THE UNIVERSAL FACTORIAL HALL–LITTLEWOOD P- AND Q-FUNCTIONS

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Dedicated to the memory of Professor Piotr Pragacz

Abstract. In this paper, we introduce *factorial* analogues of the ordinary Hall–Littlewood P- and Q-polynomials, which we call the *factorial Hall–Littlewood P- and Q-polynomials*. Using the *universal* formal group law, we further generalize these polynomials to the *universal factorial Hall–Littlewood P- and Q-functions*. We show that these functions satisfy the *vanishing property* which the ordinary factorial Schur S-, P-, and Q-polynomials have. By the vanishing property, we derive the Pieri-type formula and a certain generalization of the classical hook formula. We then characterize our functions in terms of Gysin maps from flag bundles in the complex cobordism theory. Using this characterization and Gysin formulas for flag bundles, we obtain generating functions for the universal factorial Hall–Littlewood P- and Q-functions. Using our generating functions, we show that our factorial Hall–Littlewood P- and Q-polynomials have a certain *cancellation property*. Further applications such as Pfaffian formulas for K-theoretic factorial Q-polynomials are also given.

1. Introduction

Let $\boldsymbol{x}_n = (x_1, \ldots, x_n)$ and t be independent indeterminates over $\mathbb{Z}$, and $\lambda = (\lambda_1, \ldots, \lambda_n)$ a partition of length $\leq n$. Then the ordinary *Hall–Littlewood P- and Q-polynomials*, denoted by $P_\lambda(\boldsymbol{x}_n; t)$ and $Q_\lambda(\boldsymbol{x}_n; t)$ respectively, are symmetric polynomials with coefficients in $\mathbb{Z}[t]$. When $t = 0$, both the polynomials $P_\lambda(\boldsymbol{x}_n; t)$ and $Q_\lambda(\boldsymbol{x}_n; t)$ reduce to the ordinary Schur (S-) polynomial $s_\lambda(\boldsymbol{x}_n)$, and when $t = -1$, to the ordinary Schur P-polynomial $P_\lambda(\boldsymbol{x}_n)$ and Q-polynomial $Q_\lambda(\boldsymbol{x}_n)$ respectively. Thus the polynomials $P_\lambda(\boldsymbol{x}_n; t)$, $Q_\lambda(\boldsymbol{x}_n; t)$ serve to interpolate between the Schur polynomials and the Schur P- and Q-polynomials, and play a crucial role in the symmetric function theory, representation theory, and combinatorics. In the context of *Schubert calculus*, it is well-known that the ordinary Schur S-, P-, and Q-polynomials appear as the Schubert classes in the ordinary cohomology rings of the various Grassmannians (Fulton [7, §9.4], Pragacz

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[31, §6]). Moreover, their *factorial* analogues, namely, the factorial Schur S-, P-, and Q-polynomials play an analogous role in *equivariant* Schubert calculus (Knutson–Tao [17], Ikeda [11], Ikeda–Naruse [13]). As for the Hall–Littlewood polynomials, it is known that there are some geometric or representation theoretic inetepretations of them related to flag varieties or flag bundles (readers are referred to e.g., De Concini–Procesi [5], Garsia–Procesi [9], Pragacz [32]). In the context of Schubert calculus, there seems no obvious geometric meaning of the Hall–Littlewood polynomials at present, which needs to be investigated further. In fact, in [35], Totaro considered the *coinvariant ring* $F(e,n)$ of the complex reflection group $G(e,1,n) = \mathbb{Z}/e\mathbb{Z} \wr S_n$ (the wreath product) for $e \geq 2$, and suggested to think of the ring $F(e,n)$ as the cohomology of a certain "flag manifold". Then he considered a subring $C(e,n)$ of $F(e,n)$, and described a basis for the ring $C(e,n)$ given by the Hall–Littlewood Q-polynomials. For $e = 2$, the inclusion $C(2,n) \subset F(2,n)$ is the inclusion of the cohomology of the Lagrangian Grassmannian in that of the isotropic flag manifold of the symplectic group, and Totaro's result is interpreted as a generalization of the classical result in Schubert calculus for Lagrangian Grassmannians (Józefiak [16], Pragacz [31, §6]). It is natural to consider a generalization of the above theory to the *double coinvariant rings* (or *equivariant coinvariant rings*) of complex reflection groups (cf. recent work of McDaniel [20]). From a geometric or topological point of view, one expects that these rings would be related to *torus-equivariant cohomology* of certain "flag manifolds", and factorial version of the Hall–Littlewood polynomials would play a crucial role. Moreover we notice that all the results stated above are formulated in the ordinary cohomology theory $H^*(-)$. In topology, it is classical that a *complex-oriented generalized cohomology theory* $h^*(-)$ gives rise to a *formal group law* $F^h(u,v)$ over the coefficient ring $h^* := h^*(\mathrm{pt})$, where pt is a single point. Three typical examples are the ordinary cohomology theory $H^*(-)$, the (topological) complex K-theory $K^*(-)$, and the complex cobordism theory $MU^*(-)$, which correspond to the *additive* formal group law $F_a(u,v) = u+v$, the *multiplicative* formal group law $F_m(u,v) = u \oplus v = u + v - \beta uv$, and the *universal* formal group law $F_{\mathbb{L}}(u,v) = u +_{\mathbb{L}} v$, respectively. By the classical result of Quillen [34, Proposition 1.10], the complex cobordism theory is *universal* among all complex-oriented generalized cohomology theories. Therefore it is also quite natural to ask whether one can generalize the above results formulated in the ordinary cohomology theory to the complex cobordism theory.

Motivated by these facts and the above preceding results, in this paper, we introduce *factorial* and *universal* analogues of the ordinary Hall–Littlewood P- and Q-polynomials, which we call the *universal factorial Hall–Littlewood P- and Q-functions* (for notation, see §2.1):

Definition 1.1 (Definition 3.1, cf. Naruse [28])**.** For a sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_r)$ with $r \leq n$, we define

$$HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) := \sum_{\overline{w} \in S_n/(S_1)^r \times S_{n-r}} w \cdot \left[[\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}} \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i +_{\mathbb{L}} [t](\overline{x}_j)}{x_i +_{\mathbb{L}} \overline{x}_j} \right],$$

$$HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) := \sum_{\overline{w} \in S_n/(S_1)^r \times S_{n-r}} w \cdot \left[[[\boldsymbol{x}; t|\boldsymbol{b}]]^{\lambda}_{\mathbb{L}} \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i +_{\mathbb{L}} [t](\overline{x}_j)}{x_i +_{\mathbb{L}} \overline{x}_j} \right].$$

To the best of our knowledge, even a factorial version of the ordinary Hall–Littlewood polynomials has not appeared in the literature. Here we emphasize the importance of these factorial Hall–Littlewood polynomials. In fact, they will be needed in describing the torus-equivariant cohomology of *p-compact flag variety* corresponding to $G(e, 1, n)$ (cf. recent work of Ortiz [30]). In this context, the "deformation parameters" $\boldsymbol{b}$ are interpreted as the torus-equivariant parameters. We will discuss this new aspect of the Hall–Littlewood functions in more detail in our forthcoming paper [27].

Then, we show that our factorial Hall–Littlewood P- and Q-functions have the so-called *vanishing property* (see Propositions 3.7, 3.8). We emphasize that this vanishing property will be useful in describing the so-called *GKM description* of the torus-equivariant cohomology ring of the p-compact flag variety corresponding to $G(e, 1, n)$ ([27]). By the vanishing property, we can derive a Pieri-type formula for factorial Hall–Littlewood P-polynomials (see Proposition 3.9). Moreover, a simple recursive argument based on the associativity of factorial Hall–Littlewood P-polynomials, we can derive a certain generalization of the hook formula (see Proposition 3.10). We then give a characterization of them in terms of Gysin maps from full flag bundles in the complex cobordism theory (Proposition 3.5). Using this characterization, we can derive *generating functions* for the universal factorial Hall–Littlewood P- and Q-functions. The idea of getting our result is to apply the Gysin formula for a projective bundle repeatedly to the full flag bundle since a full flag bundle is constructed as a sequence of projective bundles. However, the existence of the deformation parameter $\boldsymbol{b} = (b_1, b_2, \dots)$ prevent us from a direct application of the Gysin formula. To circumvent this difficulty, we developed a specific modification in each step (for details, see §4.1). Then, carrying out an argument carefully, we succeeded in getting the required result. To state our result, we prepare some notation from §2.1, 2.2, and 4.1: For a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $r \le n$, we set

$$\begin{aligned}\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1, u_2, \ldots, u_i|\boldsymbol{b}) &:= \frac{u_i}{u_i +_{\mathbb{L}} [t](\overline{u}_i)} \cdot \frac{1}{\mathscr{P}^{\mathbb{L}}(u_i)} \\ &\times \left(\prod_{j=1}^{n} \frac{u_i +_{\mathbb{L}} [t](\overline{x}_j)}{u_i +_{\mathbb{L}} \overline{x}_j} \prod_{j=1}^{i-1} \frac{u_i +_{\mathbb{L}} \overline{u}_j}{u_i +_{\mathbb{L}} [t](\overline{u}_j)} \prod_{j=1}^{\lambda_i} \frac{u_i +_{\mathbb{L}} b_j}{u_i} - t^{n-i+1} \prod_{j=1}^{\lambda_i} \frac{b_j}{u_i} \right), \\ \widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b}) = \widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(u_1, u_2, \ldots, u_r|\boldsymbol{b}) &:= \prod_{i=1}^{r} \widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1, u_2, \ldots, u_i|\boldsymbol{b}).\end{aligned}$$

Then, our main result in this paper is stated as follows:

Theorem 1.2 (Theorem 4.3)**.** *For a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $r \le n$, the universal factorial Hall–Littlewood P-function $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$ is the coefficient of $\boldsymbol{u}^{-\lambda} = u_1^{-\lambda_1} \cdots u_r^{-\lambda_r}$ in $\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(u_1, \ldots, u_r|\boldsymbol{b})$. Thus*

$$HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) = [\boldsymbol{u}^{-\lambda}] \left(\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b}) \right).$$

Using similar, but simpler technique, we can also obtain a generating function for $HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$ (see Theorem 4.5). Here we stress the usefulness of a technique of generating functions. For instance, it is easy to derive Pfaffian formulas for factorial K-theoretic Q-polynomials in a simple and uniform manner (see Theorem 5.3). Moreover, a certain *cancellation property* (cf. Pragacz [31, §2]) of the factorial Hall–Littlewood P- and Q-polynomials can be verified immediately (see Proposition 5.1). For further applications of generating functions such as the so-called Pieri rule for K-theoretic P- and Q-polynomials, see also Naruse [28].

1.1. **Organization of the paper.** The paper is organized as follows: In Section 2, we prepare notation and conventions concerning the universal formal group law, a Gysin formula for a projective bundle, which will be used throughout the paper. In Section 3, the universal factorial Hall–Littlewood P- and Q-functions are introduced, and a characterization of them by means of a Gysin map is given. The vanishing property of these functions are also discussed. By the vanishing property, a Pieri-type formula and a generalization of the hook formula are derived. Using Gysin formulas for flag bundles and characterizations of the Hall–Littlewood functions by means of Gysin maps, in Section 4, we obtain generating functions for these universal factorial Hall–Littlewood functions. In Section 5, using our generating functions, we shall show that the factorial Hall–Littlewood P- and Q-polynomials satisfy certain cancellation property. Pfaffian formulas for factorial K-theoretic Q-polynomials can be obtained as a by-product. In Appendix (Section 6), we deal with the topic closely related to the current work, namely, generating functions for the *dual Grothendieck polynomials* and the *dual K-theoretic Schur Q-polynomials*.

2. Notation, conventions, and preliminary results

For notation and conventions, we shall follow those used in our previous papers [24], [26]. However, to make the exposition self-contained as much as possible, we collect some of them frequently used in this paper.

2.1. Lazard ring $\mathbb{L}$ and the universal formal group law $F_{\mathbb{L}}$.

Let

$$F_{\mathbb{L}}(u,v) = u + v + \sum_{i,j\geq 1} a^{\mathbb{L}}_{i,j} u^i v^j \in \mathbb{L}[[u,v]]$$

be the *universal formal group law*, where $\mathbb{L}$ is the *Lazard ring*. Namely, $F_{\mathbb{L}}(u,v)$ is a formal power series in two indeterminates u, v with coefficients $a^{\mathbb{L}}_{i,j} \in \mathbb{L}$ which satisfies the axioms of the formal group law. For the universal formal group law, we shall use the following notation:

$$\begin{aligned}
&u +_{\mathbb{L}} v = F_{\mathbb{L}}(u,v) && \text{(formal sum)},\\
&\overline{u} = [-1]_{\mathbb{L}}(u) = \chi_{\mathbb{L}}(u) && \text{(formal inverse of } u),\\
&u -_{\mathbb{L}} v = u +_{\mathbb{L}} [-1]_{\mathbb{L}}(v) = u +_{\mathbb{L}} \overline{v} && \text{(formal subtraction)}.
\end{aligned}$$

Furthermore, we define $[0]_{\mathbb{L}}(u) := 0$, and inductively, $[n]_{\mathbb{L}}(u) := [n-1]_{\mathbb{L}}(u) +_{\mathbb{L}} u$ for a positive integer $n \geq 1$. We also define $[-n]_{\mathbb{L}}(u) := [n]_{\mathbb{L}}([-1]_{\mathbb{L}}(u))$ for $n \geq 1$. We call $[n]_{\mathbb{L}}(u)$ the *n-series* in the sequel. Denote by $\ell_{\mathbb{L}}(u) \in \mathbb{L} \otimes \mathbb{Q}[[u]]$ the *logarithm* of $F_{\mathbb{L}}$, i.e., a unique formal power series with leading term u such that

$$\ell_{\mathbb{L}}(u +_{\mathbb{L}} v) = \ell_{\mathbb{L}}(u) + \ell_{\mathbb{L}}(v).$$

Using the logarithm $\ell_{\mathbb{L}}(u)$, one can rewrite the n-series $[n]_{\mathbb{L}}(u)$ for a non-negative integer n as $\ell_{\mathbb{L}}^{-1}(n \cdot \ell_{\mathbb{L}}(u))$, where $\ell_{\mathbb{L}}^{-1}(u)$ is the formal power series inverse to $\ell_{\mathbb{L}}(u)$. This formula allows us to define

$$[t]_{\mathbb{L}}(x) = [t](x) := \ell_{\mathbb{L}}^{-1}(t \cdot \ell_{\mathbb{L}}(x))$$

for an indeterminate t. This is a natural extension of $t \cdot x$ as well as the n-series $[n]_{\mathbb{L}}(x)$.

Next we shall introduce various generalizations of the ordinary power of variables. Let $\boldsymbol{x} = (x_1, x_2, \ldots)$ be a countably infinite sequence of independent variables. We also introduce another set of independent variables

$\boldsymbol{b} = (b_1, b_2, \ldots)$. Then, for a positive integer $k \geq 1$, we define a generalization of the ordinary k-th power x^k of one variable x by

$$[x|\boldsymbol{b}]^k_{\mathbb{L}} := \prod_{j=1}^{k}(x +_{\mathbb{L}} b_j) = (x +_{\mathbb{L}} b_1)(x +_{\mathbb{L}} b_2)\cdots(x +_{\mathbb{L}} b_k).$$

We set $[x|\boldsymbol{b}]^0_{\mathbb{L}} := 1$. For a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$, we set

$$[\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}} := \prod_{i=1}^{r}[x_i|\boldsymbol{b}]^{\lambda_i}_{\mathbb{L}} = \prod_{i=1}^{r}\prod_{j=1}^{\lambda_i}(x_i +_{\mathbb{L}} b_j).$$

Similarly, we define

$$[[x|\boldsymbol{b}]]^k_{\mathbb{L}} := (x +_{\mathbb{L}} x)[x|\boldsymbol{b}]^{k-1}_{\mathbb{L}} = (x +_{\mathbb{L}} x)(x +_{\mathbb{L}} b_1)(x +_{\mathbb{L}} b_2)\cdots(x +_{\mathbb{L}} b_{k-1}).$$

For a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$, we set

$$[[\boldsymbol{x}|\boldsymbol{b}]]^{\lambda}_{\mathbb{L}} := \prod_{i=1}^{r}[[x_i|\boldsymbol{b}]]^{\lambda_i}_{\mathbb{L}} = \prod_{i=1}^{r}(x_i +_{\mathbb{L}} x_i)[x_i|\boldsymbol{b}]^{\lambda_i - 1}_{\mathbb{L}}.$$

Moreover, for indeterminates x and t, we define

$$[[x; t|\boldsymbol{b}]]^k_{\mathbb{L}} := (x +_{\mathbb{L}} [t](\overline{x}))[x|\boldsymbol{b}]^{k-1}_{\mathbb{L}}$$

for a positive integer $k \geq 1$. For a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$, we define

$$[[\boldsymbol{x}; t|\boldsymbol{b}]]^{\lambda}_{\mathbb{L}} := \prod_{i=1}^{r}[[x_i; t|\boldsymbol{b}]]^{\lambda_i}_{\mathbb{L}} = \prod_{i=1}^{r}(x_i +_{\mathbb{L}} [t](\overline{x}_i))[x_i|\boldsymbol{b}]^{\lambda_i - 1}_{\mathbb{L}}.$$

2.2. **Gysin formula for a projective bundle in complex cobordism.** Recall from Quillen [33, Theorem 1] the Gysin formula for a projective bundle in complex cobordism. We shall state his result in a manner suitable for our purpose (for more details, see Nakagawa–Naruse [26, §3.1]): Let $E \longrightarrow X$ be a complex vector bundle of rank n. For any integer $m \in \mathbb{Z}$, denote by $\mathscr{S}^{\mathbb{L}}_m(E) = \mathscr{S}^{MU}_m(E)$ the Segre class of E in complex cobordism, and

$$\mathscr{S}^{\mathbb{L}}(E; u) := \sum_{m \in \mathbb{Z}} \mathscr{S}^{\mathbb{L}}_m(E) u^m$$

its Segre series. The explicit expression of $\mathscr{S}^{\mathbb{L}}(E; u)$ is given by

(2.1)

$$\mathscr{S}^{\mathbb{L}}(E; u) = \frac{1}{\mathscr{P}^{\mathbb{L}}(z)}\prod_{j=1}^{n}\frac{z}{z +_{\mathbb{L}} \overline{x}_j}\Bigg|_{z=u^{-1}} = \frac{1}{\mathscr{P}^{\mathbb{L}}(z)}\frac{z^n}{\prod_{j=1}^{n}(z +_{\mathbb{L}} \overline{x}_j)}\Bigg|_{z=u^{-1}},$$

where $\mathscr{P}^{\mathbb{L}}(z) := 1 + \sum_{i=1}^{\infty} a^{\mathbb{L}}_{i,1} z^i$, and $x_1, \ldots, x_n$ are the Chern roots of E in complex cobordism.

Now consider the Grassmann bundle $\pi^1 : G^1(E) \longrightarrow X$ of *hyperplanes* in E. Denote by Q^1 the tautological *quotient* bundle on $G^1(E)$. Put $x_1 :=$

$c_1^{MU}(Q^1) \in MU^2(G^1(E))$. For a monomial m of a formal Laurent series F, we denote by $[m](F)$ the coefficient of m in F. Note that the Grassmann bundle $G^1(E)$ of hyperplanes in E is canonically isomorphic to the projective bundle $P(E^\vee) = G_1(E^\vee)$ of lines in the dual bundle $E^\vee$. Then, by dualizing the formula [26, (3.4)], we have the following form of Quillen's Gysin formula:

Proposition 2.1. *For a polynomial $f(u) \in MU^*(X)[u]$, the Gysin map $\pi_*^1 : MU^*(G^1(E)) \longrightarrow MU^*(X)$ is described by the following formula*:

$$\pi_*^1(f(x_1)) = [u^{n-1}](f(u) \cdot \mathscr{S}^{\mathbb{L}}(E; 1/u)). \tag{2.2}$$

This is the fundamental formula for establishing more general Gysin formulas for general flag bundles.

Here we shall fix some notation concerning flag bundles[1]: Let $E \longrightarrow X$ be a complex vector bundle of rank n. For a positive integer $r = 1, 2, \ldots, n$, denote by $\pi^{r,r-1,\ldots,1} : \mathcal{F}\ell^{r,r-1,\ldots,1}(E) = \mathcal{F}\ell_{n-r,n-r+1,\ldots,n-1}(E) \longrightarrow X$ the associated flag bundle. Thus a point in $\mathcal{F}\ell^{r,r-1,\ldots,1}(E)$ is written as a pair $(x, (W_\bullet)_x)$, where $(W_\bullet)_x$ is a flag, i.e., nested subspaces of the form $(W_1)_x \subset (W_2)_x \subset \cdots \subset (W_r)_x$, $\operatorname{codim}(W_i)_x = r + 1 - i$, in the fiber E_x of E over each point $x \in X$. Following Darondeau–Pragacz [4, §1.2], we shall call the flag bundle of the form $\pi^{r,r-1,\ldots,1} : \mathcal{F}\ell^{r,r-1,\ldots,1}(E) \longrightarrow X$ the *full* flag bundle in this paper. When $r = n$, we call $\pi^{n,n-1,\ldots,1} : \mathcal{F}\ell^{n,n-1,\ldots,1}(E) \longrightarrow X$ the *complete* flag bundle, and just write $\pi : \mathcal{F}\ell(E) \longrightarrow X$. On $\mathcal{F}\ell(E)$, there is the universal flag of subbundles

$$0 = U_0 \subset U_1 \subset \cdots \subset U_i \subset \cdots \subset U_{n-1} \subset U_n = \pi^*(E),$$

where $\operatorname{rank} U_i = i$ $(i = 0, 1, \ldots, n)$. and we put

$$x_i := c_1^{MU}(U_{n+1-i}/U_{n-i}) \in MU^2(\mathcal{F}\ell(E)) \quad (i = 1, 2, \ldots, n), \tag{2.3}$$

which are the MU^*-theory Chern roots of E. It is well-known (see e.g., Darondeau–Pragacz [4, §1.2]) that the full flag bundle $\mathcal{F}\ell^{r,r-1,\ldots,1}(E)$ is constructed as a sequence of Grassmann bundles of codimension one hyperplanes[2]:

(2.4)

$$\pi^{r,\ldots,1} : \mathcal{F}\ell^{r,r-1,\ldots,1}(E) = G^1(U_{n-r+1}) \xrightarrow{\pi^r} \cdots \longrightarrow G^1(U_{n-1}) \xrightarrow{\pi^2} G^1(E) \xrightarrow{\pi^1} X.$$

[1] The notation concerning flag bundles or flag manifolds varies depending on the authors. We followed basically that used in Nakagawa–Naruse [24, §4.1], Darondeau–Pragacz [4, §1].

[2] Note that, in [4, §1.2], the full flag bundle is constructed as a sequence of projective bundles of lines.

3. Universal factorial Hall–Littlewood P- and Q-functions

In this section, we shall introduce our main object to study, the *universal factorial Hall–Littlewood P- and Q-functions*, which are universal as well as factorial analogues of the ordinary Hall–Littlewood polynomials.

3.1. Universal factorial Hall–Littlewood P- and Q-functions.

3.1.1. *Definition of the universal factorial Hall–Littlewood P- and Q-functions.* We shall use the notation introduced in §2.1. We provide the variables $\boldsymbol{x} = (x_1, x_2, \ldots)$ and $\boldsymbol{b} = (b_1, b_2, \ldots)$ with $\deg(x_i) = \deg(b_i) = 1$ for $i = 1, 2, \ldots$. Then we make the following definition:

Definition 3.1 (Universal factorial Hall–Littlewood P- and Q-functions)**.** For a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $r \leq n$, we define

$$HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) := \sum_{\overline{w} \in S_n/(S_1)^r \times S_{n-r}} w \cdot \left[[\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}} \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i +_{\mathbb{L}} [t](\overline{x}_j)}{x_i +_{\mathbb{L}} \overline{x}_j} \right],$$

$$HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) := \sum_{\overline{w} \in S_n/(S_1)^r \times S_{n-r}} w \cdot \left[[[\boldsymbol{x}; t|\boldsymbol{b}]]^{\lambda}_{\mathbb{L}} \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i +_{\mathbb{L}} [t](\overline{x}_j)}{x_i +_{\mathbb{L}} \overline{x}_j} \right],$$

where the symmetric group S_n acts naturally on the variables $\boldsymbol{x}_n = (x_1, \ldots, x_n)$ by permuting them. We also define

$$HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t) := HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{0}) \quad \text{and} \quad HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t) := HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{0}).$$

In what follows, $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t)$ and $HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t)$ will be called the *universal Hall–Littlewood P- and Q-functions* respectively.

In the above definition, the action of the subgroup $(S_1)^r \times S_{n-r}$ of S_n on the first factors $[\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}}$ and $[[\boldsymbol{x}; t|\boldsymbol{b}]]^{\lambda}_{\mathbb{L}}$ is trivial, and the second factor $\displaystyle\prod_{1 \leq i \leq r,\, i < j \leq n} \frac{x_i +_{\mathbb{L}} [t](\overline{x}_j)}{x_i +_{\mathbb{L}} \overline{x}_j}$ is invariant under this action. Therefore, the action of the symmetric group does not depend on the choice of a representative w of the coset $\overline{w} \in S_n/(S_1)^r \times S_{n-r}$. Note that when $t = -1$ in the definition, then $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; -1|\boldsymbol{b})$ (resp. $HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; -1|\boldsymbol{b})$) coincides with the universal factorial Schur P-function $P^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n|\boldsymbol{b})$ (resp. Q-function $Q^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n|\boldsymbol{b})$), for a strict partition λ, which have been introduced in our previous paper [23, Definition 4.1]. In contrast to this, when $t = 0$, both $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; 0|\boldsymbol{b})$ and $HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; 0|\boldsymbol{b})$ are different from the universal factorial Schur functions $s^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n|\boldsymbol{b})$ ([23, Definition 4.10]), $\mathbb{S}^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n|\boldsymbol{b})$ ([24, Definition 5.1]).

3.1.2. *Factorial Hall–Littlewood P- and Q-polynomials.* The specialization from $F_{\mathbb{L}}(u,v) = u +_{\mathbb{L}} v$ to $F_a(u,v) = u+v$ is of particular importance. Under this specialization, the generalized powers $[x|\boldsymbol{b}]^k_{\mathbb{L}}$, $[[x;t|\boldsymbol{b}]]^k_{\mathbb{L}}$ reduce to $[x|\boldsymbol{b}]^k = \prod_{j=1}^k (x+b_j)$, $[[x;t|\boldsymbol{b}]]^k = (x-tx)[x|\boldsymbol{b}]^{k-1}$ respectively, and we obtain *new* symmetric polynomials denoted by $HP_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b})$ and $HQ_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b})$ respectively. More explicitly, these are defined as follows:

Definition 3.2 (Factorial Hall–Littlewood P- and Q-polynomials)**.** For a sequence of positive integers $\lambda = (\lambda_1,\dots,\lambda_r)$ with $r \leq n$, we define

$$\begin{aligned}
HP_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b}) &:= \sum_{\overline{w}\in S_n/(S_1)^r\times S_{n-r}} w\cdot\left[[\boldsymbol{x}|\boldsymbol{b}]^\lambda \prod_{i=1}^r\prod_{j=i+1}^n \frac{x_i-tx_j}{x_i-x_j}\right],\\
&= \sum_{\overline{w}\in S_n/(S_1)^r\times S_{n-r}} w\cdot\left[\prod_{i=1}^r\prod_{j=1}^{\lambda_i}(x_i+b_j)\times\prod_{i=1}^r\prod_{j=i+1}^n \frac{x_i-tx_j}{x_i-x_j}\right],\\
HQ_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b}) &:= \sum_{\overline{w}\in S_n/(S_1)^r\times S_{n-r}} w\cdot\left[[[\boldsymbol{x};t|\boldsymbol{b}]]^\lambda \prod_{i=1}^r\prod_{j=i+1}^n \frac{x_i-tx_j}{x_i-x_j}\right]\\
&= (1-t)^r\times\\
&\quad \sum_{\overline{w}\in S_n/(S_1)^r\times S_{n-r}} w\cdot\left[\prod_{i=1}^r\prod_{j=1}^{\lambda_i-1}x_i(x_i+b_j)\times\prod_{i=1}^r\prod_{j=i+1}^n \frac{x_i-tx_j}{x_i-x_j}\right].
\end{aligned}$$

We also define

$$HP_\lambda(\boldsymbol{x}_n;t) := HP_\lambda(\boldsymbol{x}_n;t|\boldsymbol{0}) \quad\text{and}\quad HQ_\lambda(\boldsymbol{x}_n;t) := HQ_\lambda(\boldsymbol{x}_n;t|\boldsymbol{0}),$$

and will be called the *Hall–Littlewood P- and Q-polynomials* respectively.

Note that, by definition, we have $HQ_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b}) = (1-t)^{\ell(\lambda)}HP_\lambda(\boldsymbol{x}_n;t|0,\boldsymbol{b})$. For a strict partition λ, if t specializes to be -1, then $HP_\lambda(\boldsymbol{x}_n;-1|\boldsymbol{b})$ and $HQ_\lambda(\boldsymbol{x}_n;-1|\boldsymbol{b}) = 2^{\ell(\lambda)}HP_\lambda(\boldsymbol{x}_n;-1|0,\boldsymbol{b})$ coincide with the factorial Schur P- and Q-polynomials (by replacing $\boldsymbol{b}$ with $-\boldsymbol{b} = (-b_1,-b_2,\dots)$) (for their definition, see Ikeda–Mihalcea–Naruse [12, §4.2]). However, for a partition λ, both $HP_\lambda(\boldsymbol{x}_n;0|\boldsymbol{b})$ and $HQ_\lambda(\boldsymbol{x}_n;0|\boldsymbol{b})$ do not coincide with the factorial Schur polynomial (for its definition, see Molev–Sagan [21, §2, (3)]).

Example 3.3. Direct computation from Definition 3.2 gives some examples:

$$\begin{aligned}
&HP_{(1)}(\boldsymbol{x}_n;t|\boldsymbol{b}) = x_1+x_2+\cdots+x_n+\frac{1-t^n}{1-t}b_1,\\
&HP_{(1^2)}(\boldsymbol{x}_n;t|\boldsymbol{b})\\
&\qquad = (1+t)\left[m_{(1^2)}(\boldsymbol{x}_n)+\frac{1-t^{n-1}}{1-t}b_1m_{(1)}(\boldsymbol{x}_n)+\frac{(1-t^{n-1})(1-t^n)}{(1-t)(1-t^2)}b_1^2\right],\\
&HP_{(2)}(\boldsymbol{x}_n;t|\boldsymbol{b}) = (s_{(2)}(\boldsymbol{x}_n)-ts_{(1^2)}(\boldsymbol{x}_n)+(b_1+b_2)s_{(1)}(\boldsymbol{x}_n)+b_1b_2\frac{1-t^m}{1-t}.
\end{aligned}$$

Here $m_\lambda(\boldsymbol{x}_n)$ and $s_\lambda(\boldsymbol{x}_n)$ are respectively the monomial symmetric polynomials and Schur polynomials corresponding to λ.

If λ is a partition of length $\ell(\lambda) = r \le n$, i.e., $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r > 0$, our factorial Hall–Littlewood P- and Q-polynomials are related to Macdonald's Hall–Littlewood P- and Q-polynomials in the following way: We rewrite λ as $\lambda = (n_1^{p_1}\ n_2^{p_2}\ \cdots n_{d-1}^{p_{d-1}}\ n_d^{p_d})$, where $n_1 > n_2 > \cdots > n_{d-1} > n_d = 0$, each $p_i > 0$, $p_d = n - r$, and $\sum_{i=1}^d p_i = n$. We put $\nu(k) := \sum_{i=1}^k p_i$ for $k = 1, \ldots, d$ and $\nu(0) := 0$. Denote by S_{p_k} the symmetric group on m_k letters $\nu(k-1)+1, \ldots, \nu(k)$ for $k = 1, \ldots, d$. Thus the stabilizer subgroup S_n^λ of λ under the action of S_n on λ is given by $S_n^\lambda = \prod_{k=1}^d S_{p_k}$. For an integer $k \ge 0$, let $v_k(t) := \prod_{i=1}^k \frac{1-t^i}{1-t}$, and for the above partition λ, we define[3]

$$v_{\lambda>0}(t) := \prod_{k=1}^{d-1} v_{p_k}(t).$$

Using the identity

$$\sum_{w \in S_n} w \cdot \left[\prod_{1 \le i < j \le n} \frac{x_i - t x_j}{x_i - x_j} \right] = v_n(t) \tag{3.1}$$

in [19, Chapter III, (1.4)], one can prove the following fact along the same line as the case of the usual Hall–Littlewood polynomials ([19, Chapter III, (1.5)]):

$$HP_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b}) = v_{\lambda>0}(t) \times \sum_{\overline{w} \in S_n/S_n^\lambda} w \cdot \left[[\boldsymbol{x}|\boldsymbol{b}]^\lambda \cdot \prod_{\substack{1 \le i < j \le n \\ \lambda_i > \lambda_j}} \frac{x_i - t x_j}{x_i - x_j} \right]. \tag{3.2}$$

Thus $HP_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$ is divisible by $v_{\lambda>0}(t)$. Taking this fact into account, we define

$$P_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b}) := \frac{1}{v_{\lambda>0}(t)} HP_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b}), \tag{3.3}$$

or equivalently,

$$P_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b}) := \sum_{\overline{w} \in S_n/S_n^\lambda} w \cdot \left[[\boldsymbol{x}|\boldsymbol{b}]^\lambda \cdot \prod_{\substack{1 \le i < j \le n \\ \lambda_i > \lambda_j}} \frac{x_i - t x_j}{x_i - x_j} \right]. \tag{3.4}$$

It is this polynomial that can be considered as a factorial version of Macdonald's Hall–Littlewood P-polynomial $P_\lambda(\boldsymbol{x}_n; t)$. Putting $\boldsymbol{b} = \boldsymbol{0}$ in (3.3), we have $HP_\lambda(\boldsymbol{x}_n; t) = v_{\lambda>0}(t) P_\lambda(\boldsymbol{x}_n; t)$. In particular, for λ strict, $HP_\lambda(\boldsymbol{x}_n; t)$

[3]Do not confuse $v_{\lambda>0}(t)$ with $v_\lambda(t) := \prod_{i \ge 0} v_{m_i}(t)$ in Macdonald [19, Chapter III, §1], where $m_i = m_i(\lambda)$ means the *multiplicity* for each $i \ge 0$.

coincides with $P_\lambda(\boldsymbol{x}_n;t)$. On the other hand, by the argument in Macdonald' book [19, pp.210–211], we see that $HQ_\lambda(\boldsymbol{x}_n;t)$ equals to the ordinary Hall–Littlewood Q-polynomial $Q_\lambda(\boldsymbol{x}_n;t)$.

Remark 3.4. (1) The universal analogue of the left hand side of (3.1), namely,

$$\sum_{w\in S_n} w\cdot\left[\prod_{1\le i<j\le n}\frac{x_i +_{\mathbb{L}} [t](\overline{x}_j)}{x_i +_{\mathbb{L}} \overline{x}_j}\right]$$

is no longer a polynomial in t alone (it contains the variables $x_1,\dots,x_n$). Therefore an analogous formula of (3.2) does not hold in this case.

(2) For a general sequence of positive integers λ, $HP_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b})$ may not be divisible by $v_{\lambda>0}(t)$.

3.2. **Characterization of the universal factorial Hall–Littlewood P- and Q-functions.** Geometrically, the universal factorial Hall–Littlewood P- and Q-functions are characterized by means of the Gysin map for certain flag bundles. (We learned this idea from the work [32] by Pragacz.) Let $E\longrightarrow X$ be a complex vector bundle of rank n, and $x_1,\dots,x_n$ are the MU^*-theory Chern roots of E as in (2.3). Consider the associated full flag bundle $\pi^{r,r-1,\dots,1}:\mathcal{F}\ell^{r,r-1,\dots,1}(E)\longrightarrow X$. Then the Gysin homomorphism $(\pi^{r,\dots,1})_*: MU^*(\mathcal{F}\ell^{r,\dots,1}(E))\longrightarrow MU^*(X)$ is described as the following type of a symmetrizing operator (see Nakagawa–Naruse [24, Theorem 4.10], also Brion [3, Proposition 1.1] for cohomology): For an $(S_1)^r\times S_{n-r}$-invariant polynomial $f(X_1,\dots,X_n)\in MU^*(X)[X_1,\dots,X_n]^{(S_1)^r\times S_{n-r}}$, one has

$$(\pi^{r,\dots,1})_*(f(x_1,\dots,x_n)) = \sum_{\overline{w}\in S_n/(S_1)^r\times S_{n-r}} w\cdot\left[\frac{f(x_1,\dots,x_n)}{\prod_{1\le i\le r,\, i<j\le n}(x_i +_{\mathbb{L}} \overline{x}_j)}\right].$$

Then it follows from Definition 3.1 and the above description of the Gysin homomorphism $(\pi^{r,\dots,1})_*$ that the following formula holds:

Proposition 3.5 (Characterization of the universal factorial Hall–Littlewood P- and Q-functions)**.**

$$(\pi^{r,\dots,1})_*\left([\boldsymbol{x}|\boldsymbol{b}]^\lambda_{\mathbb{L}}\prod_{i=1}^r\prod_{j=i+1}^n(x_i +_{\mathbb{L}} [t](\overline{x}_j))\right) = HP^{\mathbb{L}}_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b}), \tag{3.5}$$

$$(\pi^{r,\dots,1})_*\left([[\boldsymbol{x};t|\boldsymbol{b}]]^\lambda_{\mathbb{L}}\prod_{i=1}^r\prod_{j=i+1}^n(x_i +_{\mathbb{L}} [t](\overline{x}_j))\right) = HQ^{\mathbb{L}}_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b}). \tag{3.6}$$

Here $\boldsymbol{b}=(b_1,b_2,\dots)$ is a sequence of elements in $MU^*(X)$.

This characterization seems merely a paraphrase of Definition 3.1 at first sight. However, this geometric interpretation will be crucial in our current work. In fact, as shown in the subsequent section §4, a careful application of the fundamental Gysin formula (2.2) to the left hand side of (3.5), (3.6) enables us to obtain the generating functions for the universal factorial Hall–Littlewood P- and Q-functions.

Remark 3.6. As a special case of the above result, the factorial Hall–Littlewood P-polynomial $HP_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$ is characterized by the cohomology Gysin map, i.e., we have

$$(\pi^{r,\dots,1})_* \left([\boldsymbol{x}|\boldsymbol{b}]^\lambda \prod_{i=1}^{r} \prod_{j=i+1}^{n} (x_i - tx_j) \right) = HP_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b}).$$

A factorial version of Macdonald's Hall–Littlewood P-polynomial $P_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$ can be also characterized by the Gysin map: Consider the partial flag bundle $\pi^\lambda : \mathcal{F}\ell^\lambda(E) := \mathcal{F}\ell^{\nu(d-1),\nu(d-2),\dots,\nu(1)}(E) \longrightarrow X$. Here we write $\lambda = (n_1^{p_1} \cdots n_d^{p_d})$ and $\nu(k) = \sum_{i=1}^{k} p_i$ as in §3.1. Then the following formula holds:

$$(\pi^\lambda)_* \left([\boldsymbol{x}|\boldsymbol{b}]^\lambda \prod_{i=1}^{r} \prod_{j=i+1}^{n} (x_i - tx_j) \right) = P_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b}).$$

3.3. **Vanishing properties of factorial Hall–Littlewood P- and Q-polynomials.** It is known that the factorial Schur S-, P-, and Q-polynomials have the remarkable property called *vanishing property* (see Molev–Sagan [21, Theorem 2.1], Ivanov [15, Theorem 5.3]). In this subsection, we shall show that our factorial Hall–Littlewood P- and Q-polynomials have this property. Let $\boldsymbol{b} = (b_1, b_2, \dots)$ be a sequence of indeterminates, and t be an indeterminate. For a partition $\mu = (\mu_1, \mu_2, \dots)$, let $m_i = m_i(\mu)$ be the multiplicity of i $(1 \le i \le \mu_1)$, i.e., the number of components in μ whose size is equal to i. We define

$$-\boldsymbol{b}_\mu(t) := (-\boldsymbol{b}_{\mu_1}^{m_{\mu_1}}(t), \dots, -\boldsymbol{b}_2^{m_2}(t), -\boldsymbol{b}_1^{m_1}(t)),$$

where $-\boldsymbol{b}_i^k(t) := (-b_i, -tb_i, \dots, -t^{k-1}b_i)$ (we set $-\boldsymbol{b}_i^0(t) = (\ )$, the empty sequence). Let us consider to substitute the variables $\boldsymbol{x}_n = (x_1, \dots, x_n)$ with the sequence $-\boldsymbol{b}_\mu(t)$ for a partition μ of length $\ell(\mu) \le n$. We sometimes write $\boldsymbol{x}_n \to -\boldsymbol{b}_\mu(t)$, or more specifically, say, $x_1 \to -b_{\mu_1}$ when we make such substitution. After the substitution $\boldsymbol{x}_n \to -\boldsymbol{b}_\mu(t)$ was made, denote by $\mathrm{ev}_\mu(x_i)$ $(i = 1, \dots, n)$ the i-th entry of $-\boldsymbol{b}_\mu(t)$. Therefore we have

$$(\mathrm{ev}_\mu(x_1), \dots, \mathrm{ev}_\mu(x_n)) = -\boldsymbol{b}_\mu(t).$$

We also use the notation $\mathrm{ev}_\mu(f(x_1, \dots, x_n)) = f(\mathrm{ev}_\mu(x_1), \dots, \mathrm{ev}_\mu(x_n))$ in the following. For example, if $\mu = (5, 5, 5, 4, 1, 1)$, then $m_1(\mu) = 2$, $m_2(\mu) = 0$, $m_3(\mu) = 0$, $m_4(\mu) = 1$, $m_5(\mu) = 3$, and

$-\boldsymbol{b}_\mu(t) = (-b_5, -tb_5, -t^2b_5, -b_4, -b_1, -tb_1)$, $\mathrm{ev}_\mu(x_1) = -b_5$, $\mathrm{ev}_\mu(x_2) = -tb_5$, $\mathrm{ev}_\mu(x_2 - tx_1) = -tb_5 - t\cdot(-b_5) = 0$, etc. With these notations, we can prove the following:

Proposition 3.7 (Vanishing property)**.** *Let λ, μ be partitions of length at most n and set $\hat{\mu} := \mu + (1^n) = (\mu_1+1, \mu_2+1, \ldots, \mu_n+1)$. Then the factorial Hall–Littlewood P- and Q-polynomials satisfy the following vanishing property*:

(1) If $\mu \not\supset \lambda$, we have

$$HQ_\lambda(-\boldsymbol{b}_\mu(t), \underbrace{0,\ldots,0}_{n-\ell(\mu)}; t|\boldsymbol{b}) = 0 \quad \textit{and} \quad HP_\lambda(-\boldsymbol{b}_{\hat{\mu}}(t); t|\boldsymbol{b}) = 0.$$

(2) If $\mu = \lambda$, we have

$$\begin{aligned} HQ_\lambda(-\boldsymbol{b}_\lambda(t), \underbrace{0,\ldots,0}_{n-\ell(\lambda)}; t|\boldsymbol{b}) &= \prod_{q=1}^{\lambda_1}\prod_{k=1}^{m_q(\lambda)}\left(\prod_{p=1}^{q}(-t^{k-1}b_q + t^{m_p(\lambda)}b_p)\right), \textit{and} \\ HP_\lambda(-\boldsymbol{b}_{\hat{\lambda}}(t); t|\boldsymbol{b}) &= v_{\lambda>0}(t)\prod_{q=2}^{\hat{\lambda}_1}\prod_{k=1}^{m_q(\hat{\lambda})}\left(\prod_{p=1}^{q-1}(-t^{k-1}b_q + t^{m_p(\hat{\lambda})}b_p)\right). \end{aligned}$$

Proof. We only prove the case of $HP_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$. The case of $HQ_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$ can be proved similarly.

(1) As $\lambda \not\subset \mu$, we can find minimal k such that $\lambda_k > \mu_k$ $(1 \le k \le \ell(\lambda) = r)$. For each choice w of $\overline{w} \in S_n/(S_1)^r \times S_{n-r}$, we will show the corresponding summand in (3.2) vanishes, i.e.,

$$\left(w \cdot \left[[x_1|\boldsymbol{b}]^{\lambda_1}\cdots[x_r|\boldsymbol{b}]^{\lambda_r}\prod_{1\le i\le r,\ i<j\le n}\frac{x_i - tx_j}{x_i - x_j}\right]\right)_{\boldsymbol{x}_n \to -\boldsymbol{b}_{\hat{\mu}}(t)} = 0.$$

For the permutation w, take minimal d $(1 \le d \le k)$ such that $w(d) \ge k$. Then we divide the discussion into two cases:
Case 1. $w(d) = 1$ or $[w(d) > 1$ and $\mu_{w(d)-1} > \mu_{w(d)}]$. In this case,

$$\left([x_{w(d)}|\boldsymbol{b}]^{\lambda_d}\right)_{x_{w(d)} \to \mathrm{ev}_{\hat{\mu}}(x_{w(d)})} = 0$$

because $\mathrm{ev}_{\hat{\mu}}(x_{w(d)}) = -b_{\mu_{w(d)}+1}$ and $\lambda_d \ge \lambda_k > \mu_k \ge \mu_{w(d)}$.
Case 2. $w(d) > 1$ and $\mu_{w(d)-1} = \mu_{w(d)}$. In this case, we claim that

$$\mathrm{ev}_{\hat{\mu}}\left(\prod_{1\le i\le r,\ i<j\le n}\frac{x_{w(i)} - tx_{w(j)}}{x_{w(i)} - x_{w(j)}}\right) = 0.$$

First note that, by the minimality of the choice of k, we have $w(d) > k$. Let p $(1 \le p \le n)$ be an integer such that $w(p) = w(d) - 1$. Then, by the minimality of d, we have $d < p \le n$. Since $\mu_{w(p)} = \mu_{w(d)}$ and $w(d) = w(p)+1$, we have $\mathrm{ev}_{\hat{\mu}}(x_{w(d)}) = t\cdot \mathrm{ev}_{\hat{\mu}}(x_{w(p)})$. As $1 \le d \le r$, and $d < p \le n$, the factor $\mathrm{ev}_{\hat{\mu}}(x_{w(d)} - tx_{w(p)})$ vanishes, and therefore our claim follows.

(2) When $\mu = \lambda$, we first show that each summand corresponding to $\overline{w} \in S_n/(S_1)^r \times S_{n-r}$ vanishes under the evaluation $\mathrm{ev}_{\hat{\lambda}}$, except for $\overline{w} = \overline{e}$ (e the identity element). In fact, if $\overline{w} \neq \overline{e}$, we can find minimal d such that $1 \le d \le r$ and $w(d) > d$. Then, by dividing the argument into two cases Case 1. $\lambda_{w(d)-1} > \lambda_{w(d)}$, and Case 2. $\lambda_{w(d)-1} = \lambda_{w(d)}$, we can show that the corresponding summand vanishes under the evaluation $\mathrm{ev}_{\hat{\lambda}}$.

For $w = e$, we can evaluate the term as follows. For each i $(1 \le i \le r)$, we can write $\mathrm{ev}_{\hat{\lambda}}(x_i) = t^{k-1}b_q$ $(k \ge 1,\ q = \lambda_i + 1 \ge 2)$. Then, the direct computation yields

$$\mathrm{ev}_{\hat{\lambda}}\left([x_i|\boldsymbol{b}]^{\lambda_i}\prod_{j=i+1}^{n}\frac{x_i - tx_j}{x_i - x_j}\right) = \frac{1 - t^{m_q(\hat{\lambda})-k+1}}{1-t}\prod_{p=1}^{q-1}(-t^{k-1}b_q + t^{m_p(\hat{\lambda})}b_p).$$

We then take all the product of these evaluations for $1 \le i \le r$. Since we have $\displaystyle\prod_{q=2}^{\hat{\lambda}_1}\prod_{k=1}^{m_q(\hat{\lambda})}\frac{1 - t^{m_q(\hat{\lambda})-k+1}}{1-t} = v_{\lambda>0}(t)$, we get the desired formula. $\square$

More generally, we can prove the vanishing property of the universal factorial Hall–Littlewood P- and Q-functions by the similar way. We only exhibit the result. To state the result, we prepare some notations. For a partition μ, we define

$$\overline{\boldsymbol{b}}_\mu[t] := (\overline{\boldsymbol{b}}_{\mu_1}^{m_{\mu_1}}[t], \overline{\boldsymbol{b}}_{\mu_1-1}^{m_{\mu_1-1}}[t], \dots, \overline{\boldsymbol{b}}_2^{m_2}[t], \overline{\boldsymbol{b}}_1^{m_1}[t]),$$

where $\overline{\boldsymbol{b}}_i^k[t] := (\overline{b}_i, [t](\overline{b}_i), \dots, [t^{k-1}](\overline{b}_i))$ (we set $\overline{\boldsymbol{b}}_i^0[t] = (\ )$ i.e., the empty sequence).

Proposition 3.8 (Vanishing property)**.** *Let λ, μ be partitions of length at most n and set $\hat{\mu} = \mu + (1^n) = (\mu_1+1, \mu_2+1, \dots, \mu_n+1)$. Then the universal factorial Hall–Littlewood P- and Q-functions satisfy the following vanishing property*:

(1) If $\mu \not\supset \lambda$, we have

$$HQ_\lambda^{\mathbb{L}}(\overline{\boldsymbol{b}}_\mu[t], \underbrace{0, \dots, 0}_{n-\ell(\mu)}; t|\boldsymbol{b}) = 0 \quad \text{and} \quad HP_\lambda^{\mathbb{L}}(\overline{\boldsymbol{b}}_{\hat{\mu}}[t]; t|\boldsymbol{b}) = 0.$$

(2) If $\mu = \lambda$, we have

$$HQ^{\mathbb{L}}_\lambda(\overline{\boldsymbol{b}}_\lambda[t], \underbrace{0,\dots,0}_{n-\ell(\lambda)}; t|\boldsymbol{b}) = \prod_{q=1}^{\lambda_1}\prod_{k=1}^{m_q(\lambda)}\left(\prod_{p=1}^{q}([t^{k-1}](\overline{b}_q) +_{\mathbb{L}} [t^{m_p(\lambda)}](b_p))\right), \textit{ and}$$

$$HP^{\mathbb{L}}_\lambda(\overline{\boldsymbol{b}}_{\hat{\lambda}}[t]; t|\boldsymbol{b}) = v_{\lambda>0}(t)\prod_{q=2}^{\hat{\lambda}_1}\prod_{k=1}^{m_q(\hat{\lambda})}\left(\prod_{p=1}^{q-1}([t^{k-1}](\overline{b}_q) +_{\mathbb{L}} [t^{m_p(\hat{\lambda})}](b_p))\right).$$

3.4. **Pieri-type formula and Hook formula.** The vanishing property established in the previous section is so useful that one can derive several interesting results of factorial Hall–Littlewood polynomials from this. Denote by $\Lambda(\boldsymbol{x}_n) = \mathbb{Z}[x_1,\dots,x_n]^{S_n}$ the ring of symmetric polynomials of n variables, and $\mathcal{P}_n$ the set of partitions of length $\leq n$. Then, it is known that the usual Hall–Littlewood P-polynomials $P_\lambda(\boldsymbol{x}_n; t)$ ($\lambda \in \mathcal{P}_n$) form a $\mathbb{Z}[t]$-basis of $\Lambda(\boldsymbol{x}_n)[t] \cong \mathbb{Z}[t] \otimes_{\mathbb{Z}} \Lambda(\boldsymbol{x}_n)$ (cf. Macdonald [19, III, (2.7)]). Therefore there exist polynomials $c^\nu_{\lambda,\mu}(t) = c^{\nu,(n)}_{\lambda,\mu}(t) \in \mathbb{Z}[t]$ such that

$$P_\lambda(\boldsymbol{x}_n; t)P_\mu(\boldsymbol{x}_n; t) = \sum_\nu c^\nu_{\lambda,\mu}(t)P_\nu(\boldsymbol{x}_n; t) \quad (\lambda, \mu, \nu \in \mathcal{P}_n).$$

It is known that (see Macdonald [19, III, (5.7)]) the following Pieri-type formula holds:

$$P_{(1)}(\boldsymbol{x}_n; t)P_\mu(\boldsymbol{x}_n; t) = \sum_{\mu\subset\nu,\ |\nu/\mu|=1} \alpha_{\nu/\mu}(t)P_\nu(\boldsymbol{x}_n; t), \tag{3.7}$$

where polynomial $\alpha_{\nu/\mu}(t) = \alpha^{(n)}_{\nu/\mu}(t)$ is given by $\dfrac{1-t^{m_j(\nu)}}{1-t}$ if ν/μ has a box in the jth column. As for the factorial version of Macdonald's Hall–Littlewood P-polynomials $P_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$ (see (3.3)), one can consider a similar problem: First we see that factorial Hall–Littlewood P-polynomials $P_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})$ ($\lambda \in \mathcal{P}_n$) form a $\mathbb{Z}[t] \otimes_{\mathbb{Z}} \mathbb{Z}[\boldsymbol{b}]$-basis of $\Lambda(\boldsymbol{x}_n|\boldsymbol{b})[t] := \mathbb{Z}[t] \otimes_{\mathbb{Z}} \mathbb{Z}[\boldsymbol{b}] \otimes_{\mathbb{Z}} \Lambda(\boldsymbol{x}_n)$, where $\mathbb{Z}[\boldsymbol{b}] = \mathbb{Z}[b_1, b_2, \dots]$ is a polynomial ring of indeterminates $\boldsymbol{b} = (b_1, b_2, \dots)$. Therefore there exist polynomials $c^\nu_{\lambda,\mu}(t|\boldsymbol{b}) = c^{\nu,(n)}_{\lambda,\mu}(t|\boldsymbol{b}) \in \mathbb{Z}[t] \otimes \mathbb{Z}[\boldsymbol{b}]$ such that

$$P_\lambda(\boldsymbol{x}_n; t|\boldsymbol{b})P_\mu(\boldsymbol{x}_n; t|\boldsymbol{b}) = \sum_\nu c^\nu_{\lambda,\mu}(t|\boldsymbol{b})P_\nu(\boldsymbol{x}_n; t|\boldsymbol{b}) \quad (\lambda, \mu, \nu \in \mathcal{P}_n). \tag{3.8}$$

By definition, the "structure constant" $c^\nu_{\lambda,\mu}(t|\boldsymbol{b})$ is a homogeneous polynomial of degree $|\lambda| + |\mu| - |\nu|$ in the indeterminates $\boldsymbol{b} = (b_1, b_2, \dots)$ with coefficients in $\mathbb{Z}[t]$. Comparing the highest homogeneous components in $\boldsymbol{x}_n = (x_1, \dots, x_n)$ on both sides of (3.8), we see that

$$c^\nu_{\lambda,\mu}(t|\boldsymbol{b}) = \begin{cases} c^\nu_{\lambda,\mu}(t) & \text{if } |\lambda| + |\mu| = |\nu|, \\ 0 & \text{if } |\lambda| + |\mu| < |\nu|. \end{cases}$$

From the commutativity of the product in the left hand side of (3.8), the symmetry $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) = c^{\nu}_{\mu,\lambda}(t|\boldsymbol{b})$ holds obviously. Furthermore, using the vanishing property[4], Proposition 3.7, we claim that $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b})$ is zero unless $\lambda \subset \nu$ and $\mu \subset \nu$. The proof proceeds as follows (cf. Molev–Sagan [21, p.4434]): Let ν be minimal with respec to containment relation among all partitions ρ in (3.8) such that $c^{\rho}_{\lambda,\mu}(t|\boldsymbol{b}) \neq 0$. Suppose that $\mu \not\subset \nu$. We set $\boldsymbol{x}_n = -\boldsymbol{b}_{\hat{\nu}}(t)$ in (3.8). Then, by Proposition 3.7 (1), we have

$$0 = c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) P_{\nu}(-\boldsymbol{b}_{\hat{\nu}}(t); t|\boldsymbol{b}).$$

By Proposition 3.7 (2), we have $P_{\nu}(-\boldsymbol{b}_{\hat{\nu}}(t); t|\boldsymbol{b}) \neq 0$, and hence $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) = 0$. However this contradicts to $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) \neq 0$, and hence $\mu \subset \nu$ holds. From this and the symmetry relation $c^{\nu}_{\lambda,\mu}(t|\boldsymbol{b}) = c^{\nu}_{\mu,\lambda}(t|\boldsymbol{b})$, our claim follows.

Now we consider the case where $\lambda = (1)$ in (3.8). Then, by the known properties of the structure constants, we only need to consider those ν with $\mu \subset \nu$ and $|\nu| \le |\mu| + 1$, Thus (3.8) takes the following form:

$$P_{(1)}(\boldsymbol{x}_n; t|\boldsymbol{b}) P_{\mu}(\boldsymbol{x}_n; t|\boldsymbol{b}) = c^{\mu}_{(1),\mu}(t|\boldsymbol{b}) P_{\mu}(\boldsymbol{x}_n; t|\boldsymbol{b}) + \sum_{\mu\subset\nu,\ |\nu/\mu|=1} c^{\nu}_{(1),\mu}(t|\boldsymbol{b}) P_{\nu}(\boldsymbol{x}_n; t|\boldsymbol{b}).$$

Setting $\boldsymbol{x}_n = -\boldsymbol{b}_{\hat{\mu}}(t)$ and using the vanishing property, we see that $c^{\mu}_{(1),\mu}(t|\boldsymbol{b}) = P_{(1)}(-\boldsymbol{b}_{\hat{\mu}}(t); t|\boldsymbol{b})$. On the other hand, by the degree reason, we have $c^{\nu}_{(1),\mu}(t|\boldsymbol{b}) = c^{\nu}_{(1),\mu}(t) = \alpha_{\nu/\mu}(t)$ when $\mu \subset \nu$ and $|\nu/\mu| = 1$. Therefore we obtain the following formula:

Proposition 3.9 (Pieri-type formula for factorial Hall–Littlewood P-polynomials)**.**

$$P_{(1)}(\boldsymbol{x}_n; t|\boldsymbol{b}) P_{\mu}(\boldsymbol{x}_n; t|\boldsymbol{b}) = P_{(1)}(-\boldsymbol{b}_{\hat{\mu}}(t); t|\boldsymbol{b}) P_{\mu}(\boldsymbol{x}_n; t|\boldsymbol{b}) + \sum_{\mu\subset\nu,\ |\nu/\mu|=1} \alpha_{\nu/\mu} P_{\nu}(\boldsymbol{x}_n; t|\boldsymbol{b}).$$

Using Proposition 3.9, we can derive a generalization of the so-called *hook (length) formula*. We argue as follows (the following argument is essentially the same as that given in Molev–Sagan [21, Proposition 3.2] for factorial Schur polynomials, although they did not mention the relation to the hook formula. For this type of argument, see also Naruse–Okada [29, Lemma 4.5]). For simplicity, we shall use the abbreviated notation P_{λ}, $c^{\nu}_{\lambda,\mu}$, and $\alpha_{\lambda/\mu}$ for $P_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$, $c^{\nu,(n)}_{\lambda,\mu}(t|\boldsymbol{b})$, and $\alpha^{(n)}_{\lambda/\mu}(t)$ respectively in the following. Then our hook formula is stated as follows:

Proposition 3.10 (Hook formula for factorial Hall–Littlewood P-polynomials)**.** *Let μ be a partition of length $\ell(\mu) \le n$ and size $|\mu| = k$, a positive integer.*

[4]By the definition (3.3), $P_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$'s also satisfy the vanishing property.

Then we have the following formula:

(3.9)

$$\sum_{\mu=\mu^{(0)}\supsetneq\mu^{(1)}\supsetneq\mu^{(2)}\supsetneq\cdots\supsetneq\mu^{(k)}=\emptyset} \frac{\alpha_{\mu^{(k-1)}/\mu^{(k)}}}{c^{\mu}_{(1),\mu}-c^{\mu^{(k)}}_{(1),\mu^{(k)}}}\cdot\cdots\cdot\frac{\alpha_{\mu^{(1)}/\mu^{(2)}}}{c^{\mu}_{(1),\mu}-c^{\mu^{(2)}}_{(1),\mu^{(2)}}}\cdot\frac{\alpha_{\mu^{(0)}/\mu^{(1)}}}{c^{\mu}_{(1),\mu}-c^{\mu^{(1)}}_{(1),\mu^{(1)}}} = \frac{1}{P_\mu(-\boldsymbol{b}_{\hat{\mu}}(t);t|\boldsymbol{b})}.$$

Proof. We consider the associativity of the product

$$(P_{(1)}P_\lambda)P_\mu = P_{(1)}(P_\lambda P_\mu),$$

and take the coefficient of P_μ on both sides. Using the fact that $c^{\gamma}_{\alpha,\beta}$ is zero unless $\alpha\subset\gamma$ and $\beta\subset\gamma$, and Proposition 3.9, we have

$$c^{\lambda}_{(1),\lambda}c^{\mu}_{\lambda,\mu} + \sum_{\mu\supset\nu\supsetneq\lambda,\ |\nu/\lambda|=1} \alpha_{\nu/\lambda}\, c^{\mu}_{\nu,\mu} = c^{\mu}_{(1),\mu}c^{\mu}_{\lambda,\mu},$$

and therefore we have

$$(c^{\mu}_{(1),\mu}-c^{\lambda}_{(1),\lambda})c^{\mu}_{\lambda,\mu} = \sum_{\mu\supset\nu\supsetneq\lambda,\ |\nu/\lambda|=1} \alpha_{\nu/\lambda}\, c^{\mu}_{\nu,\mu}.$$

By definition and Example 3.3, we know that $P_{(1)}(\boldsymbol{x}_n;t|\boldsymbol{b}) = x_1+\cdots+x_n+\frac{1-t^n}{1-t}b_1$. Therefore, if $\mu\supsetneq\lambda$, we see that $c^{\mu}_{(1),\mu}-c^{\lambda}_{(1),\lambda} = P_{(1)}(-\boldsymbol{b}_{\hat{\mu}}(t);t|\boldsymbol{b}) - P_{(1)}(-\boldsymbol{b}_{\hat{\lambda}}(t);t|\boldsymbol{b}) \neq 0$. Thus we have the following recurrence formula:

$$c^{\mu}_{\lambda,\mu} = \sum_{\mu\supset\nu\supsetneq\lambda,\ |\nu/\lambda|=1} \frac{\alpha_{\nu/\lambda}}{c^{\mu}_{(1),\mu}-c^{\lambda}_{(1),\lambda}} c^{\mu}_{\nu,\mu}.$$

Using this recurrence formula repeatedly, we obtain

$$c^{\mu}_{\emptyset,\mu} = \sum_{\mu=\mu^{(0)}\supsetneq\mu^{(1)}\supsetneq\mu^{(2)}\supsetneq\cdots\supsetneq\mu^{(k)}=\emptyset} \frac{\alpha_{\mu^{(k-1)}/\mu^{(k)}}}{c^{\mu}_{(1),\mu}-c^{\mu^{(k)}}_{(1),\mu^{(k)}}}\cdot\cdots\cdot\frac{\alpha_{\mu^{(0)}/\mu^{(1)}}}{c^{\mu}_{(1),\mu}-c^{\mu^{(1)}}_{(1),\mu^{(1)}}} c^{\mu}_{\mu,\mu}.$$

The fact that $c^{\mu}_{\emptyset,\mu}=1$ is obvious from the definition of structure constants. The value of $c^{\mu}_{\mu,\mu}$ equals to $P_\mu(-\boldsymbol{b}_{\hat{\mu}}(t);t|\boldsymbol{b})$ by virtue of the vanishing property, Proposition 3.7. Therefore, we have the desired equation. □

As mentioned before the proposition, one can obtain a similar hook formula by [21, Proposition 3.2]. More concretely, under their notation, one has the following formula:

(3.10)

$$\sum_{\emptyset=\rho^{(0)}\to\rho^{(1)}\to\cdots\to\rho^{(l-1)}\to\rho^{(l)}=\nu} \frac{1}{(|a_\nu|-|a_{\rho^{(0)}}|)\cdots(|a_\nu|-|a_{\rho^{(l-1)}}|)} = \frac{1}{s_\nu(a_\nu|a)}.$$

We remark that this formula can be interpreted as a special case of Nakada's *colored hook formula* ([22, Corollary 7.2]), which is a generalization of the famous hook formula due to Frame–Robinson–Thrall [6]. As an example, let us take $\nu = (2,2)$ and $n = 2$, the number of variables. Then the above formula leads to

$$\begin{aligned}&\frac{1}{(a_3-a_2)(a_3-a_1)(a_4-a_1)(a_4+a_3-a_2-a_1)}\\ &\qquad+\frac{1}{(a_3-a_2)(a_4-a_2)(a_4-a_1)(a_4+a_3-a_2-a_1)}\\ =\ &\frac{1}{(a_3-a_2)(a_3-a_1)(a_4-a_2)(a_4-a_1)}.\end{aligned}$$

Now consider the simple root system $\{\alpha_1, \alpha_2, \alpha_3\}$ of type A_3. If one represent the simple root α_i as $a_i - a_{i+1}$ for $i = 1, 2, 3$, then the above identity becomes

(3.11)
$$\begin{aligned}&\frac{1}{\alpha_2(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+2\alpha_2+\alpha_3)}\\ &\qquad+\frac{1}{\alpha_2(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+2\alpha_2+\alpha_3)}\\ =\ &\frac{1}{\alpha_2(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)},\end{aligned}$$

which agrees with the example given in [22, p.1088]. When we specialize t to be 0, our factorial Hall–Littlewood P-polynomial $HP_\lambda(\boldsymbol{x}_n; 0|\boldsymbol{b}) = P_\lambda(\boldsymbol{x}_n; 0|\boldsymbol{b})$ does not coincide with the factorial Schur polynomial $s_\lambda(\boldsymbol{x}_n|\boldsymbol{b})$.[5] Thus $t = 0$ specialization of our hook formula (3.9) yields *another* colored hook formula (see the example below). It is well-known that the classical hook formula and its *shifted* analogue have geometric background known as Schubert calculus, and are closely related to combinatorics of Grassmannians, root systems, and Weyl groups (see e.g., Hiller [10]). In our forthcoming paper ([27]), we shall discuss geometric or topological background of our hook formula, in relation to complex reflection groups $G(e, 1, n)$ and $G(e, e, n)$ (for root systems of these groups, see Bremke–Malle [1] [2]).

Example 3.11. For the partition $\mu = (2, 2)$, the explicit form of our hook length formula is given as follows: First note that there exist "two paths" from $\mu = (2, 2)$ to $\emptyset = (\ )$. Namely,

$$\mu = (2,2) \supsetneq (2,1) \supsetneq (2) \supsetneq (1) \supsetneq (\ ) \text{ and } \mu = (2,2) \supsetneq (2,1) \supsetneq (1,1) \supsetneq (1) \supsetneq (\ ).$$

[5]In the definition of the factorial Schur polynomial $s_\lambda(x|a)$ given by Molev–Sagan [21, §2, (3)], we replaced a doubly-infinite variable sequence $a = (a_i)$, $i \in \mathbb{Z}$, by $\boldsymbol{b} = (b_1, b_2, \ldots)$.

From the fact that $c^{\nu}_{(1),\nu} = c^{\nu,(n)}_{(1),\nu}(t|\boldsymbol{b}) = P^{(n)}_{(1)}(-\boldsymbol{b}_{\hat{\nu}}(t);t|\boldsymbol{b})$, we get the following result directly:

$$\begin{aligned}
c^{(\,)}_{(1),(\,)} &= 0,\\
c^{(1)}_{(1),(1)} &= -b_2 + t^{n-1}b_1,\\
c^{(1,1)}_{(1),(1,1)} &= (1+t)(-b_2 + t^{n-2}b_1),\\
c^{(2)}_{(1),(2)} &= -b_3 + t^{n-1}b_1,\\
c^{(2,1)}_{(1),(2,1)} &= -b_3 - b_2 + (1+t)t^{n-2}b_1,\\
c^{(2,2)}_{(1),(2,2)} &= (1+t)(-b_3 + t^{n-2}b_1).
\end{aligned}$$

Similarly, $\alpha_{\nu/\lambda} = \alpha^{(n)}_{\nu/\lambda}(t)$ can be computed directly from the definition, and we get

$$\alpha_{(2,2)/(2,1)} = 1+t,\ \alpha_{(2,1)/(2)} = 1,\ \alpha_{(2,1)/(1,1)} = 1,\ \alpha_{(2)/(1)} = 1,\ \alpha_{(1,1)/(1)} = 1+t,\ \alpha_{(1)/(\,)} = 1.$$

By Proposition 3.7, we have, for $\mu = (2,2)$,

$$P^{(n)}_{\mu}(-\boldsymbol{b}_{\hat{\mu}}(t);t|\boldsymbol{b}) = (-b_3 + t^{n-2}b_1)(-tb_3 + t^{n-2}b_1)(-b_3 + b_2)(-tb_3 + b_2).$$

Therefore our hook formula gives the following identity:

$$\begin{aligned}
&\frac{1+t}{-tb_3 + b_2}\cdot\Bigg(\frac{1}{-tb_3 + t^{n-2}b_1}\cdot\frac{1}{-b_3 - tb_3 + b_2 + t^{n-2}b_1}\\
&\qquad\qquad + \frac{1}{-b_3 - tb_3 + b_2 + tb_2}\cdot\frac{1+t}{-b_3 - tb_3 + b_2 + t^{n-2}b_1}\Bigg)\\
&\qquad\qquad\qquad\qquad \times\frac{1}{-b_3 - tb_3 + t^{n-2}b_1 + t^{n-1}b_1}\\
= \;&\frac{1}{(-b_3 + t^{n-2}b_1)(-tb_3 + t^{n-2}b_1)(-b_3 + b_2)(-tb_3 + b_2)}.
\end{aligned}$$

4. Generating functions for the universal factorial Hall–Littlewood P- and Q-functions

In this section, by utilizing a Gysin formula in complex cobordism, Proposition 2.1, we shall derive the generating functions for the universal factorial Hall–Littlewood P- and Q-functions.

4.1. **Generating function for $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b})$.** Basic idea is to apply the fundamental formula (2.2) repeatedly to the characterization (3.5) to obtain the generating function. Here we remark that the formula (2.2) still holds for a formal power series $f(u) \in MU^*(X)[[u]]$ as well, and we shall use such an extended form of (2.2). However, we will be confronted with some difficulty when we apply the formula to (3.5). In order to clarify the difficulty, let us consider the simplest case $\lambda = (\lambda_1)$ with $\lambda_1 \geq 1$ (and hence $r = 1$) of (3.5). We wish to push-forward the expression $[x_1|\boldsymbol{b}]^{\lambda_1}_{\mathbb{L}} \prod_{j=2}^n (x_1 +_{\mathbb{L}} [t](\overline{x}_j))$ via the Gysin map $\pi^1_* : MU^*(G^1(E)) \longrightarrow MU^*(X)$. Naively, setting

$$f(u) := [u|\boldsymbol{b}]^{\lambda_1}_{\mathbb{L}} \cdot \prod_{j=2}^n (u +_{\mathbb{L}} [t](\overline{x}_j)),$$

we wish to compute $\pi^1_*(f(x_1))$. However, one cannot regard $f(u)$ as an element of $MU^*(X)[[u]]$ as it is. Therefore we consider the following expression instead:

$$f_1(u) := \frac{[u|\boldsymbol{b}]^{\lambda_1}_{\mathbb{L}}}{u +_{\mathbb{L}} [t](\overline{u})} \cdot \prod_{j=1}^n (u +_{\mathbb{L}} [t](\overline{x}_j)).$$

Since symmetric functions in $x_1, \dots, x_n$ can be regarded as elements of $MU^*(X)$ ($x_1, \dots, x_n$ are the Chern roots of E), the coefficients of $f_1(u)$ with respect to u are actually in $MU^*(X)$. Moreover, we have $f(x_1) = f_1(x_1)$ obviously. However, it is not a formal power series in u because of the constant term $b_1 b_2 \cdots b_{\lambda_1}$ in the numerator, and therefore the formula (2.2) does not apply directly. We further modify $f_1(u)$, and consider the following expression:

$$f_2(u) := \frac{[u|\boldsymbol{b}]^{\lambda_1}_{\mathbb{L}}}{u +_{\mathbb{L}} [t](\overline{u})} \left\{ \prod_{j=1}^n (u +_{\mathbb{L}} [t](\overline{x}_j)) - \prod_{j=1}^n [t](u +_{\mathbb{L}} \overline{x}_j) \right\}. \tag{4.1}$$

The effect of subtracting the term $\prod_{j=1}^n [t](u +_{\mathbb{L}} \overline{x}_j)$ (hereafter we call it the "correction term") is two-fold: Firstly, the expression $\prod_{j=1}^n (u +_{\mathbb{L}} [t](\overline{x}_j)) - \prod_{j=1}^n [t](u +_{\mathbb{L}} \overline{x}_j)$ is divisible by u, and therefore $f_2(u)$ becomes indeed a formal power series in u with coefficients in $MU^*(X)$. Secondly, when we substitute x_1 for u, we have $f(x_1) = f_2(x_1)$ by the obvious identity $\prod_{j=1}^n [t](x_1 +_{\mathbb{L}} \overline{x}_j) = 0$. Therefore the fundamental Gysin formula (2.2) does apply to $f_2(u)$, and the result is given as follows:

$$\begin{aligned}
&HP^{\mathbb{L}}_{(\lambda_1)}(\boldsymbol{x}_n; t|\boldsymbol{b}) = \pi^1_*(f_2(x_1)) = [u^{n-1}](f_2(u) \times \mathscr{S}^{\mathbb{L}}(E; 1/u)) \\
&= [u^{n-1}] \left[\frac{[u|\boldsymbol{b}]^{\lambda_1}_{\mathbb{L}}}{u +_{\mathbb{L}} [t](\overline{u}_1)} \left\{ \prod_{j=1}^n (u +_{\mathbb{L}} [t](\overline{x}_j)) - \prod_{j=1}^n [t](u +_{\mathbb{L}} \overline{x}_j) \right\} \cdot \mathscr{S}^{\mathbb{L}}(E; 1/u) \right] \\
&= [u^{-\lambda_1}] \left[\frac{1}{\mathscr{P}^{\mathbb{L}}(u)} \frac{u}{u +_{\mathbb{L}} [t](\overline{u})} \left\{ \prod_{j=1}^n \frac{u +_{\mathbb{L}} [t](\overline{x}_j)}{u +_{\mathbb{L}} \overline{x}_j} - \prod_{j=1}^n \frac{[t](u +_{\mathbb{L}} \overline{x}_j)}{u +_{\mathbb{L}} \overline{x}_j} \right\} \cdot \prod_{j=1}^{\lambda_1} \frac{u +_{\mathbb{L}} b_j}{u} \right].
\end{aligned}$$

Example 4.1. As a special case of the above formula, the ordinary factorial Hall-Littlewood P-polynomial corresponding to the one-row (λ_1) is given by

$$HP_{(\lambda_1)}(\boldsymbol{x}_n; t|\boldsymbol{b}) = [u^{-\lambda_1}] \left[\frac{1}{1-t} \left(\prod_{j=1}^n \frac{u - tx_j}{u - x_j} - t^n \right) \times \prod_{j=1}^{\lambda_1} \frac{u + b_j}{u} \right].$$

In particular, we have

$$
\begin{aligned}
HP_{(1)}(\boldsymbol{x}_n;t|\boldsymbol{b}) &= [u^{-1}]\left[\frac{1}{1-t}\left(\prod_{j=1}^{n}\frac{u-tx_j}{u-x_j}-t^n\right)\times\frac{u+b_1}{u}\right]\\
&= \frac{1}{1-t}q_1(\boldsymbol{x}_n;t)+\frac{1-t^n}{1-t}b_1\\
&= x_1+x_2+\cdots+x_n+(1+t+t^2+\cdots+t^{n-1})b_1.
\end{aligned}
$$

Here $q_r(\boldsymbol{x}_n;t)$ $(r=0,1,2,\ldots)$ are given by the following generating functions:

$$
\prod_{j=1}^{n}\frac{z-tx_j}{z-x_j}\bigg|_{z=u^{-1}}=\prod_{j=1}^{n}\frac{1-tx_ju}{1-x_ju}=\sum_{r=0}^{\infty}q_r(\boldsymbol{x}_n;t)u^r.
$$

For a general sequence of positive integers $\lambda=(\lambda_1,\ldots,\lambda_r)$ with $r\leq n$, we need to compute the push-forward image of $[\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}}\prod_{i=1}^{r}\prod_{j=i+1}^{n}(x_i+_{\mathbb{L}}[t](\overline{x}_j))$ under the Gysin map $(\pi^{r,r-1,\ldots,1})_*: MU^*(\mathcal{F}\ell^{r,\ldots,1}(E))\longrightarrow MU^*(X)$. The image of $(\pi^{r,r-1,\ldots,1})_*$ can be computed by applying $\pi^r_*,\pi^{r-1}_*,\ldots,\pi^1_*$ successively. In each step, we use the modification such as (4.1), i.e., subtracting the "correction term". This technique enables us to apply the fundamental Gysin formula (2.2), and we are able to show the following result:

Lemma 4.2. *For a sequence of positive integers $\lambda=(\lambda_1,\ldots,\lambda_r)$ with $r\leq n$, we have the following formula*:

(4.2)

$$
\begin{aligned}
&(\pi^{r,r-1,\ldots,1})_*\left([\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}}\prod_{i=1}^{r}\prod_{j=i+1}^{n}(x_i+_{\mathbb{L}}[t](\overline{x}_j))\right)=\left[\prod_{i=1}^{r}u_i^{-\lambda_i}\right]\\
&\left[\prod_{i=1}^{r}\frac{u_i}{u_i+_{\mathbb{L}}[t](\overline{u}_i)}\cdot\frac{1}{\mathscr{P}^{\mathbb{L}}(u_i)}\right.\\
&\qquad\times\left\{\prod_{j=1}^{n}\frac{u_i+_{\mathbb{L}}[t](\overline{x}_j)}{u_i+_{\mathbb{L}}\overline{x}_j}-\prod_{j=1}^{i-1}\frac{u_i+_{\mathbb{L}}[t](\overline{u}_j)}{[t](u_i+_{\mathbb{L}}\overline{u}_j)}\prod_{j=1}^{n}\frac{[t](u_i+_{\mathbb{L}}\overline{x}_j)}{u_i+_{\mathbb{L}}\overline{x}_j}\right\}\\
&\qquad\left.\times\prod_{1\leq i<j\leq r}\frac{u_j+_{\mathbb{L}}\overline{u}_i}{u_j+_{\mathbb{L}}[t](\overline{u}_i)}\times\prod_{i=1}^{r}\prod_{j=1}^{\lambda_i}\frac{u_i+_{\mathbb{L}}b_j}{u_i}\right].
\end{aligned}
$$

Proof. Let us compute the push-forward image of $[\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}}\prod_{i=1}^{r}\prod_{j=i+1}^{n}(x_i+_{\mathbb{L}}[t](\overline{x}_j))$ under the Gysin map $(\pi^{r,r-1,\ldots,1})_*=\pi^1_*\circ\cdots\circ\pi^{r-1}_*\circ\pi^r_*$. As we explained above, we carry out the computation inductively. For a $(=1,2,\ldots,r-$

1), we assume the following result:

(4.3)

$$
\begin{aligned}
&(\pi^{r-a+1}\circ\cdots\circ\pi^{r-1}\circ\pi^r)_*\left([\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}}\prod_{i=1}^{r}\prod_{j=i+1}^{n}(x_i+_{\mathbb{L}}[t](\overline{x}_j))\right)\\
&=[u^{n-1}_{r-a+1}\cdots u^{n-1}_{r-1}u^{n-1}_r]\left[\prod_{i=1}^{r-a}[x_i|\boldsymbol{b}]^{\lambda_i}_{\mathbb{L}}\prod_{j=i+1}^{n}(x_i+_{\mathbb{L}}[t](\overline{x}_j))\cdot\prod_{i=r-a+1}^{r}\frac{[u_i|\boldsymbol{b}]^{\lambda_i}_{\mathbb{L}}}{u_i+_{\mathbb{L}}[t](\overline{u}_i)}\right.\\
&\times\left\{\prod_{j=r-a+1}^{n}(u_i+_{\mathbb{L}}[t](\overline{x}_j))-\prod_{j=r-a+1}^{i-1}\frac{u_i+_{\mathbb{L}}[t](\overline{u}_j)}{[t](u_i+_{\mathbb{L}}\overline{u}_j)}\prod_{j=r-a+1}^{n}[t](u_i+_{\mathbb{L}}\overline{x}_j)\right\}\\
&\left.\times\prod_{i=r-a+1}^{r}\prod_{j=1}^{r-a}(u_i+_{\mathbb{L}}\overline{x}_j)\cdot\prod_{r-a+1\le i<j\le r}\frac{u_j+_{\mathbb{L}}\overline{u}_i}{u_j+_{\mathbb{L}}[t](\overline{u}_i)}\cdot\prod_{i=r-a+1}^{r}\mathscr{S}^{\mathbb{L}}(E;1/u_i)\right].
\end{aligned}
$$

We would like to push-forward this formula via the Gysin map

$$
\pi^{r-a}_*: MU^*(G^1(U_{n-r+a+1}))\longrightarrow MU^*(G^1(U_{n-r+a+2})).
$$

Taking (4.1) into account, we modify the right-hand side of (4.3) as

$$
\begin{aligned}
&[u^{n-1}_{r-a+1}\cdots u^{n-1}_{r-1}u^{n-1}_r]\left[\prod_{i=1}^{r-a-1}[x_i|\boldsymbol{b}]^{\lambda_i}_{\mathbb{L}}\prod_{j=i+1}^{n}(x_i+_{\mathbb{L}}[t](\overline{x}_j))\right.\\
&\times\frac{[x_{r-a}|\boldsymbol{b}]^{\lambda_{r-a}}_{\mathbb{L}}}{x_{r-a}+_{\mathbb{L}}[t](\overline{x}_{r-a})}\left\{\prod_{j=r-a}^{n}(x_{r-a}+_{\mathbb{L}}[t](\overline{x}_j))-\prod_{j=r-a}^{n}[t](x_{r-a}+_{\mathbb{L}}\overline{x}_j)\right\}\\
&\times\prod_{i=r-a+1}^{r}\frac{[u_i|\boldsymbol{b}]^{\lambda_i}_{\mathbb{L}}}{u_i+_{\mathbb{L}}[t](\overline{u}_i)}\left\{\frac{1}{u_i+_{\mathbb{L}}[t](\overline{x}_{r-a})}\prod_{j=r-a}^{n}(u_i+_{\mathbb{L}}[t](\overline{x}_j))\right.\\
&\qquad\left.-\prod_{j=r-a+1}^{i-1}\frac{u_i+_{\mathbb{L}}[t](\overline{u}_j)}{[t](u_i+_{\mathbb{L}}\overline{u}_j)}\cdot\frac{1}{[t](u_i+_{\mathbb{L}}\overline{x}_{r-a})}\prod_{j=r-a}^{n}[t](u_i+_{\mathbb{L}}\overline{x}_j)\right\}\\
&\times\prod_{i=r-a+1}^{r}\prod_{j=1}^{r-a-1}(u_i+_{\mathbb{L}}\overline{x}_j)\times\prod_{i=r-a+1}^{r}(u_i+_{\mathbb{L}}\overline{x}_{r-a})\times\prod_{r-a+1\le i<j\le r}\frac{u_j+_{\mathbb{L}}\overline{u}_i}{u_j+_{\mathbb{L}}[t](\overline{u}_i)}\\
&\left.\qquad\times\prod_{i=r-a+1}^{r}\mathscr{S}^{\mathbb{L}}(E;1/u_i)\right].
\end{aligned}
$$

Then, apply the fundamental Gysin formula (2.2). In the above modification, we divide both denominator and numerator of $\dfrac{1}{u_i+_{\mathbb{L}}[t](\overline{x}_{r-a})}$ by u_i, and consider it as a formal power series in x_{r-a}. We also treat $\dfrac{1}{[t](u_i+_{\mathbb{L}}\overline{x}_{r-a})}$ in the same manner. Under this remark, the result is just replacing x_{r-a} by the formal variable u_{r-a}, and multiplying by $\mathscr{S}^{\mathbb{L}}(U_{n-r+a+1};1/u_{r-a})$. Then, we extract the coefficient of u_{r-a}^{n-r+a}. Since we know from (2.1)

$$\mathscr{S}^{\mathbb{L}}(U_{n-r+a+1};1/u_{r-a}) = u_{r-a}^{-(r-a-1)}\prod_{j=1}^{r-a-1}(u_{r-a}+_{\mathbb{L}}\overline{x}_j)\times \mathscr{S}^{\mathbb{L}}(E;1/u_{r-a}),$$

we see directly that the formula (4.3) holds for $a+1$. Therefore, when $a=r$, we have

$$(\pi^{r,r-1,\dots,1})_*\left([\boldsymbol{x}|\boldsymbol{b}]^{\lambda}_{\mathbb{L}}\prod_{i=1}^{r}\prod_{j=i+1}^{n}(x_i+_{\mathbb{L}}[t](\overline{x}_j))\right) = [u_1^{n-1}\dots u_r^{n-1}]$$
$$\left[\prod_{i=1}^{r}\frac{[u_i|\boldsymbol{b}]^{\lambda_i}_{\mathbb{L}}}{u_i+_{\mathbb{L}}[t](\overline{u}_i)}\left\{\prod_{j=1}^{n}(u_i+_{\mathbb{L}}[t](\overline{x}_j)) - \prod_{j=1}^{i-1}\frac{u_i+_{\mathbb{L}}[t](\overline{u}_j)}{[t](u_i+_{\mathbb{L}}\overline{u}_j)}\prod_{j=1}^{n}[t](u_i+_{\mathbb{L}}\overline{x}_j)\right\}\right.$$
$$\left.\times\prod_{1\le i<j\le r}\frac{u_j+_{\mathbb{L}}\overline{u}_i}{u_j+_{\mathbb{L}}[t](\overline{u}_i)}\times\prod_{i=1}^{r}\mathscr{S}^{\mathbb{L}}(E;1/u_i)\right].$$

Then, using the Segre series (2.1), we obtain the required formula. $\square$

By a characterization (3.5), the left-hand side of (4.2) is $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b})$, and hence the right-hand side gives a generating function for $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b})$. Let us simplify this generating function in the following way: First note that

$$\prod_{1\le i<j\le r}\frac{u_j+_{\mathbb{L}}\overline{u}_i}{u_j+_{\mathbb{L}}[t](\overline{u}_i)} = \prod_{1\le j<i\le r}\frac{u_i+_{\mathbb{L}}\overline{u}_j}{u_i+_{\mathbb{L}}[t](\overline{u}_j)} = \prod_{i=1}^{r}\prod_{j=1}^{i-1}\frac{u_i+_{\mathbb{L}}\overline{u}_j}{u_i+_{\mathbb{L}}[t](\overline{u}_j)}.$$

Therefore if we put

$$\mathcal{HP}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1,u_2,\dots,u_i|\boldsymbol{b}) := \frac{u_i}{u_i+_{\mathbb{L}}[t](\overline{u}_i)}\cdot\frac{1}{\mathscr{P}^{\mathbb{L}}(u_i)}$$
$$\times\left(\prod_{j=1}^{n}\frac{u_i+_{\mathbb{L}}[t](\overline{x}_j)}{u_i+_{\mathbb{L}}\overline{x}_j}\prod_{j=1}^{i-1}\frac{u_i+_{\mathbb{L}}\overline{u}_j}{u_i+_{\mathbb{L}}[t](\overline{u}_j)}\prod_{j=1}^{\lambda_i}\frac{u_i+_{\mathbb{L}}b_j}{u_i}\right.$$
$$\left.-\prod_{j=1}^{n}\frac{[t](u_i+_{\mathbb{L}}\overline{x}_j)}{u_i+_{\mathbb{L}}\overline{x}_j}\prod_{j=1}^{i-1}\frac{u_i+_{\mathbb{L}}\overline{u}_j}{[t](u_i+_{\mathbb{L}}\overline{u}_j)}\prod_{j=1}^{\lambda_i}\frac{u_i+_{\mathbb{L}}b_j}{u_i}\right),$$

$$\mathcal{HP}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b}) = \mathcal{HP}^{\mathbb{L},(n)}_{\lambda}(u_1,u_2,\dots,u_r|\boldsymbol{b}) := \prod_{i=1}^{r}\mathcal{HP}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1,u_2,\dots,u_i|\boldsymbol{b}),$$

then, one has

$$HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;t|\boldsymbol{b}) = [\boldsymbol{u}^{-\lambda}]\left(\mathcal{HP}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b})\right). \tag{4.4}$$

Moreover, observe that

- $\dfrac{u_i}{u_i+_{\mathbb{L}}[t](\overline{u}_i)}\cdot\dfrac{1}{\mathscr{P}^{\mathbb{L}}(u_i)}$ is a formal power series in u_i.

- $\displaystyle\prod_{j=1}^{n}\frac{[t](u_i +_{\mathbb{L}} \overline{x}_j)}{u_i +_{\mathbb{L}} \overline{x}_j}\prod_{j=1}^{i-1}\frac{u_i +_{\mathbb{L}} \overline{u}_j}{[t](u_i +_{\mathbb{L}} \overline{u}_j)}$ is regarded as a formal power series in u_i with constant term t^{n-i+1}.
- $\displaystyle\prod_{j=1}^{\lambda_i}\frac{u_i +_{\mathbb{L}} b_j}{u_i}$ is a formal Laurent series in u_i whose lowest degree term is $u_i^{-\lambda_i}$ with coefficient $\displaystyle\prod_{j=1}^{\lambda_i} b_j$.

Taking the above observation into account, we put

$$\begin{aligned}\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1, u_2, \ldots, u_i|\boldsymbol{b}) &:= \frac{u_i}{u_i +_{\mathbb{L}} [t](\overline{u}_i)} \cdot \frac{1}{\mathscr{P}^{\mathbb{L}}(u_i)} \\ &\times \left(\prod_{j=1}^{n}\frac{u_i +_{\mathbb{L}} [t](\overline{x}_j)}{u_i +_{\mathbb{L}} \overline{x}_j}\prod_{j=1}^{i-1}\frac{u_i +_{\mathbb{L}} \overline{u}_j}{u_i +_{\mathbb{L}} [t](\overline{u}_j)}\prod_{j=1}^{\lambda_i}\frac{u_i +_{\mathbb{L}} b_j}{u_i} - t^{n-i+1}\prod_{j=1}^{\lambda_i}\frac{b_j}{u_i}\right),\\ \widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b}) = \widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(u_1, u_2, \ldots, u_r|\boldsymbol{b}) &:= \prod_{i=1}^{r}\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{i,\lambda_i}(u_1, u_2, \ldots, u_i|\boldsymbol{b}).\end{aligned}$$

Then, we can reduce $\mathcal{HP}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b})$ to $\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b})$, and we obtain from (4.4) the following result:

Theorem 4.3 (Generating function for $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$)**.** *For a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $r \le n$, the universal factorial Hall–Littlewood P-function $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$ is the coefficient of $\boldsymbol{u}^{-\lambda} = u_1^{-\lambda_1}u_2^{-\lambda_2}\cdots u_r^{-\lambda_r}$ in $\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(u_1, u_2, \ldots, u_r|\boldsymbol{b})$. Thus*

$$HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b}) = [\boldsymbol{u}^{-\lambda}]\left(\widetilde{\mathcal{HP}}^{\mathbb{L},(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b})\right).$$

If we specialize the universal formal group law $F_{\mathbb{L}}(u, v) = u +_{\mathbb{L}} v$ to $F_a(u, v) = u + v$, the above generating function becomes a relatively simple form:

$$\begin{aligned}\widetilde{\mathcal{HP}}^{(n)}_{i,\lambda_i}(u_1, \ldots, u_i|\boldsymbol{b}) &= \frac{1}{1-t}\\ &\quad\times\left(\prod_{j=1}^{n}\frac{u_i - tx_j}{u_i - x_j}\prod_{j=1}^{i-1}\frac{u_i - u_j}{u_i - tu_j}\prod_{j=1}^{\lambda_i}\frac{u_i + b_j}{u_i} - t^{n-i+1}\prod_{j=1}^{\lambda_i}\frac{b_j}{u_i}\right),\\ \widetilde{\mathcal{HP}}^{(n)}_{\lambda}(\boldsymbol{u}_r|\boldsymbol{b}) &= \widetilde{\mathcal{HP}}^{(n)}_{\lambda}(u_1, u_2, \ldots, u_r|\boldsymbol{b}) = \prod_{i=1}^{r}\widetilde{\mathcal{HP}}^{(n)}_{i,\lambda_i}(u_1, u_2, \ldots, u_i|\boldsymbol{b}).\end{aligned}$$

Thus we have the following corollary:

Corollary 4.4 (Generating function for $HP_{\lambda}(\boldsymbol{x}_n; t|\boldsymbol{b})$)**.** *For a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $r \le n$, the factorial Hall–Littlewood*

P-polynomial $HP_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b})$ is the coefficient of $\boldsymbol{u}^{-\lambda} = u_1^{-\lambda_1}u_2^{-\lambda_2}\cdots u_r^{-\lambda_r}$ in $\widetilde{\mathcal{HP}}_\lambda^{(n)}(u_1,u_2,\dots,u_r|\boldsymbol{b})$. Thus

$$HP_\lambda(\boldsymbol{x}_n;t|\boldsymbol{b}) = [\boldsymbol{u}^{-\lambda}]\left(\widetilde{\mathcal{HP}}_\lambda^{(n)}(\boldsymbol{u}_r|\boldsymbol{b})\right).$$

4.2. **Generating function for $HQ_\lambda^{\mathbb{L}}(\boldsymbol{x}_n;t|\boldsymbol{b})$.** Next we shall derive the generating function for $HQ_\lambda^{\mathbb{L}}(\boldsymbol{x}_n;t|\boldsymbol{b})$. In the one-row case $\lambda = (\lambda_1)$ of (3.6), we push-forward the expression

$$\begin{aligned}
[[x_1;t|\boldsymbol{b}]]_{\mathbb{L}}^{\lambda_1}\prod_{j=2}^{n}(x_1 +_{\mathbb{L}} [t](\overline{x}_j)) &= \frac{(x_1 +_{\mathbb{L}} [t](\overline{x}_1))[x_1|\boldsymbol{b}]_{\mathbb{L}}^{\lambda_1-1}}{x_1 +_{\mathbb{L}} [t](\overline{x}_1)}\prod_{j=1}^{n}(x_1 +_{\mathbb{L}} [t](\overline{x}_j)) \\
&= [x_1|\boldsymbol{b}]_{\mathbb{L}}^{\lambda_1-1}\prod_{j=1}^{n}(x_1 +_{\mathbb{L}} [t](\overline{x}_j)),
\end{aligned}$$

which is a formal power series in x_1, and therefore (2.2) applies without any problem, and one can obtain the following:

$$\begin{aligned}
&HQ_\lambda^{\mathbb{L}}(\boldsymbol{x}_n;t|\boldsymbol{b}) = [u_1^{-\lambda_1}\cdots u_r^{-\lambda_r}] \\
&\quad\left[\prod_{i=1}^{r}\frac{1}{\mathscr{P}^{\mathbb{L}}(u_i)}\prod_{j=1}^{n}\frac{u_i +_{\mathbb{L}} [t](\overline{x}_j)}{u_i +_{\mathbb{L}} \overline{x}_j}\prod_{1\le i<j\le r}\frac{u_j +_{\mathbb{L}} \overline{u}_i}{u_j +_{\mathbb{L}} [t](\overline{u}_i)}\cdot\prod_{i=1}^{r}\prod_{j=1}^{\lambda_i-1}\frac{u_i +_{\mathbb{L}} b_j}{u_i}\right].
\end{aligned}$$

For each non-negative integer k, we set

$$\mathcal{HQ}_k^{\mathbb{L},(n)}(u|\boldsymbol{b}) := \frac{1}{\mathscr{P}^{\mathbb{L}}(u)}\prod_{j=1}^{n}\frac{u +_{\mathbb{L}} [t](\overline{x}_j)}{u +_{\mathbb{L}} \overline{x}_j}\times\prod_{j=1}^{k}\frac{u +_{\mathbb{L}} b_j}{u}.$$

For a sequence of positive integers $\lambda = (\lambda_1,\dots,\lambda_r)$ with $r \le n$, we set

$$\mathcal{HQ}_\lambda^{\mathbb{L},(n)}(\boldsymbol{u}_r|\boldsymbol{b}) = \mathcal{HQ}_\lambda^{\mathbb{L},(n)}(u_1,\dots,u_r|\boldsymbol{b}) := \prod_{i=1}^{r}\mathcal{HQ}_{\lambda_i-1}^{\mathbb{L},(n)}(u_i|\boldsymbol{b})\prod_{1\le i<j\le r}\frac{u_j +_{\mathbb{L}} \overline{u}_i}{u_j +_{\mathbb{L}} [t](\overline{u}_i)}.$$

Thus we have the following result:

Theorem 4.5 (Generating function for $HQ_\lambda^{\mathbb{L}}(\boldsymbol{x}_n;t|\boldsymbol{b})$)**.** *For a sequence of positive integers $\lambda = (\lambda_1,\dots,\lambda_r)$ with $r \le n$, the universal factorial Hall–Littlewood Q-function $HQ_\lambda^{\mathbb{L}}(\boldsymbol{x}_n;t|\boldsymbol{b})$ is the coefficient of $\boldsymbol{u}^{-\lambda} = u_1^{-\lambda_1}u_2^{-\lambda_2}\cdots u_r^{-\lambda_r}$ in $\mathcal{HQ}_\lambda^{\mathbb{L},(n)}(u_1,u_2,\dots,u_r|\boldsymbol{b})$. Thus*

$$HQ_\lambda^{\mathbb{L}}(\boldsymbol{x}_n;t|\boldsymbol{b}) = [\boldsymbol{u}^{-\lambda}]\left(\mathcal{HQ}_\lambda^{\mathbb{L},(n)}(\boldsymbol{u}_r|\boldsymbol{b})\right).$$

5. Application of generating functions

5.1. **e-Cancellation property.** A symmetric polynomial $f(x_1,\dots,x_n)$ with coefficients in $\mathbb{Z}$ has the *Q-cancellation property* if the following holds: when the substitution $x_1 = a$, $x_2 = -a$, a an indeterminate, is made in f, the resulting polynomial is independent of a (Pragacz [31, §2]). It is known that the Schur P- and Q-polynomials satisfy this cancellation property. The notion of the Q-cancellation property is generalized in the following way: Let $e \geq 2$ be a fixed positive integer, and $\zeta = \zeta_e$ be a primitive eth root of unity. We define a sequence $\boldsymbol{a}^e(\zeta) := (a, \zeta a, \zeta^2 a, \ldots, \zeta^{e-1}a)$. Suppose that $e \leq n$. Then a symmetric polynomial $f(x_1, \ldots, x_n)$ with coefficients in $\mathbb{Z}[\zeta]$ has the *e-cancellation property* if $f(\boldsymbol{a}^e(\zeta), x_{e+1}, \ldots, x_n) = f(a, \zeta a, \zeta^2 a, \ldots, \zeta^{e-1}a, x_{e+1}, \ldots, x_n)$ does not depend on a. In the case $e = 2$, this property is nothing but the Q-cancellation property. By specializing t to be ζ, the factorial Hall–Littlewood polynomials $HP_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ and $HQ_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ are symmetric polynomials with coefficients in $\mathbb{Z}[\zeta] \otimes \mathbb{Z}[\boldsymbol{b}] = \mathbb{Z}[\zeta] \otimes \mathbb{Z}[b_1, b_2, \ldots]$. Thus one can ask if these symmetric polynomials have the e-cancellation property or not. In this subsection, as the first application of our generating functions, we shall show the e-cancellation property of the factorial Hall–Littlewood P- and Q-polynomials.

Proposition 5.1 (e-Cancellation property)**.** *Assume that $e \leq n$. The factorial Hall–Littlewood polyonomials $HP_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ and $HQ_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ have the e-cancellation property.*

Proof. Let r be the length of λ. Then, by Corollary 4.4, $HP_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ is given as the coefficient of $\boldsymbol{u}^{-\lambda}$ in the following generating function

$$\frac{1}{(1-\zeta)^r} \prod_{i=1}^{r} \left(\prod_{j=1}^{n} \frac{u_i - \zeta x_j}{u_i - x_j} \prod_{j=1}^{i-1} \frac{u_i - u_j}{u_i - \zeta u_j} \prod_{j=1}^{\lambda_i} \frac{u_i + b_j}{u_i} - \zeta^{n-i+1} \prod_{j=1}^{\lambda_i} \frac{b_j}{u_i} \right). \tag{5.1}$$

Substituting $(x_1, \ldots, x_e)$ with $\boldsymbol{a}^e(\zeta)$ in each factor $\displaystyle\prod_{j=1}^{n} \frac{u_i - \zeta x_j}{u_i - x_j}$, we have

$$\prod_{j=1}^{e} \frac{u_i - \zeta^j a}{u_i - \zeta^{j-1} a} \times \prod_{j=e+1}^{n} \frac{u_i - \zeta x_j}{u_i - x_j} = \prod_{j=e+1}^{n} \frac{u_i - \zeta x_j}{u_i - x_j},$$

since $\zeta^e = 1$. Therefore, (5.1) does not depend on a and $x_1, \ldots, x_e$. From this, the e-cancellation property of $HP_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ follows. By Theorem 4.5, $HQ_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ is given as the coefficient of $\boldsymbol{u}^{-\lambda}$ in the following generating function

$$\prod_{i=1}^{r} \prod_{j=1}^{n} \frac{u - \zeta x_j}{u - x_j} \times \prod_{j=1}^{\lambda_i - 1} \frac{u + b_j}{u} \times \prod_{1 \leq i < j \leq r} \frac{u_j - u_i}{u_j - \zeta u_i}.$$

From this, the e-cancellation property of $HQ_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$ follows by the same reason as that of $HP_\lambda(\boldsymbol{x}_n; \zeta|\boldsymbol{b})$. □

Remark 5.2. In the case of the universal factorial Hall–Littlewood P- and Q-functions, we consider the substitution $\boldsymbol{x}_n$ with the sequence $\boldsymbol{a}^e[\zeta] :=$

$(a, [\zeta](a), [\zeta^2](a), \ldots, [\zeta^{e-1}](a))$. Then, using Theorems 4.3 and 4.5, one can verify easily that $HP^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;\zeta|\boldsymbol{b})$ and $HQ^{\mathbb{L}}_{\lambda}(\boldsymbol{x}_n;\zeta|\boldsymbol{b})$ satisfy the e-cancellation property too.

5.2. **Pfaffian formula for $GQ_{\nu}(\boldsymbol{x}_n|\boldsymbol{b})$.** Another application of our generating functions, we shall derive the *Pfaffian formulas* for the the K-theoretic factorial Q-polynomial $GQ_{\nu}(\boldsymbol{x}_n|\boldsymbol{b})$, which seems to be new. In what follows, we assume that the length $\ell(\nu)$ of a strict partition ν is $2m$ (even). We consider the specialization from $F_{\mathbb{L}}(u,v) = u +_{\mathbb{L}} v$ to $F_m(u,v) = u \oplus v$ with $t = -1$. Then $HQ^{\mathbb{L}}_{\nu}(\boldsymbol{x}_n;t|\boldsymbol{b})$ specializes to $GQ_{\nu}(\boldsymbol{x}_n|\boldsymbol{b})$, and, the generating function $\mathcal{HQ}^{\mathbb{L},(n)}_{\nu}(\boldsymbol{u}_{2m}|\boldsymbol{b})$ reduces to

$$\mathcal{GQ}^{(n)}_{\nu}(\boldsymbol{u}_{2m}|\boldsymbol{b}) = \prod_{i=1}^{2m} \mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b}) \prod_{1\le i<j\le 2m} \frac{u_j \ominus u_i}{u_j \oplus u_i}, \tag{5.2}$$

where, for each non-negative integer k, we define

$$\begin{aligned}
\mathcal{GQ}^{(n)}_{k}(u|\boldsymbol{b}) &:= \frac{1}{1+\beta u}\prod_{j=1}^{n}\frac{u\oplus x_j}{u\ominus x_j} \times \prod_{j=1}^{k}\frac{u\oplus b_j}{u} \\
&= \frac{1}{1+\beta u}\prod_{j=1}^{n}\frac{1+(u^{-1}+\beta)x_j}{1+(u^{-1}+\beta)\overline{x}_j} \times \prod_{j=1}^{k}\{1+(u^{-1}+\beta)b_j\}.
\end{aligned}$$

This is a generating function for the factorial K-theoretic Q-polynomials $GQ_{\nu}(\boldsymbol{x}_n|\boldsymbol{b})$. Here we recall from Ikeda–Naruse [14, Lemma 2.4] the following formula:

$$\mathrm{Pf}\left(\frac{x_j - x_i}{x_j \oplus x_i}\right)_{1\le i<j\le 2m} = \prod_{1\le i<j\le 2m}\frac{x_j - x_i}{x_j\oplus x_i}.$$

Thus we can compute[6]

$$\begin{aligned}
&\mathcal{GQ}^{(n)}_{\nu}(\boldsymbol{u}_{2m}|\boldsymbol{b}) = \prod_{i=1}^{2m}\mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b}) \prod_{1\le i<j\le 2m}\frac{u_j\ominus u_i}{u_j\oplus u_i} \\
&= \prod_{i=1}^{2m}\mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b})\prod_{i=1}^{2m}\frac{1}{(1+\beta u_i)^{2m-i}}\cdot \mathrm{Pf}\left(\frac{u_j-u_i}{u_j\oplus u_i}\right)_{1\le i<j\le 2m} \\
&= \mathrm{Pf}_{2m}\left(\mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{\nu_j-1}(u_j|\boldsymbol{b})\frac{1}{(1+\beta u_i)^{2m-i}}\frac{1}{(1+\beta u_j)^{2m-j}}\cdot\frac{u_j-u_i}{u_j\oplus u_i}\right) \\
&= \mathrm{Pf}_{2m}\left((1+\beta u_i)^{i+1-2m}(1+\beta u_j)^{j-2m}\mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{\nu_j-1}(u_j|\boldsymbol{b})\cdot\frac{u_j\ominus u_i}{u_j\oplus u_i}\right).
\end{aligned}$$

[6]Below we shall use the notation $\mathrm{Pf}_{2m}(a_{i,j})$ for the abbreviation of $\mathrm{Pf}(a_{i,j})_{1\le i<j\le 2m}$ when the expression is too long.

For non-negative integers $p, q \geq 0$ and positive integers $k, l \geq 1$, we define polynomials $GQ^{(p,q)}_{(k,l)}(\boldsymbol{x}_n|\boldsymbol{b})$ to be

$$GQ^{(p,q)}_{(k,l)}(\boldsymbol{x}_n|\boldsymbol{b}) := [u_1^{-k}u_2^{-l}]\left(\mathcal{GQ}^{(n)}_{p-1}(u_1|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{q-1}(u_2|\boldsymbol{b})\cdot\frac{u_2\ominus u_1}{u_2\oplus u_1}\right).$$

Note that, by Theorem 4.5, we have $GQ_{(k,l)}(\boldsymbol{x}_n|\boldsymbol{b}) = GQ^{(k,l)}_{(k,l)}(\boldsymbol{x}_n|\boldsymbol{b})$ for positive integers $k > l > 0$. Then, by Theorem 4.5, one obtains

$$\begin{aligned}
GQ_\nu(\boldsymbol{x}_n|\boldsymbol{b}) &= \left[\prod_{i=1}^{2m}u_i^{-\nu_i}\right]\Bigg(\mathrm{Pf}\Bigg((1+\beta u_i)^{i+1-2m}(1+\beta u_j)^{j-2m}\\
&\qquad\times\mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{\nu_j-1}(u_j|\boldsymbol{b})\times\frac{u_j\ominus u_i}{u_j\oplus u_i}\Bigg)_{1\leq i<j\leq 2m}\Bigg)\\
&= \mathrm{Pf}\Bigg([u_i^{-\nu_i}u_j^{-\nu_j}]\,\big((1+\beta u_i)^{i+1-2m}(1+\beta u_j)^{j-2m}\\
&\qquad\times\mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{\nu_j-1}(u_j|\boldsymbol{b})\times\frac{u_j\ominus u_i}{u_j\oplus u_i}\big)\Bigg)_{1\leq i<j\leq 2m}\\
&= \mathrm{Pf}\Bigg([u_i^{-\nu_i}u_j^{-\nu_j}]\Bigg(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{i+1-2m}{k}\binom{j-2m}{l}\beta^{k+l}u_i^k u_j^l\\
&\qquad\times\mathcal{GQ}^{(n)}_{\nu_i-1}(u_i|\boldsymbol{b})\,\mathcal{GQ}^{(n)}_{\nu_j-1}(u_j|\boldsymbol{b})\cdot\frac{u_j\ominus u_i}{u_j\oplus u_i}\Bigg)\Bigg)_{1\leq i<j\leq 2m}\\
&= \mathrm{Pf}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{i+1-2m}{k}\binom{j-2m}{l}\beta^{k+l}GQ^{(\nu_i,\nu_j)}_{(\nu_i+k,\nu_j+l)}(\boldsymbol{x}_n|\boldsymbol{b})\right)_{1\leq i<j\leq 2m}.
\end{aligned}$$

Thus we obtained the following:

Theorem 5.3 (Pfaffian formula for $GQ_\nu(\boldsymbol{x}_n|\boldsymbol{b})$)**.** *For a strict partition ν of length $2m$, we have*

$$GQ_\nu(\boldsymbol{x}_n|\boldsymbol{b}) = \mathrm{Pf}_{2m}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{i+1-2m}{k}\binom{j-2m}{l}\beta^{k+l}GQ^{(\nu_i,\nu_j)}_{(\nu_i+k,\nu_j+l)}(\boldsymbol{x}_n|\boldsymbol{b})\right).$$

Remark 5.4. (1) Putting $\boldsymbol{b} = \boldsymbol{0}$ in (5.2), we obtain a generating function for the (non-factorial) K-theoretic Q-polynomials $GQ_\nu(\boldsymbol{x}_n)$. On the other hand, dual K-theoretic P- and Q-polynomials were introduced in our previous papers [23, §5], [25]. We have a conjecture on a generating function for the dual K-theoretic Q-polynomials, and their Pfaffian formula (see Appendix §6.2).

(2) The generating function technique can also be applied to the derivation of the determinantal formula for factorial Grothendieck polynomials $G_\lambda(\boldsymbol{x}_n|\boldsymbol{b})$. On the other hand, a generating function for the dual Grothendieck polynomials $g_\lambda(\boldsymbol{x}_n)$ (for their definition, see

Lascoux–Naruse [18]) can be obtained by a purely algebraic manner. We shall give the details in the Appendix, §6.1.

6. Appendix

6.1. **Generating function for the dual Grothendieck polynomials.** As we mentioned in Remark 5.4, we give a generating function for the dual Grothendieck polynomials. Following Lascoux–Naruse [18], let us introduce the dual Grothendieck polynomials $g_\lambda(\boldsymbol{y}_n)$, where $\boldsymbol{y}_n = (y_1, y_2, \ldots, y_n)$ is a set of independent variables and $\lambda \in \mathcal{P}_n$. First we need some notation: Given two sets of variables (called *alphabets* as usual) $\mathbf{A}$, $\mathbf{B}$, the *complete functions* $s_k(\mathbf{A}-\mathbf{B})$ $(k = 0, 1, 2, \ldots)$ are given by the following generating function:

$$\sum_{k=0}^{\infty} s_k(\mathbf{A}-\mathbf{B})z^k = \prod_{a\in\mathbf{A}} \frac{1}{1-az} \prod_{b\in\mathbf{B}}(1-bz).$$

In particular, when we add r letters specialized to 1, namely the set $\{\underbrace{1, 1, \ldots, 1}_{r}\}$, to one of the alphabets $\mathbf{A}$ or $\mathbf{B}$, we have

$$\sum_{k=0}^{\infty} s_k(\mathbf{A}-\mathbf{B}\pm r)z^k = (1-z)^{\mp r} \prod_{a\in\mathbf{A}} \frac{1}{1-az} \prod_{b\in\mathbf{B}}(1-bz).$$

Then, for the variables $\boldsymbol{y}_n = (y_1, \ldots, y_n)$ and any integer r, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} s_k(\boldsymbol{y}_n + r)z^k &= (1-z)^{-r}\prod_{i=1}^{n}\frac{1}{1-y_i z} \\
&= \left(\sum_{i=0}^{\infty}\binom{-r}{i}(-z)^i\right)\left(\sum_{j=0}^{\infty}h_j(\boldsymbol{y}_n)z^j\right) \\
&= \left(\sum_{i=0}^{\infty}(-1)^i\binom{r+i-1}{i}(-z)^i\right)\left(\sum_{j=0}^{\infty}h_j(\boldsymbol{y}_n)z^j\right) \\
&= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{r+i-1}{i}h_{k-i}(\boldsymbol{y}_n)\right)z^k,
\end{aligned}$$

and hence we have

$$s_k(\boldsymbol{y}_n + r) = \sum_{i=0}^{k}\binom{r+i-1}{i}h_{k-i}(\boldsymbol{y}_n) \quad (k = 0, 1, \ldots). \tag{6.1}$$

Using (6.1), the dual Grothendieck polynomial $g_\lambda(\boldsymbol{y}_n)$ for $\lambda \in \mathcal{P}_n$ of length r is given by (see Lascoux–Naruse [18, (3)]):

$$
\begin{aligned}
g_\lambda(\boldsymbol{y}_n) &= s_\lambda(\boldsymbol{y}_n, \boldsymbol{y}_n+1, \ldots, \boldsymbol{y}_n+n-1) \\
&= \det(s_{\lambda_i-i+j}(\boldsymbol{y}_n+i-1))_{1\le i,j\le r} \\
&= \det\left(\sum_{k=0}^{\lambda_i-i+j}\binom{(i-1)+k-1}{k} h_{\lambda_i-i+j-k}(\boldsymbol{y}_n)\right)_{1\le i,j\le r} \qquad (6.2)\\
&= \det\left(\sum_{k=0}^{\infty}\binom{i+k-2}{k} h_{\lambda_i-i+j-k}(\boldsymbol{y}_n)\right)_{1\le i,j\le r}.
\end{aligned}
$$

We set

$$
\begin{aligned}
H(z) &= H^{(n)}(z) := \prod_{j=1}^{n}\frac{1}{1-y_jz} = \sum_{k=0}^{\infty} h_k(\boldsymbol{y}_n)z^k, \\
g(\boldsymbol{z}_r) &= g(z_1,\ldots,z_r) := \prod_{i=1}^{r} H(z_i)\prod_{1\le i<j\le r}\frac{z_i \ominus z_j}{z_i}.
\end{aligned}
$$

We shall show that $g(\boldsymbol{z}_r)$ is the generating function for the dual Grothendieck polynomials, namely we have the following:

Theorem 6.1 (Generating function for $g_\lambda(\boldsymbol{y}_n)$)**.** *For a partition $\lambda = (\lambda_1,\ldots,\lambda_r)$ of length $\ell(\lambda) = r \le n$, the dual Grothendieck polynomial $g_\lambda(\boldsymbol{y}_n)$ is the coefficient of $\boldsymbol{z}^\lambda = z_1^{\lambda_1}z_2^{\lambda_2}\cdots z_r^{\lambda_r}$ in $g(z_1,z_2,\ldots,z_r)$. Thus*

$$
g_\lambda(\boldsymbol{y}_n) = [\boldsymbol{z}^\lambda](g(\boldsymbol{z}_r)).
$$

Proof. By the Vandermonde determinant formula, we have

$$
\begin{aligned}
\prod_{1\le i<j\le r}\frac{z_i\ominus z_j}{z_i} &= \prod_{i=1}^{r-1}\prod_{j=i+1}^{r}\frac{1}{1+\beta z_j}\cdot\frac{z_i-z_j}{z_i} \\
&= \prod_{i=1}^{r-1}\frac{1}{(1+\beta z_{i+1})\cdots(1+\beta z_r)}\cdot\prod_{i=1}^{r-1}\frac{1}{z_i^{r-i}}\cdot\prod_{1\le i<j\le r}(z_i-z_j) \\
&= \prod_{i=1}^{r}\frac{1}{(1+\beta z_i)^{i-1}}\cdot\prod_{i=1}^{r}\frac{1}{z_i^{r-i}}\cdot\det(z_i^{r-j})_{1\le i,j\le r} \\
&= \det((1+\beta z_i)^{1-i}z_i^{i-j})_{1\le i,j\le r}.
\end{aligned}
$$

Therefore one can compute

$$
g(z_1,\ldots,z_r) = \prod_{i=1}^{r}H(z_i)\prod_{1\le i<j\le r}\frac{z_i\ominus z_j}{z_i} = \det((1+\beta z_i)^{1-i}z_i^{i-j}H(z_i))_{1\le i,j\le r}.
$$

Extracting the coefficient of the monomial $\boldsymbol{z}^\lambda = \prod_{i=1}^r z_i^{\lambda_i}$, we obtain

$$
\begin{aligned}
[\boldsymbol{z}^\lambda](g(z_1,\dots,z_r)) &= \left[\prod_{i=1}^r z_i^{\lambda_i}\right]\left(\det((1+\beta z_i)^{1-i} z_i^{i-j} H(z_i))_{1\le i,j\le r}\right) \\
&= \det\left([z_i^{\lambda_i}]((1+\beta z_i)^{1-i} z_i^{i-j} H(z_i)\right)_{1\le i,j\le r} \\
&= \det\left([z_i^{\lambda_i}]\left(\sum_{k=0}^\infty \binom{1-i}{k}\beta^k z_i^k \cdot z_i^{i-j} H(z_i)\right)\right)_{1\le i,j\le r} \\
&= \det\left(\sum_{k=0}^\infty \binom{1-i}{k}\beta^k h_{\lambda_i-i+j-k}(\boldsymbol{y}_n)\right)_{1\le i,j\le r} \\
&= \det\left(\sum_{k=0}^\infty \binom{i+k-2}{k}(-\beta)^k h_{\lambda_i-i+j-k}(\boldsymbol{y}_n)\right)_{1\le i,j\le r}.
\end{aligned}
$$

Here we used the following identity:

$$
\binom{1-i}{k} = \binom{-(i-1)}{k} = (-1)^k\binom{i+k-2}{k},
$$

for integers $i \ge 1$, $k \ge 0$. This is the dual Grothendieck polynomial $g_\lambda(\boldsymbol{y}_n)$ introduced in (6.2) with $\beta = -1$. $\square$

6.2. **Conjecture on a generating function for $gq_\nu(\boldsymbol{y}_n)$.** In [14, §3.4], Ikeda–Naruse introduced the K-theoretic P- and Q-functions $GP_\nu(\boldsymbol{x})$ and $GQ_\nu(\boldsymbol{x})$ in countably many variables $\boldsymbol{x} = (x_1, x_2, \dots)$. Let $G\Gamma'(\boldsymbol{x})$ denote the ring of symmetric functions satisfying the K-theoretic Q-cancellation property (see [14, Definition 1.1]). Similarly, let $G\Gamma(\boldsymbol{x})$ denote the subring of $G\Gamma'(\boldsymbol{x})$ consisting of functions f satisfying the condition:$f(t, x_2, \dots) - f(0, x_2, \dots)$ is divisible by $t \oplus t$.[7] Then, they showed that $GP_\nu(\boldsymbol{x})$'s and $GQ_\nu(\boldsymbol{x})$'s (ν strict) form a formal $\mathbb{Z}[\beta]$-basis of $G\Gamma'(\boldsymbol{x})$ and $G\Gamma(\boldsymbol{x})$ respectively. Using this "basis theorem" and the following "Cauchy kernel"

$$
\Delta(\boldsymbol{x};\boldsymbol{y}) = \prod_{i=1}^\infty \prod_{j=1}^\infty \frac{1-\overline{x}_i y_j}{1-x_i y_j},
$$

where $\boldsymbol{y} = (y_1, y_2, \dots)$ is another set of independent variables, we can define the *dual K-theoretic P- and Q-functions*, denoted by $gp_\nu(\boldsymbol{y})$ and $gq_\nu(\boldsymbol{y})$, as follows (see also Nakagawa–Naruse [23, Definition 5.3, Remark 5.4]):

Definition 6.2 (Dual K-theoretic Schur P- and Q-functions)**.** Let $\mathcal{SP}$ denote the set of all strict partitions. We define $gp_\nu(\boldsymbol{y})$ and $gq_\nu(\boldsymbol{y})$ for a strict

[7] We slightly changed the notation from that used in [14]. In that paper, $G\Gamma'(\boldsymbol{x})$ and $G\Gamma(\boldsymbol{x})$ are written as $G\Gamma$ and $G\Gamma_+$ respectively.

partition $\nu \in \mathcal{SP}$ by the following identities:

(6.3)
$$\Delta(\boldsymbol{x};\boldsymbol{y}) = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\frac{1-\overline{x}_i y_j}{1-x_i y_j} = \sum_{\nu\in\mathcal{SP}} GP_\nu(\boldsymbol{x}) gq_\nu(\boldsymbol{y}) = \sum_{\nu\in\mathcal{SP}} GQ_\nu(\boldsymbol{x}) gp_\nu(\boldsymbol{y}).$$

One can check that $gp_\nu(\boldsymbol{y})$ and $gq_\nu(\boldsymbol{y})$ are actually symmetric functions, i.e., they are elements of $\Lambda(\boldsymbol{y})\otimes\mathbb{Z}[\beta]$, where $\Lambda(\boldsymbol{y})$ is the ring of symmetric functions in the variables $\boldsymbol{y} = (y_1, y_2, \ldots)$ over $\mathbb{Z}$. For each positive integer n, one can define a surjective ring homomorphism $\rho^{(n)} : \Lambda(\boldsymbol{y}) \twoheadrightarrow \Lambda(\boldsymbol{y}_n)$ by putting $y_{n+1} = y_{n+2} = \cdots = 0$. Here $\Lambda(\boldsymbol{y}_n) = \mathbb{Z}[y_1, \ldots, y_n]^{S_n}$ is the ring of symmetric polynomials in $\boldsymbol{y}_n = (y_1, \ldots, y_n)$ under the usual action by the symmetric group S_n. We also denote by $\rho^{(n)}$ its extension over $\mathbb{Z}[\beta]$. Then we define the *dual K-theoretic Schur P- and Q-polynomials*, denoted by $gp_\nu(\boldsymbol{y}_n)$ and $gq_\nu(\boldsymbol{y}_n)$ for a strict partition ν of length $\leq n$, by $gp_\nu(\boldsymbol{y}_n) = \rho^{(n)}(gp_\nu(\boldsymbol{y}))$ and $gq_\nu(\boldsymbol{y}_n) = \rho^{(n)}(gq_\nu(\boldsymbol{y}))$ respectively.

Next we set
$$gq(z) = \prod_{j=1}^{n}\frac{1-y_j\overline{z}}{1-y_j z} = \sum_{k=0}^{\infty} gq_k(\boldsymbol{y}_n) z^k,$$
$$gq(\boldsymbol{z}_r) = gq(z_1, \ldots, z_r) := \prod_{i=1}^{r} gq(z_i) \prod_{1\leq i<j\leq r}\frac{z_i \ominus z_j}{z_i \oplus z_j}.$$

Then we make the following conjectures:

Conjecture 6.3 (Generating function for $gq_\nu(\boldsymbol{y}_n)$)**.** For a strict partition $\nu = (\nu_1, \ldots, \nu_r)$ of length $\ell(\nu) = r \leq n$, the dual K-theoretic Q-polynomial $gq_\nu(\boldsymbol{y}_n)$ is the coefficient of $\boldsymbol{z}_r = z_1^{\nu_1} z_2^{\nu_2}\cdots z_r^{\nu_r}$ in $gq(z_1, \ldots, z_r)$. Thus
$$gq_\nu(\boldsymbol{y}_n) = [\boldsymbol{z}^\nu](gq(\boldsymbol{z}_r)).$$

We have checked that the above conjecture holds for $r \leq 2$. As a corollary to the above conjecture, we immediately obtain the following formula:

Corollary 6.4 (Pfaffian formula for $gq_\nu(\boldsymbol{y}_n)$)**.** *For a strict partition ν of length $2m$, we have*
$$gq_\nu(\boldsymbol{y}_n) = \mathrm{Pf}\left(\sum_{k=0}^{i-1}\sum_{l=0}^{j}\beta^{k+l}\binom{i-1}{k}\binom{j}{l} gq_{(\nu_i-k,\nu_j-l)}(\boldsymbol{y}_n)\right)_{1\leq i<j\leq 2m}.$$

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