Wait, Why Gauge?

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Abstract

Philosophers of physics have spent much effort unpacking the structure of gauge theories. But surprisingly, little attention has been devoted to the question of why we should require our best theories to be locally gauge invariant in the first place. Drawing on Steven Weinberg's works in the mid-1960s, I argue that the principle of local gauge invariance follows from Lorentz invariance and other natural assumptions in the context of perturbative relativistic quantum field theory. On this view, gauge freedom is a mere accidental feature of an already highly constrained set of quantities; the distinctive structure of our best gauge theories, in turn, traces back to the peculiar space-time transformation properties of particles like photons, gluons, and gravitons. I conclude by drawing a few interpretative lessons for the philosophy of gauge theories.

1 Introduction

Gauge theories are well known for being particularly puzzling. On the one hand, they seem to owe their remarkable success to a peculiar and highly constraining symmetry principle, the principle of local gauge invariance, which requires their dynamics to remain invariant when we shift in some prescribed way the value of their physical variables by a different amount at different space-time points. On the other hand, this principle seems to make their physical content—and thus what ultimately underwrites their success—highly underdetermined insofar as the world appears to be equally well described by means of their 'original' physical variables or locally transformed ones.

As expected, the puzzling structure of these theories has not failed to attract philosophers. In particular, much effort has been devoted to clarifying the dubious gauge argument, which is supposed to justify the introduction of gauge fields from the requirement that the dynamics of matter fields be invariant under local rotations in classical field theory (or that the time evolution of a wave function be covariant under local phase transformations in non-relativistic quantum mechanics).[1] Likewise, much has been

[1] See (Brown [1999]; Teller [2000]; Lyre [2001]; Martin [2002], [2003]; Healey [2007], sec. 6.3; Rickles [2008], sec. 3.2.1; Wallace [2009], sec. 2, 7; Hetzroni [2021]; Gomes *et al.* [2022]). A crucial warning here: for conceptual clarity and contrary to widespread usage, it is best to distinguish the gauge argument from the gauge principle, i.e., the principle (or constraint) of local gauge invariance, which also applies to pure gauge field theories without matter fields or wave function.

said about the set of interpretative issues regarding determinism, objectivity, realism, locality, and empirical significance, which arises from the use of local transformations and the presence of descriptive redundancies (or surplus structure or descriptive fluff) in gauge theories.[2]

But surprisingly, little attention has been devoted in comparison to the question of why we should require our best theories to be locally gauge invariant in the first place, especially in the context which arguably matters the most, to wit, relativistic quantum field theory (QFT). Agreed: One finds in the literature various justifications appealing to the theoretical virtues of gauge theories.[3] Yet these justifications usually have little foundational oomph (e.g., mathematical tractability and computational efficiency) or look less compelling once we remember that our best current gauge theories are unlikely to remain empirically reliable at very high energies (e.g., renormalisability). Agreed again: One also finds in the literature some healthy scepticism and more ambitious attempts to account for the origin of local gauge invariance (see Martin [2002], [2003]; Rovelli [2014], [2020]). But even in the latter case, when the plausibly derivative character of local gauge invariance is not just mentioned in passing, we do not really get any non-circular, principled, and convincing explanation as to why we should not include irreducibly gauge-variant (or gauge-dependent) terms in the dynamics of gauge theories (I will briefly discuss Rovelli's account in Section 2). And overall, one is left with the impression that their structure of invariance has just been taken for granted in foundational investigations.

My goal in this paper is to show that there is a specific yet powerful way of looking at perturbative relativistic QFTs in which the principle of local gauge invariance follows in large part from Lorentz invariance. More precisely, I will argue in the general case that for a large class of realistic Lagrangian QFT models $\mathcal{L}$ in the perturbative setting, if the action associated with $\mathcal{L}$ is Lorentz invariant and satisfies other natural assumptions, then $\mathcal{L}$ is locally gauge invariant. As Steven Weinberg was the first to champion the 'Lorentz invariance implies local gauge invariance' claim in the 1960s and made essential use of Eugene Wigner's earlier achievements, I will first take the occasion to revisit their work, clarifying a number of important subtleties along the way and using as little mathematics as possible to keep the account accessible to a general audience (see Sections 3–4).[4] Then, besides explaining how the locally gauge invariant structure of our

[2]See (Healey [1997], [2007]; Belot [1998], [2003]; Maudlin [1998], [2018]; Earman [2002], [2004]; Brading and Brown [2003]; Castellani [2003]; Nounou [2003]; Redhead [2003]; Mattingly [2006]; Rickles [2008], chap. 3; Catren [2008]; Lyre [2009]; Myrvold [2011]; Wallace [unpublished], [2022a], [2022b]; Pitts [2014]; Friederich [2015]; Teh [2016]; Weatherall [2016]; Shech [2018]; Nguyen *et al.* [2020]; Berghofer *et al.* [unpublished]; Dougherty [2021b]; Gomes [2021]; Mulder [2021]; Murgueitio Ramirez and Teh [forthcoming]). See also (Healey [2007], sec. 6.5–6.6; Gomes [2019]; Dougherty [2020]; Gomes and Riello [unpublished]) for the important topic of topology and boundary conditions in gauge field theories, which I will not engage with here.

[3]See (Aitchison and Hey [2003], pp. 178–179; 't Hooft [2007], sec. 4.2) on the physics side; (Belot [2003], sec. 13; Guay [2008]; Gomes *et al.* [2022], sec. 4) on the philosophy side.

[4]See especially (Wigner [1939]; Weinberg [1964a], [1964b], [1964c], [1964d], [1965a], [1965b],

best theories arises from a demand on their spatio-temporal structure, I will address two closely related puzzles. To the question 'Why gauge freedom?' I will respond that it is a mere accident: the dynamics of a gauge theory is already sufficiently constrained by Lorentz invariance and other natural assumptions to ensure that a gauge field and any of its local transforms evolve in the same way (all else being equal). To the question 'Why gauge theories?' I will respond that the distinctive structure of our best quantum gauge field theories ultimately traces back to the peculiar space-time transformation properties of massless particles like photons, gluons, and gravitons.

Three disclaimers before I begin. First, I will use 'gauge' in its restricted field-theoretic sense, that is, exclusively in relation to the number-valued or operator-valued field variables traditionally associated with the electromagnetic, weak, strong, and gravitational forces without making further assumptions about their transformation rules or abstract mathematical structure. Second, I will mainly focus on the dynamics of the photon field for simplicity and only briefly explain how the argument extends to more complicated gauge theories. Small adjustments will be required depending on the class of models considered: for instance, infinitesimal Lorentz invariance for models with non-linear couplings in the gauge field variables. Third, I will appeal to substantial assumptions usually endorsed in the standard perturbative relativistic QFT setting besides Lorentz invariance, including the masslessness of photons and a Fock space structure for multi-particle states. There is, of course, much to be said about whether this is the right framework to engage with the foundations of relativistic quantum gauge field theories.[5] I will assume here that it is unproblematic, at least insofar as we are interested in learning more about the structure of our best gauge theories as they are successfully brought into contact with experiments.

The paper is organised as follows. Section 2 briefly rehearses how local gauge invariance constrains the dynamics of the photon field with and without matter fields before quantization. Section 3 makes a first step towards understanding the dynamical structure of the photon field by taking a detour through the peculiar space-time transformation properties of photons. Section 4 shows that natural attempts to construct a covariant Lorentz four-vector field for physical photons (and nothing more) are doomed to fail. Yet empirically viable models appear to require some kind of elementary photon field. Section 5 constructs the most natural version of such a field in this setting and explains how demanding its dynamics to be Lorentz invariant for a large class of models gets us

[1965c], [1969]). See also (Weinberg [1972], chap. 10.8, [1995]; Han and Kim [1981]; Scaria [2003]; Zee [2010], sec. III.4; Nicolis [unpublished]; Schwichtenberg [unpublished]; Arkani-Hamed *et al.* [2021]; Osborn [unpublished], sec. 4). The relevance of Weinberg's works for the gauge principle is briefly mentioned by Martin ([2002], p. S231, [2003], sec. 3.3) and more recently by Salimkhani ([2020], p. 2). The argument has also some conceptual affinity with Jauch's theorem in the non-relativistic quantum mechanical case (Jauch [1964], [1968], chap. 13; see also Gomes *et al.* [2022], sec. 3) and similar works in relativistic classical mechanics (Barandes [unpublished], [2021]).

[5] See (Wallace [2006]; Fraser [2009]; Miller [2018]; Williams [2019]; Rivat [2019], [2021]; Fraser [2020]) for a discussion in the broader context of QFT.

local gauge invariance for free. Section 6 highlights the main assumptions involved in the argument and explains how it extends to quantum electrodynamics (QED) and more complicated gauge theories. Section 7 concludes with a few interpretative lessons for the philosophy of gauge theories.

2 Two Gauge Principles

Let me begin by spelling out how the gauge principle in its two most paradigmatic forms shapes the dynamics of our best field theories.

Suppose first that you live in a world of pure light. The dynamics of the photon four-vector field A_μ before quantization is usually encoded in the following Lagrangian density:

$$\mathcal{L}_{\text{light}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.1}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the electromagnetic tensor field. Equation 2.1 satisfies what is sometimes called the 'restricted gauge principle', which requires $\mathcal{L}_{\text{light}}$ to be invariant under the local (or space-time-dependent or malleable) gauge transformation:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x), \tag{2.2}$$

with $\lambda(x)$ some arbitrary scalar function ('gauge freedom' or 'gauge redundancy'). If the photon field had a mass and we decided to account directly for it by adding a term $m^2 A_\mu A^\mu$ in Equation 2.1, $\mathcal{L}_{\text{light}}$ would not be invariant under the transformation specified by Equation 2.2. According to the restricted gauge principle, we should exclude such terms.

Suppose now that you live in a world of light filled with electrons and positrons. It is standard practice to describe the dynamics of this system before quantization with the following Lagrangian density:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i\gamma^\mu(\partial_\mu - ieA_\mu) - m_e]\psi, \tag{2.3}$$

with ψ the electron spinor field, m_e its mass, e its charge, and γ^μ the usual Dirac matrices. Equation 2.3 satisfies what is sometimes called the 'extended gauge principle', which requires $\mathcal{L}_{\text{QED}}$ to be jointly (or 'simultaneously') invariant under a local gauge transformation and a local phase transformation:

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) + \partial_\mu \lambda(x) \\ \psi(x) &\to e^{ie\lambda(x)}\psi(x). \end{aligned} \tag{2.4}$$

Once again, this principle imposes stringent constraints on the type of interaction terms that may figure in $\mathcal{L}_{\text{QED}}$ (e.g., arbitrary polynomial terms in A^2 and $(\partial A)^2$ are forbidden).[6]

For the record, the extended gauge principle is usually motivated by running first the 'gauge argument': (i) demand that the dynamics of matter spinor fields be invariant under arbitrary number-valued local phase transformations; (ii) introduce some appropriate dynamical four-vector field $A_\mu(x)$ defined up to some prescribed derivative term and couple it appropriately to matter fields to compensate for the new terms that arise under local phase transformations; (iii) identify this field with the photon field and adjust the parameters of the dynamics if needed. Philosophers and physicists alike widely agree that the gauge argument is not up to scratch (see Martin [2003]; Healey [2007], sec. 6.3; Gomes *et al.* [2022], for critical philosophical assessments). But even if we were to take the local phase invariance of the matter field dynamics for granted and find a way to save the argument, it would still not be sufficient to account for the origin of local gauge invariance in pure gauge field theories.

Let me now give two familiar pieces of textual evidence to testify to the long-standing importance of the restricted and extended gauge principles:

> Perhaps the most important discovery of modern particle physics is the gauge principle. According to it, all interactions in nature arise from the claim that the Lagrangian has to be invariant under local symmetry transformations, i.e. symmetry rotations that may be different at different space-time points. (Polyakov [1987], p. 5)

> The gauge principle is generally regarded as the most fundamental cornerstone of modern theoretical physics. In my view, its elucidation is the most pressing problem in current philosophy of physics. (Redhead [2003], p. 138)

One would expect philosophers to jump on the occasion to dig into the origins of local gauge invariance (and not just focus on the well-foundedness of the gauge argument). Unfortunately, this has not really been the case in comparison to other philosophical topics surrounding gauge theories. The dominant attitude seems to boil down to admitting that 'local gauge symmetry' is a primitive feature of the standard formulation of our best gauge theories (Healey [2007], p. 167; Dougherty [2021a]) while recognising that different contexts will probably require different answers to the question 'Why gauge?' (Earman [2003], p. 159).

Rovelli ([2014], [2020]) provides a welcome exception in his attempt to explain in physical terms why gauge theories have such a peculiar structure. He argues that the gauge-dependent variables used to describe a system reflect its ability to be coupled

[6] Note that the distinction between the restricted and extended gauge principles also applies to the non-abelian case.

to other systems—in his own terms, gauge variables like $A_\mu(x)$ are 'handles' through which to couple different systems with one another and the presence of gauge freedom in a theory is a 'formalization of the relational nature of [its] physical degrees of freedom' (Rovelli [2020], p. 1). On this view, the ubiquity and success of gauge theories in physics come from the fact that they have appropriate formal resources to account for how different systems relate to each other. And although Rovelli does not say much about this, the gauge principle is naturally interpreted on his account as a demand to treat the system of interest in isolation, whether elementary (as in Equation 2.1) or composite (as in Equation 2.3).

Of course, much more would need to be said to do justice to Rovelli's account and I will restrict myself here to raising some doubts about its ability to get to the bottom of local gauge invariance (see also Teh [unpublished], for a discussion). Note first that Rovelli's relationalist interpretation depends on the assumption that at least some interaction term makes explicit and indispensable use of a gauge-dependent variable. Otherwise, we might as well eliminate such variables altogether and treat them as mere mathematical redundancies. Then, the issue is that the *gauge-dependent character* of such variables has nothing to do with the ability of the corresponding system to be coupled through such special interaction terms. Or, to put it differently, gauge freedom does not bring any new information whatsoever about possible dynamical couplings. For instance, we do need the elementary interaction term $\bar{\psi}\gamma^\mu A_\mu \psi$ and thus the variable A_μ to make empirically viable predictions at low energies. But this is unrelated to gauge freedom and gauge invariance (I will provide more detail in Section 5). To take a straightforward case, we would still need and be able to use this interaction term at low energies even if we break local gauge invariance by including some arbitrary higher-order interaction term in Equation 2.3 (with some sufficiently small physical parameter and perhaps some fine-tuning for other renormalized parameters to keep things empirically equivalent at low energies).

More generally, the link between gauge freedom and dynamical couplings is too thin to be compelling. Many QFT models describing physically salient low-energy systems interacting with each other do not make any use of gauge-dependent variables and local gauge invariance (e.g., Fermi-type models). Even when they do, the gauge-dependent character of a field variable does not seem to play any role in the ability of the corresponding system to interact with others beyond the special cases mentioned above. For instance, we may perfectly well couple photons to electrons and positrons through interaction terms like $\bar{\psi}F^{\mu\nu}\sigma_{\mu\nu}\psi$, with $\sigma_{\mu\nu}$ the usual commutator of Dirac matrices. And so coming back to what is at stake here, it does not seem that there is any special reason to believe that the gauge dependence of variables like A_μ stands for their relational character contrary to what Rovelli claims. Nor does it seem, in turn, that the gauge principle boils down to a demand to treat a given system in isolation.

I will take another route to explain the mysterious origin of local gauge invariance in the context of relativistic QFT. Following Redhead's plea, the proposal here is to 'elucidate' the gauge principle(s) by showing that local gauge invariance arises in large part from Lorentz invariance. I will side with Rovelli in suggesting that the photon field variable A_μ does have a definite physical meaning. But I will depart from him by showing that gauge freedom is a mere accidental mathematical redundancy in this setting.

3 But Who Is Really the Photon?

To get there, we first need to take a step back and look more closely at the most elementary properties of free photons (without yet dressing them up with a field structure or speaking about their dynamics).

There are two ways to go here. We might first try to identify directly the kind-constitutive properties of the free particle of interest, which, in relativistic quantum theories, amounts to determining the set of its invariant properties under arbitrary space-time transformations. In typical cases, this set is fully specified by the eigenvalues of the so-called 'Casimir operators', that is, the invariant operators associated with the algebraic structure of space-time transformations. Then, once we have identified the particle of interest, we may safely ask about what it looks like in various circumstances and how its different states relate to each other.

Unfortunately, this method does not work for the photon.[7] But there is a somewhat more general and iterative way of finding its invariant properties and the structure of its state space, the so-called 'method of induced representations'. The basic idea here is to examine the behaviour of an arbitrary particle under space-time transformations and decompose its possible states into invariant subsets whose labels stand for particle properties until we are able to identify every possible kind of particle and exhaust everything there is to say about each of them. Of course, for this to work, we need a preliminary set of quantum numbers to specify a general state space. For standard space-time symmetries, the most natural choice is to begin with the four-momentum p^μ of a particle and denote all of its other internal quantum numbers with some unspecified label σ.

Consider then the behaviour of some unspecified photon in the state $|p, \sigma\rangle$ under arbitrary space-time transformations. The momentum of our photon does not change when we translate it around (i.e., the original and translated states have the same momentum eigenvalue). But this fails to be the case for uniform boosts and rotations. In fact, the only functions of p^μ that remain invariant under such transformations in the general case are functions of p^2.[8] We may thus use $m^2 = p^2$ as our first tag to specify the identity of

[7] The eigenvalues of the Casimir operators fail to be in one-to-one correspondence with distinct irreducible unitary representations of the Poincaré group in the massless case.

[8] Note that for $p^2 \geq 0$ and $p^0 \neq 0$, the sign of p^0 also remains invariant under arbitrary Lorentz transformations. I will choose $p^0 > 0$ here, using the metric signature $(+, -, -, -)$.

a particle. The photon, in turn, is assumed to be massless ($m^2 = 0$), that is, to be a kind of particle that cannot be put to rest.

Our photon might still have other invariant properties. Suppose then (as we have assumed so far) that its states are characterised by some additional independent quantum number σ. By construction, since we take our photon to have some definite momentum p^μ and thus $|p, \sigma\rangle$ to be an eigenstate of the four-momentum operator, σ does not change when we translate the photon around. We can thus restrict our focus to boosts and rotations. Performing an arbitrary transformation Λ of this kind on the photon shifts its momentum from p^μ to some definite value $(\Lambda p)^\mu$. But for σ, the photon typically ends up in a superposition of states $|\Lambda p, \sigma'\rangle$ with coefficients depending on σ and σ'.

Let us dig a little bit deeper into the transformation properties of our photon with respect to σ. To this end, we can decompose the action $U(\Lambda)$ of the transformation Λ on the state of the photon into a part that affects σ but not its momentum and other parts that affect only its momentum but not σ. It is also convenient to pick a particular momentum k^μ for which the set of transformations that leave it invariant is as transparent as possible (more about this below). Then, if we start with a photon with arbitrary momentum p^μ, we can boost it into a state of momentum k^μ without affecting its label σ, examine the most general ways of transforming σ without affecting k^μ, and boost the photon to a state of momentum $(\Lambda p)^\mu$ without affecting σ. This amounts to decomposing Λ as follows:

$$\Lambda = L(\Lambda p)W(\Lambda, p)L^{-1}(p),$$

with $L(p)$ some standard Lorentz transformation that brings the momentum of the photon from k^μ to p^μ without affecting σ, and $W(\Lambda, p) := L^{-1}(\Lambda p)\Lambda L(p)$ a Lorentz transformation that leaves k^μ invariant. The set of such transformations $W(\Lambda, p)$ is called the 'little group' of k^μ after Wigner ([1939]).[9]

What momentum k^μ should we choose? If we were dealing with a massive particle, it would be easy. We could put the particle to rest, i.e., $k^\mu = (m, 0, 0, 0)$, and directly find that the only transformations that leave this momentum invariant are rotations in three dimensions. Unfortunately, things are not as neat in the massless case (assuming $k^\mu \neq 0$). But we can still pick some sufficiently simple reference frame, say, some 'rest frame' $\mathcal{R}$ in which the photon moves upward with energy k, i.e., $k^\mu = (k, 0, 0, k)$, and look for the subset of transformations $W(\Lambda, p)$ that leave this momentum invariant.

There are two different kinds of transformations in store here. Most obviously, k^μ remains invariant if we rotate the photon in the xy-plane by some arbitrary angle θ. Less obviously, we can also perform arbitrary boosts with parameters β_1 and β_2 in the

[9]The structure of the little group or stabiliser is purely a matter of geometry and remains invariant for different k^μ belonging to the same orbit $k^2 = m^2$. By contrast, the choice of $L(p)$ and k^μ is purely conventional and not unique. See (Woit [2017], chap. 20) for more mathematical details and (Schuster and Toro [2013b], p. 9) for some insightful remarks about the significance of $L(p)$.

xy-plane, composed with rotations around the x- and y-axes to bring back the direction of the momentum along z and a boost along z to bring back its value to k. We can thus decompose any transformation $W(\Lambda, p)$ into the product of a little group rotation $R(\theta)$ around z and a little group boost $S(\beta_1, \beta_2)$ in the two-dimensional xy-plane (with S including some additional appropriate rotation-boost).

Now, this little group is isomorphic to the group of rotations and translations in the two-dimensional Euclidean plane. We can thus directly represent the action of a little group transformation on the state $|k, \sigma\rangle$ of the photon by the operator $U[W(\theta, \beta_1, \beta_2)] = \exp(i\theta J_3) \exp(i\beta_1 B_1 + i\beta_2 B_2)$, with J_3 and (B_1, B_2) the generators of rotations and translations, respectively. These operators can be used, in turn, to specify further $|k, \sigma\rangle$. In close analogy with space-time translations, a natural choice here is to select the eigenvalues of B_1 and B_2 for σ and thus express the states of the photon with the help of two additional 'internal momentum' parameters as $|k, b_1, b_2\rangle$. Likewise, the 'internal mass' $r^2 = b_1^2 + b_2^2$ remains invariant when we perform little group rotations and boosts on the photon, and we can thus use r, which is usually called the 'spin scale', to distinguish again between different massless particle species.

The most general choice $r \neq 0$ corresponds to what is called today the 'continuous spin representation' of massless particles. As in the massive case, each non-zero value of r is associated with a distinct kind of massless particle. And in the same way as the external momentum p^μ of these particles shifts to $(\Lambda p)^\mu$ under a Lorentz transformation Λ, their internal momentum (b_1, b_2) shifts to (b_1^θ, b_2^θ) under a rotation of angle θ along z. These particles thus have a continuous number of internal degrees of freedom in this basis (their wave function varies independently for each value of b_1^θ and b_2^θ). But there are other options. For instance, we can move to the so-called 'spin basis', in which states are labelled by the eigenvalues h of J_3. Since the operators $B_\pm = B_1 \pm iB_2$ increase or decrease the value of h by one unit, we end up with an infinite series of integer-spaced states $|k, h\rangle$. Either way, the most generic types of massless particles ($r \neq 0$) have an infinite number of internal degrees of freedom.[10]

In practice, all the particles we have detected so far appear to have only a finite number of internal degrees of freedom. We thus seem to be forced to turn to the second choice, i.e., the representation of the little group with $r = 0$, which corresponds to the trivial representation ($b_1 = b_2 = 0$), and specify the states of massless particles with J_3, i.e., $J_3|k, h\rangle = h|k, h\rangle$. Like her cousin the spin j in the massive case, the helicity number h turns out to be invariant under space-time transformations. In particular, the operators B_1 and B_2 now act trivially on $|k, h\rangle$ and do not mix helicity states compared to the continuous spin case. We can thus use h to distinguish again between different massless particle species with zero spin scale. The photon (resp. graviton) is assumed to be a kind of massless particle with helicity $h = \pm 1$ (resp. $h = \pm 2$).[11]

[10] For more technical details, see (Brink *et al.* [2002]; Schuster and Toro [2013a], [2013b]; Bekaert and Skvortsov [2017]).

[11] Note that neutrinos are now considered to be massive and that, as far as the argument goes, gluons

Closing the discussion, we obtain the transformation properties of the photon under arbitrary (homogeneous) Lorentz transformations by boosting it back to the frame in which it has momentum $(\Lambda p)^\mu$:

$$U(\Lambda)|p, h\rangle = e^{ih\theta}|\Lambda p, h\rangle, \tag{3.1}$$

where $\theta \in [0, 2\pi[$ depends on Λ and p^μ since $W(\Lambda, p)$ depends on Λ and $L(p)$.

4 Don't Tell Me About the Photon Field

Let me now connect the discussion back to the standard account of electromagnetism and try to construct a local Lorentz four-vector field operator $A^\mu(x)$ for photons as they have been identified so far. (No assumption is made at this stage about whether the dynamics of the photon field is locally gauge invariant.)

We face two main constraints here. First, we need to make sure that $A^\mu(x)$ accounts for the specific kind of particle we are interested in, namely, massless particles with helicity $h = \pm 1$, and does not allow for unwanted creatures to pop out of the vacuum. The simplest choice is to take the photon field to create and annihilate free photons one at a time and account for states with multiple photons by means of appropriate combinations of elementary field excitations. Translated in mathematical terms, this amounts to constructing $A^\mu(x)$ in terms of a linear combination of creation and annihilation operators $a_h^\dagger(p)$ and $a_h(p)$ for free photons of arbitrary momenta p^μ and helicity $h = \pm 1$, with $a_h^\dagger(p)|0\rangle = |p, h\rangle$, $a_h(p)|p, h\rangle = |0\rangle$, and $|0\rangle$ some vacuum state, and using operator products to deal with more complicated states. If we put other constraints aside for now, the most general form of the photon field operator satisfying this first requirement is:

$$A^\mu(x) = \sum_{h=\pm 1} \int d^4p[c_1^\mu(x, h, p)a_h(p) + c_2^\mu(x, h, p)a_h^\dagger(p)],$$

with $c_{1,2}^\mu$ some coefficients carrying the 'local field' and 'four-vector' properties of the photon field, i.e., $c_1^\mu(x, h, p) = \langle 0|A^\mu(x)|p, h\rangle$ and $c_2^\mu(x, h, p) = \langle p, h|A^\mu(x)|0\rangle$.[12]

Second, we need to make sure that the photon field has appropriate space-time transformation properties. In line with the standard QFT framework, let us assume that the vacuum state remains unaffected by space-time transformations: $U(\Lambda, \alpha)|0\rangle = |0\rangle$, with Λ some Lorentz transformation and α^μ some translation parameter. In this case, the creation and annihilation operators automatically inherit the transformation properties of elementary photon states: for instance, $Ua_h^\dagger(p)U^{-1} = e^{i(h\theta+\alpha p)}a_h^\dagger(\Lambda p)$, assuming that we

are special kinds of photons with some extra dummy label.

[12]Multi-photon states need to have the right sort of discrete symmetries to ensure, for instance, that the annihilation and creation operators satisfy appropriate commutation relations. There are also stronger reasons than simplicity for using sums of products of creation and annihilation operators, such as to ensure the cluster decomposition and Lorentz invariance properties of the S-matrix (see Weinberg [1995], chap. 3–5; Duncan [2012], chap. 4–5, for more detail and subtleties).

perform the translation first. Then, if we require the 'sum' of creation and annihilation operators over momenta to behave appropriately under translations and have both some appropriate Lorentz invariant measure and normalisation constant, we end up with the following expression for the photon field:

$$A^\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_{h=\pm1} \int \frac{d^3p}{\sqrt{2p^0}} [\epsilon_h^\mu(p) a_h(p) e^{-ip.x} + \epsilon_h^{\mu*}(p) a_h^\dagger(p) e^{ip.x}], \tag{4.1}$$

with $\epsilon_h^\mu(p)$ some 'four-vector' coefficients (more about this shortly). If we also require $A^\mu(x)$ to transform appropriately under boosts and rotations, i.e.,

$$U(\Lambda) A^\mu(x) U^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x), \tag{4.2}$$

we obtain the constraint:

$$\Lambda^\mu{}_\nu \epsilon_h^\nu(p) = e^{ih\theta} \epsilon_h^\mu(\Lambda p), \tag{4.3}$$

which links the Lorentz transformation properties of the photon field (left-hand side of Equation 4.3) to those of the photon states (right-hand side).

If all was well, the story after this would be well known. We would be able to construct a Lagrangian (or Hamiltonian) density $\mathcal{L}$ out of this (interaction picture) photon field and roll out the standard perturbative treatment of QFTs (see Weinberg [1964b], [1964c], [1965b], [1995]).

Now for the troubles. Equation 4.3 shows that the four-component functions $\epsilon_h^\mu(p)$ do not transform as Lorentz four-vectors under arbitrary transformations, which suggests that $A^\mu(x)$ is a very unusual kind of Lorentz four-vector field. In fact, the situation is far worse. It is not just that the solutions $\epsilon_h^\mu(p)$ of Equation 4.3 have peculiar transformation properties. In general, if we take Equation 4.3 to hold for arbitrary Lorentz transformations and not just a subset thereof, this equation has no solution at all, which means that we cannot even force $A^\mu(x)$ (as constructed so far) to transform as a Lorentz four-vector field.

To make this explicit, suppose that $A^\mu(x)$ does transform like a Lorentz four-vector field and consider the amplitude for the annihilation of an arbitrary photon of momentum k^μ as specified in Section 3, i.e., $\langle 0|A^\mu(x)|k, h\rangle = \epsilon_h^\mu(k)e^{-ikx}$. As in the general case, the transformation properties of the photon field $A^\mu(x)$ and photon states $|k, h\rangle$ uniquely determine those of $\epsilon_h^\mu(k)$. In particular, if we take Λ to be a little group rotation $R(\theta)$, the polarisation vectors $\epsilon_h^\mu(k)$ pick up a phase factor $e^{ih\theta}$ from the photon states $|k, h\rangle$: $R^\mu{}_\nu \epsilon_h^\nu(k) = e^{ih\theta} \epsilon_h^\mu(k)$. The only available unit vectors satisfying this constraint with $h = \pm 1$ are:

$$\epsilon_\pm^\mu = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \tag{4.4}$$

which looks very much like the usual circular polarisation vectors for the vector potential.[13]

On the other hand, if we take Λ to be a little group boost $S(\beta_1,\beta_2)$, the polarisation vectors $\epsilon_h^\mu(k)$ do not pick up anything from the photon states: $S^\mu{}_\nu \epsilon_h^\nu(k) = \epsilon_h^\mu(k)$ (recall that $U(S)|k,h\rangle = |k,h\rangle$). But this is in direct contradiction with what we obtain when we examine how the unit vectors $\epsilon_\pm^\mu$ in Equation 4.4 transform under the little group boost:

$$S(\beta_1,\beta_2)^\mu{}_\nu \epsilon_\pm^\nu = \epsilon_\pm^\mu + (\beta_1 \pm i\beta_2)\frac{k^\mu}{k\sqrt{2}}. \tag{4.5}$$

This means that there are no $\epsilon_h^\mu(k)$ consistent with Equation 4.3 in the general case (and thus no $\epsilon_h^\mu(p)$ either) and that we got ourselves into a fix somewhere.

As it turns out, there is a principled algebraic explanation for this, which, broadly speaking, comes from a fundamental mismatch between the finite-dimensional field representations and infinite-dimensional state-space representations of the Lorentz group that are commonly used in the massless case (see Weinberg [1964c], sec. III, for more detail).[14] Fortunately, there is no need to delve further into this here; we will only need the peculiar transformation properties of a suitably defined photon field to get local gauge invariance for free in Section 5.

Before that, it will be instructive to play around a little while longer with the issue at hand to get some photon field off the ground (we do not have any so far) and compare our plight with the miseries encountered in standard approaches to electromagnetism (Jackson [1998]; Aitchison and Hey [2003]).

Since photons with momentum k^μ do not feel anything when we perform a little group boost on them, i.e., $U(S)|k,h\rangle = |k,h\rangle$, let us restrict the discussion to Lorentz transformations involving only a little group rotation, i.e., $\Lambda = L(\Lambda p)R(\theta)L^{-1}(p)$. We can then define a photon field $A^\mu(x)$ by requiring it to behave appropriately only under such transformations. In this case, the polarisation vectors $\epsilon_\pm^\mu(k)$ must be again transverse (as in Equation 4.4). We also obtain $\epsilon_\pm^\mu(p) = L^\mu{}_\nu(p)\epsilon_\pm^\nu$ by using the constraint of Lorentz covariance for $L(p)$ (the little group rotation is trivial in this case). Now, recall that the choice of $L(p)$ is conventional. In particular, we can preserve two key properties of $\epsilon_\pm^\mu$ in Equation 4.4 by using a boost along z and rotations in the xz- and yz-planes: $\epsilon_\pm^0(p) = \epsilon_\pm^0 = 0$ and $\vec{p}.\vec{\epsilon}_\pm(p) = \vec{k}.\vec{\epsilon}_\pm = 0$ (see Weinberg [1964c], p. 884, for more detail). And if we go back to the expression of the photon field in Equation 4.1 and keep in mind that these two constraints hold for the polarisations of any photon of

[13] For the explicit matrix representation of $R^\mu{}_\nu$ and $S^\mu{}_\nu$, see (Weinberg [1964d], appendix A, [1995], sec. 2.5).

[14] In short, a field representation (A, B) of the Lorentz group has to satisfy the algebraic relation $B - A = h$ in the massless case, which means that we can use a field representation for photons with $h = 1$ transforming according to $(0, 1)$ and $(1/2, 3/2)$ but not according to $(1/2, 1/2)$ for instance, i.e., not as a Lorentz four-vector field.

arbitrary momentum p^μ, we obtain:

$$A^0(x) = 0 \tag{4.6}$$

$$\vec{\partial}.\vec{A}(x) = 0, \tag{4.7}$$

which looks very much like the usual radiation (or Coulomb) gauge in the absence of a charge distribution. So we now have at least some kind of photon field. But once gain, Equations 4.6–4.7 make it clear that it would not transform like a Lorentz four-vector if we were to perform an arbitrary Lorentz transformation.[15]

All of this should of course ring a bell. By constraining $\epsilon^\mu_\pm(k)$ and $\epsilon^\mu_\pm(p)$ through Equation 4.3, we have indirectly reduced the number of degrees of freedom of the photon field operator at each space-time point to two. And if we try to keep only two physical polarisations for photons in standard approaches to classical and quantum electrodynamics, that is, one independent four-vector for each h and p^μ, we face the same issue. For instance, if we impose the Coulomb gauge $\vec{\partial}.\vec{A} = 0$ or the axial gauge $A^3 = 0$ plus some subsidiary condition on A^0 in Equation 2.1, we directly end up in the same situation as above. The Lorenz gauge $\partial_\mu A^\mu = 0$ allows us to save manifest Lorentz covariance for a little while longer. But if we ultimately want to work only with two physical polarisations, we have no choice but to impose some condition breaking Lorentz covariance at the end, such as $\epsilon^0 = 0$ for instance. Similar issues arise in the quantum case too (Itzykson and Zuber [1980], sec. 3.2.1; Mandl and Shaw [2010], chap. 5). And so whether we follow Weinberg's constructive approach or the standard path, we seem to run into trouble if we try to formulate a relativistic QFT of photons in terms of a Lorentz four-vector field $A^\mu(x)$ involving only creation and annihilation operators for massless helicity $h = \pm 1$ particles.

One might wonder at this point whether we might not be better served by simply giving up on the photon field A^μ and using the electromagnetic field $F^{\mu\nu}$ instead. $F^{\mu\nu}$ is indeed the simplest covariant tensor field representation of free massless helicity $h = \pm 1$ particles (Weinberg [1964c]). We can use more complicated tensor field representations with more independent components to account for photons. But we cannot use elementary ones like $A^\mu(x)$ as we just saw above. So, if we want to account for light in the simplest quantum field-theoretic terms, we do not even need to know anything about the classical electric and magnetic fields. We would be automatically led to the covariant and gauge-indifferent tensor $F^{\mu\nu}$.

Unfortunately, as was briefly mentioned in Section 2, this option does not seem to work in the world in which we live. In the context of perturbative relativistic QFTs,

[15] Note that the 'gauge-fixing conditions' in Equations 4.6–4.7 are not directly implemented on the general solutions of a locally gauge invariant dynamics. Rather, these conditions result from the partial constraint of Lorentz covariance imposed on $\epsilon^\mu_\pm(k)$ and $\epsilon^\mu_\pm(p)$ through Equation 4.3 for a particular choice of $L(p)$. Different choices of $L(p)$ merely correspond to different coordinate definitions of the polarisation vectors: $\tilde{\epsilon}_\pm(p) = \tilde{L}(p)L^{-1}(p)\epsilon_\pm(p)$, with $\tilde{\epsilon}_\pm(p) = \tilde{L}(p)\epsilon_\pm$. As we will see below, the argument in Sections 5–6 is independent of any such choice.

we need the simple interaction term $A^\mu \bar{\psi}\gamma_\mu \psi$ and thus the photon field A^μ to account at least for long-range electromagnetic forces. Consider, for instance, the electrostatic force between two charged sources. The long-range behaviour of this force is encoded in the fact that the Coulomb potential $V(r)$ varies as $1/r$, with r the distance between the two sources. We can compute the amplitude for the scattering between two charged particles at tree level and recover the expression of this potential $V(r) \propto \int d^3 p e^{i\vec{p}.\vec{r}}/|\vec{p}^2|$ in the non-relativistic limit. The crucial point is that if we use the simple interaction term $A^\mu \bar{\psi}\gamma_\mu \psi$ to characterise photon exchanges, the variation of $V(r)$ in $1/r$ is directly related to the variation of the propagator for the photon field A^μ in $1/p^2$. If we were to use instead some interaction term involving the electromagnetic tensor field $F^{\mu\nu}$, such as $\bar{\psi}F^{\mu\nu}\sigma_{\mu\nu}\psi$ for instance, we would be naturally led to use the propagator for $F^{\mu\nu}$, which varies in $(p^2)^0$ (i.e., it does not depend on $|p|$). This, in turn, would give rise to a potential $V(r)$ in $1/r^3$, which is too weak at large distances to account for long-range electrostatic phenomena.

To wrap up, the real mystery of gauge theories in the simple example taken so far comes from the apparent tension between the following two claims: (i) we need some kind of four-vector photon field in our most empirically successful relativistic QFT models; (ii) the photon field with only two physical polarisations does not transform, as a matter of principle, like a Lorentz four-vector.

5 Getting Local Gauge Invariance for Free

The previous section suggests that we have no choice but to get some elementary photon field off the ground. A first attempt was made above by requiring A^μ to behave appropriately at least under a subset of Lorentz transformations and defining it in terms of transverse polarisation vectors $\epsilon^\mu_\pm(p)$. However, as Equations 4.6–4.7 make it clear, the situation now looks even worse than expected. As far as the argument goes, there is, for instance, no reason to believe that the electromagnetic field $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ built out of such A^μ transforms in general like a (rank 2) Lorentz tensor. But perhaps the outcome would not be so disastrous if we were to take into account the effects of the remaining part of the little group on the photon field (as it has been defined so far). As we will now see, this is indeed the case. We will even be able to get the restricted gauge principle for free by requiring the Lagrangian density $\mathcal{L}$ built out of A_μ in the interaction picture to remain invariant under arbitrary Lorentz transformations (together with a mild restriction on the class of models considered). Section 6 will explain how the argument extends to more complicated gauge theories.

Suppose then that we constrain $\epsilon^\mu_h(k)$ as we have done so far, so as to absorb the helicity phase factors $e^{\pm ih\theta}$ obtained from the annihilation and creation operators when we perform an arbitrary Lorentz transformation on the photon field operator (see Equations 4.3–4.4). As we have seen in Equation 4.5, the resulting polarisation vectors ϵ^μ_h pick up an additional longitudinal contribution under arbitrary little group boosts $S(\beta_1, \beta_2)$. This did not look good above. But we can now use this peculiar transformation property to our advantage.[16]

Note first that we are free to take a different $L(p)$ and thus define $\epsilon_h^\mu(p)$ differently for arbitrary p^μ such that Equations 4.6–4.7 do not hold in general. In particular, we can take $L'(p) = L(p)S$, with S some little group boost (since $U(L')|k, h\rangle = |p, h\rangle$). This amounts to defining $\epsilon_h^\mu(p)$ as $L(p)^\mu{}_\nu(\epsilon_h^\nu + ck^\nu)$, with c some complex number. As a result, we obtain a photon field A^μ with non-vanishing temporal and non-transverse spatial components (since $\epsilon_\pm^0(p) \neq 0$ and $\vec{p}.\vec{\epsilon}_\pm(p) \neq 0$ in this case). For simplicity, I will keep the original choice of $L(p)$ in what follows, which is akin to the radiation gauge in the standard approach. But nothing hinges on this choice in the sequel, and especially not the constraints imposed by Lorentz invariance on the dynamics (see Equations 5.1–5.2 below).

More importantly, the photon field acquires gauge-like transformation properties once we consider the full little group. To make this explicit, suppose that we perform an arbitrary Lorentz transformation on the photon field operator:

$$A^\mu(x) \to U(\Lambda)A^\mu(x)U^{-1}(\Lambda),$$

with the decomposition $\Lambda = L(\Lambda p)S(\beta_1, \beta_2)R(\theta)L^{-1}(p)$ for arbitrary momenta p^μ. Again, the only component parts of $A^\mu(x)$ that do not commute with $U(\Lambda)$ are the creation and annihilation operators. As we have seen, they pick up a momentum shift $(\Lambda p)^\mu$ and a phase factor $e^{\pm ih\theta}$ under such a transformation. On the other hand, if we use the little group decomposition of Λ and $\epsilon_h^\mu(p) = L^\mu{}_\nu(p)\epsilon_h^\nu$ together with Equation 4.5, we obtain:

$$\Lambda^\mu{}_\nu \epsilon_h^\nu(p) = e^{ih\theta}[\epsilon_h^\mu(\Lambda p) + (\beta_1 + ih\beta_2)\frac{(\Lambda p)^\mu}{k\sqrt{2}}], \tag{5.1}$$

keeping in mind that it does not matter whether we perform a little group boost first (this only affects the value of β_1 and β_2) or choose a different $L(p)$ (the transformation rule in Equation 5.1 would have the same form).

This last equation shows that we can absorb the two phase factors $e^{\pm ih\theta}$ in the expansion of $UA^\mu(x)U^{-1}$ if we replace $e^{-ih\theta}\epsilon_h^\mu(p)$ by $(\Lambda^{-1})^\mu{}_\nu\epsilon_h^\nu(\Lambda p)$ and $e^{ih\theta}\epsilon_h^{\mu*}(p)$ by $(\Lambda^{-1})^\mu{}_\nu\epsilon_h^{\nu*}(\Lambda p)$. But in this case, we obtain additional terms depending on $(\beta_1 \pm ih\beta_2)p^\mu e^{\pm ipx}$ in the expansion. That is, the photon field operator picks up an additional derivative term under

[16] We could also first require A^μ to transform covariantly under Lorentz transformations involving only a little group boost and constrain the polarisation vectors for k^μ and p^μ accordingly: $S(\beta_1, \beta_2)^\mu{}_\nu\epsilon_h^\nu(k) = \epsilon_h^\mu(k)$ and $\epsilon_h^\mu(p) = L^\mu{}_\nu(p)\epsilon_h^\nu(k)$, for some $L(p)$. In this case, however, we would not be able to absorb the helicity phase factors $e^{\pm ih\theta}$ obtained from arbitrary Lorentz transformations and thus recover appropriate transformation properties for $F^{\mu\nu}$.

an arbitrary Lorentz transformation:

$$U(\Lambda)A^\mu(x)U^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x) + \partial^\mu \lambda_\Lambda(x), \tag{5.2}$$

where $\lambda_\Lambda(x)$ involves an integral over the creation and annihilation operators of the photon field (as in Equation 4.1) and depends implicitly on Λ through its dependence on β_1 and β_2 (and θ in general).

Needless to say, the photon field still does not transform as a Lorentz four-vector. But we can do well out of this peculiar transformation rule. Note first that the electromagnetic field $F^{\mu\nu}$ now does transform appropriately under arbitrary Lorentz transformations. The additional terms $\partial^\mu\partial^\nu\lambda_\Lambda$ and $\partial^\nu\partial^\mu\lambda_\Lambda$ obtained by using Equation 5.2 together with $\partial^\mu A^\nu - \partial^\nu A^\mu$ directly cancel each other. We can thus construct a Lorentz invariant dynamics with $F^{\mu\nu}$ to describe a world filled with massless helicity $h = \pm 1$ particles. We can even introduce Lorentz invariant self-interaction terms like $(F^{\mu\nu}F_{\mu\nu})^2$ if needed.

Even more fundamental, this construction shows that we get the restricted gauge principle for free if we require the dynamics of the photon field to be Lorentz invariant (i.e., again, the Hamiltonian or Lagrangian density built out of A_μ in the interaction picture). Since A^μ is still not a Lorentz four-vector, garden-variety terms like $m^2 A_\mu A^\mu$ and $(A_\mu A^\mu)^3$ are not Lorentz scalars; they would come along with additional terms under the transformation specified by Equation 5.2 compared to F^2 for instance. But if we require the dynamics to be Lorentz invariant, we are forced to keep only Lorentz scalars, including terms and groups of terms that behave effectively as Lorentz scalars, and thus a set of terms that is in general automatically locally gauge invariant. In fact, it is even possible to prove that the dynamics of the photon field without matter fields is locally gauge invariant in the traditional sense if we restrict ourselves to Lorentz transformations with infinitesimal little group parameters, infinitesimal local gauge transformations, and models without higher-order derivative variables and for which $\partial_\mu A_\nu$ couples only to antisymmetric tensors.[17]

Let me conclude this section with one key remark. Since λ_Λ is a non-trivial operator, we cannot eliminate this new kind of gauge freedom by performing a traditional local gauge transformation (as specified by Equations 2.2 or 2.4 for instance, with $\lambda(x) \equiv \lambda(x)\mathbb{1}$ for the local gauge transformation). The pure dynamics of the photon field is indeed invariant under a more general type of operator-valued local gauge transformation:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \hat{\lambda}(x),$$

[17]Consider first the variation under infinitesimal Lorentz transformations $\delta\mathcal{L}_{\text{Lorentz}} = (\delta\mathcal{L}/\delta A_\mu)\partial_\mu\lambda_\Lambda + (\delta\mathcal{L}/\delta\partial_\mu A_\nu)\partial_\mu\partial_\nu\lambda_\Lambda$. The second term on the right-hand side vanishes because of the model restriction. Then, Lorentz invariance $\delta\mathcal{L}_{\text{Lorentz}} = 0$ implies $\delta\mathcal{L}/\delta A_\mu = 0$ (the little group parameters in $\partial_\mu\lambda_\Lambda$ are arbitrary). Hence, the variation under infinitesimal local gauge transformations automatically vanishes: $\delta\mathcal{L}_{\text{gauge}} = 0$.

with $\hat{\lambda}(x)$ some arbitrary scalar operator.

6 Beyond the Pure Photon Field Dynamics

I will now highlight the main assumptions involved in the argument so far and discuss how it extends to QED and more complicated gauge theories.

Section 3 makes essential use of **(A1)** standard quantum and relativity principles, **(A2)** the assumption that photons are massless helicity $h = \pm 1$ particles, and **(A3)** the natural choice of basis $|p, h\rangle$ for one-particle photon states. But note that their covariant transformation rule in Equation 3.1 is independent of a particular choice of momentum k^μ, Lorentz boost $L(p)$, and little group decomposition for Λ.

The crucial assumption in Section 4 is that **(A4)** the dynamics of the photon field in perturbation theory is constructed out of free local field operators in the interaction picture. This makes the additional requirements of **(A5)** a Fock space structure for multi-particle states, **(A6)** a Poincaré invariant vacuum state, and **(A7)** a local, translation invariant, linear combination of creation and annihilation operators for free massless helicity $h = \pm 1$ particles (and nothing more) rather natural. Except for the requirement of homogeneous Lorentz covariance, all the ingredients for a perturbative relativistic local QFT of physical photons are in place at this stage.

Section 5 brings us as close as possible to this last requirement by assuming that **(A8)** the photon field transforms covariantly under the subset of Lorentz transformations involving only a little group rotation. This partial constraint of Lorentz covariance offers a natural way to fix the form of the polarisation vectors (since photon states are unaffected by little group boosts) and is, in fact, the only way to obtain a covariant kinetic term for the photon field in this setting. Note that the little group decomposition of Λ for arbitrary p^μ is essential at this stage. By contrast, the choice of k^μ and $L(p)$ still does not play any role here.

Finally, Section 5 also assumes that **(A9)** the class of models $\mathcal{L}$ considered involve no higher-order derivative photon field variables $\partial_{\mu_1}...\partial_{\mu_i}A_\nu$ ($i \geq 2$) and couples $\partial_\mu A_\nu$ (if any) only to some antisymmetric tensor. This restriction is rather mild (e.g., higher-order interaction terms like $(F^{\mu\nu}F_{\mu\nu})^2$ are still allowed) but required for a strict infinitesimal derivation of the restricted gauge principle for the pure photon field dynamics.

The extension of the argument to QED and more complicated gauge theories requires both additional assumptions and weaker constraints of Lorentz invariance. Consider first the case of QED. As we have just seen, the constraint of Lorentz invariance at the level of the Lagrangian density $\mathcal{L}$ for the photon field is very stringent. In fact, this constraint is so stringent that it forbids any interaction term between A^μ and some external covariant current J_μ. The infinitesimal variation $\delta\mathcal{L} = \partial^\mu \lambda_\Lambda J_\mu$ indeed vanishes only if $J_\mu = 0$. We are thus forced to take a step back in more realistic models and require only the action $S = \int d^4\mathcal{L}$ to remain invariant under arbitrary Lorentz transformations. Yet this weaker constraint is still remarkably stringent. If we use again (A9) and assume that **(A10)** the boundary conditions of the system are trivial, the constraint of Lorentz invariance at the level of the action implies $\partial_\mu(\delta\mathcal{L}/\delta A_\mu) = 0$ (after integrating $\partial_\mu\lambda_\Lambda(\delta\mathcal{L}/\delta A_\mu)$ by parts and ignoring the total derivative term). Hence, in the simple case of QED and more generally for any model involving only linear couplings in A^μ, this constraint implies that A^μ may couple only to a *conserved* covariant external current J_μ.[18]

The discussion so far tells us nothing about the exact form of J_μ or the matter field content of the model under consideration. Although this is far from being a proof, there seem to be only two non-trivial cases. (a) A^μ couples to the gradient of some antisymmetric tensor $H^{\mu\nu}$: $\delta\mathcal{L}/\delta A_\mu = \partial_\nu H^{\mu\nu}$. We automatically obtain $\partial_\mu(\delta\mathcal{L}/\delta A_\mu) = 0$ in this case. But the resulting models are empirically equivalent to models involving only $F^{\mu\nu}$-couplings in this setting (since we may integrate again by parts in the action and rewrite $\partial_\nu A_\mu H^{\mu\nu}$ as $F_{\mu\nu}H^{\mu\nu}$). (b) A^μ couples to the currents associated with the global symmetries of $\mathcal{L}$, which enables us to consider elementary interaction terms between the photon field and matter fields as in QED.

The good news is that Lorentz invariance still imposes remarkably stringent constraints on the dynamics in case (b). Note first that it is non-trivial to find matter fields and global symmetries such that A^μ linearly couples to the corresponding currents J_μ. If we restrict ourselves to massive spinor fields $\psi_i = \psi, \bar{\psi}$ as in Section 2 and consider infinitesimal global variations $\psi_i \to \psi_i + \Delta\psi_i\lambda$, with λ some constant parameter, this amounts to finding a Lagrangian density $\mathcal{L}$ that satisfies the condition:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m_e)\psi + gA^\mu\frac{\delta\mathcal{L}}{\delta\partial^\mu\psi_i}\Delta\psi_i, \tag{6.1}$$

using here some arbitrary coupling parameter g and the standard expression for the current J_μ arising from global infinitesimal transformations. One may verify that this condition is highly restrictive by testing it with different interaction terms.

It is also possible to prove that $\mathcal{L}$ satisfies the extended gauge principle. Since (a)-(b) is not a strict consequence of Lorentz invariance, we need to assume that **(A11)** $\mathcal{L}$ contains matter fields, exhibits global symmetries, and involves only linear interaction terms between A_μ and the corresponding currents, i.e., $gA^\mu J_\mu$. But I should emphasise right out here that this assumption is forced upon us by Lorentz invariance if we want to consider non-trivial models with matter fields like QED. Consider then the variation of $\mathcal{L}$ under infinitesimal local gauge and phase transformations $A_\mu \to A_\mu + g'\partial_\mu\lambda(x)\mathbb{1}$

[18]Note that in his original treatment, Weinberg ([1964a], [1965b]) relies on the invariance of perturbative S-matrix elements under arbitrary Lorentz transformations to determine the form of interaction terms. Since the interaction parts of the Hamiltonian density $\mathcal{H}$ and Lagrangian density $\mathcal{L}$ are the same in the interaction picture, the constraint of Lorentz invariance at this level also implies that the covariant currents to which the photon field may couple are conserved (Weinberg [1965b], p. B994).

and $\psi_i \to \psi_i + \Delta\psi_i\lambda(x)$, with g' some constant:

$$\begin{aligned}\delta\mathcal{L}_{\text{gauge}} =& [\frac{\delta\mathcal{L}}{\delta\psi_i}\Delta\psi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi_i}\Delta\partial_\mu\psi_i]\lambda(x)\\ &+[g'\frac{\delta\mathcal{L}}{\delta A_\mu} + \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi_i}\Delta\psi_i]\partial_\mu\lambda(x)\\ &+g'\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\partial_\mu\partial_\nu\lambda(x).\end{aligned} \tag{6.2}$$

The global symmetries of $\mathcal{L}$ make the first line in Equation 6.2 vanish. (A9) makes the third line vanish. The second line vanishes for $g' = -1/g$. In a word, (A1)-(A10) and the Lorentz invariance of S naturally lead us to (A11), which, together with the other assumptions, ensures that $\mathcal{L}$ satisfies the extended gauge principle with $g' = -1/g$.

Two comments are in order. First, the argument does not require the photon field and matter fields to be minimally coupled. We may perfectly include Lorentz-invariant interaction terms like $(F^{\mu\nu}F_{\mu\nu})^2$, $\bar{\psi}F^{\mu\nu}\sigma_{\mu\nu}\psi$, or $\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi$. Second, we do not need to assume that the equations of motion for the matter fields are satisfied. In fact, this follows from $\partial_\mu J^\mu = 0$ and the invariance of S under global symmetries. The conservation law $\partial_\mu J^\mu = 0$, in turn, follows from the Lorentz invariance of S and (A10).

What about more complicated gauge theories? I have again good news here: the argument works for our best quantum gauge field theories for the weaker constraint of infinitesimal Lorentz invariance. The action of gauge theories involving non-linear interaction terms in the gauge field variables indeed fails to be invariant under Lorentz transformations with non-infinitesimal little group parameters. One may verify this with the complex scalar QED model by computing explicitly $\Delta S = \int d^4x[\mathcal{L}(A_\mu + \partial_\mu\lambda_\Lambda) - \mathcal{L}(A_\mu)]$ (see Itzykson and Zuber [1980], pp. 30–31, for the expression of $\mathcal{L}$). We are thus forced to retreat to infinitesimal Lorentz transformations. But I should emphasise again that this restriction does not affect the thrust of the argument: in more complicated gauge theories, combining infinitesimal Lorentz invariance at the level of the action with other natural assumptions provides sufficiently strong constraints to single out locally gauge invariant models and thus account for their structure of invariance.

Consider, for instance, the case of pure quantum chromodynamics (QCD).[19] Gluons are usually assumed to be massless particles with helicity $h = \pm 1$ and, as such, to behave exactly like photons under arbitrary Lorentz transformations. We may thus use the analogue of the electromagnetic tensor field to construct a kinetic term $F^{\mu\nu a}F^a_{\mu\nu}$, with $F^{\mu\nu a} = \partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}$, $A^{\mu a}$ the gluon fields, and $a = 1, ..., N$ some unspecified labels for now ($N = 8$ for QCD). Then, if we consider Lorentz transformations with infinitesimal little group parameters, the constraint of Lorentz invariance at the level of the action

[19] See also Weinberg ([1965b]) for the application of the constraint of Lorentz invariance at the level of the S-matrix for the linearized version of general relativity, understood as a relativistic QFT of massless helicity $h = \pm 2$ particles.

implies:

$$\int d^4x \Big[\frac{\delta \mathcal{L}}{\delta A^a_\mu} \partial_\mu \lambda^a_\Lambda + \frac{\delta \mathcal{L}}{\delta(\partial_\mu A^a_\nu)} \partial_\mu \partial_\nu \lambda^a_\Lambda \Big] = 0, \tag{6.3}$$

ignoring again higher-order derivative variables and using the same notation as before (λ^a_Λ involves an integral over the creation and annihilation operators of the gluon field A^a_μ). Together with (A9)-(A10) for gluon fields, Equation 6.3 implies that A^a_μ may couple only to conserved currents $\delta\mathcal{L}/\delta A^a_\mu$.

Again, if we want to understand how Lorentz invariance constrains realistic models, we are naturally led to consider models with global symmetries. Suppose then that **(B11)** the gauge fields A^a_μ couple exclusively to the currents associated with the global gauge symmetries of $\mathcal{L}$. Global gauge invariance forces us to take A^a_μ and $\delta\mathcal{L}/\delta A^a_\mu$ to transform according to the same representation of the global gauge group G, i.e., the adjoint representation of SU(3) for pure QCD. One may verify again that (B11) imposes remarkably stringent constraints on the dynamics (as in Equation 6.1). It is also possible to prove that the dynamics is locally gauge invariant by considering infinitesimal gauge transformations and thus recover the restricted gauge principle in the non-abelian case (as in Equation 6.2).

As a final remark, the argument also extends to on-shell perturbative S-matrix elements beyond Lagrangian (or Hamiltonian) field-theoretic models. Consider, for instance, the absorption of a soft photon of momentum p^μ in a given scattering process. In the perturbative setting, the full S-matrix element for this process takes the form $\epsilon^\mu_h(p)M_\mu$, with $\epsilon^\mu_h(p)$ the polarisation vectors of the incoming photon (as specified above) and M_μ the Lorentz covariant amplitude for the rest of the process without incoming photons (see Weinberg [1964d], [1995], chap. 13, for more detail). Then, if we require this S-matrix element to be Lorentz invariant and use the transformation rule for $\epsilon^\mu_h(p)$ in Equation 5.1, we obtain a more mundane yet essential type of local gauge invariance for free: the Ward identity at the level of on-shell amplitudes, i.e., $p^\mu M_\mu = 0$. And we may speak in this case of 'mass-shell gauge invariance' (Weinberg [1964d], p. 1049) or 'weak gauge invariance' (Weinberg [1964a], p. 358, f. 5).[20]

[20] There is a small caveat related to the use of standard covariant perturbative methods when the demand of Lorentz invariance is imposed at the level of the S-matrix. If we define the free photon field propagator off-shell, we are forced to include a non-covariant Coulomb interaction term in the Hamiltonian density to cancel problematic non-covariant parts of the propagator (Weinberg [1965b], p. B989). One might think that this undermines any strict demand of Lorentz invariance at the level of the action. We have to keep in mind, however, that the issue arises only if we use off-shell quantities. And, as such, showing how Lorentz invariance at the level of the action constrains the dynamics is independent of how it is employed to compute perturbative S-matrix elements.

7 Conclusion

> You could have fun faking Feynman's mannerism and accent, saying, 'Aw shucks, all that fancy talk about little groups! Who needs it? Those experimentalists won't ever be able to prove that the photon mass is mathematically zero anyway'. (Zee [2010], p. 188)

We have seen that combining Lorentz invariance with other natural assumptions is sufficient to make the standard perturbative field-theoretic descriptions of massless discrete helicity particles locally gauge invariant. But as Zee felicitously puts it, if photons, gluons, and gravitons had an exceedingly small non-zero mass, we would not have to worry about their peculiar Lorentz transformation properties and local gauge invariance in the first place. We would not have any trouble with special relativity either (apart from changing the name of the natural constant c). And so one might wonder at this point whether the argument is really significant.

I would like to conclude by highlighting its conceptual payoff. First and foremost, the discussion so far provides non-circular and principled answers to three fundamental questions at the foundations of relativistic quantum gauge field theories.

(i) To the question 'Why local gauge invariance?' (or, equivalently, 'Why local gauge symmetry?'): it is a consequence of a principle of Lorentz invariance and other natural assumptions applied either (a) directly at the level of the dynamics in the pure photon field case, (b) at the level of the action in all other realistic field-theoretic cases, or (c) at the level of scattering amplitudes for on-shell processes.

(ii) To the related question 'Why gauge freedom?': it is an accidental feature of an already highly constrained dynamics and set of S-matrix elements. We may accordingly speak of local gauge symmetry as an accidental symmetry within this setting, in close analogy with baryon and lepton number symmetries in the Standard Model of particle physics for instance.

(iii) To the somewhat even more fundamental question 'Why relativistic quantum gauge field theories?': we ultimately owe their existence to the peculiar space-time transformation properties of massless particles like photons, gluons, and gravitons. If every known particle were massive, the demand of Lorentz invariance would indeed not lead us to consider perturbative relativistic QFTs with a locally gauge invariance structure. We only need such theories to account for a rather atypical subset of relativistic quantum particles. And in the same way as the structure of classical gauge field theories guides the construction of quantum gauge field theories, the structure of perturbative relativistic QFTs for massless discrete helicity particles provides a rationale for working with locally gauge invariant relativistic QFTs in the non-perturbative setting.

Second, the argument has important implications for popular interpretative assumptions and strategies in the philosophy of gauge theories. To begin with, we need to qualify the assumption that gauge freedom is ultimately about variable redundancy, which, as Gomes *et al.* ([2022]) rightly emphasise in my view, is one of the most central interpretative assumptions held among philosophers engaging with gauge theories. Strictly speaking, the photon field in Section 4 has only two physical internal degrees of freedom at each space-time point, that is, two independent polarisation vectors for each p^{μ}.[21] Performing arbitrary Lorentz transformations on the photon field does not 'add' new degrees of freedom to it; it merely reveals the peculiar space-time transformation rule of its polarisation vectors. By contrast, once the dynamics is constrained by Lorentz invariance, we are free to redefine the photon field up to the derivative of an arbitrary scalar operator and thus 'add' new internal degrees of freedom to the field variable. But there is no reason to take this accidental type of variable redundancy to play any significant interpretative role insofar as the unitary evolution of photon states, the dynamical structure of the photon field, and the physical meaning of both are already unambiguously determined. This, in turn, suggests that the philosophical debate about how one should go on with constraining the dynamical variables of a gauge theory somewhat loses its bite. This also suggests that the set of attempts to provide a gauge-invariant interpretation of our best theories are somewhat less interpretatively perspicuous than it is usually assumed.

Likewise, we need to be careful about attributing too much significance to local gauge transformations and local gauge variables (see Redhead [2003]; Brading and Brown [2003]; Rickles [2008], chap. 3). What counts as a relevant local transformation, gauge variable, and gauge-invariant quantity is highly sensitive to the dynamics we pick in the first place (and more generally to the set of quantities we are interested in). To take the most straightforward case, if we were to add a mass term in the dynamics of the photon field, what counts as a traditionally gauge-variant observable would be naturally regarded as 'gauge-invariant' in this setting. In the same vein, the specific types of number-valued local phase transformations usually considered for matter fields are somewhat arbitrary; the dynamics could allow for much more complicated ones, including space-time-dependent operator-valued phase transformations in the quantum case. In a word, the discussion so far suggests that the most philosophically significant aspect of gauge theories lies in their highly constrained dynamics. And if we keep an open mind about what their dynamics might look like, as when we include mass or higher-order interaction terms, it does not seem that we should attribute excessive interpretative weight to gauge symmetries and gauge redundancies.

Let me end by giving one philosophically significant example to make these interpretative lessons more concrete. The accidental character of local gauge symmetries in this setting suggests that they do not have any physical or empirical significance,

[21] Note that the argument above did not require us to get rid of redundant degrees of freedom in some arbitrary fashion. In particular, when we pick the trivial representation ($b_1, b_2 = 0$) for massless particles, we are not eliminating anything redundant in the formalism. On the other hand, the partial constraint of Lorentz covariance imposed on $\epsilon^{\mu}_{h}(k)$ and $\epsilon^{\mu}_{h}(p)$ in Section 5 is forced upon us if we want to absorb helicity phase factors and define a covariant electromagnetic tensor field.

not even of an indirect kind. For a start, the photon case shows that they do not relate physical particle states to each other and cannot be implemented as unitary transformations, which suggests that the label ‘gauge symmetry’ is somewhat of a misnomer in the quantum case. Local gauge transformations only apply at the field-theoretic level in the discussion above. Yet they do not play any role in constraining the dynamics or ensuring that the relevant currents (and charges) are conserved in realistic cases (e.g., global internal symmetries and space-time symmetries are enough in QED). This suggests, in turn, that local gauge transformations do not even have any indirect physical or empirical bearing.

All of this, of course, remains only one perspective on gauge theories within the context of perturbative relativistic quantum field theory. More work is required to understand the deep yet intricate link between Lorentz invariance and local gauge invariance beyond this context.

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