AKADÉMIAI KIADÓ

Bounds for the electrical resistance for non-homogeneous conducting body

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ABSTRACT

A mathematical model is developed to determine the steady-state electric current flow through in non-homogeneous isotropic conductor whose shape has a three-dimensional hollow body. The equations of the Maxwell's theory of electric current flow in a non-homogeneous isotropic solid conductor body are used to formulate the corresponding electric boundary value problem. The determination of the steady motion of charges is based on the concept of the electrical conductance. The derivation of the upper and lower bound formulae for the electrical conductance is based on Cauchy-Schwarz inequality. Two numerical examples illustrate the applications of the derived upper and lower bound formulae.

KEYWORDS

electrical resistance, hollow body, lower and upper bounds, steady-state

1. INTRODUCTION

Electrical resistance of an electrical conductor is a measure of the difficulty to pass a steady electric current through the conductor. The well-known elementary form of Ohm's law states that when the conductor carries a current I from a point P_1 at potential U_1 to a point P_2 at potential U_2 then $U_1–U_2 = RI$, where R is the resistance of the conductor between points P_1 and P_2, it depends only on the shape and temperature and the material of the conductor. The inverse of electric resistance is the electric conductance $G = 1/R$. This paper deals with the electric resistance of a three-dimensional non-homogeneous conductor body. Examination of non-homogeneous structural elements is a very important task. Maróti's study [1] deals with the bending vibration of axially non-homogeneous beams. The buckling problem of axially functionally graded beams is considered in paper [2]. For prescribed frequency and buckling loads Maróti and Elishakoff [2] determined the Young's modulus in axial direction as a function of axial coordinate. The non-homogeneous isotropic hollow conductor is bounded by two closed surfaces ∂V_1 and ∂V_2, which have no common point. The current flows inside the conductor from inner boundary surface ∂V_1 whose potential is U_1 to the outer boundary surface ∂V_2 whose potential is U_2, $U_1>U_2$. Two-side estimation will be proven for the electrical conductance of non-homogeneous isotropic hollow three-dimensional conductor. The mathematical formalism follows the methods, which were used in papers [3–5]. In paper [3] upper and lower bounds are proven for the electrical resistance of homogeneous isotropic ring like axisymmetric conductor. In paper [4] the capacitance of two-dimensional cylindrical capacitor, which consists of non-homogeneous dielectric materials is studied. Examples illustrate the applications of the derived bounding formulae of capacitance [4]. A mathematical heat transfer model is developed for the steady-state heat transfer problem for homogeneous isotropic body of rotation in [5] and it is used to obtain estimations of thermal heat transfer conductance.

Let us consider the steady motion of charges in the non-homogeneous hollow conductor shown in Fig. 1. The conductor body occupies the space domain V and its boundary surfaces

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are ∂V_1 and ∂V_2. The electric potential U on the boundary surfaces ∂V_1 and ∂V_2 are prescribed, so the following boundary conditions are valid [6–8],

$$U(\boldsymbol{r}) = U_i = \text{constant}, \quad \boldsymbol{r} \in \partial V_i \ (i = 1, 2), \tag{1}$$

where $\boldsymbol{r}$ denotes the position vector (Fig. 1). According to Maxwell's theory [6–8] the steady motion of charges is described by the next equations:

$$\boldsymbol{j} = \sigma \boldsymbol{E}, \ \nabla \cdot \boldsymbol{j} = 0, \quad \boldsymbol{E} = -\nabla U. \tag{2}$$

Differential form of Ohm's law formulates that at constant temperature in isotropic conductor the current density vector $\boldsymbol{j}$ is proportional to the electric field vector $\boldsymbol{E}$. Here $\sigma = \sigma(\boldsymbol{r})$ is the conductivity of the non-homogenous hollow conductor. In Eq. (2) ∇ is the del operator and the dot between two vectors denotes the scalar product [9]. From the above equations it follows that

$$\sigma(\boldsymbol{r})\Delta U + \nabla\sigma \cdot \nabla U = 0, \quad \boldsymbol{r} \in V, \ \Delta = \nabla \cdot \nabla. \tag{3}$$

Introducing a new function $u = u(\boldsymbol{r})$ by the next definition,

$$U(\boldsymbol{r}) = (U_1 - U_2)\, u(\boldsymbol{r}) + U_2 \quad U_1 \neq U_2. \tag{4}$$

It is evident that $u = u(\boldsymbol{r})$ satisfies the following boundary value problem,

$$\begin{aligned} &\sigma(\boldsymbol{r})\Delta u + \nabla\sigma \cdot \nabla u = 0, \quad \boldsymbol{r} \in V, \\ &u = 1, \quad \boldsymbol{r} \in \partial V_1, \quad u = 0, \quad \boldsymbol{r} \in \partial V_2 \, . \end{aligned} \tag{5}$$

The function $u = u(x, y)$ plays crucial role in the expressions of electrical resistance and electrical conductance. An electric current in the conductor is the continuous passage of the current along that conductor. The constant potential difference between the closed surfaces ∂V_1 and ∂V_2 maintains the steady flow of the electric current. The amount of charge flowing through surface ∂V_1 per unit time is I. The determination of I is based on the next equation

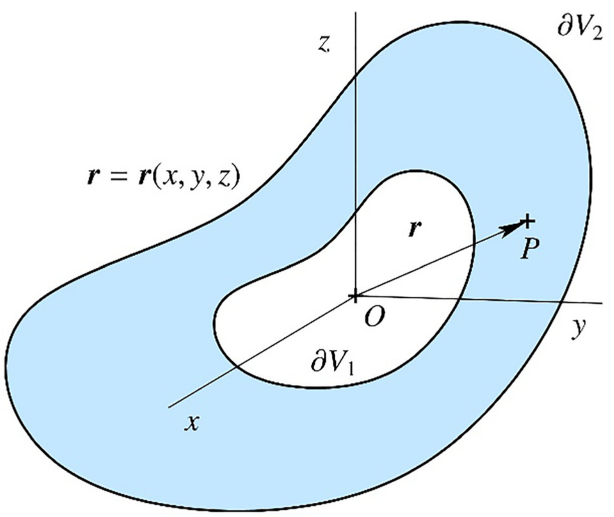

Fig. 1. Hollow non-homogeneous conductor body bounded by closed surfaces

$$\begin{aligned} I = -\int_{\partial V_1} \boldsymbol{j} \cdot \boldsymbol{n} \, dA = (U_1 - U_2) \int_{\partial V_1} \sigma(\boldsymbol{r}) \, \boldsymbol{n} \cdot \nabla u \, dA = \\ (U_1 - U_2) \int_{\partial V_1} \sigma(\boldsymbol{r}) \frac{\partial u}{\partial n} dA \, . \end{aligned} \tag{6}$$

In Eq. (6), $\boldsymbol{n}$ is the outer normal unit vector of the inner boundary surface ∂V_1 and dA is the area element of ∂V_1. The electrical resistance R and the conductance G of the hollow conductor is defined as [6, 8],

$$\begin{aligned} R &= \frac{U_1 - U_2}{I} = \frac{1}{\int_{\partial V_1} \sigma(\boldsymbol{r}) \frac{\partial u}{\partial n} dA}, \\ G &= \frac{I}{U_1 - U_2} = \int_{\partial V_1} \sigma(\boldsymbol{r}) \frac{\partial u}{\partial n} dA \, . \end{aligned} \tag{7}$$

From Eq. (5) it follows that

$$\begin{aligned} \int_V u[\sigma(\boldsymbol{r})\Delta u + \nabla\sigma \cdot \nabla u] dV = &\int_{\partial V_1} u\sigma(\boldsymbol{r}) \, \boldsymbol{n} \cdot \nabla u dA \\ &- \int_V \sigma(\boldsymbol{r}) |\nabla u|^2 dV \\ &= 0, \end{aligned} \tag{8}$$

$$G = \int_V \sigma(\boldsymbol{r}) |\nabla u|^2 dV, \quad R = \frac{1}{\int_V \sigma(\boldsymbol{r}) |\nabla u|^2 dV}. \tag{9}$$

Note that if

$$\nabla\sigma \cdot \nabla u = 0, \quad \boldsymbol{r} \in V, \tag{10}$$

then $u(\boldsymbol{r}) = u_0(\boldsymbol{r})$, where $u_0(\boldsymbol{r})$ is a unique solution of the following Dirichlet type boundary-value problem

$$\Delta u_0 = 0, \ \boldsymbol{r} \in V, \ u_0 = 1, \ \boldsymbol{r} \in \partial V_1, \ u_0 = 0, \ \boldsymbol{r} \in \partial V_2. \tag{11}$$

In this case

$$G = \int_V \sigma(\boldsymbol{r}) |\nabla u_0|^2 dV. \tag{12}$$

There are several approximation methods to get the solution of the boundary-value problem Eq. (5), most of which use the results of variational calculus for example as Ritz method, finite element method [8, 9]. Other methods are also known and they used, for example finite difference methods, method of weighted residuals, boundary element method [10]. It must be mentioned that, many numerical-analytical method are used R-functions to solve the boundary value problems of electrodynamics [11–13]. The efficiency of the R-Function Method (RFM) to solving the boundary value problems of electrostatics in very complicated domain is illustrated in paper by Kravchenko and Basarab [14]. They considered a boundary-value problem of electrodynamics in the fractal regions of the Sierpiski carpet and the Koch island types [14]. Iványi solved a number of two-dimensional boundary value problems of static and stationary electromagnetisms by variational method connecting of them with the use of R-functions

[12, 13, 15, 16]. It is not the aim of this paper is to give a detailed list of different analytical and numerical methods, which are used widespread in electrical engineering calculations.

2. UPPER BOUND FOR *G* AND LOWER BOUND FOR *R*

Theorem 1. If the function $F = F(\boldsymbol{r})$ which is continuously differentiable in $V \cup \partial V$ satisfies the boundary conditions (13) then the inequality relation (14) is valid

$$F(\boldsymbol{r}) = 1, \quad \boldsymbol{r} \in \partial V_1, \quad F(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in \partial V_2, \tag{13}$$

$$G \le \int_V \sigma(\boldsymbol{r}) |\nabla F|^2 dV. \tag{14}$$

Proof. The proof of inequality $(14)_3$ can be derived by the Cauchy-Schwarz inequality relation (15),

$$\left(\int_V \sigma(\boldsymbol{r}) \nabla F \cdot \nabla u \; dV \right)^2 \le \int_V \sigma(\boldsymbol{r}) |\nabla F|^2 dV \cdot \int_V \sigma(\boldsymbol{r}) |\nabla u|^2 dV. \tag{15}$$

A simple computation leads to the result

$$\int_V \sigma(\boldsymbol{r}) \nabla F \cdot \nabla u \; dV = \int_{\partial V_1} \sigma(\boldsymbol{r}) \boldsymbol{n} \cdot \nabla u \; dA = \int_V \sigma(\boldsymbol{r}) |\nabla u|^2 dV. \tag{16}$$

The combination of the inequality relation (15) with Eq. (16) and using formula (9) gives (14). A brief discussion shows that the sign of equality in relation (14) is valid only if $F(\boldsymbol{r}) \equiv u(\boldsymbol{r})$.

3. LOWER BOUND FOR *G*, UPPER BOUND FOR *R*

Theorem 2. Let $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{r})$ be a vector field defined in the hollow space domain $V \cup \partial V$, which satisfies the following equations

$$\nabla \cdot \boldsymbol{q} = 0, \; \boldsymbol{r} \in V, \; \boldsymbol{n} \cdot \boldsymbol{q} = 0, \; \boldsymbol{r} \in \partial V_1, \tag{17}$$

in this case

$$G \ge \frac{\left(\int_{\partial V_1} \sigma(\boldsymbol{r}) \; \boldsymbol{n} \cdot \boldsymbol{q} dA \right)^2}{\int_V \sigma(\boldsymbol{r}) \boldsymbol{q}^2 dV}, \quad \int_V \boldsymbol{q}^2 dV \ne 0. \tag{18}$$

In lower bound formula (18) equality is reached only if $\boldsymbol{q} \equiv \lambda \nabla u$, where λ is an arbitrary constant which is different from zero.

Proof. The proof of lower bound formula (18) is based on the Cauchy-Schwarz inequality relation (19)

$$\left(\int_V \sigma(\boldsymbol{r}) \; \boldsymbol{p} \cdot \boldsymbol{q} dV \right)^2 \le \int_V \sigma(\boldsymbol{r}) \boldsymbol{p}^2 dV \int_V \sigma(\boldsymbol{r}) \boldsymbol{q}^2 dV. \tag{19}$$

Let

$$\boldsymbol{p} = \nabla u \tag{20}$$

be in inequality relation (19). A simple calculation yields the result

$$\int_V \sigma(\boldsymbol{r}) \nabla u \cdot \boldsymbol{q} dV = \int_{\partial V} u \sigma(\boldsymbol{r}) \boldsymbol{n} \cdot \boldsymbol{q} dA - \int_V u \nabla \cdot [\sigma(\boldsymbol{r}) \boldsymbol{q}] dV = \int_{\partial V_1} \sigma(\boldsymbol{r}) \boldsymbol{n} \cdot \boldsymbol{q} dA. \tag{21}$$

Substitution Eq. (21) into Cauchy-Schwarz inequality (19) gives

$$\left(\int_{\partial V_1} \sigma(\boldsymbol{r}) \; \boldsymbol{n} \cdot \boldsymbol{q} dA \right)^2 \le \int_V \sigma(\boldsymbol{r}) |\nabla u|^2 dV \int_V \sigma(\boldsymbol{r}) \boldsymbol{q}^2 dV. \tag{22}$$

From inequality relation (22) the proof of lower bound formula, (18) can be obtained immediately.

Theorem 3. Let $f = f(\boldsymbol{r})$ be non-identically constant function in $V \cup \partial V$, which satisfies the Laplace equation in V

$$\nabla \cdot \nabla f = \Delta f = 0, \; \boldsymbol{r} \in V. \tag{23}$$

The following lower bound formula is valid for G

$$G \ge \frac{\left(\int_{\partial V_1} \frac{\partial f}{\partial n} dA \right)^2}{\int_V \frac{|\nabla f|^2}{\sigma(r)} dV}. \tag{24}$$

Proof. The proof of (24) can be obtained from (18) with under-mentioned $\boldsymbol{q}(\boldsymbol{r})$

$$\boldsymbol{q}(\boldsymbol{r}) = \frac{\nabla f}{\sigma(\boldsymbol{r})}, \; \boldsymbol{r} \in V \cup \partial V. \tag{25}$$

4. NUMERICAL EXAMPLES

In the numerical examples the spherical coordinate system is used. The connection between the Cartesian coordinates x, y, z and the spherical coordinates r, ϕ, ϑ is $x = r\cos\phi\sin\vartheta$, $y = r\sin\phi\sin\vartheta$, $z = r\cos\vartheta$. Developed numerical example relate to axisymmetric electrical problems.

Example 1. The meridian section of hollow spherical domain is shown in Fig. 2. The specific conductivity is a given function of the radial coordinate

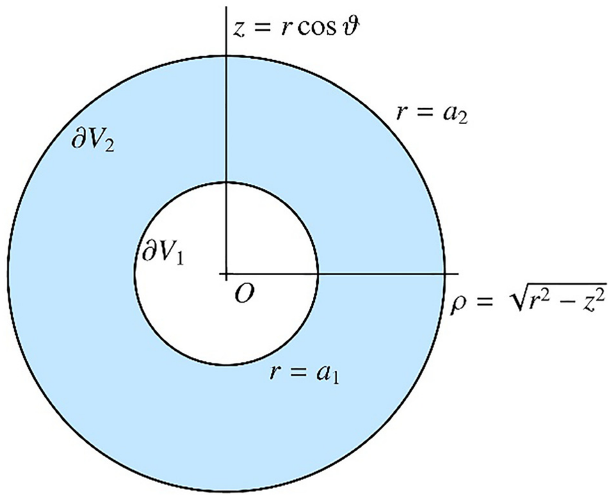

Fig. 2. Meridian section of hollow spherical domain

$$\sigma(r,\alpha) = \sigma_0 \exp\left(\alpha \frac{r}{a_1}\right).$$

Application of Theorem 1 to the function $F(\boldsymbol{r}) = F(r) = 1 - \frac{1/r - 1/a_1}{1/a_2 - 1/a_1}$ gives

$$G(\alpha) \leq G_U(\alpha) =$$
$$-\frac{1}{(a_1 - a_2)^2}\left\{4\pi\sigma_0 a_1 a_2 \left[\begin{array}{l} a_1 \exp\left(\frac{\alpha a_2}{a_1}\right) + E_i\left(1, -\frac{\alpha a_2}{a_1}\right)\alpha a_2 \\ -a_2 \exp(\alpha) - a_2 E_i(1,-\alpha)\alpha \end{array}\right]\right\}, \tag{26}$$

here $E_i(1,x)$ is the exponential integral [17, 18]. Putting the following function $f(\boldsymbol{r}) = f(r) = r^{-1}$ in the lower bound formula (24) gives

$$G(\alpha) \geq G_L(\alpha) =$$
$$\frac{4\pi\sigma_1 a_1 a_2}{-a_1 \exp\left(-\frac{\alpha a_2}{a_1}\right) + E_i\left(1, \frac{\alpha a_2}{a_1}\right)\alpha a_2 + a_2 \exp(-\alpha) - a_2 E_i(1,\alpha)\alpha}. \tag{27}$$

Lengthy, but elementary calculations shows that, the exact value of electrical conductivity which is obtained from the solution of boundary value problem (5) is

$$G(\alpha) =$$
$$\frac{4\pi\sigma_1 a_1 a_2}{-a_1 \exp\left(-\frac{\alpha a_2}{a_1}\right) + a_2 \exp(-\alpha) + \alpha a_2 E_i\left(1, \frac{\alpha a_2}{a_1}\right) - \alpha a_2 E_i(1,\alpha)} \tag{28}$$

that in this case is $G(\alpha) = G_L(\alpha)$. The validity of upper bound (14) for $\alpha = -2.5$ and $\alpha = 2.5$ examplifies as follows

$$G_L(-2.5) = G(-2.5) = 2.651169 \times 10^6 \frac{1}{\Omega} \leq G_U(-2.5)$$
$$= 3.301497 \times 10^6 1/\Omega, \tag{29}$$

$$G_L(2.5) = G(2.5) = 1.59105 \times 10^6 \frac{1}{\Omega} \leq G_U(2.5)$$
$$= 1.981333 \times 10^6 1/\Omega. \tag{30}$$

Figure 3 shows the upper and the lower bounds of the conductance G as a function of α for $-2 \leq \alpha \leq 2$. In this example $a_1 = 0.3$ m, $a_2 = 0.5$ m, $\sigma_0 = 7.69 \times 10^6 1/\Omega$m.

Figure 4 shows the plot of function $F = F(r)$, the plot of exact analytical solution $u = u(r)$ and the plot of $u_F = u_F(r)$ which is obtained from Finite Element (FE) approximation in the case of Example 1 for $\alpha = 2.5$. The FE model is developed in ABAQUS with DC3D8 elements (node numbers are 706 356) and for definition of the nonlinear material a special user subroutine usdfld() is applied.

Example 2. The meridian section of axisymmetric hollow domain bounded by two spherical surfaces as it is shown in Fig. 5. The following data are used $a_1 = 0.3$ m, $a_2 = 0.5$ m, $b = 0.025$, $\sigma_1 = 7.69 \times 10^6$ $1/m\Omega$, $\sigma(r,n) = \sigma_1 \left(\frac{r}{a_1}\right)^n$.

Let $F(r,\vartheta) = \ln\frac{R(\vartheta)}{r}\left(\ln\frac{R(\vartheta)}{a_1}\right)^{-1}$ be in Theorem 1 and in Theorem 3 $f(r) = r^{-1}$.

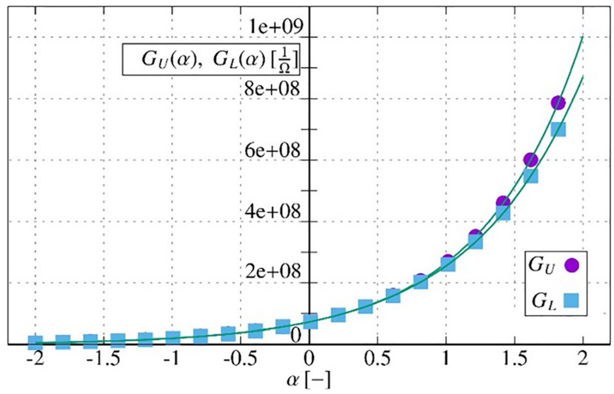

Fig. 3. Upper and lower bounds for the conductance of non-homogeneous spherical conductor as a function of α for $-1.5 \leq \alpha \leq 1.5$

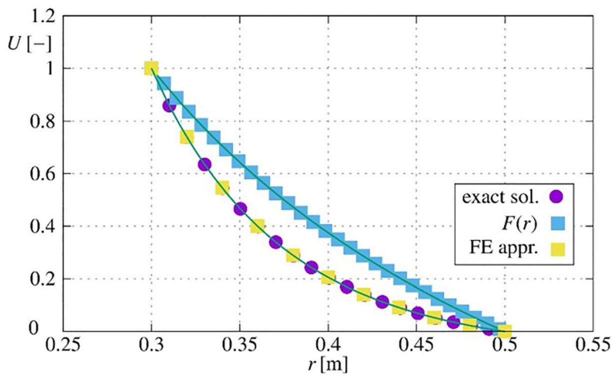

Fig. 4. Comparison of exact solution, approximate solution and FE approximation

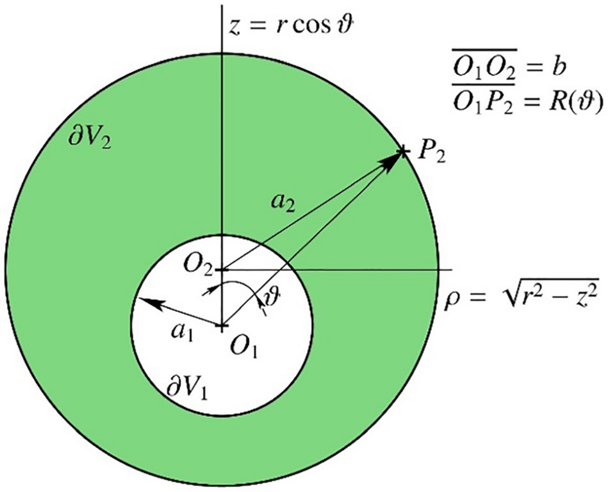

Fig. 5. The meridian section of axisymmetric hollow domain bounded by two spherical surfaces with different centre points

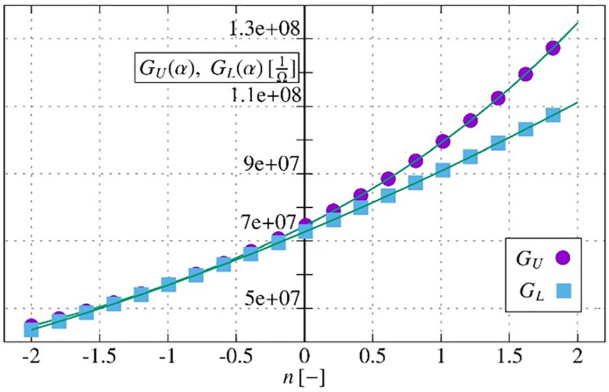

Fig. 6. Upper and lower bounds for the conductance as a function of power index n

Figure 6 shows the upper and the lower bounds as a function of power index for $-2 \leq n \leq 2$.

5. CONCLUSIONS

A mathematical model is developed to determine the steady-state electric current flow through in non-homogeneous isotropic conductor whose shape is a three-dimensional hollow body. The hollow body considered is bounded by two closed surfaces which have no common points. The derivation of the upper and lower bound formulae for the electrical conductance is based on the two types of Cauchy-Schwarz inequality. Two numerical examples illustrate the applications of the derived upper and lower bounds for the conductance. The derived upper and lower bound formulae of electric conductance can be used to check the results of numerical computations obtained by finite element method, boundary element method and by any other numerical methods.

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