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Dynamic Stability of Cooperative Investment under Uncertainty*

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Abstract

This article models the inherent cooperative and non-cooperative incentives of stakeholders in investment projects in a novel way by combining concepts from cooperative game theory and real options theory. As stakeholders have outside options, in the sense that they may terminate negotiations with the current coalition and join another, we introduce and analyze a coalitional and dynamic stability concept. We show that investment projects, in which cooperation between stakeholders is necessary, are more prone to coalitional instability when there are insufficient synergies between the stakeholders. We characterize the proportional investment scheme as the investment scheme that maximizes the total project value and that results in the earliest investment timing. A failure to implement proportional investing leads to the formation of a smaller, less efficient, coalition. The vulnerability to fail is exacerbated in a market that is characterized by high profit growth and low profit uncertainty, or vice versa. Finally, we explicitly consider one-leader investment projects and characterize the prioritized investment scheme that maximizes the value of the leader. We show that the same market conditions govern the stability of the prioritized investment scheme, which contributes to the robustness of our results.

Keywords: cooperative investment projects, synergies between stakeholders, investment schemes, dynamic stability.

JEL Classification Number: C70, G11, L24.

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1 Introduction

Cooperation between governments, companies, non-profit organizations, or private individuals is often essential for the realization of investment projects. The construction of an airport, a railroad station, a bridge, a dam, or a sports stadium, typically involves cooperation between several stakeholders. The premise is that cooperation creates *synergy*, which usually manifests itself in increased efficiency by working together as opposed to working alone. However, poor communication between stakeholders is a common denominator for many failed infrastructure projects, such as: *Feyenoord City* in the Netherlands which was canceled in 2022;[1] the *Eurostadium* in Belgium which was canceled in 2020;[2] several EU-backed gas pipelines connecting European Union member states, for example, one connecting Portugal and Spain which was canceled in 2020;[3] Berlin's new airport in Germany, which, after missing seven opening dates, opened in 2020, nearly a decade behind schedule, marking an ignominious failure for Germany.[4]

As failed infrastructure projects have dire financial consequences and cause reputational harm, we aim to explain the driving forces behind cooperative investment failures from a theoretical perspective. A failed infrastructure project can manifest itself in various ways. An infrastructure project evidently fails when it is canceled completely. Although also when the project will be realized in a scaled-down form, as is the case with Feyenoord City,[5] or when it is significantly delayed, as is the case with Berlin's airport, we speak of a failure.

Many real life investment situations are *dynamic* and exhibit both *cooperative* and *non-cooperative* characteristics. In practice, stakeholders have to form *coalitions* in which they negotiate about the terms of the partnership, which are then formalized in a legal document. By allowing for the formation of coalitions, each stakeholder has outside options because it may, at any time, terminate negotiations with the current coalition and join another. If such a threat is credible, then it causes instability of the negotiation process. Stakeholders also have non-cooperative incentives in the sense that they choose their investment strategy as to maximize their own profit resulting from the project. Furthermore, negotiations between stakeholders, and their investment decisions are dynamic processes, which are influenced by market uncertainty. By combining concepts from cooperative game theory and real options theory, we are able to capture these key elements in a fairly general way.

A real options perspective lends itself for cooperative investment projects (see, e.g., Kogut (1991)). We employ standard real options theory (cf. Dixit and Pindyck (1994)) to value the project for each stakeholder in each coalition that it is a member of. This allows us to put the model in a dynamic stochastic framework in which we can determine the optimal investment timing for each stakeholder for any given investment cost and for any coalition

[1]Raad van State haalt streep door bestemmingsplan Feyenoord City. (2022, October 26). *Nederlandse Omroep Stichting*. Retrieved from `https://nos.nl/` (article in Dutch).

[2]Definitief einde voor Eurostadion: Ghelamco vangt bot bij Raad van State. (2020, October 15). *BRUZZ*. Retrieved from `https://bruzz.be/` (article in Dutch).

[3]EU companies burn fossil gas and taxpayer cash. (2021, February 22). *Global Witness*. Retrieved from `https://www.globalwitness.org/en`.

[4]Berlin's new airport: A story of failure and embarrassment. (2020, October 31). *Deutsche Welle*. Retrieved from `https://www.dw.com/en`.

[5]Gemeente Rotterdam. (2023, March 29). *Nieuwe ruimtelijke visie Feyenoord City 2.0* [Press release]. Retrieved from `https://persberichtenrotterdam.nl/` (press release in Dutch).

that it is a member of. Upon realization of the project by a coalition, each stakeholder in this coalition receives a stakeholder-specific profit stream. Synergies are thus modeled by allowing for coalition-specific profit streams. In this way, a stakeholder joining a coalition can positively affect the profitability of the individual stakeholders in that coalition.

As noted by Azevedo and Paxson (2014), the vast majority of real option game models are non-cooperative in nature, whereas only a very few consider cooperation. In fact, also the vast majority of subsequent real option game models are non-cooperative in nature (e.g., Huisman and Kort (2015), Hellmann and Thijssen (2018), and Sunar, Yu, and Kulkarni (2021)). Typically, when cooperation between stakeholders is allowed, the optimal investment decisions are chosen with respect to the sum of their values. For example, Weeds (2002) compares the optimal R&D investment timing decisions of two competing firms to an optimal cooperative benchmark in which the firms essentially merge and plan their investments cooperatively. This cooperative benchmark is determined under the assumption that side payments may be used to ensure that neither firm has an incentive to deviate. Weeds (2002) finds that in the cooperative optimum the firms invest sequentially. Sequential investments by the stakeholders is not allowed in our model, because in our model we restrict stakeholders to invest at the same investment threshold — after all, a given investment project can only take place if all stakeholders are willing to invest. Weeds (2002) additionally analyzes investment behavior under the restriction of simultaneous investment and, like us, finds that the individual unrestricted stakeholder invests earlier compared to the case in which it is restricted to cooperate and invest at the same trigger point. More specifically, in our model the investment threshold of a coalition is determined by the largest optimal investment threshold of all stakeholders in that coalition.

We focus our attention on the coalition with the largest total project profit; this coalition is called the *grand coalition*. Stakeholders within the grand coalition negotiate about their individual investments and if they come to an agreement, their investment proposal is implemented and the project is realized. In general, *investment proposals* are proposals put forward by coalitions that contain the individual contribution of each stakeholder to the cost of the project. Nevertheless, an investment proposal by the grand coalition need not always be *stable* against coalitional deviations. We call an investment proposal by the grand coalition stable if there does not exist an investment proposal by a smaller coalition such that all stakeholders within that coalition have a strict incentive to deviate to that coalition. In other words, stability means that, for all investment proposals by all other coalitions, there exists at least one stakeholder that disagrees with that proposal and rather stays in the grand coalition. The notion of stability we introduce closely relates to the solution concepts of the *core* and the *bargaining set* in cooperative game theory. The core contains all feasible outcomes that no stakeholder or group of stakeholders can improve upon by acting for themselves (cf. Kannai (1992)). The bargaining set contains all feasible outcomes to which no stakeholder can rightfully object because each objection is met with a counter-objection (cf. Maschler (1992)). Because negotiations are dynamic in nature, we distinguish between stability at a specific moment in time and stability at all points in time, which we term *dynamic stability*. For instance, we show that, although an investment proposal is stable at one point in time, it need not be stable at a later point in time. In this way, we differ from the static concepts of the core and the bargaining set. There nonetheless exist dynamic bargaining procedures for cooperative games (see, e.g., Maschler (1992), and Peleg and Sudhölter (2007)). Furthermore, there exist dynamic concepts of the core, for instance in dynamic cooperative games (see, e.g., Kranich, Perea ý Monsuwé, and Peters (2005), Haurie and Zaccour (2005), and Lehrer and Scarsini (2013)). Unlike these dynamic cooperative game models, our model does not rely on a characteristic function that assigns to each coalition one joint monetary value. Instead, as stakeholders have non-cooperative incentives as well, we work with stakeholder-specific values in each coalition.

We find that the cooperative solution ensues if stakeholders implement the *proportional investment scheme*, which entails that the project cost is shared proportionally with respect to the coalition-specific profits of the individual stakeholders. The proportional investment scheme is the collection of all coalition-specific *proportional investment proposals*. For example, if a stakeholder receives 50% of the total profits of the coalition it is part of, then this stakeholder pays 50% of the project cost. We call it the cooperative solution because it is the investment for which the sum of the individual project values of the stakeholders is maximized at each point in time. Moreover, a proportional investment by the stakeholders balances their individual optimal investment thresholds such that each of them wants to invest at the same moment in time, which in turn implies that investment for this project takes place at the earliest moment in time. The proportional investment scheme is also desirable from a practical perspective because its implementation relies only on the ratios of the individual project profits to the total project profits. Precise knowledge of the market characteristics, such as the growth rate of the profits, the profit uncertainty, and the discount rate, is therefore not necessary.

In this article, we particularly focus on the stability of the proportional investment scheme because the implementation of a proportional investment by the grand coalition maximizes the sum of the stakeholders' value and leads to investment at the earliest moment in time.[6] Instability of a proportional investment, and thus a failure of the grand coalition to implement it, means that a smaller, less efficient, coalition may form in which the sum of the stakeholders' value is smaller and where investment takes place later. We show that a proportional investment by the grand coalition is dynamically stable if all stakeholders experience synergy with respect to the grand coalition. A stakeholder is said to experience synergy with respect to the grand coalition if its project profit is not strictly larger in any other coalition that this stakeholder can join. However, if some stakeholders experience no synergy with respect to the grand coalition, then stability is contingent on the market characteristics.

We use the *synergy effect* and the *timing effect* to analyze the stability of a proportional investment by the grand coalition. The synergy effect represents the gain, or loss, in net present value from investing now at the threshold of the grand coalition, where a positive synergy effect means that a player gains from investing now. The timing effect represents the impatience of a player with investment, where a positive timing effect means that a player is impatient with investment and prefers to invest now. The proportional investment proposal of the grand coalition is stable if, for each other coalition, there exists a stakeholder for which the gain in value from undertaking the investment now at the threshold of the

[6] For ease of exposition, we do not always make an explicit distinction between stability and dynamic stability in the introduction, even though we consider both forms of stability separately in our analysis.

grand coalition is larger than the *value of waiting* for that other coalition.

We show that, irrespective of the synergies that stakeholders experience, the proportional investment proposal of the grand coalition is guaranteed to be stable prior to and at the moment of investment if the market is characterized by low project profit uncertainty and low growth rate of the project profits. In such a situation, the expected discounted net profits are low in all coalitions, which weakens the synergy effect. At the same time, there is an additional component that makes deviation to another coalition less attractive. Because the investment threshold under a proportional investment by the grand coalition is the smallest one possible, any other coalition can only act on its deviation after the opportunity to invest in the grand coalition has passed. A low volatility of the profits coupled with a low growth rate means that profits accumulate relatively slowly which reduces the value of the option to invest in another coalition even further.

Furthermore, stability prior to and at the moment of investment is also guaranteed if the market is volatile, the growth rate of the profits is high, and, for each other coalition, at least one stakeholder has a higher level of profitability in the grand coalition. Contrary to the previous case, the effect of synergies is stronger because the expected discounted net profits are high. Moreover, project profits accumulate faster which weakens the timing effect. Here, the synergy effect dominates the timing effect and because, for each coalition, at least one stakeholder has a larger project profit in the grand coalition, this stakeholder will block any deviation attempts. Nevertheless, the other side of the coin is that the grand coalition is particularly unstable if there exists a coalition in which no stakeholder has a larger project profit in the grand coalition, the market is volatile, and the growth rate of the profits is high.

Our stability analysis also shows that incentives to deviate are stronger in a promising market characterized by a high expected profit growth and a low level of profit uncertainty. Kogut (1989) obtains the same result, albeit from an empirical analysis on the stability of joint ventures. Even though a growing market makes the grand coalition more attractive, a growing market also implies that outside options in which stakeholders have a larger project profit become more attractive, which may lead to conflict between the stakeholders in the grand coalition. Kogut (1989) consequently posits that the incentive to cooperate becomes weaker once market uncertainties are resolved. Indeed, our theoretical model confirms this conjecture, which demonstrates the benefit of including market uncertainty in our analysis. Conversely, our stability analysis shows that incentives to deviate are also stronger in an unpromising market characterized by a low expected profit growth and a high level of profit uncertainty. In an unpromising market, stakeholders prefer to delay their investment such that, when there is a lack of synergies or when synergies are relatively small, outside options become more attractive. In a press conference, the CEO of the Dutch football club Feyenoord implied that they were facing such a market, which is why they decided to pull the plug on the construction of the new football stadium.[7]

Instability of a proportional investment by the grand coalition can nonetheless be overcome by side payments in which stakeholders propose an investment that is more advantageous to stakeholders that have an incentive to deviate. The caveat is that side payments lead to the implementation of a less efficient investment proposal in the sense that the in-

[7]Feyenoord gaat niet door met bouw nieuw stadion. (2022, April 21). *Nederlandse Omroep Stichting.* Retrieved from `https://nos.nl/` (article in Dutch).

vestment is delayed and that the sum of the stakeholders' value is not maximized.

In practice, there is often one *leader* in an investment project which means that only coalitions that the leader is part of can undertake the project. One such example is Feyenoord City in which the original plan could not be continued with a smaller coalition when the football club Feyenoord withdrew from the project. Arguably, the leader has more negotiation power in these situations and can therefore propose to be prioritized by investing less than proportionally whereas the remaining players invest more than proportionally. In addition, we therefore introduce another type of investment scheme, namely one in which exactly one stakeholder is prioritized in the sense that the investment is chosen as to maximize its value prior to and at the moment of investment. However, *one-leader cooperative investment projects* may also lead to inefficiencies as the implementation of a prioritized investment entails that the sum of the players' value is not maximized and that the investment takes place later. It is in the leader's best interest to invest as early as possible if the value of waiting with investment is low. Because investment takes place at the earliest moment in time under a proportional investment, the prioritized investment gets closer to the proportional one as the value of waiting becomes smaller. In fact, the two investment schemes meet in the limit when the value of waiting vanishes, which happens when the market uncertainty tends to zero and when the growth rate of the project profits tends to minus infinity. We show that conditions for stability of a prioritized investment are alike those for the proportional investment, which contributes to the robustness of our results.

Our results stress the importance of *establishing* synergies between stakeholders. This starts with bringing stakeholders together that want to work together and by ensuring that their incentives align. Here lies a role for the (local) government as it is usually one of the stakeholders in large infrastructure projects. As a matter of fact, in doing so, one should not overlook intangible forms of synergies such as those arising from corporate cultures. Indeed, Weber and Camerer (2003) shows that the influence of conflicting corporate cultures on the failure of mergers is underestimated.

A large strand of literature studies the instability and failure of joint ventures from an empirical perspective. Our focus is not on joint ventures *per se*, because we allow stakeholders to act as separate entities within the cooperative contract, instead of forcing them to pool their resources and form a common legal entity. Despite this difference, some results of these empirical studies align with ours. A key finding of Park and Russo (1996) is that partnerships between competitors are significantly more likely to fail. A lack of synergies between partners is apparent when they are competitors outside of the agreement, which is why cooperation between competing stakeholders is also more to prone to failure in our model. Competition between stakeholders need not be the only source of conflict. Weber and Camerer (2003) shows that mergers are more likely to fail when corporate cultures of merging firms are different.

Surprisingly, despite the importance of synergies between stakeholders and the stochastic market environment, there is limited research that studies their effect on cooperative investment projects from the perspective of cooperative game theory and real options theory. We find that combining these two fields of research yields a tractable *model* that allows us to pinpoint the factors that contribute to the failure of infrastructure projects in which cooperation between stakeholders in the grand coalition is necessary. In doing so, we also provide a *theoretical* substantiation of results found in empirical studies on instability and failure of cooperative investment projects, for instance those by Kogut (1989), Park and Russo (1996), and Weber and Camerer (2003).

The article is organized as follows. Section 2 introduces our cooperative investment model and presents the concept of (dynamic) stability. Section 3 introduces the proportional investment scheme, derives its properties, and analyzes its stability. Section 4 provides a similar analysis but for the prioritized investment scheme in one-leader cooperative investment projects. Section 5 concludes, and includes ideas for future research. All proofs are presented in the appendix.

2 Cooperative investment projects

Let N be a finite set of profit-maximizing stakeholders, which will henceforth be referred to as a set of *players*. A *coalition* of players is a subset of N and is denoted by S. The collection of all subsets of N is denoted by 2^N. The players can form coalitions to undertake an investment project, the cost of which is fixed and is equal to $C > 0$.

For all $S \in 2^N$, the instantaneous *project profit* of player $i \in S$ at time t is given by

$$\pi_i^S(t) = D_i^S X(t),$$

in which $X(t)$ is the economy-wide stochastic component of the profit that follows a geometric Brownian motion with drift, given by

$$dX(t) = \mu X(t)dt + \sigma X(t)dz(t), \tag{2.1}$$

in which $\mu \in \mathbb{R}$ is the growth rate parameter, $\sigma > 0$ is the variance parameter, and $dz(t)$ is the increment of a Wiener process. The starting value of the geometric Brownian motion, $X(0)$, is denoted by X and is strictly positive.

Players are risk neutral and discount project profits against positive rate r for which we assume that $r > \mu$. If this requirement is not satisfied, players delay their investment indefinitely and are thus never willing to undertake the investment (see, e.g., Dixit and Pindyck (1994)).

The non-negative vectors $D^S = (D_i^S)_{i \in S}$ for $S \in 2^N$ are given. From the outset each player does not know the project profits of the other players. Moreover, in general, for each $i \in N$, we impose no restrictions on the relationship between D_i^S and D_i^T for coalitions $S \neq T$ with $S \ni i$ and $T \ni i$.

A *cooperative investment project* thus consists of a finite set of profit-maximizing players in which these players can form coalitions to undertake an investment project against a given cost. Upon realization of the project by a coalition, each player receives a coalition-specific profit stream that is governed by a dynamic stochastic process, which is a geometric Brownian motion.

Players have an incentive to cooperate if their cooperation creates *synergy*. As is formalized in the following definition, a player experiences synergy with respect to a coalition if its project profit is not strictly larger in any smaller coalition that this player is a member of.

Definition 2.1. Let $S \in 2^N$. A player $i \in S$ *experiences synergy with respect to* S if, for all $T \subset S$ with $T \ni i$, it holds that $D_i^S \geq D_i^T$.[8]

However, the fact that one player experiences synergy with respect to a coalition does not necessarily imply that another player experiences synergy with respect to that coalition.

If coalition $S \in 2^N$ undertakes the project at X, then the expected discounted project profit of player $i \in S$ at X is equal to

$$R_i^S(X) = \mathbb{E}\left[\int_{t=0}^{\infty} D_i^S X(t) e^{-rt} dt \,\middle|\, X(0) = X\right] = \frac{D_i^S X}{r - \mu}. \tag{2.2}$$

We assume that only one coalition can undertake the project. For that reason we focus our attention on the coalition with the largest total expected discounted project profit, which we assume to be the *grand coalition* N, and investigate under which circumstances it is able to realize the project. Formally, we assume that

$$\sum_{i \in N} R_i^N(X) > \sum_{i \in S} R_i^S(X) \tag{2.3}$$

for all $S \subset N$ and all X. The assumption (2.3) is equivalent to $\sum_{i \in N} D_i^N > \sum_{i \in S} D_i^S$ for all $S \subset N$.

The players in coalition N need to reach a consensus on the individual contributions to the project, because a failure to do so results in a failure to realize the project which implies that a smaller, possibly less efficient, coalition forms. We assume that the investment is irreversible and takes place only once, whereupon the project is realized and players receive project profits.[9] Any investment proposal by a coalition satisfies *investment boundedness*, *investment efficiency*, and a *zero investment property*, which are given in the following definition, respectively.

Definition 2.2. Let $S \in 2^N$. The vector $I^S = (I_i^S)_{i \in S}$, in which I_i^S denotes the investment of player $i \in S$, is an *investment proposal* if

(i) $0 \leq I_i^S \leq C$ for all $i \in S$,

(ii) $\sum_{i \in S} I_i^S = C$, and

(iii) $I_i^S = 0$ for all $i \in S$ with $D_i^S = 0$.

The set of investment proposals of S is denoted by $\mathcal{I}^S$.[10]

[8] The notation $T \subset S$ means that T is a proper subset of S, that is, for all $T \in 2^S$, $T \cap S = T$ and $T \neq S$. The notation $T \subseteq S$ allows $T = S$.

[9] In practice, there is usually a construction period before operations commence and stakeholders make profits. Here, we implicitly assume that the duration of the construction period is the same for each coalition, which implies that the assumption that profits are instantaneous upon realization can be made without loss of generality. Allowing coalition-specific construction periods is beyond the scope of this work, but makes for interesting future research.

[10] It follows from $C > 0$ that $\mathcal{I}^S = \emptyset$ if $\sum_{i \in S} D_i^S = 0$.

The first condition implies that each player in a coalition can neither make a negative investment nor invest more than the project cost C; the second condition states that the joint contribution of the players in S equals the cost of the project; the third condition implies that a player with no profitability in coalition S will not invest anything.

Our model allows for side payments through the choice of an investment proposal rather than also allowing players in a coalition to transfer their individual project profits among members of that coalition. For example, to induce agreement between players in a coalition, players with a relatively low project profit could contribute less, so that, by investment efficiency, other players, for example, those with a relatively large project profit, contribute more.

The investment choice of a player affects its optimal investment timing. We first consider the investment problem a player faces when it is not constrained by the investment decisions taken by the other players. Suppose that player $i \in S$ invests $I_i^S \geq 0$. Then, given I_i^S, the investment problem player $i \in S$ solves is given by

$$F_i^S(X, I_i^S) = \max_{\tau_i^S \geq 0} \mathbb{E}\left[\int_{t=\tau_i^S}^{\infty} D_i^S X(t) e^{-rt} dt - e^{-r\tau_i^S} I_i^S \middle| X(0) = X\right], \tag{2.4}$$

in which τ_i^S is the time player i undertakes the investment in the unconstrained setting. Let $\hat{X}_i^S(I_i^S)$ denote the optimal investment threshold of player $i \in S$ in the unconstrained setting if it invests I_i^S. More specifically, if $X < \hat{X}_i^S(I_i^S)$, player i does not invest; if $X \geq \hat{X}_i^S(I_i^S)$, player i invests. Correspondingly, the optimal investment timing (i.e., the solution to (2.4)) is equal to $\hat{\tau}_i^S = \inf\{t \,|\, X(t) \geq \hat{X}_i^S(I_i^S)\}$.[11] The following proposition provides the value and the investment threshold of player $i \in S$ if it invests I_i^S.

Proposition 2.1. *Let $S \in 2^N$. The value of player $i \in S$ if it invests $I_i^S \geq 0$ is equal to*

$$F_i^S(X, I_i^S) = \begin{cases} \left(\dfrac{X}{\hat{X}_i^S(I_i^S)}\right)^{\beta} \left(\dfrac{D_i^S \hat{X}_i^S(I_i^S)}{r-\mu} - I_i^S\right) & \text{if } X < \hat{X}_i^S(I_i^S), \\ \dfrac{D_i^S X}{r-\mu} - I_i^S & \text{if } X \geq \hat{X}_i^S(I_i^S), \end{cases} \tag{2.5}$$

in which the investment threshold, $\hat{X}_i^S(I_i^S)$, is given by

$$\hat{X}_i^S(I_i^S) = \frac{I_i^S}{D_i^S} \frac{\beta}{\beta - 1}(r - \mu), \tag{2.6}$$

and

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1. \tag{2.7}$$

[11] The investment problem is trivial if $D_i^S = 0$ as it implies that $I_i^S = 0$. In that case, the solution is equal to $\hat{\tau}_i^S = 0$ (i.e., $\hat{X}_i^S(I_i^S) = 0$).

The term $(X/\hat{X}_i^S(I_i^S))^\beta$ in the value function (2.5) is the stochastic discount factor[12] and the other term is equal to the expected discounted net profit of player $i \in S$, $R_i^S(X) - I_i^S$, evaluated at $X \geq \hat{X}_i^S(I_i^S)$ (see also (2.2)).

The parameter β given in (2.7) contains the parameters μ and σ that govern the geometric Brownian motion $X(t)$ (see (2.1)), and the discount rate r. It holds that $\frac{\partial \beta}{\partial \sigma} < 0$, $\frac{\partial \beta}{\partial \mu} < 0$, and $\frac{\partial \beta}{\partial r} > 0$ (see, e.g., Dixit and Pindyck (1994)). Furthermore, using the explicit expressions for these partial derivatives, one can show that

$$\frac{\partial \hat{X}_i^S(I_i^S)}{\partial \sigma} > 0, \frac{\partial \hat{X}_i^S(I_i^S)}{\partial \mu} < 0, \text{ and } \frac{\partial \hat{X}_i^S(I_i^S)}{\partial r} > 0. \tag{2.8}$$

Therefore, keeping all else constant, an increase in the uncertainty of the project profits raises the investment threshold; an increase in the expected growth of the project profits accelerates investment; an increase in the discount rate delays investment.

However, the project can only be undertaken if all players are willing to invest; therefore, the investment timing of coalition S is equal to $\tau_S^* = \max_{i \in S} \hat{\tau}_i^S$. Or, equivalently, the *investment threshold of coalition* S is given by

$$X_S^*(I^S) = \max_{i \in S} \hat{X}_i^S(I_i^S),$$

in which $I^S \in \mathcal{I}^S$ is the investment proposal of coalition S, and $\hat{X}_i^S(I_i^S)$ is given by (2.6). A change in the parameters σ, μ, and r shifts the optimal investment thresholds $\hat{X}_i^S(I_i^S)$ uniformly, because shocks to the project profit are economy-wide. Therefore, $X_S^*(I^S)$ is influenced in the same way as given in (2.8).

The following example illustrates the dynamics of the investment thresholds.

Example 2.1. Let $S = \{1, 2\}$, $D^S = (1, 2)$, and $C = 4$. Then, see (2.6),

$$\hat{X}_1^S(I_1^S) = \frac{I_1^S}{1} \frac{\beta}{\beta - 1}(r - \mu), \text{ and } \hat{X}_2^S(I_2^S) = \frac{I_2^S}{2} \frac{\beta}{\beta - 1}(r - \mu).$$

Suppose that both players invest the same amount, that is, $I^S = (2, 2)$. Then, $\hat{X}_2^S(2) < \hat{X}_1^S(2)$, meaning that player 2 prefers to start the project earlier, as is also shown in Figure 2.1. The project can only be initiated when both players are willing to do so, which is the case if $X \geq \hat{X}_1^S(2)$. Hence, it holds that $X_S^*(I^S) = \hat{X}_1^S(2)$. From Figure 2.1 it can also be deduced that the investment threshold $X_S^*(I^S)$ is minimal when $\hat{X}_1^S(I_1^S) = \hat{X}_2^S(I_2^S)$, which corresponds to $I^S = (1\frac{1}{3}, 2\frac{2}{3})$.

[12]The stochastic discount factor is given by $\mathbb{E}[e^{-rT}] = (X/\hat{X}_i^S(I_i^S))^\beta$, in which the stochastic variable T is the first time the geometric Brownian motion hits $\hat{X}_i^S(I_i^S)$ starting at $X(0) = X$ (see, e.g., pages 315-316 of Dixit and Pindyck (1994) for a derivation of the stochastic discount factor).

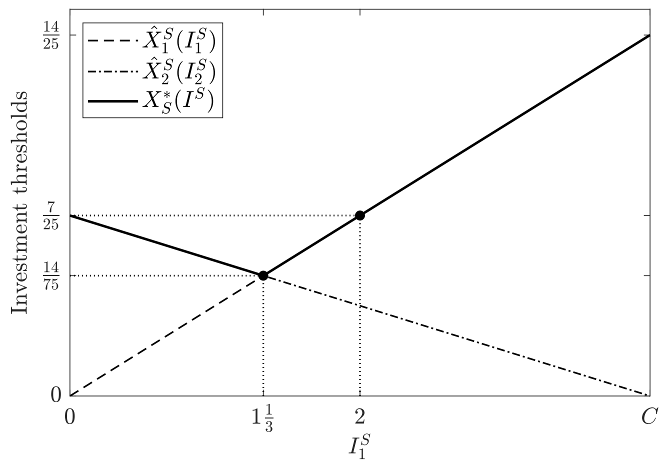


Figure 2.1: Optimal investment thresholds $\hat{X}_1^S(I_1^S)$ and $\hat{X}_2^S(C - I_1^S)$ for players 1 and 2, respectively, as a function of I_1^S. Parameter values are $\sigma = 0.20$, $\mu = 0.03$, and $r = 0.10$ such that $\beta = 2$.

△

For each coalition $S \in 2^N$, let $V_i^S(X, I^S)$ denote the value of player $i \in S$ at X with respect to investment proposal $I^S \in \mathcal{I}^S$. It directly follows from Proposition 2.1 that this value, which is either equal to the option value of investment or the expected discounted net profit, is given by

$$V_i^S(X, I^S) = \begin{cases} \left(\frac{X}{X_S^*(I^S)}\right)^\beta \left(\frac{D_i^S X_S^*(I^S)}{r-\mu} - I_i^S\right) & \text{if } X < X_S^*(I^S), \\ \frac{D_i^S X}{r-\mu} - I_i^S & \text{if } X \geq X_S^*(I^S), \end{cases} \tag{2.9}$$

in which

$$X_S^*(I^S) = \left(\max_{i \in S} \frac{I_i^S}{D_i^S}\right) \frac{\beta}{\beta - 1}(r - \mu).$$

If $X \leq X_S^*(I^S)$, the value of investment of player $i \in S$ can be decomposed as follows:

$$V_i^S(X, I^S) = \left(\frac{D_i^S X}{r-\mu} - I_i^S\right) + W_i(X, I^S),$$

in which the first term is the net present value of the investment at X and $W_i(X, I^S)$ is the *value of waiting* at X with respect to I^S, which is equal to

$$W_i(X, I^S) = \left(\frac{X}{X_S^*(I^S)}\right)^\beta \left(\frac{D_i^S X_S^*(I^S)}{r-\mu} - I_i^S\right) - \left(\frac{D_i^S X}{r-\mu} - I_i^S\right). \tag{2.10}$$

The value of waiting is the difference between the value obtained from investing at $X_S^*(I^S)$ and the value obtained from investing at X. A positive (negative) value of waiting implies that waiting with investment is worth more (less) than undertaking the investment now. A coalition S undertakes the investment at $X_S^*(I^S)$ for which it holds that $W_i(X_S^*(I^S), I^S) = 0$ for all $i \in S$, so the value of waiting for each player is zero.

The value of waiting before investment has taken place, is strictly positive for players for which their individual optimal investment threshold coincides with that of the coalition, which is in accordance with standard real options theory. On the other hand, if a player's individual optimal investment threshold is lower than that of the coalition, this player's value of waiting is strictly negative for the moments in time where it wants to invest but is unable to do so. The following proposition formalizes these two observations.

Proposition 2.2. *Let $S \in 2^N$, let $I^S \in \mathcal{I}^S$, and let $i \in S$. Then,*

(i) if $\hat{X}_i^S(I_i^S) = X_S^*(I^S)$, then $W_i(X, I^S) > 0$ for all $X < X_S^*(I^S)$;

(ii) if $\hat{X}_i^S(I_i^S) < X_S^*(I^S)$, then $W_i(X, I^S) < 0$ for all $\hat{X}_i^S(I_i^S) \leq X < X_S^*(I^S)$.

Players in a coalition $S \in 2^N$ negotiate about their individual investment. Given $I^S \in \mathcal{I}^S$, it is in the best interest of each player $i \in S$ to report its true $\hat{X}_i^S(I_i^S)$, from which other players can subsequently deduce D_i^S. If player $i \in S$ deviates from $\hat{X}_i^S(I_i^S)$ such that $X_S^*(I^S)$ changes, then this implies that investment takes place at a moment that is suboptimal for player $i \in S$.

We are interested under which conditions an investment agreed on by the grand coalition, I^N, is *stable* against coalitional deviations. Stability of the investment I^N at X entails that, for each investment proposal by a coalition $S \subset N$, there exists at least one player in S who disagrees with that proposal in the sense that this player has no strict incentive to deviate. In other words, a coalition S will only deviate from N at X if there exists an investment I^S such that all players in S are strictly better off when coalition S forms. Moreover, stability of I^N at X can be directly translated to stability at points in time, because $X(0) = X$ equals the level of the geometric Brownian motion at $t = 0$. Hence, stability of I^N at X implies that I^N is stable at all moments in time t for which $X(t) = X$. Without loss of generality, we formulate stability with respect to the current time, which corresponds to $X(0) = X$. However, from the outset it is not guaranteed that stability of I^N at X implies stability at all time points. For this reason we also consider a stronger form of stability, which we call *dynamic stability*.

Definition 2.3. An investment proposal $I^N \in \mathcal{I}^N$ is called *stable* at X if, for all $S \subset N$ and all $I^S \in \mathcal{I}^S$, there exists at least one player $i \in S$ such that $V_i^N(X, I^N) \geq V_i^S(X, I^S)$. An investment proposal $I^N \in \mathcal{I}^N$ is called *dynamically stable* if it is stable for all X.

One can also employ an equivalent min-min-max formulation of stability. The investment proposal I^N is stable at X if and only if

$$\min_{S \subset N} \min_{I^S \in \mathcal{I}^S} \max_{i \in S} \{V_i^N(X, I^N) - V_i^S(X, I^S)\} \geq 0. \tag{2.11}$$

The maximum operator in the above formulation guarantees that for each $S \subset N$ and $I^S \in \mathcal{I}^S$ there exists at least one player that is unwilling to deviate to S. Even in the coalition S for which I^S is such that the difference between the value in N and S is smallest, such a player prefers the grand coalition. On the other hand, if the value of the optimization problem in (2.11) is negative, then there exists a coalition $S \subset N$ and a corresponding investment $I^S \in \mathcal{I}^S$ such that all players $i \in S$ prefer S over N (i.e., I^N is not stable at X).

Stability of an investment proposal I^N is most relevant at its respective moment of investment, $X_N^*(I^N)$. The following theorem states that an investment proposal by a coalition S, I^S, which induces all members of S to deviate at $X_N^*(I^N)$ is *credible*, because members in S can act on their deviation at the points in time for which $X \leq X_S^*(I^S)$.

Theorem 2.1. *If the investment proposal $I^N \in \mathcal{I}^N$ is not stable at $X_N^*(I^N)$ with respect to S and $I^S \in \mathcal{I}^S$, then I^N is not stable with respect to S and I^S for all $X \leq X_S^*(I^S)$.*

In other words, Theorem 2.1 states that, if a coalition S has an incentive to deviate at $X_N^*(I^N)$ with respect to I^S, then it also has an incentive to deviate at $X_S^*(I^S)$, thereby making its threat to deviate at $X_N^*(I^N)$ credible.

3 The proportional investment scheme

In the proportional investment scheme, for each coalition S, the investment cost C is shared proportionally with respect to the expected project profits of the individual players in that coalition. For example, if player $i \in N$ is expected to receive half of the total project profits in N, that is, $\frac{R_i^N(X)}{\sum_{j \in N} R_j^N(X)} = \frac{1}{2}$, then this player incurs half of the investment cost.

Definition 3.1. The *proportional investment scheme* $\rho = \{\rho^S \mid S \neq \emptyset\}$ consists of *proportional investment proposals* ρ^S and is defined by setting, for all $S \in 2^N$ with $S \neq \emptyset$,

$$\rho_i^S = \frac{D_i^S}{\sum_{j \in S} D_j^S} C$$

for all $i \in S$.

A proportional investment proposal ρ^S has the property that $\hat{X}_i^S(\rho_i^S) = \hat{X}_j^S(\rho_j^S)$ for all $i, j \in S$ (see also (2.6)), which means that the investment threshold of coalition S is equal to

$$X_S^*(\rho^S) = \frac{C}{\sum_{i \in S} D_i^S} \frac{\beta}{\beta - 1}(r - \mu). \tag{3.1}$$

Contributing to the success of infrastructure projects is the ease of implementation of an investment scheme. For the proportional investment scheme only the ratios of the individual project profits to the total project profits are relevant, and not their actual values which are not always known beforehand. This additionally implies that precise knowledge of the market characteristics, such as the growth rate of the profits and the profit uncertainty, as well as the discount rate, are not necessary for the implementation of a proportional investment.

The proportional investment scheme exhibits desirable properties, which are of interest from both a private and social perspective. The proportional investment scheme leads to a cooperative solution in the sense that the investment is chosen as to maximize, at each point in time, the sum of the individual project values of the players. Furthermore, by investing *pro rata*, the individual investment thresholds of the players are equal, which in turn implies that investment for this coalition takes place at the earliest moment in time (see also Figure 2.1). The following theorem provides this characterization of the proportional investment scheme.

Theorem 3.1. *Let $S \in 2^N$. The proportional investment proposal ρ^S satisfies*

(i) $X^*_S(\rho^S) < X^*_S(I^S)$ *for all* $I^S \in \mathcal{I}^S$ *with* $I^S \neq \rho^S$

(ii) and $\sum_{i \in S} V_i^S(X, \rho^S) = \max_{I^S \in \mathcal{I}^S} \sum_{i \in S} V_i^S(X, I^S)$.

In particular, considering expression (3.1), we see that, under the proportional investment scheme, a coalition invests earlier if it generates more project profit. In particular, as the grand coalition generates the largest project profit by assumption, its corresponding investment threshold is smallest. Correspondingly, a proportional investment by the grand coalition maximizes the overall sum of the values of the players.

Theorem 3.2. *For all $S \subset N$ and all $I^S \in \mathcal{I}^S$, the proportional investment proposal ρ^N satisfies*

(i) $X^*_N(\rho^N) < X^*_S(I^S)$

(ii) and $\sum_{i \in N} V_i^N(X, \rho^N) > \sum_{i \in S} V_i^S(X, I^S)$.

Hence, instability of ρ^N implies a deviation from ρ^N to a smaller coalition which results in an efficiency loss because the investment takes place at a later moment in time and the investment yields a lower total value. Moreover, Theorem 3.1 implies that there is also an efficiency loss if the grand coalition implements $I^N \neq \rho^N$ as the investment takes place at a later moment in time. Consequently, if $X < X^*_N(\rho^N)$, the sum of the individual values of the players is strictly smaller under $I^N \neq \rho^N$, which implies an additional efficiency loss.

The following theorem states that the proportional investment proposal ρ^N is dynamically stable if each player experiences synergy with respect to the grand coalition. The intuition behind the result is that each player potentially earns more by joining the grand coalition, which follows from $D_i^N \geq D_i^S$ for all $S \subset N$ and all $i \in S$, and has to contribute less because the investment cost C is shared among more players.

Theorem 3.3. *Let $D_i^N \geq D_i^S$ for all $S \subset N$ and all $i \in S$. Then, the proportional investment proposal ρ^N is dynamically stable.*

In practice, however, there may exist players that do not experience synergy with respect to the grand coalition. In what follows, we will show that under the right market conditions the proportional investment proposal ρ^N can nonetheless be stable.

Consider $S \subset N$ and $I^S \in \mathcal{I}^S$. A player $i \in S$ prefers the grand coalition with investment proposal ρ^N at $X_N^*(\rho^N)$ if

$$\Delta_i^S(I^S;\beta) \equiv V_i^N(X_N^*(\rho^N),\rho^N) - V_i^S(X_N^*(\rho^N), I^S) \geq 0,$$

that is, if its expected discounted net profit under ρ^N, which is given by $V_i^N(X_N^*, \rho^N)$, is at least equal to the value of the option to invest in S with respect to I^S, which is given by $V_i^S(X_N^*(\rho^N), I^S)$. Using the decomposition of the value of the option to invest in S into the net present value and the value of waiting (see (2.10)), we obtain, for all $i \in S$,

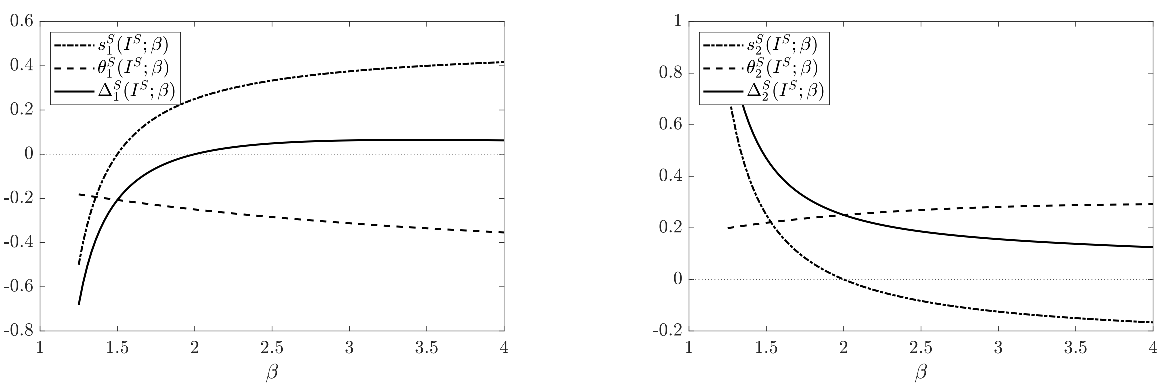
$$\Delta_i^S(I^S;\beta) = \underbrace{\frac{(D_i^N - D_i^S)X_N^*(\rho^N)}{r-\mu} - (\rho_i^N - I_i^S)}_{\text{(1) ``synergy effect''}} \underbrace{-W(X_N^*(\rho^N), I^S)}_{\text{(2) ``timing effect''}}, \tag{3.2}$$

where (1) represents the gain, or loss, in net present value from investing at $X_N^*(\rho^N)$, which we call the *synergy effect*, and is denoted by $s_i^S(I^S;\beta)$. That is, player $i \in S$ gains from investing at $X_N^*(\rho^N)$ as opposed to $X_S^*(I^S)$ if there is a positive synergy effect. In other words, the net present value of player $i \in S$ would be strictly lower in S with respect to I^S if coalition S is able to invest at $X_N^*(\rho^N)$. The *timing effect* (2) of player $i \in S$ in (3.2) is denoted by $\theta_i^S(I^S;\beta)$, which represents the *impatience with investment* of player $i \in S$. More specifically, player $i \in S$ is willing to wait with investment in S if $\theta_i^S(I^S;\beta) < 0$, but is unwilling to wait if $\theta_i^S(I^S;\beta) > 0$. Thus, a player's willingness to wait with investment in S grows when the timing effect decreases.

We distinguish between players that have a strictly higher level of profitability in N than in S and those that do not. For each $S \subset N$, the set of players that make strictly more project profit in N than in S is given by

$$\mathcal{Y}(S) = \{i \in S \mid D_i^N > D_i^S\}.$$

Figure 3.1 shows the synergy and timing effects as a function of β for a specific example where player 1 makes more project profit in S compared to N, whereas player 2 does not.



(a) Synergy effect and timing effect for player 1 for which $D_1^N < D_1^S$ and $I_1^S = C$.

(b) Synergy effect and timing effect for player 2 for which $D_2^N > D_2^S$ and $I_2^S = 0$.

Figure 3.1: Parameter values are $N = \{1,2,3\}$, $D^N = (1,2,1)$, $S = \{1,2\}$, $D^S = (2,1)$, $C = 1$, $\rho^N = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, and $I^S = (1,0)$. Here, $X_N^*(\rho^N) = \frac{1}{4}\frac{\beta}{\beta-1}(r-\mu)$ and $X_S^*(I^S) = \frac{1}{2}\frac{\beta}{\beta-1}(r-\mu)$.

As seen in Figure 3.1a, the synergy effect of player 1 increases as β increases and is positive if $\beta > 1\frac{1}{2}$. A large β corresponds to a situation with low project profit uncertainty, σ, and a low project profit growth rate, μ.[13] In such a situation the expected discounted net profits are low, so the effect of making more project profit in S diminishes which makes N more attractive, especially considering the fact that $I_1^S = C$. On the other hand, as seen in Figure 3.1b, the synergy effect of player 2 decreases as β increases and is negative if $\beta > 2$. Here, the effect of making more project profit in N diminishes as β increases, and since $I_2^S = 0$, coalition S becomes more attractive. In other words, from Figure 3.1 it follows that the synergy effect is strong if β is small, which happens if σ and μ are large.

In general, as the synergy effect of player $i \in S$ can be written as

$$s_i^S(I^S; \beta) = I_i^S - \frac{\beta(D_i^S/D_i^N) - 1}{\beta - 1}\rho_i^N,$$

it follows that the synergy effect of player $i \in S$ is non-negative if

$$I_i^S \geq \frac{\beta(D_i^S/D_i^N) - 1}{\beta - 1}\rho_i^N. \tag{3.3}$$

The right-hand side of (3.3) converges to $(D_i^S/D_i^N)\rho_i^N$ as $\beta \to \infty$. If $i \in \mathcal{Y}(S)$, the right-hand side of (3.3) increases as β increases and is smaller than ρ_i^N. Hence, coalition S is more attractive for player $i \in \mathcal{Y}(S)$ only if I_i^S is sufficiently smaller than ρ_i^N. If $i \notin \mathcal{Y}(S)$, the right-hand side of (3.3) decreases as β increases and is larger than ρ_i^N, so coalition N is more attractive for player i only if I_i^S is sufficiently larger than ρ_i^N.

Figure 3.1 additionally shows that the timing effect of player 1 is negative, meaning that player 1 is patient with investment, whereas it is positive for player 2, meaning that player 2 is impatient with investment. In fact, from Proposition 2.2 it follows that the timing effect of player $i \in S$ is guaranteed to be positive if I_i^S is such that $\hat{X}_i^S(I_i^S) \leq X_N^*(\rho^N)$, which happens if

$$I_i^S \leq \frac{D_i^S}{D_i^N}\rho_i^N, \tag{3.4}$$

that is, player i prefers to invest at $X_N^*(\rho^N)$ with N instead of waiting for $X_S^*(I^S)$ so it can undertake the investment with S. Proposition 2.2 also implies that the timing effect for player $i \in S$ is always negative if I_i^S is such that $\hat{X}_i^S(I_i^S) = X_S^*(I^S)$. Nevertheless, the timing effect of player $i \in S$ may either be positive or negative if I_i^S is such that $X_N^*(\rho^N) \leq \hat{X}_i^S(I_i^S) < X_S^*(I^S)$. In particular, the timing effect decreases when I_i^S increases because then a player is more patient with investment. In relation to Figure 3.1, player 1 is patient with investment since $I_1^S = C$ such that $\hat{X}_i^S(I_i^S) = X_S^*(I^S)$. Player 2, on the other hand, is impatient with investment since $I_2^S = 0$ such that $\hat{X}_i^S(I_i^S) = 0$.

From (3.3) and (3.4) it follows that there are two opposing effects at play with respect to the investment in S, namely that the synergy effect increases whereas the timing effect

[13]We keep the discount rate, r, fixed throughout the stability analysis because it has no influence on the geometric Brownian motion and thus the rate of change of the project profits.

decreases when one has to invest more in S.

The timing effect of player $i \in S$, being equal to the difference between the expected discounted net profit when investing now at $X_N^*(\rho^N)$ and the expected discounted net profit when investing later at $X_S^*(I^S)$, is given by

$$\begin{aligned}\theta_i^S(I^S;\beta) &= -W(X_N^*(\rho^N), I^S) \\ &= \left(\frac{D_i^S X_N^*(\rho^N)}{r-\mu} - I_i^S\right) - \left(\frac{X_N^*(\rho^N)}{X_S^*(I^S)}\right)^\beta \left(\frac{D_i^S X_S^*(I^S)}{r-\mu} - I_i^S\right). \end{aligned} \tag{3.5}$$

The effect of β on $\theta_i^S(I^S;\beta)$ is twofold. The first term in (3.5) decreases as β increases because a small σ and μ imply a low expected discounted net profit, so this incentivizes a player to wait with investment. Nevertheless, as β increases, the expected discounted net profit when investing at $X_S^*(I^S)$ also decreases, which incentivizes a player to invest now. In particular, in addition to the decrease in expected discounted net profit at $X_S^*(I^S)$ when β increases, the stochastic discount factor $(X_N^*(\rho^N)/X_S^*(I^S))^\beta$ decreases as β increases. That is, if σ and μ are small, it takes relatively long for $X(t)$ to grow from $X_N^*(\rho^N)$ to $X_S^*(I^S)$. As β tends to infinity, it takes infinitely long for profits to accumulate, so the second term in (3.5) vanishes and thus the first term in (3.5) will dominate if β is sufficiently large. Hence, when β is large, so when the market is relatively stable with low uncertainty and low growth rate of the project profits, players rather invest now with N than wait for the investment opportunity with S. This is also what Figure 3.1a illustrates. If β is larger than two, player 1 prefers to invest now with the grand coalition despite making more project profit in S.

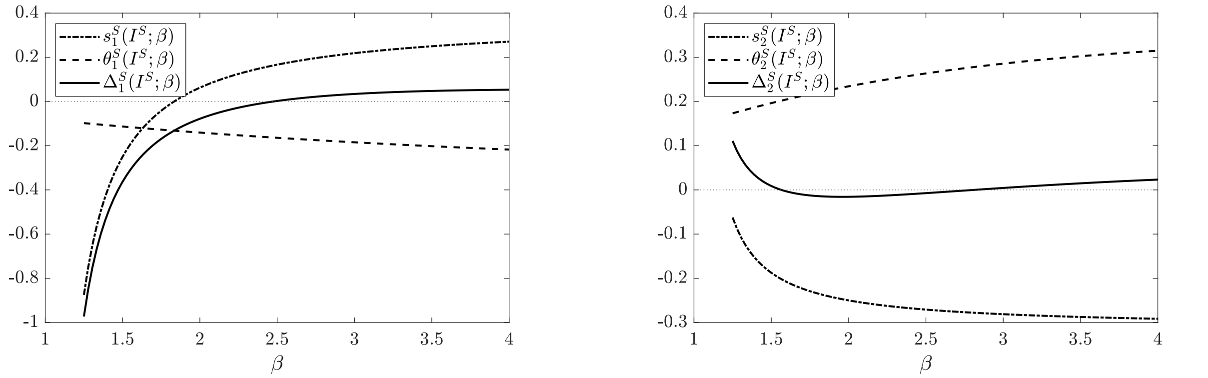
The following theorem formalizes the observation that the value of the option to invest in another coalition decreases as β increases, and therefore players always prefer to undertake the investment with the grand coalition if β is sufficiently large. As a matter of fact, stability of the proportional investment proposal ρ^N at $X_N^*(\rho^N)$ guarantees stability at $X \leq X_N^*(\rho^N)$, because $X_N^*(\rho^N)$ is the smallest possible investment threshold (see Theorem 3.2). Therefore, any threat by coalition S to deviate is not credible as they cannot act on it for $X \leq X_N^*(\rho^N)$.

Theorem 3.4. *Stability of ρ^N for $X \leq X_N^*(\rho^N)$ is guaranteed if β is sufficiently large.*[14]

Theorem 3.4 states that a proportional investment by the grand coalition is guaranteed to be stable if the market is sufficiently stable. On the other hand, as we will show next, a proportional investment by the grand coalition can also be stable if the market is growing yet volatile.

Figure 3.2 visualizes the situation like that in Figure 3.1 except for the fact that D_2^N is smaller, which makes the difference between coalition N and S smaller for player 2. As can be seen in Figure 3.2, ρ^N is stable at $X_N^*(\rho^N)$ with respect to I^S if β is sufficiently large.

[14]We refrain from quantifying how large β should be to guarantee stability because it does not yield valuable additional economic insights.



(a) Synergy effect and timing effect for player 1 for which $D_1^N < D_1^S$ and $I_1^S = C$.

(b) Synergy effect and timing effect for player 2 for which $D_2^N > D_2^S$ and $I_2^S = 0$.

Figure 3.2: Parameter values are $N = \{1, 2, 3\}$, $D^N = (1, 1\frac{1}{5}, 1)$, $S = \{1, 2\}$, $D^S = (2, 1)$, $C = 1$, $\rho^N = (\frac{5}{16}, \frac{3}{8}, \frac{5}{16})$, and $I^S = (1, 0)$. Here, $X_N^*(\rho^N) = \frac{5}{16}\frac{\beta}{\beta-1}(r-\mu)$ and $X_S^*(I^S) = \frac{1}{2}\frac{\beta}{\beta-1}(r-\mu)$.

In fact, Figure 3.2b shows that ρ^N is also stable at $X_N^*(\rho^N)$ with respect to I^S if β is sufficiently small. A small β corresponds to a situation with large project profit uncertainty, σ, and high project profit growth rate, μ. Contrary to the case in which β is large, the synergy effect is stronger because the expected discounted net profits are larger in all coalitions. At the same time, the timing effect is weaker because $X(t)$ grows faster which increases the value of waiting and the option value to invest in another coalition. The synergy effect dominates as β becomes sufficiently close to one, which makes deviation more attractive for a player $i \notin \mathcal{Y}(S)$ because this player has a higher level of profitability in S than in N, as can be seen in Figure 3.2a. Conversely, the grand coalition becomes more attractive for a player $i \in \mathcal{Y}(S)$ as this player makes strictly more project profit in N than in S, as can be seen in Figure 3.2b, which consequently prevents coalition S from deviating. Nevertheless, note that $\mathcal{Y}(S) = \emptyset$ may happen for some $S \subset N$. For instance, $D^N = (1, 2, 4)$ with $N = \{1, 2, 3\}$ and $D^{\{1,2\}} = (3, 3)$. In that case, provided that β is not sufficiently large as per Theorem 3.4, there exist situations for which deviation pays.

Theorem 3.5. *If $\mathcal{Y}(S) \neq \emptyset$ for all $S \subset N$, then stability of ρ^N for $X \leq X_N^*(\rho^N)$ is guaranteed if β is sufficiently close to one.*

So far we have discussed stability when β is either sufficiently large or small. However, in Figure 3.2 it can be seen that it pays for players 1 and 2 to deviate from the grand coalition if β is neither sufficiently small nor sufficiently large, which occurs when μ is relatively small and σ relatively large or vice versa. When both σ and μ are large, the grand coalition is preferred by player 2 as $D_2^N > D_2^S$ even though $\rho_2^N = \frac{3}{8} > I_2^S = 0$. However, when μ decreases and the market worsens in the sense that players prefer to delay the investment in general (see also (2.8)), the discounted project profit at $X_N^*(\rho^N)$ in N decreases such that the effect of making more project profit in N diminishes. As a result, player 2 puts more emphasis on I_2^S being zero such that player 2 prefers S. Although the synergy effect falls relatively quickly when μ decreases, the timing effect only grows relatively slowly, as is illustrated in

Figure 3.2b. As μ decreases, the grand coalition becomes less attractive, but at the same time coalition S also becomes less attractive. The fact that the synergy effect falls relatively quickly and the fact that player 2 has to invest zero dominate such that player 2 is willing to deviate from the grand coalition. Similarly, also when σ decreases and the market becomes more promising in the sense that players prefer to accelerate investment in general, the fact that player 2 has to invest zero is the decisive factor in why player 2 favors S. Nevertheless, as mentioned before, eventually, as β increases, the value of waiting for S diminishes so that the grand coalition will always form.

The following example illustrates Theorem 3.4 and Theorem 3.5, and shows that the proportional investment proposal ρ^N can be unstable if β is neither small nor large. Nevertheless, the example suggests a potential remedy for this instability.

Example 3.1. Consider the cooperative investment project in which $N = \{1, 2, 3\}$, $D^N = (2, 0, 8)$, $D^{\{1,2\}} = (1, 8)$, $D^S = 0$ for all other coalitions $S \subset N$ with $S \neq \{1, 2\}$, and $C = 1$. Under a proportional investment,

$$\rho^N = \frac{1}{10}(2, 0, 8),$$

the investment threshold of the grand coalition is equal to

$$X^*_N(\rho^N) = \frac{1}{10}\frac{\beta}{\beta - 1}(r - \mu).$$

As player 2 has a project profit of zero in the grand coalition, it will choose its investment in $\{1, 2\}$ strategically in an attempt to convince player 1 to switch to $\{1, 2\}$. Let $\delta = (1-\delta_2, \delta_2)$ with $\delta_2 \in [0, 1]$ be the investment vector in $\{1, 2\}$. The corresponding investment threshold of $\{1, 2\}$ is given by

$$X^*_{\{1,2\}}(\delta) = \max\{1 - \delta_2, \frac{\delta_2}{8}\}\frac{\beta}{\beta - 1}(r - \mu).$$

The proportional investment in $\{1, 2\}$ is given by $\rho^{\{1,2\}} = \frac{1}{9}(1, 8)$. Let $X < X^*_{\{1,2\}}(\delta)$. The value of player 1 in $\{1, 2\}$ is equal to

$$V_1^{\{1,2\}}(X, \delta) = \begin{cases} \left(\frac{X(\beta - 1)}{\beta(r - \mu)(1 - \delta_2)}\right)^\beta \frac{1 - \delta_2}{\beta - 1} & \text{if } 0 \leq \delta_2 \leq \rho_2^{\{1,2\}}, \\ \left(\frac{X(\beta - 1)8}{\beta(r - \mu)\delta_2}\right)^\beta \left(\frac{\delta_2}{8}\frac{\beta}{\beta - 1} - (1 - \delta_2)\right) & \text{if } \rho_2^{\{1,2\}} \leq \delta_2 \leq 1. \end{cases}$$

Player 2 chooses δ_2 as to maximize $V_1^{\{1,2\}}(X, \delta)$. The derivative of $V_1^{\{1,2\}}(X, \delta)$ with respect to δ_2 is strictly increasing for $\delta_2 \leq \rho_2^{\{1,2\}}$. Therefore, player 2 chooses a $\delta_2 \in [\rho_2^{\{1,2\}}, 1]$. For such values of δ_2, player 2 can make coalition $\{1, 2\}$ more attractive to player 1 by investing more, that is, by increasing δ_2 (cf. (3.3)). However, in doing so the investment threshold of $\{1, 2\}$ increases, which implies that investment takes place at a later moment in time, thereby making $\{1, 2\}$ less attractive to player 1 (cf. (3.4)). Hence, due to these opposing effects, the optimal δ_2 will be in between $\rho_2^{\{1,2\}}$ and 1. As a result, the optimal investment is given by $\delta^*(\beta) = (1 - \delta_2^*(\beta), \delta_2^*(\beta))$, in which

$$\delta_2^*(\beta) = \begin{cases} 1 & \text{if } 1 < \beta < 8, \\ \dfrac{8\beta}{9\beta - 8} & \text{if } \beta \geq 8. \end{cases} \tag{3.6}$$

From (3.6) it follows that player 2 pays the full investment cost if $\beta < 8$. If β is relatively small, which happens if σ is large, μ is large, and r is small, the option value of waiting with investment is large. Hence, player 1 prefers to invest zero at the expense of waiting longer for the investment at $X^*_{\{1,2\}}(\delta^*(\beta))$ to materialize. On the other hand, player 2 will invest less than one if $\beta \geq 8$ in order to speed up the coalitional investment. In fact, we obtain the proportional investment if β tends to infinity, that is, $\lim_{\beta \to \infty} \delta^*(\beta) = \frac{1}{9}(1, 8) = \rho^{\{1,2\}}$. If β is relatively large, the option value of waiting with investment is small, which implies that player 2 should bring the investment timing of coalition $\{1, 2\}$ forward by investing less, up to the point at which it invests proportionally. The optimal value of player 1 in $\{1, 2\}$ is given by

$$V_1^{\{1,2\}}(X, \delta^*(\beta)) = \begin{cases} \left(\dfrac{X(\beta - 1)8}{\beta(r - \mu)}\right)^{\beta} \dfrac{1}{8} \dfrac{\beta}{\beta - 1} & \text{if } 1 < \beta < 8, \\ \left(\dfrac{(\beta - 1)X(9\beta - 8)}{\beta^2(r - \mu)}\right)^{\beta} \dfrac{1}{\beta - 1} & \text{if } \beta \geq 8. \end{cases} \tag{3.7}$$

We now focus our attention on stability of ρ^N at $X_N^*(\rho^N)$. The value of player 1 in N at $X_N^*(\rho^N)$ equals

$$V_1^N(X_N^*(\rho^N), \rho^N) = \frac{D_1^N X_N^*(\rho^N)}{r - \mu} - \rho_1^N = \frac{1}{5} \frac{1}{\beta - 1}.$$

For ρ^N to be stable at $X_N^*(\rho^N)$ we require that

$$V_1^N(X_N^*(\rho^N), \rho^N) \geq V_1^{\{1,2\}}(X_N^*(\rho^N), \delta^*(\beta)).$$

Figure 3.3 shows the value of player 1 in N and in $\{1, 2\}$ at $X_N^*(\rho^N)$. From the figure it follows that ρ^N is stable if β is sufficiently small or sufficiently large, which is in accordance with Theorem 3.4 and Theorem 3.5.

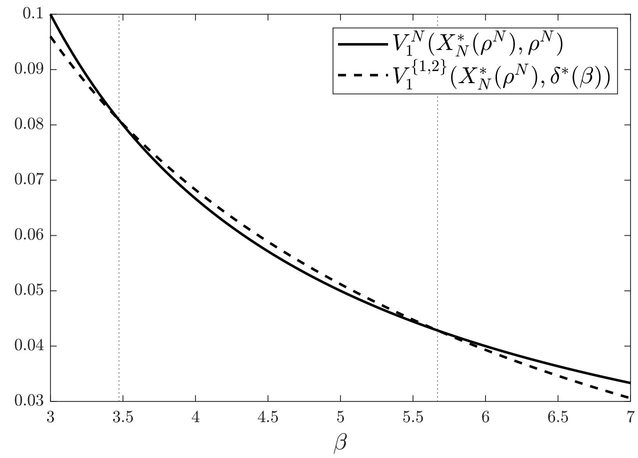


Figure 3.3: The value of player 1 at $X_N^*(\rho^N)$ for different values of β in both N and in $\{1, 2\}$. The value of player 1 is larger in N than in $\{1, 2\}$ if $\beta < 3.47$ or $\beta > 5.67$.

Nevertheless, Figure 3.3 additionally shows that player 1 has an incentive to deviate if β is neither small nor large. In that case, player 3 can prevent player 1 from deviating by proposing a suitably chosen investment $\gamma^N = (1 - \gamma_3, 0, \gamma_3)$, in which $\gamma_3 \in (\frac{4}{5}, 1]$ such that $\gamma_1^N < \rho_1^N$. For example, if player 3 proposes that it pays the full investment cost (i.e., $\gamma_3 = 1$), then the investment threshold of N with respect to γ^N and the investment threshold of $\{1, 2\}$ with respect to $\delta^*(\beta)$ coincide (i.e., $X_N^*(\gamma^N) = X_{\{1,2\}}^*(\delta^*(\beta))$). As player 1 makes more project profit in N than in $\{1, 2\}$ (i.e., $D_1^N > D_1^{\{1,2\}}$), and has to invest less under γ^N than under ρ^N (i.e., $\gamma_1^N < \rho_1^N$), the investment proposal γ^N prevents player 1 from deviating to S. In this way, γ^N can be seen as a side payment. △

As the following example illustrates, stability of the proportional investment proposal ρ^N at the moment of investment does not guarantee stability at a later point in time. In fact, there may exist an $S \subset N$ and $I^S \in \mathcal{I}^S$ for which ρ^N is stable at $X_N^*(\rho^N)$, not stable at $X_S^*(I^S)$, but then stable again if X is sufficiently large.

Example 3.2. Reconsider the cooperative investment project of Example 3.1. Let $\sigma = 0.10$, $\mu = 0.01$, and $r = 0.06$ such that $\beta = 3$ and $\delta^*(3) = (0, 1)$. The value of player 1 in $\{1, 2\}$ at $X \geq X_N^*(\rho^N)$ is given by (see (3.7))

$$V_1^{\{1,2\}}(X, \delta^*(\beta)) = \begin{cases} \left(\dfrac{X}{X_S^*(\delta^*(\beta))}\right)^{\beta} \dfrac{1}{8} \dfrac{\beta}{\beta - 1} & \text{if } X_N^*(\rho^N) \leq X < X_S^*(\delta^*(\beta)), \\ \dfrac{X}{r - \mu} & \text{if } X \geq X_S^*(\delta^*(\beta)). \end{cases}$$

The value of player 1 in N at $X \geq X_N^*(\rho^N)$ is given by

$$V_1^N(X, \rho^N) = \frac{D_1^N X}{r - \mu} - \rho_1^N = \frac{2X}{r - \mu} - \frac{1}{5}.$$

As seen in Figure 3.3 in Example 3.1, ρ^N is stable at $X_N^*(\rho^N)$ if $\beta = 3$. However, Figure 3.4 shows that ρ^N is not stable at $X_{\{1,2\}}^*(\delta^*(\beta))$. The option value of investment in $\{1,2\}$ grows non-linearly in X, and therefore it may occur that the value in $\{1,2\}$ exceeds the value in N at $X_{\{1,2\}}^*(\delta^*(\beta))$. In spite of this, as Figure 3.4 shows, the proportional investment proposal ρ^N becomes stable once X is sufficiently large because $D_1^N > D_1^{\{1,2\}}$ implies that the expected discounted net profit for player 1 grows faster in N than in $\{1,2\}$.

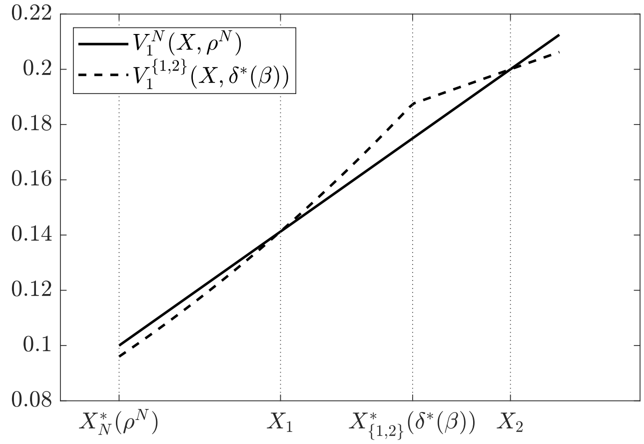


Figure 3.4: The value of player 1 for different values of X in both N and in $\{1,2\}$. Parameter values are $\sigma = 0.10$, $\mu = 0.01$, and $r = 0.06$ such that $\beta = 3$.

It follows that whether N or coalition $\{1,2\}$ forms is history-dependent, because it is dependent on the initial value of the geometric Brownian motion, X. If $X < X_N^*(\rho^N)$, the geometric Brownian motion approaches $X_N^*(\rho^N)$ from below so that investment takes place by N at $X_N^*(\rho^N)$. If $X \geq X_N^*(\rho^N)$ but $X < X_1$ (see Figure 3.4), there is immediate investment by N at X. However, if $X \geq X_1$ but $X < X_{\{1,2\}}^*(\delta^*(\beta))$, either coalition may form. Even though the project value of player 1 is larger in $\{1,2\}$ than in N for such X, coalition $\{1,2\}$ currently holds an option to invest, which it will exercise at its investment threshold. If the project profit, that is, the level of the geometric Brownian motion, grows until it reaches $X_{\{1,2\}}^*(\delta^*(\beta))$, coalition $\{1,2\}$ forms, whereas N forms if the project profit drops to a level just below X_1. The point X_1 is an indifference point at which the values of player 1 in N and $\{1,2\}$ are equal. An infinitesimal downward shock at X_1 implies that N forms; note that it need not be true that $\{1,2\}$ forms for an infinitesimal upward shock, because then X is still below the threshold $X_{\{1,2\}}^*(\delta^*(\beta))$. Finally, if $X \geq X_{\{1,2\}}^*(\delta^*(\beta))$ but $X < X_2$ (see Figure 3.4), there is immediate investment by $\{1,2\}$; if $X > X_2$ there is immediate investment by N. The point X_2 is an indifference point similar to X_1. An infinitesimal downward shock at X_2 implies that $\{1,2\}$ forms, whereas an infinitesimal upward shock implies that N forms. △

4 One-leader cooperative investment projects

In the previous section, we established that the proportional investment scheme is a scheme for which the investment threshold is smallest; additionally, it is such that the sum of the players' value is maximized. Subsequently, we analyzed the stability of a proportional investment by the grand coalition. In this section, we consider one-leader cooperative investment projects and characterize an investment scheme that maximizes the value of the leader. We then use this investment scheme to show that the stability analysis of the previous section is robust.

In practice, it is common that a project may only be realized if one specific player is part of the coalition. A recent example of such a project is *Feyenoord City*, which is a large infrastructure project that initially started with plans for a new football stadium for the Dutch football club Feyenoord in the city of Rotterdam. In 2017, it turned into a master plan for the rejuvenation of the corresponding neighborhood, including the construction of 3550 new housing units, restaurants, hotels, offices, and shops.[15] On the 21st of April 2022, more than a decade after negotiations started, the football club Feyenoord pulled the plug on the project by announcing that it will not proceed with the construction of a new stadium. The municipality of Rotterdam intended to proceed with realizing the remaining components of the Feyenoord City master plan, but, on the 26th of October 2022, the Dutch Council of State put a stop to their ambitions by rejecting the development plan.[16] The Council of State argued that the stadium is such an integral part of the plan that continuation of the project without it is injudicious. Clearly, Feyenoord acted as a leader within this project.

A *one-leader cooperative investment project* is a cooperative investment project as outlined in Section 2 with the additional assumption that there is exactly one leader in the sense that a coalition can undertake a project only if the leader is part of that coalition.

Definition 4.1. A *one-leader cooperative investment project* with leader $k \in N$ is characterized by $D^S = 0$ for all $S \in 2^N$ with $S \not\ni k$.

The leader has more negotiation power in these situations and can therefore propose to be prioritized by investing less than proportionally whereas the remaining players invest more than proportionally.

Furthermore, investment proposals in which one player is prioritized are also relevant in situations in which players in $S \subset N$ want to entice a player to break away from the grand coalition, just like in Example 3.1. The best way of doing so is by prioritizing this player by letting it pay less than proportionally. By means of an example, consider coalition $S = \{1, 2, 3, 4\}$ and suppose that player 1 is the leader. We then consider $I^S \in \mathcal{I}^S$ with the property that

$$\hat{X}_2^S(I_2^S) = \hat{X}_3^S(I_3^S) = \hat{X}_4^S(I_4^S) > \hat{X}_1^S(I_1^S). \tag{4.1}$$

[15] See the website of the Office of Metropolitan Architecture for more information: `https://www.oma.com/projects/feyenoord-city`.

[16] The full statement of the Council of State (in Dutch) can be accessed here: `https://www.raadvanstate.nl/@133494/202101596-1-r3/`.

Hence, player $i \in S \setminus \{1\}$ invests more than proportionally, that is, $I_i^S > \rho_i^S$, while player 1 invests less than proportionally, that is, $I_1^S < \rho_1^S$.[17]

Investment proposals adhering to the condition like in (4.1) with $k = 1$ are called *k-prioritized investment proposals*. We define such a proposal only for coalitions that the leader k is a member of, because if $S \not\ni k$, then $D^S = 0$ and $\mathcal{I}^S = \emptyset$. Moreover, we also disregard $S \in 2^N$ with $k \in S$ and $D_k^S = 0$ because $D_k^S = 0$ implies that $I_k^S = 0$ for all $I^S \in \mathcal{I}^S$ and thus $V_k^S(X, I^S) = 0$ for all X. Hence, in defining the *k-prioritized investment scheme*, we will restrict ourselves to k-prioritized investment proposals for coalitions $S \in 2^N$ with $k \in S$ and $D_k^S > 0$.

Definition 4.2. The *k-prioritized investment scheme* $\varepsilon(\beta) = \{\varepsilon^{S,k}(\beta) \,|\, S \neq \emptyset, k \in S, D_k^S > 0\}$ consists of *k-prioritized investment proposals* $\varepsilon^{S,k}(\beta)$ and is defined by setting, for all $S \in 2^N$ with $S \neq \emptyset$, $k \in S$, and $D_k^S > 0$,

$$\varepsilon_i^{S,k}(\beta) = \min\left\{\frac{D_i^S}{\sum_{j\in S\setminus\{k\}} D_j^S}, \frac{\beta D_i^S}{\beta(\sum_{j\in S} D_j^S) - \sum_{j\in S\setminus\{k\}} D_j^S}\right\} C \tag{4.2}$$

for all $i \in S \setminus \{k\}$, and $\varepsilon_k^{S,k}(\beta) = C - \sum_{i\in S\setminus\{k\}} \varepsilon_i^{S,k}(\beta)$.

Prioritizing player $k \in S$ additionally entails choosing the investment as to maximize the value of player $k \in S$ prior to and at the moment investment takes place. The following theorem states that, indeed, for each $S \in 2^N$ with $k \in S$ and $D_k^S > 0$, the prioritized investment proposal $\varepsilon^{S,k}(\beta)$ maximizes the value of player k for each $X \leq X_S^*(\varepsilon^{S,k}(\beta))$.

Theorem 4.1. *Let $S \in 2^N$ with $k \in S$ and $D_k^S > 0$. Then, for all $X \leq X_S^*(\varepsilon^{S,k}(\beta))$, the k-prioritized investment proposal $\varepsilon^{S,k}(\beta)$ satisfies*

$$V_k^S(X, \varepsilon^{S,k}(\beta)) = \max_{I^S\in\mathcal{I}^S} V_k^S(X, I^S).$$

Unlike the proportional investment scheme, the k-prioritized investment scheme depends on β. If β is below a sufficiently small value, then a k-prioritized investment proposal with respect to S is a boundary proposal in the sense that

$$\varepsilon_i^{S,k}(\beta) = \frac{D_i^S}{\sum_{j\in S\setminus\{k\}} D_j^S} C \text{ for all } i \in S \setminus \{k\}, \text{ and } \varepsilon_k^{S,k}(\beta) = 0. \tag{4.3}$$

Moreover, in that case it holds that $X_S^*(\varepsilon^{S,k}(\beta)) = X_{S\setminus\{k\}}^*(\rho^{S\setminus\{k\}})$. As argued in Example 3.1, if β is relatively small, the investment option has a large value so that the value of waiting with investment is large too. Therefore, player k prefers a zero investment at the expense of investing later. On the other hand, as β increases the value of waiting is lower and player k prefers to invest earlier. Therefore, the investment of player k increases and approaches the point at which player k invests proportionally. More specifically, $\varepsilon^{S,k}(\beta)$ approaches the proportional investment proposal ρ^S monotonically as β tends to infinity, that is,

$$\lim_{\beta\to\infty} \varepsilon^{S,k}(\beta) = \rho^S, \tag{4.4}$$

[17] This follows from the fact that, for all $S \in 2^N$, $I^S \in \mathcal{I}^S$, and $i \in S$, if $\hat{X}_i^S(I_i^S) = X_S^*(I^S) \geq X_S^*(\rho^S)$, then $\frac{I_i^S}{D_i^S} = \max_{k\in S} \frac{I_k^S}{D_k^S} \geq \frac{C}{\sum_{j\in N} D_j^S}$ which implies that $I_i^S \geq \rho_i^S$.

and consequently $X_S^*(\varepsilon^{S,k}(\beta)) > X_S^*(\rho^S)$. From (4.3) and (4.4) it follows that, for all β, the investment threshold of coalition S with respect to (4.2) has the property that

$$X_S^*(\rho^S) < X_S^*(\varepsilon^{S,k}(\beta)) \leq X_{S\setminus\{k\}}^*(\rho^{S\setminus\{k\}}).$$

One-leader cooperative investment projects may thus lead to inefficiencies because the implementation of a k-prioritized investment entails that the sum of the players' value is not maximized and that the investment takes place later.

Furthermore, unlike Theorem 3.1 on the optimality of the proportional investment scheme with respect to the sum of the players' values, Theorem 4.1 does not hold for all X. To see this, recall that the investment of the leader is positive if β is sufficiently large. If we subsequently consider any other investment proposal in which the investment of the leader is zero, and if we consider an X for which investment has taken place under both investment proposals, then the value of the leader is larger in case its investment is zero. For the same reason a k-priority investment proposal need not be stable if X is sufficiently large, even if the leader experiences synergy with respect to the grand coalition. We demonstrate this in the following example.

Example 4.1. Reconsider the cooperative investment project of Example 3.1 with $N = \{1, 2, 3\}$, $D^N = (2, 0, 8)$, $D^{\{1,2\}} = (1, 8)$, and $C = 1$. However, we let $D^{\{1,3\}} = (1, 1)$, $D^{\{1\}} = 1$, and $D^S = 0$ for all $S \subset N$ with $S \not\ni 1$, such that player 1 is the leader and experiences synergy with respect to N. Let $\sigma = 0.10$, $\mu = -0.05$, and $r = 0.06$ such that $\beta = 12$. The 1-priority investment proposal with respect to N is equal to $\varepsilon^{N,1}(12) = (\frac{1}{7}, 0, \frac{6}{7})$. Consider $I^N = (0, 0, 1)$ and $I^{\{1,2\}} = (0, 1)$. Under these specific investment choices, it holds that $X_N^*(I^N) = X_{\{1,2\}}^*(I^{\{1,2\}})$, because $D_3^N = 8 = D_2^{\{1,2\}}$. Figure 4.1 shows that, for $X > X_N^*(\varepsilon^{N,1}(\beta))$, $\varepsilon^{N,1}(\beta)$ need neither be optimal with respect to I^N nor stable with respect to $I^{\{1,2\}}$.

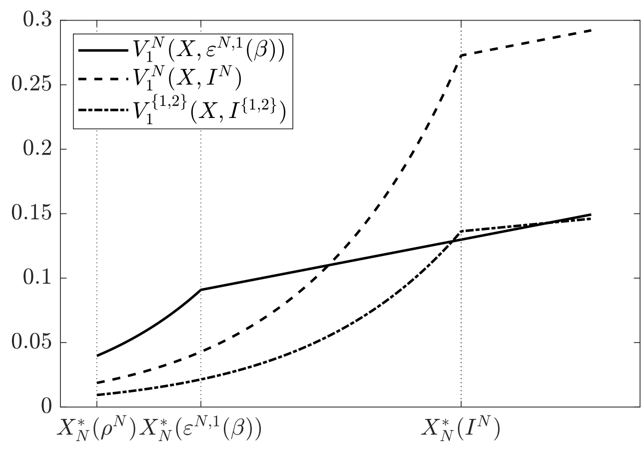


Figure 4.1: The value of player 1 for different values of X in N with respect to $\varepsilon^{N,1}(\beta) = (\frac{1}{7}, 0, \frac{6}{7})$, in N with respect to $I^N = (0, 0, 1)$, and in $\{1, 2\}$ with respect to $I^{\{1,2\}} = (0, 1)$. Parameter values are $\sigma = 0.10$, $\mu = -0.05$, and $r = 0.06$ such that $\beta = 12$.

△

The leader can exert pressure on the other players to demand the k-prioritized investment proposal $\varepsilon^{N,k}(\beta)$ to be implemented. Indeed, as the following theorem states, the k-prioritized investment proposal with respect to N is stable if the leader experiences synergy with respect to the grand coalition. In particular, the k-prioritized investment proposal is dynamically stable if β is sufficiently small because then the leader has no investment cost, whereas the k-prioritized investment proposal is stable prior to and at the moment investment takes place if β is sufficiently large because then the leader has a positive investment cost.

Theorem 4.2. *Let $D_k^N \geq D_k^S$ for all $S \subset N$ with $k \in S$, and let $D^S = 0$ for all $S \in 2^N$ with $S \not\ni k$. Then, the k-prioritized investment proposal $\varepsilon^{N,k}(\beta)$ is dynamically stable if $\beta \leq (\sum_{i \in N\setminus\{k\}} D_i^N)/D_k^N$, and stable for $X \leq X_N^*(\varepsilon^{N,k}(\beta))$ if $\beta > (\sum_{i\in N\setminus\{k\}} D_i^N)/D_k^N$.*

Both Theorem 3.3 on the stability of a proportional investment and Theorem 4.2 on the stability of a priority investment with respect to the leader, state that experiencing synergy is a sufficient condition for stability of the respective investment proposal. Returning to the case of Feyenoord City, besides serious doubts and uncertainty about the required financial means to build the stadium, the fact that it was not clear whether the football club Feyenoord was going to use the new stadium was the foremost concern of the Dutch Council of State. Their statement indicates that the football club Feyenoord experienced little to no synergy with respect to grand the coalition. Indeed, the lack of synergy contributed to the failure of the project as it gave the football club Feyenoord an incentive to unilaterally deviate.

Theorem 4.2 on the stability of $\varepsilon^{N,k}(\beta)$ for sufficiently small β is analogous to Theorem 3.5, which states that the proportional investment proposal ρ^N is guaranteed to be stable prior to and at the moment of investment if β is sufficiently small and if there is at least one player with synergy with respect to the grand coalition. Indeed, if β is sufficiently small, the leader's synergy effect, which is amplified by the leader having an investment cost of zero in N, outweighs the value of waiting with investment and undertaking the investment with another coalition.

In general, the k-prioritized investment, $\varepsilon^{N,k}(\beta)$, is guaranteed to be stable prior to and at the moment of investment if β is sufficiently large, because $\varepsilon^{N,k}(\beta)$ approaches the proportional investment proposal ρ^N as β approaches infinity (see also (4.4)).

5 Concluding remarks

This article models the inherent cooperative and non-cooperative incentives of stakeholders in investment projects in a novel way by combining concepts from cooperative game theory and real options theory. The analysis allowed us to pinpoint the factors that contribute to the failure of infrastructure projects in which cooperation between stakeholders in the grand coalition is necessary. An absence of synergy with respect to the grand coalition is a crucial contributing factor to instability of investment projects. Additional contributing factors are the growth rate of the project profits and the uncertainty level of the project profits. We use the synergy effect and the timing effect to explain why investment proposals can be (dynamically) stable. These effects typically move in opposite directions when the growth rate of the profits and the uncertainty level of the profits change.

Under the proportional investment scheme in which the project cost is shared proportionally with respect to the coalition-specific profits of the individual stakeholders, the investment takes place at the earliest moment in time and the sum of the stakeholders' value is maximized. If the grand coalition fails to implement a proportional investment, then a smaller coalition may form that is less efficient as the investment takes place later and the sum of the stakeholders' value is lower. Furthermore, the implementation of another investment proposal by the grand coalition, such as a priority investment proposal in case there is exactly one leader, or because stakeholders use side payments, leads to inefficiencies for the same reasons.

As there are only very few real options game models that consider cooperation, as noted by Azevedo and Paxson (2014), we would like to list some possible extensions of our model. One extension is to allow for heterogeneity of the coalitions, which can be done in various ways. In our current model, we implicitly assume that coalitions are homogeneous in the sense that each coalition can undertake the same investment project with the same cost. A way to introduce heterogeneity of the coalitions is to allow the cost of the project to be positively correlated with the size of a coalition. One can also allow coalition-specific construction periods which may depend on the cost of the project and which may be of uncertain duration. Moreover, the one-stakeholder coalition can be used as a reference point in the sense that it represents the situation without investment in which the stakeholder continues its usual business operations. Deviation from the grand coalition to the one-stakeholder coalition thus means that this stakeholder rejects the partnership and wants to maintain the status quo.

Another extension is to drop the assumption that the grand coalition is the coalition with the largest total expected discounted project profit, and approach cooperative investment from a social perspective. In practice, governments may want the coalition with the largest expected discounted total welfare to form, or governments may want to meet certain environmental policy targets and objectives. For example, in an attempt to accelerate the transition to a climate-neutral energy system, the Federal Cabinet of Germany adopted a comprehensive package of legislation on 6 April 2022, among which is the ambition to expand the onshore wind production with 10 gigawatt per year.[18]

[18]Bundesministerium für Wirtschaft und Klimaschutz. (2022, April 6). *Federal Minister Robert Habeck says Easter package is accelerator for renewable energy as the Federal Cabinet adopts key amendment to accelerate the expansion of renewables* [Press release]. Retrieved from `https://www.bmwk.de/en`.

A Proofs

Proof of Proposition 2.1. Let $i \in S$ and let $I_i^S \geq 0$. Denote the investment threshold by $\hat{X}_i^S(I_i^S)$. On the basis of $\hat{X}_i^S(I_i^S)$ two regions can be distinguished, namely the waiting region and the investment region.

Waiting region. Let $X < \hat{X}_i^S(I_i^S)$. The value of the option to invest is given by (see, e.g., Dixit and Pindyck (1994))

$$H(X) = AX^{\beta} \tag{A.1}$$

in which A is some to be determined constant, and

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

is the positive root of the quadratic equation

$$Q(\beta) = \frac{1}{2}\sigma^2\beta^2 + (\mu - \frac{1}{2}\sigma^2)\beta - r = 0.$$

It follows that $\beta > 1$ because $Q(\beta)$ is a strictly convex function with $Q(0) = -r < 0$ and $Q(1) = -(r - \mu) < 0$.

Investment region. Let $X \geq \hat{X}_i^S(I_i^S)$. Denote the expected discounted project profit if player i invests at X given I_i^S by $R(X, I_i^S)$. Then,

$$R(X, I_i^S) = \mathbb{E}\left[\int_{t=0}^{\infty} D_i^S X(t) e^{-rt} dt - I_i^S \,\middle|\, X(0) = X\right] = \frac{D_i^S X}{r - \mu} - I_i^S. \tag{A.2}$$

The investment threshold $\hat{X}_i^S(I_i^S)$ follows from the value matching and smooth pasting conditions at $\hat{X}_i^S(I_i^S)$, which are given by

$$\begin{aligned} H(\hat{X}_i^S(I_i^S)) &= R(\hat{X}_i^S(I_i^S), I_i^S), \\ \frac{\partial H(X)}{\partial X}\Big|_{X=\hat{X}_i^S(I_i^S)} &= \frac{\partial R(X, I_i^S)}{\partial X}\Big|_{X=\hat{X}_i^S(I_i^S)}, \end{aligned} \tag{A.3}$$

respectively. Substituting (A.1) and (A.2) into (A.3) and solving for $\hat{X}_i^S(I_i^S)$ and A gives

$$\hat{X}_i^S(I_i^S) = \frac{I_i^S}{D_i^S}\frac{\beta}{\beta - 1}(r - \mu) \text{ and } A = (\hat{X}_i^S(I_i^S))^{-\beta}\left(\frac{D_i^S \hat{X}_i^S(I_i^S)}{r - \mu} - I_i^S\right).$$

□

Proof of Proposition 2.2. First, let $I_i^S = 0$. Then, $0 = \hat{X}_i^S(I_i^S) < X_S^*(I^S)$, and, for all $\hat{X}_i^S(I_i^S) \leq X < X_S^*(I^S)$,

$$W_i(X, I^S) = \frac{D_i^S X}{r - \mu}\left(\left(\frac{X}{X_S^*(I^S)}\right)^{\beta - 1} - 1\right) < 0.$$

Second, let $I_i^S > 0$. It holds that

$$\frac{\partial W_i(X, I^S)}{\partial X} = \frac{D_i^S}{r-\mu}\left(\left(\frac{X}{X_S^*(I^S)}\right)^{\beta-1} \frac{\beta(X_S^*(I^S) - \hat{X}_i^S(I_i^S)) + \hat{X}_i^S(I_i^S)}{X_S^*(I^S)} - 1\right),$$

which implies that $W_i(X, I^S)$ is minimized at

$$Y^* = \left(\frac{X_S^*(I^S)}{\beta(X_S^*(I^S) - \hat{X}_i^S(I_i^S)) + \hat{X}_i^S(I_i^S)}\right)^{\frac{1}{\beta-1}} X_S^*(I^S),$$

for which it holds that $Y^* \leq X_S^*(I^S)$. Moreover, it holds that $W_i(0, I^S) = I_i^S > 0$ and $W(X_S^*(I^S), I^S) = 0$.

We will now prove parts (i) and (ii).

(i). Let $\hat{X}_i^S(I_i^S) = X_S^*(I^S)$. Then, $\frac{\partial W_i(X,I^S)}{\partial X} < 0$ for all $X < Y^* = X_S^*(I^S)$, so $W_i(X, I^S) > 0$ for all $X < X_S^*(I^S)$.

(ii). Let $\hat{X}_i^S(I_i^S) < X_S^*(I^S)$. Then, $Y < X_S^*(I^S)$. We will show that $W(\hat{X}_i^S(I_i^S), I^S) < 0$ so that $W(Y^*, I^S) < 0$, which implies that $W_i(X, I^S) < 0$ for all $\hat{X}_i^S(I_i^S) \leq X < X_S^*(I^S)$. It holds that

$$\begin{aligned} W(\hat{X}_i^S(I_i^S), I^S) &= -\frac{I_i^S}{\beta-1} + \left(\frac{\hat{X}_i^S(I_i^S)}{X_S^*(I^S)}\right)^{\beta}\left(\frac{D_i^S X_S^*(I^S)}{r-\mu} - I_i^S\right) \\ &= -\frac{I_i^S}{\beta-1}\left(1 - \beta\left(\frac{\hat{X}_i^S(I_i^S)}{X_S^*(I^S)}\right)^{\beta-1} + (\beta-1)\left(\frac{\hat{X}_i^S(I_i^S)}{X_S^*(I^S)}\right)^{\beta}\right). \end{aligned}$$

Let $a = \left(\frac{\hat{X}_i^S(I_i^S)}{X_S^*(I^S)}\right)^{\beta} \in (0,1)$, and define

$$f(\beta) = 1 - \beta a^{\beta-1} + (\beta-1)a^{\beta}.$$

To show that $W(\hat{X}_i^S(I_i^S), I^S) < 0$, it suffices to show that $f(\beta) > 0$ for all $\beta > 1$. It holds that $f(1) = 0$, and

$$f'(\beta) = a^{\beta-1}((a-1)(1+\beta\ln(a)) - a\ln(a)),$$

so $f'(\beta) > 0$ if $\beta > \overline{\beta}(a)$, in which

$$\overline{\beta}(a) = 1 + \frac{(1-a)+\ln(a)}{-\ln(a)(1-a)}.$$

It is well known that $(1-a)+\ln(a) < 0$ for all $a > 0$, so $\overline{\beta}(a) < 1$ for all $a \in (0,1)$. This implies that $f(\beta) > 0$ for all $\beta > 1$. □

Proof of Theorem 2.1. Let $I^N \in \mathcal{I}^N$, let $S \subset N$, and let $I^S \in \mathcal{I}^S$ be such that $V_i^N(X_N^*(I^N), I^N) < V_i^S(X_N^*(I^N), I^S)$ for all $i \in S$. Recall that the value function of a player is given by (2.9). We will show that, for all $X \leq X_S^*(I^S)$ and all $i \in S$, it holds that

$$V_i^N(X, I^N) < V_i^S(X, I^S).$$

To this end, we distinguish between two cases for I^S, namely I^S such that $X_S^*(I^S) \geq X_N^*(I^N)$ and I^S such that $X_S^*(I^S) < X_N^*(I^N)$.

First, let I^S be such that $X_S^*(I^S) \geq X_N^*(I^N)$. Let $i \in S$.

If $X < X_N^*(I^N)$, then

$$\begin{aligned}
V_i^N(X, I^N) &= \left(\frac{X}{X_N^*(I^N)}\right)^\beta V_i^N(X_N^*(I^N), I^N) \\
&< \left(\frac{X}{X_N^*(I^N)}\right)^\beta V_i^S(X_N^*(I^N), I^S) \\
&= \left(\frac{X}{X_N^*(I^N)}\right)^\beta \left(\frac{X_N^*(I^N)}{X_S^*(I^S)}\right)^\beta V_i^S(X_S^*(I^S), I^S) \\
&= V_i^S(X, I^S),
\end{aligned}$$

in which the inequality follows from the fact that $V_i^N(X_N^*(I^N), I^N) < V_i^S(X_N^*(I^N), I^S)$.

Let $X_N^*(I^N) \leq X < X_S^*(I^S)$. Then,

$$\begin{aligned}
\frac{\partial}{\partial X} V_i^S(X, I^S) &= \frac{\beta}{X}\left(\frac{X}{X_S^*(I^S)}\right)^\beta \left(\frac{D_i^S X_S^*(I^S)}{r-\mu} - I_i^S\right) \\
&\geq \frac{\beta}{X_N^*(I^N)}\left(\frac{X_N^*(I^N)}{X_S^*(I^S)}\right)^\beta \left(\frac{D_i^S X_S^*(I^S)}{r-\mu} - I_i^S\right) \\
&= \frac{\beta}{X_N^*(I^N)} V_i^S(X_N^*(I^N), I^S) \\
&> \frac{\beta}{X_N^*(I^N)} V_i^N(X_N^*(I^N), I^N) \\
&= \frac{\beta}{X_N^*(I^N)}\left(\frac{D_i^N X_N^*(I^N)}{r-\mu} - I_i^N\right) \\
&\geq \frac{\beta}{\hat{X}_i^N(I_i^N)}\left(\frac{D_i^N \hat{X}_i^N(I_i^N)}{r-\mu} - I_i^N\right) \\
&= \frac{D_i^N}{r-\mu} \\
&= \frac{\partial}{\partial X} V_i^N(X, I^N).
\end{aligned}$$

The first inequality follows from the fact that $\beta > 1$ and $X \geq X_N^*(I^N)$; the second inequality follows from the fact that $V_i^N(X_N^*(I^N), I^N) < V_i^S(X_N^*(I^N), I^S)$; the third inequality follows from the fact that $\hat{X}_i^N(I_i^N) \leq X_N^*(I^N)$. Therefore, the value of player i at X grows faster in

S than in N. This holds for any $X_N^*(I^N) \leq X < X_S^*(I^S)$, so for all $Y \in [X_N^*(I^N), X_S^*(I^S))$, it holds that

$$\frac{\partial}{\partial X} V_i^S(X, I^S)\Big|_{X=Y} > \frac{\partial}{\partial X} V_i^N(X, I^N)\Big|_{X=Y}. \tag{A.4}$$

In particular, because $V_i^N(X_N^*(I^N), I^N) < V_i^S(X_N^*(I^N), I^S)$, we must also have that

$$V_i^N(X, I^N) < V_i^S(X, I^S)$$

for all $X_N^*(I^N) \leq X < X_S^*(I^S)$.

Let $X = X_S^*(I^S)$. Suppose that $V_i^N(X_S^*(I^S), I^N) \geq V_i^S(X_S^*(I^S), I^S)$. Then, this implies that there exist an $Y \in [X_N^*(I^N), X_S^*(I^S))$ such that

$$\frac{\partial}{\partial X} V_i^S(X, I^S)\Big|_{X=Y} \leq \frac{\partial}{\partial X} V_i^N(X, I^N)\Big|_{X=Y},$$

which contradicts (A.4). Hence, we must also have that $V_i^N(X_S^*(I^S), I^N) < V_i^S(X_S^*(I^S), I^S)$.

Second, let I^S be such that $X_S^*(I^S) < X_N^*(I^N)$.

We will start by showing that, for all $X_S^*(I^S) \leq X < X_N^*(I^N)$ and all $i \in S$, it holds that

$$V_i^N(X, I^N) < V_i^S(X, I^S). \tag{A.5}$$

Let $X_S^*(I^S) \leq X < X_N^*(I^N)$. Suppose, on the contrary, that there exists an $i \in S$ such that

$$V_i^N(X, I^N) \geq V_i^S(X, I^S).$$

Let player $k \in S$ be such a player, and let $X \leq Z < X_N^*(I^N)$. Then, using similar arguments as in the previous case, one can show that the value of player k at Z grows at least as fast in N than in S, that is, for all all $Y \in [X, X_N^*(I^N))$,

$$\frac{\partial}{\partial X} V_k^N(X, I^N)\Big|_{X=Y} \geq \frac{\partial}{\partial X} V_k^S(X, I^S)\Big|_{X=Y}.$$

For this reason we must have that $V_k^N(X_N^*(I^N), I^N) \geq V_k^S(X_N^*(I^N), I^S)$, which contradicts the assumption that I^N is not stable with respect to I^S at $X_N^*(I^N)$.

Finally, let $X < X_S^*(I^S)$ and let $i \in S$. Then, in a similar fashion as before,

$$V_i^N(X, I^N) = \left(\frac{X}{X_S^*(I^S)}\right)^\beta V_i^N(X_S^*(I^S), I^N) < \left(\frac{X}{X_S^*(I^S)}\right)^\beta V_i^S(X_S^*(I^S), I^S) = V_i^S(X, I^S).$$

The inequality follows from (A.5) with the value functions evaluated at $X_S^*(I^S)$. □

Proof of Theorem 3.1. (i). Consider the following optimization problem:

$$\min_{I^S \in \mathcal{I}^S} \max_{i \in S} \hat{X}_i^S(I_i^S),$$

or equivalently,

$$\min_{I^S \in \mathcal{I}^S} \max_{i \in S} \left\{ \frac{I_i^S}{D_i^S} \frac{\beta}{\beta - 1}(r - \mu) \right\}. \tag{A.6}$$

We will show that ρ^S is the unique solution to (A.6) with corresponding optimal value $X_S^*(\rho^S)$.

Let $I^S \in \mathcal{I}^S$ with $I^S \neq \rho^S$. We will show that

$$\max_{i \in S} \frac{I_i^S}{D_i^S} > \max_{i \in S} \frac{\rho_i^S}{D_i^S} = \frac{C}{\sum_{j \in S} D_j^S},$$

which implies the desired result that ρ^S is the unique solution to (A.6). Suppose that

$$\max_{i \in S} \frac{I_i^S}{D_i^S} \leq \max_{i \in S} \frac{\rho_i^S}{D_i^S} = \frac{C}{\sum_{j \in S} D_j^S}.$$

Then, $I_i^S \leq \rho_i^S$ for all $i \in S$ with $I_i^S < \rho_i^S$ for some $i \in S$ because $I^S \neq \rho^S$. Consequently,

$$C = \sum_{i \in S} I_i^S < \sum_{i \in S} \rho_i^S = C,$$

which is a contradiction.

(ii). First, let $X < X_S^*(\rho^S)$. The objective function is given by

$$f(X, I^S) = \sum_{i \in S} V_i^S(X, I^S) = \left(\frac{X}{X_S^*(I^S)} \right)^\beta \left(\frac{(\sum_{i \in S} D_i^S) X_S^*(I^S)}{r - \mu} - C \right), \tag{A.7}$$

in which

$$X_S^*(I^S) = \left(\max_{i \in S} \frac{I_i^S}{D_i^S} \right) \frac{\beta}{\beta - 1}(r - \mu).$$

Taking the derivative of (A.7) with respect to $X_S^*(I^S)$ gives

$$\frac{\partial f(X, I^S)}{\partial X_S^*(I^S)} = \left(\frac{X}{X_S^*(I^S)} \right)^\beta \left(\frac{C\beta}{X_S^*(I^S)} - \frac{(\beta - 1)}{r - \mu} (\sum_{i \in S} D_i^S) \right),$$

which is strictly negative for

$$X_S^*(I^S) > \frac{C}{\sum_{i \in S} D_i^S} \frac{\beta}{\beta - 1}(r - \mu)$$

and strictly positive for

$$0 < X_S^*(I^S) < \frac{C}{\sum_{i \in S} D_i^S} \frac{\beta}{\beta - 1}(r - \mu).$$

Therefore, the investment threshold $X_S^*(I^S)$ that maximizes (A.7) is equal to

$$X_S^*(I^S) = \frac{C}{\sum_{i\in S} D_i^S}\frac{\beta}{\beta-1}(r-\mu). \tag{A.8}$$

The (unique) investment vector I^S that corresponds to (A.8) is given by

$$I^S = \frac{C}{\sum_{i\in S} D_i^S}\left(D_i^S\right)_{i\in S},$$

which is the proportional investment proposal ρ^S. So far we have shown that, for all $X < X_S^*(\rho^S)$,

$$\sum_{i\in S} V_i^S(X,\rho^S) \geq \sum_{i\in S} V_i^S(X,I^S)$$

for all $I^S \in \mathcal{I}^S$.

Second, let $X \geq X_S^*(\rho^S)$. Then, define the set of investment proposals for which investment takes place after X by

$$\hat{\mathcal{I}}^S = \{I^S \in \mathcal{I}^S \,|\, X < X_S^*(I^S)\}.$$

If $I^S \notin \hat{\mathcal{I}}^S$, then

$$\begin{aligned}\sum_{i\in S} V_i^S(X,I^S) &= \sum_{i\in S}\left(\frac{D_i^S X}{r-\mu} - I_i^S\right)\\ &= \frac{(\sum_{i\in S} D_i^S)X}{r-\mu} - C\\ &= \sum_{i\in S}\left(\frac{D_i^S X}{r-\mu} - \rho_i^S\right)\\ &= \sum_{i\in S} V_i^S(X,\rho^S).\end{aligned}$$

The second equality follows from efficiency of I^S.

If $I^S \in \hat{\mathcal{I}}^S$, then

$$\begin{aligned}\sum_{i\in S} V_i^S(X,I^S) &= \left(\frac{X}{X_S^*(I^S)}\right)^\beta\left(\frac{(\sum_{i\in S} D_i^S)X_S^*(I^S)}{r-\mu} - C\right)\\ &< \left(\frac{X}{X}\right)^\beta\left(\frac{(\sum_{i\in S} D_i^S)X}{r-\mu} - C\right)\\ &= \sum_{i\in S}\left(\frac{D_i^S X}{r-\mu} - \rho_i^S\right)\\ &= \sum_{i\in S} V_i^S(X,\rho^S).\end{aligned}$$

The inequality follows from the fact that $X < X_S^*(I^S)$ and $\beta > 1$. □

Proof of Theorem 3.2. Let $S \subset N$ and $I^S \in \mathcal{I}^S$.

(i). From $\sum_{i\in N} D_i^N > \sum_{i\in S} D_i^S$, it follows that

$$X_N^*(\rho^N) = \frac{C}{\sum_{i\in N} D_i^N}\frac{\beta}{\beta-1}(r-\mu) < \frac{C}{\sum_{i\in S} D_i^S}\frac{\beta}{\beta-1}(r-\mu) = X_S^*(\rho^S) \leq X_S^*(I^S),$$

where the last inequality follows from Theorem 3.1.

(ii). Let $X < X_N^*(\rho^N)$. Then,

$$\begin{aligned}\sum_{i\in N} V_i^N(X,\rho^N) &= \left(\frac{X}{X_N^*(\rho^N)}\right)^{\beta}\frac{C}{\beta-1} \\ &> \left(\frac{X}{X_S^*(\rho^S)}\right)^{\beta}\frac{C}{\beta-1} \\ &= \sum_{i\in S} V_i^S(X,\rho^S) \\ &\geq \sum_{i\in S} V_i^S(X,I^S).\end{aligned}$$

The first inequality follows from $X_N^*(\rho^N) < X_S^*(\rho^S)$; the second inequality follows from Theorem 3.1.

Let $X_N^*(\rho^N) \leq X < X_S^*(I^S)$. Then,

$$\begin{aligned}\sum_{i\in N} V_i^N(X,\rho^N) &\geq \sum_{i\in N} V_i^N(X_N^*(\rho^N),\rho^N) \\ &= \frac{C}{\beta-1} \\ &> \left(\frac{X}{X_S^*(\rho^S)}\right)^{\beta}\frac{C}{\beta-1} \\ &= \sum_{i\in S} V_i^S(X,\rho^S) \\ &\geq \sum_{i\in S} V_i^S(X,I^S).\end{aligned}$$

The last inequality follows from Theorem 3.1.

Let $X \geq X_S^*(I^S)$. Then, from $\sum_{i\in N} D_i^N \geq \sum_{i\in S} D^S$ it follows that

$$\sum_{i\in N} V_i^N(X,\rho^N) = \frac{(\sum_{i\in N} D_i^N)X}{r-\mu} - C > \frac{(\sum_{i\in S} D_i^S)X}{r-\mu} - C = \sum_{i\in S} V_i^S(X,I^S).$$

□

Proof of Theorem 3.3. Let $S \subset N$ and consider $I^S \in \mathcal{I}^S$. Let player $k \in S$ be such that $\hat{X}_k^S(I_k^S) = X_S^*(I^S)$.

First, let $X = X_N^*(\rho^N)$. The values of the players in N at $X_N^*(\rho^N)$ under the proportional investment proposal ρ^N are equal to

$$V^N(X_N^*(\rho^N), \rho^N) = \frac{D^N X_N^*(\rho^N)}{r-\mu} - \rho^N = \frac{C/\sum_{i\in N} D_i^N}{\beta - 1} D^N.$$

The value of player $k \in S$ at $X_N^*(\rho^N)$ is equal to

$$\begin{aligned} V_k^S(X_N^*(\rho^N), I^S) &= \left(\frac{X_N^*(\rho^N)}{\hat{X}_k^S(I_k^S)}\right)^\beta \frac{I_k^S}{\beta-1} \\ &= \left(\frac{C/\sum_{i\in N} D_i^N}{I_k^S/D_k^S}\right)^\beta \frac{I_k^S}{\beta-1} \\ &\leq \frac{C/\sum_{i\in N} D_i^N}{\beta-1} D_k^S \\ &\leq \frac{C/\sum_{i\in N} D_i^N}{\beta-1} D_k^N \\ &= V_k^N(X_N^*(\rho^N), \rho^N). \end{aligned}$$

The first inequality follows from the fact that $\left(\frac{C/\sum_{i\in N} D_i^N}{I_k^S/D_k^S}\right)^\beta < \left(\frac{C/\sum_{i\in N} D_i^N}{I_k^S/D_k^S}\right)$ because $\beta > 1$, and because I^S can be such that $\hat{X}_k^S(I_k^S) = X_S^*(I^S) = X_N^*(\rho^N)$; the second inequality follows from the assumption $D_k^N \geq D_k^S$. Hence, it holds that $V_k^N(X_N^*(\rho^N), \rho^S) \geq V_k^S(X_N^*(\rho^N), I^S)$, meaning that, under the investment I^S, player $k \in S$ has no incentive to deviate at $X_N^*(\rho^N)$. Therefore, the proportional investment proposal ρ^N is stable at $X_N^*(\rho^N)$.

Second, let $X < X_N^*(\rho^N)$. Then, player k also has no incentive to deviate at X because

$$\begin{aligned} V_k^N(X, \rho^N) &= \left(\frac{X}{X_N^*(\rho^N)}\right)^\beta V_k^N(X_N^*(\rho^N), \rho^N) \\ &\geq \left(\frac{X}{X_N^*(\rho^N)}\right)^\beta V_k^S(X_N^*(\rho^N), I^S) \\ &= \left(\frac{X}{X_N^*(\rho^N)}\right)^\beta \left(\frac{X_N^*(\rho^N)}{X_S^*(\rho^S)}\right)^\beta V_k^S(X_S^*(I^S), I^S) \\ &= V_k^S(X, I^S). \end{aligned}$$

Third, let $X > X_N^*(\rho^N)$. The derivative of $V_k^S(X)$ with respect to X is equal to

$$\frac{\partial V_k^S(X, I^S)}{\partial X} = \begin{cases} \left(\frac{X}{\hat{X}_k^S(I_k^S)}\right)^{\beta-1} \frac{D_k^S}{r-\mu} & \text{if } X_N^*(\rho^N) \leq X < \hat{X}_k^S(I_k^S), \\ \frac{D_k^S}{r-\mu} & \text{if } X \geq \hat{X}_k^S(I_k^S). \end{cases} \tag{A.9}$$

The derivative $V_k^N(X, \rho^N)$ with respect to X is, for all $X \geq X_N^*(\rho^N)$, equal to

$$\frac{\partial V_k^N(X, \rho^N)}{\partial X} = \frac{D_k^N}{r-\mu},$$

which, as a result of $D_k^N \geq D_k^S$, is at least equal to the derivative given in (A.9). Therefore, the value of player k grows at least as fast in coalition N than in coalition S. Hence, because $V_k^N(X_N^*(\rho^N), \rho^N) > V_k^S(X_N^*(\rho^N), I^S)$, we must also have that $V_k^N(X, \rho^N) > V_k^S(X, I^S)$. □

Proof of Theorem 3.4. Let $S \subset N$ and $I^S \in \mathcal{I}^S$. Recall from Theorem 3.2 that $X_N^*(\rho^N) < X_S^*(I^S)$.

We will show that ρ^N is guaranteed to be stable at $X_N^*(\rho^N)$ with respect to I^S if β is sufficiently large. It suffices to restrict ourselves to stability of ρ^N at $X_N^*(\rho^N)$ because $V_k^N(X_N^*(\rho^N), \rho^N) \geq V_k^S(X_N^*(\rho^N), I^S)$ for some $k \in S$ implies that, for all $X < X_N^*(\rho^N)$,

$$\begin{aligned} V_k^N(X, \rho^N) &= \left(\frac{X}{X_N^*(\rho^N)}\right)^\beta V_k^N(X_N^*(\rho^N), \rho^N) \\ &\geq \left(\frac{X}{X_N^*(\rho^N)}\right)^\beta V_k^S(X_N^*(\rho^N), I^S) \\ &= \left(\frac{X}{X_N^*(\rho^N)}\right)^\beta \left(\frac{X_N^*(\rho^N)}{X_S^*(I^S)}\right)^\beta V_k^S(X_S^*(I^S), I^S) \\ &= V_k^S(X, I^S). \end{aligned}$$

Let $i \in S$. Set

$$\Delta_i^S(I^S; \beta) \equiv V_i^N(X_N^*(\rho^N), \rho^N) - V_i^S(X_N^*(\rho^N), I^S).$$

First, if $i \in \mathcal{Y}(S)$, then $D_i^N > D_i^S$, that is, $\frac{D_i^N}{D_i^S} > 1$. If I^S is such that $\hat{X}_i^S(I_i^S) = X_S^*(I^S)$, then

$$\Delta_i^S(I^S; \beta) = \frac{C}{\beta - 1} \frac{D_i^S}{\sum_{j \in N} D_j^N} \left(\frac{D_i^N}{D_i^S} - \left(\frac{X_N^*(\rho^N)}{X_S^*(I^S)}\right)^{\beta - 1} \right). \quad \text{(A.10)}$$

The condition $\Delta_i^S(I^S; \beta) \geq 0$ is equivalent to

$$\frac{D_i^N}{D_i^S} \geq \left(\frac{X_N^*(\rho^N)}{X_S^*(I^S)}\right)^{\beta - 1}. \quad \text{(A.11)}$$

The left-hand side of (A.11) is strictly larger than one, whereas the right-hand side is at most one. Therefore, $\Delta_i^S(I^S; \beta) \geq 0$ for all $\beta > 1$.

If I^S is such that $\hat{X}_i^S(I_i^S) < X_S^*(I^S)$, then

$$\Delta_i^S(I^S; \beta) = \frac{C}{\beta - 1} \frac{D_i^S}{\sum_{j \in N} D_j^N} \left(\frac{D_i^N}{D_i^S} - \left(\frac{X_N^*(\rho^N)}{X_S^*(I^S)}\right)^{\beta - 1} \left(\beta - (\beta - 1)\frac{\hat{X}_i^S(I_i^S)}{X_S^*(I^S)}\right) \right). \quad \text{(A.12)}$$

Now, the condition $\Delta_i^S(I^S; \beta) \geq 0$ is equivalent to

$$g(\beta) \equiv \left(\frac{X_N^*(\rho^N)}{X_S^*(I^S)}\right)^{\beta - 1} \left(\beta - (\beta - 1)\frac{\hat{X}_i^S(I_i^S)}{X_S^*(I^S)}\right) \leq \frac{D_i^N}{D_i^S}. \quad \text{(A.13)}$$

If $\beta = 1$, then $g(\beta) = 1 < \frac{D_i^N}{D_i^S}$. It holds that $g(\beta)$ is strictly increasing for $\beta < \beta^*$ and strictly decreasing for $\beta > \beta^*$, in which

$$\beta^* = \frac{1}{\ln(X_S^*(I^S)/X_N^*(\rho^N))} - \frac{\hat{X}_i^S(I_i^S)}{X_S^*(I^S) - \hat{X}_i^S(I_i^S)}. \tag{A.14}$$

Hence, $g(\beta)$ is maximized at $\beta = \beta^*$. Moreover, $g(\beta) \downarrow 0$ as $\beta \to \infty$. If either $\beta^* \leq 1$ or $\beta^* > 1$ and $g(\beta^*) \leq \frac{D_i^N}{D_i^S}$, then $g(\beta) \leq \frac{D_i^N}{D_i^S}$ for all $\beta > 1$. Otherwise, if $\beta^* > 1$ and $g(\beta^*) > \frac{D_i^N}{D_i^S}$, then there must exist two roots, $\overline{\beta}_1$ and $\overline{\beta}_2$ with $\overline{\beta}_1 < \overline{\beta}_2$, for which $g(\overline{\beta}_1) = g(\overline{\beta}_2) = \frac{D_i^N}{D_i^S}$. In such a case, it holds that $\Delta_i^S(I^S;\beta) \geq 0$ if $\beta \in (1, \overline{\beta}_1] \cup [\overline{\beta}_2, \infty)$. In other words, $\Delta_i^S(I^S;\beta) \geq 0$ if β is sufficiently small or sufficiently large.

Second, if $i \notin \mathcal{Y}(S)$, then $D_i^N \leq D_i^S$, that is, $\frac{D_i^N}{D_i^S} \leq 1$. If I^S is such that $\hat{X}_i^S(I_i^S) = X_S^*(I^S)$, then $\Delta_i^S(I^S;\beta)$ is given by (A.10). It follows that $\Delta_i^S(I^S;\beta) \geq 0$ if

$$\beta \geq 1 + \frac{\ln(D_i^N/D_i^S)}{\ln(X_N^*(\rho^N)/X_S^*(I^S))}. \tag{A.15}$$

The right-hand side of (A.15) is at least one because $\ln(D_i^N/D_i^S) \leq 0$ and $\ln(X_N^*(\rho^N)/X_S^*(I^S)) < 0$, which follow from $D_i^N \leq D_i^S$ and $X_N^*(\rho^N) < X_S^*(I^S)$, respectively. Therefore, $\Delta_i^S(I^S;\beta) \geq 0$ if β is sufficiently large.

If I^S is such that $\hat{X}_i^S(I_i^S) < X_S^*(I^S)$, then $\Delta_i^S(I^S;\beta)$ is given by (A.12). The condition $\Delta_i^S(I^S;\beta) \geq 0$ is equivalent to $g(\beta) \leq \frac{D_i^N}{D_i^S}$, in which the function $g(\beta)$ is given by (A.13). The function $g(\beta)$ is strictly increasing for $\beta < \beta^*$ and strictly decreasing for $\beta > \beta^*$, in which β^* is given by (A.14). Therefore, $g(\beta)$ is maximized at β^*. Moreover, $g(\beta) \downarrow 0$ as $\beta \to \infty$, and it holds that $g(\beta^*) \geq g(1) = 1 \geq \frac{D_i^N}{D_i^S}$. Hence, $\Delta_i^S(I^S;\beta) \geq 0$ if β is sufficiently large. □

Proof of Theorem 3.5. Let $\mathcal{Y}(\mathcal{S}) \neq \emptyset$ for all $S \subset N$. Let $S \subset N$ and let $i \in \mathcal{Y}(S)$. The result then follows from the proof of Theorem 3.4 where it was shown that, for all $I^S \in \mathcal{I}^S$, either $\Delta_i^S(I^S;\beta) \geq 0$ for all $\beta > 1$, or $\Delta_i^S(I^S;\beta) \geq 0$ if β is sufficiently small or large. □

Proof of Theorem 4.1. Without loss of generality, let $S = N = \{1, 2, \ldots, n\}$ with $n = |N|$ and without loss of generality let the first player be the player whose value is maximized, that is, $k = 1$. Let the vector of investments be given by $I^N = (C - \sum_{i=2}^n \delta_i, \delta_2, \ldots, \delta_n) \in \mathcal{I}^N$.

We will show that

$$V_1^N(X, \varepsilon^{N,1}(\beta)) \geq V_1^N(X, I^N)$$

for all $I^N \in \mathcal{I}^N$ and all $X \leq X_N^*(\varepsilon^{N,1}(\beta))$. To this end, we distinguish between two cases with respect to X, namely $X < X_N^*(\rho^N)$ and $X_N^*(\rho^N) \leq X \leq X_N^*(\varepsilon^{N,1}(\beta))$.

First, let $X < X_N^*(\rho^N)$. Then, for all $I^N \in \mathcal{I}^N$, investment has not taken place, which implies that the optimization problem is given by

$$\max_{\delta=(\delta_2,\ldots,\delta_n)} \quad \left(\frac{X}{X_N^*(\delta)}\right)^\beta \left(\frac{D_1^N X_N^*(\delta)}{r-\mu} - (C - \sum_{i=2}^n \delta_i)\right)$$

subject to

$$0 \le \delta_i \le C \quad \forall i \in \{2, \dots, n\},$$
$$\sum_{i=2}^{n} \delta_i \le C,$$

in which

$$X_N^*(\delta) = \max \left\{ \frac{C - \sum_{i=2}^{n} \delta_i}{D_1^N}, \frac{\delta_2}{D_2^N}, \dots, \frac{\delta_n}{D_n^N} \right\} \frac{\beta}{\beta - 1}(r - \mu).$$

We proceed by showing that in this case the optimal solution δ^* must satisfy

$$\hat{X}_1^N(C - \sum_{i=2}^{n} \delta_i^*) < \hat{X}_2^N(\delta_2^*) = \dots = \hat{X}_n^N(\delta_n^*) = X_N^*(\delta^*), \tag{A.16}$$

or equivalently,

$$\frac{C - \sum_{i=2}^{n} \delta_i^*}{D_1^N} < \frac{\delta_2^*}{D_2^N} = \dots = \frac{\delta_n^*}{D_n^N}.$$

Let δ be such that

$$\hat{X}_1^N(C - \sum_{i=2}^{n} \delta_i) = \hat{X}_k^N(\delta_k) > \hat{X}_j^N(\delta_j), \tag{A.17}$$

in which $k \in \mathcal{K}$ and $j \in \mathcal{J}$ with $\mathcal{K} \cup \mathcal{J} = \{2, \dots, n\}$ and $\mathcal{K} \cap \mathcal{J} = \emptyset$. We treat the special case $|\mathcal{J}| = 0$, that is, $|\mathcal{K}| = n - 1$ and $\hat{X}_1^N(C - \sum_{i=2}^{n} \delta_i) = \hat{X}_k^N(\delta_k)$ for all $k = 2, \dots, n$, separately later on. If (A.17) holds, then,

$$X_N^*(\delta) = \frac{C - \sum_{i=2}^{n} \delta_i}{D_1^N} \frac{\beta}{\beta - 1}(r - \mu) = \frac{\delta_k}{D_k^N} \frac{\beta}{\beta - 1}(r - \mu)$$

for all $k \in \mathcal{K}$, thereby reducing the dimension of the optimization problem to $|\mathcal{J}|$. That is, for all $k \in \mathcal{K}$, it holds that

$$\delta_k = (C - \sum_{j \in \mathcal{J}} \delta_j) \frac{D_k^N}{D_1^N + \sum_{k \in \mathcal{K}} D_k^N},$$

so that the objective function is equal to

$$f(\delta) = c^\beta (C - \sum_{j \in \mathcal{J}} \delta_j)^{-(\beta - 1)} \frac{D_1^N}{D_1^N + \sum_{k \in \mathcal{K}} D_k^N} \frac{1}{\beta - 1},$$

in which c is some constant. For each $j \in \mathcal{J}$, it holds that

$$\frac{\partial f(\delta)}{\partial \delta_j} = c^\beta (C - \sum_{j \in \mathcal{J}} \delta_j)^{-\beta} \frac{D_1^N}{D_1^N + \sum_{k \in \mathcal{K}} D_k^N} > 0.$$

Therefore, the objective value strictly increases if at least one δ_j with $j \in \mathcal{J}$ increases. In doing so, the value $(C - \sum_{j\in\mathcal{J}} \delta_j)$ strictly decreases. Hence, we must eventually reach the point at which $|\mathcal{J}| = 0$ such that

$$X_N^*(\delta) = \frac{C - \sum_{i=2}^n \delta_i}{D_1^N} \frac{\beta}{\beta - 1}(r - \mu) = \frac{\delta_2}{D_2^N} \frac{\beta}{\beta - 1}(r - \mu) = \cdots = \frac{\delta_n}{D_n^N} \frac{\beta}{\beta - 1}(r - \mu).$$

The vector $\delta_c = (\delta_i)_{i=2}^n$ that satisfies this requirement is given by

$$\delta_i = \frac{D_i^N}{\sum_{j\in N} D_j^N} C \tag{A.18}$$

for all $i \in \{2, 3, \ldots, n\}$. Thus, the objective value is strictly larger at δ_c than at any other δ that satisfies (A.17). Nevertheless, we will later show that δ_c is not optimal.

Now, without loss of generality, let δ be such that

$$\hat{X}_2^N(\delta_2) > \hat{X}_1^N(C - \sum_{i=2}^n \delta_i) \text{ and } \hat{X}_2^N(\delta_2) = \hat{X}_k^N(\delta_k) > \hat{X}_j^N(\delta_j),$$

in which $k \in \mathcal{K}$ and $j \in \mathcal{J}$ with $\mathcal{K} \cup \mathcal{J} = \{3, 4, \ldots, n\}$, $\mathcal{K} \cap \mathcal{J} = \emptyset$, and $|\mathcal{J}| \geq 1$. The case $|\mathcal{J}| = 0$ is given in (A.16). Hence,

$$X_N^*(\delta) = \frac{\delta_2}{D_2^N} \frac{\beta}{\beta - 1}(r - \mu) = \frac{\delta_k}{D_k^N} \frac{\beta}{\beta - 1}(r - \mu)$$

for all $k \in \mathcal{K}$, thereby reducing the dimension of the optimization problem to $1 + |\mathcal{J}|$. That is, for all $k \in \mathcal{K}$, it holds that

$$\delta_k = \delta_2 \frac{D_k^N}{D_2^N},$$

so that the objective function is equal to

$$f(\delta) = c^\beta (\delta_2)^{-\beta} \left(\delta_2 \left(\frac{D_1^N}{D_2^N} \frac{\beta}{\beta - 1} + \frac{\sum_{k\in\mathcal{K}} D_k^N}{D_2^N} + 1 \right) - (C - \sum_{j\in\mathcal{J}} \delta_j) \right)$$

in which c is some constant. For each $j \in \mathcal{J}$, it holds that

$$\frac{\partial f(\delta)}{\partial \delta_j} = c^\beta (\delta_2)^{-\beta} > 0.$$

Therefore, provided that we keep $\delta_2 > 0$ fixed, the objective value strictly increases if at least one δ_j with $j \in \mathcal{J}$ increases. Consequently, two cases may arise. In the first case, we reach the boundary of the feasible set, that is, $\sum_{i=2}^n \delta_i = C$, which implies that the objective function becomes equal to

$$f(\delta_2) = c^\beta (\delta_2)^{-(\beta-1)} \frac{D_1^N}{D_2^N} \frac{\beta}{\beta - 1}.$$

Correspondingly, it readily follows that $f'(\delta_2) < 0$, which means that we should decrease δ_2 to strictly increase the objective value. We therefore eventually reach the point at which $|\mathcal{J}| = 0$ such that

$$X_N^*(\delta) = \hat{X}_2^N(\delta_2) = \cdots = \hat{X}_n^N(\delta_n) > \hat{X}_1^N(C - \sum_{i=2}^{n} \delta_i) \tag{A.19}$$

and $\sum_{i=2}^{n} \delta_i = C$. The vector $\delta_b = (\delta_i)_{i=2}^{n}$ that satisfies these requirements is given by

$$\delta_i = \frac{D_i^N}{\sum_{j=2}^{n} D_j^N} C \tag{A.20}$$

for all $i \in \{2, \ldots, n\}$. In the other case, we also eventually reach the point at which $|\mathcal{J}| = 0$ such that (A.19) holds, but with $\sum_{i=2}^{n} \delta_i < C$.

Thus, all candidate solutions have been reduced to the vector δ_c, as given by (A.18), the vector δ_b, as given by (A.20), and any vector δ satisfying (A.19) with $\sum_{i=2}^{n} \delta_i < C$.

Let δ be such that it satisfies (A.19), that is,

$$\delta_k = \delta_2 \frac{D_k^N}{D_2^N}$$

for all $k = 3, \ldots, n$. The optimization problem then reduces to a one-dimensional optimization problem, which can be formulated as

$$\max_{\delta_2} \quad \delta_2^{-\beta} \left(\delta_2 \left(D_1^N \frac{\beta}{\beta - 1} + \sum_{i=2}^{n} D_i^N \right) - C D_2^N \right)$$

subject to

$$0 \leq \delta_2 \leq \frac{D_2^N}{\sum_{i=2}^{n} D_i^N} C.$$

The derivative of the objective function with respect to δ_2 is given by

$$\delta_2^{-\beta} \left(-\beta (\sum_{i \in N} D_i^N) + \sum_{i=2}^{n} D_i^N + \frac{C \beta D_2^N}{\delta_2} \right)$$

which is strictly positive for

$$\delta_2 < \frac{C \beta D_2^N}{\beta (\sum_{i \in N} D_i^N) - \sum_{i=2}^{n} D_i^N}$$

and strictly negative for

$$\delta_2 > \frac{C \beta D_2^N}{\beta (\sum_{i \in N} D_i^N) - \sum_{i=2}^{n} D_i^N}.$$

This implies that the unconstrained maximum is obtained at

$$\overline{\delta}_2(\beta) = \frac{C\beta D_2^N}{\beta(\sum_{i\in N} D_i^N) - \sum_{i=2}^n D_i^N}.$$

If $\overline{\delta}_2(\beta)$ is not feasible, which happens if

$$\beta < \frac{\sum_{i=2}^n D_i^N}{D_1^N},$$

then the constrained maximum is obtained at the boundary of the feasible set, given by δ_b. Furthermore, $\overline{\delta}_2(\beta)$ decreases with β and

$$\lim_{\beta\to\infty} \overline{\delta}_2(\beta) = \frac{D_2^N}{\sum_{i\in N} D_i^N} C,$$

which implies that δ_c cannot be an optimal solution.

Therefore, the vector $\delta^* = (\delta_i^*)_{i=2}^n$ that solves the optimization problem is given by

$$\delta_i^* = \min\left\{\frac{D_i^N}{\sum_{i=2}^n D_i^N}, \frac{\beta D_i^N}{\beta(\sum_{i\in N} D_i^N) - \sum_{i=2}^n D_i^N}\right\} C.$$

for all $i \in \{2, \ldots, n\}$, which corresponds to $\varepsilon^{N,1}(\beta)$.

Thus far we have shown that

$$V_1^N(X, \varepsilon^{N,1}(\beta)) > V_1^N(X, I^N)$$

for all $I^N \in \mathcal{I}^N$ with $I^N \neq \varepsilon^{N,1}(\beta)$ and all $X < X_N^*(\rho^N)$. What remains to be shown is

$$V_1^N(X, \varepsilon^{N,1}(\beta)) \geq V_1^N(X, I^N)$$

for all $I^N \in \mathcal{I}^N$ and all $X_N^*(\rho^N) \leq X \leq X_N^*(\varepsilon^{N,1}(\beta))$.

Let $I^N = (I_i^N)_{i\in N} \in \mathcal{I}^N$. We will first let I^N be such that $X_N^*(\rho^N) \leq X_N^*(I^N) \leq X_N^*(\varepsilon^{N,1}(\beta))$ and show that

$$V_1^N(X, \varepsilon^{N,1}(\beta)) \geq V_1^N(X, I^N)$$

for all $X_N^*(\rho^N) \leq X \leq X_N^*(\varepsilon^{N,1}(\beta))$. We subsequently show the same but for the case in which I^N is such that $X_N^*(I^N) > X_N^*(\varepsilon^{N,1}(\beta))$.

Let $I^N \neq \varepsilon^{N,1}(\beta)$ be such that $X_N^*(\rho^N) \leq X_N^*(I^N) \leq X_N^*(\varepsilon^{N,1}(\beta))$. Consider the following optimization problem:

$$\max_{\delta^N = (\delta_i^N)_{i\in N}} \quad V_1^N(X, \delta^N)$$

subject to

$$\delta^N \in \hat{\mathcal{I}}^N = \{\delta^N \in \mathcal{I}^N |\, X_N^*(I^N) \leq X_N^*(\delta^N)\},$$

in which $X < X_N^*(I^N)$. The feasible set of this optimization problem is a subset of the one considered before (i.e., $\hat{\mathcal{I}}^N \subseteq \mathcal{I}^N$). Moreover, the objective is similar as the investment has not taken place at X for all $\delta^N \in \hat{\mathcal{I}}^N$. However, it holds that $\varepsilon^{N,1}(\beta) \in \hat{\mathcal{I}}^N$ so that $\varepsilon^{N,1}(\beta)$ is again the optimal solution. Hence, we have

$$V_1^N(X, \varepsilon^{N,1}(\beta)) > V_1^N(X, \delta^N)$$

for all $\delta^N \in \hat{\mathcal{I}}^N$ with $\delta^N \neq \varepsilon^{N,1}(\beta)$ and all $X < X_N^*(I^N)$. In particular, as $I^N \in \hat{\mathcal{I}}^N$, we have that

$$V_1^N(X, \varepsilon^{N,1}(\beta)) > V_1^N(X, I^N) \tag{A.21}$$

for all $X < X_N^*(I^N)$. Inequality (A.21), continuity of $V_1^N(X, I^N)$ in X for all $I^N \in \mathcal{I}^N$, and the fact that $V_1^N(X, I^N)$ is strictly increasing in X for all $I^N \in \mathcal{I}^N$, imply that

$$V_1^N(X_N^*(I^N), \varepsilon^{N,1}(\beta)) \geq V_1^N(X_N^*(I^N), I^N). \tag{A.22}$$

It remains to show that

$$V_1^N(X, \varepsilon^{N,1}(\beta)) \geq V_1^N(X, I^N)$$

for all $X_N^*(I^N) < X \leq X_N^*(\varepsilon^{N,1}(\beta))$.

Consider $X = X_N^*(\varepsilon^{N,1}(\beta))$. We first show that $I_1^N \geq \varepsilon_1^{N,1}(\beta)$. To see this, recall that the investment proposal $\varepsilon^{N,1}(\beta)$ has the property that

$$X_N^*(\varepsilon^{N,1}(\beta)) = \hat{X}_i^N(\varepsilon_i^{N,1}(\beta)) > \hat{X}_1^N(\varepsilon_1^{N,1}(\beta)),$$

for all $i \in N \setminus \{1\}$. Consequently, $X_N^*(I^N) \leq X_N^*(\varepsilon^{N,1}(\beta))$ implies that $I_i^N \leq \varepsilon_i^{N,1}(\beta)$ for all $i \in N \setminus \{1\}$. Therefore,

$$I_1^N = C - \sum_{i \in N \setminus \{1\}} I_i^N \geq C - \sum_{i \in N \setminus \{1\}} \varepsilon_i^{N,1}(\beta) = \varepsilon_1^{N,1}(\beta).$$

Hence, for $X = X_N^*(\varepsilon^{N,1}(\beta))$, it holds that

$$V_1^N(X, \varepsilon^{N,1}(\beta)) = \frac{D_1^N X}{r - \mu} - \varepsilon_1^{N,1}(\beta) \geq \frac{D_1^N X}{r - \mu} - I_1^N = V_1^N(X, I^N). \tag{A.23}$$

The inequality follows from the fact that $I_1^N \geq \varepsilon_1^{N,1}(\beta)$. The functions $V_1^N(X, \varepsilon^{N,1}(\beta))$ and $V_1^N(X, I^N)$ are strictly increasing in X for all X. Furthermore, as we have shown before (see (A.22) and (A.23), respectively),

$$V_1^N(X_N^*(I^N), \varepsilon^{N,1}(\beta)) \geq V_1^N(X_N^*(I^N), I^N)$$

and

$$V_1^N(X_N^*(\varepsilon^{N,1}(\beta)), \varepsilon^{N,1}(\beta)) \geq V_1^N(X_N^*(\varepsilon^{N,1}(\beta)), I^N).$$

Thus, by continuity of the value function in X it must also hold that

$$V_1^N(X, \varepsilon^{N,1}(\beta)) \geq V_1^N(X, I^N)$$

for all $X_N^*(I^N) < X \leq X_N^*(\varepsilon^{N,1}(\beta))$.

Finally, let $I^N \neq \varepsilon^{N,1}(\beta)$ be such that $X_N^*(I^N) > X_N^*(\varepsilon^{N,1}(\beta))$. Consider the following optimization problem:

$$\max_{\delta^N = (\delta_i^N)_{i \in N}} \quad V_1^N(X, \delta^N)$$

subject to

$$\delta^N \in \hat{\mathcal{I}}^N = \{\delta^N \in \mathcal{I}^N |\, X_N^*(\varepsilon^{N,1}(\beta)) \leq X_N^*(\delta^N)\},$$

in which $X < X_N^*(\varepsilon^{N,1}(\beta))$. By applying similar arguments as before, it follows that $\varepsilon^{N,1}(\beta)$ is the optimal solution. Hence, we have

$$V_1^N(X, \varepsilon^{N,1}(\beta)) > V_1^N(X, \delta^N)$$

for all $\delta^N \in \hat{\mathcal{I}}^N$ with $\delta^N \neq \varepsilon^{N,1}(\beta)$ and all $X < X_N^*(\varepsilon^{N,1}(\beta))$. In particular, as $I^N \in \hat{\mathcal{I}}^N$, we have that

$$V_1^N(X, \varepsilon^{N,1}(\beta)) > V_1^N(X, I^N) \tag{A.24}$$

for all $X < X_N^*(\varepsilon^{N,1}(\beta))$. Moreover, inequality (A.24), continuity of $V_1^N(X, I^N)$ in X for all $I^N \in \mathcal{I}^N$, and the fact that $V_1^N(X, I^N)$ is strictly increasing in X for all $I^N \in \mathcal{I}^N$, imply that

$$V_1^N(X_N^*(\varepsilon^{N,1}(\beta)), \varepsilon^{N,1}(\beta)) \geq V_1^N(X_N^*(\varepsilon^{N,1}(\beta)), I^N).$$

□

Proof of Theorem 4.2. Without loss of generality, let $k = 1$. Then, for all $S \subset N$ with $S \not\ni \{1\}$, it holds that $D^S = 0$ so that $I^S = 0$ and $V_i^S(X, 0) = 0$ for all $i \in S$ and all X. Therefore, the prioritized investment proposal $\varepsilon^{N,1}(\beta)$ is stable with respect to coalitions $S \subset N$ with $S \not\ni \{1\}$.

Let $S \subset N$ with $S \ni \{1\}$, and consider $I^S = (I_i^S)_{i \in S} \in \mathcal{I}^S$. Define $\tilde{I}^N = (\tilde{I}_i^N)_{i \in N} \in \mathcal{I}^N$ by

$$\tilde{I}_i^N = \begin{cases} I_i^S \dfrac{D_i^N}{D_i^S} & \text{for all } i \in S, \\ 0 & \text{for all } i \notin S. \end{cases}$$

We proceed by showing that under this choice of $\tilde{I}^N$ it holds that

$$V_1^N(X, \tilde{I}^N) \geq V_1^S(X, I^S)$$

for all X.

The choice of $\tilde{I}^N$ is such that its corresponding investment threshold is equal to that of I^S, that is,

$$X_N^*(\tilde{I}^N) = \left(\max_{i\in N} \frac{\tilde{I}_i^N}{D_i^N}\right) \frac{\beta}{\beta-1}(r-\mu) = \left(\max_{i\in S} \frac{I_i^S}{D_i^S}\right) \frac{\beta}{\beta-1}(r-\mu) = X_S^*(I^S).$$

Then,

$$V_1^N(X_N^*(\tilde{I}^N), \tilde{I}^N) - V_1^S(X_S^*(I^S), I^S) = (D_1^N - D_1^S)\left(\frac{\beta}{\beta-1}\left(\max_{i\in S} \frac{I_i^S}{D_i^S}\right) - \frac{I_1^S}{D_1^S}\right) \geq 0,$$

in which the inequality follows from the assumption that $D_1^N \geq D_1^S$. Moreover, using this result, we find that, for $X < X_N^*(\tilde{I}^N) = X_S^*(I^S)$, it holds that

$$\begin{aligned} V_1^N(X, \tilde{I}^N) &= \left(\frac{X}{X_N^*(\tilde{I}^N)}\right)^\beta V_1^N(X_N^*(\tilde{I}^N), \tilde{I}^N) \\ &\geq \left(\frac{X}{X_S^*(I^S)}\right)^\beta V_1^S(X_S^*(I^S), I^S) \\ &= V_1^S(X, I^S), \end{aligned}$$

and for $X > X_N^*(\tilde{I}^N) = X_S^*(I^S)$ it holds that $V_1^N(X, \tilde{I}^N) \geq V_1^S(X, I^S)$ because

$$\frac{\partial V_1^N(X, \tilde{I}^N)}{\partial X} = \frac{D_1^N}{r-\mu} \geq \frac{D_1^S}{r-\mu} = \frac{\partial V_1^S(X, I^S)}{\partial X}$$

for $X \geq X_N^*(\tilde{I}^N) = X_S^*(I^S)$. Thus,

$$V_1^N(X, \tilde{I}^N) \geq V_1^S(X, I^S) \tag{A.25}$$

for all X.

Consequently, for all $X \leq X_N^*(\varepsilon^{N,1}(\beta))$, it holds that

$$V_1^N(X, \varepsilon^{N,1}(\beta)) \geq V_1^N(X, \tilde{I}^N) \geq V_1^S(X, I^S).$$

The first inequality follows from Theorem 4.1; the second inequality follows from (A.25). Hence, $\varepsilon^{N,1}(\beta)$ is stable for all β and all $X \leq X_N^*(\varepsilon^{N,1}(\beta))$.

Finally, if $\beta < \frac{\sum_{i\in N\setminus\{1\}} D_i^N}{D_1^N}$, then $\varepsilon_1^{N,1}(\beta) = 0$. Consequently, for all $X > X_N^*(\varepsilon^{N,1}(\beta))$, it also holds that

$$V_1^N(X, \varepsilon^{N,1}(\beta)) = \frac{D_1^N X}{r-\mu} - \varepsilon_1^{N,1}(\beta) \geq \frac{D_1^N X}{r-\mu} - \tilde{I}_1^N = V_1^N(X, \tilde{I}^N) \geq V_1^S(X, I^S).$$

The first inequality follows from the fact that $\tilde{I}_1^N \geq 0 = \varepsilon_1^{N,1}(\beta)$; the second inequality follows from (A.25). Hence, $\varepsilon^{N,1}(\beta)$ is dynamically stable if $\beta < \frac{\sum_{i\in N\setminus\{1\}} D_i^N}{D_1^N}$. □

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