}}\right)\right] &= G(s,y) + \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}}} \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \ge \beta(s+u)\right) \mathrm{d}u\right].
\end{aligned}$$

After recalling that $W^\beta \le W$, we can merge the two previous equalities into

$$\mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}}} \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \ge \beta(s+u)\right) \mathrm{d}u\right] \le \mathbb{E}_y\left[\int_0^{\sigma_{\mathcal{C}}} \partial_t G\left(s+u, Y_u\right) \mathrm{d}u\right],$$

which, alongside the fact that $\partial_t G(s,y) < 0$ for all $(s,y) \in \mathcal{D}$ (otherwise we get from (4.32) that the first exit time from a ball around (s,y) small enough will yield a better strategy than stopping immediately) and the continuity of β, implies that $b \ge \beta$.

Suppose that there exists a point $s \in \mathbb{R}_+$ such that $b(s) > \beta(s)$ and fix $y \in (\beta(s), b(s))$. Consider the stopping time $\sigma^* = \sigma^*(s,y)$ and plug it into both (4.35) and (4.38) to obtain

$$\begin{aligned}
\mathbb{E}_y\left[W^\beta\left(s+\sigma^*, Y_{\sigma^*}\right)\right] &= \mathbb{E}_y\left[G\left(s+\sigma^*, Y_{\sigma^*}\right)\right] \\
&= W^\beta(s,y) + \mathbb{E}_y\left[\int_0^{\sigma^*} \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \ge \beta(s+u)\right) \mathrm{d}u\right]
\end{aligned}$$

and

$$\mathbb{E}_y\left[W\left(s+\sigma^*, Y_{\sigma^*}\right)\right] = \mathbb{E}_y\left[G\left(s+\sigma^*, Y_{\sigma^*}\right)\right] = W(s,y).$$

Thus, since $W^\beta \le W$, we get

$$\mathbb{E}_y\left[\int_0^{\sigma^*} \partial_t G\left(s+u, Y_u\right) \mathbb{1}\left(Y_u \ge \beta(s+u)\right) \mathrm{d}u\right] \ge 0.$$

Using the fact that $y < b(s)$, the continuity of b, and the time-continuity of the process Y, we can state that $\sigma^* > 0$ $\mathbb{P}_y$-a.s. Therefore, since $\partial_t G(s,y) < 0$ for all $(s,y) \in \mathcal{D}_\beta$ (the same arguments used to prove that $\partial_t G < 0$ in $\mathcal{D}$ lead to this conclusion) the previous inequality can only stand if $\mathbb{1}\left(Y_u \ge \beta(s+u)\right) = 0$ for all $0 \le u \le \sigma^*$, meaning that $b(s+u) \le \beta(s+u)$ in the same interval, which contradicts the assumption $b(s) > \beta(s)$ due to the continuity of both b and β. $\square$

4.5 Solution of the original problem and some extensions

Recall that the OSPs (4.6) and (4.3) are equivalent, meaning that the value functions and the OSTs of both problems are linked through a homeomorphic transformation. Details on how to actually translate one problem into the other were given in Proposition 4.1. It then follows that the stopping time $\tau^*(t,x)$ defined in (4.9) is optimal for (4.6) and it admits the following alternative representation under $\mathbb{P}_x$:

$$\tau^*(t,x) = \inf\left\{u \geq 0 : X_{t+u} \geq \beta(t+u)\right\}, \quad \beta(t) = \frac{z}{c_z} G_{c_z}\left(s, b(s)\right), \tag{4.40}$$

where β is the OSB associated to (4.3), and $s = \upsilon(t)$ and c_z are defined in Proposition 4.1. We can obtain both V and β without requiring the computation of W and b. Indeed, consider the infinitesimal generator of $\{(t, X_t)\}_{t\in[0,1]}$, $\mathbb{L}_X$, and set $y = c_x$, $s_\varepsilon = s + \varepsilon$, and $t_\varepsilon = \upsilon^{-1}(s_\varepsilon)$ for $\varepsilon \in \mathbb{R}$. By means of (4.8) and the chain rule, we get that

$$\begin{aligned}
\frac{z}{c_z}\left(\mathbb{L}W_{c_z}\right)(s,y) &:= \lim_{\varepsilon\to 0} \varepsilon^{-1}\left(\mathbb{E}_y\left[\frac{z}{c_z} W_{c_z}\left(s_\varepsilon, Y_\varepsilon\right)\right] - \frac{z}{c_z} W_{c_z}(s,y)\right) \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-1}\left(\mathbb{E}_{t,x}\left[V(t_\varepsilon, X_{t_\varepsilon})\right] - V(t,x)\right) \\
&= \left(\mathbb{L}_X V\right)(t,x)(\upsilon^{-1})'(s).
\end{aligned}$$

Hence, after multiplying both sides of (4.32) by z/c_z, integrating with respect to $\upsilon^{-1}(u)$ instead of u, and recalling that $\mathbb{L}_X V(t,x) = 0$ for all $x \leq \beta(t)$ and $V(t,x) = x$ for all $x \geq \beta(t)$, we get the pricing formula

$$\begin{aligned}
V(t,x) &= z - \mathbb{E}_{t,x}\left[\int_0^{1-t} (\mathbb{L}_X V)(t+u, X_{t+u})\,\mathrm{d}u\right] \\
&= z - \mathbb{E}_{t,x}\left[\int_0^{1-t} \mu(t+u, X_{t+u})\mathbb{1}(X_{t+u} \geq \beta(t+u))\,\mathrm{d}u\right].
\end{aligned} \tag{4.41}$$

In the same fashion we obtained (4.36), we can take advantage of the linearity of $x \mapsto \mu(t,x)$ and the Gaussian marginal distributions of X to come up with the following refined version of (4.41):

$$V(t,x) = z - \int_t^1 K(t,x,u,\beta(u))\,\mathrm{d}u, \tag{4.42}$$

where, for $x_1, x_2 \in \mathbb{R}$ and $0 \leq t_1 \leq t_2 \leq 1$,

$$K(t_1,x_1,t_2,x_2) := \alpha\frac{z\widetilde{\Phi}_{t_1,x_1,t_2,x_2} - \cosh(\alpha(1-t_2))(m_{t_2}(t_1,x_1)\widetilde{\Phi}_{t_1,x_1,t_2,x_2} + \upsilon_{t_2}(t_1)\widetilde{\phi}_{t_1,x_1,t_2,x_2})}{\sinh(\alpha(1-t_2))}, \tag{4.43}$$

with

$$\widetilde{\Phi}_{t_1,x_1,t_2,x_2} := \bar{\Phi}\left(\frac{x_2 - m_{t_2}(t_1,x_1)}{\upsilon_{t_2}(t_1)}\right), \quad \widetilde{\phi}_{t_1,x_1,t_2,x_2} := \phi\left(\frac{x_2 - m_{t_2}(t_1,x_1)}{\upsilon_{t_2}(t_1)}\right)$$

and

$$m_{t_2}(t_1,x_1) := \mathbb{E}_{t_1,x_1}\left[X_{t_2}\right] = \frac{x_1\sinh(\alpha(1-t_2)) + z\sinh(\alpha(t_2-t_1))}{\sinh(\alpha(1-t_1))},$$

$$v_{t_2}(t_1) := \sqrt{\mathbb{V}\mathrm{ar}_{t_1}\left[X_{t_2}\right]} = \sqrt{\frac{\gamma^2}{\alpha}\frac{\sinh(\alpha(1-t_2))\sinh(\alpha(t_2-t_1))}{\sinh(\alpha(1-t_1))}}.$$

Consequently, by taking $x \downarrow \beta(t)$ in (4.41) (or by directly transforming (4.37) in the same way we obtained (4.41) from (4.35)), we get the free-boundary equation

$$\begin{aligned}\beta(t) &= z - \mathbb{E}_{t,\beta(t)}\left[\int_0^{1-t} (\mathbb{L}_X V)(t+u, X_{t+u})\,\mathrm{d}u\right] \\ &= z - \mathbb{E}_{t,\beta(t)}\left[\int_0^{1-t} \mu(t+u, X_{t+u})\mathbb{1}(X_{t+u} \geq \beta(t+u))\,\mathrm{d}u\right],\end{aligned}$$

which may also be expressed as

$$\beta(t) = z - \int_t^1 K(t, \beta(t), u, \beta(u))\,\mathrm{d}u. \tag{4.44}$$

The next three remarks broaden the scope of applicability of the OUB as the underlying model in (4.3). In particular, the two first reveal that setting the terminal time to 1 and the pulling level (coming from the asymptotic mean of the OU process underneath) to 0 does not take a toll on generality, while the last one shows that the OSP for the BB arises as a limit case when $\alpha \to 0$.

Remark 4.1 (OUB with a general pulling level)**.** *Let $\widetilde{X}^\theta = \{\widetilde{X}_t^\theta\}_{t\in[0,1]}$ be an OU process satisfying the SDE $\mathrm{d}\widetilde{X}_t^\theta = \alpha(\widetilde{X}_t^\theta - \theta)\,\mathrm{d}t + \gamma\,\mathrm{d}B_t$. That is, $X^{\theta,z}$ is pulled towards θ with a time-dependent strength dictated by α. Denote by $X^{\theta,z} = \{X_t^{\theta,z}\}_{t\in[0,1]}$ the OUB process built on top of $\widetilde{X}^\theta$ and such that $X_1^{\theta,z} = z$. It is easy to check that $X^{\theta,z} = X^{0,z-\theta} + \theta$, whenever $X_0^{0,z-\theta} = X_0^{\theta,z} - \theta$. Denote by $V^{\theta,z}$ and $\beta^{\theta,z}$ the value function and the OSB related to the OSP (4.3) with X replaced by $X^{\theta,z}$. Then $V^{\theta,z}(t,x) = V^{0,z-\theta}(t, x-\theta) + \theta$ and $b^{\theta,z}(t) = b^{0,z-\theta}(t) + \theta$.*

Remark 4.2 (OUB with a general horizon)**.** *Denote by $X^{\alpha,\gamma,T} = \{X_t^{\alpha,\gamma,T}\}_{t\in[0,T]}$ an OUB with slope α, volatility γ, and horizon T. Likewise, let $V^{\alpha,\gamma,T}$ and $\beta^{\alpha,\gamma,T}$ be the corresponding value function and the OSB. By relying on the scaling property of a Brownian motion, one can easily verify that $X_t^{\alpha r,\gamma,T} = X_{rt}^{\alpha,\gamma r^{-1/2},rT}$ $\mathbb{P}_x$-a.s. for any $r > 0$. Consequently, $V^{\alpha r,\gamma,T}(t,x) = V^{\alpha,\gamma r^{-1/2},rT}(rt,x)$ and $\beta^{\alpha r,\gamma,T}(t) = \beta^{\alpha,\gamma r^{-1/2},rT}(rt)$. Thereby, by taking $r = 1/T$, one can derive $V^{\alpha,\gamma,T}$ and $\beta^{\alpha,\gamma,T}$ for any set of values α, γ, and T from the solution of the OSP in (4.3).*

Remark 4.3 (BB from an OUB)**.** *To emphasize the dependence on α, denote by $X(\alpha)$, V_α, and β_α, respectively, the OUB solving (4.1), the value function in (4.13), and the corresponding OSB. The process $X_t(\alpha)$ has the following integral representation under $\mathbb{P}_x$ (Barczy and Kern, 2013):*

$$X_t = x\frac{\sinh(\alpha(1-t))}{\sinh(\alpha)} + z\frac{\sinh(\alpha t)}{\sinh(\alpha)} + \sigma\int_0^t \frac{\sinh(\alpha(1-t))}{\sinh(\alpha(1-u))}\,\mathrm{d}B_u,$$

from where we can conclude, after taking $\alpha \to 0$ and using the DCT, that $X_t(\alpha) \to \widetilde{X}_t$ $\mathbb{P}_x$-a.s. for all $t \in [0,1)$, where $\widetilde{X}$ is a BB process with final value $\tilde{X}_1 = z$. Then, by applying Theorem 5 from Coquet and Toldo (2007) we have that $V_\alpha \to \widetilde{V}$, and hence $\beta_\alpha \to \widetilde{\beta}$, as $\alpha \to 0$, where $\widetilde{V}$ and $\widetilde{\beta}$ are the value function and the OSB related to $\widetilde{X}$.

4.6 Numerical results

The free-boundary equation (4.44) does not admit a closed-form solution and thus numerical procedures are needed to compute an approximate boundary. By exploiting the fact that the OSB at a given time t depends only on its shape from t up to the horizon, one can discretize the integral in (4.44) by means of a right Riemann sum and, since the terminal value $\beta(1)$ is known, the entire boundary can be computed in a backward form. This method of backward induction is detailed in Detemple (2005, Chapter 8) and examples of its implementation can be found, e.g., in Pedersen and Peskir (2002). Another approach to solve (4.44) is by using Picard iterations, that is, by treating (4.44) as a fixed-point problem in which the entire boundary is updated in each step. The works of Detemple and Kitapbayev (2020) and De Angelis and Milazzo (2020) use this approach to solve the associated Volterra-type integral equation characterizing the OSB. To the best of our knowledge, when it comes to non-linear integral equations arisen from OSPs, the convergence of both the Picard scheme and the backward induction technique are numerically checked rather than formally proved. Therefore, we chose to use the Picard scheme since empirical tests suggested a faster convergence rate while keeping a similar accuracy compared to the backward induction approach.

Define a partition of $[0, 1]$, namely, $0 = t_0 < t_1 < \cdots < t_N = 1$ for $N \in \mathbb{N}$. Given that $\beta(1) = z$, we will initialize the Picard iterations by starting with the constant boundary $\beta^{(0)} : [0, 1] \to \mathbb{R}$ with $\beta^{(0)} \equiv z$. The updating mechanism that generates subsequent boundaries is laid down in the following formula, which comes after discretizing the integral in (4.44) by using a right Riemann sum:

$$\beta_i^{(k)} = z - \sum_{j=i}^{N-2} K\left(t_i, \beta_i^{(k-1)}, t_{j+1}, \beta_{j+1}^{(k-1)}\right)(t_{j+1} - t_j), \quad k = 1, 2, \ldots$$

We neglect the $(N-1)$-addend and allow the sum to run only until $N-2$ since $K(t, x, 1, z)$ is not well defined, and therefore the last integral piece cannot be included in the right Riemann sum. As the overall integral is finite, the last piece vanishes as t_{N-1} approaches 1.

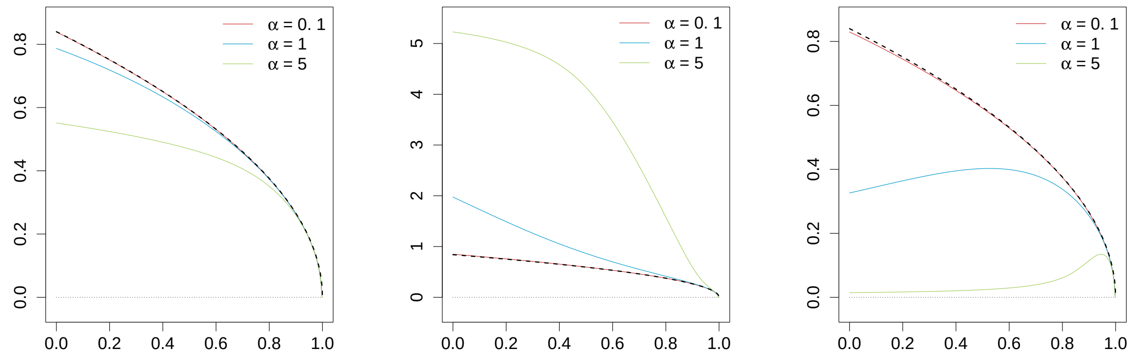
We chose to stop the fixed-point Picard algorithm after the m-th iteration if $m = \min\{k > 0 : \max_{i=1,\ldots,N} |\beta_i^{k-1} - \beta_i^k| < \varepsilon\}$ for $\varepsilon = 10^{-4}$. Empirical evidence suggested that the best performance of the algorithm was achieved when using a non-uniform mesh whose distances $t_i - t_{i-1}$ smoothly decrease as i increases. In our computations, we used the logarithmically-spaced partition $t_i = \ln(1 + i(e-1)/N)$, where $N = 500$ unless otherwise specified.

Figures 4.1, 4.2, and 4.3 reveal how the OSB's shape is affected by different sets of values for the slope α, the volatility γ, and the anchor point z.

The code implementing the boundary computation is available at `https://github.com/aguazz/OSP_OUB`.

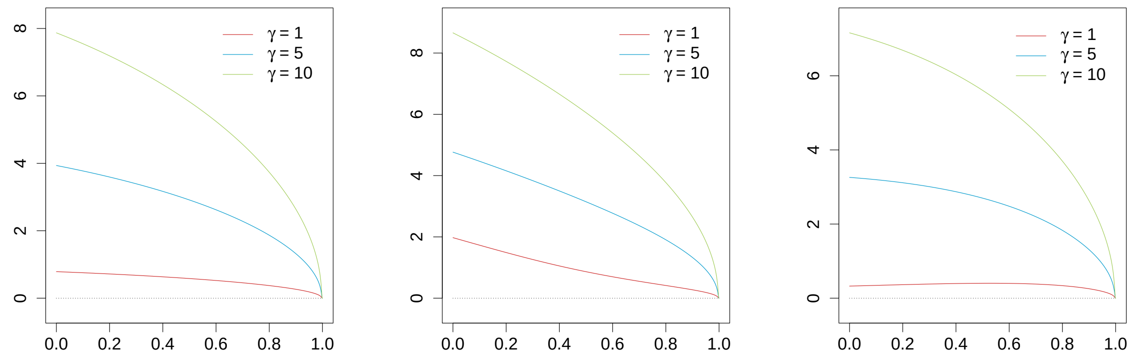
4.7 Conclusions

In this paper we solved the finite-horizon OSP for an OUB process with the identity as the gain function. To the best of our knowledge, so far the only Markov bridge addressed by the optimal stopping literature has been the BB and some slight variations of it (see, e.g., Shepp (1969); Föllmer (1972); Ekström and Wanntorp (2009); Ernst and Shepp (2015); Leung et al. (2018); De Angelis and Milazzo (2020); Glover (2020); Ekström and Vaicenavicius (2020); D'Auria et al.



(a) $z = 0, \gamma = 1$ (b) $z = -5, \gamma = 1$ (c) $z = 5, \gamma = 1$

Figure 4.1: Optimal stopping boundary estimation for different values of α. The boundary is pulled towards 0 with a strength that increases as both $|\alpha|$ (values of α with equal absolute values yield the same boundary) and the residual time to the horizon $1-t$ increases. As $\alpha \to 0$, the boundary estimation is shown to converge towards the OSB of a BB (dashed line), which is known to be $z + L\sqrt{1-t}$, for $L \approx 0.8399$.

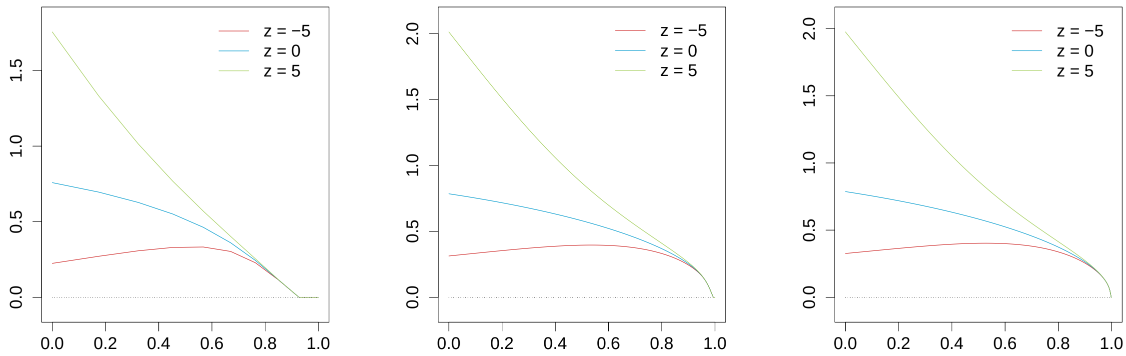


(a) $z = 0, \alpha = 1$ (b) $z = -5, \alpha = 1$ (c) $z = 5, \alpha = 1$

Figure 4.2: Optimal stopping boundary estimation for different values of γ. The boundary exhibits an increasing proportional relationship with respect to γ.

(2020)). Markov bridges are potentially useful in mathematical finance as they allow including additional information at some terminal time.

Arguing as Shepp (1969) for the BB, we worked out the OUB case by coming up with an equivalent OSP having a Brownian motion as the underlying process after time-space transforming the OUB. Contrary to Shepp (1969), the complexity of our problem did not allow us to guess a candidate solution, and we directly characterized the value function and the OSB by means of the pricing formula and the free-boundary equation. However, the equivalence between both OSPs was used only to facilitate technicalities along the proofs, and it is not necessary to compute the solution, since both the pricing formula and the free-boundary equation are also provided in the original formulation. We discussed how to use a Picard iteration algorithm to numerically approximate the OSB and displayed some examples to illustrate how different sets



(a) $\alpha = \gamma = 1, N = 10$ (b) $\alpha = \gamma = 1, N = 100$ (c) $\alpha = \gamma = 1, N = 500$

Figure 4.3: Optimal stopping boundary estimation for different values of z and N. We display $t \mapsto \beta(t) - z$ to allow a clearer comparison across the different values of z. As N increases the boundary estimation is seen to converge.

of values for the OUB's parameters rule the shape of the OSB.

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Chapter 5

Optimal stopping of Gauss–Markov bridges

Abstract

We solve the non-discounted, finite-horizon optimal stopping problem of a Gauss–Markov bridge by using a time-space transformation approach. The associated optimal stopping boundary is proved to be Lipschitz continuous on any closed interval that excludes the horizon, and it is characterized by the unique solution of an integral equation. A Picard iteration algorithm is discussed and implemented to exemplify the numerical computation and geometry of the optimal stopping boundary for some illustrative cases.

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Contents

5.1 Introduction . . . 111
5.2 Gauss–Markov bridges . . . 114
5.3 Two equivalent formulations of the OSP . . . 119
5.4 Solution of the infinite-horizon OSP . . . 120
5.5 Solution of the original OSP . . . 132
5.6 Numerical results . . . 134
5.7 Concluding remarks . . . 137
References . . . 137

5.1 Introduction

The problem of optimally stopping a Markov process to attain a maximum mean reward dates back to Wald's sequential analysis (Wald, 1947) and is consolidated in the work of Dynkin (1963). Ever since, it has received increasing attention from numerous theoretical and practical perspectives, as it is comprehensively compiled in the book of Peskir and Shiryaev (2006). However, Optimal Stopping Problems (OSPs) are mathematically complex objects, which makes it difficult to obtain sound results in general settings, and typically lead to requiring smoothness conditions and simplifying assumptions for their solution, being time-homogeneity of the Markovian process among the most popular ones.

Time-inhomogeneous diffusions can be cast back to time-homogeneity (see, e.g., Taylor (1968), Dochviri (1995), Shiryaev (2008)) at the cost of increasing the dimension of the OSP, which results in an increased complexity, hampering subsequent derivations or limiting studies to tackle specific, simplified time dependencies. Take, as examples, the works of Krylov and Aries (1980), Oshima (2006), and Yang (2014), which proved different types of continuities and characterizations of the value function; or those of Friedman (1975b) and Jacka and Lynn (1992), which shed light on the shape of the stopping set; and Friedman (1975a) and Peskir (2019), who studied the smoothness of the associated free boundary. To mitigate the burden of time-inhomogeneity, many of these works ask for the process's coefficients to be Lipschitz continuous or at least bounded. This usual assumption excludes important classes of time-dependent processes, such as that of diffusion bridges, whose drifts explode as time approaches a terminal point.

In a broad and rough sense, bridge processes, or bridges for short, are stochastic processes "anchored" to deterministic values at some initial and terminal time points. Formal definitions and potential applications of different classes of bridges have been extensively studied. Bessel and Lévy bridges are respectively described by Pitman and Yor (1982) and Salminen (1984), and by Hoyle et al. (2011) and Erickson and Steck (2022). A canonical reference for Gaussian bridges can be found in the work of Gasbarra et al. (2007), while Markov bridges are addressed in great generality by Fitzsimmons et al. (1993), Chaumont and Bravo (2011), and Çetin and Danilova (2016).

In finance, diffusion bridges are appealing models from the perspective of a trader who wants to incorporate his beliefs about future events, like in trading perishable commodities, modeling the presence of arbitrage, incorporating algorithms' forecasts and experts' predictions, or trading mispriced assets that could rapidly return to their fair price. Works that consider models based on a Brownian Bridge (BB) to address these and other insider trading situations include Kyle (1985), Brennan and Schwartz (1990), Back (1992), Liu and Longstaff (2004), Campi and Çetin (2007), Campi et al. (2011), Campi et al. (2013), Cetin and Xing (2013), Sottinen and Yazigi (2014), Cartea et al. (2016), Angoshtari and Leung (2019), and Chen et al. (2021). The early work of Boyce (1970) had already suggested the use of a BB after taking the perspective of an investor who wants to optimally sell a bond. Recently, D'Auria et al. (2020) applied a BB to optimally exercise an American option in the presence of the so-called stock-pinning effect (see Krishnan and Nelken (2001), Ni et al. (2005), Golez and Jackwerth (2012), and Ni et al. (2021)), obtaining competitive empirical results when compared to the classic Black–Scholes model. Taking distance from the BB model, Hilliard and Hilliard (2015) used an Ornstein–Uhlenbeck Bridge (OUB) to model the effect of short-lived arbitrage opportunities in pricing an American option, although they recurred to a binomial-tree numerical method instead of providing analytical results.

Non-financial applications of BBs include their usual adoption to model animal movement (see Horne et al. (2007); Venek et al. (2016), Kranstauber (2019), and Krumm (2021)), and their construction as a limit case of sequentially drawing elements without replacement from a large population (see Rosén (1965)). The latter connection makes BBs good asymptotic models for classical statistical problems, like variations of the urn's problem (see Ekström and Wanntorp (2009), Andersson (2012), and Chen et al. (2015)).

Whenever the goal is to optimize the time to take an action, all the previous situations in which a BB, an OUB, or diffusion bridges have applications can be intertwined with optimal stopping theory. However, within the time-inhomogeneous realm, diffusion bridges are particularly challenging to treat with classical optimal-stopping tools, as they feature explosive drifts. It comes as no surprise, hence, that the literature addressing this topic is scarce when compared with its non-bridge counterpart. The first incursion into OSPs with diffusion bridges is by Shepp's work (Shepp, 1969), who solved the OSP of a BB by linking it to that of a simpler Brownian Motion (BM) representation. After Shepp's result, the more recent studies of OSPs with diffusion bridges still revolve around modifications of the BB. Ekström and Wanntorp (2009) and Ernst and Shepp (2015) revisited Shepp's problem with novel solution methods. Ekström and Wanntorp (2009) and De Angelis and Milazzo (2020) widened the class of gain functions; D'Auria et al. (2020) considered the (exponentially) discounted version; while Föllmer (1972), Leung et al. (2018), Glover (2020), and Ekström and Vaicenavicius (2020), introduced randomization in either the terminal time or the pinning point. To the best of our knowledge, the only solution to an OSP with diffusion bridges that steps outside the BB, came recently in D'Auria et al. (2021), which extends Shepp's technique to embrace an OUB.

Both the BB and the OUB belong to the class of Gauss–Markov Bridges (GMBs), that is, bridges that simultaneously exhibit the Markovian and Gaussian properties. Due to their enhanced tractability and wide applicability, these processes have been in the spotlight for some decades, especially in recent years. A good compendium of works related to GMBs can be found in Abrahams and Thomas (1981), Buonocore et al. (2013), Barczy and Kern (2013a), Barczy and Kern (2013b), Barczy and Kern (2011), Chen and Georgiou (2016), and Hildebrandt and Rœlly (2020).

In this paper we solve the finite-horizon OSP of a GMB. In doing so, we generalize not only Shepp's result for the BB case, but also its methodology. Indeed, the same type of transformation that casts a BB into a BM is embedded in a more general change-of-variable method to solve OSPs, which is detailed in Peskir and Shiryaev (2006, Section 5.2) and exemplary used in Pederson and Peskir (2000) for non-linear OSPs. When the GM process is also a bridge, such a representation presents regularities that we show useful to overcome the bridges' explosive drifts. Loosely, the drift's divergence is equated to that of a time-transformed BM, and then explained in terms of the laws of iterated logarithms. This trick allows working out the solution of an equivalent infinite-horizon OSP with a time-spaced transformed BM underneath, and then casting the solution back into original terms. The solution is attained, in a probabilistic fashion, by proving that both the value function and the Optimal Stopping Boundary (OSB) are regular enough to meet the premises of a relaxed Itô's lemma that allows deriving the free-boundary equation. Among these regularities, the local Lipschitz continuity of the OSB off the horizon stands out, which implies its differentiability almost everywhere. We prove that such a degree of smoothness of the OSB suffices to obtain the smooth-fit condition. The free-boundary equation is given in terms of a Volterra-type integral equation with a unique solution. For enriched perspectives and full sight of the reach of GMBs, we provide, besides the BM representation, a third angle from which GMBs can be seen: as time-inhomogeneous OUBs. Hence, our work also extends the work of D'Auria et al. (2021) for a time-independent OUB. This OUB representation is arguably more appealing to numerically explore the OSB's shape, which is done by using a Picard iteration algorithm that solves the free-boundary equation. The OSB exhibits a trade-off between two pulling forces, the one towards the mean-reverting level of the OUB representation, and that which anchors the process at the horizon.

The rest of this paper is organized as follows. Section 5.2 establishes four equivalent definitions of GMBs, including the time-spaced transformed BM representation. Section 5.3 introduces the finite-horizon OSP of a GMB and proves its equivalence to that of an infinite-horizon, time-dependent gain function, and a BM underneath. The auxiliary OSP is then treated in Section 5.4 as a standalone problem. This section also accounts for the main technical work of the paper, where classical and new techniques of optimal stopping theory are combined to obtain the solution of the OSP. This solution is then translated back into original terms in Section 5.5, where the free-boundary equation is provided. Section 5.6 discusses the practical aspects of numerically solving the free-boundary equation, and shows computer drawings of the OSB. Final remarks are given in Section 5.7.

5.2 Gauss–Markov bridges

Both Gaussian and Markovian processes exhibit features that are appealing from a theoretical, computational, and applicable viewpoint. Gauss–Markov (GM) processes, that is, processes that are Gaussian and Markovian at the same time, merge the advantages of these two classes. They inherit the convenient Markovian lack of memory and the Gaussian processes' property of being characterized by their mean and covariance functions. Additionally, the Markovianity of Gaussian processes is equivalent for their covariances to admit a certain "factorization". The following lemma collects such a useful characterization, whose proof follows from the lemma on page 863 from Borisov (1983), and Theorem 1 and Remarks 1–2 in Mehr and McFadden (1965).

Lemma 5.1 (Characterization of non-degenerated GM processes).
A function $R : [0, T]^2 \to \mathbb{R}$ such that $R(t_1, t_2) \neq 0$ for all $t_1, t_2 \in (0, T)$ is the covariance function of a non-degenerated GM process in $(0, T)$ if and only if there exist functions $r_1, r_2 : [0, T] \to \mathbb{R}$, that are unique up to a multiplicative constant, such that

(i) $R(t_1, t_2) = r_1(t_1 \wedge t_2) r_2(t_1 \vee t_2)$;

(ii) $r_1(t) \neq 0$ *and* $r_2(t) \neq 0$ *for all* $t \in (0, T)$;

(iii) r_1/r_2 *is positive and strictly increasing.*

Moreover, r_1 *and* r_2 *take the form*

$$r_1(t) = \begin{cases} R(t, t'), & t \leq t', \\ R(t, t)R(t', t')/R(t', t), & t > t', \end{cases} \quad r_2(t) = \begin{cases} R(t, t)/R(t, t'), & t \leq t', \\ R(t', t)/R(t', t'), & t > t', \end{cases} \tag{5.1}$$

for some $t' \in (0, T)$*. Changing* t' *is equivalent to scaling* r_1 *and* r_2 *by a constant factor.*

We say that the functions r_1 and r_2 in Lemma 5.1 are a factorization of the covariance function R. The lemma provides a simple technique to construct GM processes with ad hoc covariance functions that are not necessarily time-homogeneous. This is particularly useful given the complexity of proving the positive-definiteness of an arbitrary function to check its validity as a covariance function. GM processes also admit a simple representation by means of time-space transformed BMs (see, e.g., Mehr and McFadden (1965)), which results in higher tractability. Moreover, viewed through the lens of diffusions, GM processes account for space-linear drifts and space-independent volatilities, both coefficients being time-dependent (see, e.g., Buonocore et al. (2013)).

We call Gauss–Markov Bridge (GMB) a process that results after "conditioning" (for a formal definition see, e.g., Gasbarra et al. (2007)) a GM process to start and end at some initial and terminal points. It is straightforward to see that the Markovian property is preserved after conditioning. So it is the Gaussianity (see, e.g., Williams and Rasmussen (2006, Formula A.6), or Buonocore et al. (2013)). Hence, the above-mentioned conveniences of GM processes are inherited by GMBs. In particular, the time-space transformed BM representation adopts a specific form that characterizes GMBs and forms the backbone of our main results. The following proposition sheds light on that representation and serves to formally define a GMB as well as to offer different characterizations.

Proposition 5.1 (Gauss–Markov bridges).
Let $X = \{X_u\}_{u\in[0,T]}$ be a GM process defined on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$, for some $T > 0$. The following statements are equivalent:

(i) There exists a time-continuous, non-degenerated GM process defined on $(\Omega, \mathcal{F}, \mathsf{P})$, denoted by $\widetilde{X} = \{\widetilde{X}_u\}_{u\in\mathbb{R}_+}$, whose mean and covariance functions are twice continuously differentiable, and such that

$$\mathrm{Law}(X, \mathsf{P}) = \mathrm{Law}(\widetilde{X}, \mathsf{P}_{x,T,z}),$$

with $\mathsf{P}_{x,T,z}(\cdot) = \mathsf{P}(\cdot\ |\widetilde{X}_0 = x, \widetilde{X}_T = z)$ for some $x \in \mathbb{R}$ and $(T, z) \in \mathbb{R}_+ \times \mathbb{R}$.

(ii) Let $m(t) := \mathsf{E}[X_t]$ and $R(t_1, t_2) := \mathsf{Cov}[X_{t_1}, X_{t_2}]$, where E and Cov are the mean and covariance operators related to P. Then, $t \mapsto m(t)$ is twice continuously differentiable, and there exist functions r_1 and r_2 that are unique up to multiplicative constants and such that:

(ii.1) $R(t_1, t_2) = r_1(t_1 \wedge t_2) r_2(t_1 \vee t_2)$;

(ii.2) $r_1(t) \neq 0$ *and* $r_2(t) \neq 0$ *for all* $t \in (0, T)$;

(ii.3) r_1/r_2 *is positive and strictly increasing;*

(ii.4) $r_1(0) = r_2(T) = 0$;

(ii.5) r_1 *and* r_2 *are twice continuously differentiable;*

(ii.6) $r_1(T) \neq 0$ *and* $r_2(0) \neq 0$.

(iii) X admits the representation

$$\begin{cases} X_t = \alpha(t) + \beta_T(t)\left((z - \alpha(T))\gamma_T(t) + \left(B_{\gamma_T(t)} + \dfrac{x - \alpha(0)}{\beta_T(0)}\right)\right), \ t \in [0, T), \\ X_T = z. \end{cases} \tag{5.2}$$

where $\{B_u\}_{u\in\mathbb{R}_+}$ is a standard BM, and $\alpha : [0, T] \to \mathbb{R}$, $\beta_T : [0, T] \to \mathbb{R}_+$, and $\gamma_T : [0, T) \to \mathbb{R}_+$ are twice continuously differentiable functions such that:

(iii.1) $\beta_T(T) = \gamma_T(0) = 0$;

(iii.2) γ_T *is monotonically increasing;*

(iii.3) $\lim_{t\to T} \gamma_T(t) = \infty$ *and* $\lim_{t\to T} \beta_T(t)\gamma_T(t) = 1$.

(iv) X is the unique strong solution of the OUB Stochastic Differential Equation (SDE)

$$\mathrm{d}X_t = \theta(t)(\kappa(t) - X_t)\,\mathrm{d}t + \nu(t)\,\mathrm{d}B_t, \quad t \in (0, T), \tag{5.3}$$

with initial condition $X_0 = x$. $\{B_t\}_{t\in\mathbb{R}_+}$ is a standard BM, and $\theta : [0, T) \to \mathbb{R}_+$, $\kappa : [0, T] \to \mathbb{R}$, and $\nu : [0, T] \to \mathbb{R}_+$ are continuously differentiable functions such that:

(iv.1) $\lim_{t\to T}\int_0^t \theta(u)\,\mathrm{d}u = \infty$;

(iv.2) $\nu^2(t) = \theta(t)\exp\left\{-\int_0^t \theta(u)\,\mathrm{d}u\right\}$ *or, equivalently,* $\theta(t) = \nu^2(t)/\int_t^T \nu^2(u)\,\mathrm{d}u$.

Proof. (*i*) $\Longrightarrow$ (*ii*). X is a non-degenerated GM process in $(0,T)$, as it arises by conditioning a process with the same qualities to take deterministic values at $t=0$ and $t=T$. Hence, Lemma 5.1 guarantees that $\widetilde{R}(t_1,t_2) := \mathsf{Cov}\left[\widetilde{X}_{t_1},\widetilde{X}_{t_2}\right]$ meets conditions (*ii.1*)–(*ii.3*). Since X degenerates at $t=0$ and $t=T$, and due to (*ii.1*), condition (*ii.4*) holds true. From the twice continuous differentiability (with respect to both variables) of the covariance function of X, alongside representation (5.1), it follows (*ii.5*).

We now prove (*ii.6*). Let $\widetilde{m},\widetilde{r}_1,\widetilde{r}_2 : [0,T]\to\mathbb{R}$ be the mean and the covariance factorization of $\widetilde{X}$. Hence (see, e.g., Williams and Rasmussen (2006, Formula A.6) or Buonocore et al. (2013)),

$$m(t) = \widetilde{m}(t) + (x-\widetilde{m}(0))\frac{r_2(t)}{r_2(0)} + (z-\widetilde{m}(T))r_1(t), \quad t\in[0,T), \tag{5.4}$$

and

$$\begin{cases} r_1(t) = \dfrac{\widetilde{r}_1(t)\widetilde{r}_2(0)-\widetilde{r}_1(0)\widetilde{r}_2(t)}{\widetilde{r}_1(T)\widetilde{r}_2(0)-\widetilde{r}_1(0)\widetilde{r}_2(T)}, \\ r_2(t) = \widetilde{r}_1(T)\widetilde{r}_2(t)-\widetilde{r}_1(t)\widetilde{r}_2(T). \end{cases} \tag{5.5}$$

Property (*ii.3*) implies that $\widetilde{r}_1(T)\widetilde{r}_2(0)-\widetilde{r}_1(0)\widetilde{r}_2(T)>0$ and, hence, (5.5) results in $r_1(T)=1$ and $r_2(0)>0$. This does not mean that $r_1(T)$ and $r_2(0)$ must be positive, as $-r_1$ and $-r_2$ are also a factorization of R, but it does imply (*ii.6*).

(*ii*) $\Longrightarrow$ (*i*). Consider the functions

$$\widetilde{m}(t) := m(t) - (x-m_1)\frac{r_2(t)}{r_2(0)} - (z-m_2)r_1(t), \quad t\in(0,T), \tag{5.6}$$

with $\widetilde{m}(0) := m_1$ and $\widetilde{m}(T) := m_2$ for $m_1,m_2\in\mathbb{R}$, and

$$\widetilde{r}_1(t) := ar_1(t)+br_2(t); \quad \widetilde{r}_2(t) := cr_1(t)+dr_2(t), \quad t\in[0,T], \tag{5.7}$$

for $a,b,c>0$ and $d=(1+ac)/a$, such that $bc/a - da/c > 0$. This relation is met, for instance, by setting $a=b=1$ and $c=2$. Note that dividing by $r_2(0)$ in (5.6) is allowed since (*ii.6*) is assumed. Let $h(t) := r_1(t)/r_2(t)$ and $\widetilde{h}(t) := \widetilde{r}_1(t)/\widetilde{r}_2(t)$. We get that $\widetilde{h}(t) = \frac{a+(b/a)h(t)}{c+(d/c)h(t)}$ from (5.7). Hence,

$$\widetilde{h}'(t) > 0 \Longleftrightarrow h'(t)\left(\frac{b}{a}c - \frac{d}{c}a\right) > 0.$$

Condition (*ii.3*) along with our choice of a, b, c, and d guarantees that the right hand side of the equivalence holds. Therefore, $\widetilde{h}(t)$ is strictly increasing. Since $\widetilde{h}$ is also continuous and positive, $\widetilde{R}(t_1,t_2) := \widetilde{r}_1(t_1\wedge t_2)\widetilde{r}_2(t_1\vee t_2)$ is the covariance function of a non-degenerated GM process, as Lemma 5.1 states, which we denote by $\widetilde{X} = \{\widetilde{X}_t\}_{t\in[0,T]}$ and whose mean is set to be equal to $\widetilde{m}(t)$. From the twice continuous differentiability of m, r_1, and r_2, alongside (5.6) and (5.7), it follows that of $\widetilde{m}$, $\widetilde{r}_1$, and $\widetilde{r}_2$ (and $\widetilde{R}$).

One can check, after some straightforward algebra and in alignment with (5.4)–(5.5), that the mean and covariance functions of the GMB derived from conditioning $\widetilde{X}$ to go from $(0,x)$ to (T,z) coincide with m and R.

(*i*) $\Longrightarrow$ (*iii*). Let $\widetilde{m}(t) := \mathsf{E}[\widetilde{X}_t]$ and $\widetilde{R}(t_1, t_2) := \mathsf{Cov}\left[\widetilde{X}_{t_1}, \widetilde{X}_{t_2}\right]$. As a result of conditioning $\widetilde{X}$ to have initial and terminal points $(0, x)$ and (T, z), X is a GM process with mean m given by (5.4) and covariance factorization r_1 and r_1 given by (5.5). Although not explicitly denoted, recall that m depends on x, T, and z, and r_1 and r_2 depend on T.

Therefore, X admits the representation

$$X_t = m(t) + r_2(t)B_{h(t)}, \quad 0 \leq t < T, \tag{5.8}$$

where $t \mapsto h(t) := r_1(t)/r_2(t)$ is a strictly increasing function such that $h(0) = 0$ and $\lim_{t\to T} h(t) = \infty$. Since $\lim_{t\to T} r_2(t)h(t) = r_1(T) = 1$ (see (5.5)), the law of the iterated logarithm allows us to continuously extend X_t to T as the P-a.s. limit $X_T := \lim_{t\to T} X_t = z$. Then, representation (5.2) and properties (*iii.1*)–(*iii.3*) follow after taking $\alpha = \widetilde{m}$, $\beta_T = r_2$, and $\gamma_T = h$. It also follows that α, β_T, and γ_T are twice continuously differentiable, like $\widetilde{m}$, $\widetilde{r}_1$, and $\widetilde{r}_2$ as well.

(*iii*) $\Longrightarrow$ (*ii*). Assuming that $X = \{X_t\}_{t\in[0,T]}$ admits representation (5.2) and that properties (*iii.1*)–(*iii.3*) hold, then X is a GMB with covariance factorization given by $r_1(t) = \beta_T(t_1)\gamma_T(t_1)$ and $r_2(t) = \beta_T(t)$. It readily follows that $r_1(t)$ and $r_2(t)$ meet conditions (*ii.1*)–(*ii.6*). It is also trivial to note that X has a twice continuously differentiable mean.

(*i*) $\Longrightarrow$ (*iv*). Let $\mathsf{E}_{t,x}$ be the mean operator with respect to the probability $\mathsf{P}_{t,x}$ such that $\mathsf{P}_{t,x}(\cdot) = \mathsf{P}(\cdot|X_t = x)$. Recall that X admits representation (5.8), with m, r_1, and r_2 coming from (5.4) and (5.5), and $h = r_2/r_1$. Then,

$$\begin{aligned}\lim_{u\downarrow 0} u^{-1}\mathsf{E}_{t,x}\left[X_{t+u} - x\right] &= m'(t) + (x - m(t))r_2'(t)/r_2(t),\\ \lim_{u\downarrow 0} u^{-1}\mathsf{E}_{t,x}\left[(X_{t+u} - x)^2\right] &= r_2^2(t)h'(t).\end{aligned}$$

By comparing the drift and volatility terms, X is the unique strong solution (see Example 2.3 by Çetin and Danilova (2016)) of the SDE (5.3) for

$$\begin{cases}\theta(t) = -r_2'(t)/r_2(t),\\ \kappa(t) = m(t) - m'(t)r_2(t)/r_2'(t),\\ \nu(t) = r_2(t)\sqrt{h'(t)}.\end{cases} \tag{5.9}$$

It follows from (5.9) (or by directly deriving it from (5.3)) that

$$m(t) = \varphi(t)\left(x + \int_0^t \frac{\kappa(u)\theta(u)}{\varphi(u)}\,\mathrm{d}u\right) \tag{5.10}$$

$$= \varphi(t)\left(x + \int_0^t \frac{\widetilde{m}(u)\theta(u) - \widetilde{m}'(u)}{\varphi(u)}\,\mathrm{d}u + (z - \widetilde{m}(T))\int_0^t \frac{r_1(u)\theta(u) - r_1'(u)}{\varphi(u)}\,\mathrm{d}u\right) \tag{5.11}$$

and

$$r_1(t) = \varphi(t)\int_0^t \frac{\nu^2(u)}{\varphi^2(u)}\,\mathrm{d}u, \quad r_2(t) = \varphi(t), \tag{5.12}$$

for $t \in [0, T)$, with $\varphi(t) = \exp\left\{-\int_0^t \theta(u)\,\mathrm{d}u\right\}$. Since X is degenerated at $t = T$, $r_2(T) = 0$, which implies (*iv.1*). By comparing (5.11) with (5.4),

$$r_1(t) = \varphi(t)\int_0^t \frac{r_1(u)\theta(u) - r_1'(u)}{\varphi(u)}\,\mathrm{d}u = 2\varphi(t)\int_0^t \frac{r_1(u)\theta(u)}{\varphi(u)}\,\mathrm{d}u - r_1(t),$$

which, after using (5.12), leads to

$$\int_0^t \frac{\nu^2(u)}{\varphi^2(u)}\,\mathrm{d}u = \int_0^t \frac{r_1(u)\theta(u)}{\varphi(u)}\,\mathrm{d}u.$$

Differentiating with respect to t both sides of the equation above, and relying again on (5.12), we get

$$\frac{\nu^2(t)}{\varphi^2(t)} = \theta(t)\int_0^t \frac{\nu^2(u)}{\varphi^2(u)}\,\mathrm{d}u.$$

The expression above is an ordinary differential equation in $f(t) = \int_0^t \nu^2(u)/\varphi^2(u)\,\mathrm{d}u$ whose solution is $f(t) = C_1 + 1/\varphi(t)$ for some constant C_1. Hence, $f'(t) = \theta(t)/\varphi(t)$. Therefore, some straightforward algebra leads us to the first equality in (*iv.2*), which implies that

$$\int_0^t \nu^2(u)\,\mathrm{d}u = C_2 + \int_0^t \theta(u)\varphi(u)\,\mathrm{d}u = C_2 + 1 - \varphi(t),$$

for a constant $C_2 \in \mathbb{R}$. Since $\lim_{t\to T}\varphi(t) = 0$, then $C_2 = \int_0^T \nu^2(u)\,\mathrm{d}u - 1$. Hence,

$$\int_0^t \theta(u)\,\mathrm{d}u = -\ln\left(C_2 + 1 - \int_0^t \nu^2(u)\,\mathrm{d}u\right),$$

from where it follows the second equality in (*iv.2*) after differentiating.

Finally, from the smoothness of $\widetilde{m}$, $\widetilde{r}_1$, and $\widetilde{r}_2$, which implies that of m, r_1, and r_2, it follows that θ, κ, and ν are continuously differentiable.

(*iv*) $\Longrightarrow$ (*ii*). Functions θ, κ, and ν are sufficiently regular to prove, by using Itô's lemma, that

$$X_t = \varphi(t)\left(X_0 + \int_0^t \frac{\kappa(u)\theta(u)}{\varphi(u)}\,\mathrm{d}u + \int_0^t \frac{\nu(u)}{\varphi(u)}\,\mathrm{d}B_u\right)$$

is the unique strong solution (see Example 2.3 by Çetin and Danilova (2016)) of (5.3), where again $\varphi(t) = \exp\left\{-\int_0^t \theta(u)\,\mathrm{d}u\right\}$. That is, X is a GM process with mean m and covariance factorization r_1 and r_2 given by (5.10) and (5.12), respectively.

Relations (*ii.2*) and (*ii.3*) are trivial to check. From (*iv.1*), it follows (*ii.4*). The continuous differentiability of θ, κ, and ν implies (*ii.5*). Using (*iv.2*) and integrating by parts we get that

$$r_1(t) = 1 - \varphi(t). \tag{5.13}$$

Then, (*ii.6*) holds, as $r_1(T) := \lim_{t\to T} r_1(t) = 1$ and $r_2(0) = 1$. □

Remark 5.1. *Notice that, after condition (iv.2) and relation (5.9), we get that $r_2'(t)r_2(t) < 0$ for all $t \in (0,T)$. Hence, since r_2 is continuous and does not vanish in $[0,T)$, it can be chosen as either positive and decreasing, or negative and increasing. In (5.5), the positive decreasing version is chosen, which is reflected by the fact that $\beta_T > 0$ is assumed in representation (5.8). Since $\beta_T = r_2$, then β_T is also decreasing. Likewise, (5.5) and (5.13) indicate that r_1 is chosen as positive and increasing.*

One could argue that defining a GMB should only require the process to degenerate at $t = 0$ and $t = T$, which is equivalent to (*ii.1*)–(*ii.4*). GMBs defined in this way are not necessarily derived from conditioning a GM process, as it is assumed in representation (*i*). Indeed, consider the Gaussian process $X = \{X_t\}_{t\in[0,1]}$ with zero mean and covariance function $R(t_1, t_2) = r_1(t_1 \wedge t_2)r_2(t_1 \vee t_2)$ for all $t_1, t_2 \in [0, 1]$, where $r_1(t) = t^2(1 - t)$ and $r_2(t) = t(1-t)$. Lemma 5.1 entails that R is a valid covariance function and X is Markovian. Moreover, since $r_1(0) = r_2(1) = 0$, X is a bridge from $(0, 0)$ to $(1, 0)$. However, $r_1(0) = r_2(0) = 0$. That is, (*ii.6*) fails and, hence, X does not satisfy definition (*ii*). Recognizing the differences between both definitions of GMBs, we adopt that in which a GM process is conditioned to take deterministic values at some initial and future time, since representation (5.2) is key to our results in Section 5.4. It reveals the (linear) dependence of the mean with respect to x and z, and it clarifies the relation between OUBs and GMBs in (*iv*).

Notice that a higher smoothness of the GMB mean and covariance factorization is assumed in all alternative characterizations in Proposition 5.1. Clearly, this is a useful assumption to define GMBs, but not necessary. We discuss it in Remark 5.3. Finally, although easily obtainable from (5.9), for the sake of reference we write down the explicit relation between the BM representations (5.2) and the OUB representation (5.3), namely:

$$\begin{cases} \theta(t) = -\beta_T'(t)/\beta_T(t), \\ \kappa(t) = \alpha(t) - \beta_T(t)/\beta_T'(t)(\alpha'(t) + (z - \alpha(T))\beta_T(t)\gamma_T'(t)), \\ \nu(t) = \beta_T(t)\sqrt{\gamma_T'(t)}. \end{cases} \tag{5.14}$$

It is also worth mentioning that condition (*iv.2*), which is necessary and sufficient for an OU process to be an OUB, was also recently found in Hildebrandt and Rœlly (2020, Theorem 3.1) for the case in which κ is assumed constant.

5.3 Two equivalent formulations of the OSP

For $0 \le t < T$, let $X = \{X_u\}_{u\in[0,T]}$ be a real-valued, time-continuous GMB with $X_T = z$, for some $z \in \mathbb{R}$. Define the finite-horizon OSP

$$V_{T,z}(t, x) := \sup_{\tau \le T-t} \mathsf{E}_{t,x}\left[X_{t+\tau}\right], \tag{5.15}$$

where $V_{T,z}$ is the value function and $\mathsf{E}_{t,x}$ is the mean operator with respect to the probability measure $\mathsf{P}_{t,x}$ such that $\mathsf{P}_{t,x}(X_t = x) = 1$. The supremum in (5.15) is taken among all random times τ such that $t + \tau$ is a stopping time for X, although, for simplicity, we will refer to τ as a stopping time from now on.

Likewise, for $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$ and a BM $Y = \{Y_{s+u}\}_{u\in\mathbb{R}_+}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, define the infinite-horizon OSP

$$W_{T,z}(s, y) := \sup_{\sigma \ge 0} \mathbb{E}_{s,y}\left[G_{T,z}\left(s + \sigma, Y_{s+\sigma}\right)\right], \tag{5.16}$$

where $\mathbb{P}_{s,y}$, and $\mathbb{E}_{s,y}$ have analogous definitions to those of $\mathsf{P}_{t,x}$ and $\mathsf{E}_{t,x}$, and the supremum is taken among the stopping times of Y. The (gain) function $G_{T,z}$ takes the form

$$G_{T,z}(s, y) := \alpha(\gamma_T^{-1}(s)) + \beta_T(\gamma_T^{-1}(s))\left((z - \alpha(T))s + y\right), \tag{5.17}$$

for α, β_T, and γ_T as in (*iii.1*)–(*iii.3*) from Proposition 5.1.

Solving (5.15) and (5.16) means both providing a tractable expression for $V(t,x)$ and $W(s,y)$, as well as finding stopping times (if they exist) $\tau^* = \tau^*(t,x)$ and $\sigma^* = \sigma^*(s,y)$ such that

$$V_{T,z}(t,x) = \mathbb{E}_{t,x}\left[X_{t+\tau^*}\right], \quad W_{T,z}(s,y) = \mathbb{E}_{s,y}\left[G_{T,z}\left(s+\sigma^*, Y_{s+\sigma^*}\right)\right].$$

In such a case, τ^* and σ^* are called Optimal Stopping Times (OSTs) for (5.15) and (5.16), respectively.

We claim that the OSPs (5.15) and (5.16) are equivalent in the sense specified in the following proposition.

Proposition 5.2 (Equivalence of the OSPs)**.**
Let V and W be the value functions in (5.15) and (5.16). For $(t,x) \in [0,T] \times \mathbb{R}$, let $s = \gamma_T(t)$ and $y = \left(x - \alpha(t) - \beta_T(t)\gamma_T(t)(z - \alpha(T))\right)/\beta_T(t)$. Then,

$$V_{T,z}(t,x) = W_{T,z}(s,y). \tag{5.18}$$

Moreover, $\tau^* = \tau^*(t,x)$ is an OST for $V_{T,z}$ if and only if $\sigma^* = \sigma^*(s,y)$, defined such that $s + \sigma^* = \gamma_T(t+\tau^*)$, is an OST for W.

Proof. For every stopping time τ of $\{X_{t+u}\}_{u\in[0,T-t]}$, consider the stopping time σ of $\{Y_{s+u}\}_{u\in\mathbb{R}_+}$ such that $s + \sigma = \gamma_T(t+\tau)$. Representation (5.2) implies that

$$\mathrm{Law}\left(\{X_{t+u}\}_{u\in\mathbb{R}_+}, \mathsf{P}_{t,x}\right) = \mathrm{Law}\left(\left\{G_{T,z}\left(\gamma_T(t+u), Y_{\gamma_T(t+u)}\right)\right\}_{u\in\mathbb{R}_+}, \mathbb{P}_{s,y}\right).$$

Hence, (5.18) follows from the following sequence of equalities:

$$V_{T,z}(t,x) = \sup_{\tau\le T-t} \mathsf{E}_{t,x}\left[X_{t+\tau}\right] = \sup_{\sigma\ge 0} \mathbb{E}_{s,y}\left[G_{T,z}\left(s+\sigma, Y_{s+\sigma}\right)\right] = W_{T,z}(s,y).$$

Furthermore, suppose that $\tau^* = \tau^*(t,x)$ is an OST for (5.15) and that there exists a stopping time $\sigma' = \sigma'(s,y)$ that performs better than $\sigma^* = \sigma^*(s,y)$ in (5.16). Consider $\tau' = \tau'(t,x)$ such that $t + \tau' = \gamma_T^{-1}(s+\sigma')$. Then,

$$\mathsf{E}_{t,x}\left[X_{t+\tau'}\right] = \mathbb{E}_{s,y}\left[G_{t,T}(s+\sigma', Y_{s+\sigma'})\right] > \mathbb{E}_{s,y}\left[G_{t,T}(s+\sigma^*, Y_{s+\sigma*})\right] = \mathsf{E}_{t,x}\left[X_{t+\tau^*}\right],$$

which contradicts the fact that τ^* is optimal. Using similar arguments, we can obtain the reverse implication, that is, if σ^* is an OST for (5.16), then τ^* is an OST for (5.15). □

5.4 Solution of the infinite-horizon OSP

We have shown that solving (5.15) is equivalent to solving (5.16), which is expressed in terms of a simpler BM. In this section we leverage that advantage to solve (5.16) but, first, we rewrite it with a cleaner notation that hides its explicit connection with the original OSP, and allows us to treat (5.16) as a standalone problem.

Let $Y := \{Y_{s+u}\}_{u\in\mathbb{R}_+}$ be a standard BM on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the probability measure $\mathbb{P}_{s,y}$ such that $\mathbb{P}_{s,y}(Y_s = y) = 1$. Consider the OSP

$$W(s,y) := \sup_{\sigma\ge 0} \mathbb{E}_{s,y}\left[G(s+\sigma, Y_{s+\sigma})\right] = \sup_{\sigma\ge 0} \mathbb{E}\left[G(s+\sigma, Y_\sigma + y)\right], \tag{5.19}$$

where $\mathbb{E}$ and $\mathbb{E}_{s,y}$ are the mean operators with respect to $\mathbb{P}$ and $\mathbb{P}_{s,y}$, respectively. The supremum in (5.19) is taken among the stopping times of Y. The (gain) function G takes the form

$$G(s, y) = a_1(s) + a_2(s)\,(c_0 s + y)\,, \tag{5.20}$$

where $c_0 \in \mathbb{R}$ and $a_1, a_2 : \mathbb{R}_+ \to \mathbb{R}$ are assumed to be such that:

$$a_1 \text{ and } a_2 \text{ are twice continuously differentiable,} \tag{5.21a}$$
$$a_1, a_1', a_1'', a_2, a_2', \text{ and } a_2'' \text{ are bounded;} \tag{5.21b}$$
$$\text{there exists } c_1 \in \mathbb{R} \text{ such that } \lim_{s\to\infty} a_1(s) = c_1; \tag{5.21c}$$
$$\text{for all } s \in \mathbb{R}, a_2(s) > 0; \tag{5.21d}$$
$$\text{there exists } c_2 \in \mathbb{R} \text{ such that } \lim_{s\to\infty} a_2(s)s = c_2; \tag{5.21e}$$
$$\text{for all } s \in \mathbb{R}, a_2'(s) < 0. \tag{5.21f}$$

Remark 5.2. *Equation (5.20), as well as assumptions (5.21c)–(5.21e), come after (5.17) and (iii.1)–(iii.3) from Proposition 5.1. Indeed, the constant c_0 and the functions a_1 and a_2 are taken such that $c_0 = z - \alpha(T)$, $a_1(s) = \alpha(\gamma_T^{-1}(s))$, and $a_2(s) = \beta_T(\gamma_T^{-1}(s))$.*

Remark 5.3. *Although (5.21a) and (5.21b) are derived from the twice continuous differentiability of α, β_T, and γ_T, this degree of smoothness is not required to define GMBs. These assumptions are only used to prove smoothness properties of the value function and the OSB. The assumptions on the first derivatives are used to prove the Lipschitz continuity of the value function (see Proposition 5.3), while the ones on the second derivatives are required to prove the local Lipschitz continuity of the OSB (see Proposition 5.7).*

Remark 5.4. *The following relation, which we use recurrently throughout the paper, comes after (5.21a), (5.21b), and (5.21e):*

$$\lim_{s\to\infty} a_2'(s)s = 0. \tag{5.22}$$

Alternatively, (5.22) can be directly derived from (5.5) and the fact that $\lim_{s\to\infty} a_2(s) = 0$. Indeed,

$$\begin{aligned}\lim_{s\to\infty} a_2'(s)s &= \lim_{s\to\infty} a_2'(s)s + a_2(s) = \lim_{s\to\infty} \partial_s\left[a_2(s)s\right] = \lim_{s\to\infty} \partial_s r_1(\gamma_T^{-1}(s)) = \lim_{t\to T} \frac{r_1'(t)}{\gamma_T'(t)} \\ &= \lim_{t\to T} \frac{r_1'(t)r_2^2(t)}{r_1'(t)r_2(t) - r_1(t)r_2'(t)} = 0,\end{aligned}$$

where ∂_s denotes the derivative with respect to the variable $s \in \mathbb{R}_+$. In the last equality we used that $0 \leq r_1'(t)/r_2'(t) \leq r_1(t)/r_2(t)$, which comes after r_1 and r_2 being, respectively, an increasing and a decreasing function (see Remark 5.1).

Likewise, (5.22) along with the L'Hôpital rule implies that

$$\lim_{s\to\infty} a_2''(s)s^2 = -\lim_{s\to\infty} a_2'(s)s = 0. \tag{5.23}$$

Again, (5.23) can be obtained from its representation in terms of the covariance factorization given by r_1 and r_2,

$$\lim_{s\to\infty} a_2''(s)s^2 = \lim_{s\to\infty} \partial_{ss}\left[a_2(s)s\right]s = \lim_{s\to\infty} \partial_{ss} r_1(\gamma_T^{-1}(s))\gamma_T(\gamma_T^{-1}(s))$$

$$
\begin{aligned}
&= \lim_{s\to\infty} \partial_s \frac{r_1'(\gamma_T^{-1}(s))}{\gamma_T'(\gamma_T^{-1}(s))}\gamma_T(\gamma_T^{-1}(s)) = \lim_{t\to T}\left(\frac{r_1''(t)}{(\gamma_T'(t))^2} - \frac{r_1(t)\gamma_T''(t)}{(\gamma_T'(t))^3}\right)\gamma_T(t) \\
&= \lim_{t\to T}\left(\frac{r_1^2(t)r_2^3(t)\left(r_1'(t)r_2''(t) - r_1''(t)r_2'(t)\right)}{\left(r_1'(t)r_2(t) - r_1(t)r_2^3(t)\right)} - \frac{2r_1(t)r_2^2(t)\left(r_1'(t)\right)^2}{\left(r_1'(t)r_2(t) - r_1(t)r_2'(t)\right)^2}\right) \\
&= 0,
\end{aligned}
$$

where ∂_{ss} indicates the second derivative with respect to s.

Remark 5.5. *Assumption (5.21f) is needed to derive the boundedness of the OSB (see Proposition 5.6 and Remark 5.6). Similarly to Assumptions (5.21a)–(5.21e), Assumption (5.21f) can be obtained from the regularity of the underlying GMB already used in Section 5.2, and does not impose any further restrictions. Specifically, Assumption (5.21f) is equivalent to condition $\theta(t) > 0$ for all $t \in [0, T]$, in the OUB representation (iv) from Proposition 5.1, and to $\beta_T(t) = r_2(t) > 0$ and $\beta_T'(t) = r_2'(t) < 0$, in terms of representations (iii) and (ii) (see Remark 5.1).*

Notice that (5.21c), (5.21e), and (5.22), together with the law of the iterated logarithm, imply that, for all $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$,

$$
\mathbb{P}_{s,y}\text{-}\lim_{u\to\infty} G(s+u, Y_{s+u}) = c_1 + c_0 c_2. \tag{5.24}
$$

For later reference, let us introduce the notation

$$
\left.
\begin{aligned}
A_1 &:= \sup_{s\in\mathbb{R}_+} |a_1(s)|, \quad & A_1' &:= \sup_{s\in\mathbb{R}_+} |a_1'(s)|, \quad & A_1'' &:= \sup_{s\in\mathbb{R}_+} |a_1''(s)|, \\
A_2 &:= \sup_{s\in\mathbb{R}_+} |a_2(s)|, \quad & A_2' &:= \sup_{s\in\mathbb{R}_+} |a_2'(s)|, \quad & A_2'' &:= \sup_{s\in\mathbb{R}_+} |a_2''(s)|, \\
A_3 &:= \sup_{s\in\mathbb{R}_+} |a_2(s)s|, \quad & A_3' &:= \sup_{s\in\mathbb{R}_+} |a_2'(s)s|, \quad & A_3'' &:= \sup_{s\in\mathbb{R}_+} |a_2''(s)s|.
\end{aligned}
\right\} \tag{5.25}
$$

In addition, we will often require the expression of the partial derivatives of G, namely,

$$
\partial_t G(s, y) = a_1'(s) + c_0 a_2(s) + a_2'(s)(c_0 s + y), \tag{5.26}
$$

$$
\partial_x G(s, y) = a_2(s). \tag{5.27}
$$

Here and thereafter, ∂_t and ∂_x stand, respectively, for the differential operator with respect to time and space.

Notice that (5.21e) guarantees the existence of $m > 0$ such that $|a_2(s)| \le (1+m)/s$ for all $s \ge 1$, which, combined with the boundedness of a_1, a_2, and $s \mapsto a_2(s)s$, entails the following bound with $A = \max\{A_1 + |c_0|A_3, A_2\}$:

$$
\begin{aligned}
&\mathbb{E}_{s,y}\left[\sup_{u\in\mathbb{R}_+} |G(s+u, Y_{s+u})|\right] \\
&\quad \le \sup_{u\in\mathbb{R}_+} |a_1(u) + a_2(u)(c_0 u + y)| + \mathbb{E}\left[\sup_{u\in\mathbb{R}_+} |a_2(s+u)Y_u|\right] \\
&\quad \le A(1+|y|) + \mathbb{E}\left[\sup_{u\in\mathbb{R}_+} |a_2(s+u)Y_u|\right] \\
&\quad \le A(1+|y|) + \max_{u\le 1\vee(1-s)} |a_2(s+u)|\, \mathbb{E}\left[\sup_{u\le 1\vee(1-s)} |Y_u|\right] + \mathbb{E}\left[\sup_{u\ge 1\vee(1-s)} |a_2(s+u)Y_u|\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq A(1+|y|) + \max_{u\leq 1} |a_2(u)|\,\mathbb{E}\left[\sup_{u\leq 1} |Y_u|\right] + (1+m)\mathbb{E}\left[\sup_{u\geq 1} |Y_u|/u\right] \\
&= A\left(1 + \left(|y| + \mathbb{E}\left[\sup_{u\leq 1} |Y_u|\right]\right)\right) + (1+m)\mathbb{E}\left[\sup_{u\geq 1} \left|Y_{1/u}\right|\right] \\
&= A\left(1 + \left(|y| + \mathbb{E}\left[\sup_{u\leq 1} |Y_u|\right]\right)\right) + (1+m)\mathbb{E}\left[\sup_{u\leq 1} |Y_u|\right] < \infty. \qquad (5.28)
\end{aligned}
$$

In the last equality, the time-inversion property of the BM was used.

The continuity of G alongside (5.28) implies the continuity of W. However, given the assumptions (5.21a)–(5.21e), one can obtain higher smoothness for the value function, namely its Lipschitz continuity, as shown in the proposition below.

Proposition 5.3 (Lipschitz continuity of the value function)**.**
For any bounded set $\mathcal{R} \subset \mathbb{R}$ there exists $L_{\mathcal{R}} > 0$ such that

$$
|W(s_1, y_2) - W(s_2, y_2)| \leq L_{\mathcal{R}}(|s_1 - s_2| + |y_1 - y_2|), \qquad (5.29)
$$

for all $(s_1, y_1), (s_2, y_2) \in \mathbb{R}_+ \times \mathcal{R}$.

Proof. For any $(s_1, y_1), (s_2, y_2) \in \mathbb{R}_+ \times \mathcal{R}$, the following equality holds:

$$
\begin{aligned}
W(s_1, y_1) - W(s_2, y_2) = {} & \sup_{\sigma\geq 0} \mathbb{E}_{s_1,y_1}\left[G\left(s_1+\sigma, Y_{s_1+\sigma}\right)\right] - \sup_{\sigma\geq 0} \mathbb{E}_{s_1,y_2}\left[G\left(s_1+\sigma, Y_{s_1+\sigma}\right)\right] \\
& + \sup_{\sigma\geq 0} \mathbb{E}_{s_1,y_2}\left[G\left(s_1+\sigma, Y_{s_1+\sigma}\right)\right] - \sup_{\sigma\geq 0} \mathbb{E}_{s_2,y_2}\left[G\left(s_1+\sigma, Y_{s_2+\sigma}\right)\right].
\end{aligned}
$$

Since $|\sup_\sigma a_\sigma - \sup_\sigma b_\sigma| \leq \sup_\sigma |a_\sigma - b_\sigma|$, and due to Jensen's inequality,

$$
\begin{aligned}
&\left|\sup_{\sigma\geq 0} \mathbb{E}_{s_1,y_1}\left[G\left(s_1+\sigma, Y_{s_1+\sigma}\right)\right] - \sup_{\sigma\geq 0} \mathbb{E}_{s_1,y_2}\left[G\left(s_1+\sigma, Y_{s_1+\sigma}\right)\right]\right| \\
&\qquad \leq \mathbb{E}\left[\sup_{u\geq 0} \left|G\left(s_1+u, Y_u+y_1\right) - G\left(s_1+u, Y_u+y_2\right)\right|\right] \\
&\qquad = \sup_{u\geq 0} |a_2(s_1+u)(y_1-y_2)| \\
&\qquad \leq A_2\,|y_1 - y_2|. \qquad (5.30)
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\left|\sup_{\sigma\geq 0} \mathbb{E}_{s_1,y_2}\, G\left(s_1+\sigma, Y_{s_1+\sigma}\right)\right] - \sup_{\sigma\geq 0} \mathbb{E}_{s_2,y_2}\left[G\left(s_2+\sigma, Y_{s_2+\sigma}\right)\right]\right| \\
&\qquad \leq \mathbb{E}\left[\sup_{u\geq 0} \left|G\left(s_1+u, Y_u+y_2\right) - G\left(s_2+u, Y_u+y_2\right)\right|\right] \\
&\qquad = \mathbb{E}\left[\sup_{u\geq 0} \left|\partial_t G\left(\eta_u, Y_u+y_2\right)(s_1-s_2)\right|\right] \\
&\qquad \leq \left(A_1' + (A_3' + A_2)|c_0| + A_2'\left(\sup_{y\in\mathcal{R}}\{y\} + \mathbb{E}\left[\sup_{u\geq 0} |Y_u|\right]\right)\right)|s_1 - s_2|, \qquad (5.31)
\end{aligned}
$$

where $\eta_u \in (s_1 \wedge s_2 + u, s_1 \vee s_2 + u)$ comes from the mean value theorem, which, along with (5.26), was used to derive the last inequality. Constants A_1', A_2, A_2', and A_3' were defined in (5.25). We finally get (5.29) after merging (5.30) and (5.31). ☐

Define $\sigma^* = \sigma^*(s, y) := \inf \{u \in \mathbb{R}_+ : (s + u, Y_{s+u}) \in \mathcal{D}\}$, where the closed set

$$\mathcal{D} := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R} : W(s, y) = G(s, y)\},$$

is called the *stopping set*. The continuity of W and G (it suffices lower semi-continuity of W and upper semi-continuity of G) along with (5.28) and (5.24), guarantees that σ^* is an OST for (5.19) (see Corollary 2.9 and Remark 2.10 in Peskir and Shiryaev (2006)), meaning that

$$W(s, y) = \mathbb{E}_{s,y}\left[G(s + \sigma^*, Y_{s+\sigma^*})\right]. \tag{5.32}$$

We get the following alternative representations of W after applying Itô's lemma to (5.19) and (5.32):

$$\begin{aligned} W(s, y) - G(s, y) &= \sup_{\sigma \geq 0} \mathbb{E}_{s,y}\left[\int_0^{\sigma} \mathbb{L}G\left(s + u, Y_{s+u}\right) \mathrm{d}u\right] \\ &= \mathbb{E}_{s,y}\left[\int_0^{\sigma^*(s,y)} \mathbb{L}G\left(s + u, Y_{s+u}\right) \mathrm{d}u\right], \end{aligned} \tag{5.33}$$

where $\mathbb{L} := \partial_t + \frac{1}{2}\partial_{xx}$ is the infinitesimal generator of the process $\{(s, Y_s)\}_{s\in\mathbb{R}_+}$ and the operator ∂_{xx} is a shorthand for $\partial_x\partial_x$. Note that $\mathbb{L}G = \partial_t G$.

Denote by $\mathcal{C}$ the complement of $\mathcal{D}$,

$$\mathcal{C} := \{(s, y) \in \mathbb{R}_+ \times \mathbb{R} : W(s, y) > G(s, y)\},$$

which is called the *continuation set*. The boundary between $\mathcal{D}$ and $\mathcal{C}$ is the OSB and it determines the OST σ^*.

In addition to the Lipschitz continuity, higher smoothness of the value function is achieved away from the OSB, as stated in the next proposition. We also determine the connection between the OSP (5.19) and its associated free-boundary problem. For further details on this connection in a more general setting we refer to Section 7 of Peskir and Shiryaev (2006).

Proposition 5.4 (Higher smoothness of the value function and the free-boundary problem). *$W \in C^{1,2}(\mathcal{C})$, that is, the functions $\partial_t W$, $\partial_x W$, and $\partial_{xx} W$ exist and are continuous on $\mathcal{C}$. Additionally, $y \mapsto W(s, y)$ is convex for all $s \in \mathbb{R}_+$ and $\mathbb{L}W = 0$ on $\mathcal{C}$.*

Proof. The convexity of W with respect to the space coordinate is a straightforward consequence of the linearity of Y_{s+u} with respect to y under $\mathbb{P}_{s,y}$, for all $s \in \mathbb{R}_+$. Indeed, it follows from (5.19) that $W(s, ry_1 + (1 - r)y_2) \leq rW(s, y_1) + (1 - r)W(s, y_2)$, for all $y_1, y_2 \in \mathbb{R}$ and $r \in [0, 1]$.

Since W is continuous on $\mathcal{C}$ (see Proposition 5.3) and the coefficients in the parabolic operator $\mathbb{L}$ are smooth enough (it suffices to require local α-Hölder continuity), then standard theory of parabolic partial differential equations (Friedman, 1964, Section 3, Theorem 9) guarantees that, for an open rectangle $\mathcal{R} \subset \mathcal{C}$, the initial-boundary value problem

$$\begin{cases} \mathbb{L}f = 0 & \text{in } \mathcal{R}, \\ f = W & \text{on } \partial\mathcal{R}, \end{cases} \tag{5.34}$$

where $\partial\mathcal{R}$ refers to the boundary of $\mathcal{R}$, has a unique solution $f \in C^{1,2}(\mathcal{R})$. Therefore, we can use Itô's formula on $f(s + u, Y_{s+u})$ at $u = \sigma_{\mathcal{R}}$, that is, the first time $(s + u, Y_{s+u})$ exits $\mathcal{R}$, and then take $\mathbb{P}_{s,y}$-expectation with $(s, y) \in \mathcal{R}$, which guarantees the vanishing of the martingale term and yields, along with (5.34) and the strong Markov property, the equalities $W(s, y) = \mathbb{E}_{s,y}\left[W(s + \sigma_{\mathcal{R}}, Y_{s+\sigma_{\mathcal{R}}})\right] = f(s, y)$. Since $W = G$ on $\mathcal{D}$, it follows that $W \in C^{1,2}(\mathcal{D})$. ☐

In addition to the partial differentiability of W, it is possible to provide relatively explicit forms for $\partial_t W$ and $\partial_x W$ by relying on representation (5.33) and the fact that a_1 and a_2 are differentiable functions.

Proposition 5.5 (Partial derivatives of the value function).
For any $(s,y) \in \mathcal{C}$, consider the OST $\sigma^* = \sigma^*(s,y)$. Then,

$$\partial_t W(s,y) = \partial_t G(s,y) + \mathbb{E}_{s,y}\left[\int_s^{s+\sigma^*} \left(a_1''(u) + 2c_0 a_2'(u) + a_2''(u)(c_0 u + Y_u)\right) \mathrm{d}u\right] \tag{5.35}$$

and

$$\partial_x W(s,y) = \mathbb{E}_{s,y}\left[a_2(s+\sigma^*)\right]. \tag{5.36}$$

Proof. Since $\sigma^* = \sigma^*(s,y)$ is suboptimal for any initial condition other than (s,y), then

$$\varepsilon^{-1}\left(W(s,y) - W(s-\varepsilon,y)\right) \leq \varepsilon^{-1}\mathbb{E}\left[G(s+\sigma^*, Y_{\sigma^*}+y) - G(s-\varepsilon+\sigma^*, Y_{\sigma^*}+y)\right],$$

for any $0 < \varepsilon \leq s$. Hence, by letting $\varepsilon \to 0$ and recalling that $W \in C^{1,2}(\mathcal{C})$ (see Proposition 5.4), we get that, for $(s,y) \in \mathcal{C}$,

$$\partial_t W(s,y) \leq \mathbb{E}_{s,y}\left[\partial_t G(s+\sigma^*, Y_{s+\sigma^*})\right] = \partial_t G(s,y) + \mathbb{E}_{s,y}\left[\int_0^{\sigma^*} \mathbb{L}\partial_t G(s+u, Y_{s+u})\,\mathrm{d}u\right]. \tag{5.37}$$

In the same fashion, we obtain that

$$\varepsilon^{-1}\left(W(s+\varepsilon,y) - W(s,y)\right) \geq \varepsilon^{-1}\mathbb{E}\left[G(s+\varepsilon+\sigma^*, Y_{\sigma^*}+y) - G(s+\sigma^*, Y_{\sigma^*}+y)\right],$$

which, after letting $\varepsilon \to 0$, yields (5.37) in the reverse direction. Therefore (5.35) is proved after computing $\mathbb{L}\partial_t G(s+u, Y_{s+u}) = \partial_{tt} G(s+u, Y_{s+u})$.

To get the analog result for the space coordinate, notice that

$$\begin{aligned}
\varepsilon^{-1}\left(W(s,y) - W(s,y-\varepsilon)\right) &\leq \varepsilon^{-1}\mathbb{E}\left[W(s+\sigma^*, Y_{\sigma^*}+y) - W(s+\sigma^*, Y_{\sigma^*}+y-\varepsilon)\right] \\
&\leq \varepsilon^{-1}\mathbb{E}\left[G(s+\sigma^*, Y_{\sigma^*}+y) - G(s+\sigma^*, Y_{\sigma^*}+y-\varepsilon)\right] \\
&= \mathbb{E}_{s,y}\left[a_2(s+\sigma^*)\right],
\end{aligned}$$

while the same reasoning yields the inequality

$$\varepsilon^{-1}\left(W(s,y+\varepsilon) - W(s,y)\right) \geq \mathbb{E}_{s,y}\left[a_2(s+\sigma^*)\right],$$

and then, by letting $\varepsilon \to 0$, (5.36) follows. □

Besides the regularity of the value function, that of the OSB is also key to solving the OSP. However, defined as the boundary between $\mathcal{D}$ and $\mathcal{C}$, the OSB admits little space for technical manipulations. The next proposition gives a handle on the OSB by showing that it is the graph of a bounded function of time, above which $\mathcal{D}$ lies.

Proposition 5.6 (Shape of the OSB).
There exists a function $b : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\mathcal{D} = \left\{(s,y) \in \mathbb{R}_+ \times \mathbb{R} : y \geq b(s)\right\}.$$

Moreover, $g(s) < b(s) < \infty$ for all $s \in \mathbb{R}_+$, where $g(s) := (-a_1'(s) - c_0(a_2(s) + a_2'(s)s))/a_2'(s)$.

Proof. Define b as

$$b(s) := \inf\{y : (s, y) \in \mathcal{D}\}, \quad s \in \mathbb{R}_+. \tag{5.38}$$

The claimed shape for the stopping set is a straightforward consequence of the decreasing behavior of $y \mapsto (W - G)(s, y)$ for all $s \in \mathbb{R}_+$, which comes after (5.21f), (5.26), and (5.33).

To derive the lower bound for b, notice that, for all (s, y) such that $\partial_t G(s, y) > 0$, we can pick a ball $\mathcal{B}$ such that $(s, y) \in \mathcal{B}$ and $\partial_t G > 0$ on $\mathcal{B}$. After recalling (5.33) and by letting $\sigma_{\mathcal{B}} = \sigma_{\mathcal{B}}(s, y)$ to be the first exit time of $Y^{s,y}$ from $\mathcal{B}$, we get that

$$W(s, y) - G(s, y) \geq \mathbb{E}_{s,y}\left[\int_0^{\sigma_{\mathcal{B}}} \partial_t G(s + u, Y_{s+u})\, \mathrm{d}u\right] > 0,$$

which means that $(s, y) \in \mathcal{C}$. Finally, the claimed lower bound for b comes after using (5.26) and (5.21f) to realize that $\partial_t G(s, y) > 0$ if and only if $y < g(s)$.

We now prove that $b(s) < \infty$ for all $s \in \mathbb{R}_+$. Let $X = \{X_t\}_{t\in[0,T]}$ be the OUB representation of the process $s \mapsto G(s, Y_s)$, given in (*iv*) from Proposition 5.1, with drift $\mu(t, x) = \theta(t)(\kappa(t) - x)$ and volatility (function) ν. In addition, define the OUBs $X^{(i)}$, for $i = 1, 2$, with volatility ν and drifts

$$\mu^{(1)}(t, x) = \theta(t)(K - x), \quad \mu^{(2)}(t, x) = \frac{\underline{\nu}}{\overline{\nu}(T - t)}(K - x),$$

respectively, where $K := \max\{\kappa(t) : t \in [0, T]\}$, $\overline{\nu} := \max\{\nu(t) : t \in [0, T]\}$, and $\underline{\nu} := \min\{\nu(t) : t \in [0, T]\}$. Consider the OSPs

$$\begin{aligned}
V^{(0)}(t, x) &:= \sup_{\tau \leq T-t} \mathbb{E}_{t,x}[X_{t+\tau}], \\
V^{(1)}(t, x) &:= \sup_{\tau \leq T-t} \mathbb{E}_{t,x}\left[X^{(1)}_{t+\tau}\right], \\
V^{(2)}_K(t, x) &:= \sup_{\tau \leq T-t} \mathbb{E}_{t,x}\left[K + |X^{(2)}_{t+\tau} - K|\right],
\end{aligned}$$

alongside their respective stopping sets $\mathcal{D}^{(0)}$, $\mathcal{D}^{(1)}$, and $\mathcal{D}^{(2)}_K$.

Notice that $\mu(t, x) \leq \mu^{(1)}(t, x)$ for all $(t, x) \in [0, T) \times \mathbb{R}$. Hence, $X_{t+u} \leq X^{(1)}_{t+u}$ $\mathbb{P}_{t,x}$-a.s. for all $u \in [0, T - t]$, as Corollary 3.1 in Peng and Zhu (2006) states. This implies that $\mathcal{D}^{(1)} \subset \mathcal{D}^{(0)}$.

On the other hand, it follows from (*iv.2*) that $\theta(t) \geq \underline{\nu}/(\overline{\nu}(T - t))$, meaning that $\mu(t, x) \leq \mu^{(2)}(t, x)$ if and only if $x \geq K$. By using the same comparison result in Peng and Zhu (2006), we get the second inequality in the following sequence of relations:

$$X^{(1)}_{t+u} \leq K + |X^{(1)}_{t+u} - K| \leq K + |X^{(2)}_{t+u} - K|,$$

$\mathbb{P}_{t,x}$-a.s. for all $u \in [0, T - t]$. Hence, for a pair $(t, x) \in \mathcal{D}^{(2)}_K$, we get that $V^{(0)}(t, x) \leq V^{(2)}_K(t, x) = x$, that is, $(t, x) \in \mathcal{D}^{(1)}$ and, therefore, $\mathcal{D}^{(2)}_K \subset \mathcal{D}^{(0)}$. The OSP related to $V^{(2)}_K$ can be shown to account for a finite OSB. Specifically, it is a multiple of that of a BB (see Section 5 from D'Auria and Ferriero (2020)). Then, $\mathcal{D}^{(0)} \cap (\{t\} \times \mathbb{R})$ is non-empty for all $t \in [0, T)$, and the equivalence result in Proposition 5.2 guarantees that so are the sets of the form $\mathcal{D} \cap (\{t\} \times \mathbb{R})$, meaning that the OSB b is bounded from above. □

Remark 5.6. *Note that the same reasoning we used to derive the lower bound of b in the proof of Proposition 5.6 also implies that, if $a_2'(s) > 0$ for some $s \in \mathbb{R}_+$, then $(s, y) \in \mathcal{C}$ for all $y > (-a_1'(s) - c(a_2(s) + a_2'(s)s))/a_2'(s)$, meaning that $b(s) = \infty$. To avoid this explosion of the OSB we impose $a_2'(s) < 0$ for all $s \in \mathbb{R}_+$ in (5.21f).*

Summarizing, we have proved that W satisfies the free-boundary problem

$$\begin{aligned} \mathbb{L}W(s,y) &= 0 && \text{for } y < b(t), \\ W(s,y) &> G(s,y) && \text{for } y < b(t), \\ W(s,y) &= G(s,y) && \text{for } y \geq b(t). \end{aligned}$$

In order to guarantee the uniqueness of its solution, since $b(t)$ is unknown, an additional condition is needed. When b is regular enough, this *smooth-fit condition* comes by making the value and the gain function coincide smoothly at the free boundary.

The works of De Angelis (2015), Peskir (2019), and De Angelis and Stabile (2019) address the smoothness of the free boundary. For one-dimensional, time-homogeneous processes with locally Lipschitz-continuous drift and volatility, De Angelis (2015) provides the continuity of the free boundary. Peskir (2019) works with the two-dimensional case in a fairly general setting, proving the impossibility of first-type discontinuities (second-type discontinuities are not addressed). De Angelis and Stabile (2019) go further by proving the local Lipschitz continuity of the free boundary in a higher-dimensional framework. In particular, local Lipschitz continuity suffices for the smooth-fit condition to hold (see Proposition 5.8 ahead), which is the main drive to tailor the method of De Angelis and Stabile (2019) to fit our settings in the next proposition. Specifically, the relation between the partial derivatives imposed on Assumption (D) by De Angelis and Stabile (2019) excludes our gain function, but Equation (5.43) overcomes this issue.

Proposition 5.7 (Lispchitz continuity of the OSB)**.**
The OSB b is Lipschitz continuous on any closed interval of $\mathbb{R}_+$.

Proof. Let $H(s,y) := W(s,y) - G(s,y)$, and consider the closed interval $I = [\underline{s}, \bar{s}] \subset \mathbb{R}_+$. Proposition 5.6 guarantees that b is bounded from below and, hence, we can choose $r < \inf\{b(s) : s \in I\}$. Then, $I \times \{r\} \subset \mathcal{C}$, meaning that $H(s,r) > 0$ for all $s \in I$. Since H is continuous (see Proposition 5.3) on $\mathcal{C}$, there exists a constant $a > 0$ such that $H(s,r) \geq a$ for all $s \in I$. Therefore, for all δ such that $0 < \delta \leq a$, and all $s \in I$, there exists $y \in \mathbb{R}$ such that $H(s,y) = \delta$. Such a value of y is unique, as $\partial_x H < 0$ on $\mathcal{C}$ (see (5.36)). Hence, we can denote it by $b_\delta(s)$ and define the function $b_\delta : I \to (b(s), r]$. H is regular enough away from the boundary to apply the implicit function theorem, which states the differentiability of b_δ along with

$$b'_\delta(s) = -\partial_t H(s, b_\delta(s)) / \partial_x H(s, b_\delta(s)). \tag{5.39}$$

Since the function b_δ decreases in δ and is upper-bounded uniformly in $s \in I$, it converges pointwise to some limit function b_0 as $\delta \to 0$. It follows that $b_0 \leq b$ on I, as $b_\delta < b$ for all δ. The reverse inequality follows from

$$H(s, b_0(s)) = \lim_{\delta \to 0} H(s, b_\delta(s)) = \lim_{\delta \to 0} \delta = 0,$$

meaning that $(s, b_0(s)) \in \mathcal{D}$. Hence, $b_0 = b$ on I.

For $(s,y) \in \mathcal{C}$ such that $s \in I$ and $y > r$, consider the stopping times $\sigma^* = \sigma^*(s,y)$ and

$$\sigma_r = \sigma_r(s,y) = \inf\{s \geq 0 : (s+u, Y_{s+u}) \notin I \times (r, \infty)\}.$$

By recalling (5.35), it readily follows that

$$|\partial_t H(s,y)| \leq K^{(1)}\, m(s,y) \tag{5.40}$$

for $K^{(1)} = \max\{A_1'' + 2c_0 A_2' + c_0 A_3'', 1\}$ and

$$m(s,y) := \mathbb{E}_{s,y}\left[\int_0^{\sigma^*} \left(1 + |a_2''(s+u)Y_{s+u}|\right)\, \mathrm{d}u\right].$$

Due to the tower property of conditional expectation and the strong Markov property, we have that

$$m(s,y) = \mathbb{E}_{s,y}\left[\int_0^{\sigma^*\wedge\sigma_r} \left(1 + |a_2''(s+u)Y_{s+u}|\right)\, \mathrm{d}u + \mathbb{1}\left(\sigma_r \le \sigma^*\right) m(s+\sigma_r, Y_{s+\sigma_r})\right]. \tag{5.41}$$

On the set $\{\sigma_r \le \sigma^*\}$, $(s+\sigma_r, Y_{s+\sigma_r}) \in \Gamma_s$ $\mathbb{P}_{s,y}$-a.s. whenever $r < y < b(s)$, with $\Gamma_s := ((s,\bar{s})\times\{r\}) \cup (\{\bar{s}\}\times[r, b(\bar{s})])$. Hence, if $\sigma_r \le \sigma^*$, then

$$\begin{aligned}
m\left(s+\sigma_r, Y_{\sigma_r}\right) &\le \sup_{(s',y')\in\Gamma_s} m(s',y') \\
&\le \sup_{(s',y')\in\Gamma_s} \mathbb{E}_{s',y'}\left[\int_0^\infty \left(1 + |a_2''(s'+u)Y_{s+u}|\right)\, \mathrm{d}u\right] \\
&\le \sup_{(s',y')\in\Gamma_s} \int_0^\infty \left(1 + |a_2''(s'+u)y'|\right)\, \mathrm{d}u + \int_0^\infty \mathbb{E}\left[|a_2''(s'+u)Y_u|\right]\, \mathrm{d}u \\
&\le \int_0^\infty \left(1 + |a_2''(u)b(\bar{s})|\right)\, \mathrm{d}u + \int_0^\infty a_2''(u)\sqrt{2u/\pi}\, \mathrm{d}u < \infty.
\end{aligned} \tag{5.42}$$

We can guarantee the convergence of both integrals since (5.23) implies that $|a_2''(s)|$ is asymptotically equivalent to s^{-2}. By plugging (5.42) into (5.41), recalling (5.40), and noticing that $1 + |a_2''(s+u)Y_{s+u}| \le 1 + A_2'' \max\left[|\sup_{s\in I} b(s)|, |r|\right]$ whenever $u \le \sigma^*\wedge\sigma_r$, we obtain that there exists $K_I^{(2)} > 0$ such that

$$|\partial_t H(s,y)| \le K_I^{(2)}\, \mathbb{E}_{s,y}\left[\sigma_\delta \wedge \sigma_r + \mathbb{1}\left(\sigma_r \le \sigma_\delta\right)\right]. \tag{5.43}$$

Arguing as in (5.41) and relying on (5.27), (5.36), and (5.21f), we get that

$$\begin{aligned}
&|\partial_x H(s,y)| \\
&\quad = \mathbb{E}_{s,y}\left[a_2(s) - a_2(s+\sigma^*)\right] = \mathbb{E}_{s,y}\left[\int_0^{\sigma^*} -a_2'(s+u)\, \mathrm{d}u\right] \\
&\quad = \mathbb{E}_{s,y}\left[\int_0^{\sigma^*\wedge\sigma_r} -a_2'(s+u)\, \mathrm{d}u + \mathbb{1}\left(\sigma_r \le \sigma^*\right)|\partial_x H(s+\sigma_r, Y_{s+\sigma_r})|\right] \\
&\quad \ge \mathbb{E}_{s,y}\left[\int_0^{\sigma^*\wedge\sigma_r} -a_2'(s+u)\, \mathrm{d}u + \mathbb{1}\left(\sigma_r \le \sigma^*, \sigma_r < \bar{s}-s\right)|\partial_x H(s+\sigma_r, r)|\right].
\end{aligned} \tag{5.44}$$

Since $I\times\{r\} \subset \mathcal{C}$, we can take $\varepsilon > 0$ such that $\mathcal{R}_\varepsilon := [\underline{s}, \bar{s}+\varepsilon]\times(r-\varepsilon, r+\varepsilon) \subset \mathcal{C}$. Thereby, $\sigma^* > \sigma_\varepsilon$ $\mathbb{P}_{s,r}$-a.s. for all $s \in I$, where

$$\sigma_\varepsilon = \sigma_\varepsilon(s,r) := \inf\left\{u \ge 0 : (s+u, Y_{s+u}) \notin \mathcal{R}_\varepsilon\right\}.$$

Hence,

$$\begin{aligned}
|\partial_x H(s+\sigma_r, r)| &\ge \inf_{s\in I}|\partial_x H(s,r)| = \inf_{s\in I}\mathbb{E}_{s,r}\left[a_2(s) - a_2(s+\sigma^*)\right] \\
&\ge \inf_{s\in I}\mathbb{E}_{s,r}\left[a_2(s) - a_2(s+\sigma_\varepsilon)\right]
\end{aligned}$$

$$\geq \inf_{s\in I} \left(a_2(s) - a_2(\overline{s}+\varepsilon)\right) \mathbb{P}_{s,r}\left(\sigma_\varepsilon = \overline{s}+\varepsilon - s\right)$$

$$\geq \left(a_2(\overline{s}) - a_2(\overline{s}+\varepsilon)\right) \mathbb{P}\Bigg(\sup_{u\leq \overline{s}+\varepsilon-\underline{s}} |Y_u| < \varepsilon\Bigg) > 0, \tag{5.45}$$

where we use that a_2 is decreasing. Recalling that a_2' is a bounded function and plugging (5.45) into (5.44), we get that, for a constant $K_{I,\varepsilon}^{(3)} > 0$,

$$|\partial_x H(s,y)| \geq K_{I,\varepsilon}^{(3)}\, \mathbb{E}_{s,y}\left[\sigma^* \wedge \sigma_r + \mathbb{1}\left(\sigma_r \leq \sigma^*, \sigma_r < \overline{s} - s\right)\right]. \tag{5.46}$$

Substituting (5.43) and (5.46) into (5.39) we get the following bound for the derivative of b by some constant $K_{I,\varepsilon}^{(4)} > 0$, $y_\delta = b_\delta(s)$, and $\sigma_\delta = \sigma^*(s, y_\delta)$:

$$\begin{aligned}
|b_\delta'(s)| &\leq K_{I,\varepsilon}^{(4)} \frac{\mathbb{E}_{s,y_\delta}\left[\sigma_\delta \wedge \sigma_r + \mathbb{1}\left(\sigma_r \leq \sigma_\delta\right)\right]}{\mathbb{E}_{s,y_\delta}\left[\sigma_\delta \wedge \sigma_r + \mathbb{1}\left(\sigma_r \leq \sigma_\delta, \sigma_r < \overline{s}-s\right)\right]} \\
&\leq K_{I,\varepsilon}^{(4)} \left(1 + \frac{\mathbb{P}_{s,y_\delta}\left(\sigma_r \leq \sigma_\delta\right)}{\mathbb{E}_{s,y_\delta}\left[\sigma_\delta \wedge \sigma_r + \mathbb{1}\left(\sigma_r \leq \sigma_\delta, \sigma_r < \overline{s}-s\right)\right]}\right) \\
&\leq K_{I,\varepsilon}^{(4)} \left(1 + \frac{\mathbb{P}_{s,y_\delta}\left(\sigma_r \leq \sigma_\delta, \sigma_r = \overline{s}-s\right)}{\mathbb{E}_{s,y_\delta}\left[\sigma_\delta \wedge \sigma_r\right]} + \frac{\mathbb{P}_{s,y_\delta}\left(\sigma_r \leq \sigma_\delta, \sigma_r < \overline{s}-s\right)}{\mathbb{E}_{s,y_\delta}\left[\mathbb{1}\left(\sigma_r \leq \sigma_\delta, \sigma_r < \overline{s}-s\right)\right]}\right) \\
&\leq K_{I,\varepsilon}^{(4)} \left(2 + \frac{\mathbb{P}_{s,y_\delta}\left(\sigma_r \leq \sigma_\delta, \sigma_r = \overline{s}-s\right)}{\mathbb{E}_{s,y_\delta}\left[\mathbb{1}\left(\sigma_r \leq \sigma_\delta, \sigma_r = \overline{s}-s\right)\left(\sigma_\delta \wedge \sigma_r\right)\right]}\right) \\
&\leq K_{I,\varepsilon}^{(4)} \left(2 + \frac{1}{\overline{s}-s}\right).
\end{aligned} \tag{5.47}$$

Let $I_\varepsilon = [\underline{s}, \overline{s}-\varepsilon]$ for $\varepsilon > 0$ small enough. By (5.47), there exists a constant $L_{I,\varepsilon} > 0$, independent from δ, such that $|b_\delta'(s)| < L_{I,\varepsilon}$ for all $s \in I_\varepsilon$ and $0 < \delta \leq a$. Hence, Arzelà–Ascoly's theorem guarantees that b_δ converges to b uniformly in $\delta \in I_\varepsilon$. $\square$

For locally Lipschitz continuous OSBs, the proof of the smooth-fit condition comes relatively easy from the law of the iterated logarithm (see Remark 4.5 by De Angelis and Stabile (2019)) and the work of De Angelis and Peskir (2020). The proposition below provides the details.

Proposition 5.8 (The smooth-fit condition)**.**
For all $s \in \mathbb{R}_+$, $y \mapsto W(s,y)$ is differentiable at $y = b(s)$. Moreover, $\partial_x W(s, b(s)) = \partial_x G(s, b(s))$.

Proof. On the one hand, since $W = G$ on $\mathcal{D}$, we get that $\partial_x W(s, b(s)^+) = \partial_x G(s, b(s)) = a_2(s)$. On the other hand, the law of the iterated logarithm alongside the local Lipschitz continuity of b yield the following for all $s \in \mathbb{R}_+$ and some constant $L_s > 0$ that depends on s:

$$\begin{aligned}
&\mathbb{P}_{s,b(s)}\left(\inf\left\{u > 0 : Y_{s+u} < b(s+u)\right\} = 0\right) \\
&= \lim_{\varepsilon\downarrow 0} \mathbb{P}_{s,b(s)}\left(\inf\left\{u > 0 : Y_{s+u} < b(s+u)\right\} < \varepsilon\right) = \lim_{\varepsilon\downarrow 0} \mathbb{P}_{s,b(s)}\left(\inf_{u\in(0,\varepsilon)} \left(Y_{s+u} - b(s+u)\right) < 0\right) \\
&= \lim_{\varepsilon\downarrow 0} \mathbb{P}_{s,b(s)}\left(\inf_{u\in(0,\varepsilon)} \frac{Y_{s+u} - b(s+u)}{\sqrt{2u\ln(\ln(1/u))}} < 0\right) \geq \lim_{\varepsilon\downarrow 0} \mathbb{P}_{s,b(s)}\left(\inf_{u\in(0,\varepsilon)} \frac{Y_{s+u} - b(s) + L_s u}{\sqrt{2u\ln(\ln(1/u))}} < 0\right) \\
&= \mathbb{P}_{s,b(s)}\left(\liminf_{u\downarrow 0} \frac{Y_{s+u} - b(s) + L_s u}{\sqrt{2u\ln(\ln(1/u))}} < 0\right) = 1,
\end{aligned}$$

that is, $\{(s+u, Y_{s+u})\}_{u\in\mathbb{R}_+}$ immediately enters the interior of $\mathcal{D}$ $\mathbb{P}_{s,b(s)}$-a.s. and, hence, Corollary 6 from De Angelis and Peskir (2020) guarantees that $\sigma^*(s, b(s)^-) = \sigma^*(s, b(s)) = 0$ $\mathbb{P}$-a.s. Therefore, the dominated convergence theorem and (5.36) allow concluding the proof by showing $\partial_x W(s, b(s)^-) = a_2(s) = \partial_x G(s, b(s))$. □

Finally, we are able to provide the solution for the OSP (5.19). Indeed, so far we have gathered all the regularity conditions needed to apply an extended Itô's formula to $W(s+u, Y_{s+u})$ to obtain characterizations of the value function and the OSB. The former is given in terms of an integral of the OSB, while the latter is proved to be the unique solution of a type-two, nonlinear, Volterra integral equation. Both characterizations benefit from the Gaussianity of the BM, yielding relatively explicit integrands. Theorem 5.1 dives into details. Its proof needs the following lemma.

Lemma 5.2. *For all* $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\lim_{u\to\infty} \mathbb{E}_{s,y}\left[W(s+u, Y_{s+u})\right] = c_1 + c_0 c_2.$$

Proof. Let $s_u := s + u$ for $s, u \in \mathbb{R}_+$. Denote by $\widehat{Y}$ a version of Y. Hence, the Markov property of Y implies that

$$\begin{aligned}
&\lim_{u\to\infty} \mathbb{E}_{s,y}\left[W(s_u, Y_{s_u})\right] \\
&\quad = \lim_{u\to\infty} \mathbb{E}_{s,y}\left[\sup_{\sigma\geq 0} \mathbb{E}_{s_u, Y_{s_u}}\left[G\left(s_u+\sigma, Y_{s_u+\sigma}\right)\right]\right] \\
&\quad \leq \lim_{u\to\infty} \mathbb{E}_{s,y}\left[\mathbb{E}_{s_u, Y_{s_u}}\left[\sup_{r\geq 0} G\left(s_u+r, Y_{s_u+r}\right)\right]\right] \\
&\quad = \lim_{u\to\infty} \mathbb{E}_{s,y}\left[\sup_{r\geq 0}\left\{a_1(s_u+r) + c_0 a_2(s_u+r)(s_u+r) + c_0 a_2(s_u+r) Y_{s_u+r}\right\}\right] \\
&\quad = \mathbb{E}_{s,y}\left[\lim_{u\to\infty}\sup_{r\geq 0}\left\{a_1(s_u+r) + c_0 a_2(s_u+r)(s_u+r) + c_0 a_2(s_u+r) Y_{s_u+r}\right\}\right] \\
&\quad = \mathbb{E}_{s,y}\left[\limsup_{u\to\infty}\left\{a_1(s_u) + c_0 a_2(s_u) s_u + c_0 a_2(s_u) Y_{s_u}\right\}\right] \\
&\quad = c_1 + c_0 c_2,
\end{aligned}$$

where the interchangeability of the limit and the mean operator is justified by the monotone convergence theorem. The last equality comes after (5.21c) and (5.21e), along with the law of the iterated logarithm, which guarantees that $\limsup_{u\to\infty} a_2(u) Y_{s_u} = 0$.

Likewise, we have that

$$\begin{aligned}
\lim_{u\to\infty} \mathbb{E}_{s,y}\left[W(s_u, Y_{s_u})\right] &\geq \lim_{u\to\infty} \mathbb{E}_{s,y}\left[\mathbb{E}_{s_u, Y_{s_u}}\left[\inf_{r\geq 0} G\left(s_u+r, Y_{s_u+r}\right)\right]\right] \\
&= \mathbb{E}_{s,y}\left[\liminf_{u\to\infty}\left\{a_1(s_u) + c_0 a_2(s_u) s_u + c_0 a_2(s_u) Y_{s_u}\right\}\right] \\
&= c_1 + c_0 c_2,
\end{aligned}$$

which concludes the proof. □

Theorem 5.1 (Solution of the OSP).
The OSB related to the OSP (5.19) satisfies the free-boundary (integral) equation

$$G(s, b(s)) = c_1 + c_0c_2 - \int_s^\infty K(s, b(s), u, b(u))\,\mathrm{d}u, \tag{5.48}$$

where the kernel K is defined as

$$\begin{aligned}K(s_1, y_1, s_2, y_2) := \;& ((a_1'(s_2) + c_0a_2(s_2) + c_0a_2'(s_2)(s_2 + y_1))\,\bar{\Phi}_{s_1,y_1,s_2,y_2} \\ & + c_0a_2'(s_2)\sqrt{s_2 - s_1}\phi_{s_1,y_1,s_2,y_2}\end{aligned}$$

with $0 \leq s_1 \leq s_2$, $y_1, y_2 \in \mathbb{R}$, and

$$\bar{\Phi}_{s_1,y_1,s_2,y_2} := \bar{\Phi}\left(\frac{y_2 - y_1}{\sqrt{s_2 - s_1}}\right), \quad \phi_{s_1,y_1,s_2,y_2} := \phi\left(\frac{y_2 - y_1}{\sqrt{s_2 - s_1}}\right).$$

The functions ϕ and $\bar{\Phi}$ are, respectively, the density and survival functions of a standard normal random variable. In addition, the integral equation (5.48) admits a unique solution among the class of continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ of bounded variation such that $f(s) > c_1 + c_0c_2$ for all $s \in \mathbb{R}_+$.

The value function is given by the formula

$$W(s, y) = c_1 + c_0c_2 - \int_s^\infty K(s, y, u, b(u))\,\mathrm{d}u. \tag{5.49}$$

Proof. Propositions 5.3–5.8 provide the required regularity to apply an extended Itô's lemma (see Peskir (2005a) for an original derivation and Lemma A2 in D'Auria et al. (2020) for a reformulation that better suits our settings) to $W(s + h, Y_h)$ for $s, h \geq 0$. Since $\mathbb{L}W = 0$ on $\mathcal{C}$ and $W = G$ on $\mathcal{D}$, after taking $\mathbb{P}_{s,y}$-expectation (which cancels the martingale term) it follows that

$$\begin{aligned}W(s, y) &= \mathbb{E}_{s,y}\left[W(s + h, Y_h)\right] - \mathbb{E}_{s,y}\left[\int_0^h (\mathbb{L}W)(s + u, Y_{s+u})\,\mathrm{d}u\right] \\ &= \mathbb{E}_{s,y}\left[W(s + h, Y_h)\right] - \mathbb{E}_{s,y}\left[\int_0^h \partial_t G(s + u, Y_{s+u})\,\mathbb{1}(Y_{s+u} \geq b(s + u))\,\mathrm{d}u\right],\end{aligned} \tag{5.50}$$

where the local time term does not appear due to the smooth-fit condition. Hence, by taking $h \to \infty$ in (5.50) and relying on Lemma 5.2, we get the following formula for the value function:

$$\begin{aligned}W(s, y) &= c_1 + c_0c_2 - \mathbb{E}_{s,y}\left[\int_0^\infty (\mathbb{L}W)(s + u, Y_{s+u})\,\mathrm{d}u\right] \\ &= c_1 + c_0c_2 - \mathbb{E}_{s,y}\left[\int_0^\infty \partial_t G(s + u, Y_{s+u})\,\mathbb{1}(Y_{s+u} \geq b(s + u))\,\mathrm{d}u\right].\end{aligned} \tag{5.51}$$

We can obtain a more tractable version of (5.51) by exploiting the linearity of $y \mapsto \partial_t G(s, y)$ (see (5.26)) as well as the fact that $Y_{s+u} \sim \mathcal{N}(y, u)$ under $\mathbb{P}_{s,y}$. Then,

$$\mathbb{E}_{s,y}\left[Y_{s+u}\mathbb{1}(Y_{s+u} \geq x)\right] = \bar{\Phi}((x - y)/\sqrt{u})y + \sqrt{u}\phi((x - y)/\sqrt{u}).$$

Hence, by right-shifting the integrating variable s units, we get equation (5.49).

Take now $y \downarrow b(s)$ in both (5.51) and (5.49) to derive the free-boundary equation

$$G(s, b(s)) = c_1 + c_0c_2 - \mathbb{E}_{s,b(s)}\left[\int_0^\infty \partial_t G(s + u, Y_{s+u})\,\mathbb{1}(Y_{s+u} \geq b(s + u))\,\mathrm{d}u\right], \tag{5.52}$$

alongside its more explicit expression (5.48).

The uniqueness of the solution of equation (5.52) follows a well-known methodology first developed by Peskir (2005b, Theorem 3.1) that we omit here for the sake of briefness. □

5.5 Solution of the original OSP

In this section we continue with the notation used in Section 5.3.

Recall that Proposition 5.2 dictates the equivalence between the OSPs (5.15) and (5.16), and gives explicit formulae to link their value functions and OSTs. Consequently, it follows that the stopping time $\tau^*(t,x)$ defined in Proposition 5.2 in terms of $\sigma^*(s,y)$ is not only optimal for (5.15), but it holds the following representation under $\mathsf{P}_{t,x}$:

$$\tau^*(t,x) = \inf\left\{u \geq 0 : X_{t+u} \geq \mathsf{b}_{T,z}(t+u)\right\}, \quad \mathsf{b}_{T,z}(t) := G_{T,z}(s, b_{T,z}(s)), \tag{5.53}$$

where $\mathsf{b}_{T,z}$ and $b_{T,z}$ are, respectively, the OSBs related to (5.15) and (5.16), and s is defined, in terms of t, in Proposition 5.2. Note that $b_{T,z}$ coincides with the function defined in (5.38), with constants c_0, c_1, and c_2, from (5.20), (5.21c), and (5.21e), taking the values

$$c_0 = z - \alpha(T), \quad c_1 = \alpha(T), \quad c_2 = 1. \tag{5.54}$$

Moreover, it is not necessary to compute $W_{T,z}$ and $b_{T,z}$ to obtain $V_{T,z}$ and $\mathsf{b}_{T,z}$. By considering the infinitesimal generator of $\{(t,X_t)\}_{t\in[0,T]}$, L, letting $s_\varepsilon = s+\varepsilon$ and $t_\varepsilon = \gamma_T^{-1}(s_\varepsilon)$ for $\varepsilon > 0$, and using (5.18) alongside the chain rule, we get that

$$\begin{aligned}
(\mathbb{L}W_{T,z})(s,y) &:= \lim_{\varepsilon\to 0} \varepsilon^{-1}\left(\mathbb{E}_{s,y}\left[W_{T,z}\left(s_\varepsilon, Y_{s_\varepsilon}\right)\right] - W_{T,z}(s,y)\right) \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-1}\left(\mathsf{E}_{t,x}\left[V_{T,z}(t_\varepsilon, X_{t_\varepsilon})\right] - V_{T,z}(t,x)\right) \\
&= \left(\mathrm{L}V_{T,z}\right)(t,x)\left[\gamma_T^{-1}\right]'(s).
\end{aligned} \tag{5.55}$$

We recall the relation between s and t, and y and x, in Proposition 5.2. After integrating with respect to $\gamma_T^{-1}(u)$ instead of u in (5.50), keeping in mind (5.54) and (5.55), and recalling that $\mathrm{L}V_{T,z}(t,x) = 0$ for all $x \leq \mathsf{b}_{T,z}(t)$ and $V_{T,z}(t,x) = x$ for all $x \geq \mathsf{b}_{T,z}(t)$, we get the formula

$$\begin{aligned}
V_{T,z}(t,x) &= z - \mathsf{E}_{t,x}\left[\int_0^{T-t} (\mathrm{L}V_{T,z})(t+u, X_{t+u})\,\mathrm{d}u\right] \\
&= z - \mathsf{E}_{t,x}\left[\int_0^{T-t} \mu(t+u, X_{t+u})\mathbb{1}(X_{t+u} \geq \mathsf{b}_{T,z}(t+u))\,\mathrm{d}u\right],
\end{aligned} \tag{5.56}$$

where, in alignment to (5.14),

$$\begin{aligned}
\mu(t,x) &:= \lim_{u\downarrow 0} u^{-1}\mathsf{E}_{t,x}\left[X_{t+u} - x\right] = \theta(t)(\kappa(t) - x) \\
&= \alpha'(t) + (x - \alpha(t))\frac{\beta_T'(t)}{\beta_T(t)} + (z - \alpha(T))\beta_T(t)\gamma_T'(t).
\end{aligned}$$

As we did to obtain (5.49), the linearity of $x \mapsto \mu(t,x)$ and the Gaussian marginal distributions of X, allow us to produce a refined version of (5.56):

$$V_{T,z}(t,x) = z - \int_t^T \mathsf{K}(t,x,u,\mathsf{b}_{T,z}(u))\,\mathrm{d}u, \tag{5.57}$$

where

$$\mathsf{K}(t_1,x_1,t_2,x_2)$$

$$:= \theta(t_2)\left((\kappa(t_2) - \mathsf{E}_{t_1,x_1}[X_{t_2}])\Phi_{t_1,x_1,t_2,x_2} - \sqrt{\mathrm{Var}_{t_1}[X_{t_2}]}\frac{\beta_T'(t_2)}{\beta_T(t_2)}\phi_{t_1,x_1,t_2,x_2}\right) \tag{5.58}$$

$$= \left(\alpha'(t_2) + (\mathsf{E}_{t_1,x_1}[X_{t_2}] - \alpha(t_2))\frac{\beta_T'(t_2)}{\beta_T(t_2)} + (z-\alpha(T))\beta_T(t_2)\gamma_T'(t_2)\right)\Phi_{t_1,x_1,t_2,x_2}$$

$$+ \sqrt{\mathrm{Var}_{t_1}[X_{t_2}]}\frac{\beta_T'(t_2)}{\beta_T(t_2)}\phi_{t_1,x_1,t_2,x_2}, \tag{5.59}$$

with $0 \leq t_1 \leq t_2 < T$, $x_1, x_2 \in \mathbb{R}$, and

$$\Phi_{t_1,x_1,t_2,x_2} := \bar{\Phi}\left(\frac{x_2 - \mathsf{E}_{t_1,x_1}[X_{t_2}]}{\sqrt{\mathrm{Var}_{t_1}[X_{t_2}]}}\right), \quad \phi_{t_1,x_1,t_2,x_2} := \phi\left(\frac{x_2 - \mathsf{E}_{t_1,x_1}[X_{t_2}]}{\sqrt{\mathrm{Var}_{t_1}[X_{t_2}]}}\right),$$

and, as stated in (5.10), (5.12), and (5.14),

$$\mathsf{E}_{t_1,x_1}[X_{t_2}] = \varphi(t_2)\left(\frac{x}{\varphi(t_1)} + \int_{t_1}^{t^2}\frac{\kappa(u)\theta(u)}{\varphi(u)}\,\mathrm{d}u\right) \tag{5.60}$$

$$= \alpha(t_2) + \beta_T(t_2)\left((z-\alpha(T))\gamma_T(t_2) - \frac{x_1 - \alpha(t_1) - \beta_T(t_1)\gamma_T(t_1)(z-\alpha(T))}{\beta_T(t_1)}\right),$$

$$\mathrm{Var}_{t_1}[X_{t_2}] = \varphi^2(t_2)\int_{t_1}^{t_2}\frac{\nu^2(u)}{\varphi^2(u)}\,\mathrm{d}u \tag{5.61}$$

$$= \beta_T(t_1)\gamma_T(t_1)\beta_T(t_2),$$

with $\varphi(t) = \exp\left\{-\int_0^t \theta(u)\,\mathrm{d}u\right\}$. Hence, after taking $x \downarrow \mathsf{b}(t)$ in (5.56) (or by directly expressing (5.52) in terms of the original OSP, as we did to obtain (5.56) from (5.51)), we get the free-boundary equation

$$\mathsf{b}_{T,z}(t) = z - \mathsf{E}_{t,\mathsf{b}_{T,z}(t)}\left[\int_0^{T-t}(\mathbb{L}_X V_{T,z})(t+u, X_{t+u})\,\mathrm{d}u\right]$$

$$= z - \mathsf{E}_{t,\mathsf{b}_{T,z}(t)}\left[\int_0^{T-t}\mu(t+u, X_{t+u})\mathbb{1}(X_{t+u} \geq \mathsf{b}_{T,z}(t+u))\,\mathrm{d}u\right],$$

which is also expressible as

$$\mathsf{b}_{T,z}(t) = z - \int_t^T \mathsf{K}(t, \mathsf{b}_{T,z}(t), u, \mathsf{b}_{T,z}(u))\,\mathrm{d}u. \tag{5.62}$$

The uniqueness of the solution of the Volterra-type integral equation (5.62) comes after that of (5.48).

Remark 5.7. *We highlight some smoothness properties that the value function V and the OSB b inherit from W and b, based on the equivalences (5.18) and (5.53).*

From the Lipschitz continuity of W on compact sets of $\mathbb{R}_+ \times \mathbb{R}$ (see Proposition 5.3), it follows that of V in compact sets of $[0,T) \times \mathbb{R}$. Higher smoothness of V is also attained away from the boundary, $(t, \mathsf{b}(t))$ for all $t \in [0,T)$, as it follows from Proposition 5.4. The smooth-fit condition proved in Proposition 5.8 implies that of V, namely, $\partial_x V(t, \mathsf{b}(t)) = \mathsf{b}(t)$, for all $t \in [0,T)$.

The OSB b is Lipschitz continuous on any closed subinterval of $[0,T)$, which is a consequence of Proposition 5.7.

5.6 Numerical results

In this section we shed light on the OSB's shape by using a Picard iteration algorithm to solve the free-boundary equation (5.62). This approach is commonly used in the optimal stopping literature; see, e.g., the works of Detemple and Kitapbayev (2020) and De Angelis and Milazzo (2020).

A Picard iteration scheme approaches (5.62) as a fixed-point problem. From an initial candidate boundary, it produces a sequence of functions by iteratively computing the integral operator in the right-hand side of (5.62), until the error between consecutive boundaries is below a prescribed threshold. More precisely, for a partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0, T]$, $N \in \mathbb{N}$, the updating mechanism that generates subsequent boundaries follows after the discretization of the integral in (5.62) by using a right Riemann sum:

$$\mathsf{b}_i^{(k)} = z - \sum_{j=i}^{N-2} \mathsf{K}\left(t_i, \mathsf{b}_i^{(k-1)}, t_{j+1}, \mathsf{b}_{j+1}^{(k-1)}\right)(t_{j+1} - t_j), \quad i = 0, 1, \dots, N-2, \tag{5.63}$$

$$\mathsf{b}_{N-1}^{(k)} = \mathsf{b}_N^{(k)} = z, \tag{5.64}$$

for $k = 1, 2, \dots$ and with $\mathsf{b}_i^{(k)}$ standing for the value of the boundary at t_i output after the k-th iteration. We neglect the $(N-1)$-addend of the sum, and instead consider (5.64), since $\mathsf{K}(t, x, T, z)$ is not well defined. As the integral in (5.62) is finite, the last piece vanishes as t_{N-1} approaches T. Given that $\mathsf{b}(T) = z$, we set the initial constant boundary $\mathsf{b}_i^{(0)} = z$ for all $i = 0, \dots, N$. We stop the fixed-point algorithm when the relative (squared) L_2-distance between the consecutive discretized boundaries, defined as

$$d_k := \frac{\sum_{i=1}^{N} \left(\mathsf{b}_i^{(k)} - \mathsf{b}_i^{(k-1)}\right)^2 (t_i - t_{i-1})}{\sum_{i=1}^{N} \left(\mathsf{b}_i^{(k)}\right)^2 (t_i - t_{i-1})},$$

is lower than 10^{-3}.

To the best of our knowledge, no formal proof has been provided to address the convergence of Picard iterations within the context of the free-boundary equations that typically arise when solving OSPs. We thereby show empirical evidence of its convergence in Figures 5.1–5.2. For each computer drawing of the OSB, we provide smaller images at the bottom with the (logarithmically-scaled) errors d_k, which tend to decrease at a steep pace, making the algorithm converge ($d_k < 10^{-3}$) after few iterations.

We perform all boundary computations by relying on the SDE representation of the kernel K defined at (5.58), (5.60), and (5.61), since we adopted the viewpoint of a GMB derived from conditioning a time-dependent OU process to degenerate at the horizon. The relation between the "parent" OU process and the resulting OUB is neatly stated in Buonocore et al. (2013, Section 3), although we include here a modified version that fits our notation better. That is, if $\widetilde{X} = \{\widetilde{X}_t\}_{t\in[0,T]}$ solves the SDE

$$\mathrm{d}\widetilde{X}_t = \widetilde{\theta}(t)(\widetilde{\kappa}(t) - \widetilde{X}_t)\,\mathrm{d}t + \widetilde{\nu}(t)\,\mathrm{d}B_t, \quad t \in [0, T], \tag{5.65}$$

then, the corresponding GMB is an OUB that solves the SDE

$$\mathrm{d}X_t = \theta(t)(\kappa(t) - X_t)\,\mathrm{d}t + \nu(t)\,\mathrm{d}B_t, \quad t \in (0, T), \tag{5.66}$$

with

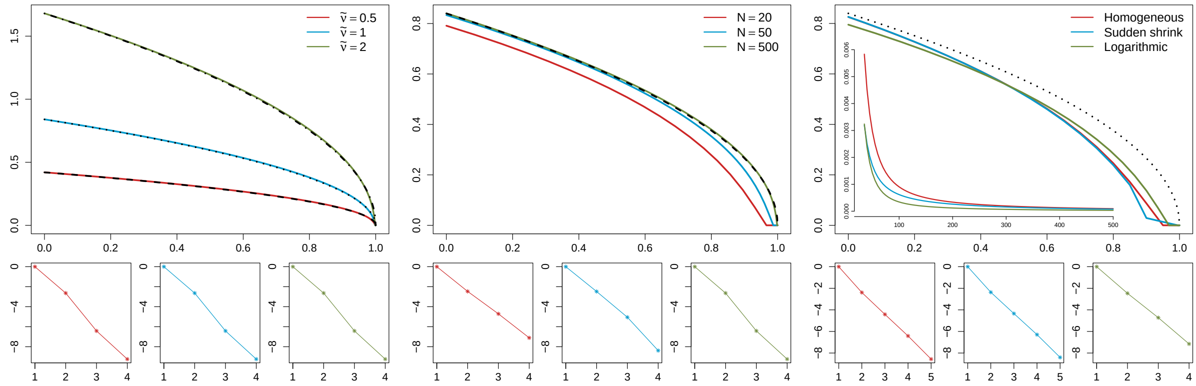
$$\begin{cases} \theta(t) = \widetilde{\theta}(t) + \dfrac{\widetilde{\nu}^2(t)}{\widetilde{\varphi}^2(t)\int_t^T \widetilde{\nu}^2(u)/\widetilde{\varphi}(u)\,\mathrm{d}u}, \\ \kappa(t) = \widetilde{\kappa}(t) + \dfrac{\widetilde{\nu}^2(t)}{\theta(t)}\dfrac{x - \widetilde{\varphi}(T)\int_t^T \widetilde{\kappa}(u)\widetilde{\theta}(u)/\widetilde{\varphi}(u)\,\mathrm{d}u}{\widetilde{\varphi}(t)\widetilde{\varphi}(T)\int_t^T \widetilde{\nu}^2(u)/\widetilde{\varphi}(u)\,\mathrm{d}u}, \\ \nu(t) = \widetilde{\nu}(t), \end{cases} \tag{5.67}$$

and where $\widetilde{\varphi}(t) = \exp\{-\int_0^t \widetilde{\theta}(u)\,\mathrm{d}u\}$. We choose representations (5.65) and (5.66) for GM processes and GMBs, over those given in Lemma 5.1 and (*iii*) from Proposition 5.1, as they have a more intuitive meaning. Indeed, recall that θ ($\widetilde{\theta}$) indicates the strength with which the underlying process is pulled towards the mean-reverting level κ ($\widetilde{\kappa}$), while ν ($\widetilde{\nu}$) regulates the intensity of the white-noise.

Figure 5.1(a) empirically validates the Picard algorithm's accuracy, as it is tested against the OSB of a BB, whose closed-form solution is given by $z + K\sigma\sqrt{T-t}$, for $K \approx 0.8399$. This result was originally due to Shepp (1969). Notice in Figure 5.1(b) how the numerical boundary approaches the real one as the time partition becomes thinner.

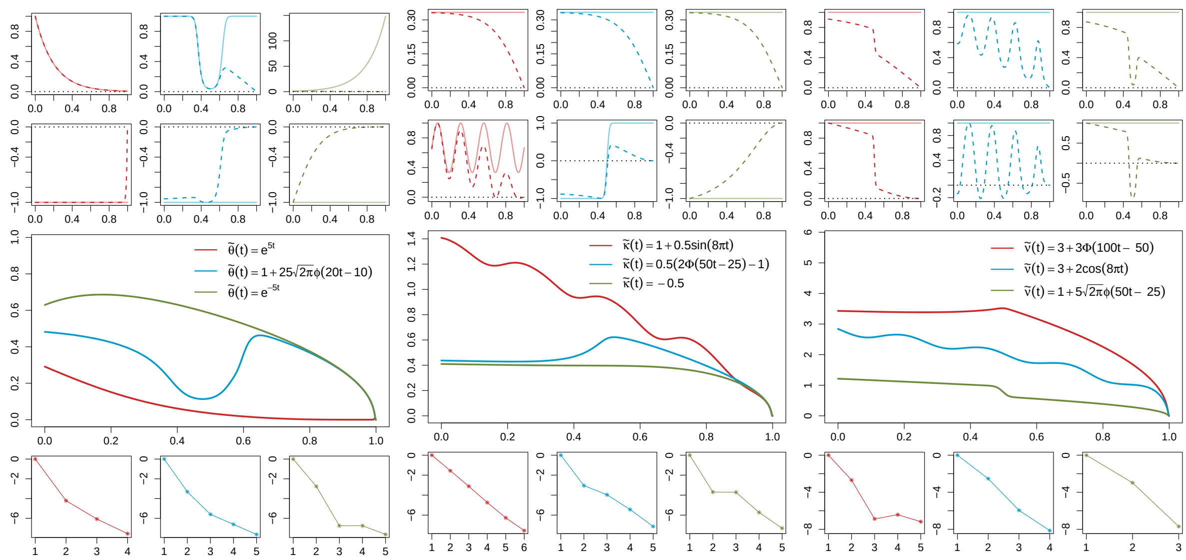
For all boundary computations, $T = 1$ and $N = 500$ were set unless otherwise stated. We used the logarithmically-spaced partition $t_i = \ln(1 + i(e-1)/N)$, since numerical tests suggested that the best performance is achieved when using a non-uniform mesh whose distances $t_i - t_{i-1}$ smoothly decrease. Figure 5.1(c) illustrates such an effect of the mesh increments by comparing the performance of the logarithmically-spaced partition against an equally-spaced one and another that is also equally spaced until the second last node, where it suddenly shrinks the distance to a fourth of the regular space. Note how the first partition significantly outperforms the other two with a lower overall L_2-error due to its better accuracy near the horizon. Intuition might dictate that introducing the sudden shrink at the horizon may derive in a better performance by diminishing the error that arises when considering (5.64), yet Figure 5.1(c) indicates otherwise.

Figure 5.2 shows how changing the coefficients of the process affects the OSB shape. In the first two rows of images, we visually represent the transformation of coefficients (5.67). The volatility is excluded as it remains the same after "bridging" the OU process. To compare the slopes we rely on $1/\widetilde{\theta}(t)$ and $1/\theta(t)$, as $\theta(t) \to \infty$ as $t \to \infty$ (see (*iv*) in Proposition 5.1) and, thus, its explosion would have obscured the shape of the bounded function $\widetilde{\theta}$, had they been plotted in the same graph. In alignment with the meaning behind each time-dependent coefficient, the OSB is pulled towards $\widetilde{\kappa}$ with a strength directly proportional to $\widetilde{\theta}$. This pulling force conflicts with the much stronger one towards the pinning point of the bridge process, resulting in an attraction toward the "bridged" mean-reverting level κ with strength dictated by θ. We recall that modifying $\widetilde{\nu}$, and thus ν, is equivalent to change θ due to (*iv.2*). We remind that the functions Φ and ϕ in Figure 5.2 stand for the distribution and the density of a standard normal random variable. The former is used to smoothly represent sudden changes of regimes, while the latter introduces smooth temporal anomalies. For instance, $\widetilde{\kappa}(t) = 2\Phi(50t - 25) - 1$ rapidly changes the mean-reverting level of the underlying process from -1 to 1 around $t = 0.5$, and $\widetilde{\nu}(t) = 1 + \sqrt{2\pi}\phi(100t - 25)$ introduces a brief period of increased volatility around $t = 0.25$, before and after which the volatility remains at (constant) baseline levels. Periodic fluctuations of the parameters were also considered, as they typically arise in problems that account for seasonality.



(a) $\widetilde{\theta} \equiv 0$. (b) $\widetilde{\theta} \equiv 0$, $\widetilde{\nu} \equiv 1$. (c) $\widetilde{\theta} \equiv 0$, $\widetilde{\nu} \equiv 1$, $N = 20$.

Figure 5.1: For the images on top, the solid colored lines represent the computed OSBs for the different choices of the volatility coefficient $\widetilde{\nu}$ (image (a)), the partition length N (image (b)), and the type of partition considered (image (c)). Black dashed, dotted, and dashed-dotted lines stand for the OSB of a BB associated with the different values of $\widetilde{\nu}$. Specifications are shown in the legend and caption of each image. Image (c) accounts for a subplot that shows, as a function of the partition size N (x axis), the evolution of the relative L_2 error between the different computed boundaries and the true one (y axis). The smaller images below display the log-errors $\log_{10}(d_k)$ between consecutive boundaries for each iteration $k = 1, 2, \dots$ of the Picard algorithm.



(a) $\widetilde{\kappa} \equiv -1$, $\widetilde{\nu} \equiv 1$. (b) $\widetilde{\theta} \equiv 3$, $\widetilde{\nu} \equiv 1$. (c) $\widetilde{\theta} \equiv 1$, $\widetilde{\kappa} \equiv 1$.

Figure 5.2: The first row of three plots shows $1/\widetilde{\theta}$ (continuous line) versus $1/\theta$ (dashed line) for the different choices of the slope $\widetilde{\theta}$ (image (a)), the mean-reverting level $\widetilde{\kappa}$ (image (b)), and the volatility $\widetilde{\nu}$ (image (c)) functions. Specifications of the functions are given in the legend and caption of each image. The second row does the same for $\widetilde{\kappa}$ and κ. The main plot, in the third row, shows in solid colored lines the computed OSBs. The smaller images at the bottom display the log-errors $\log_{10}(d_k)$ between consecutive boundaries for each iteration $k = 1, 2, \dots$ of the Picard algorithm.

Notice that, after Proposition 5.1, it readily follows that all coefficients θ, κ, and ν used in this section meet assumptions (5.21a)–(5.21f), as they are twice continuous derivable, $\theta(t) > 0$ for all $t \in [0, T)$, and satisfy conditions (*iv.1*) and (*iv.2*).

The R code in the public GitHub repository `https://github.com/aguazz/OSP_GMB` implements the Picard iteration algorithm (5.63)–(5.64). The repository allows for full replicability of the above numerical examples.

5.7 Concluding remarks

We solved the finite-horizon OSP of a GMB by proving that its OSB uniquely solves the Volterra-type integral equation (5.62).

GMBs were comprehensively studied in Section 5.2, where four equivalent definitions were presented, making it easier to identify, create, and understand them from different perspectives. One of these representations allows bypassing the challenge of working with diffusions with non-bounded drifts and, instead, working with an equivalent infinite-horizon OSP with a BM underneath. Equations (5.53) explicitly relate both OSTs and OSBs, while (5.57) and (5.62) give the value formula and free-boundary equation in the original OSP.

The method of solving the alternative OSP consisted in solving the associated free-boundary problem. To do so, several regularity properties about the value function and the OSB were obtained in Section 5.4, among which the local Lipschitz continuity of the OSB stands out as a remarkable property.

Finally, we approached the free-boundary equation as a fixed-point problem in Section 5.6 to numerically explore the geometry of the OSB. This provided insights about its shape for different sets of coefficients of the underlying GMB, seen as bridges derived from conditioning a time-dependent OU process to hit a pinning point at the horizon. The OSB shows an attraction toward the mean-reverting level, which fades away as time approaches the horizon, where the boundary hits the OUB's pinning point.

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Chapter 6

Final thoughts and extensions

6.1 Conclusions

In this thesis we contribute to the optimal stopping theory literature, in the time-inhomogeneous framework, by solving Optimal Stopping Problems (OSPs) related to Gauss–Markov (GM) processes, both when they are non-degenerated, and when they are pinned to a deterministic value at a terminal time. For pinned processes, we bypassed the challenge of their explosive drifts by equating them to a time-space-transformed Brownian Motion (BM). For each OSP, we characterized the free-boundary equation as the unique solution of a type-two Volterra integral equation. The value functions were, then, expressed as an integral of the OSBs.

We used a solution methodology in the spirit of Peskir (2005). That is, a direct, probabilistic approach that harvests sufficient smoothness of the value function and the Optimal Stopping Boundary (OSB) to solve the associated free-boundary problem by using an extended Itô's lemma. In doing so, we proved the Lipschitz continuity of the OSB away from the horizon. This result extends the technique in De Angelis and Stabile (2019) and blueprints a methodology to obtain similar smoothness on other OSPs. Another highly customizable technique was the one we employed to obtain the OSB's boundedness. By comparing the non-degenerated GM process and the Gauss Markov Bridge (GMB) with a BM and a Brownian Bridge (BB), respectively, we found bounds for the OSBs of the former two processes from those of the latter two.

Two different fixed-point algorithms were presented and implemented to solve the free-boundary equation. One based on backward induction (see Section 3.4) and one based on the Picard iteration method (see Sections 2.5, 4.6, and 5.6). With the aid of these algorithms, we illustrated the geometry of the OSB for different forms of the processes' drift and volatility (see Figures 2.1, 3.1, 4.1–4.3, and 5.2).

It is worth mentioning the statistical inference study we perform on the OSB in the BB case (see Section 3.4), as this is not a typical subject addressed in optimal stopping theory, and it is potentially extensible to tackle more general settings where likelihood theory is worked out. Indeed, the methodology consists in using the asymptotic normality of the BB volatility's maximum-likelihood estimate to extend, by using the delta method, such property to the OSB plugin estimator. This allowed us to provide (point-wise) confidence curves for the OSB.

We also offer a financial perspective of our work in Chapters 2 and 3, by linking the OSPs to the problem of optimally exercising American options. Remarkably, in Section 3.5, we show the competitiveness of the BB model against the geometric BM in this regard, when the option is written on IBM's and Apple's stocks, and in the presence of the pinning-at-the-strike effect. In addition, the confidence curves computed in Section 3.4 provide traders with a mechanism to introduce a risk-preference element.

6.2 Future work

We conclude this thesis by discussing some specific extensions that are worthwhile to mention as future research paths.

All of the OSPs we addressed live in the realm of non-monotonic OSBs, which makes it complicated to prove the smooth-fit condition. Our workaround to solve this issue was to prove the local Lipschitz continuity of the OSBs and then rely on the law of the iterated logarithm to obtain their probabilistic regularity (see Corollary 6 from De Angelis and Peskir (2020)) for the interior of the stopping set. Lipschitz continuity of the OSB, however, is a demanding property and might become difficult to obtain under certain settings. An alternative way to prove the smooth-fit condition is based on the continuity and piecewise monotonicity of the OSB (see Example 7 from De Angelis and Peskir (2020) and Corollary 8 from Cox and Peskir (2015)). However, all our attempts to try to obtain the boundary's piecewise monotonicity failed. We consider that further investigations into this smoothing property are worthwhile, as they might result in extending some of the results hereby presented to a wider class of processes, such as general diffusion bridges.

The likelihood-based inferential methodology used in Section 3.4 falls apart when the estimators of the process' parameters cannot be readily obtained and proved to be asymptotically normal. This disables the application of the delta method and increases the difficulty of obtaining confidence curves. Non-readily usable likelihood inference is ubiquitous in diffusive models, with the notable exceptions of the OU and other Gaussian models. For these cases, a similar approach to Section 3.4, with added technical complexities on the estimators and the asymptotic variances, could be thus achieved. More flexible inferential approaches, like bootstrap resampling, could be explored to address the non-tractable likelihood inference case, especially to tackle time-dependent GM processes and GMBs in a non-parametric fashion. That approach would imply performing multiple computations of the (numerically-computed) OSB, which considerably increases the associated computational load.

In line with the previous point, the advance of statistical inference for the OSB of the time-dependent OU, OUB, and GMB processes (Chapters 2, 4, and 5) would allow their empirical study in real-data applications, similar to that in Section 3.5 for the OSB of the BB. In particular, such real data applications would help to illustrate the practical usability of these OSBs, their flexibilities, their different stopping strategies once uncertainty is incorporated, and their potential benefits against, for example, the OSB of the classical geometric BM or other standard models.

Due to the aforementioned computational cost, and as a stand-alone problem with value in itself, efficient methods to compute the OSB are required. We saw the effect of considering a partition that gradually gets thinner as time approaches the horizon, which drove us to work with a logarithmically-spaced partition. However, other settings could yield more accurate boundary approximations, and a theoretical and numerical study on optimal partition settings could be carried out. Likewise, the fixed-point algorithms used to solve the Volterra integral equation characterizing the OSB are the state-of-the-art method to address this numerical problem, yet their convergence remains an open problem in general frameworks. It would be worthwhile to treat this problem through the lens of the Banach fixed-point theorem by proving that the integral operator is a contraction mapping.

From a financial point of view, Figures 3.5 and 3.6 reveal the importance of including the variance, and not only the mean, in the analysis of optimally exercising an American option, as strategies similar to the optimal one could be found in average profit terms, but with a significantly lower variance, meaning less risk. To this end, risk-adjusted measures could be used as the gain function, including the Sharpe and Sortino ratios and the risk-adjusted return on capital. The quadratic non-linearity of the variance makes Mean-variance OSPs fundamentally more challenging than classical ones. These types of problems have been studied in Pedersen and Peskir (2016) and Pedersen and Peskir (2017). The first one works out the solution for a geometric BM, while the latter does it for a wealth process that results from holding a risky stock (whose price follows a geometric BM) and a riskless bond with exponential growth.

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