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A strategic approach to bankruptcy problems based on the TAL family of rules

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Abstract

This paper analyzes bankruptcy problems from a strategic perspective using the parameterized TAL family of bankruptcy rules. We construct a strategic game where every player selects a parameter, and the rule from the TAL family that corresponds to the mean of the chosen parameters is used to divide the estate. We prove the existence of Nash equilibria for this strategic game. In particular, we provide the set of all Nash equilibria for two players, and for more players, we prove existence by constructing a Nash equilibrium of a particular form based on the notion of a pivotal player.

JEL Classification Number: C71, C72

1 Introduction

If an entity has a monetary estate that is insufficient to cover its monetary obligations to its claimants, all with justifiable claims, this leads to a so-called *bankruptcy problem*. A systematic procedure to solve every bankruptcy problem is called a *bankruptcy rule*, or, in short, *rule*. For every bankruptcy problem, a rule prescribes an allocation of the estate, such that no claimant receives more than his claim. The formal game-theoretic study of bankruptcy problems and rules began with O'Neill (1982). For an excellent survey, see Thomson (2019).

Several bankruptcy rules have been proposed in the literature, one of them being CEA (constrained equal awards). The CEA rule allocates the estate as equally as possible, under the condition that no claimant receives more than his claim. Similarly, the CEL (constrained equal losses) rule distributes the losses, i.e. the difference between the claim and the awards, as equal as possible such that no claimant receives a negative amount. Aumann and Maschler (1985) introduced the TAL (Talmud) rule, which is a combination of CEA and CEL, where

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claimants receive no more than half their claim if the sum of claims is greater than twice the estate, and no less than half their claim otherwise. In particular, TAL coincides with the nucleolus of the associated bankruptcy game, as introduced by O'Neill (1982). The TAL *family* of bankruptcy rules, first defined by Moreno-Ternero and Villar (2006), contains CEA, CEL, and TAL. Every rule in the TAL family is characterized by a parameter $\theta \in [0,1]$, representing a measure of the distributive power of the rule. Moreno-Ternero and Villar (2006) studied the axiomatic foundations of the TAL family as a whole, in light of the standard properties of rules in the literature.

Besides the axiomatic approach to bankruptcy problems, there is the *strategic approach* (also called the *Nash program*, see Serrano (2021)). If the allocations of a bankruptcy rule correspond to the payoffs of a Nash equilibrium of an adequate strategic game for every bankruptcy problem, one can consider this to be a strategic justification of this rule. O'Neill (1982) proposed a strategic game in which the Nash equilibria correspond to the minimum overlap bankruptcy rule. Chun (1989) introduced a strategic game where each player is allowed to propose a solution concept. Dagan et al. (1997) capture a strategic dimension of the consistency property of bankruptcy rules. Furthermore, García-Jurado et al. (2006) provide a strategic game where each player declares how much he is willing to concede. Recently, Moreno-Ternero et al. (2022) have provided a strategic justification for each individual rule of the TAL family.

In this paper, we construct a strategic game, called the *strategic TAL game*, in which each player selects a parameter between 0 and 1, and the resulting payoffs are determined by the member of the TAL family corresponding to the *mean* of the selected parameters. The aim of this paper is to get insight into the Nash equilibria of this strategic TAL game. For two-player strategic TAL games, the payoff functions are monotone in their arguments, which enables us to characterize all Nash equilibria. Furthermore, we show that all Nash equilibrium payoffs are equal to the allocation of the Talmud rule, for every bankruptcy problem.

For strategic TAL games with three or more players, the situation is more complicated, as the payoff functions are in general not monotone for every player. We show that the existence of Nash equilibria does not readily follow from the Kakutani fixed point theorem. Instead, a constructive approach is used to prove the existence of Nash equilibria for every strategic TAL game. The Nash equilibrium we construct is based on a pivotal player, such that every player whose claim is lower than this player selects the parameter 1, and every player whose claim is higher selects the parameter 0. This Nash equilibrium reflects the intuition that players with large claims prefer CEL to CEA and vice versa.

Section 2 introduces the TAL family of bankruptcy rules, after which Section 3 formally defines the strategic TAL game. Section 4 fully describes the set of Nash equilibria for every two-player TAL game. Lastly, in Section 5 we construct a specific type of Nash equilibrium, based on pivotal players, for every strategic TAL game.

2 The TAL family of bankruptcy rules

Let N be a finite set of claimants or players. A *bankruptcy problem* with player set N is a tuple (E, c) where $E \in \mathbb{R}_+$ denotes the *estate* and $c \in \mathbb{R}_+^N$ a vector of *claims*, such that $\sum_{j \in N} c_j \geq E$. Let $\mathcal{B}^N$ denote the class of all bankruptcy problems with a fixed player set N.

In this paper, we assume without loss of generality that $N = \{1, \ldots, n\}$ and

$$c_1 \leq \ldots \leq c_n.$$

A *bankruptcy rule* is a map $R : \mathcal{B}^N \to \mathbb{R}^N$ that assigns a vector of *awards* to every bankruptcy problem in such a way that, for every $(E, c) \in \mathcal{B}^N$,

- $0 \leq R_i(E, c) \leq c_i$ for every $i \in N$,
- $\sum_{i \in N} R_i(E, c) = E$.

One way of allocating the estate is to award everyone as equally as possible, under the condition that no player receives more than his claim.

Definition. The *constrained equal awards* (CEA) rule is defined by

$$CEA_i(E, c) = \min\{c_i, \lambda\},$$

for every $(E, c) \in \mathcal{B}^N$ and $i \in N$, where $\lambda \in \mathbb{R}$ is such that $\sum_{j \in N} \min\{c_j, \lambda\} = E$.

In the same way, one can distribute the losses (i.e. the difference between the claim and the awards of each player) as equally as possible under the condition that no player is allocated a negative amount.

Definition. The *constrained equal losses* (CEL) rule is defined by

$$CEL_i(E, c) = \max\{c_i - \lambda, 0\},$$

for every $(E, c) \in \mathcal{B}^N$ and $i \in N$, where $\lambda \in \mathbb{R}$ is such that $\sum_{j \in N} \max\{c_j - \lambda, 0\} = E$.

The *Talmud rule* is a combination of CEA and CEL. If the estate is insufficient to guarantee everyone half his claim, CEA is used as the allocation principle. If the estate is sufficient to cover all half-claims, every player gets half of his claim, and the remainder is allocated using CEL.

Definition (Aumann and Maschler (1985))**.** The *Talmud (TAL)* rule is defined by

$$TAL(E, c) = \begin{cases} CEA(E, \frac{1}{2}c) & \text{if } \frac{1}{2}\sum_{i \in N} c_i \geq E, \\ \frac{1}{2}c + CEL(E - \frac{1}{2}\sum_{i \in N} c_i, \frac{1}{2}c) & \text{else,} \end{cases}$$

for every $(E, c) \in \mathcal{B}^N$.

Using the fact that CEA and CEL are dual rules, i.e. it holds for every $(E,c) \in \mathcal{B}^N$ that

$$CEL(E,c) = c - CEA(\sum_{i\in N} c_i - E, c), \tag{2.1}$$

we can also write

$$TAL(E,c) = \begin{cases} CEA(E, \frac{1}{2}c) & \text{if } \frac{1}{2}\sum_{i\in N} c_i \geq E, \\ c - CEA(\sum_{i\in N} c_i - E, \frac{1}{2}c) & \text{else,} \end{cases}$$

for every $(E,c) \in \mathcal{B}^N$.

Extending the Talmud rule, Moreno-Ternero and Villar (2006) introduced the TAL *family*. Each member of the TAL family is characterized by a parameter $\theta \in [0,1]$, representing a degree of the distributive power of the bankruptcy rule.

Definition (Moreno-Ternero and Villar (2006))**.** Let $\theta \in [0,1]$. The rule TAL^θ is defined by

$$TAL^\theta(E,c) = \begin{cases} CEA(E, \theta c) & \text{if } \theta\sum_{i\in N} c_i \geq E, \\ \theta c + CEL(E - \theta\sum_{i\in N} c_i, (1-\theta)c) & \text{else,} \end{cases}$$

for every $(E,c) \in \mathcal{B}^N$.

Using (2.1), one can also write

$$TAL^\theta(E,c) = \begin{cases} CEA(E, \theta c) & \text{if } \theta\sum_{i\in N} c_i \geq E, \\ c - CEA(\sum_{i\in N} c_i - E, (1-\theta)c) & \text{else,} \end{cases} \tag{2.2}$$

for every $(E,c) \in \mathcal{B}^N$.

Clearly, $TAL^0 = CEL$, $TAL^1 = CEA$ and $TAL^{\frac{1}{2}} = TAL$. Furthermore, it is readily seen that $TAL^\theta(E,c)$ is continuous in θ for every $(E,c) \in \mathcal{B}^N$. In fact, we show that the function $TAL_i^\theta(E,c)$ is piecewise linear in θ for every $i \in N$ and $(E,c) \in \mathcal{B}^N$. To this end, for every bankruptcy problem $(E,c) \in \mathcal{B}^N$, we define the points $a_1(E,c), \ldots, a_n(E,c)$ and $b_n(E,c), \ldots, b_1(E,c)$, with

$$a_1(E,c) \leq \ldots \leq a_n(E,c) \leq b_n(E,c) \leq \ldots \leq b_1(E,c), \tag{2.3}$$

given by

$$\begin{aligned} a_k(E,c) &= 1 - \frac{C_n - E}{C_k + (n-k)c_k}, \\ b_k(E,c) &= \frac{E}{C_k + (n-k)c_k}, \end{aligned} \tag{2.4}$$

for every $k \in \{1, \ldots, n\}$. Here, we use the notation $C_k = \sum_{j=1}^k c_j$. Notice that not all the points in (2.4) need to be contained in the interval $[0,1]$. We will often abbreviate $a_k = a_k(E,c)$ and $b_k = b_k(E,c)$ to ease notation.

Note that

$$a_n(E, c) = b_n(E, c) = \frac{E}{C_n}$$

and

$$\begin{cases} a_1(E, c) = 1 - \frac{C_n - E}{nc_1}, \\ b_1(E, c) = \frac{E}{nc_1}, \end{cases}$$

for every $(E, c) \in \mathcal{B}^N$. Furthermore, if $(E, c) \in \mathcal{B}^N$ is such that $c_i = c_j$ for some $i, j \in N$, we have $a_i(E, c) = a_j(E, c)$ and $b_i(E, c) = b_j(E, c)$.

The points $b_n(E, c), \ldots, b_1(E, c)$ can be used to rewrite the function $CEA_i(E, \theta c)$ for every $(E, c) \in \mathcal{B}^N$, $i \in N$ and $\theta \in [0, 1]$.

Lemma 2.1. *Let $(E, c) \in \mathcal{B}^N$, $i \in N$ and $\theta \in [0, 1]$. Then,*

$$CEA_i(E, \theta c) = \begin{cases} \theta c_i & \text{if } \theta \leq b_i(E, c), \\ \frac{E - \theta C_k}{n-k} & \text{if } b_{k+1}(E, c) \leq \theta \leq b_k(E, c) \text{ and } k \in \{1, \ldots, i-1\}, \\ \frac{E}{n} & \text{if } \theta \geq b_1(E, c). \end{cases}$$

Proof. Let $\theta \leq b_i$. Then $\theta(C_i + (n-i)c_i) \leq E$, and hence it is possible to allocate every player $j \in \{1, \ldots, i\}$ his claim θc_j, and the remaining players $k \in \{i+1, \ldots, n\}$ are guaranteed at least θc_i. Consequently,

$$CEA_i(E, \theta c) = \theta c_i. \tag{2.5}$$

Next, let $\theta \in [b_{k+1}, b_k]$ and $k \in \{1, \ldots, i-1\}$. Then

$$\theta \left[C_k + (n-k)c_k\right] \leq E \leq \theta \left[C_{k+1} + (n-(k+1))c_k\right].$$

Hence it is possible to allocate every player $j \in \{1, \ldots, k\}$ his claim θc_k, and the remaining players are allocated equally a value between θc_k and θc_{k+1}. In other words,

$$CEA_j(E, \theta c) = \begin{cases} \theta c_j & \text{if } j \in \{1, \ldots, k\}, \\ \frac{E - C_k}{n-k} & \text{if } j \in \{k+1, \ldots, n\}. \end{cases} \tag{2.6}$$

Consequently, $CEA_i(E, \theta c) = \frac{E - C_k}{n-k}$.

Finally, let $\theta \geq b_1$. Then $\theta(nc_1) \geq E$, which means no player can be allocated his full claim. Therefore

$$CEA_j(E, c) = \frac{E}{n}, \tag{2.7}$$

for all $j \in N$. Consequently, $CEA_i(E, \theta c) = \frac{E}{n}$. $\square$

Using Lemma 2.1, and the points $a_1(E,c),\ldots a_n(E,c)$ for the case where $\theta\sum_{j\in N} c_j \le E$ in equation (2.2), we are now able to provide a full piecewise linear description of $TAL_i^\theta(E,c)$ for every $(E,c)\in\mathcal{B}^N$, $i\in N$ and $\theta\in[0,1]$.

Proposition 2.2. *Let $(E,c)\in\mathcal{B}^N$, $i\in N$ and $\theta\in[0,1]$. Then,*

$$TAL_i^\theta(E,c)=\begin{cases} c_i-\frac{C_n-E}{n} & \text{if } \theta\le a_1,\\ c_i-\frac{(C_n-C_k)-E}{n-k}-\theta\frac{C_k}{n-k} & \text{if } \theta\in[a_k,a_{k+1}] \text{ and } k\in\{1,\ldots,i-1\},\\ \theta c_i & \text{if } \theta\in[a_i,b_i],\\ \frac{E-\theta C_k}{n-k} & \text{if } \theta\in[b_{k+1},b_k] \text{ and } k\in\{1,\ldots,i-1\},\\ \frac{E}{n} & \text{if } \theta\ge b_1.\end{cases}$$

Proof. Note that, if $\theta\ge b_n=\frac{E}{\sum_{j\in N}c_j}$, we have $\theta\sum_{j\in N}c_j\ge E$, and due to Lemma 2.1

$$TAL_i^\theta(E,c)=\begin{cases}\theta c_i & \text{if } \theta\in[b_n,b_i],\\ \frac{E-\theta C_k}{n-k} & \text{if } \theta\in[b_{k+1},b_k] \text{ and } k\in\{1,\ldots,i-1\},\\ \frac{E}{n} & \text{if } \theta\ge b_1.\end{cases}$$

So, let $\theta\sum_{j\in N}c_j\le E$ (i.e. $\theta\le a_n=b_n$). Then,

$$TAL_i^\theta(E,c)=c_i-CEA_i\left(\sum_{j\in N}c_j-E,(1-\theta)c\right), \tag{2.8}$$

and, using Lemma 2.1, it follows that

$$CEA_i\left(\sum_{j\in N}c_j-E,(1-\theta)c_j\right)=\begin{cases}(1-\theta)c_i & \text{if } (1-\theta)\le\tilde{b}_i,\\ \frac{C_n-E-(1-\theta)C_k}{n-k} & \text{if } (1-\theta)\in[\tilde{b}_{k+1},\tilde{b}_k]\\ & \text{and } k\in\{1,\ldots,i-1\},\\ \frac{C_n-E}{n} & \text{if } (1-\theta)\ge\tilde{b}_1,\end{cases} \tag{2.9}$$

where $\tilde{b}_k=b_k\left(\sum_{j\in N}c_j-E,c\right)$ for all $k\in\{1,\ldots,n\}$.

Now, note that for all $k\in\{1,\ldots,n\}$

$$1-\tilde{b}_k=1-b_k\left(\sum_{j\in N}c_j-E,c\right)=1-\frac{C_n-E}{C_k+(n-k)}=a_k(E,c).$$

Consequently, with $a_k=a_k(E,c)$ for every $k\in\{1,\ldots,n\}$, (2.8) and (2.9) imply

$$TAL_i^\theta(E,c)=\begin{cases} c_i-\frac{C_n-E}{n} & \text{if } \theta\le a_1,\\ c_i-\frac{(C_n-C_k)-E}{n-k}-\theta\frac{C_k}{n-k} & \text{if } \theta\in[a_k,a_{k+1}],\\ \theta c_i & \text{if } \theta\ge a_i.\end{cases}$$

□

Note that Proposition 2.2 implies that $TAL_i^\theta(E,c)$ is strictly increasing on $[a_i,b_i]\cap[0,1]$, and $TAL_i^\theta(E,c)$ is weakly decreasing on $[0,a_i]$ and $[b_i,1]$, for every $i\in N$ and $(E,c)\in\mathcal{B}^N$.

We now illustrate the previous results by means of an example, and, in particular, we use a picture that turns out to be helpful in our further analysis.

Example 2.1. Consider the bankruptcy problem $(E,c)\in\mathcal{B}^N$, with $N=\{1,2,3,4\}$, $E=550$ and $c=(150,200,250,350)$. First, note that

$$\begin{aligned}
a_1 &= 1-\frac{400}{600}=\frac{1}{3} & b_4 &= \frac{550}{950}=\frac{11}{19}\\
a_2 &= 1-\frac{400}{750}=\frac{7}{15} & b_3 &= \frac{550}{850}=\frac{11}{17}\\
a_3 &= 1-\frac{400}{850}=\frac{9}{17} & b_2 &= \frac{550}{750}=\frac{11}{15}\\
a_4 &= 1-\frac{400}{950}=\frac{11}{19} & b_1 &= \frac{550}{600}=\frac{11}{12}.
\end{aligned}$$

Using Proposition 2.2, we are able to calculate $TAL_i^\theta(E,c)$, for every $i\in N$, as a function of $\theta\in[0,1]$. For example, we have

$$TAL_2^\theta(E,c)=\begin{cases}100 & \text{if } 0\le\theta\le a_1,\\ \frac{350}{3}-50\cdot\theta & \text{if } \theta\in[a_1,a_2],\\ 200\cdot\theta & \text{if } \theta\in[a_2,b_2],\\ \frac{550}{3}-50\cdot\theta & \text{if } \theta\in[b_2,b_1],\\ \frac{550}{4} & \text{if } b_1\le\theta\le 1.\end{cases}$$

It follows from the above description that $TAL_2^\theta(E,c)$ is weakly decreasing on $[0,a_2]$, then strictly increasing on $[a_2,b_2]$ and lastly weakly decreasing on $[b_2,1]$. In Figure 1 the function $TAL_i^\theta(E,c)$ is displayed in a graph for every player $i\in N$.

Finally, note that $TAL_1^\theta(E,c)$ is weakly increasing, and $TAL_4^\theta(E,c)$ is weakly decreasing on $[0,1]$. $\diamond$

The final remark in Example 2.1 can be generalized. Using Proposition (2.2), we can write

$$TAL_1^\theta(E,c)=\begin{cases}c_1-\frac{C_n-E}{n} & \text{if } 0\le\theta\le a_1,\\ \theta c_1 & \text{if } \theta\in[a_1,b_1],\\ \frac{E}{n} & \text{if } b_1\le\theta\le 1,\end{cases}$$

and, similarly, for every $(E,c)\in\mathcal{B}^N$

$$TAL_n^\theta(E,c)=\begin{cases}c_n-\frac{C_n-E}{n} & \text{if } 0\le\theta\le a_1,\\ c_n-\frac{(C_n-C_k)-E}{n-k}-\theta\frac{C_k}{n-k} & \text{if } \theta\in[a_k,a_{k+1}] \text{ and } k\in\{1,\ldots,n-1\},\\ \theta c_n & \text{if } \theta\in[a_n,b_n],\\ \frac{E-\theta C_k}{n-k} & \text{if } \theta\in[b_{k+1},b_k] \text{ and } k\in\{1,\ldots,n-1\},\\ \frac{E}{n} & \text{if } b_1\le\theta\le 1,\end{cases}$$

From this, one readily obtains the following result.

Proposition 2.3. *$TAL_1^\theta(E,c)$ is weakly increasing in θ and $TAL_n^\theta(E,c)$ is weakly decreasing in θ for every $(E,c) \in \mathcal{B}^N$.*

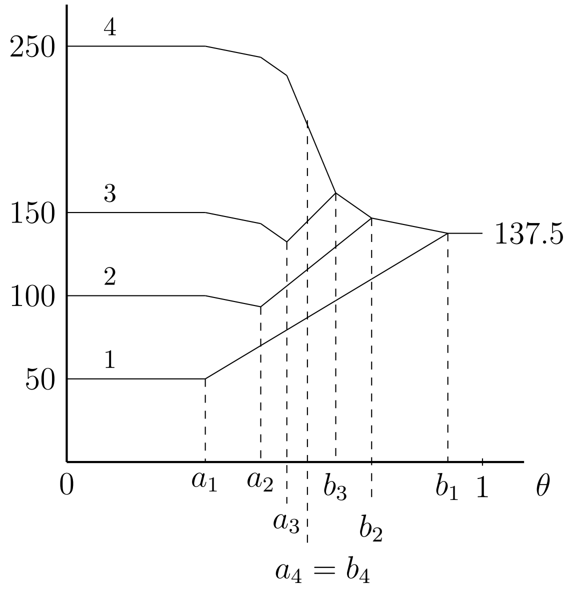

Figure 1: $TAL^\theta(E,c)$ in Example 2.1.

3 Strategic TAL games

The strategic approach we propose in this paper is the following. Given a bankruptcy problem, we let each player select a parameter between 0 and 1. The rule of the TAL family corresponding to the *mean* of the selected parameters is then used to allocate the estate.

Definition. For every bankruptcy problem $(E,c) \in \mathcal{B}^N$, we define the corresponding *strategic TAL game* $G^{TAL}(E,c)$ by

$$G^{TAL}(E,c) = ((\Theta_i, \pi_i))_{i \in N},$$

where $\Theta_i = [0,1]$ is player i's strategy space, for every $i \in N$, and

$$\pi_i(\theta_1, \ldots, \theta_n) = TAL_i^{\frac{1}{n}\sum_{j \in N} \theta_j}(E,c)$$

is player i's payoff function, for every $i \in N$.

The focus of this paper is to gain insight into the *Nash equilibria* (Nash (1951)) of strategic TAL games. A *Nash equilibrium* of a strategic TAL game $G^{TAL}(E,c) = ((\Theta_i, \pi_i))_{i \in N}$ is a strategy profile $(\theta_i)_{i \in N} \in \prod_{i \in N} \Theta_i$ such that for every $i \in N$ and $\theta_i' \in \Theta_i$ it holds that

$$\pi_i(\theta_i, \theta_{-i}) \geq \pi_i(\theta_i', \theta_{-i}).$$

That is, for every player $i \in N$, the strategy $\theta_i \in \Theta_i$ is a *best reply* to the strategies θ_{-i} played by the other players.[1] For every player $i \in N$ and $\theta_{-i} \in \prod_{j \in N \setminus \{i\}} \Theta_j$, we write

$$B_i(\theta_{-i}) = \left\{\theta_i \in \Theta_i \mid \pi_i(\theta_i, \theta_{-i}) \geq \pi_i(\theta_i', \theta_{-i}) \text{ for all } \theta_i' \in \Theta_i\right\}$$

as the set of best replies. Clearly, a Nash equilibrium is a strategy profile $(\theta_j)_{j \in N}$ such that $\theta_i \in B_i(\theta_{-i})$ for every $i \in N$. Denote by $\mathcal{N}\left(G^{TAL}(E,c)\right)$ the set of Nash equilibria of the game $G^{TAL}(E,c)$.

Proposition 2.3, together with the fact that $\pi_i((\theta_j)_{j \in N}) = TAL_i^{\sum_{j \in N} \theta_j}(E,c)$, for every $(E,c) \in \mathcal{B}^N$, implies

$$\begin{cases} 1 \in B_1(\theta_{-1}) & \text{for all } \theta_{-1} \in \prod_{j \in N \setminus \{1\}} \Theta_j, \\ 0 \in B_n(\theta_{-n}) & \text{for all } \theta_{-n} \in \prod_{j \in N \setminus \{n\}} \Theta_j. \end{cases} \tag{3.1}$$

Next, we provide a full description of the set of Nash equilibria of two-player strategic TAL games. By construction, every two-player strategic TAL game is a constant-sum game. This implies that all Nash equilibria lead to the same payoff vector. Combining this with (3.1) we obtain the following result.

Proposition 3.1. *Let $(E,c) \in \mathcal{B}^N$ with $N = \{1,2\}$. Then, $(1,0) \in \mathcal{N}\left(G^{TAL}(E,c)\right)$. As a consequence, all Nash equilibria of $G^{TAL}(E,c)$ lead to the payoff vector given by $TAL^{\frac{1}{2}}(E,c) = TAL(E,c)$.*

The following bankruptcy problem is an example where the corresponding two-player strategic TAL game has more than one Nash equilibrium.

Example 3.1. Consider $(E,c) \in \mathcal{B}^N$ with $N = \{1,2\}$, $E = 450$ and $c = (250, 400)$. Clearly, we have

$$a_1 = 1 - \frac{200}{500} = \frac{3}{5} \qquad b_2 = \frac{450}{650} = \frac{9}{13}$$
$$a_2 = 1 - \frac{200}{650} = \frac{9}{13} \qquad b_1 = \frac{450}{500} = \frac{9}{10}.$$

It follows from Proposition 2.2 that

$$TAL_1^{\theta}(E,c) = \begin{cases} 150 & \text{if } 0 \leq \theta \leq \frac{3}{5}, \\ 250 \cdot \theta & \text{if } \frac{3}{5} \leq \theta \leq \frac{9}{10}, \\ 225 & \text{if } \frac{9}{10} \leq \theta \leq 1, \end{cases}$$

$$TAL_2^{\theta}(E,c) = \begin{cases} 300 & \text{if } 0 \leq \theta \leq \frac{3}{5}, \\ 450 - 250 \cdot \theta & \text{if } \frac{3}{5} \leq \theta \leq \frac{9}{10}, \\ 225 & \text{if } \frac{9}{10} \leq \theta \leq 1, \end{cases}$$

[1] For a vector $x \in \mathbb{R}^N$ and a subset of players $S \subset N$, we denote by x_{-S} the restriction of x to $N \setminus S$. Moreover, $x_{-\{i\}}$ is abbreviated to x_{-i}.

as depicted in Figure 2. The set $B_1(\theta_2)$ of all the best replies of Player 1 against θ_2 is given by

$$B_1(\theta_2) = \begin{cases} [0,1] & \text{if } 0 \leq \theta_2 \leq \frac{1}{5}, \\ \{1\} & \text{if } \frac{1}{5} < \theta_2 \leq \frac{4}{5}, \\ [\frac{9}{5} - \theta_2, 1] & \text{if } \frac{4}{5} \leq \theta_2 \leq 1, \end{cases}$$

which is depicted in Figure 3.

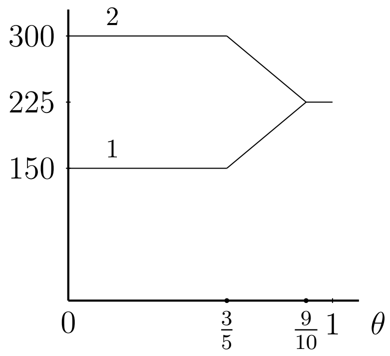

Figure 2: $TAL^{\theta}(E,c)$ in Example 3.1 as a function of θ.

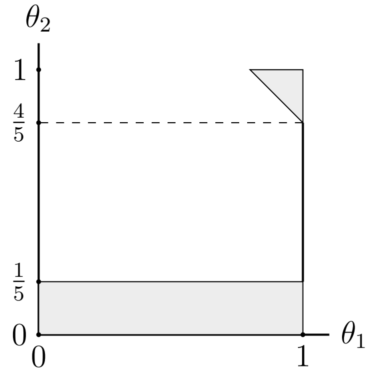

Figure 3: The best reply correspondence for Player 1 in Example 3.1.

To see this, the following intuitive reasoning can be applied.
If $\theta_2 \leq \frac{1}{5}$, then all replies $\theta_1 \in [0,1]$ lead to a mean below $\frac{3}{5}$, and therefore all lead to a payoff to player 1 of 150.

If $\frac{1}{5} < \theta_2 \leq \frac{4}{5}$, then $\theta_1 = 1$ leads to a mean between $\frac{3}{5}$ and $\frac{9}{10}$. Clearly, replies $\theta_1 \in [0,1)$ lead to a strictly lower mean, and therefore lead to a strictly lower payoff to player 1.

Finally, if $\frac{4}{5} \leq \theta_2 \leq 1$, then exactly all replies $\theta_1 \in [\frac{9}{5} - \theta_2, 1]$ lead to a mean of at least $\frac{9}{10}$ and therefore to the highest possible payoff to player 1 of 225.

Similarly, the set $B_2(\theta_1)$ of all best replies of Player 2 against θ_1 is given by

$$B_2(\theta_1) = \begin{cases} [0,1] & \text{if } 0 \le \theta_1 \le \frac{1}{5}, \\ [0, \frac{6}{5} - \theta_1] & \text{if } \frac{1}{5} \le \theta_1 \le 1, \end{cases}$$

as depicted in Figure 4.

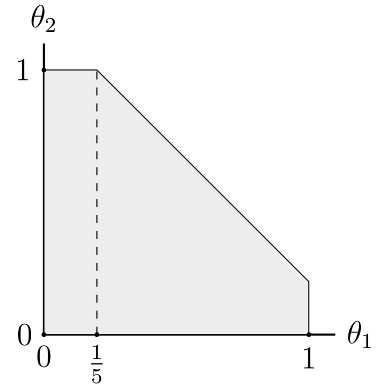

Figure 4: The best reply correspondence for Player 2 in Example 3.1.

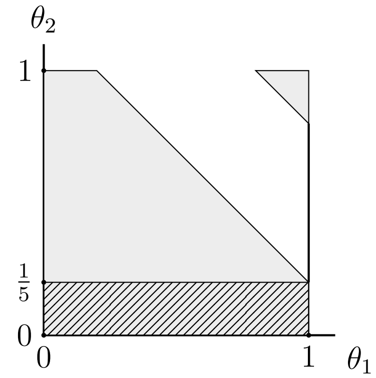

Figure 5: The set of Nash equilibria from Example 3.1, indicated by the dashed area.

From Figure 5, in which the intersection of two best reply correspondences is depicted, it can be concluded that

$$\mathcal{N}\left(G^{TAL}(E,c)\right) = [0,1] \times [0, \frac{1}{5}]. \qquad \diamond$$

The next theorem shows that the set of Nash equilibria of a strategic TAL game depends on the exact position of the estate with respect to the individual claims.

Theorem 3.2. *Let $(E,c) \in \mathcal{B}^N$ with $N = \{1,2\}$. Then,*

$$\mathcal{N}\left(G^{TAL}(E,c)\right) = \begin{cases} [0,1] \times [0,1] & \text{if } c_1 = c_2, \\ [0,1] \times [0, \frac{E-c_2}{c_1}] & \text{if } c_1 < c_2 \leq E, \\ [\frac{E}{c_1}, 1] \times [0,1] & \text{if } E \leq c_1 < c_2, \\ \{(1,0)\} & \text{if } c_1 < E < c_2. \end{cases}$$

Proof. Let $G^{TAL}(E,c) = (([0,1], \pi_1), ([0,1], \pi_2))$ be the strategic TAL game corresponding to the bankruptcy problem (E,c). Then,

$$\pi_1(\theta_1, \theta_2) = \begin{cases} c_1 - \frac{c_1+c_2-E}{2} & \text{if } \frac{\theta_1+\theta_2}{2} \leq a_1, \\ \left(\frac{\theta_1+\theta_2}{2}\right) c_1 & \text{if } a_1 \leq \frac{\theta_1+\theta_2}{2} \leq b_1, \\ \frac{E}{2} & \text{if } b_1 \leq \frac{\theta_1+\theta_2}{2}, \end{cases}$$

and, since $E = \pi_1(\theta_1, \theta_2) + \pi_2(\theta_1, \theta_2)$,

$$\pi_2(\theta_1, \theta_2) = \begin{cases} c_2 - \frac{c_1+c_2-E}{2} & \text{if } \frac{\theta_1+\theta_2}{2} \leq a_1, \\ E - \left(\frac{\theta_1+\theta_2}{2}\right) c_1 & \text{if } a_1 \leq \frac{\theta_1+\theta_2}{2} \leq b_1, \\ \frac{E}{2} & \text{if } b_1 \leq \frac{\theta_1+\theta_2}{2}, \end{cases}$$

for all $\theta_1, \theta_2 \in [0,1]$.

If $c_1 = c_2$, then $\mathcal{N}\left(G^{TAL}(E,c)\right) = [0,1] \times [0,1]$, since in this case $a_1 = (a_2 = b_2) = b_1$ and therefore

$$\pi_1(\theta_1, \theta_2) = \pi_2(\theta_1, \theta_2) = \frac{E}{2},$$

for all $\theta_1, \theta_2 \in [0,1]$.

Next, let $c_1 < c_2 \leq E$. Then, $0 \leq \frac{E-c_2}{c_1} \leq 1$. To prove that

$$\mathcal{N}\left(G^{TAL}(E,c)\right) = [0,1] \times [0, \frac{E-c_2}{c_1}],$$

it suffices to show that $B_2(\theta_1) = [0, \frac{E-c_2}{c_1}]$ for all $\theta_1 \in [0,1]$, and that $B_1(\theta_2) = [0,1]$ for all $\theta_2 \in [0, \frac{E-c_2}{c_1}]$.

Let $\theta_1 \in [0,1]$. Then every $\theta_2 \in [0, \frac{E-c_2}{c_1}]$ leads to a mean less or equal to than $a_1 = 1 - \frac{c_1+c_2-E}{2c_1}$ and consequently to the same payoff to player 2. Moreover, every $\theta_2 \in (\frac{E-c_2}{c_1}, 1]$ leads to a mean strictly higher than a_1, and therefore to a lower payoff to player 2. This implies $B_2(\theta_1) = [0, \frac{E-c_2}{c_1}]$ for every $\theta_1 \in [0,1]$.
Next, let $\theta_2 \in [0, \frac{E-c_2}{c_1}]$. Then, for every $\theta_1 \in [0,1]$, the mean is smaller than a_1. This means that the payoff to player 1 is constant. Therefore $B_1(\theta_2) = [0,1]$ for every $\theta_2 \in [0, \frac{E-c_2}{c_1}]$.

We omit the proof of the case where $E \leq c_1 < c_2$, because it is analogous to the case where $c_1 < c_2 \leq E$.

Finally, let $c_1 < E < c_2$. Then $a_1 < \frac{1}{2} < b_1$. We will prove that

$$\mathcal{N}\left(G^{TAL}(E,c)\right) = \{(1,0)\}.$$

Proposition 3.1 guarantees that $(1,0) \in \mathcal{N}\left(G^{TAL}(E,c)\right)$. To see that this is the only Nash equilibrium, note that all equilibria must have the same payoff vector $TAL^{\frac{1}{2}}(E,c) = (c_1/2, E - c_1/2)$. Since $a_1 < \frac{1}{2} < b_1$, we have that $TAL_1^{\theta}(E,c)$ is strictly increasing in a neighborhood of $\theta = \frac{1}{2}$. Similarly, $TAL_2^{\theta}(E,c)$ is strictly decreasing in a neighborhood of $\theta = \frac{1}{2}$. This, together with the fact that $TAL_1^{\theta}(E,c)$ is weakly increasing and $TAL_2^{\theta}(E,c)$ is weakly decreasing, implies that $TAL^{\theta}(E,c) \neq TAL^{\frac{1}{2}}(E,c)$ for $\theta \in [0,1] \setminus \{\frac{1}{2}\}$. Now, let $(\theta_1, \theta_2) \in \mathcal{N}\left(G^{TAL}(E,c)\right)$. The argument above implies

$$\frac{\theta_1 + \theta_2}{2} = \frac{1}{2}.$$

If $\theta_1 < 1$, then player 1 can improve his payoff by selecting a parameter θ_1' close to θ_1 with $\theta_1' > \theta_1$, because $TAL_1^{\theta}(E,c)$ is strictly increasing around $\theta = \frac{1}{2}$. Therefore, $\theta_1 = 1$. Similarly, if $\theta_2 > 0$, player 2 can improve his payoff by selecting a parameter θ_2' close to θ_2 with $\theta_2' < \theta_2$. Hence, $\theta_2 = 0$. $\square$

4 Constructing a Nash equilibrium in n-player TAL games

In this section, a specific type of Nash equilibrium is constructed for every strategic TAL game with three or more players. As a first illustration, we analyze a three-player strategic TAL game.

Example 4.1. Consider the bankruptcy problem $(E,c) \in \mathcal{B}^N$ with $N = \{1,2,3\}$, $E = 600$ and $(c_1, c_2, c_3) = (150, 300, 450)$.

By (3.1) we have that $1 \in B_1(\theta_2, \theta_3)$ and $0 \in B_3(\theta_1, \theta_2)$ for all $\theta_1, \theta_2, \theta_3 \in [0,1]$. Therefore,

$$\left\{ (1, \theta_2^*, 0) \;\middle|\; \theta_2^* \in \underset{\theta_2 \in [0,1]}{\arg\max}\, \pi_2(1, \theta_2, 0) \right\} \subseteq \mathcal{N}\left(G^{TAL}(E,c)\right). \tag{4.1}$$

Note that the set in (4.1) is non-empty because $\pi_2(1, \theta_2, 0)$ is continuous in θ_2 and the maximum is taken over a compact interval.

Using Proposition 2.2,

$$TAL_2^{\theta}(E,c) = \begin{cases} 200 & \text{if } 0 \leq \theta \leq \frac{1}{3}, \\ 225 - 75\theta & \text{if } \frac{1}{3} \leq \theta \leq \frac{3}{5}, \\ 300\theta & \text{if } \frac{3}{5} \leq \theta \leq \frac{4}{5}, \\ 300 - 75\theta & \text{if } \frac{4}{5} \leq \theta \leq 1, \end{cases}$$

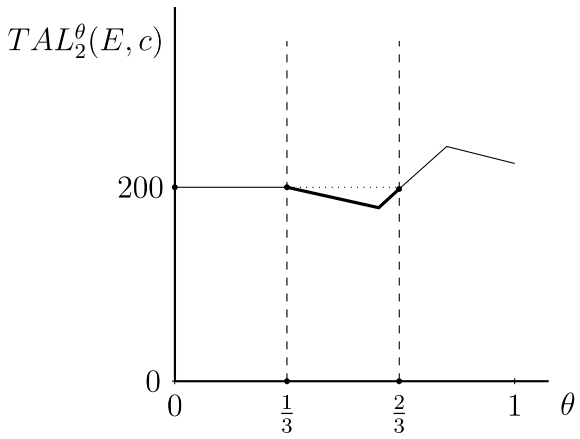

Figure 6: $TAL_2^\theta(E,c)$ in Example 4.1.

as depicted in Figure 6. For strategy combinations of the type $(1,\theta_2,0)$ the mean will be between $\frac{1}{3}$ (when $\theta_2=0$) and $\frac{2}{3}$ (when $\theta_2=1$). Since $TAL_2^{\frac{1}{3}}(E,c)=TAL_2^{\frac{2}{3}}(E,c)=200$, it follows that

$$\underset{\theta_2\in[0,1]}{\arg\max}\,\pi_2(1,\theta_2,0)=\{0,1\}. \tag{4.2}$$

Hence, $\{(1,0,0),(1,1,0)\}\subseteq\mathcal{N}\left(G^{TAL}(E,c)\right)$. $\diamond$

In particular, Example 4.1 illustrates that the set of best replies need not be convex. This means that the Kakutani fixed point theorem, which is commonly used to prove the existence of Nash equilibria, cannot be applied in the conventional way.

In Example 4.1, the constructed equilibria are such that the players with a claim lower than player 2 select the strategy $\theta=1$, and players with a claim higher than player 2 select the strategy $\theta=0$. Here, player 2 acts as a 'pivotal player'. In Example 4.2 we construct a similar type of Nash equilibrium, using a pivotal player, for a strategic TAL game with four players, but as the example shows, a more subtle approach is needed.

Example 4.2. Consider the bankruptcy problem $(E,c)\in\mathcal{B}^N$ with $N=\{1,2,3,4\}$, $E=600$ and $c=(100,200,350,500)$. We claim that

$$\left(1,1,\frac{2}{5},0\right)\in\mathcal{N}\left(G^{TAL}(E,c)\right). \tag{4.3}$$

We show (4.3) by showing that all the strategies involved are mutual best replies.

From (3.1) it immediately follows that the strategy $\theta=1$ for player 1 and the strategy $\theta=0$ for player 4 are best replies.

For player 2, strategy combinations of the type $(1,\theta_2,\frac{2}{5},0)$ lead to a mean between $\frac{2}{5}$ (when $\theta_2=0$) and $\frac{13}{20}$ (when $\theta_2=1$). From Figure 7 it can be seen that $\theta_2=1$ is indeed a best reply.

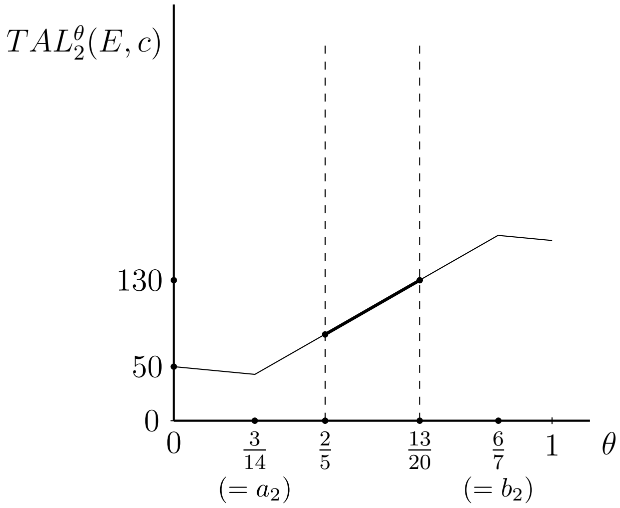

Figure 7: $TAL_2^\theta(E,c)$ in Example 4.2.

Lastly, for player 3, strategy combinations of the type $(1, 1, \theta_3, 0)$ lead to a mean between $\frac{1}{2}$ (if $\theta_3 = 0$) and $\frac{3}{4}$ (if $\theta_3 = 1$). From Figure 8 it can be seen that playing $\theta_3 = \frac{2}{5}$ is indeed a best reply, leading to an average of $b_3 = \frac{3}{5}$. $\diamond$

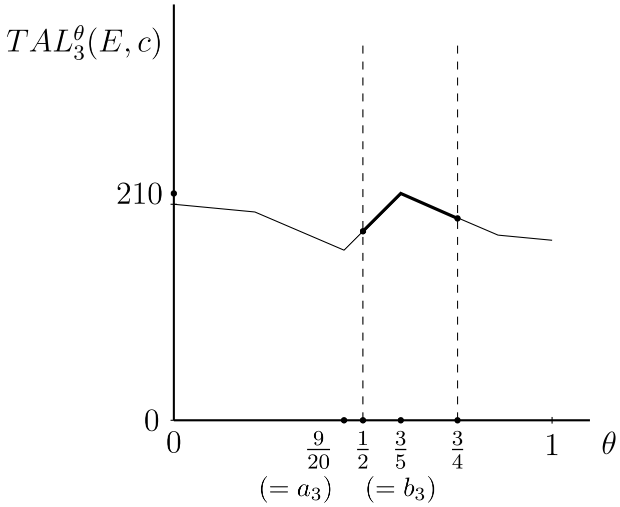

Figure 8: $TAL_3^\theta(E,c)$ in Example 4.2.

Let $(E, c) \in \mathcal{B}^N$. In the remainder of this section, we will show the existence of a Nash equilibrium of $G^{TAL}(E, c)$ of the form[2]

$$(1^{m-1}, \theta_m^*, 0^{n-m}), \tag{4.4}$$

[2]We write 1^j and 0^j for a repetition of j ones and j zeros respectively, where the case $j = 0$ corresponds to an empty repetition.

with a pivotal player $m = m(E, c) \in N$ and $\theta_m^* \in [0, 1]$. In particular, in Example 4.2 player 3 acts as the pivotal player.

The following lemma provides a starting point in the search for the pivotal player.

Lemma 4.1. *Let $(E, c) \in \mathcal{B}^N$. Then, there exists a player $k \in N$ such that*

$$\frac{k-1}{n} \leq b_k \leq \frac{k}{n} \tag{4.5}$$

or such that

$$b_k < \frac{k-1}{n} < b_{k-1}, \tag{4.6}$$

where, for notational convenience, we denote $b_0 = \infty$.

Proof. Let

$$U = \left\{ i \in N \;\middle|\; b_i \leq \frac{i}{n} \right\},$$
$$V = \left\{ i \in N \;\middle|\; \frac{i-1}{n} \leq b_i \right\}.$$

Clearly, if $k \in U \cap V$, (4.5) is satisfied. Note that $U \neq \emptyset$ because $n \in U$ since $b_n = \frac{E}{C_n} \leq 1$.

Next, let $U \cap V = \emptyset$. If $1 \notin V$, then $b_1 < 0 < b_0 = \infty$, and (4.6) is satisfied for $k = 1$. So assume $1 \in V$. Since $n \in U$ and $U \cap V = \emptyset$, we know that $n \notin V$, and therefore there must be a $k \in N \setminus \{1\}$ such that $k - 1 \in V$ and $k \notin V$. Because $U \cap V = \emptyset$, this implies that $k - 1 \notin U$. Thus, (4.6) is satisfied for player k. □

For $(E, c) \in \mathcal{B}^N$, define $k(E, c) \in N$ as the player with the smallest index that satisfies (4.5) or (4.6).

In Example 4.2 it is seen that $k(E, c) = 3$, which is equal to the pivotal player in the constructed Nash equilibrium. However, this is not always the case, as the following example shows.

Example 4.3. Consider the bankruptcy problem $(E, c) \in \mathcal{B}^N$ with $N = \{1, 2, 3, 4\}$, $E = 400$ and $c = (100, 200, 200, 200)$. Proposition 2.2 implies

$$TAL_1^\theta(E, c) = \begin{cases} 25 & \text{if } 0 \leq \theta \leq \frac{1}{4}, \\ 100 \cdot \theta & \text{if } \frac{1}{4} \leq \theta \leq 1. \end{cases}$$

and for every $j \in \{2, 3, 4\}$

$$TAL_j^\theta(E, c) = \begin{cases} 125 & \text{if } 0 \leq \theta \leq \frac{1}{4}, \\ \frac{400}{3} - \frac{100}{3} \cdot \theta & \text{if } \frac{1}{4} \leq \theta \leq 1. \end{cases}$$

It is straightforward to check that $k(E, c) = 3$ (in this case $\frac{1}{2} \leq b_3 \leq \frac{3}{4}$). Since $B_2(1, \theta_3, 0) = \{0\}$ for every $\theta_3 \in [0, 1]$, the profile $(1, 1, \theta_3, 0)$ cannot be a Nash equilibrium for any $\theta_3 \in [0, 1]$. In this case, since $TAL_1^\theta(E, c)$ is weakly increasing and $TAL_j^\theta(E, c)$, for $j \in \{2, 3, 4\}$, is weakly decreasing on $[0, 1]$, the strategy profile $(1, 0, 0, 0)$ is a Nash equilibrium of $G^{TAL}(E, c)$. ◇

Let $(E,c) \in \mathcal{B}^N$. We define

$$\ell(E,c) = \min\big\{i \in N \mid 0 \in B_i(1^{i-1}, 0^{n-i})\big\}, \tag{4.7}$$

where B_i is the best reply correspondence of player i in the game $G^{TAL}(E,c)$. Note that (3.1) implies that $\ell(E,c)$ is well-defined. We have that in Example 4.2 $\ell(E,c) = 4$ and in Example 4.3 $\ell(E,c) = 2$.

For $\ell = \ell(E,c)$ we have $0 \in B_\ell(1^{\ell-1}, 0^{n-\ell})$. In fact, the following lemma states that for every player j with an index higher than ℓ, the best reply structure exhibits the same feature: $0 \in B_j(1^{\ell-1}, 0^{n-\ell})$.

Lemma 4.2. *Let $(E,c) \in \mathcal{B}^N$, with $|N| \geq 3$. Let $\ell = \ell(E,c) \in N$ be defined as in (4.7). For every $j \in N$, let B_j be the best reply correspondence of player j for $G^{TAL}(E,c)$. Then, for every $j \in \{\ell, \dots, n\}$*

$$0 \in B_j(1^{\ell-1}, 0^{n-\ell}).$$

Proof. Let $j \in \{\ell, \dots, n\}$. To ease notation, we write $TAL_j^\theta = TAL_j^\theta(E,c)$. By assumption, $0 \in B_\ell(1^{\ell-1}, 0^{n-\ell})$, i.e.

$$TAL_\ell^{\frac{\ell-1}{n}} \geq TAL_\ell^{\frac{\ell-1+\theta_\ell}{n}} \text{ for all } \theta_\ell \in [0,1]. \tag{4.8}$$

If $\ell = n$, there is nothing to prove.

Assume, for the sake of contradiction, that $0 \notin B_j(1^{\ell-1}, 0^{n-\ell})$. In other words, suppose there is a strategy $\theta' \in (0,1]$ for player j such that

$$TAL_j^{\frac{\ell-1}{n}} < TAL_j^{\frac{\ell-1+\theta'}{n}}. \tag{4.9}$$

Note that TAL_j^θ is weakly decreasing for $\theta \geq b_j$. Therefore, (4.9) implies that $\frac{\ell-1}{n} < b_j$, and consequently, since $b_j \leq b_\ell$, we have $\frac{\ell-1}{n} < b_\ell$. Then, as TAL_ℓ^θ is strictly increasing on $[a_\ell, b_\ell] \cap [0,1]$, it follows from (4.9) that $\frac{\ell-1}{n} < a_\ell$. Proposition 2.2 now implies that

$$TAL_\ell^{\frac{\ell-1}{n}} = \begin{cases} c_\ell - \frac{C_n - E}{n} & \text{if } \frac{\ell-1}{n} \leq a_1, \\ c_\ell - \frac{(C_n - C_r) - E}{n-r} - \frac{\ell-1}{n}\frac{C_r}{n-r} & \text{if } \frac{\ell-1}{n} \in [a_r, a_{r+1}], \end{cases}$$

and, since $a_\ell \leq a_j$,

$$TAL_j^{\frac{\ell-1}{n}} = \begin{cases} c_j - \frac{C_n - E}{n} & \text{if } \frac{\ell-1}{n} \leq a_1, \\ c_j - \frac{(C_n - C_r) - E}{n-r} - \frac{\ell-1}{n}\frac{C_r}{n-r} & \text{if } \frac{\ell-1}{n} \in [a_r, a_{r+1}]. \end{cases}$$

Consequently,

$$TAL_j^{\frac{\ell-1}{n}} - TAL_\ell^{\frac{\ell-1}{n}} = c_j - c_\ell. \tag{4.10}$$

Note that (4.9) implies that

$$a_j < \frac{\ell - 1 + \theta'}{n}, \tag{4.11}$$

because TAL_j^{θ} is weakly decreasing for $\theta \leq a_j$.

We distinguish between two cases.

Case 1: $\frac{\ell-1+\theta'}{n} \leq b_j$.

In this case, (4.11) implies that $\frac{\ell-1+\theta'}{n} \in [a_j, b_j]$, and therefore $\frac{\ell-1+\theta'}{n} \in [a_\ell, b_\ell]$. Consequently, Proposition 2.2, together with (4.8), implies

$$TAL_\ell^{\frac{\ell-1}{n}} \geq TAL_\ell^{\frac{\ell-1+\theta'}{n}} = \left(\frac{\ell - 1 + \theta'}{n}\right) c_\ell,$$

and, together with (4.9), it follows that

$$TAL_j^{\frac{\ell-1}{n}} < TAL_j^{\frac{\ell-1+\theta'}{n}} = \left(\frac{\ell - 1 + \theta'}{n}\right) c_j.$$

Finally, using (4.10), the above inequalities imply

$$c_j - c_\ell = TAL_j^{\frac{\ell-1}{n}} - TAL_\ell^{\frac{\ell-1}{n}} < \left(\frac{\ell - 1 + \theta'}{n}\right)(c_j - c_\ell),$$

which is clearly a contradiction, since $\frac{\ell-1+\theta'}{n} < 1$ and $c_\ell \leq c_j$.

Case 2: $\frac{\ell-1+\theta'}{n} > b_j$.

Since TAL_j^{θ} is weakly decreasing for $\theta \geq b_j$, Proposition 2.2 implies

$$TAL_j^{\frac{\ell-1}{n}} < TAL_j^{\frac{\ell-1+\theta'}{n}} \leq TAL_j^{b_j} = b_j c_j.$$

Furthermore, $a_\ell \leq a_j \leq b_j \leq b_\ell$, and therefore $b_j \in [a_\ell, b_\ell]$. Hence, Proposition 2.2 implies $TAL_\ell^{b_j} = b_j c_\ell$. Because $b_j \in [\frac{\ell-1}{n}, \frac{\ell}{n}]$, from (4.8) it follows that

$$TAL_\ell^{\frac{\ell-1}{n}} \geq TAL_\ell^{b_j} = b_j c_\ell.$$

Similar to the previous case, the above arguments imply that

$$c_j - c_\ell = TAL_j^{\frac{\ell-1}{n}} - TAL_\ell^{\frac{\ell-1}{n}} < b_j(c_j - c_\ell) \leq \left(\frac{\ell - 1 + \theta'}{n}\right)(c_j - c_\ell),$$

which is a contradiction. □

Now we have available all ingredients to formulate our constructive main equilibrium existence result based on a pivotal player.

Let $(E,c) \in \mathcal{B}^N$. We define the *pivotal player* $m(E,c) \in N$ as

$$m(E,c) = \min\{k(E,c), \ell(E,c)\}, \tag{4.12}$$

and we will show that the pivotal player $m(E,c)$ is indeed the player m such that the proposed strategy profile in (4.4) is a Nash equilibrium for the game $G^{TAL}(E,c)$.

Note that in Example 4.2 $m(E,c) = k(E,c) = 3$, while in Example 4.3 $m(E,c) = \ell(E,c) = 2$.

Theorem 4.3. *Let $(E,c) \in \mathcal{B}^N$ with $|N| \geq 3$ and let $G^{TAL}(E,c)$ be the corresponding strategic TAL game. Let $m(E,c) \in N$ be the pivotal player as defined in (4.12). Then, there is a $\theta^*_{m(E,c)} \in [0,1]$ such that*

$$\left(1^{m(E,c)-1}, \theta^*_{m(E,c)}, 0^{n-m(E,c)}\right) \in \mathcal{N}\left(G^{TAL}(E,c)\right).$$

Proof. We write $m = m(E,c)$, $\ell = \ell(E,c)$, $k = k(E,c)$ and $TAL^\theta_j = TAL^\theta_j(E,c)$.

We first treat the case where $m = 1$. If $m = k = 1$, then $b_1 < \frac{1}{n}$. This implies that $0 \in B_j(1, 0^{n-2})$, for every $j \in \{2,\dots,n\}$, since TAL^θ_j is constant for $\theta \geq b_1$. By (3.1), we have that $1 \in B_1(0^{n-1})$, and we can choose θ^*_1, as $(1,0,\dots,0)$ is a Nash equilibrium.

If $m = \ell = 1$, then $0 \in B_1(0^{n-1})$. Lemma 4.2 now implies that $0 \in B_j(0^{n-1})$, for every $j \in \{2,\dots,n\}$. Consequently, $(0,\dots,0)$ is a Nash equilibrium.

For the remainder of this proof, assume $m \geq 2$. We distinguish two cases.

Case 1: $k < \ell$.

In this case, $m = k$. Note that $m < \ell$ implies that player m satisfies (4.5) from Lemma 4.1, because the alternative (4.6) would imply $0 \in B_m(1^{m-1}, 0^{n-m})$ since $TAL_m(E,c)$ is weakly decreasing for $\theta \geq b_m$. So

$$\frac{m-1}{n} \leq b_m \leq \frac{m}{n}. \tag{4.13}$$

Let $\theta^*_m = nb_m - m + 1$. Clearly, using (4.13), we have $0 \leq \theta^*_m \leq 1$. We claim that $(1^{m-1}, \theta^*_m, 0^{n-m})$ is a Nash equilibrium of the game $G^{TAL}(E,c)$. Note that the mean of the strategy profile $(1^{m-1}, \theta^*_m, 0^{n-m})$ equals b_m.

Claim 1a: $1 \in B_j(1^{m-2}, \theta^*_m, 0^{n-m})$ *for every* $j \in \{1,\dots,m-1\}$.

Clearly, (3.1) implies that $1 \in B_1(1^{m-2}, \theta^*_m, 0^{n-m})$. Next, suppose that for some $j \in \{2,\dots,m-1\}$ we have that $1 \notin B_j(1^{m-2}, \theta^*_m, 0^{n-m})$. Then, there is a $\theta'_j \in [0,1)$ such that

$$TAL_j^{\frac{m-2+\theta'_j+\theta^*_m}{n}} > TAL_j^{\frac{m-1+\theta^*_m}{n}} = TAL_j^{b_m}. \tag{4.14}$$

It follows that $\frac{m-2+\theta'_j+\theta^*_m}{n} < a_j$, since TAL^θ_j is increasing on $[a_j, b_j] \cap [0,1]$. In particular, $\frac{j-1}{n} \leq \frac{m-2+\theta'_j+\theta^*_m}{n} < a_j$.

From (4.14) and the fact that TAL^θ_j is decreasing for $\theta \leq a_j$, it follows that

$$TAL_j^{\frac{j-1}{n}} > TAL_j^{b_m}. \tag{4.15}$$

Next, we show that for all $\theta' \in [0,1]$

$$TAL_j^{\frac{j-1}{n}} \geq TAL_j^{\frac{j-1+\theta'}{n}}, \tag{4.16}$$

which implies $0 \in B_j(1^{j-1}, 0^{n-j})$. This is a contradiction, since $j < m \leq \ell$.

Let $\theta' \in [0,1]$. Then, $\frac{j-1+\theta'}{n} \leq \frac{m-1+\theta^*_m}{n} = b_m$. If $\frac{j-1+\theta'}{n} \leq a_j$, then $TAL_j^{\frac{j-1}{n}} \geq TAL_j^{\frac{j-1+\theta'}{n}}$, since TAL^θ_j is decreasing for $\theta \leq a_j$. If $\frac{j-1+\theta'}{n} \in (a_j, b_m]$, then (4.15) implies

$$TAL_j^{\frac{j-1}{n}} > TAL_j^{b_m} \geq TAL_j^{\frac{j-1+\theta'}{n}},$$

because TAL^θ_j is increasing for $\theta \in (a_j, b_m] \cap [0,1]$. This shows (4.16), a contradiction, and therefore $1 \in B_j(1^{m-2}, \theta^*_m, 0^{n-m})$.

Claim 1b: $0 \in B_j\left(1^{m-1}, \theta^*_m, 0^{n-m-1}\right)$ *for every* $j \in \{m+1, \ldots, n\}$.

For every $j \in \{m+1, \ldots, n\}$ we have $b_j \leq b_m$. This implies that for every $\theta'_j \in (0,1]$

$$TAL_j^{b_m} = TAL_j^{\frac{m-1+\theta^*_m}{n}} \geq TAL_j^{\frac{m-1+\theta^*_m+\theta'_j}{n}},$$

since TAL^θ_j is decreasing for $\theta \geq b_j$.

Claim 1c: $\theta^*_m \in B_m\left(1^{m-1}, 0^{n-m}\right)$.

It is clear that no deviation $\theta'_m > \theta^*_m$ can be profitable, because TAL^θ_m is decreasing for $\theta \geq b_m = \frac{m-1+\theta^*_m}{n}$.

Next, suppose there is a $\theta'_m < \theta^*_m$ such that

$$TAL_m^{\frac{m-1+\theta'_m}{n}} > TAL_m^{\frac{m-1+\theta^*_m}{n}} = TAL_m^{b_m}. \tag{4.17}$$

This implies $\frac{m-1+\theta'_m}{n} < a_m$, because TAL^θ_m is increasing for $\theta \in [a_m, b_m] \cap [0,1]$. In particular, $\frac{m-1}{n} \leq \frac{m-1+\theta'_m}{n} < a_m$. By (4.17), and the fact that TAL^θ_m is decreasing for $\theta \leq a_m$, we have

$$TAL_m^{\frac{m-1}{n}} \geq TAL_m^{\frac{m-1+\theta'_m}{n}} > TAL_m^{b_m}. \tag{4.18}$$

Next, we show that for all $\theta' \in [0,1]$

$$TAL_m^{\frac{m-1}{n}} \geq TAL_m^{\frac{m-1+\theta'}{n}}, \tag{4.19}$$

which implies $0 \in B_m(1^{m-1}, 0^{n-m})$, a contradiction.

Let $\theta' \in [0,1]$. If $\frac{m-1+\theta'}{n} \le a_m$, we have

$$TAL_m^{\frac{m-1}{n}} \ge TAL_m^{\frac{m-1+\theta'}{n}},$$

since TAL_m^{θ} is decreasing for $\theta \le a_m$. If $\frac{m-1+\theta'}{n} > a_m$, then (4.18) implies

$$TAL_m^{\frac{m-1}{n}} > TAL_m^{b_m} \ge TAL_m^{\frac{m-1+\theta'}{n}}, \tag{4.20}$$

since TAL^{θ} is increasing on $[a_m, b_m] \cap [0,1]$, and decreasing for $\theta \ge b_m$. This proves (4.19), a contradiction, and therefore $\theta_m^* \in B_m(1^{m-1}, 0^{n-m})$.

Case 2: $k \ge \ell$.

In this case $m = \ell$. We claim that in this case, the strategy profile $(1^{m-1}, 0^{n-m+1})$ is a Nash equilibrium.

For sure, Lemma 4.2 implies that $0 \in B_j(1^{m-1}, 0^{n-m})$, for every $j \in \{m, \ldots, n\}$. Next, let $j \in \{1, \ldots, m-1\}$. First we show that

$$a_j < \frac{j}{n} \le \frac{m-1}{n} < b_j. \tag{4.21}$$

Clearly, since $m = \ell$, we have $0 \notin B_{m-1}(1^{m-2}, 0^{n-m+1})$. Therefore, $\frac{m-2}{n} < b_{m-1}$, because TAL_{m-1}^{θ} is weakly decreasing for $\theta \ge b_{m-1}$. Since $m-1 < k$, the definition of k (and (4.5) in particular) implies that $\frac{m-1}{n} < b_{m-1}$, and consequently $\frac{m-1}{n} < b_j$.

Clearly, $\frac{j}{n} \le \frac{m-1}{n}$. To show that (4.21) holds, it suffices to show that $a_j < \frac{j}{n}$. Suppose $a_j \ge \frac{j}{n}$. Then, for all $\theta \in [0,1]$

$$TAL_j^{\frac{j-1}{n}} \ge TAL_j^{\frac{j-1+\theta}{n}},$$

which follows from the fact that TAL_j^{θ} is decreasing for $\theta \le a_j$. However, this would imply that $0 \in B_j(1^{j-1}, 0^{n-j})$, contradicting the fact that $j < m = \ell$.

Next, we will show that $1 \in B_j(1^{m-2}, 0^{n-m+1})$. We proceed by contradiction. Let $1 \notin B_j(1^{m-2}, 0^{n-m+1})$. Then, there exists $\theta_j' \in [0,1)$ such that

$$TAL_j^{\frac{m-2+\theta_j'}{n}} > TAL_j^{\frac{m-1}{n}}. \tag{4.22}$$

Since TAL_j^{θ} is increasing for $\theta \in [a_j, b_j] \cap [0,1]$, we have that (4.22) implies that $\frac{m-2+\theta_j'}{n} < a_j$. Consequently,

$$TAL_j^{\frac{j-1}{n}} > TAL_j^{\frac{m-1}{n}}, \tag{4.23}$$

since $\frac{j-1}{n} \le \frac{m-2+\theta_j'}{n} < a_j$, and TAL_j^{θ} is decreasing for $\theta \le a_j$.

Finally, we show that for all $\theta' \in [0,1]$

$$TAL_j^{\frac{j-1}{n}} \geq TAL_j^{\frac{j-1+\theta'}{n}}, \tag{4.24}$$

which implies $0 \in B_j(1^{j-1}, 0^{n-j})$. Note that this is a contradiction since $j < m = \ell$.

Let $\theta' \in [0,1]$. If $\frac{j-1+\theta'}{n} \leq a_j$, then

$$TAL_j^{\frac{j-1}{n}} > TAL_j^{\frac{j-1+\theta'}{n}},$$

because TAL_j^{θ} is decreasing for $\theta \leq a_j$. Next, if $\frac{j-1+\theta'}{n} > a_j$, then in particular $a_j < \frac{j-1+\theta'}{n} \leq \frac{m-1}{n} < b_j$. Consequently, (4.23) implies that

$$TAL_j^{\frac{j-1}{n}} > TAL_j^{\frac{m-1}{n}} \geq TAL_j^{\frac{j-1+\theta'}{n}},$$

since TAL_j^{θ} is increasing on $[a_j, b_j] \cap [0,1]$. This proves (4.24), a contradiction, and therefore $1 \in B_j(1^{m-2}, 0^{n-m+1})$. □

It is worth mentioning that Example 4.2 aligns with Case 1 in the proof of Theorem 4.3, while Example 4.3 corresponds to Case 2.

Theorem 4.3 shows that for strategic TAL games with four or more players there exists a Nash equilibrium similar to the one constructed in Example 4.1. However, the proof of Theorem 4.3 shows that constructing a Nash equilibrium for strategic TAL games with four or more players is notably more difficult than for games with only two or three players, as the construction of the pivotal player is quite intricate. It is well-known that CEA benefits players with lower claims, and CEL favors those with higher claims. Interestingly, this fact is reflected in the constructed Nash equilibrium, where the pivotal player neatly separates the two groups.

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