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Publication date:
2023

Document Version
Early version, also known as pre-print

No. 2023-021

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31 August 2023

ISSN 0924-7815
ISSN 2213-9532

Induced Rules for Minimum Cost Spanning Tree Problems: towards Merge-proofness and Coalitional Stability

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Abstract

This paper examines cost allocation rules for minimum cost spanning tree (MCST) problems, focusing on the properties of merge-proofness and coalitional stability. Merge-proofness ensures that no coalition of agents has the incentive to merge before participating in the cost allocation process. On the other hand, coalitional stability ensures that no coalition has the incentive to withdraw from the cost allocation process after the cost allocation proposal is made. We propose a novel class of rules called induced rules, which are derived recursively from base rules designed for two-person MCST problems. We demonstrate that induced rules exhibit both merge-proofness and coalitional stability within a restricted domain, provided that the corresponding base rules satisfy specific conditions.

JEL classification: C71;D61;D79
Keywords: Minimum cost spanning tree problems; Cooperative games; Induced cost allocation rules; Merge-proofness; Coalitional stability.

1 Introduction

The study of minimum cost spanning tree (MCST) problems is motivated by situations where there is a set of agents and a source which supplies some good or service that all agents need. Every agent can choose to connect to the source directly or indirectly through cooperating with other agents. For each connection, there is a related cost. The overall cost to connect all agents to the source is minimized via an MCST, which is a spanning tree with minimum cost. Finding an MCST for a given instance can be done in polynomial time by algorithms proposed in Kruskal (1956) and Prim (1957). A natural associated question is how to allocate the minimum total cost among the agents in a fair and stable way. Various cost allocation rules are proposed and studied in the literature regarding this issue.

A cost allocation rule assigns to every MCST problem within a certain domain an allocation vector among the agents. Many of the rules in the literature are characterized in a cooperative game-theoretic way. In Bird (1976), a construct and charge rule that fits with the Prim-Dijkstra (Prim (1957)) is proposed. An MCST problem is associated with a cooperative MCST game with transferable utility where the cost related to each coalition S is equal to the minimal cost to connect them

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to the source, without cooperating with agents outside of S. Bird's rule always selects an allocation vector in the core of that game. The core and the nucleolus (Schmeidler (1969)) of the MCST game are further studied in Granot and Huberman (1984). Kar (2002) looks into the Shapley value (Shapley (1953)) of the MCST game as a rule for MCST problems. The folk solution, also known as the equal remaining obligation rule, is proposed in Feltkamp et al. (1994). It is later called the Potters-value in Branzei et al. (2003) and can be interpreted as the Shapley value of the irreducible MCST game in Bergantiños and Vidal-Puga (2007)). Some follow-up works on the folk solution include Bogomolnaia and Moulin (2010) and Norde (2019). The folk solution decides how to divide the cost of an edge among the agents as it is chosen by Kruskal's algorithm (Kruskal (1956)), and the allocated cost of an agent is determined when the algorithm terminates. Different rules are associated with different properties, while each property represents a kind of understanding of fair allocations.

This paper focuses on *merge-proofness*, as first considered in the MCST context in Gómez-Rúa and Vidal-Puga (2011). This property is related to a specific type of coalitional consideration: Agents in set N plan to allocate the cost of the MCST that connects them to the source according to a specific rule. The agents in a coalition $S \subseteq N$ may merge together before they participate in the cost allocation procedure. In other words, the agents in coalition S can first connect themselves to each other, and let one of them enter the cost allocation procedure under the same rule. Since the agents in S are already internally connected, they can all access the source as long as the single representative agent does. Within this merger, the cost born by the agents in S consists of two parts: the cost needed for agents in S to be connected internally and the cost allocated to the single representative agent under the specific rule in the new instance where the subset S becomes a single agent. The agents in set S will be incentivized to merge in advance if they are charged less in this way. A rule is called merge-proofness if it can provide cost allocations that prevent such coalitional considerations. Merge-proofness of the cost allocation rules for MCST problems is studied in Ozsoy (2006), Gómez-Rúa and Vidal-Puga (2011) and Gómez-Rúa and Vidal-Puga (2017). Among other things, Bird's rule has been characterized on the basis of a set of properties including merge-proofness.

Merge-proofness is related to some well-studied properties for solutions to other types of cost allocation problems. Among them, there are the no-advantageous-merging property for bankruptcy problems (O'Neill (1982), Moulin (1987)), the group strategy-proofness property for social choice problems (Barberà et al. (2012), Manjunath (2012)), and the merge-proofness property for taxation problems (Ju and Moreno-Ternero (2011)) and scheduling problems (Moulin (2008)).

Another widely studied property related to coalitional considerations is *coalitional stability*. In an MCST context, a rule is said to be coalitionally stable if it provides cost allocation vectors that belong to the core of the corresponding cooperative MCST games. In other words, under a coalitionally stable rule, no group of agents would pay more than the minimum cost needed to connect them to the source using internal edges only. A distinct difference between merge-proofness and coalitional stability is that merge-proofness can prevent any coalition of agents from merging into a single agent *before* participating in cooperation, while a rule which satisfies coalitional stability would avoid any coalitional deviations from cooperation *after* the cost allocation proposal is made.

In general, merge-proofness is difficult to obtain. Ozsoy (2006) shows that there is no merge-proof rule on the domain of all MCST problems. In Gómez-Rúa and Vidal-Puga (2011), Bird's rule is found to be the only rule that satisfies merge-proofness, coalitional stability and independence of extreme null points on the domain of MCST problems which allow for a unique MCST and any two edges in the MCST have different costs. In comparison with Gómez-Rúa and Vidal-Puga (2011), the current paper focuses on a smaller domain that allows for a richer class of rules that satisfy merge-proofness and coalitional stability.

This paper introduces a new class of cost allocation rules for MCST problems: *induced rules* which are derived from *base rules* that are defined for two person MCST problems. For every MCST problem, the induced allocation vector is computed recursively and ensures that the sum of costs allocated to an agent together with all of its followers is equal to the cost they have to bear if they form a coalition in advance. Another interpretation of induced rules based on the concept of *compensation value functions* is provided. This leads to a direct, non-recursive characterization of induced rules.

By design, any induced rule prevents mergers between coalitions of agents that are connected to each other through edges in the MCST. Yet more conditions are needed to produce a merge-proof induced rule. Interestingly, it is found that a variety of induced rules satisfy merge-proofness on the domain of MCST problems which allow for a unique MCST, with all edges having different costs and with exactly one agent directly connected to the source, such that every edge not in the MCST has a cost larger than or equal to the cost of the unique edge in the MCST connected to the source. We prove that this class of induced rules is induced by base rules whose compensation values are in between those of Bird's rule and the folk solution. Furthermore, it is shown that all such induced rules would satisfy coalitional stability as well.

The paper is organized as follows. In section 2, we present the notions of MCST problems and corresponding cost allocation rules. In section 3, the definitions of merge-proofness and coalitional stability properties are provided. In section 4, induced rules are introduced using base rules. A recursive procedure and a direct characterization of the rules on the basis of compensation value functions are proposed. In section 5, the induced rules that satisfy both merge-proofness and coalitional stability are presented. Some technical results about MCST problems with fixed edge sets are provided in an appendix.

2 MCST Problems and Rules

2.1 MCST problems

An MCST problem studies the situation where there is a finite group N of agents and a source 0 to which each agent needs to be connected. Each agent is able to connect to the source directly or indirectly by cooperating with other agents. Define the node set $N_0 = N \cup \{0\}$, where agents and the source are all regarded as nodes, and the edge set $E^{N_0} = \{\{i,j\} \mid i,j \in N_0 \text{ and } i \neq j\}$, where each edge represents a connection that can be built. There is a cost function $c : E^{N_0} \to \mathbb{R}_+$, where $c(\{i,j\})$ is the cost of building the connection between nodes i and j. For simplicity, we use c_{ij} or $c_{i,j}$ to denote $c(\{i,j\})$. In short, an MCST problem is identified by a triple $(N,0,c)$. The set of all MCST problems is denoted by $\mathcal{M}$. Then we use $\mathcal{M}^* \subseteq \mathcal{M}$ to denote the set of MCST problems for which there exists a unique MCST, and $\mathcal{M}_0^* \subseteq \mathcal{M}^*$ to denote the set of MCST problems for which there is only one node connected to the source in the unique MCST.

Given an MCST problem $(N,0,c)$, the minimal cost to connect all agents in N to the source 0 is given by:

$$m(N,0,c) = \min\Big\{\sum_{e\in E} c(e) \mid (N_0,E) \text{ is a connected graph}\Big\},$$

where the minimum will be obtained in a tree called an MCST.

Similarly, for a coalition $S \subseteq N$, define $S_0 = S \cup \{0\}$. The cost of aan MCST connecting all agents in S to the source 0 is given by:

$$m(S,0,c) = \min\Big\{\sum_{e\in E} c(e) \mid (S_0,E) \text{ is a connected graph}\Big\}.$$

The minimum cost for a coalition $S \subseteq N$ to connect themselves, is denoted by $M(S,c)$:

$$M(S,c) = \min\{\sum_{e\in E} c(e) \mid (S,E) \text{ is a connected graph}\}.$$

Note that we have $M(S,c) = 0$ if $|S| = 1$.

Several algorithms can solve an MCST problem in polynomial time, such as Kruskal's algorithm, the reverse-delete algorithm (Kruskal (1956)) and Prim's algorithm (Prim (1957)).

2.2 Cost allocation rules

A cost allocation rule ψ on domain $D \subseteq \mathcal{M}$ assigns to each MCST problem $(N,0,c) \in D$ a cost allocation vector $\psi(N,0,c) \in \mathbb{R}^N$.

Here we present two widely studied cost allocation rules in the literature of MCST problems, Bird's rule (Bird (1976)) and the folk solution (Feltkamp et al. (1994), Bergantiños and Vidal-Puga (2007)).

1. Bird's rule B is defined on $\mathcal{M}^*$ as follows.

   Take $(N,0,c) \in \mathcal{M}^*$, let $\Gamma = (N_0,E)$ be the MCST for $(N,0,c)$ and let $i \in N$. Let $p_\Gamma(i) \in N$ be the immediate predecessor of i on the path from i to 0 in Γ. The cost allocated to player i according to Bird's rule is given by the cost of the edge between i and $p_{\Gamma(i)}$:

   $$B_i(N,0,c) = c_{p_\Gamma(i),i}$$

   For a 2-person MCST problem with $N = \{1,2\}$ and a unique MCST $\Gamma = (N_0, \big\{\{0,1\},\{1,2\}\big\})$, this leads to:

   $$B_1(N,0,c) = c_{01},$$
   $$B_2(N,0,c) = c_{12}.$$

2. The folk solution F on $\mathcal{M}$ can be defined in multiple ways with various interpretations. Among others, the allocation vector $F(N,0,c)$ for $(N,0,c) \in \mathcal{M}$ equals the Shapley value of the irreducible cost game associated with $(N,0,c)$ (cf. Bergantiños and Vidal-Puga (2007)).

   For a 2-person MCST problem $(N,0,c) \in \mathcal{M}$ with $N = \{1,2\}$ and an MCST $\Gamma = (N_0, \big\{\{0,1\},\{1,2\}\big\})$, this leads to:

   $$F_1(N,0,c) = \begin{cases} c_{01}, & \text{if } c_{01} \le c_{12} \\ \frac{1}{2}c_{01} + \frac{1}{2}c_{12} & \text{if } c_{01} > c_{12} \end{cases}$$

   and

   $$F_2(N,0,c) = \begin{cases} c_{12}, & \text{if } c_{01} \le c_{12} \\ \frac{1}{2}c_{01} + \frac{1}{2}c_{12} & \text{if } c_{01} > c_{12}. \end{cases}$$

3 Merge-proofness and Coalitional Stability

In this section, we introduce the properties of MCST rules which are the main focus of this paper: merge-proofness and coalitional stability. We first illustrate the concept of mergers, then present the definitions of the properties merge-proofness and coalitional stability and finally, we discuss the difficulty of obtaining merge-proofness for cost allocation rules.

The concept of mergers is introduced to describe the cooperative incentives of a group of agents to form a coalition in advance and to be treated as a single agent. This kind of incentives is different from the one related to coalitional stability (known as the core selection property in Bergantiños and Vidal-Puga (2007)): a cost allocation rule is coalitionally stable if, after the allocation rule has been applied, no group of agents has an incentive to only connect themselves to the source.

A merger takes place between a set of agents $S \subseteq N$ before they participate in the cost allocation procedure. After merging together, the set S of agents are represented by a single agent m^S before they enter the cost allocation procedure. In the context of the resulting MCST problem after the merger, the new agent set is $(N \backslash S) \cup \{m^S\}$ and the new cost function c^S is defined as:

$$c^S_{i,j} = \begin{cases} c_{i,j}, & \text{if } i, j \in N_0 \backslash S \\ \min_{k \in S} c_{k,j} & \text{if } i = m^S \text{ and } j \in N_0 \backslash S. \end{cases} \tag{3.1}$$

Clearly, $((N \backslash S) \cup \{m^S\}, 0, c^S)$ is the remaining MCST problem after the merger. An example is presented below to better illustrate mergers and the new cost functions after mergers.

Example 3.1. Consider an MCST problem $(N, 0, c)$ with $N = \{1, 2, 3, 4\}$ as shown on the left hand side of Figure 3.1, with the costs represented by the numbers on the edges. Now suppose agents 1, 3 and 4 merge together, and are represented by m^S, where $S = \{1, 3, 4\}$. The resulting MCST problem $((N \backslash S) \cup \{m^S\}, 0, c^S)$ after the merger is shown on the right hand side of Figure 3.1. ◁

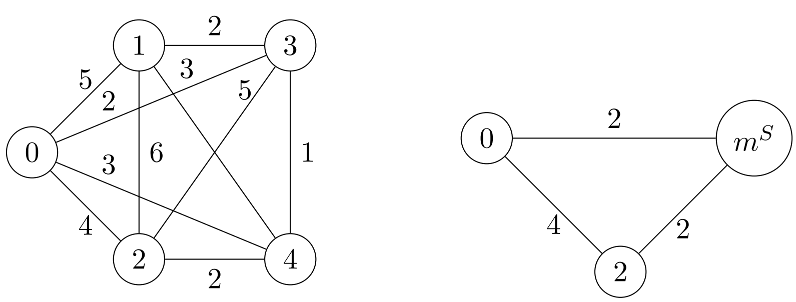

Figure 3.1: $(N, 0, c)$ and $(N \backslash S \cup \{m^S\}, 0, c^S)$ in Example 3.1

Mergers can happen simultaneously. Consider disjoint agent sets $S_1, ..., S_l$ ($l \in \mathbb{N}$). After the simultaneous mergers of these sets, let the set S_k ($k \in \{1, ..., l\}$) be represented by a single agent m^{S_k}. The remaining MCST problem after the mergers is denoted by $([N \backslash (\cup_{k \in \{1,...,n\}} S_k)] \cup \{m^{S_1}, ..., m^{S_l}\}, 0, c^{S_1,...,S_l})$, where the new cost function $c^{S_1,...,S_l}$ is defined as:

$$c^{S_1,...,S_l}_{i,j} = \begin{cases} c_{i,j}, & \text{if } i, j \in N_0 \backslash (\cup_{k \in \{1,...l\}} S_k) \\ \min_{k \in S_u} c_{k,j}, & \text{if } j \in N_0 \backslash (\cup_{k \in \{1,...l\}} S_k) \text{ and } i = m^{S_u} \text{ for some } u \in \{1, ..., l\} \\ \min_{k \in S_u, l \in S_v} c_{k,l} & \text{if } i = m^{S_u}, j = m^{S_v} \text{ for some } u, v \in \{1, ..., l\}. \end{cases} \tag{3.2}$$

In the notation $c^{S_1,\ldots S_l}$, we assume that all players $i \in N \backslash (\cup_{k=1}^{l} S_k)$ remain singletons. It could also be that some set S_k, $k \in \{1, ..., l\}$, consists of a singleton. For a singleton merger w.r.t. $i \in N$, the notation $m^{\{i\}}$ and $\{i\}$ are used interchangeably.

A property of MCST allocation rules that prevents mergers among groups of agents is called merge-proofness and is defined in Gómez-Rúa and Vidal-Puga (2011). The merge-proofness property is always defined on a domain of problems that is closed under mergers, i.e., for any MCST problem in this domain, any possible remaining MCST problem after merging remains in the domain.

Definition (Merge-proofness). Let D be a domain of MCST problems that is closed under mergers. A rule ψ satisfies merge-proofness on D if for all MCST problems $(N, 0, c) \in D$ and for all $S \subseteq N$,

$$\sum_{i \in S} \psi_i(N, 0, c) \leq \psi_{m^S}((N \backslash S) \cup \{m^S\}, 0, c^S) + M(S, c). \tag{3.3}$$

On the left hand side of inequality (3.3) is the sum of costs allocated to agents in the set S, which is the total cost carried by the agents in S in the original problem without the merger between agents in S. On the right hand side is the minimum cost needed for agents in S to connect themselves internally, plus the cost allocated to the single agent that represents S in the remaining problem after the agents in S have merged. Thus the right hand side is the total cost assigned to the agents in S if they choose to form a coalition in advance. The inequality (3.3) indicates that the set S of agents can not be better off if they decide to conduct a merger.

The merge-proofness property is not easy to obtain. On the general domain of all MCST problems, no cost allocation rule is merge-proof (Ozsoy (2006)). In Gómez-Rúa and Vidal-Puga (2011), an example is presented to show that most cost allocation rules in the literature do not satisfy merge-proofness, even on the domain of MCST problems where there is a unique minimum cost spanning tree. Among the rules considered, there are the Shapley value (Kar (2002)), Dutta and Kar's rule (Dutta and Kar (2004)), the nucleolus (Granot and Huberman (1984)), and the folk solution (Feltkamp et al. (1994), Bergantiños and Vidal-Puga (2007)).

Next we turn to coalitional stability.

Definition (Coalitional Stability). Let D be some domain of MCST problems. A rule ψ satisfies coalitional stability on D if for any MCST problem $(N, 0, c) \in D$ and any subset $S \subseteq N$,

$$\sum_{i \in S} \psi_i(N, 0, c) \leq m(S, 0, c).$$

Bird's rule satisfies merge-proofness on the domain of MCST problems for which there is a unique MCST and any two edges in the MCST have different cost values. In fact, Gómez-Rúa and Vidal-Puga (2011) show that on the domain where no two edges in all possible MCST's have the same cost, Bird's rule is the only rule satisfying merge-proofness, coalitional stability and the property of independence of extreme null points which we will not discuss in detail.

4 Induced Rules

This section introduces a new type of rules called induced rules, which are defined on $\mathcal{M}^*$ (the domain of MCST problems with a unique MCST). They are called induced rules because they are induced by base rules for 2-person MCST problems. The allocation vector according to an induced rule is determined recursively and is designed to ensure that the sum of costs allocated to a node and all its followers in the unique MCST (w.r.t. the source) is equal to the cost they have to carry if they form a coalition in advance. The concepts of base rules and corresponding compensation value functions are introduced in section 4.1. The details of the recursive procedure are shown in section 4.2, in which we also discuss the characterization of induced rules based on compensation value functions.

4.1 Base rules

Let $\mathcal{M}_2^* \subseteq \mathcal{M}^*$ be the domain of all MCST problems $(N, 0, c)$ with $|N| = 2$, for which there is a unique MCST.

Definition 4.1. Let ϕ be a cost allocation rule on $\mathcal{M}_2^*$. Then ϕ is called a base rule if

1. For all $(N, 0, c) \in \mathcal{M}_2^*$ with $N = \{1, 2\}$ and a unique MCST $\Gamma = (N_0, \{\{0, 1\}, \{0, 2\}\})$, we have
$$\phi_1(N, 0, c) = c_{01} \text{ and } \phi_2(N, 0, c) = c_{02}.$$
2. For all $(N, 0, c) \in \mathcal{M}_2^*$ with $N = \{1, 2\}$ and a unique MCST $\Gamma = (N_0, \{\{0, 1\}, \{1, 2\}\})$, and for all $(N, 0, c') \in \mathcal{M}_2^*$ with $N = \{1, 2\}$ such that $c'_{01} = c_{01}$, $c'_{12} = c_{12}$ and a unique MCST $(N_0, \{\{0, 1\}, \{1, 2\}\})$, we have
$$\phi(N, 0, c) = \phi(N, 0, c').$$

Focusing on 2-person MCST problems with a unique MCST, when the connection between agents is not included in the MCST, the two agents do not need to cooperate in order to build the most efficient network. In this case, any base rule will assign to each agent the costs of the connection between the source and himself according to condition 1. Condition 2 requires that the allocation provided by any base rule only depends on the MCST, and not on the exact cost of the edge not included in the MCST.

From the expressions in Section 2.2, one can directly see that Bird's rule and the folk solution restricted on $\mathcal{M}_2^*$ are both base rules. In Example 4.1, we show that the Shapley value restricted to $\mathcal{M}_2^*$ is not a base rule.

Example 4.1. Two 2-person MCST problems are presented in Figure 4.1 Both problems have a unique MCST composed of edges $\{0, 1\}$ and $\{1, 2\}$. The allocation vectors provided by Shapley value are given by (2,3) and (-1,6) respectively.

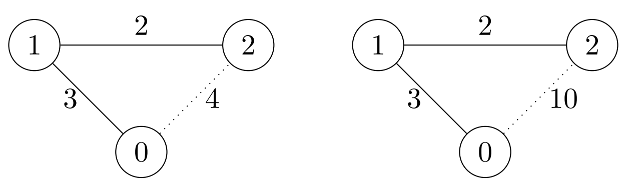

Figure 4.1: The two MCST problems in Example 4.1

◁

For the purpose of getting new *merge-proof* rules, some rules other than those studied in the literature can be used as base rules. As an illustration, the quarter rule is presented in Example 4.2.

Example 4.2. The quarter rule Q for any 2-person MCST problem $(N, 0, c)$ with $N = \{1, 2\}$ and a unique MCST $\Gamma = (N_0, \{\{0, 1\}, \{1, 2\}\})$ is given by:

$$Q_1(N, 0, c) = \begin{cases} c_{01} & \text{if } c_{01} \leq c_{12} \\ c_{01} - \frac{1}{4}(c_{01} - c_{12}) & \text{if } c_{01} > c_{12} \end{cases}$$

$$Q_2(N, 0, c) = \begin{cases} c_{12} & \text{if } c_{01} \leq c_{12} \\ c_{12} + \frac{1}{4}(c_{01} - c_{12}) & \text{if } c_{01} > c_{12}. \end{cases}$$

Moreover, for any $(N, 0, c)$ with $N = \{1, 2\}$ and a unique MCST $\Gamma = (N_0, \{\{0, 1\}, \{0, 2\}\})$, the quarter rule is given by:

$$Q_1(N, 0, c) = c_{01} \text{ and } Q_2(N, 0, c) = c_{02}.$$

Clearly, the quarter rule is a base rule.

$\triangleleft$

Next, we present another way to represent base rules by introducing the concept of compensation value function.

A base rule ϕ on $\mathcal{M}_2^*$ can be identified with a *compensation value function* $cv^\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that for all MCST problems $(N, 0, c) \in \mathcal{M}_2^*$ with $N = \{1, 2\}$ and a unique MCST $(N_0, \{\{0, 1\}, \{1, 2\}\})$,

$$\phi_1(N, 0, c) = c_{01} - cv^\phi(c_{01}, c_{12}),$$

and

$$\phi_2(N, 0, c) = c_{12} + cv^\phi(c_{01}, c_{12}).$$

Note that, for the Bird's rule B, restricted to $\mathcal{M}_2^*$,

$$cv^B(a, b) = 0 \text{ for all } a \geq 0, b \geq 0,$$

while, for the folk solution F, restricted to $\mathcal{M}_2^*$,

$$cv^F(a, b) = \frac{1}{2} \max\{a - b, 0\} \text{ for all } a \geq 0, b \geq 0.$$

4.2 Induced rules

Given a base rule ϕ on $\mathcal{M}_2^*$, we obtain its corresponding induced rule I^ϕ on $\mathcal{M}^*$ through the following recursive procedure.

Consider an MCST problem $(N, 0, c) \in \mathcal{M}^*$ with a unique MCST Γ. Take $SC(\Gamma)$ as the source connection set, which consists of all the nodes directly connected to the source in Γ. For $i \in N$, let $F(i)$ denote the set of the followers of i in Γ w.r.t. the source 0 and denote $V_i = \{i\} \cup F(i)$. The allocation $I^\phi(N, 0, c)$ will be determined branch by branch.

Fix $s \in SC(\Gamma)$ and consider the restricted MCST problem $(V_s, 0, c_s)$ on the branch of Γ corresponding to s. Here $c_s(\{i, j\}) = c(\{i, j\})$ for any $i, j \in V_s \cup \{0\}$ with $i \neq j$.

Let $i \in V_s$. The idea of how to obtain $I_i^\phi(N,0,c)$ in a recursive way is the following. When i is considered in the procedure, he will take the role of *representative* for the set V_i consisting of himself and all of his followers, under the assumption that all of his followers have already been considered earlier in the procedure, i.e., for all $j \in F(i)$, $I_j^\phi(N,0,c)$ has already been determined.

First of all, agent i will be an *internal representative* in taking care of connecting all agents in V_i internally, making use of the already established contributions $I_j^\phi(N,0,c)$ for $j \in F(i)$. This will lead to internal representation costs $r_i^\phi(N,0,c)$ for agent i given by

$$r_i^\phi(N,0,c) = M(V_i,c) - \sum_{j\in F(i)} I_j^\phi(N,0,c).$$

Secondly, agent i will be an *external representative* for V_i, in the negotiations with $V_s \backslash V_i$ about allocating the costs of the part of Γ that is within the branch including s. In the negotiation, we assume $V_s \backslash V_i$ to act as one merger. If $V_s \backslash V_i \neq \varnothing$, the external allocation problem is a 2-person allocation problem to which we apply our base rule ϕ. If $V_s \backslash V_i = \varnothing$, or equivalently if $i = s$, then i will carry the costs of connecting himself to the source 0. This will lead to external representation costs $R_i^\phi(N,0,c)$ for agent i given by

$$R_i^\phi(N,0,c) = \begin{cases} \phi_{m^{V_i}}(\{m^{V_i}, m^{V_s\backslash V_i}\}, 0, c^{V_i, V_s\backslash V_i}) & \text{if } V_s\backslash V_i \neq \varnothing \\ c_{0,s} & \text{if } V_s\backslash V_i = \varnothing. \end{cases}$$

Finally, we define

$$I_i^\phi(N,0,c) = r_i^\phi(N,0,c) + R_i^\phi(N,0,c),$$

as the sum of the internal representation costs $r_i^\phi(N,0,c)$ and external representation costs $R_i^\phi(N,0,c)$.

To illustrate the computation of the cost allocation vectors prescribed by an induced rule, we present two examples.

Example 4.3. Consider an MCST problem $(N,0,c) \in \mathcal{M}^*$ with $N = \{1,2,...,6\}$ and a unique MCST $\Gamma = (N_0, E)$ as shown in Figure 4.2. In this example, the folk solution F is used as the base rule. Its corresponding induced rule is denoted by I^F.

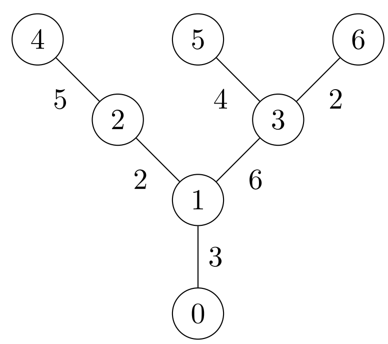

Figure 4.2: the MCST Γ for the MCST problem in Example 4.3

For end node 6, obviously the internal representation costs $r_6^\phi(N,0,c)$ are equal to 0. To determine the external representation cost $R_6^\phi(N,0,c)$ we consider the 2-person MCST problem as provided in Figure 4.3.

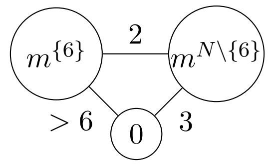

Figure 4.3: the 2-person MCST problem for node 6 in Example 4.3

The costs of the edges between $m^{\{6\}}$ and $m^{N\backslash\{6\}}$ and between 0 and $m^{N\backslash\{6\}}$ follow from the uniqueness of the MCST Γ. Since Γ is the unique MCST for $(N, 0, c)$, the costs of the edge between 0 and $m^{\{6\}}$ are strictly higher than the maximal cost of an edge on the unique path in Γ from 0 to 6, given by $\max\{c_{01}, c_{13}, c_{36}\} = 6$. Applying the folk solution as a base rule to this 2-person MCST problem we obtain that

$$R_6^F(N, 0, c) = 2\frac{1}{2},$$

and hence that

$$I_6^F(N, 0, c) = r_6^F(N, 0, c) + R_6^F(N, 0, c) = 2\frac{1}{2}.$$

In a similar way, for end nodes 4 and 5, we obtain

$$I_4^F(N, 0, c) = 5 \text{ and } I_5^F(N, 0, c) = 4.$$

Next consider node 3. Clearly

$$r_3^F(N, 0, c) = M(\{3, 5, 6\}, c) - I_5^F(N, 0, c) - I_6^F(N, 0, c) = 6 - 4 - 2\frac{1}{2} = -\frac{1}{2},$$

where $M(\{3, 5, 6\}, c) = 6$, again because Γ is the unique MCST for $(N, 0, c)$.

To determine $R_3^F(N, 0, c)$, consider the 2-person MCST problem as provided in Figure 4.4

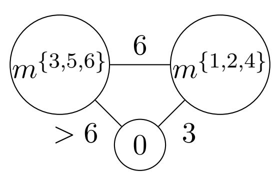

Figure 4.4: the 2-person MCST problem for node 3 in Example 4.3

Hence

$$R_3^F(N, 0, c) = 6,$$

and, consequently,

$$I_3^F(N, 0, c) = 6 - \frac{1}{2} = 5\frac{1}{2}.$$

In a similar way, one finds that

$$I_2^F(N, 0, c) = 2\frac{1}{2}.$$

Finally, consider the source connection node 1. Clearly

$$r_1^F(N, 0, c) = M(N, c) - \sum_{j=2}^{6} I_j^F N, 0, c = 19 - 19\frac{1}{2} = -\frac{1}{2}.$$

Moreover, $R_1^F(N,0,c) = c_{01} = 3$. Hence

$$I_1^F(N,0,c) = -\frac{1}{2} + 3 = 2\frac{1}{2}.$$

In summary, the induced rule based on the folk solution for $(N,0,c)$ leads to the allocation vector

$$I^F(N,0,c) = (2\frac{1}{2}, 2\frac{1}{2}, 5\frac{1}{2}, 5, 4, 2\frac{1}{2}).$$

The cost allocation vector obtained by the direct application of the folk solution on $(N,0,c)$ is given by

$$F(N,0,c) = (2\frac{1}{2}, 2\frac{1}{2}, 3\frac{2}{3}, 5, 4\frac{2}{3}, 3\frac{2}{3}).$$

Note that $F(N,0,c) \neq I^F(N,0,c)$.

$\lhd$

Example 4.4. Reconsider the MCST problem in Example 4.3. Performing a similar analysis with the quarter rule Q as a base rule we find that

$$r^Q(N,0,c) = (-\frac{1}{4}, 0, -\frac{1}{4}, 0, 0, 0),$$

and

$$R^Q(N,0,c) = (3, 2\frac{1}{4}, 6, 5, 4, 2\frac{1}{4}),$$

and hence that

$$I^Q(N,0,c) = (2\frac{3}{4}, 2\frac{1}{4}, 5\frac{3}{4}, 5, 4, 2\frac{1}{4}).$$

$\lhd$

Next, we provide an alternative direct, non-recursive, interpretation of an induced rule I^ϕ based on the description of a base rule via compensation value functions.

Consider an MCST problem $(N,0,c) \in \mathcal{M}^*$ with a unique MCST Γ. For all $i \in N \backslash SC(\Gamma)$, let $p(i) \in N$ denote the immediate predecessor of $i \in N$ w.r.t. the source. Moreover, let $IF(i)$ be the set of immediate followers of i in Γ. Finally, for all $i \in N \backslash SC(\Gamma)$, let $sc(i)$ denote the unique source connection node such that $i \in F(s(i))$.

The compensation value function interpretation of induced rules is stated in Theorem 4.1. In the proof, we use three lemmas. Lemma 4.1 indicates that the sets V_i, $i \in N$, play a special role in the recursive procedure.

Lemma 4.1. Let $(N,0,c) \in \mathcal{M}^*$ be an MCST problem with a unique MCST Γ and let ϕ be a base rule. Then, for all $i \in N$,

1. $M(V_i,c) = \sum_{j \in IF(i)} M_j(V_j,c) + \sum_{j \in IF(i)} c_{i,j}$.
2. $\sum_{j \in V_i} I_j^\phi(N,0,c) = R_i^\phi(N,0,c) + M(V_i,c)$.

Proof. The first statement holds trivially due to the uniqueness of the MCST. Next, let $i \in N$. The second statement follows from

$$\begin{aligned} I_i^\phi(N,0,c) &= r_i^\phi(N,0,c) + R_i^\phi(N,0,c) \\ &= M(V_i,c) - \sum_{j \in F(i)} I_j^\phi(N,0,c) + R_i^\phi(N,0,c), \end{aligned}$$

which implies that

$$\sum_{j \in V_i} I_j^\phi(N,0,c) = R_i^\phi(N,0,c) + M(V_i,c).$$

□

Furthermore, Lemma 4.2 shows that the internal representation costs can be removed from the description of the induced rule in the following way.

Lemma 4.2. Let $(N,0,c) \in \mathcal{M}^*$ be an MCST problem with a unique MCST Γ and let ϕ be a base rule, then, for all $i \in N$,

$$I_i^\phi(N,0,c) = R_i^\phi(N,0,c) - \sum_{j \in IF(i)} (c_{i,j} - R_j^\phi(N,0,c)).$$

Proof. Let $i \in N$. Then we have

$$\begin{aligned} I_i^\phi(N,0,c) &= \sum_{k \in V_i} I_k^\phi(N,0,c) - \sum_{j \in IF(i)} \sum_{u \in V_j} I_u^\phi(N,0,c) \\ &= R_i^\phi(N,0,c) + M(V_i,c) - \sum_{j \in IF(i)} (R_j^\phi(N,0,c) + M(V_j,c)) \\ &= R_i^\phi(N,0,c) - \sum_{j \in IF(i)} (c_{i,j} - R_j^\phi(N,0,c)), \end{aligned}$$

where the second equation follows from the second statement of Lemma 4.1 and the third equation follows from the first statement of Lemma 4.1. □

Finally, Lemma 4.3 shows that the external representation costs can be expressed into compensation values. This result is a direct consequence of the definition of compensation functions.

Lemma 4.3. Let $(N,0,c) \in \mathcal{M}^*$ be an MCST problem with a unique MCST Γ and let ϕ be a base rule. Then for all $i \in N$,

$$R_i^\phi(N,0,c) = \begin{cases} c_{p(i),i} + cv^\phi(c_{0,sc(i)}, c_{p(i),i}) & \text{if } i \notin SC(\Gamma) \\ c_{0,i} & \text{if } i \in SC(\Gamma). \end{cases}$$

Theorem 4.1. Let ϕ be a base rule. Let cv^ϕ be its corresponding compensation value function. Then,

$$I_s^\phi(N,0,c) = c_{0,s} - \sum_{j \in IF(s)} cv^\phi(c_{0,s}, c_{s,j}) \text{ for all } s \in SC(\Gamma),$$

and

$$I_i^\phi(N,0,c) = c_{p(i),i} + cv^\phi(c_{0,sc(i)}, c_{p(i),i}) - \sum_{j \in IF(i)} cv^\phi(c_{0,sc(i)}, c_{i,j}) \text{ for all } i \in N \backslash SC(\Gamma).$$

Proof. By Lemma 4.2 we have that for any $i \in N$,

$$I_i^{\phi}(N,0,c) = R_i^{\phi}(N,0,c) - \sum_{j \in IF(i)} (c_{i,j} - R_j^{\phi}(N,0,c)).$$

With Lemma 4.3, for $s \in SC(\Gamma)$,

$$\begin{aligned} I_s^{\phi}(N,0,c) &= R_s^{\phi}(N,0,c) - \sum_{j \in IF(s)} (c_{s,j} - R_j^{\phi}(N,0,c)) \\ &= c_{0,s} - \sum_{j \in IF(s)} cv^{\phi}(c_{0,s}, c_{s,j}). \end{aligned}$$

Moreover, for any $i \in N \backslash SC(\Gamma)$,

$$\begin{aligned} I_i^{\phi}(N,0,c) &= R_i^{\phi}(N,0,c) - \sum_{j \in IF(i)} (c_{i,j} - R_j^{\phi}(N,0,c)) \\ &= c_{p(i),i} + cv^{\phi}(c_{0,sc(i)}, c_{p(i),i}) - \sum_{j \in IF(i)} (c_{i,j} - R_j^{\phi}(N,0,c)). \end{aligned}$$

□

Let I^B denote the induced rule corresponding to the Bird's rule on $\mathcal{M}_2^*$. Since $cv^B(a,b) = 0$ for all $a, b \in \mathbb{R}_+$, Theorem 4.1 directly implies:

Corollary 4.1. Let $(N,0,c) \in \mathcal{M}^*$. Then $B(N,0,c) = I^B(N,0,c)$.

The following example illustrates how to use Theorem 4.1 to determine an induced rule in a non-recursive way.

Example 4.5. Reconsider the MCST problem $(N,0,c)$ where the agent set is $N = \{1,2,...,6\}$ and whose MCST is indicated in Figure 4.2. The folk solution F is used as the base rule. Hence, its corresponding compensation value function satisfies $cv^F(a,b) = 0$ if $b \geq a$ and $cv^F(a,b) = (a-b)/2$ if $b < a$. Using Theorem 4.1, we have that

$$\begin{aligned} I_1^F(N,0,c) &= c_{01} - cv^F(c_{01}, c_{12}) - cv^F(c_{01}, c_{13}) = 3 - 1/2 - 0 = 5/2, \\ I_2^F(N,0,c) &= c_{12} + cv^F(c_{01}, c_{12}) - cv^F(c_{01}.c_{24}) = 2 + 1/2 - 0 = 5/2, \\ I_3^F(N.0.c) &= c_{13} + cv^F(c_{01}, c_{13}) - cv^F(c_{01}, c_{35}) - cv^F(c_{01}, c_{36}) = 6 + 0 - 0 - 1/2 = 11/2, \\ I_4^F(N,0,c) &= c_{24} + cv^F(c_{01}, c_{24}) = 5 + 0 = 5, \\ I_5^F(N,0,c) &= c_{35} + cv^F(c_{01}, c_{35}) = 4 + 0 = 4, \\ I_6^F(N,0,c) &= c_{36} + cv^F(c_{01}, c_{36}) = 2 + 1/2 = 5/2. \end{aligned}$$

◁

5 Merge-proof and Coalitionally Stable Induced Rules

As mentioned in the previous sections, the recursive procedure is designed in a way that the sum of costs allocated by an induced rule to an agent and all its followers is equal to the cost needed for them to merge in advance plus the cost assigned to the merger by the induced rule. Such a design guarantees that every $\Gamma-$branch-connected set of agents has no incentive to merge when an induced rule is used, as is seen in Lemma 5.1. A $\Gamma-$branch-connected set is a connected set of agents within a branch of Γ. A formal definition is presented below.

Definition 5.1. Let $(N,0,c) \in \mathcal{M}$ be an MCST problem with an MCST Γ. Let $S \subseteq N$. S is called a $\Gamma-$branch-connected set if there exists $s \in S$ such that for all $i \in S\backslash\{s\}$, $i \in F(s)$ and $p(i) \in S$. A subset $S \subseteq N$, $S' \subseteq S$ is called a $\Gamma-$branch-connected component of S if S' is a $\Gamma-$branch-connected set, and $\nexists S''$ s.t. $S' \subsetneq S'' \subseteq N$ and S'' is a $\Gamma-$branch-connected set.

Lemma 5.1. Let I^ϕ be the induced rule corresponding to a base rule ϕ. Let $(N,0,c) \in \mathcal{M}^*$ be an MCST problem with a unique MCST Γ. For any $S \subseteq N$ that is $\Gamma-$branch-connected, we have

$$\sum_{k\in S} I_k^\phi(N,0,c) = I_{m^S}^\phi((N\backslash S)\cup\{m^S\},0,c^S) + M(S,c). \tag{5.1}$$

Proof. Note that the right hand side of (5.1) is well-defined because $((N\backslash S)\cup\{m^S\},0,c^S) \in \mathcal{M}^*$. Let cv^ϕ be the compensation value function corresponding to the base rule ϕ.

First, Assume that $S = V_j$ for some $j \in N$.

According to Lemma 4.1, the left hand side of (5.1) is given by

$$\sum_{k\in S} I_k^\phi(N,0,c) = \sum_{k\in V_j} I_k^\phi(N,0,c) = R_j^\phi(N,0,c) + M(V_j,c). \tag{5.2}$$

The right hand side of (5.1) is given by

$$\begin{aligned} I_{m^S}^\phi((N\backslash S)\cup\{m^S\},0,c^S) + M(S,c) &= I_{m^{V_j}}^\phi((N\backslash V_j)\cup\{m^{V_j}\},0,c^{V_j}) + M(V_j,c) \\ &= r_{m^{V_j}}^\phi((N\backslash V_j)\cup\{m^{V_j}\},0,c^{V_j}) + R_{m^{V_j}}^\phi((N\backslash V_j)\cup\{m^{V_j}\},0,c^{V_j}) \\ &\quad + M(V_j,c) \\ &= R_{m^{V_j}}^\phi((N\backslash V_j)\cup\{m^{V_j}\},0,c^{V_j}) + M(V_j,c) \\ &= R_j^\phi(N,0,c) + M(V_j,c), \end{aligned} \tag{5.3}$$

where the third equation follows from the fact that m^{V_j} is a leaf node in $((N\backslash V_j)\cup\{m^{V_j}\},0,c^{V_j})$, so we have $r_{m^{V_j}}^\phi((N\backslash V_j)\cup\{m^{V_j}\},0,c^{V_j}) = 0$. The fourth equation follows from the fact that the external representative cost of m^{V_j} in problem $((N\backslash V_j)\cup\{m^{V_j}\},0,c^{V_j})$ and that of j in problem $(N,0,c)$ are identical, i.e., $R_j^\phi(N,0,c) = R_{m^{V_j}}^\phi((N\backslash V_j)\cup\{m^{V_j}\},0,c^{V_j})$. By (5.2) and (5.3), (5.1) follows.

Now consider an arbitrary $\Gamma-$branch-connected set $S \subseteq N$. Let $s \in S$ be the node such that all the other nodes in the set S are followers of the node s, with $H = \{i \in V_s\backslash S \mid p(i) \in S\}$. Clearly, $V_s\backslash(\cup_{i\in H}V_i) = S$. In this case, the left hand side of (5.1) is given by:

$$\sum_{k\in S} I_k^\phi(N,0,c) = \sum_{k\in V_s} I_k^\phi(N,0,c) - \sum_{i\in H}\sum_{k\in V_i} I_k^\phi(N,0,c). \tag{5.4}$$

The first term on the right hand side of (5.4) is given by:

$$\begin{aligned} \sum_{k\in V_s} I_k^\phi(N,0,c) &= M(V_s,c) + R_s^\phi(N,0,c) \\ &= \begin{cases} M(V_s,c) + c_{p(s),s} + cv^\phi(c_{0,sc(s)}, c_{p(s),s}) & \text{if } s \notin sc(\Gamma) \\ M(V_s,c) + c_{0,s} & \text{if } s \in sc(\Gamma), \end{cases} \end{aligned}$$

where the first equation follows from Lemma 4.1, part 2 and the second equation follows from Lemma 4.3. Similarly, we have that for all $i \in H$

$$\sum_{k \in V_i} I_k^{\phi}(N, 0, c) = M(V_i, c) + c_{p(i),i} + cv^{\phi}(c_{0,sc(i)}, c_{p(i),i}).$$

If $s \notin sc(\Gamma)$, the left hand side of (5.1) is given by:

$$\begin{aligned}
\sum_{k \in S} I_k^{\phi}(N, 0, c) &= M(V_s, c) + c_{p(s),s} + cv^{\phi}(c_{0,sc(s)}, c_{p(s),s}) \\
&\quad - \sum_{i \in H} \big(M(V_i, c) + c_{p(i),i} + cv^{\phi}(c_{0,sc(i)}, c_{p(i),i})\big) \\
&= \big[M(V_s, c) - \sum_{i \in H} \big(M(V_i, c) + c_{p(i),i}\big)\big] + c_{p(s),s} + cv^{\phi}(c_{0,sc(s)}, c_{p(s),s}) \\
&\quad - \sum_{i \in H} cv^{\phi}(c_{0,sc(i)}, c_{p(i),i}) \\
&= M(S, c) + c_{p(s),s} + cv^{\phi}(c_{0,sc(s)}, c_{p(s),s}) - \sum_{i \in H} cv^{\phi}(c_{0,sc(s)}, c_{p(i),i}),
\end{aligned}$$

where the last equation follows from the fact that $sc(i) = sc(s)$ for every $i \in H$, because i and s are within the same branch.

Similarly, if $s \in sc(\Gamma)$, the left hand side of (5.1) is given by:

$$\begin{aligned}
\sum_{k \in S} I_k^{\phi}(N, 0, c) &= M(V_s, c) + c_{0,s} - \sum_{i \in H} \big(M(V_i, c) + c_{p(i),i} + cv^{\phi}(c_{0,sc(i)}, c_{p(i),i})\big) \\
&= M(S, c) + c_{0,s} - \sum_{i \in H} cv^{\phi}(c_{0,s}, c_{p(i),i}).
\end{aligned}$$

If $s \notin sc(\Gamma)$, the right hand side of (5.1) is given by

$$\begin{aligned}
M(S, c) + I_{m^S}^{\phi}((N \backslash S) \cup \{m^S\}, 0, c^S) &= M(S, c) + c^S_{p(s),m^S} + cv^{\phi}\Big(c^S_{0,sc(s)}, c^S_{p(s),m^S}\Big) \\
&\quad - \sum_{i \in H} cv^{\phi}\Big(c^S_{0,sc(s)}, c^S_{m^S,i}\Big) \\
&= M(S, c) + c_{p(s),s} + cv^{\phi}(c_{0,sc(s)}, c_{p(s),s}) - \sum_{i \in H} cv^{\phi}(c_{0,sc(s)}, c_{p(i),i}),
\end{aligned}$$

where the first equation follows from Theorem 4.1 and the fact that the source connection node of m^S in the remaining problem is exactly the source connection node of s in Γ. The last equation follows from the fact that $c^S\big(\{p(s), m^S\}\big) = c_{p(s),s}$, $c^S_{0,sc(s)} = c_{0,sc(s)}$, and for all $i \in H$, that $c^S(\{m^S, i\}) = \min_{k \in S} c(\{i, k\}) = c_{p(i),i}$.

Similarly, if $s \in sc(\Gamma)$, the right hand side of (5.1) is given by:

$$\begin{aligned}
M(S, c) + I_{m^S}^{\phi}((N \backslash S) \cup \{m^S\}, 0, c^S) &= M(S, c) + c^S_{0,m^S} - \sum_{i \in H} cv^{\phi}\Big(c^S_{0,sc(s)}, c^S_{m^S,i}\Big) \\
&= M(S, c) + c_{0,s} - \sum_{i \in H} cv^{\phi}(c_{0,s}, c_{p(i),i}).
\end{aligned}$$

It is straightforward to see that (5.1) holds for both cases. $\square$

Lemma 4.1 guarantees that an induced rule satisfies merge-proofness when mergers take place within Γ-branch-connected sets. However, to obtain merge-proofness of induced rules, we need some additional restrictions on the domain.

Firstly, there is a problem regarding the application of an induced rule when considering mergers of sets that are not Γ−branch-connected. For an MCST problem with a unique MCST, a merger between a set that is not Γ−branch-connected may lead to multiple MCST's in the remaining problem, as shown in Example 5.1 below. This causes problems since induced rules are only defined for MCST problems with a unique MCST. For an MCST problem with a unique MCST and distinct costs for different edges in the MCST, the remaining problem after an arbitrary set $S \subseteq N$ merging together still has a unique MCST (cf. Gómez-Rúa and Vidal-Puga (2011)). If we apply this restriction on the domain, we can apply the induced rule to both the original problem and the remaining problem to further investigate merge-proofness.

Example 5.1. Consider $(N, 0, c) \in \mathcal{M}^*$ as shown in Figure 5.1 on the left hand side, whose MCST is indicated in solid lines. All unpresented costs of edges are sufficiently large so that the MCST is unique. Consider the non-Γ-branch-connected set $\{2, 3\}$ and the remaining problem $((N\backslash\{2,3\}) \cup \{m^{\{2,3\}}\}, 0, c^{\{2,3\}})$ as presented on the right hand side in Figure 5.1. It is straightforward to see that $\{\{0,1\},\{1,m^{\{2,3\}}\},\{m^{\{2,3\}},4\}\}$ forms an MCST of this remaining problem, as well as $\{\{0,1\},\{1,4\},\{m^{\{2,3\}},4\}\}$. Hence, the remaining problem $((N\backslash\{2,3\}) \cup m^{\{2,3\}}, 0, c^{\{2,3\}})$ does not belong to $\mathcal{M}^*$, preventing the application of an induced rule.

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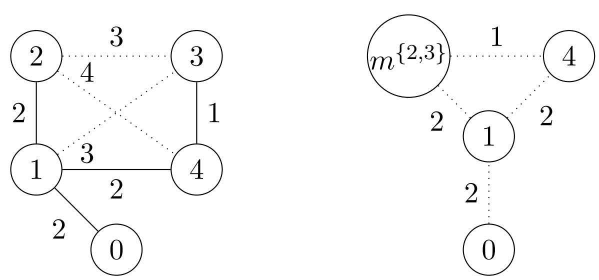

Figure 5.1: MCST problems $(N, 0, c)$ and $((N\backslash\{2,3\}) \cup m^{\{2,3\}}, 0, c^{\{2,3\}})$ in Example 5.1

Secondly, it is worth noting that, even when the remaining problem after merging has a unique MCST, agents in a set that is not Γ−branch-connected may have the incentive to merge when applying an induced rule, as is illustrated in Example 5.2 and Example 5.3. In both examples, the folk solution F, which gives an advantage to the agents closer to the source in the MCST, is used as a base rule.

Example 5.2. Consider $(N, 0, c) \in \mathcal{M}^*$ as shown in Figure 5.2 on the left hand side, whose MCST is indicated in solid lines. All unindicated costs of edges are sufficiently large so that the MCST is unique. Note that all costs on the MCST are distinct. We use the folk solution F as the base rule. Then, using Theorem 4.1,

$$I_1^F(N, 0, c) = c_{01} - cv^F(c_{01}, c_{12}) = 4 - 1 = 3,$$

and

$$I_3^F(N, 0, c) = c_{23} + cv^F(c_{01}, c_{23}) = 1 + 3/2 = 5/2.$$

Now consider the problem $((N\backslash\{1,3\})\cup\{m^{\{1,3\}}\},0,c^{\{1,3\}})$ after agents 1 and 3 in $(N,0,c)$ merge together. In this case,

$$I^F_{m^{\{1,3\}}}((N\backslash\{1,3\})\cup\{m^{\{1,3\}}\},0,c^{\{1,3\}}) = 5/2.$$

Hence, we have

$$I^F_1(N,0,c) + I^F_3(N,0,c) = 11/2,$$

while

$$I^F_{m^{\{1,3\}}}((N\backslash\{1,3\})\cup\{m^{\{1,3\}}\},0,c^{\{1,3\}}) + M(\{1,3\},c) = 5.$$

$\triangleleft$

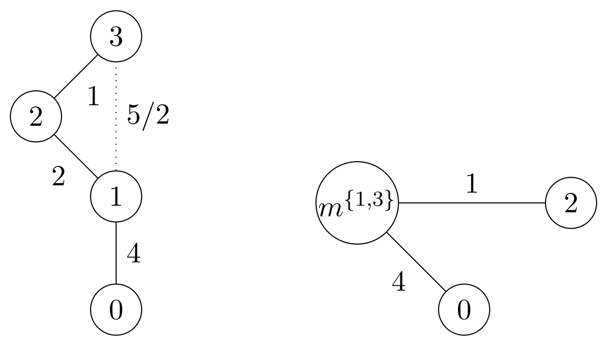

Figure 5.2: MCST problems $(N,0,c)$ and $((N\backslash\{1,3\})\cup\{m^{\{1,3\}}\},0,c^{\{1,3\}})$ in Example 5.2

Example 5.3. Consider $(N,0,c)\in\mathcal{M}^*$ as shown in Figure 5.3 on the left hand side. The MCST is indicated in solid lines. Again, all unpresented costs of edges are sufficiently large so that the MCST is unique. Note that all costs on the MCST are distinct. We use the folk solution F as the base rule. Then, using Theorem 4.1,

$$I^F_2(N,0,c) = c_{02} = 3,$$
$$I^F_3(N,0,c) = c_{13} + cv^F(c_{01},c_{13}) - cv^F(c_{01},c_{34}) - cv^F(c_{01},c_{35}) = 4 + 0 - 1/4 - 1/2 = 13/4.$$

Now consider the remaining problem $((N\backslash\{2,3\})\cup\{m^{\{2,3\}}\},0,c^{\{2,3\}})$ after agents 2 and 3 merge together. In this case,

$$I^F_{m^{\{2,3\}}}((N\backslash\{2,3\})\cup\{m^{\{2,3\}}\},0,c^{\{2,3\}}) = 3 - cv^F(3,3/2) - cv^F(3,1) = 5/4.$$

Hence,

$$I^F_2(N,0,c) + I^F_3(N,0,c) = 25/4,$$

while

$$I^F_{m^{\{2,3\}}}((N\backslash\{2,3\})\cup\{m^{\{2,3\}}\},0,c^{\{2,3\}}) + M(\{2,3\},c) = 23/4.$$

$\triangleleft$

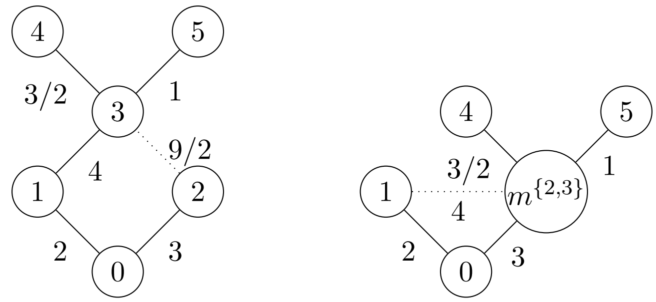

Figure 5.3: MCST problems $(N, 0, c)$ and $((N \backslash \{2,3\}) \cup \{m^{\{2,3\}}\}, 0, c^{\{2,3\}})$ in Example 5.3

In Example 5.2 we observe that if some edges between agents that are not included in the MCST have relatively small costs, these agents may have an incentive to merge. On the other hand, we observe in Example 5.3 that if there is more than one source-connection node, agents may want to merge with other agents to switch branches. Following the ideas of previous examples, we propose a more restricted domain $\mathcal{M}_D^*$, as presented in Definition 5.2, to further analyze merge-proofness of induced rules.

Definition 5.2. $\mathcal{M}_D^*$ is the class of all MCST problems $(N, 0, c)$ with a unique MCST $\Gamma = (N_0, E)$ in which any two edges have different costs, and there exists a unique node $sc \in N$ such that $sc(\Gamma) = \{sc\}$ and $c(e) \geq c_{0,sc}$ for all $e \in E^{N_0} \backslash E$.

In Theorem 5.1, we obtain a large class of induced rules satisfying merge-proofness. This class consists of induced rules I^ϕ whose corresponding base rule ϕ satisfies the additional condition (5.5) below. Consider a 2-person MCST problem $(N, 0, c)$ with $N = \{1, 2\}$ and a unique MCST Γ formed by edges $\{1, 2\}$ and $\{0, 1\}$. Condition (5.5) requires that agent 2, after paying c_{12}, may additionally pay agent 1 a compensation, which should not be too high. Example 5.4 illustrates that when condition (5.5) does not hold, merge-proofness on $\mathcal{M}_D^*$ is not guaranteed.

Example 5.4. Consider $(N, 0, c) \in \mathcal{M}_D^*$ as shown in Figure 5.4 on the left hand side. The MCST is indicated in solid lines. All unpresented costs of edges are sufficiently large so that the MCST is unique and the conditions in Definition 5.2 are satisfied. We use an induced rule I^ϕ with $cv^\phi(a, b) = a - 1$. It is easy to verify that condition (5.5) is not satisfied. By Theorem 4.1,

$$I_1^\phi(N, 0, c) = 4 - cv^\phi(4, 1) = 1,$$

$$I_3^\phi(N, 0, c) = 2 + cv^\phi(4, 2) = 5,$$

$$I_{m^{\{1,3\}}}^\phi((N \backslash \{1,3\}) \cup \{m^{\{1,3\}}\}, 0, c^{\{1,3\}}) = 4 - cv^\phi(4, 1) = 1.$$

Hence, we have,

$$I_1^\phi(N, 0, c) + I_3^\phi(N, 0, c) = 6,$$

while

$$I_{m^{\{1,3\}}}^\phi((N \backslash \{1,3\}) \cup \{m^{\{1,3\}}\}, 0, c^{\{1,3\}}) + M(\{1,3\}, c) = 11/2.$$

We can conclude that agents 1 and 3 have an incentive to merge.

$\triangleleft$

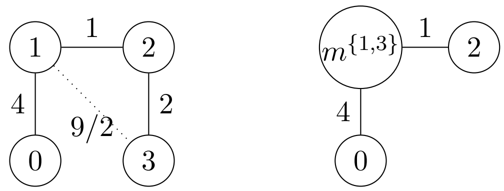

Figure 5.4: MCST problems $(N, 0, c)$ and $(N\backslash\{1,3\} \cup \{m^{\{1,3\}}\}, 0, c^{\{1,3\}})$ in Example 5.4

It is easy to check that both Bird's rule B and the folk solution F satisfy condition (5.5). In fact, they actually form the "boundaries" for the class of induced rules presented in Theorem 5.1. If the first inequality in (5.5) is an equality, then the base rule ϕ will coincide with Bird's rule B. If the second inequality is an equality, the base rule ϕ will coincide with the folk solution F.

Theorem 5.1. Let ϕ be a base rule and cv^ϕ be the corresponding compensation value function. If cv^ϕ satisfies,

$$0 \leq cv^\phi(a, b) \leq \frac{1}{2}\max\{a - b, 0\}, \text{ for every } a, b \in \mathbb{R}_+, \tag{5.5}$$

then the induced rule I^ϕ satisfies merge-proofness on $\mathcal{M}^*_D$.

Proof. To prove the merge-proofness property of I^ϕ on $\mathcal{M}^*_D$, we need to show that for every $(N, 0, c) \in \mathcal{M}^*_D$, and every $S \subseteq N$,

$$\sum_{i \in S} I^\phi_i(N, 0, c) \leq I^\phi_{m^S}((N\backslash S) \cup \{m^S\}, 0, c^S) + M(S, c). \tag{5.6}$$

Let $(N, 0, c) \in \mathcal{M}^*_D$ be an MCST problem with a unique MCST $\Gamma = (N_0, E)$. Let $sc(\Gamma) = \{sc\}$. By Lemma 5.1, we only need to consider the case where S is not a $\Gamma-$branch-connected set.

We first provide a result that is frequently used in the analysis. Consider $e \in E^{N_0}\backslash E$ and $j \in N$. Then, since $(N, 0, c) \in \mathcal{M}^*_D$, we have that $c_e \geq c_{0,sc}$. Now assume $c_e > c_{p(j),j}$.Then, $c_e \geq \max\{c_{0,sc}, c_{p(j),j}\}$. Hence, using condition (5.5), we have that:

$$\begin{aligned} cv^\phi(c_{0,sc}, c_{p(j),j}) &\leq \frac{1}{2}\max\{c_{0,sc} - c_{p(j),j}, 0\} \\ &\leq \frac{1}{2}(c_e - c_{p(j),j}), \end{aligned}$$

Consequently, for all $e \in E^{N_0}\backslash E$ and $j \in N$,

$$c(e) > c_{p(j),j} \Longrightarrow c(e) \geq c_{p(j),j} + 2cv^\phi(c_{0,sc}, c_{p(j),j}). \tag{5.7}$$

Case 1: Consider $S = \{i, j\}$ such that $\{i, j\} \notin E$. Let $P_{ij} = \{\{p_0, p_1\}, \{p_1, p_2\}, ..., \{p_t, p_{t+1}\}\} \subseteq E$, with $p_0 = i$ and $p_{t+1} = j$, be the unique path in Γ from i to j. According to Proposition A.1, the MCST for $((N\backslash\{i, j\}) \cup \{m^{\{i,j\}}\}, 0, c^{\{i,j\}})$ can only be obtained from Γ by deleting the edge $e \in P_{ij}$ such that $c_e \geq c_{p_q,p_{q+1}}$, for all $q \in \{0, 1, ..., t\}$. For simplicity of notation, let $N' = (N\backslash\{i, j\}) \cup \{m^{\{i,j\}}\}$, $c' = c^{\{i,j\}}$ and $\Gamma' = (N', E')$ be the MCST of problem $(N', 0, c')$. Note that $E' = E\backslash\{e\}$. For $k \in N'$, let $IF'(k)$ denote the immediate follower set of k and $p'(k)$ denote the immediate predecessor of k in Γ'.

Case 1.1: $i \notin F(j)$, $j \notin F(i)$.

In this case, we have that $i \neq sc$ and $j \neq sc$. By Theorem 4.1,

$$I_i^{\phi}(N,0,c) = c_{p(i),i} + cv^{\phi}(c_{0,sc}, c_{p(i),i}) - \sum_{k \in IF(i)} cv^{\phi}(c_{0,sc}, c_{p(k),k}),$$

$$I_j^{\phi}(N,0,c) = c_{p(j),j} + cv^{\phi}(c_{0,sc}, c_{p(j),j}) - \sum_{k \in IF(j)} cv^{\phi}(c_{0,sc}, c_{p(k),k}).$$

The left hand side of (5.6) in this case is:

$$\begin{aligned} I_i^{\phi}(N,0,c) + I_j^{\phi}(N,0,c) = c_{p(i),i} &+ cv^{\phi}(c_{0,sc}, c_{p(i),i}) - \sum_{k \in IF(i) \cup IF(j)} cv^{\phi}(c_{0,sc}, c_{p(k),k}) \\ &+ c_{p(j),j} + cv^{\phi}(c_{0,sc}, c_{p(j),j}). \end{aligned} \tag{5.8}$$

Here we use the fact that $IF(i) \cap IF(j) = \varnothing$.

After i and j merge, assume that $p^{'}(m^{\{i,j\}}) = p(i)$ without loss of generality. Then either $IF^{'}(m^{\{i,j\}}) = IF(i) \cup IF(j)$ or $IF^{'}(m^{\{i,j\}}) = IF(i) \cup IF(j) \cup \{p(j)\}$. If $p(j) \in IF^{'}(m^{\{i,j\}})$, then $c^{'}_{p(j),m^{\{i,j\}}} = c_{p(j),j}$. As a consequence,

$$\begin{aligned} I^{\phi}_{m^{\{i,j\}}}(N^{'},0,c^{'}) &= c^{'}_{p(i),m^{\{i,j\}}} + cv^{\phi}(c^{'}_{0,sc}, c^{'}_{p(i),m^{\{i,j\}}}) - \sum_{k \in IF^{'}(m^{\{i,j\}})} cv^{\phi}(c^{'}_{0,sc}, c^{'}_{m^{\{i,j\}},k}) \\ &= \begin{cases} c^{'}_{p(i),m^{\{i,j\}}} + cv^{\phi}(c^{'}_{0,sc}, c^{'}_{p(i),m^{\{i,j\}}}) & \\ -\sum_{k \in IF(i) \cup IF(j)} cv^{\phi}(c^{'}_{0,sc}, c^{'}_{m^{\{i,j\}},k}) & \text{if } IF^{'}(m^{\{i,j\}}) = IF(i) \cup IF(j) \\ c^{'}_{p(i),m^{\{i,j\}}} + cv^{\phi}(c^{'}_{0,sc}, c^{'}_{p(i),m^{\{i,j\}}}) & \\ -\sum_{k \in IF(i) \cup IF(j) \cup \{p(j)\}} cv^{\phi}(c^{'}_{0,sc}, c^{'}_{m^{\{i,j\}},k}) & \text{if } IF^{'}(m^{\{i,j\}}) = IF(i) \cup IF(j) \cup \{p(j)\} \end{cases} \\ &\geq c^{'}_{p(i),m^{\{i,j\}}} + cv^{\phi}(c^{'}_{0,sc}, c^{'}_{p(i),m^{\{i,j\}}}) - cv^{\phi}(c^{'}_{0,sc}, c_{p(j),j}) \\ &- \sum_{k \in IF(i) \cup IF(j)} cv^{\phi}(c^{'}_{0,sc}, c^{'}_{m^{\{i,j\}},k}) \\ &= c_{p(i),i} + cv^{\phi}(c_{0,sc}, c_{p(i),i}) - cv^{\phi}(c_{0,sc}, c_{p(j),j}) - \sum_{k \in IF(i) \cup IF(j)} cv^{\phi}(c_{0,sc}, c_{p(k),k}). \end{aligned} \tag{5.9}$$

The first equation follows from Theorem 4.1. The inequality follows from $cv^{\phi}(\cdot,\cdot) \geq 0$. The last equation follows from the fact that $c^{'}_{0,sc} = c_{0,sc}$, $c^{'}_{p(i),m^{\{i,j\}}} = c_{p(i),i}$ and $c^{'}_{m^{\{i,j\}},k} = c_{p(k),k}$ for every $k \in IF(i) \cup IF(j)$.

Starting from the right hand side of (5.6),

$$\begin{aligned} I^{\phi}_{m^{\{i,j\}}}(N^{'},0,c^{'}) + M(\{i,j\},c) &= I^{\phi}_{m^{\{i,j\}}}(N^{'},0,c^{'}) + c_{ij} \\ &\geq c_{p(i),i} + cv^{\phi}(c_{0,sc}, c_{p(i),i}) - \sum_{k \in IF(i) \cup IF(j)} cv^{\phi}(c_{0,sc}, c_{p(k),k}) \\ &+ c_{ij} - cv^{\phi}(c_{0,sc}, c_{p(j),j}) \\ &\geq c_{p(i),i} + cv^{\phi}(c_{0,sc}, c_{p(i),i}) - \sum_{k \in IF(i) \cup IF(j)} cv^{\phi}(c_{0,sc}, c_{p(k),k}) \\ &+ c_{p(j),j} + cv^{\phi}(c_{0,sc}, c_{p(j),j}) \\ &= I_i^{\phi}(N,0,c) + I_j^{\phi}(N,0,c). \end{aligned}$$

The first inequality follows from (5.9), the second inequality follows from $c_{ij} > c_{p(j),j}$ and (5.7), and the last equality from (5.8).

Case 1.2: $j \in F(i)$ or $i \in IF(j)$.

Without loss of generality, we assume that $j \in F(i)$. By Theorem 4.1, if $i \neq sc$, the left hand side of (5.6) is:

$$I_i^{\phi}(N,0,c)+I_j^{\phi}(N,0,c) = c_{p(i),i}+cv^{\phi}(c_{0,sc},c_{p(i),i})+c_{p(j),j}+cv^{\phi}(c_{0,sc},c_{p(j),j})-\sum_{k\in IF(i)\cup IF(j)} cv^{\phi}(c_{0,sc},c_{p(k),k}). \tag{5.10}$$

When $i = sc$, by condition (5,5), $cv^{\phi}(c_{0,sc},c_{0,sc}) = 0$. So if $i = sc$, (5.10) is still satisfied.

After i and j merge, $p^{'}(m^{\{i,j\}}) = p(i)$. Let $f(i)$ be the immediate follower of i on the path P_{ij}. Then,

$$IF^{'}(m^{\{i,j\}}) = \begin{cases} IF(i)\cup IF(j), & \text{if } e = \{p(j),j\} \\ IF(i)\cup IF(j)\cup\{p(j)\}, & \text{if } e \neq \{p(j),j\} \text{ and } e \neq \{i,f(i)\} \\ \big(IF(i)\cup IF(j)\backslash\{f(i)\}\big)\cup\{p(j)\} & \text{if } e = \{i,f(i)\}. \end{cases}$$

According to Theorem 4.1 and the fact that $p^{'}(m^{\{i,j\}}) = p(i)$,

$$\begin{aligned} I^{\phi}_{m^{\{i,j\}}}(N^{'},0,c^{'}) &= c^{'}_{p^{'}(m^{\{i,j\}}),m^{\{i,j\}}} + cv^{\phi}\Big(c^{'}_{0,sc},c^{'}_{p^{'}(m^{\{i,j\}}),m^{\{i,j\}}}\Big) \\ &\quad - \sum_{k\in IF^{'}(m^{\{i,j\}})} cv^{\phi}\big(c^{'}_{0,sc},c^{'}_{p^{'}(k),k}\big) \\ &= c_{p(i),i} + cv^{\phi}(c_{0,sc},c_{p(i),i}) - \sum_{k\in IF^{'}(m^{\{i,j\}})} cv^{\phi}\big(c^{'}_{0,sc},c^{'}_{p^{'}(k),k}\big). \end{aligned} \tag{5.11}$$

For any $k \in \big(IF(i)\cup IF(j)\big)\backslash\{f(i)\}$, we have that $c^{'}_{p^{'}(k),k} = c_{p(k),k}$. Moreover, if $p(j) \in IF^{'}(m^{\{i,j\}})$, then $c^{'}_{p^{'}(p(j)),p(j)} = c_{j,p(j)}$.

Given condition (5.5), we have $cv^{\phi}(\cdot,\cdot) \geq 0$. As a result

$$\begin{aligned} \sum_{k\in IF^{'}(m^{\{i,j\}})} cv^{\phi}\big(c^{'}_{0,sc},c^{'}_{p^{'}(k),k}\big) &\leq \sum_{k\in IF(i)\cup IF(j)\cup\{p(j)\}} cv^{\phi}\big(c^{'}_{0,sc},c^{'}_{p^{'}(k),k}\big) \\ &\leq \sum_{k\in IF(i)\cup IF(j)} cv^{\phi}(c_{0,sc},c_{p(k),k}) + cv^{\phi}(c_{0,sc},c_{p(j),j}), \end{aligned} \tag{5.12}$$

Hence, starting from the right hand side of (5.6),

$$\begin{aligned} I^{\phi}_{m^{\{i,j\}}}(N^{'},0,c^{'}) + M(\{i,j\},c) &= I^{\phi}_{m^{\{i,j\}}}(N^{'},0,c^{'}) + c_{ij} \\ &\geq c_{p(i),i} + cv^{\phi}(c_{0,sc},c_{p(i),i}) - \sum_{k\in IF(i)\cup IF(j)} cv^{\phi}(c_{0,sc},c_{p(k),k}) \\ &\quad - cv^{\phi}(c_{0,sc},c_{p(j),j}) + c_{ij} \\ &\geq c_{p(i),i} + cv^{\phi}(c_{0,sc},c_{p(i),i}) - \sum_{k\in IF(i)\cup IF(j)} cv^{\phi}(c_{0,sc},c_{p(k),k}) \\ &\quad + c_{p(j),j} + cv^{\phi}(c_{0,sc},c_{p(j),j}) \\ &= I_i^{\phi}(N,0,c) + I_j^{\phi}(N,0,c), \end{aligned}$$

where the first inequality combines (5.11) and (5.12), the second inequality follows from (5.7) and $c_{ij} > c_{p(j),j}$, and the last equation follows from (5.10).

Case 2: Now consider $S \subseteq N$ where $E \cap E^S = \varnothing$.

Let Q denote the set of nodes in S that are not the followers of any other nodes in S. We can see that after agents in S merges together, in the new MCST Γ', the immediate predecessor of m^S belongs to the set $\{p(q) \mid q \in Q\}$. Let $\bar{q} \in Q$ s.t. $p(\bar{q}) = p'(m^S)$. Set $\bar{p} = p(\bar{q})$. If for $i \in N \backslash S$, $\{i, m^S\} \in \Gamma'$, then there is an $s \in S$, $\{i, s\} \in \Gamma$. Hence,

$$IF'(m^S) \subseteq \big(\cup_{u \in S} IF(u)\big) \cup \big(\cup_{k \in S \backslash \{\bar{q}\}} \{p(k)\}\big). \tag{5.13}$$

For simplicity, define $N' = (N \backslash S) \cup \{m^S\}$ and $c' = c^S$. Let $G_S = (S, E_{G_S})$ with $E_{G_S} \subseteq E^S$ be a tree such that $\sum_{e \in E_{G_S}} c(e) = M(S, c)$.

By Theorem 4.1 and (5.5), the left hand side of (5.6) is given by:

$$\sum_{u \in S} I_u^{\phi}(N, 0, c) = \sum_{u \in S} \big[c_{p(u),u} + cv^{\phi}(c_{0,sc}, c_{p(u),u})\big] - \sum_{u \in S} \sum_{k \in IF(u)} cv^{\phi}(c_{0,sc}, c_{p(k),k}). \tag{5.14}$$

Within the merger, the merging cost is:

$$M(S, c) = c(E_{G_S}) = \sum_{e \in E_{G_S}} c_e.$$

By Theorem 4.1 and (5.13s), we also have that

$$\begin{aligned} I_{m^S}^{\phi}(N', 0, c') &= c'\big(\{p^{m^S}, m^S\}\big) + cv^{\phi}\Big(c'_{0,sc}, c'\big(\{p^{m^S}, m^S\}\big)\Big) - \sum_{k \in IF'(m^S)} cv^{\phi}(c'_{0,sc}, c'_{p'(k),k}) \\ &\geq c_{p(\bar{q}),\bar{q}} + cv^{\phi}(c_{0,sc}, c_{p(\bar{q}),\bar{q}}) - \sum_{u \in S} \sum_{k \in IF(u)} cv^{\phi}(c_{0,sc}, c_{p(k),k}) - \sum_{k \in S \backslash \{\bar{q}\}} cv^{\phi}(c_{0,sc}, c_{p(k),k}). \end{aligned} \tag{5.15}$$

In the remainder of the part of the proof, we use the following claim which can be seen as a variation of (5.7). For a separate prof of Claim 1, we refer to the Appendix.

Claim 1. There exists a bijection $\eta : S \backslash \{\bar{q}\} \longrightarrow E_{G_S}$, such that for each $k \in S \backslash \{\bar{q}\}$, there is $\eta(k) \in E_{G_S}$ satisfies that $c_{\eta(k)} \geq c_{p(k),k} + 2cv^{\phi}(c_{0,sc}, c_{p(k),k})$.

By Claim 1, we have that

$$M(S, c) = \sum_{e \in E_{G_S}} c(e) \geq \sum_{k \in S \backslash \{\bar{q}\}} \big(c_{p(k),k} + 2cv^{\phi}(c_{0,sc}, c_{p(k),k})\big). \tag{5.16}$$

By combining (5.15) and (5.16), we have that:

$$\begin{aligned} M(S, c) + I_{m^S}^{\phi}(N', 0, c') &\geq \sum_{u \in S} \big\{c_{p(u),u} + cv^{\phi}(c_{0,sc}, c_{p(u),u})\big\} - \sum_{u \in S} \sum_{k \in IF(u)} cv^{\phi}(c_{0,sc}, c_{p(k),k}) \\ &= \sum_{u \in S} I_u^{\phi}(N, 0, c), \end{aligned}$$

where the equality follows from (5.14).

Case 3: Finally, consider $S \subseteq N$ that is not Γ-branch-connected. Let S_1,...,S_t be its $\Gamma-$branch-connected components. Let $N^* = (N \backslash S) \cup \{m^{S_1}, ..., m^{S_2}\}$, and $c^* = c^{S_1,...,S_t}$. By repeated use of Lemma 5.1, for all $u \in \{1, ..., t\}$,

$$\sum_{k \in S_u} I_k^{\phi}(N, 0, c) = M(S_u, c) + I_{m^{S_u}}^{\phi}(N^*, 0, c^*).$$

Hence,

$$\sum_{k \in S} I_k^{\phi}(N, 0, c) = \sum_{u \in \{1,...,t\}} \big[M(S_u, c) + I_{m^{S_u}}^{\phi}(N^*, 0, c^*)\big].$$

Let $N^{'} = (N^* \backslash \{m^{S_1}, ..., m^{S_t}\}) \cup \{m^S\}$, and $c^{'} = c^{*\{m^{S_1},...,m^{S_t}\}}$. Proposition A.1 implies that for any $i, j \in \{1, ..., t\}$, $\{m^{S_i}, m^{S_j}\}$ can not be an edge in the MCST in $(N^*, 0, c^*)$. According to case 2, we have

$$\sum_{u \in \{1,...,t\}} I_{m^{S_u}}^{\phi}(N^*, 0, c^*) \leq M(\{m^{S_1}, ..., m^{S_t}\}, c^*) + I_{m^S}^{\phi}(N^{'}, 0, c^{'}).$$

Thus,

$$\begin{aligned}
\sum_{k \in S} I_k^{\phi}(N, 0, c) &= \sum_{u \in \{1,...,t\}} \big[M(S_u, c) + I_{m^{S_u}}^{\phi}(N^*, 0, c^*)\big] \\
&\leq \sum_{u \in \{1,...,t\}} M(S_u, c) + M(\{m^{S_1}, ..., m^{S_t}\}, c^*) + I_{m^S}^{\phi}(N^{'}, 0, c^{'}) \\
&= M(S, c) + I_{m^S}^{\phi}(N^{'}, 0, c^{'}),
\end{aligned}$$

where the last equation follows from the fact that

$$\sum_{u \in \{1,...,t\}} M(S_u, c) + M(\{m^{S_1}, ..., m^{S_t}\}, c^*) = M(S, c).$$

$\square$

The following theorem states that when we focus on the domain $\mathcal{M}_D^*$, an induced rule satisfies coalitional stability on this domain if its base rule meets condition (5.17).

Theorem 5.2. Let ϕ be a base rule and cv^{ϕ} be the corresponding compensation value function. If cv^{ϕ} satisfies

$$0 \leq cv^{\phi}(a, b) \leq \max\{a - b, 0\}, \text{ for all } a \geq 0, b \geq 0, \tag{5.17}$$

then the induced rule I^{ϕ} satisfies coalitional stability on domain $\mathcal{M_D}^*$.

Proof. Let $(N, 0, c) \in \mathcal{M_D}^*$ be an MCST problem with a unique MCST $\Gamma = (N_0, E)$ and source connection node sc. To prove coalitional stability of I^{ϕ} on $\mathcal{M}_D^*$, we need to show that, for every $(N, 0, c) \in \mathcal{M}_D^*$ and every $S \subseteq N$,

$$\sum_{i \in S} I_i^{\phi}(N, 0, c) \leq m(S, 0, c).$$

First consider $S \subseteq N$ where be a set whose Γ-branch-connected components are all single nodes. According to Theorem 4.1 and (5.17), for all $i \in S$,

$$\begin{aligned} I_i^{\phi}(N,0,c) &= c_{p(i),i} + cv^{\phi}(c_{0,sc}, c_{p(i),i}) - \sum_{j \in IF(i)} cv^{\phi}(c_{0,sc}, c_{p(j),j}) \\ &\leq c_{p(i),i} + cv^{\phi}(c_{0,sc}, c_{p(i),i}) \\ &\leq \max\{c_{p(i),i}, c_{0,sc}\}, \end{aligned}$$

Let $E_{G_{S\cup\{0\}}} \subseteq E^{S\cup\{0\}}$ be such that $\sum_{e \in E_{G_{S\cup\{0\}}}} c(e) = m(S,0,c)$.

For the remainder, we need the following claim in the same style as Claim 1 in the proof of Theorem 5.1. The proof of this claim is similar to that of Claim 1, and therefore omitted.

Claim 2. There exists a bijection $\eta : \hat{S} \to E_{G_{\hat{S}\cup\{0\}}}$, such that for each $i \in \hat{S}$, $\eta(i)$ satisfies $c(\eta(i)) \geq \max\{c_{p(i),i}, c_{0,sc}\}$.

Let η be as in claim 2, we have

$$m(S,0,c) = \sum_{e \in E_{G_{S\cup\{0\}}}} c(e) = \sum_{i \in S} c(\eta(i)) \geq \sum_{i \in S} \max\{c_{p(i),i}, c_{0,sc}\} \geq \sum_{i \in S} I_i^{\phi}(N,0,c). \tag{5.18}$$

Secondly, consider an arbitrary $S \subseteq N$ with t $\Gamma-$branch-connected components ($t \in \mathbb{N}$), denoted by S_1,...,S_t. Let $N^* = (N \backslash S) \cup \{m^{S_1}, ..., m^{S_2}\}$, and $c^* = c^{S_1,...,S_t}$. By repeated use of Lemma 5.1, for all $u \in \{1, ..., t\}$,

$$\sum_{k \in S_u} I_k^{\phi}(N,0,c) = M(S_u, c) + I_{m^{S_u}}^{\phi}(N^*, 0, c^*).$$

By (5.18), we have,

$$\sum_{u \in \{1,...,t\}} I_{m^{S_u}}^{\phi}(N^*, 0, c^*) \leq m(\{m^{S_1}, ..., m^{S_t}\}, 0, c^*)$$

Hence,

$$\begin{aligned} \sum_{i \in S} I_i^{\phi}(N,0,c) &= \sum_{u \in \{1,...,t\}} \sum_{k \in S_u} I_k^{\phi}(N,0,c) \\ &= \sum_{u \in \{1,...,t\}} \left[M(S_u, c) + I_{m^{S_u}}^{\phi}(N^*, 0, c^*) \right] \\ &\leq m(\{m^{S_1}, ..., m^{S_t}\}, 0, c^*) + \sum_{u \in \{1,...,t\}} M(S_u, c) \\ &= m(S,0,c), \end{aligned}$$

where the last equation follows from Proposition A.1. □

Since condition (5.17) is weaker than condition (5.5), straightforwardly, we have the following corollary:

Corollary 5.1. An induced rule obtained from a base rule ϕ that satisfies condition (5.5) is merge-proof and coalitionally stable on $\mathcal{M}_D^*$.

References

Barberà, S., D. Berga, and B. Moreno (2012). Group strategy-proof social choice functions with binary ranges and arbitrary domains: characterization results. *International Journal of Game Theory 41*, 791–808.

Bergantiños, G. and J. J. Vidal-Puga (2007). A fair rule in minimum cost spanning tree problems. *Journal of Economic Theory 137*(1), 326–352.

Bird, C. G. (1976). On cost allocation for a spanning tree: a game theoretic approach. *Networks 6*(4), 335–350.

Bogomolnaia, A. and H. Moulin (2010). Sharing a minimal cost spanning tree: Beyond the folk solution. *Games and Economic Behavior 69*(2), 238–248.

Branzei, R., S. Moretti, H. Norde, and S. Tijs (2003). The p-value for cost sharing in minimum cost spanning tree situations. *Theory and Decision 56*, 47–61.

Dutta, B. and A. Kar (2004). Cost monotonicity, consistency and minimum cost spanning tree games. *Games and Economic Behavior 48*(2), 223–248.

Feltkamp, V., S. Tijs, and S. Muto (1994). On the irreducible core and the equal remaining obligations rule of minimum cost spanning extension problems. CentER Discussion Paper 94106, Tilburg University, the Netherlands.

Gómez-Rúa, M. and J. Vidal-Puga (2011). Merge-proofness in minimum cost spanning tree problems. *International Journal of Game Theory 40*, 309–329.

Gómez-Rúa, M. and J. Vidal-Puga (2017). A monotonic and merge-proof rule in minimum cost spanning tree situations. *Economic Theory 63*(3), 813–826.

Granot, D. and G. Huberman (1984). On the core and nucleolus of minimum cost spanning tree games. *Mathematical programming 29*, 323–347.

Hall, P. (1987). On representatives of subsets. In I. Gessel and G.-C. Rota (Eds.), *Classic Papers in Combinatorics*, pp. 58–62. Boston, MA: Birkhäuser Boston.

Ju, B.-G. and J. D. Moreno-Ternero (2011). Progressive and merging-proof taxation. *International Journal of Game Theory 40*, 43–62.

Kar, A. (2002). Axiomatization of the shapley value on minimum cost spanning tree games. *Games and Economic Behavior 38*(2), 265–277.

Kruskal, J. B. (1956). On the shortest spanning subtree of a graph and the traveling salesman problem. *Proceedings of the American Mathematical society 7*(1), 48–50.

Manjunath, V. (2012). Group strategy-proofness and voting between two alternatives. *Mathematical Social Sciences 63*(3), 239–242.

Moulin, H. (1987). Equal or proportional division of a surplus, and other methods. *International Journal of Game Theory 16*(3), 161–186.

Moulin, H. (2008). Proportional scheduling, split-proofness, and merge-proofness. *Games and Economic Behavior 63*(2), 567–587.

Norde, H. (2019). The degree and cost adjusted folk solution for minimum cost spanning tree games. *Games and Economic Behavior 113*, 734–742.

O'Neill, B. (1982). A problem of rights arbitration from the talmud. *Mathematical social sciences 2*(4), 345–371.

Ozsoy, H. (2006). A characterization of Bird's rule. Working paper, Rice University.

Prim, R. C. (1957). Shortest connection networks and some generalizations. *The Bell System Technical Journal 36*(6), 1389–1401.

Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM Journal on applied mathematics 17*(6), 1163–1170.

Shapley, L. (1953). A value for n-person games. In H. Kuhn and A. Tucker (Eds.), *Contributions to the Theory of Games II*, pp. 307–317. Princeton: Princeton University Press.

Appendix

A.1 MCST w.r.t. a Fixed Edge Set

Clearly, the problem of finding a new MCST after some coalition merges can be transformed into a problem of finding an MCST w.r.t. a fixed edge set. In this section, an algorithm adapted from the reverse-delete algorithm (Kruskal (1956)) is proposed to find an MCST w.r.t. a fixed edge set. As a consequence of this algorithm, it follows that if there is a unique MCST in which all edges have different costs, the MCST after a merger only includes edges from the original MCST.

The reverse-delete algorithm recursively deletes an edge with the highest cost under the condition that the remaining graph stays connected. The procedure of finding an MCST by the reverse delete algorithm is described in Algorithm A.1.

Algorithm A.1: Reverse-delete algorithm

Input: An MCST problem $(N, 0, c)$;
A list of all edges from E^{N_0}: $e_1, e_2, ...$ in descending order with respect to costs.
Initialization: The complete graph (N_0, E^{N_0}).
for $e = e_1, e_2, ...$ **do**
  if *Deleting e from the graph leads to a connected graph* **then**
    delete e from the graph;
  else
    continue;
  end
end

The reverse-delete algorithm can be readily modified to Algorithm A.2 to solve connection problems $(N, 0, c)$ with a fixed edge set $F \subseteq E^{N_0}$ given by

$$\min\{\sum_{e \in E} c(e) \mid E \subseteq E^{N_0} \text{ and } (N_0, E \cup F) \text{ is a connected graph}\}.$$

Algorithm A.2: Reverse-delete algorithm w.r.t. a fixed edge set F

Input: An MCST problem $(N, 0, c)$; a fixed edge set F;
A list of all edges from E^{N_0}: $e_1, e_2, ...$ in descending order regarding costs.
Initialization: Start with the complete graph (N_0, E^{N_0}).
for $e = e_1, e_2, ...$ **do**
  if *$e \notin F$ and deleting e from the graph leads to a connected graph* **then**
    delete e from the graph;
  else
    continue;
  end
end

Of course, if the fixed edge set F is a tree on some nodes in N (as in the setting of mergers), then the resulting optimal structure is a tree too. In such cases, the optimal structure is called an MCST w.r.t. F for simplicity.

Finally, we consider an MCST problem $(N, 0, c)$ with a unique MCST $\Gamma = (N_0, E_\Gamma)$ and where any two edges in the MCST have different costs. Take a set $S \subseteq N$, then the MCST for problem $((N\backslash S) \cup \{m^S\}, 0, c^S)$ after the merger of S is also unique. Take an edge set F so that edges in F form a spanning tree among S. Then one can show that the edges in the MCST w.r.t. F belong to $F \cup E_\Gamma$, as is stated in the following proposition.

Proposition A.1. Let $(N, 0, c) \in \mathcal{M}^*$ with a unique MCST $\Gamma = (N_0, E_\Gamma)$ in which any two edges in the MCST have different costs. Let $S \subseteq N$ with $|S| \geq 2$ and $F \subseteq S$ be a tree such that $\sum_{e \in F} c(e) = M(S, c)$. Moreover, let $E^* \subseteq E^{N_0}$ be the edge set of the MCST w.r.t. F. Then, $E^*\backslash F \subseteq E_\Gamma$.

Proof. Apply Algorithm A.2 w.r.t. F. It suffices to show that all edges in $E^{N_0}\backslash(F \cup E_\Gamma)$ will be deleted during the algorithm.

Consider $e = \{i, j\} \in E^{N_0}\backslash(F \cup E_\Gamma)$. There exists a unique path from i to j in Γ. By the uniqueness of Γ, all edges on this path have a strictly lower cost than e. So e is considered before any of those edges is. Hence, deleting e will not lead to a disconnected graph and e will be deleted by the algorithm. $\square$

A.2 Proof of Claim 1 for Theorem 5.1

For all $E \subseteq E_{G_S}$, $E \neq \varnothing$, let $NG(E) = \{i \in S\backslash\{\bar{q}\} \mid \exists e \in E \text{ s.t. } c(e) > c_{p(i),i}\}$. Using (5.7) and the marriage theorem (Hall (1987)), to show the existence of the desired bijection in the claim, it suffices to show that, for all $E \subseteq E_{G_S}$, $E \neq \varnothing$,

$$|NG(E)| \geq |E|.$$

Let $E \subseteq E_{G_S}$, $E \neq \varnothing$.

We recursively construct a subset $D \subseteq E$ of edges and a subset $M \subseteq NG(E)$ of agents in $|E|$ steps in the following way.

Consider an order σ on E, i.e., $\sigma : \{1, ..., |E|\} \to E$ is a bijection. Let $D^k \subseteq E$ and $M^k \subseteq NG(E)$ be the sets created after step k. By definition, $D = D^{|E|}$, $M = M^{|E|}$.

Initialize $D^0 = \varnothing$ and $M^0 = \varnothing$.

Consider step $k \in \{1, ..., |E|\}$ and let $\sigma(k) = \{i, j\} \in E$. Intuitively speaking, in this step, three things can happen:

- Both the current set of edges and the current set of agents remain unchanged.
- $\sigma(k)$ is added to the current set of edges and either i or j is added to the current set of agents.
- $\sigma(k)$ is added to the current set of edges and both i and j are added to the current set of players.

Formally, we discriminate between the following three cases:

Case 1: $j \in F(i)$.

If $j \in M^{k-1}$, then let $D^k = D^{k-1}$ and $M^k = M^{k-1}$. Otherwise, if $j \notin M^{k-1}$, then let $D^k = D^{k-1} \cup \{\sigma(k)\}$ and $M^k = M^{k-1} \cup \{j\}$.

Here, note that $j \neq \bar{q}$. Moreover, since $p(j) \neq i$ and $\{p(j), j\}$ is on the path between i and j in the unique MCST Γ, we have $c(\sigma(k)) > c_{p(j),j}$. Hence, $j \in NG(E)$.

Case 2: $i \in F(j)$.

Similar to Case 1, if $i \in M^{k-1}$, then let $D^k = D^{k-1}$ and $M^k = M^{k-1}$. Otherwise, if $i \notin M^{k-1}$, then let $D^k = D^{k-1} \cup \{\sigma(k)\}$ and $M^k = M^{k-1} \cup \{i\}$.

Case 3: $j \notin F(i)$, $i \notin F(j)$.

If $i \in M^{k-1}$ or $i = \bar{q}$ and, also, $j \in M^{k-1}$ or $j = \bar{q}$, then let $D^k = D^{k-1}$ and $M^k = M^{k-1}$. Otherwise, let $D^k = D^{k-1} \cup \{\sigma(k)\}$ and $M^k = M^{k-1} \cup \big(\{i, j\} \backslash \{\bar{q}\}\big)$.

Since $j \notin F(i)$, the edge $\{p(i), i\}$ belongs to the path between i and j in Γ, so we have $c(\sigma(k)) > c_{p(j),j}$. Hence, $i \in NG(E)$ as long as $i \neq \bar{q}$. Similarly, we have $j \in NG(E)$ as long as $j \neq \bar{q}$.

By construction, $D \neq \varnothing$, $M \neq \varnothing$ while

$$M \subseteq NG(E) \text{ and } |M| \geq |D|. \tag{A.1}$$

In fact for every connected component $T \in S/D$ in the graph (S, D), we have,

$$\text{if } |T| = 1, \text{ then } T \cap M = \varnothing, \tag{A.2}$$

and,

$$\text{if } |T| \geq 2, \text{ then } |T \cap M| \in \{|D_T|, |D_T| + 1\}, \tag{A.3}$$

where D_T is the set of edges within D between players in T.

Of course, if $D = E$, the proof is finished. From now on, assume $E \backslash D \neq \varnothing$.

Next, in $|E| - |D|$ steps we recursively add every edge in $E \backslash D$ one by one, while, if necessary, at the same time we add a new agent to $NG(E)$ in such a way that, after each step, the newly created M and D still satisfy all properties described in (A.1), (A.2), and (A.3). In this way, after $|E| - |D|$ steps we obtain the edge set E and a corresponding agent set $NG \subseteq NG(E)$ such that

$$|NG(E)| \geq |NG| \geq |E|,$$

which finishes the proof.

Consider an order π on $E \backslash D$, i.e., $\pi : \{1, ..., |E \backslash D|\} \to E \backslash D$ is a bijection. Let $E^k \subseteq E$ and $NG^k \subseteq NG(E)$ be the sets created after step k.

Initialize $E^0 = D$ and $NG^0 = M$.

Consider step $k \in \{1, ..., |E \backslash D|\}$ and let $\pi(k) = \{i, j\} \in E \backslash D$. Clearly $\pi(k)$ connects two components, say R and T, with $R, T \in S/E^{k-1}$. Define

$$F = E_R^{k-1} \cup E_T^{k-1} \cup \{\pi(k)\}$$

and

$$NG_{RT} = NG^{k-1} \cap (R \cup T).$$

Because of condition (A.3) we know

$$|NG_{RT}| \geq |F| - 1.$$

First of all, if $|NG_{RT}| \geq |F|$, we define

$$E^k = E^{k-1} \cup \{\pi(k)\} \text{ and } NG^k = NG^{k-1}.$$

Clearly, E^k and NG^k retain the properties described in (A.1), (A.2) and (A.3).

For the remainder of this proof we assume

$$|NG_{RT}| = |F| - 1.$$

Then, there exists exactly one player $i(R) \in R$ s.t. $i(R) \notin NG^{k-1}$. Similarly, there exists exactly one player $i(T) \in T$ s.t. $i(T) \notin NG^{k-1}$. Note that, since $\{i(R), i(T)\} \notin D$, $i(R) \notin M$ and $i(T) \notin M$, we have that $\{i(R), i(T)\} \notin E$, due to the way D and M are constructed.

Next choose $i^{'}(R)$, $i^{'}(T) \in R \cup T$ on the unique path in $(R \cup T, F)$ connecting $i(R)$ and $i(T)$ s.t. $\{i(R), i^{'}(R)\} \in F$ and $\{i(T), i^{'}(T)\} \in F$. Note that possibily $i^{'}(R) = i^{'}(T)$.

We distinguish between two cases depending on the location of $\bar{q}$:

Case A: $\bar{q} \notin \{i(R), i(T)\}$.

Since $i(R) \notin M$, it follows, by construction of M, that $i^{'}(R) \in F(i(R))$. Similarly, we have $i^{'}(T) \in F(i(T))$. Next, we distinguish between two subcases: $i(R) \in F(i(T))$ and $i(R) \notin F(i(T))$.

Case A1: If $i(R) \in F(i(T))$, there must be an edge e on the path within F connecting $i(R)$ and $i(T)$ between an agent in $F(i(R))$ and an agent not in $F(i(R))$. In this case, define

$$E^k = E^{k-1} \cup \{\pi(k)\} \text{ and } NG^k = NG^{k-1} \cup \{i(R)\}.$$

Note that $\{p(i(R)), i(R)\}$ is on the path between $i(R)$ and $i(T)$ in Γ, while e is not in Γ, so

$$c_e > c_{\{p(i(R)), i(R)\}} \text{ and } i(R) \in NG(E).$$

Moreover, it is readily seen that E^k and NG^k retain the properties described in (A.1), (A.2) and (A.3).

Case A2: If $i(R) \notin F(i(T))$, there must be an edge on the path connecting $i(R)$ and $i(T)$ in $(R \cup T, F)$ between an agent in $F(i(T))$ and an agent not in $F(i(T))$. In this case, define

$$E^k = E^{k-1} \cup \{\pi(k)\} \text{ and } NG^k = NG^{k-1} \cup \{i(T)\}.$$

Using similar reasoning as before, we have that $i(T) \in NG(E)$ and E^k and NG^k retain the properties described in (A.1), (A.2) and (A.3).

Case B: $\bar{q} \in \{i(R), i(T)\}$.

Assume, without loss of generality, that $\bar{q} = i(R)$. Since $i(T) \notin M$, it follows, by construction of M, that $i(T) \in F(i(T))$. Moreover, since $\bar{q} = i(R)$, we know that $i(R) \notin F(i(T))$. Hence, we can proceed in exactly the same way as in Case A2 and define

$$E^k = E^{k-1} \cup \{\pi(k)\} \text{ and } NG^k = NG^{k-1} \cup \{i(T)\}.$$