ESAIM: COCV 23 (2017) 1293–1329
DOI: 10.1051/cocv/2016053

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

DISCONTINUOUS SWEEPING PROCESS WITH PROX-REGULAR SETS

Samir Adly[1], Florent Nacry[1] and Lionel Thibault[2,3]

Abstract. In this paper, we study the well−posedness (in the sense of existence and uniqueness of a solution) of a discontinuous sweeping process involving prox-regular sets in Hilbert spaces. The variation of the moving set is controlled by a positive Radon measure and the perturbation is assumed to satisfy a Lipschitz property. The existence of a solution with bounded variation is achieved thanks to the Moreau's catching-up algorithm adapted to this kind of problem. Various properties and estimates of jumps of the solution are also provided. We give sufficient conditions to ensure the uniform prox-regularity when the moving set is described by inequality constraints. As an application, we consider a nonlinear differential complementarity system which is a combination of an ordinary differential equation with a nonlinear complementarily condition. Such problems appear in many areas such as nonsmooth mechanics, nonregular electrical circuits and control systems.

Mathematics Subject Classification. 49J52, 49J53, 34A60.

Received October 6, 2015. Accepted July 7, 2016.

1. Introduction

The paper is devoted to the study of discontinuous sweeping processes through Mordukhovich limiting normal cone to nonconvex prox-regular sets. The notion of sweeping process founds its roots back to the seminal works of Jean Jacques Moreau in the seventies. Jean Jacques Moreau wrote more than 25 papers devoted to the treatment of both theoretical and numerical aspects of the sweeping process as well as its applications in unilateral mechanics. It was first considered for modeling the quasi-static evolution of elastoplastic systems. The sweeping process consists in finding a trajectory $t \in [0,T] \mapsto u(t) \in C(t)$ satisfying the following generalized Cauchy problem

$$\begin{cases} \dot{u}(t) \in -N(C(t); u(t)) \text{ a.e. } t \in [0,T] \\ u(0) = u_0 \in C(0), \end{cases} \tag{1.1}$$

where $N(C(t); u(t))$ is the (outward) *normal cone* to the moving convex and closed set $C(t)$ at the point $u(t)$ in the sense of Convex Analysis. This is a wonderful class of evolution problems subject to unilateral constraints. In order to give an idea to the reader, let us consider a mechanical system with a finite number of degrees of freedom

Keywords and phrases. Variational analysis, measure differential inclusions, sweeping process, prox-regular set, B.V. solutions, Moreau's catching-up algorithm, nonlinear differential complementarity systems.

[1] Laboratoire XLIM, Université de Limoges, 123 Avenue Albert Thomas, 87060 Limoges, cedex, France.
`samir.adly@unilim.fr & florent.nacry@unilim.fr`

[2] Département de Mathématiques, Université Montpellier, 34095 Montpellier, cedex 5, France.
`lionel.thibault@univ-montp2.fr`

[3] Centro de Modelamiento Matematico, Universidad de Chile, Santiago, Chile.

Article published by EDP Sciences

$n \geq 1$. Let $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ be the local coordinate in the manifold of the possible positions. The motion of the system is described with the dependency of the position $u(t)$ with respect to time $t \in [0, T]$. The velocity of the system is given by the derivative $\dot{u}(t) = (\dot{u}_1(t), \ldots, \dot{u}_n(t))$ if it exists. Let us assume that the system is submitted to some unilateral constraints expressed geometrically by the following set of inequalities

$$C_0(t) = \{x \in \mathbb{R}^n \;:\; g_1(t, x) \leq 0, \ldots, g_m(t, x) \leq 0\}, \tag{1.2}$$

where each function $g_k \;: [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is supposed to be of class C^1. The gradient

$$\nabla g_k(t, x) := \nabla g_k(t, \cdot)(x) = \left(\frac{\partial g_k(t, \cdot)}{\partial x_1}(x), \ldots, \frac{\partial g_k(t, \cdot)}{\partial x_n}(x) \right)$$

is supposed to be different from zero (or at least in a neighborhood of the corresponding hypersurface $g_k(t, \cdot) = 0$). The subset $C_0(t)$ is called the moving feasible region. For simplicity of the expository, let us start with a single inequality, *i.e.*, $m = 1$. The general case of m inequality constraints will be considered at the end of the paper. In this case, the moving point $u(t)$ is required to be in the feasible region

$$C_0(t) = \{x \in \mathbb{R}^n \;:\; g(t, x) \leq 0\},\ t \in [0, T].$$

Let $t \in [0, T]$ be an instant such that the right-side velocity $v^+(t) := \dot{u}^+(t)$ exists. The right-derivative of the following scalar function $\tau \mapsto \phi(\tau) = g(\tau, u(\tau))$ at $\tau = t$ is given by

$$\phi'^+(t) = \frac{\partial g}{\partial t}(t, u(t)) + \langle v^+(t), \nabla g(t, u(t)) \rangle.$$

If $g(t, u(t)) = 0$, then it is easy to see that

$$\frac{\partial g}{\partial t}(t, u(t)) + \langle v^+(t), \nabla g(t, u(t)) \rangle = \phi'^+(t) \leq 0. \tag{1.3}$$

This leads Moreau to introduce the following set-valued mapping $\Gamma : [0, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\Gamma(t, x) = \begin{cases} \mathbb{R}^n & \text{if} \quad g(t, x) < 0 \\ \left\{ v \in \mathbb{R}^n \;:\; \frac{\partial g}{\partial t}(t, x) + \langle v, \nabla g(t, x) \rangle \leq 0 \right\} & \text{if} \quad g(t, x) \geq 0. \end{cases}$$

Using the definition of Γ, the observation (1.3) means

$$\dot{u}^+(t) \in \Gamma(t, u(t)). \tag{1.4}$$

Moreau proved a viability lemma showing that if the function $t \mapsto u(t)$ is absolutely continuous on $[0, T]$ and if $\dot{u}(t) \in \Gamma(t, u(t))$ holds for a.e. $t \in [0, T]$, then if the inequality $g(t, u(t)) \leq 0$ is satisfied at the initial instant t_0, it is satisfied for every subsequent t.

With the lazy selector $m(t, x)$ of $\Gamma(t, x)$ defined as its element of minimal norm, that is, $m(t, x) = \operatorname{proj}_{\Gamma(t,x)}(0)$, let us consider the ordinary differential equation

$$\dot{u}(t) = m(t, u(t)). \tag{1.5}$$

Moreau showed that the solution of (1.5), called the lazy solution of the differential inclusion associated with $\Gamma(\cdot, \cdot)$, is exactly the solution of the following sweeping process

$$\dot{u}(t) \in -N(C_0(t); u(t)) \text{ a.e. } t \in [0, T]. \tag{1.6}$$

This means that if $g(t, u(t)) < 0$, then $u(t)$ is in the interior of $C_0(t)$ and the normal cone is reduced to zero. If $g(t, u(t)) = 0$, then there exists a Lagrange multiplier $\ell(t) \geq 0$ such that

$$\dot{u}(t) = -\ell(t) \nabla g(t, u(t)), \tag{1.7}$$

with the following complementarity conditions

$$0 \leq \ell(t) \perp g(t, u(t)) \leq 0. \tag{1.8}$$

The lazy solution (sometimes called the slow solution) possesses some crucial properties and plays an important role in economics for the study of resource allocation mechanisms (see, *e.g.*, [10, 13, 15]) as well as in mechanics (see, *e.g.*, [26]). Let us notice that in the case when the moving set is described by an inequality constraint, the sweeping process (1.6) is connected with the steepest descent method (1.7) as well as complementarity conditions (1.8).

Translating inclusion (1.6) to a mechanical language, we obtain the following interpretation:

- if the position $u(t)$ of a particule lies in the interior of the moving set $C_0(t)$, then $\dot{u}(t) = 0$, which means that the particule remains at rest;
- when the boundary of $C_0(t)$ catches up the particle, then this latter is pushed in an inward normal direction by the boundary of $C_0(t)$ to stay inside $C_0(t)$ and satisfies the constraint. This mechanical visualization leads Moreau to call this problem the *sweeping process*: the particule is swept by the moving set.

For the general case where the feasible set $C_0(t)$ is defined by m inequality constraints (1.2), if some qualification condition is satisfied on the constraints $g_k(t, \cdot)$, then the sweeping process (1.6) is equivalent to the following Differential Complementarity System:

$$\begin{cases} \dot{u}(t) = -\displaystyle\sum_{k=1}^{m} \ell_k(t) \nabla g_k(t, u(t)) \;\text{ a.e. } t \in [0, T] \\ 0 \leq \ell_k(t) \perp g_k(t, u(t)) \leq 0, \; k = 1, 2, \ldots, m, \; t \in [0, T]. \end{cases}$$

Moreau studied, in a Hilbert space, the sweeping process under the convexity of the moving closed set $C(t)$ in the absolutely continuous situation as well as when merely bounded variation property is satisfied. The convexity of the moving set $C(t)$ is equivalent to the monotonicity of the normal cone $N(C(t); \cdot)$, which ensures that if $u_1(\cdot)$ and $u_2(\cdot)$ are two solutions of the sweeping process (1.1), then the function $t \mapsto \|u_1(t) - u_2(t)\|$ is nonincreasing. Hence, the sweeping process (1.1) with initial condition u_0 possesses at most one solution $u(\cdot)$ satisfying $u(0) = u_0$.

The existence of at least one solution of problem (1.1) can be performed by the so-called "catching-up algorithm". Let us fix $k \in \mathbb{N}$ and choose a time discretization

$$0 = t_0^k < t_1^k < \ldots < t_{N-1}^k < t_N^k = T, \text{ with } t_{i+1}^k - t_i^k \leq \frac{1}{k}, \; 0 \leq i \leq N-1.$$

Using an *implicit Euler* discretization for problem (1.1) and the fact that $[I + N(C(t); \cdot)]^{-1} = \operatorname{proj}_{C(t)}(\cdot)$ for all $t \in [0, T]$, we get

$$u_0^k = u_0, \;\; u_{i+1}^k = \operatorname{proj}_{C(t_{i+1}^k)}(u_i^k), \; i = 0, 1, \ldots, N-1. \tag{1.9}$$

Using a linear interpolation, it is possible to construct a sequence of mappings $t \mapsto u_k(t)$, which contains a subsequence converging to some $u(\cdot)$ satisfying (1.1) for a.e. $t \in [0, T]$. The key assumption for the proof is the control of the moving set $C(t)$ which is allowed to change shape with respect to time. If the set $C(t)$ moves in a Lipschitz continuous way with respect to the Hausdorff distance, then there exists a unique absolutely continuous solution to problem (1.1).

There are some situations in mechanical systems where discontinuous motions of the moving set $C(t)$ occurs. More precisely, the set-valued mapping $t \mapsto C(t)$ is only assumed to have a bounded variation with respect to the

Hausdorff distance. Taking into account the possible jumps, Moreau transformed the model (1.1) to a measure differential inclusion and proves the following existence and uniqueness result:
Assume that the sets $C(t)$ of a Hilbert space $\mathcal{H}$ are nonempty closed convex sets for which there is a positive Radon measure μ on $[0,T]$ such that, for each $y \in \mathcal{H}$,

$$d(y, C(t)) \leq d(y, C(s)) + \mu(]s,t]), \quad \text{for all } 0 \leq s \leq t \leq T.$$

Then, the measure differential evolution inclusion

$$\begin{cases} \mathrm{d}u \in -N(C(t); u(t)) \\ u(0) = u_0 \in C(0) \end{cases} \tag{1.10}$$

admits one and only one right continuous solution with bounded variation.

The convexity assumption of the moving set $C(t)$ can be too restrictive in some applications. This is the case for example when the set $C(t)$ is given by (1.2) and at least one of the sublevel sets $\{g_k(t,\cdot) \leq 0\}$ is nonconvex. In [32], Valadier studied (1.1), in the absolutely continuous framework, with nonconvex set fulfilling a regularity property for the normal cone; the main case is the complement $C(t)$ of the interior of a closed convex set $K(t)$ of the space $\mathbb{R}^n$, *i.e.*, $C(t) = \mathbb{R}^n \setminus \mathrm{Int}\,(K(t))$. This can model a material point moving outside a moving convex set $K(t)$ being pushed outwards normal direction when it is caught up by the boundary of $K(t)$. Valadier obtained an existence result for problem (1.1), which can be considered as one of the first results in the nonconvex setting. The normal cone in this case is in the sense of Clarke. We can also cite the works of Benabdellah [2] and Colombo and Goncharov [9] where an existence result of (1.1) with general nonconvex subsets $C(t) \subset \mathbb{R}^n$ was given.

Usually in mechanical systems, external forces are applied, which leads to consider the perturbed version of problem (1.10)

$$\mathrm{d}u \in -N(u(t); C(t)) - F(t, u(t)), \tag{1.11}$$

where F is a set-valued mapping from $[0,T] \times \mathbb{R}^n$ into weakly compact convex sets. Castaing and Monteiro Marques extended the result of Valadier to the perturbed problem (1.11). It seems that the setting of prox-regular subset is well appropriate for handling nonconvex sweeping process in general Hilbert spaces. The prox-regularity assumption of the sets $C(t)$ allows the use of Moreau's catching-up algorithm in choosing u_i^k in (1.9) sufficiently close to the boundary of $C(t_{i+1}^k)$. This was considered, in the absolutely continuous case, by Colombo and Goncharov [9] with $F = 0$ and by Bounkhel and Thibault [4] with general set-valued mappings F. Edmond and Thibault in [12] considered the case when the sets $C(t)$ are prox-regular and move with a bounded variation in infinite dimensional Hilbert space $\mathcal{H}$, but under the compact growth condition

$$F(t,x) \subset \beta(t)(1 + \|x\|)K, \tag{1.12}$$

where K is a compact subset of $\mathcal{H}$.

Recently, Maury and Venel [18] used the perturbed sweeping process involving prox-regular sets for the modeling of crowd motion in the case of emergency evacuation.

The existence of the compact set K in the growth condition (1.12) can be too restrictive in some situations in an infinite dimensional Hilbert space. So, our aim in this paper is to prove, for a Hilbert space $\mathcal{H}$, the existence and uniqueness of solution for the measure differential inclusion

$$\mathrm{d}u \in -N(u(t); C(t)) - f(t, u(t)), \tag{1.13}$$

where $f(t,\cdot) : \mathcal{H} \to \mathcal{H}$ is a mapping which is Lipschitz on bounded sets and $C(t)$ is a prox-regular set of the Hibert space $\mathcal{H}$ which has a bounded variation with respect to t.

In nonsmooth mechanics, the feasible region is usually expressed in the form of finite intersection of m inequalities (1.2). Traditionally each inequality corresponds to the so-called unilateral constraint. As we just mentioned above, the prox-regularity of the set $C(t)$ plays an important role in the existence and uniqueness proofs. One remaining question subsists: under which conditions on the functions $g_k : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots, m$, the set $C_0(t)$ in (1.2) is prox-regular? The answer to this question is discussed in Section 9 with $g_k : [0,T] \times \mathcal{H} \to \mathbb{R}$, where $\mathcal{H}$ is a Hilbert space.

The paper is organized as follows:

In Sections 2 and 3, we introduce notations and recall some important notions which will be used through the paper. The next section is devoted to the concept of solution for the measure differential inclusion (1.13). In Sections 5−7 we study the existence and the uniqueness of solution for (1.13). Then, we recover the classical case, *i.e.*, the absolutely continuous case. In Section 9, we give sufficient conditions to ensure the uniform prox-regularity of a moving set described by inequality constraints. Then, we provide an application of our results to the theory of nonlinear differential complementarity systems.

2. Notation and preliminaries

In this section, we recall the backgrounds and preliminaries that will be useful for the rest of the paper. Throughout, $\mathbb{N}$ is the set of positive integers $n = 1, \ldots$ For I a nonempty interval of $\mathbb{R}$, λ stands for the Lebesgue measure. In all the paper, $\mathcal{H}$ is a *real Hilbert space* endowed with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$; its closed unit ball centered at zero will be denoted by $\mathbb{B}$. For any subset S of $\mathcal{H}$, $\overline{\mathrm{co}}\, S$ stands for the closed convex hull of S and d_S is the *distance function from* S, *i.e.*,

$$d_S(x) := \inf_{s \in S} \|x - s\| \quad \text{for all } x \in \mathcal{H}.$$

For a set $A \subset \mathbb{R}$, the notation $\mathbf{1}_A$ stands for the *characteristic function* in the sense of measure theory, *i.e.*, for all $x \in \mathbb{R}$, $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ otherwise.

2.1. Support function

For any subset S of the real Hilbert space $\mathcal{H}$, its *support function* $\sigma(\cdot, S)$ is defined by

$$\sigma(\zeta, S) := \sup_{x \in S} \langle \zeta, x \rangle \quad \text{for all } \zeta \in \mathcal{H}.$$

It is well-known that, for any two closed convex subsets S_1, S_2 of $\mathcal{H}$, one has

$$S_1 \subset S_2 \iff \sigma(\cdot, S_1) \leq \sigma(\cdot, S_2). \tag{2.1}$$

2.2. Normal cone, subdifferential

In this subsection, S is a nonempty susbset of the real Hilbert space $\mathcal{H}$.

The *Clarke tangent cone* of S at $x \in S$, denoted by $T^C(S; x)$, is the set of $h \in \mathcal{H}$ such that, for every sequence $(x_n)_{n\in\mathbb{N}}$ of S with $x_n \underset{n\to+\infty}{\longrightarrow} x$ and for every sequence $(t_n)_{n\in\mathbb{N}}$ of positive reals with $t_n \underset{n\to+\infty}{\longrightarrow} 0$, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ with $h_n \underset{n\to+\infty}{\longrightarrow} h$ satisfying

$$x_n + t_n h_n \in S \quad \text{for all } n \in \mathbb{N}.$$

This set is obviously a cone containing zero and it is known to be closed and convex. The polar cone of $T^C(S; x)$ is the *Clarke normal cone* $N^C(S; x)$ of S at x, that is,

$$N^C(S; x) := \left\{ \zeta \in \mathcal{H} : \langle \zeta, h \rangle \leq 0, \forall h \in T^C(S; x) \right\}.$$

If $x \notin S$, by convention $T^C(S; x)$ and $N^C(S; x)$ are empty.

Let U be a neighborhood of a point $x \in \mathcal{H}$ and $f : U \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function which is finite at x. The *Clarke subdifferential* $\partial_C f(x)$ of f at x is defined by

$$\partial_C f(x) := \left\{\zeta \in \mathcal{H} : (\zeta, -1) \in N^C\big(\mathrm{epi} f; (x, f(x))\big)\right\},$$

where $\mathcal{H} \times \mathbb{R}$ is endowed with the usual product structure and epi f is the *epigraph* of f, that is,

$$\mathrm{epi}\, f := \{(x', r) \in \mathcal{H} \times \mathbb{R} : x' \in U,\ f(x') \le r\}.$$

If f is not finite at x, we see that $\partial_C f(x) = \emptyset$. In addition to the latter definition, there is another link between the Clarke normal cone and the Clarke subdifferential, given by

$$\partial_C \psi_S(x) = N^C(S; x),$$

where ψ_S denotes the indicator function of the subset S of $\mathcal{H}$, *i.e.*, $\psi_S(x') = 0$ if $x' \in S$ and $\psi_S(x') = +\infty$ otherwise.

When f is $\gamma-$Lipschitz near x for some real $\gamma \ge 0$, one defines its *Clarke directional derivative* at x in the direction $h \in \mathcal{H}$ by

$$f^o(x; h) := \limsup_{t \downarrow 0, x' \to x} t^{-1}\big(f(x' + th) - f(x')\big),$$

where in the whole paper $t \downarrow 0$ means $t \to 0$ with $t > 0$. Recall that under such a Lipschitz hypothesis, one has

$$\partial_C f(x) = \{\zeta \in \mathcal{H} : \forall h \in \mathcal{H},\ \langle \zeta, h\rangle \le f^o(x; h)\},$$

so, as readily seen, $\partial_C f(x) \subset \gamma\mathbb{B}$. In particular, for any $y \in \mathcal{H}$, one has $\partial_C d_S(y) \subset \mathbb{B}$.

For any $x \in \mathcal{H}$, one defines the (possibly empty) set of all nearest points of x in S by

$$\mathrm{Proj}_S(x) := \{y \in S : d_S(x) = \|x - y\|\}.$$

When $\mathrm{Proj}_S(x)$ contains one and only one point $\bar{y}$, we will denote by $P_S(x)$ or $\mathrm{proj}_S(x)$ the unique element, that is, $P_S(x) := \bar{y}$. Since $x' \in \mathrm{Proj}_S(x)$ amounts to writing $x' \in S$ and $\|x - x'\|^2 \le \|x - y\|^2$ for all $y \in S$, it is readily seen that, for all $x \in \mathcal{H}$

$$x' \in \mathrm{Proj}_S(x) \Longleftrightarrow x' \in S \text{ and } \langle x - x', y - x'\rangle \le \frac{1}{2}\|y - x'\|^2 \quad \text{for all } y \in S. \tag{2.2}$$

A vector $\zeta \in \mathcal{H}$ is said to be a *proximal normal* to S at $x \in S$ whenever there exists a real $r > 0$ such that $x \in \mathrm{Proj}_S(x + r\zeta)$. The set $N^P(S; x)$ (which is obviously a cone of $\mathcal{H}$ containing 0) of all proximal normal vectors to S at x is called the *proximal normal cone* of S at x. For $v \in \mathcal{H}$ such that $\mathrm{Proj}_S(v)$ is a singleton, it is straightforward that

$$v - P_S(v) \in N^P(S; P_S(v)). \tag{2.3}$$

Weak limits of proximal normal vectors define the *Mordukhovich limiting normal cone* $N^L(S; x)$, that is, for $\zeta \in \mathcal{H}$, $\zeta \in N^L(S; x)$ if and only if (see [19]) there are sequences $(x_n)_n$ in S convering to x and $(\zeta_n)_n$ weakly converging to ζ with $\zeta_n \in N^P(S; x_n)$ for all $n \in \mathbb{N}$. As for the Clarke normal cone, one puts $N^P(S; x) = N^L(S; x) = \emptyset$ if $x \notin S$, so

$$N^P(S; x) \subset N^L(S; x) \subset N^C(S; x) \quad \text{for all } x \in \mathcal{H}. \tag{2.4}$$

For a neighborhood U of $x \in \mathcal{H}$ and an extended real-valued function $f : U \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ which is finite at x, putting $E_f := \mathrm{epi}\, f$ and $y_x := (x, f(x))$ one defines the *proximal subdifferential* $\partial_P f(x)$ and the *Mordukhovich limiting subdifferential* $\partial_L f(x)$ of f at x by

$$\partial_P f(x) = \left\{\zeta \in \mathcal{H} : (\zeta, -1) \in N^P\big(E_f; y_x\big)\right\},\ \partial_L f(x) = \left\{\zeta \in \mathcal{H} : (\zeta, -1) \in N^L\big(E_f; y_x\big)\right\},$$

so $\partial_P f(x) = \partial_l f(x) = \emptyset$ whenever f is not finite at x. It is clear from (2.4) that

$$\partial_P f(x) \subset \partial_L f(x) \subset \partial_C f(x).$$

It is well-known that, for $\zeta \in \mathcal{H}$, $\zeta \in \partial_P f(x)$ if and only if there is some real $\sigma \geq 0$ and a neighborhood $V \subset U$ of x such that

$$\langle \zeta, u - x \rangle \leq f(u) - f(x) + \sigma \|u - x\|^2 \quad \text{for all } u \in V. \tag{2.5}$$

As for the Clarke subdifferential, one has,

$$\partial_P \psi_S(x) = N^P(S; x) \quad \text{and} \quad \partial_L \psi_S(x) = N^L(S; x) \quad \text{for all } x \in \mathcal{H}.$$

Moreover, if S is closed, the following relations hold true for all $x \in S$ (see, *e.g.*, [4,19]):

$$\partial_P d_S(x) = N^P(S; x) \cap \mathbb{B} \tag{2.6}$$

and

$$\partial_L d_S(x) \subset N^L(S; x) \cap \mathbb{B}, \; \partial_C d_S(x) \subset N^C(S; x) \cap \mathbb{B}. \tag{2.7}$$

For more details, we refer to [8,19].

2.3. Prox-regular sets

In this subsection, we will recall the definition and some properties of prox-regular sets. Let S be a nonempty closed subset of the real Hilbert space $\mathcal{H}$ and $r \in \,]0, +\infty]$. We will use the convention $\frac{1}{r} = 0$ whenever $r = +\infty$.

Definition 2.1. The nonempty closed set S is said to be r-prox-regular (or uniformly prox-regular with constant r) whenever, for all $x \in S$, for all $\zeta \in N^L(S; x) \cap \mathbb{B}$ and for all $t \in]0, r[$, one has $x \in \mathrm{Proj}_S(x + t\zeta)$.

It seems that Federer [14] was the first to consider this class of sets in the finite-dimensional framework. Concerning Theorem 2.2 and Proposition 2.3 below in the context of Hilbert spaces, we refer, for example, to the paper [28] by Poliquin, Rockafellar and Thibault.

Theorem 2.2. *The following assertions are equivalent.*
(a) *The set S is r-prox-regular.*
(b) *For all $x_1, x_2 \in S$, for all $\zeta \in N^L(S; x_1)$, one has*

$$\langle \zeta, x_2 - x_1 \rangle \leq \frac{1}{2r} \|\zeta\| \|x_1 - x_2\|^2 .$$

(c) *For all $x_1, x_2 \in S$, for all $\zeta_1 \in N^L(S; x_1)$, for all $\zeta_2 \in N^L(S; x_2)$, one has*

$$\langle \zeta_1 - \zeta_2, x_1 - x_2 \rangle \geq -\frac{1}{2} \left(\frac{\|\zeta_1\|}{r} + \frac{\|\zeta_2\|}{r} \right) \|x_1 - x_2\|^2 .$$

The following facts given by the next result are fundamental.

Proposition 2.3. *Assume that the set S is r-prox-regular and define $U_r(S) = \{u \in \mathcal{H} : d_S(x) < r\}$. The following assertions hold true.*

(a) *For any $x \in S$, one has*

$$N^P(S; x) = N^L(S; x) = N^C(S; x) \quad \textit{and} \quad \partial_P d_S(x) = \partial_L d_S(x) = \partial_C d_S(x).$$

(b) *For any $x \in U_r(S)$, the set $\mathrm{Proj}_S(x)$ is a singleton, i.e., $P_S(x)$ is well-defined.*

(c) *The well-defined mapping $P_S : U_r(S) \longrightarrow S$ is locally Lipschitz on $U_r(S)$.*

According to (a), whenever S is uniformly prox-regular, we set

$$N(S;x) := N^P(S;x) = N^L(S;x) = N^C(S;x) \quad \text{for all } x \in S.$$

We also need to recall another useful characterization of the uniform prox-regularity.

Proposition 2.4. *Let $s \in]0,+\infty]$ be an extended real. The set S is r-prox-regular if and only for all $x, x' \in S$ with $\|x - x'\| < 2r$ and for all $\zeta \in N^C(S;x)$, $\langle \zeta, x' - x\rangle \leq \frac{1}{2r}\|\zeta\|\,\|x' - x\|^2$.*

2.4. Radon measure

Since we deal with measure differential inclusions, some preliminary results about vector measure theory are necessary. Most of the results can be found in [1,11]. For the convenience of the reader, we recall them in this part of the paper. Throughout this subsection, I denotes a real interval with nonempty interior. For $\varepsilon > 0$ and $t \in I$, we will put $I(t,\varepsilon) := I \cap [t-\varepsilon, t+\varepsilon]$.

Given two positive Radon measures ν_1 and ν_2 on I, we know (see [16]) that the limit (in which we use the convention $\frac{0}{0} = 0$)

$$\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2}(t) := \lim_{\varepsilon \downarrow 0} \frac{\nu_1(I(t,\varepsilon))}{\nu_2(I(t,\varepsilon))} \tag{2.8}$$

exists and is finite for ν_2-almost every $t \in I$. This nonnegative function is the *derivative of the measure ν_1 with respect to ν_2*. When the measure ν_1 is absolutely continuous with respect to ν_2, the function $\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2}(\cdot)$ is a density of ν_1 relative to ν_2, otherwise stated the equality $\nu_1 = \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2}(\cdot)\nu_2$ holds. Under such an absolute continuity assumption, a mapping $u(\cdot) : I \longrightarrow \mathcal{H}$ is ν_1-integrable on a subinterval $J \subset I$ if and only if the mapping $t \longmapsto u(t)\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2}(t)$ is ν_2-integrable on J; furthermore, in that case,

$$\int_J u(t)\mathrm{d}\nu_1(t) = \int_J u(t)\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2}(t)\mathrm{d}\nu_2(t). \tag{2.9}$$

If the two Radon measures ν_1 and ν_2 are each one absolutely continuous with respect to the other one, it will be convenient for us to declare that they are *absolutely continuously equivalent*.

If for some $t \in I$, $\nu_2(\{t\}) > 0$, (keeping in mind that λ denotes the *Lebesgue measure*) the relation (2.8) says that $\frac{\mathrm{d}\lambda}{\mathrm{d}\nu_2}(t) = \frac{\lambda(\{t\})}{\nu_2(\{t\})} = 0$. So, for ν_2-almost every $t \in I$,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\nu_2}(t)\nu_2(\{t\}) = 0. \tag{2.10}$$

2.5. Mapping of locally bounded variation and differential measure

From now on, unless otherwise stated, I denotes a real interval of $\mathbb{R}$ with nonempty interior.

The concept of solution of a measure differential inclusion involves in general mappings of locally bounded variation. The following definition is in this sense.

Let $u : I \longrightarrow \mathcal{H}$ be a mapping from I into $\mathcal{H}$. Let $a, b \in \mathbb{R}$ with $a < b$ and $[a,b] \subset I$. *A subdivision σ of* $[a,b]$ being a finite sequence $(t_0, \ldots, t_k) \in \mathbb{R}^{k+1}$ with $k \in \mathbb{N}$ such that, $a = t_0 < \ldots < t_k = b$, one associates with σ, the real $S_\sigma := \sum_{i=1}^{k} \|u(t_i) - u(t_{i-1})\|$. The *variation of u on* $[a,b]$ is defined as the extended real $\mathrm{var}(u;a,b) := \sup_{\sigma \in \mathcal{S}} S_\sigma$, where $\mathcal{S}$ is the set of all subdivisions of $[a,b]$. The mapping u is said to be of (or, with) *bounded variation on* $[a,b]$ (BV on $[a,b]$, for short) if $\mathrm{var}(u;a,b) < +\infty$. Whenever u is of bounded variation on any compact interval included in I, one says that u is of (or, with) *locally bounded variation on I* (LBV on I, for short). If I is a compact interval of $\mathbb{R}$, it is obvious that, u is BV on I if and only if it is LBV on I.

It is well-known that a mapping $u : I \longrightarrow \mathcal{H}$ of locally bounded variation on I has one sided limits at each point of I. In such a case, one defines $u(\tau^-) := \lim\limits_{t\uparrow\tau} u(t)$, for each $\tau \in I$ which is not the left endpoint of I. For more details about mappings of locally bounded variation, we refer to [11, 22].

Consider a right-continuous mapping $u(\cdot) : I \longrightarrow \mathcal{H}$ with locally bounded variation on I. With this mapping is associated a vector measure $\mathrm{d}u$ on I with values in $\mathcal{H}$ (see Dinculeanu [11] and Moreau [22]) such that, for all $s, t \in I$ with $s \le t$,

$$u(t) = u(s) + \int_{]s,t]} \mathrm{d}u.$$

This measure $\mathrm{d}u$ is called the *differential measure* (or the *Stieltjes measure*) of $u(\cdot)$.

Reciprocally, let ν be a positive Radon measure on I, $u(\cdot) : I \longrightarrow \mathcal{H}$ a mapping and $z(\cdot) \in L^1_{\mathrm{loc}}(I, \mathcal{H}, \nu)$. Given $T_0 \in I$, if for any $t \in I$,

$$u(t) = u(T_0) + \int_{]T_0,t]} z(s)\,\mathrm{d}\nu(s),$$

then $u(\cdot)$ is of locally bounded variation, right continuous on I, and clearly $\mathrm{d}u = z(\cdot)\,\mathrm{d}\nu$. Thus, the mapping $z(\cdot)$ is a *density of the measure* $\mathrm{d}u$ *relative to* ν.

Setting $I^-(t,\varepsilon) = [t-\varepsilon, t] \cap I$ and $I^+(t,\varepsilon) = [t, t+\varepsilon] \cap I$ with $\varepsilon > 0$, for ν-almost every $t \in I$, according to Moreau and Valadier [27] the limits below exist in $\mathcal{H}$ and

$$z(t) = \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) := \lim_{\varepsilon\downarrow 0} \frac{\mathrm{d}u(I(t,\varepsilon))}{\nu(I(t,\varepsilon))} = \lim_{\varepsilon\downarrow 0} \frac{\mathrm{d}u(I^+(t,\varepsilon))}{\nu(I^+(t,\varepsilon))} = \lim_{\varepsilon\downarrow 0} \frac{\mathrm{d}u(I^-(t,\varepsilon))}{\nu(I^-(t,\varepsilon))}\cdot \tag{2.11}$$

From this, it is not difficult to verify that, one also has

$$\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) = \lim_{s\uparrow t} \frac{\mathrm{d}u(]s,t] \cap I)}{\nu(]s,t] \cap I)}\cdot \tag{2.12}$$

Above $L^1_{\mathrm{loc}}(I, \mathcal{H}, \nu)$ denotes the set of (equivalence classes of) mappings Bochner ν-integrable on every compact interval included in I. Similarly, $L^p(I, \mathcal{H}, \nu)$ will stand for the space of (equivalence classes of) measurable mappings $u(\cdot)$ from I into $\mathcal{H}$ with $\|u(\cdot)\|^p$ in the usual space $L^p(I, \mathbb{R}, \nu)$.

3. Preparatory results

This section is devoted to recall some specific results which are fundamental in the rest of the paper. We begin with a Gronwall type lemma, which is due to Monteiro Marques [17].

Lemma 3.1. *Let I be a proper interval of $\mathbb{R}$ with a real $T_0 \in I$ as its left endpoint. Let ν be a positive Radon measure on I, and $g, \varphi : I \longrightarrow [0, +\infty[$ two functions such that:*

(i) *For some fixed $\theta \in [0, +\infty[$, one has, for all $t \in I \setminus \{T_0\}$, $0 \le g(t)\nu(\{t\}) \le \theta < 1$ and $g \in L^1_{\mathrm{loc}}(I, \mathbb{R}, \nu)$ with $g(\cdot) \ge 0$;*

(ii) *for some fixed $\alpha \in [0, +\infty[$, one has, for all $t \in I$, $\varphi(t) \le \alpha + \int_{]T_0,t]} g(s)\varphi(s)\,\mathrm{d}\nu(s)$ and $\varphi \in L^\infty_{\mathrm{loc}}(I, \mathbb{R}, \nu)$ with $\varphi(\cdot) \ge 0$.*

Then, for all $t \in I$,

$$\varphi(t) \le \alpha \exp\left(\frac{1}{1-\theta} \int_{]T_0,t]} g(s)\mathrm{d}\nu(s)\right).$$

The next useful proposition is a consequence of a more general inequality due to Moreau [20, 22].

Proposition 3.2. *Let ν be a positive Radon measure on a proper real interval I and $u(\cdot) : I \longrightarrow \mathcal{H}$ be a mapping from I into the real Hilbert space $\mathcal{H}$ which is right continuous with locally bounded variation and such that the differential measure du has a density $\frac{\mathrm{d}u}{\mathrm{d}\nu}$ relative to ν. Then, the function $\Phi(\cdot) = \|u(\cdot)\|^2 : I \longrightarrow \mathbb{R}$ is a right continuous function of locally bounded variation whose differential measure $\mathrm{d}\Phi$ satisfies, in the sense of the order of real measures,*

$$\mathrm{d}\Phi \leq 2 \left\langle u(\cdot), \frac{\mathrm{d}u}{\mathrm{d}\nu}(\cdot) \right\rangle \mathrm{d}\nu.$$

The following lemma will be useful. We refer to [4] for the proof.

Lemma 3.3. *Let S a subset of the real Hilbert space $\mathcal{H}$ which is r-prox-regular, with $r \in]0, +\infty]$. Let $x \in S$ and $\zeta \in \partial_P d_S(x)$. Then, for all $z \in \mathcal{H}$ such that $d_S(z) < r$, one has*

$$\langle \zeta, z - x \rangle \leq \frac{1}{2r} \|z - x\|^2 + \frac{1}{2r} d_S^2(z) + \left(\frac{1}{r} \|z - x\| + 1 \right) d_S(z),$$

and

$$\langle \zeta, z - x \rangle \leq \frac{2}{r} \|z - x\|^2 + d_S(z).$$

The proof of existence of a solution of our measure differential sweeping process requires the following result. Because of its own interest, the proof with a positive measure μ will be given in a general form.

Proposition 3.4. *Let $C : I \rightrightarrows \mathcal{H}$ be a set-valued mapping from a proper real interval I into the real Hilbert space $\mathcal{H}$ satisfying:*

(i) *For some extended real $r \in]0, +\infty]$, all the sets $C(t)$ are r-prox-regular;*

(ii) *There exists a positive measure μ on I such that, for all $s_1, s_2 \in I$ with $s_1 \leq s_2$, for all $y \in \mathcal{H}$,*

$$d_{C(s_2)}(y) - d_{C(s_1)}(y) \leq \mu(]s_1, s_2]).$$

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of I converging to some $t \in I$ with $t_n \geq t$ for all $n \in \mathbb{N}$, $(x_n)_{n\in\mathbb{N}}$ a sequence of $\mathcal{H}$ converging to some $x \in C(t)$ with $x_n \in C(t_n)$ for all $n \in \mathbb{N}$. If there exists $N \in \mathbb{N}$ with $\mu(]t, t_N]) < +\infty$, then, for any $z \in \mathcal{H}$, one has

$$\limsup_{n\to+\infty} \sigma(z, \partial_P d_{C(t_n)}(x_n)) \leq \sigma(z, \partial_P d_{C(t)}(x)).$$

Proof. Assume there exists $N \in \mathbb{N}$ with $\mu(]t, t_N]) < +\infty$. Let us fix any $z \in \mathcal{H}$. Extracting a subsequence, we may suppose (thanks to (2.6)), without loss of generality, that $(\sigma(z, \partial_P d_{C(t_n)}(x_n))_{n\in\mathbb{N}}$ converges and then

$$\limsup_{n\to+\infty} \sigma(z, \partial_P d_{C(t_n)}(x_n)) = \lim_{n\to+\infty} \sigma(z, \partial_P d_{C(t_n)}(x_n)).$$

For every $n \in \mathbb{N}$, as $C(t_n)$ is r-prox-regular, one has $\partial_P d_{C(t_n)}(x_n) = \partial_C d_{C(t_n)}(x_n)$ and then $\partial_P d_{C(t_n)}(x_n)$ is weakly compact. So, for all $n \in \mathbb{N}$, there exists $\xi_n \in \partial_P d_{C(t_n)}(x_n)$ such that $\sigma(z, \partial_P d_{C(t_n)}(x_n)) = \langle \xi_n, z \rangle$. Since $\|\xi_n\| \leq 1$ for all $n \in \mathbb{N}$, we may suppose, without loss of generality, that $(\xi_n)_{n\in\mathbb{N}}$ converges weakly to some $\xi \in \mathcal{H}$. We are going to prove that $\xi \in \partial_C d_{C(t)}(x)$. Fix any $u \in \mathcal{H}$. As $x_n \in C(t_n)$ for all $n \in \mathbb{N}$, there exists a real $\alpha_0 > 0$ such that for all $\alpha \in]0, \alpha_0[$, one has for all $n \in \mathbb{N}$,

$$d_{C(t_n)}(x_n + \alpha u) \leq \|\alpha u\| < r.$$

Then, for all $\alpha \in]0, \alpha_0[$, one has for all $n \in \mathbb{N}$, *via* Lemma 3.3,

$$\langle \xi_n, \alpha u \rangle \leq \frac{2}{r} \alpha^2 \|u\|^2 + d_{C(t_n)}(x_n + \alpha u). \tag{3.1}$$

Using (ii), one also has for all $\alpha \in]0, \alpha_0]$, for all $n \in \mathbb{N}$,

$$d_{C(t_n)}(x_n + \alpha u) \leq d_{C(t)}(x + \alpha u) + \mu(]t, t_n]) + \|x_n - x\|.$$

Let us show that $\lim_{n\to+\infty} \mu(]t, t_n]) = 0$. Extracting a subsequence if necessary, we may suppose that $(t_n)_{n\in\mathbb{N}}$ is nonincreasing (keep in mind that $t_n \geq t$ for all $n \in \mathbb{N}$). One observes that

$$\lim_{n\to+\infty} \mu(]t, t_n]) = \mu\left(\bigcap_{k\in\mathbb{N}}]t, t_k]\right) = 0.$$

We deduce for all $\alpha \in]0, \alpha_0[$, $\limsup_{n\to+\infty} d_{C(t_n)}(x_n + \alpha u) \leq d_{C(t)}(x + \alpha u)$. Using (3.1), we obtain for all $\alpha \in]0, \alpha_0]$

$$\langle \xi, \alpha u\rangle \leq \frac{2}{r}\alpha^2 \|u\|^2 + d_{C(t)}(x + \alpha u).$$

As a result, since $d_{C(t)}(x) = 0$,

$$\langle \xi, u\rangle \leq \liminf_{\alpha\downarrow 0} \frac{1}{\alpha}\big(d_{C(t)}(x + \alpha u) - d_{C(t)}(x)\big) \leq d^\circ_{C(t)}(x; u).$$

This being true for any $u \in \mathcal{H}$, it results that $\xi \in \partial_C d_{C(t)}(x) = \partial_P d_{C(t)}(x)$. Consequently, one has

$$\lim_{n\to+\infty} \sigma(z, \partial_P d_{C(t_n)}(x_n)) = \lim_{n\to+\infty} \langle \xi_n, z\rangle = \langle \xi, z\rangle \leq \sigma(z, \partial_P d_{C(t)}(x)).$$

The proof is then complete. □

With the normal cone, we have the following property on the nearest points of a uniformly prox-regular set.

Proposition 3.5. *Let S be an r-prox-regular set of the real Hilbert space $\mathcal{H}$ with $r \in]0, +\infty]$, and let $x, x' \in \mathcal{H}$. If $x - x' \in N(S; x')$ and $\|x - x'\| \leq r$ (resp. $\|x - x'\| < r$) then $x' \in \mathrm{Proj}_S(x)$ (resp. $x' = P_S(x)$).*

Proof. Assume that, $x - x' \in N(S; x')$ and $\|x - x'\| \leq r$. The nonvacuity of $N(S; x')$ gives us $x' \in S$. Combining the r-prox-regularity of S and (b) of Theorem 2.2, we get

$$\langle x - x', y - x'\rangle \leq \frac{1}{2r}\|x - x'\| \|y - x'\|^2 \quad \text{for all } y \in S.$$

So, the inequality $\|x' - x\| \leq r$ yields

$$\langle x - x', y - x'\rangle \leq \frac{1}{2}\|y - x'\|^2 \quad \text{for all } y \in S.$$

Since $x' \in S$, the latter inequality and (2.2) entails that $x' \in \mathrm{Proj}_S(x)$. If in addition, $\|x - x'\| < r$, then $d_S(x) \leq \|x - x'\| < r$. According to Proposition 2.3, $x' = P_S(x)$. □

4. Concept of solution

Following [1, 12], we define the concept of solution for our measure differential inclusion as follows:

Definition 4.1. Let I be any (not necessarily bounded) proper interval of $\mathbb{R}$ with a real $T_0 \in I$ as its left endpoint. Let $C : I \rightrightarrows \mathcal{H}$ be a set-valued mapping from I into the nonempty closed sets of the real Hilbert space $\mathcal{H}$, and let $f : I \times \mathcal{H} \longrightarrow \mathcal{H}$ be a mapping. Assume that, there exists a positive Radon measure μ on I (thus finite on every compact subinterval of I) such that

$$|d(y, C(s)) - d(y, C(t))| \leq \mu(]s, t]) \quad \text{for all } s, t \in I \text{ with } s \leq t.$$

Given $u_0 \in C(T_0)$, a mapping $u : I \longrightarrow \mathcal{H}$ is a solution of the measure differential inclusion

$$(\mathcal{P}) \begin{cases} -\mathrm{d}u \in N(C(t); u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \end{cases}$$

whenever:

(a) the mapping $u(\cdot)$ is of locally bounded variation on I, right continuous on I and satisfies $u(T_0) = u_0$ and $u(t) \in C(t)$ for all $t \in I$;

(b) there exists a positive Radon measure ν absolutely continuously equivalent to $\mu + \lambda$ with respect to which $\mathrm{d}u$ admits a density, so $\frac{\mathrm{d}u}{\mathrm{d}\nu}(\cdot)$ is defined ν-a.e. and belongs to $L^1_{\mathrm{loc}}(I, \mathcal{H}, \nu)$ with $\mathrm{d}u = \frac{\mathrm{d}u}{\mathrm{d}\nu}(\cdot)\nu$ in the sense that
$$\mathrm{d}u(]s,t]) = \int_{]s,t]} \frac{\mathrm{d}u}{\mathrm{d}\nu}(\tau)\,\mathrm{d}\nu(\tau), \quad \text{for all } s,t \in I \text{ with } s \leq t;$$

(c) $\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \in -N(C(t); u(t)) \quad \nu\text{-a.e. } t \in I.$

As in [1], the concept of solution does not depend on the measure ν in the sense that a mapping $u(\cdot) : I \longrightarrow \mathcal{H}$ satisfying (a) above is a solution of $\mathcal{P}$ if and only if (b) and (c) hold for any positive Radon measure ν which is absolutely continuously equivalent to $\mu + \lambda$. Indeed, let $u(\cdot) : I \longrightarrow \mathcal{H}$ be a solution of $\mathcal{P}$ and let ν_0, given by the definition of a solution to $\mathcal{P}$, be an associated Radon measure absolutely continuously equivalent to $\mu + \lambda$ for which
$$\frac{\mathrm{d}u}{\mathrm{d}\nu_0}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu_0}(t) \in -N(C(t); u(t)) \quad \nu_0\text{-a.e. } t \in I. \tag{4.1}$$
Fix any other Radon measure ν absolutely continuously equivalent to $\mu + \lambda$. Then, the measures ν_0 and ν are absolutely continuously equivalent. Consequently, $\frac{\mathrm{d}\nu_0}{\mathrm{d}\nu}(\cdot)$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\nu_0}(\cdot)$ exist as densities, and for $\frac{\mathrm{d}u}{\mathrm{d}\nu}(\cdot)$ and the derivative $\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(\cdot)$ the following equalities hold
$$\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) = \frac{\mathrm{d}u}{\mathrm{d}\nu_0}(t)\frac{\mathrm{d}\nu_0}{\mathrm{d}\nu}(t), \quad \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = \frac{\mathrm{d}\lambda}{\mathrm{d}\nu_0}(t)\frac{\mathrm{d}\nu_0}{\mathrm{d}\nu}(t) \quad \nu\text{-a.e. } t \in I.$$
This yields according to (4.1)
$$\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \in -N(C(t); u(t)) \quad \nu\text{-a.e. } t \in I.$$

5. Existence result

Now, we can state and prove one of the main results of the paper.

Theorem 5.1. *Let $r \in]0, +\infty]$, $T_0, T \in \mathbb{R}$ with $T_0 < T$, $I := [T_0, T]$, and $C(\cdot) : I \rightrightarrows \mathcal{H}$ be a set-valued mapping from I into the r-prox-regular subsets of the real Hilbert space $\mathcal{H}$ for which there exists a positive Radon measure μ on I such that,*
$$|d(y, C(t)) - d(y, C(s))| \leq \mu\,(]s,t]) \quad \text{for all } y \in \mathcal{H}, \text{ for all } s,t \in I \text{ with } s \leq t. \tag{5.1}$$
Assume that, one has $\sup\limits_{s \in]T_0,T]} \mu(\{s\}) < \frac{r}{2}$. Let $f : I \times \mathcal{H} \longrightarrow \mathcal{H}$ be a mapping such that:

(i) *the mapping $f(\cdot, x)$ is Lebesgue measurable for each $x \in \mathcal{H}$ and there exists a nonnegative function $\beta : I \longrightarrow \mathbb{R}$ with $\beta \in L^1(I, \mathbb{R}, \lambda)$ such that, for all $t \in I$, $x \in \bigcup\limits_{\tau \in I} C(\tau)$,*
$$\|f(t,x)\| \leq \beta(t)(1 + \|x\|);$$

(ii) *for each real $\alpha \geq 0$, there exists some nonnegative function $L_\alpha : I \longrightarrow \mathbb{R}$ with $L_\alpha \in L^1(I, \mathbb{R}, \lambda)$ such that, for all $t \in I$, for all $x, y \in \alpha\mathbb{B}$,*
$$\|f(t,x) - f(t,y)\| \leq L_\alpha(t)\,\|x - y\|\,.$$

Then, for each $u_0 \in C(T_0)$, the following measure differential inclusion sweeping process on $[T_0, T]$

$$\begin{cases} -\mathrm{d}u \in N(C(t); u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \end{cases}$$

has at least one solution which satisfies both inequalities

$$\left\| u(t) - u(t^-) \right\| \leq \mu(\{t\}) \quad \textit{for all}\ t \in]T_0, T], \tag{5.2}$$

and

$$\left\| \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t, u(t)) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \right\| \leq \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(t) + \| f(t, u(t)) \| \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \quad \nu\textit{-a.e.}\ t \in I, \tag{5.3}$$

for any measure ν absolutely continuously equivalent to $\mu + \lambda$.

Proof. Let us give the plan of the proof. First, we consider a time discretization $T_0 = t_0^n < \ldots < t_{p(n)}^n = T$ taking into account the variation of $C(\cdot)$ in such a way which allows us to apply (thanks to the fact that $C(\cdot)$ is r-prox-regular valued) the Moreau catching-up algorithm, in order to obtain some sequences of $\mathcal{H}$, depending on our discretization. Then, we interpolate these points to get some suitable approximate solutions $u_n : I \longrightarrow \mathcal{H}$. An important part of the proof is devoted to show, for each $t \in I$, that $(u_n(t))_n$ is a Cauchy sequence, obtaining in this way some mapping $u : I \longrightarrow \mathcal{H}$. Finally, through the analysis of additional properties of convergence of the sequence $(u_n)_n$, we prove that u is a solution of the measure differential inclusion.

The proof will be divided into two cases depending whether $\int_{[T_0,T]} (\beta(s) + 1) \mathrm{d}\lambda(s)$ is $\leq \frac{1}{4}$ or $\geq \frac{1}{4}$. Fix any $u_0 \in C(T_0)$.

Case 1. Assume that

$$\int_{[T_0,T]} (\beta(s) + 1) \mathrm{d}\lambda(s) \leq \frac{1}{4} \cdot \tag{5.4}$$

In order to construct a sequence $(u_n(\cdot))_n$ of suitable right continuous with bounded variation mappings, we will need a preparatory step which will allow us to define the points of which $u_n(\cdot)$ will be the interpolation. Set

$$l = 2 \Big(\mu(]T_0, T]) + \|u_0\| + 1 \Big) \tag{5.5}$$

and consider on I the positive Radon measure

$$\nu = \mu + (l + 1)(\beta(\cdot) + 1)\lambda. \tag{5.6}$$

Put $M := \nu(]T_0, T])$ and note that the function $t \mapsto \nu(]T_0, t])$ is clearly increasing and right continuous on I. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers with $\varepsilon_n \downarrow 0$ and such that for all $n \in \mathbb{N}$,

$$\varepsilon_n + \sup_{s \in]T_0, T]} \mu(\{s\}) < r. \tag{5.7}$$

As in Moreau [23], for each $n \in \mathbb{N}$, choose $0 = M_0^n < M_1^n < \ldots < M_{q_n}^n = M$ (with $q_n \in \mathbb{N}$) such that:

(a) for all $j \in \{0, \ldots, q_n - 1\}$, $M_{j+1}^n - M_j^n \leq \varepsilon_n$;

(b) for all $k \in \mathbb{N}$, $\{M_0^k, \ldots, M_{q_k}^k\} \subset \Big\{ M_0^{k+1}, \ldots, M_{q_{k+1}}^{k+1} \Big\}$.

For every $n \in \mathbb{N}$, set $M_{1+q_n}^n := M + \varepsilon_n$. For each $n \in \mathbb{N}$, consider the partition of I associated with the subsets ($j \in \{0, \ldots, q_n\}$)

$$J_j^n := \{ t \in [T_0, T] : \ M_j^n \leq \nu(]T_0, t]) < M_{j+1}^n \}$$

and note that $(J_j^m)_{0\le j\le q_m}$ is a refinement of $(J_j^n)_{0\le j\le q_n}$ for all $m,n\in\mathbb{N}$ with $m\ge n$. Since $t\mapsto \nu(]T_0,t])$ is increasing and right continuous on I, it is easy to see that, for each $n\in\mathbb{N}$, $j\in\{0,\dots,q_n-1\}$, the set J_j^n is either empty or an interval of the form $[a,b[$ with $a<b$. Furthermore, we have $J_{q_n}^n=\{T\}$ for all $n\in\mathbb{N}$. This produces for each $n\in\mathbb{N}$, an integer $p(n)\in\mathbb{N}$ and a finite sequence

$$T_0=t_0^n<\ldots<t_{p(n)}^n=T$$

such that, for each $i\in\{0,\dots,p(n)-1\}$, there is some $j\in\{0,\dots,q_n-1\}$ satisfying $J_j^n=\left[t_i^n,t_{i+1}^n\right[$. Observe that $(p(n))_{n\in\mathbb{N}}$ is an increasing sequence. Fix for a moment any $n\in\mathbb{N}$. For each $i\in\{0,\dots,p(n)-1\}$, put

$$\eta_i^n:=t_{i+1}^n-t_i^n \ \text{ and } \ \sigma_i^n:=(l+1)\int_{\left[t_i^n,t_{i+1}^n\right]}(\beta(s)+1)\mathrm{d}\lambda(s), \tag{5.8}$$

and put also $\eta^n:=\max\limits_{0\le i\le p(n)-1}(t_{i+1}^n-t_i^n)$. For every $i\in\{0,\dots,p(n)-1\}$ and every $t\in\left[t_i^n,t_{i+1}^n\right[$, one has

$$\nu(\left]t,t_{i+1}^n\right])\le\nu(]t_i^n,t_{i+1}^n])=\nu(]T_0,t_{i+1}^n])-\nu(]T_0,t_i^n])\le M_{i+1}^n-M_i^n\le\varepsilon_n, \tag{5.9}$$

thus in particular

$$\nu(\left]t_i^n,t_{i+1}^n\right])\le\varepsilon_n. \tag{5.10}$$

Hence (since $\lambda\le\nu$), one has

$$\eta_i^n=t_{i+1}^n-t_i^n\le\varepsilon_n, \tag{5.11}$$

for all $i\in\{0,\dots,p(n)-1\}$, so we observe that $\lim\limits_{k\to+\infty}\eta^k=0$. Now, put $u_0^n=u_0$ and $y_0^n=\frac{1}{\eta_0^n}\int_{[t_0^n,t_1^n]}f(s,u_0^n)\,\mathrm{d}\lambda(s)$. Let us show that $d_{C(t_1^n)}(u_0^n-\eta_0^n y_0^n)<r$. According to the assumption (5.1) on the variation of $C(\cdot)$ and the fact $u_0^n\in C(t_0^n)$, we can write

$$d_{C(t_1^n)}(u_0^n-\eta_0^n y_0^n)\le\mu\left(]T_0,t_1^n]\right)+d_{C(t_0^n)}(u_0^n-\eta_0^n y_0^n)\le\mu\left(]T_0,t_1^n]\right)+\eta_0^n\left\|y_0^n\right\|. \tag{5.12}$$

By the choice of y_0^n, the assumption (i) on the mapping f and (5.5), we obtain

$$\begin{aligned}\eta_0^n\left\|y_0^n\right\|&=\left\|\int_{[t_0^n,t_1^n]}f(s,u_0^n)\,\mathrm{d}\lambda(s)\right\|\\&\le(1+\|u_0^n\|)\int_{\left[t_0^n,t_1^n\right]}\beta(s)\,\mathrm{d}\lambda(s)\le(1+l)\int_{\left[T_0,t_1^n\right]}\beta(s)\,\mathrm{d}\lambda(s).\end{aligned} \tag{5.13}$$

Taking the definition of ν in (5.6) into account and combining (5.12), (5.13), (5.10) and (5.7), we get

$$\begin{aligned}d_{C(t_1^n)}(u_0^n-\eta_0^n y_0^n)&\le\mu(]T_0,t_1^n])+(1+l)\int_{[T_0,t_1^n]}(\beta(s)+1)\mathrm{d}\lambda(s)\\&=\nu(]T_0,t_1^n])\le\varepsilon_n<r.\end{aligned} \tag{5.14}$$

Then, the r-prox-regularity of $C(t_1^n)$ allows us to define $u_1^n:=P_{C(t_1^n)}(u_0^n-\eta_0^n y_0^n)$. By finite induction, let us construct $(u_k^n)_{0\le k\le p(n)}$ and $(y_k^n)_{0\le k\le p(n)-1}$ such that, for all $k\in\{0,\dots,p(n)-1\}$

$$y_k^n=\frac{1}{\eta_k^n}\int_{[t_k^n,t_{k+1}^n]}f(s,u_k)\,\mathrm{d}\lambda(s)\ \text{ and }\ u_{k+1}^n=P_{C(t_{k+1}^n)}\big(u_k^n-\eta_k^n y_k^n\big). \tag{5.15}$$

The case $p(n)=1$ is obvious. Assume that $p(n)>1$. Suppose that $u_0^n,\dots,u_i^n$ and $y_0^n,\dots,y_{i-1}^n$ with $0<i<p(n)$ have been defined satisfying the above equalities. We can then set $y_i^n:=\frac{1}{\eta_i^n}\int_{[t_i^n,t_{i+1}^n]}f(s,u_i^n)\,\mathrm{d}\lambda(s)$. We claim

that $P_{C(t_{i+1}^n)}\big(u_i^n - \eta_i^n y_i^n\big)$ is well-defined. The case $i=0$ has already been studied, so we can suppose that $i>0$. As in (5.12) and (5.13), we have

$$d_{C(t_{i+1}^n)}(u_i^n - \eta_i^n y_i^n) \le \mu\left(\left]T_0, t_{i+1}^n\right[\right) + \eta_i^n \left\|y_i^n\right\|,$$

and

$$\eta_i^n \left\|y_i^n\right\| \le (1+l)\int_{\left[T_0, t_{i+1}^n\right]} \beta(s)\,\mathrm{d}\lambda(s).$$

In the same way as (5.14), we get $d_{C(t_{i+1}^n)}(u_i^n - \eta_i^n y_i^n) < r$ and this completes the induction, thanks to the r-prox-regularity of $C(t_{i+1}^n)$.

For any $i \in \{0,\dots,p(n)-1\}$, from (5.15) and (5.1), we have

$$\begin{aligned}\left\|u_{i+1}^n - u_i^n + \eta_i^n y_i^n\right\| &= d_{C(t_{i+1}^n)}(u_i^n - \eta_i^n y_i^n)\\ &\le d_{C(t_i^n)}(u_i^n - \eta_i^n y_i^n) + \mu\left(\left]t_i^n, t_{i+1}^n\right]\right)\\ &\le \eta_i^n \left\|y_i^n\right\| + \mu\left(\left]t_i^n, t_{i+1}^n\right]\right),\end{aligned} \tag{5.16}$$

and hence $\|u_{i+1}^n\| - \|u_i^n - \eta_i^n y_i^n\| \le \mu\left(\left]t_i^n, t_{i+1}^n\right]\right) + \eta_i^n \|y_i^n\|$. It follows that, for all $i \in \{0,\dots,p(n)-1\}$, $\left\|u_{i+1}^n\right\| \le \|u_i^n\| + \mu\left(\left]t_i^n, t_{i+1}^n\right]\right) + 2\eta_i^n \|y_i^n\|$. Thus, we get

$$\left\|u_{i+1}^n\right\| \le \|u_0^n\| + \sum_{k=0}^{i}\left(\mu\left(\left]t_k^n, t_{k+1}^n\right]\right) + 2\eta_k^n \left\|y_k^n\right\|\right). \tag{5.17}$$

Using the definition of y_i^n and the assumption (i), we also have, for all $i \in \{0,\dots,p(n)-1\}$,

$$\|y_i^n\| \le \frac{1}{\eta_i^n}(1+\|u_i^n\|)\int_{[t_i^n, t_{i+1}^n]} \beta(s)\,\mathrm{d}\lambda(s) \le \frac{1}{\eta_i^n}(1+\max_{0\le k\le p(n)}\|u_k^n\|)\int_{[t_i^n, t_{i+1}^n]} \beta(s)\,\mathrm{d}\lambda(s). \tag{5.18}$$

Fix any $i \in \{0,\dots,p(n)-1\}$. The inequalities (5.17) and (5.18) yield

$$\left\|u_{i+1}^n\right\| \le \|u_0^n\| + \sum_{k=0}^{i}\mu\left(\left]t_k^n, t_{k+1}^n\right]\right) + 2\left(1+\max_{0\le k\le p(n)}\|u_k^n\|\right)\int_{[t_0^n, t_{i+1}^n]} \beta(s)\,\mathrm{d}\lambda(s).$$

Therefore,

$$\left\|u_{i+1}^n\right\| \le \|u_0^n\| + \mu\left(\left]T_0, T\right]\right) + 2\left(1+\max_{0\le k\le p(n)}\|u_k^n\|\right)\int_{\left[T_0, t_{i+1}^n\right]} \beta(s)\,\mathrm{d}\lambda(s),$$

so the inequality $\int_{\left[T_0, t_{i+1}^n\right]}(\beta(s)+1)\mathrm{d}\lambda(s) \le \frac{1}{4}$ gives us

$$\max_{0\le k\le p(n)}\|u_k^n\| \le \|u_0^n\| + \mu\left(\left]T_0, T\right]\right) + \frac{1}{2}\left(1+\max_{0\le k\le p(n)}\|u_k^n\|\right).$$

Using the definition of l, we can then write

$$\max_{0\le k\le p(n)}\|u_k^n\| \le 2\left(\|u_0^n\| + \mu\left(\left]T_0, T\right]\right) + \frac{1}{2}\right) \le l, \tag{5.19}$$

which combines with (5.18) and (5.8) implies

$$\eta_i^n \left\|y_i^n\right\| \le (1+\|u_i^n\|)\int_{[t_i^n, t_{i+1}^n]} \beta(s)\,\mathrm{d}\lambda(s) \le (l+1)\int_{[t_i^n, t_{i+1}^n]}(\beta(s)+1)\mathrm{d}\lambda(s) = \sigma_i^n. \tag{5.20}$$

The latter inequality and (5.16) assure us that

$$d_{C(t_{i+1}^n)}(u_i^n - \eta_i^n y_i^n) = \left\| u_{i+1}^n - u_i^n + \eta_i^n y_i^n \right\| \le \eta_i^n \left\| y_i^n \right\| + \mu\left(\left] t_i^n, t_{i+1}^n \right] \right) \le \sigma_i^n + \mu\left(\left] t_i^n, t_{i+1}^n \right] \right). \tag{5.21}$$

This along with (5.8) and the definition of ν in (5.6) entails, for all $n \in \mathbb{N}$ and for all $i \in \{0, \ldots, p(n)-1\}$,

$$\left\| u_{i+1}^n - u_i^n + \eta_i^n y_i^n \right\| \le \nu\left(\left] t_i^n, t_{i+1}^n \right] \right). \tag{5.22}$$

Step 1. Construction of the sequence $(u_n(\cdot))_{n\in\mathbb{N}}$.
In this first step, fix any $n \in \mathbb{N}$ and define the mapping $u_n(\cdot) : I \longrightarrow \mathcal{H}$ by putting, for $t \in [t_i^n, t_{i+1}^n]$ with $i \in E_n := \{0, \ldots, p(n)-1\}$,

$$u_n(t) = u_i^n + \frac{\nu\big(\left] t_i^n, t \right] \big)}{\nu\big(\left] t_i^n, t_{i+1}^n \right] \big)} (u_{i+1}^n - u_i^n + \eta_i^n y_i^n) - \int_{[t_i^n, t]} f(s, u_i^n)\, \mathrm{d}\lambda(s). \tag{5.23}$$

We observe that u_n is well defined on I and it is right continuous on I with bounded variation on the whole interval I. Furthermore, we have by the definition of $u_n(\cdot)$

$$u_n(t) = u_n(T_0) + \int_{]T_0,t]} F_n(s) \mathrm{d}\nu(s) - \int_{]T_0,t]} f\big(s, u_n(\delta_n(s))\big) \mathrm{d}\lambda(s) \quad \text{for all } t \in I,$$

where we set for every $t \in I$,

$$F_n(t) = \sum_{i=0}^{p(n)-1} \frac{u_{i+1}^n - u_i^n + \eta_i^n y_i^n}{\nu\left(\left] t_i^n, t_{i+1}^n \right] \right)} \mathbf{1}_{]t_i^n, t_{i+1}^n]}(t),$$

and

$$\delta_n(t) = \begin{cases} t_i^n & \text{if } t \in \left[t_i^n, t_{i+1}^n \right[\text{ with } i \in E_n, \\ t_{p(n)-1}^n & \text{if } t = T. \end{cases} \tag{5.24}$$

Since by (5.6), the measure λ is absolutely continuous with respect to ν, it has $\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(\cdot)$ as a density in $L^\infty(I, [0, +\infty[, \nu)$ relative to ν and then by (2.9), for all $t \in I$,

$$u_n(t) = u_n(T_0) + \int_{]T_0,t]} \left(F_n(s) - f\big(s, u_n(\delta_n(s))\big) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(s) \right) \mathrm{d}\nu(s).$$

This tells us that the vector measure $\mathrm{d}u_n$ has the latter integrand as a density in $L^1(I, \mathcal{H}, \nu)$ relative to ν. Consequently the derivative $\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(\cdot)$ is a density of $\mathrm{d}u_n$ relative to ν and

$$\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) + f\big(t, u_n(\delta_n(t))\big) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = F_n(t) \quad \text{for } \nu\text{-a.e. } t \in I. \tag{5.25}$$

Taking (5.22) into account, it results

$$\left\| \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) + f\big(t, u_n(\delta_n(t))\big) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \right\| \le 1 \quad \text{for } \nu\text{-a.e. } t \in I. \tag{5.26}$$

On the other hand, by (5.6) again, the measure $(l+1)\beta(\cdot)\lambda$ is absolutely continuous with respect to ν, thus it has $\frac{d\big((l+1)\beta(\cdot)\lambda\big)}{\mathrm{d}\nu}$ as a density relative to ν, and

$$0 \le (l+1)\beta(t) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = \frac{d\big((l+1)\beta(\cdot)\lambda\big)}{\mathrm{d}\nu}(t) \le 1 \quad \text{for } \nu\text{-a.e. } t \in I. \tag{5.27}$$

Observing also by (5.19) and by the assumption (i) that

$$\|f(t,u_n(\delta_n(t)))\| \le (l+1)\beta(t) \quad \text{for all } t \in I,$$

it ensues that

$$\left\| f\big(t,u_n(\delta_n(t))\big)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right\| \le (l+1)\beta(t)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \le 1 \quad \text{for } \nu\text{-a.e. } t \in I. \tag{5.28}$$

This and (5.26) say that

$$\left\|\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t)\right\| \le 2 \quad \text{for } \nu\text{-a.e. } t \in I. \tag{5.29}$$

From (5.26) again and the equality $\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = 0$ for all $t \in I$ with $\nu(\{t\}) > 0$, we note that

$$\left\|\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t)\right\| \le 1 \quad \text{for all } t \in I \text{ with } \nu(\{t\}) > 0. \tag{5.30}$$

Let us define the function $\theta_n : I \longrightarrow I$ by

$$\theta_n(t) = \begin{cases} t^n_{i+1} & \text{if } t \in \left[t^n_i, t^n_{i+1}\right[\text{ with } i \in E_n, \\ T & \text{if } t = T. \end{cases} \tag{5.31}$$

Using (5.25) and (2.3), we can write

$$\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) + f\big(t,u_n(\delta_n(t))\big)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \in -N^P(C(\theta_n(t)); u_n(\theta_n(t))) \quad \text{for } \nu\text{-a.e. } t \in I.$$

By (5.26) and (2.6), we get

$$\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) + f\big(t,u_n(\delta_n(t))\big)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \in -\partial_P d_{C(\theta_n(t))}\big(u_n(\theta_n(t))\big) \quad \text{for } \nu\text{-a.e. } t \in I. \tag{5.32}$$

Step 2. We claim that $(u_n(\cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}(I,\mathcal{H})$ (the real space of bounded mappings from I to $\mathcal{H}$) endowed with the norm of the uniform convergence, which is a real Banach space.

Fix any $n, m \in \mathbb{N}$. Since $u^n_0 = u_0 \in C(t^n_0)$ and $u^n_{i+1} = P_{C(t^n_{i+1})}(u^n_i - \eta^n_i y^n_i)$ for all $i \in \{0,\dots,p(n)-1\}$, we note by (5.23) and (5.31) that

$$u_n(\theta_n(t)) \in C(\theta_n(t)) \quad \text{for all } t \in I. \tag{5.33}$$

This allows us to write, for every $t \in I$,

$$\begin{aligned} d_{C(\theta_n(t))}(u_m(t)) &= d_{C(\theta_n(t))}(u_m(t)) - d_{C(\theta_m(t))}(u_m(\theta_m(t))) \\ &\le d_{C(\theta_n(t))}(u_m(t)) - d_{C(\theta_m(t))}(u_m(t)) + \|u_m(\theta_m(t)) - u_m(t)\|. \end{aligned}$$

According to the variation assumption on $C(\cdot)$ in (5.1), we have for every $t \in I$,

$$d_{C(\theta_n(t))}(u_m(t)) \le \max\{\mu(]t,\theta_n(t)]), \mu(]t,\theta_m(t)])\} + \|u_m(\theta_m(t)) - u_m(t)\|, \tag{5.34}$$

hence by (5.29) and the equality $u_m(\theta_m(t)) - u_m(t) = \int_{]t,\theta_m(t)]} \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t)\mathrm{d}\nu(t)$,

$$d_{C(\theta_n(t))}(u_m(t)) \le \max\{\mu(]t,\theta_n(t)]), \mu(]t,\theta_m(t)])\} + 2\nu(]t,\theta_m(t)]). \tag{5.35}$$

Fix any $s \in [T_0, T[$ and choose some $i_s \in \{0,\dots,p(m)-1\}$ such that $s \in \left[t^m_{i_s}, t^m_{i_s+1}\right[$. From (5.23), we have

$$u_m(\theta_m(s)) - u_m(s) = u^m_{i_s+1} - u^m_{i_s} - \frac{\nu(\left]t^m_{i_s}, s\right])}{\nu(\left]t^m_{i_s}, t^m_{i_s+1}\right])}(u^m_{i_s+1} - u^m_{i_s} + \eta^m_{i_s} y^m_{i_s}) + (s - t^m_{i_s}) y^m_{i_s}. \tag{5.36}$$

Taking this, (5.9) and (5.22) into account, we get

$$\|u_m(\theta_m(s)) - u_m(s)\| \le \left\|u^m_{i_s+1} - u^m_{i_s} + (s - t^m_{i_s}) y^m_{i_s}\right\| + \varepsilon_m,$$

and thus

$$\|u_m(\theta_m(s)) - u_m(s)\| \le \left\|u^m_{i_s+1} - u^m_{i_s} + \eta^m_{i_s} y^m_{i_s}\right\| + \eta^m_{i_s} \left\|y^m_{i_s}\right\| + \varepsilon_m. \tag{5.37}$$

From the latter inequality, (5.22), (5.20) and (5.10), we have

$$\|u_m(\theta_m(s)) - u_m(s)\| \le \nu\big(\,]t^m_{i_s}, t^m_{i_s+1}]\,\big) + \sigma^m_{i_s} + \varepsilon_m \le \sigma^m_{i_s} + 2\varepsilon_m. \tag{5.38}$$

This is also true for $s = T$ because $\theta_m(s) = T$. Coming back to (5.34) and using (5.38), we obtain

$$\begin{aligned} d_{C(\theta_n(s))}(u_m(s)) &\le \max\{\mu(]s, \theta_n(s)]), \mu(]s, \theta_m(s)])\} + \|u_m(\theta_m(s)) - u_m(s)\| \\ &\le \varepsilon_n + \varepsilon_m + \sigma^m_{i_s} + 2\varepsilon_m = \varepsilon_n + 3\varepsilon_m + \sigma^m_{i_s}. \end{aligned}$$

Using the equalities $\lim\limits_{k\to+\infty} \varepsilon_k = 0$ and $\lim\limits_{k\to+\infty} \sup\limits_{0\le i\le p(k)-1} \sigma^k_i = 0$ (see (5.8)), there exists some $N \in \mathbb{N}$ such that, for all $t \in I$ and for all integers $n, m \ge N$,

$$d_{C(\theta_n(t))}(u_m(t)) < r. \tag{5.39}$$

For each $n \in \mathbb{N}$, for all $t \in I$, set

$$\gamma_n(t) = \mu(]t, \theta_n(t)]) + \nu(]t, \theta_n(t)]). \tag{5.40}$$

Note by (5.35) and (5.40) that, for all $n, m \in \mathbb{N}$, and all $t \in I$, we have

$$d_{C(\theta_n(t))}(u_m(t)) \le \gamma_n(t) + 2\gamma_m(t). \tag{5.41}$$

Fix any $n \in \mathbb{N}$. By (5.29) and the fact that $\frac{\mathrm{d}u_n}{\mathrm{d}\nu}$ is a density of $\mathrm{d}u_n$ relative to ν, we get

$$\|u_n(\tau_2) - u_n(\tau_1)\| \le 2\nu\big(\,]\tau_1, \tau_2]\,\big) \quad \text{for all } \tau_1, \tau_2 \in I \text{ with } \tau_1 < \tau_2. \tag{5.42}$$

Fix any $\tau \in]T_0, T]$. By (5.30), we have $\|u_n(\tau) - u_n(\tau^-)\| \le \nu(\{\tau\}) = \mu(\{\tau\})$, whenever $\nu(\{\tau\}) > 0$. We get by (5.42), $\|u_n(\tau) - u_n(\tau^-)\| \le 2\nu(\{\tau\}) = 0 = \mu(\{\tau\})$, under the hypothesis $\nu(\{t\}) = 0$. As a consequence,

$$\left\|u_n(t) - u_n(t^-)\right\| \le \mu(\{t\}) \quad \text{for all } t \in]T_0, T]. \tag{5.43}$$

Now, for ν-almost every $t \in I$, put for all $n, m \in \mathbb{N}$

$$A_n(t) := \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) + f\big(t, u_n(\delta_n(t))\big) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \ \text{ and } \ B_n(t) := f\big(t, u_n(\delta_n(t))\big), \tag{5.44}$$

and for ν-almost every $t \in I$,

$$\varphi_{n,m}(t) = \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \left\|f\big(t, u_n(\delta_n(t))\big) - f\big(t, u_n(t)\big)\right\| \|u_n(t) - u_m(t)\|. \tag{5.45}$$

Fix any integers $n \geq N$ and $m \geq N$. Using (5.32), (5.39), Lemma 3.3, (5.40), (5.42) and (5.41), it follows that, for ν-almost every $t \in I$,

$$\begin{aligned}
&\big\langle A_n(t), u_n\big(\theta_n(t)\big) - u_m(t)\big\rangle \\
&\leq \frac{1}{2r}\left\|u_m(t) - u_n\big(\theta_n(t)\big)\right\|^2 + \frac{1}{2r}d^2_{C(\theta_n(t))}(u_m(t)) \\
&\quad + \left(\frac{1}{r}\left\|u_n\big(\theta_n(t)\big) - u_m(t)\right\| + 1\right) d_{C(\theta_n(t))}(u_m(t)) \\
&\leq \frac{1}{2r}\big(\,\|u_n(t) - u_m(t)\| + \left\|u_n\big(\theta_n(t)\big) - u_n(t)\right\|\,\big)^2 + \frac{1}{2r}d^2_{C(\theta_n(t))}(u_m(t)) \\
&\quad + \left[\frac{1}{r}\big(\,\left\|u_n\big(\theta_n(t)\big) - u_n(t)\right\| + \|u_n(t) - u_m(t)\|\,\big) + 1\right] d_{C(\theta_n(t))}(u_m(t)) \\
&\leq \frac{1}{2r}\big(\,\|u_n(t) - u_m(t)\| + 2\gamma_n(t)\big)^2 + \frac{1}{2r}\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 \\
&\quad + \left[\frac{1}{r}\big(2\gamma_n(t) + \|u_n(t) - u_m(t)\|\,\big) + 1\right]\big(\gamma_n(t) + 2\gamma_m(t)\big).
\end{aligned} \tag{5.46}$$

Then, for ν-almost every $t \in I$, since $\|A_n(t)\| \leq 1$ by (5.26), from (5.42) and (5.46) we have

$$\begin{aligned}
&\langle A_n(t), u_n(t) - u_m(t)\rangle \\
&= \big\langle A_n(t), u_n(t) - u_n\big(\theta_n(t)\big)\big\rangle + \big\langle A_n(t), u_n\big(\theta_n(t)\big) - u_m(t)\big\rangle \\
&\leq \left\|u_n(t) - u_n\big(\theta_n(t)\big)\right\| + \frac{1}{2r}\big(\,\|u_m(t) - u_n(t)\| + 2\gamma_n(t)\big)^2 + \frac{1}{2r}\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 \\
&\quad + \left[\frac{1}{r}\big(2\gamma_n(t) + \|u_n(t) - u_m(t)\|\,\big) + 1\right]\big(\gamma_n(t) + 2\gamma_m(t)\big) \\
&\leq 2\nu\big(\,]t, \theta_n(t)]\,\big) + \frac{1}{2r}\big(\,\|u_m(t) - u_n(t)\| + 2\gamma_n(t)\big)^2 + \frac{1}{2r}\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 \\
&\quad + \left[\frac{1}{r}\big(2\gamma_n(t) + \|u_n(t) - u_m(t)\|\,\big) + 1\right]\big(\gamma_n(t) + 2\gamma_m(t)\big).
\end{aligned} \tag{5.47}$$

The definition of $B_n(\cdot)$ in (5.44) allows us to write, for ν-almost every $t \in I$,

$$\begin{aligned}
&\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t), u_n(t) - u_m(t)\right\rangle \\
&\leq \left\langle B_n(t)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t), u_m(t) - u_n(t)\right\rangle + 2\nu\big(\,]t, \theta_n(t)]\,\big) + \frac{1}{2r}\big(\,\|u_m(t) - u_n(t)\| + 2\gamma_n(t)\big)^2 \\
&\quad + \left[\frac{1}{r}\big(2\gamma_n(t) + \|u_n(t) - u_m(t)\|\,\big) + 1\right]\big(\gamma_n(t) + 2\gamma_m(t)\big) + \frac{1}{2r}\big(\gamma_n(t) + 2\gamma_m(t)\big)^2.
\end{aligned}$$

Interchanging m and n, we obtain

$$\begin{aligned}
&\left\langle \frac{\mathrm{d}u_m}{\mathrm{d}\nu}(t), u_m(t) - u_n(t)\right\rangle \\
&\leq \left\langle B_m(t)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t), u_n(t) - u_m(t)\right\rangle + 2\nu\big(\,]t, \theta_m(t)]\,\big) + \frac{1}{2r}\big(\,\|u_n(t) - u_m(t)\| + 2\gamma_m(t)\big)^2 \\
&\quad + \left[\frac{1}{r}\big(2\gamma_m(t) + \|u_m(t) - u_n(t)\|\,\big) + 1\right]\big(\gamma_m(t) + 2\gamma_n(t)\big) + \frac{1}{2r}\big(\gamma_m(t) + 2\gamma_n(t)\big)^2.
\end{aligned}$$

Hence, by adding both latter inequalities, it follows that, for ν-almost every $t \in I$,

$$\begin{aligned}
&\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) - \frac{\mathrm{d}u_m}{\mathrm{d}\nu}(t), u_n(t) - u_m(t) \right\rangle \\
&\leq \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \, \|B_n(t) - B_m(t)\| \, \|u_n(t) - u_m(t)\| + 2\nu\big(\,]t, \theta_n(t)] \,\big) \\
&\quad + 2\nu\big(\,]t, \theta_m(t)] \,\big) + \frac{1}{2r}\big(\|u_m(t) - u_n(t)\| + 2\gamma_n(t) \big)^2 \\
&\quad + \frac{1}{2r}\big(\|u_n(t) - u_m(t)\| + 2\gamma_m(t) \big)^2 \\
&\quad + \frac{1}{2r}\left[\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 + \big(\gamma_m(t) + 2\gamma_n(t)\big)^2 \right] \\
&\quad + \left[\frac{1}{r}\big(2\gamma_n(t) + \|u_n(t) - u_m(t)\| \,\big) + 1 \right] \big(\gamma_n(t) + 2\gamma_m(t)\big) \\
&\quad + \left[\frac{1}{r}\big(2\gamma_m(t) + \|u_m(t) - u_n(t)\| \,\big) + 1 \right] \big(\gamma_m(t) + 2\gamma_n(t)\big).
\end{aligned} \tag{5.48}$$

Now from (5.42) and the fact that $u_k(T_0) = u_0$ for all $k \in \mathbb{N}$, we note that

$$\|u_k(t)\| \leq \alpha \quad \text{for all } t \in I, \text{all } k \in \mathbb{N}, \tag{5.49}$$

where $\alpha := \|u_0\| + 2\nu\big(\,]T_0, T] \,\big)$. Consequently, for ν-almost every $t \in I$, we have

$$\begin{aligned}
\|B_n(t) - B_m(t)\| &\leq \|B_n(t) - f(t, u_n(t))\| + \|f(t, u_n(t)) - f(t, u_m(t))\| + \|f(t, u_m(t)) - B_m(t)\| \\
&\leq \|B_n(t) - f(t, u_n(t))\| + L_\alpha(t) \, \|u_n(t) - u_m(t)\| + \|f(t, u_m(t)) - B_m(t)\| \, ,
\end{aligned} \tag{5.50}$$

where the last inequality is a consequence of the Lipschitz hypothesis (ii) on f.

Thus, taking into account the definition of $\varphi_{n,m}$ in (5.45), (5.50) and (5.48), we obtain for ν-almost every $t \in I$, and for all integers $n, m \geq N$

$$\begin{aligned}
&\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) - \frac{\mathrm{d}u_m}{\mathrm{d}\nu}(t), u_n(t) - u_m(t) \right\rangle \\
&\leq L_\alpha(t) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \, \|u_n(t) - u_m(t)\|^2 + \varphi_{n,m}(t) + \varphi_{m,n}(t) + 2\nu\big(\,]t, \theta_n(t)] \,\big) + 2\nu\big(\,]t, \theta_m(t)] \,\big) \\
&\quad + \frac{1}{2r}\big(\|u_m(t) - u_n(t)\| + 2\gamma_n(t) \big)^2 + \frac{1}{2r}\big(\|u_n(t) - u_m(t)\| + 2\gamma_m(t) \big)^2 \\
&\quad + \frac{1}{2r}\left[\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 + \big(\gamma_m(t) + 2\gamma_n(t)\big)^2 \right] \\
&\quad + \left[\frac{1}{r}\big(2\gamma_n(t) + \|u_n(t) - u_m(t)\| \,\big) + 1 \right] \big(\gamma_n(t) + 2\gamma_m(t)\big) \\
&\quad + \left[\frac{1}{r}\big(2\gamma_m(t) + \|u_m(t) - u_n(t)\| \,\big) + 1 \right] \big(\gamma_m(t) + 2\gamma_n(t)\big).
\end{aligned}$$

Consequently, according to (5.49), for ν-almost every $t \in I$, for all $n, m \geq N$,

$$\begin{aligned}
&\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) - \frac{\mathrm{d}u_m}{\mathrm{d}\nu}(t), u_n(t) - u_m(t) \right\rangle \\
&\leq \left(L_\alpha(t)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) + \frac{1}{r} \right) \|u_n(t) - u_m(t)\|^2 + \varphi_{n,m}(t) + \varphi_{m,n}(t) \\
&\quad + 2\nu\big(\,]t, \theta_n(t)]\,\big) + 2\nu\big(\,]t, \theta_m(t)]\,\big) + \frac{1}{2r}\big(4\gamma_n^2(t) + 8\alpha\gamma_n(t) + 4\gamma_m^2(t) + 8\alpha\gamma_m(t)\big) \\
&\quad + \frac{1}{2r}\left[\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 + \big(\gamma_m(t) + 2\gamma_n(t)\big)^2\right] \\
&\quad + \left[\frac{1}{r}\big(2\gamma_n(t) + 2\alpha\big) + 1\right]\big(\gamma_n(t) + 2\gamma_m(t)\big) + \left[\frac{1}{r}\big(2\gamma_m(t) + 2\alpha\big) + 1\right]\big(\gamma_m(t) + 2\gamma_n(t)\big).
\end{aligned} \tag{5.51}$$

Write that, for ν-almost every $t \in I$, and for all $n \in \mathbb{N}$

$$\Big(B_n(t) - f\big(t, u_n(t)\big)\Big)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = \Big(f\big(t, u_n(\delta_n(t))\big) - f\big(t, u_n(t)\big)\Big)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t). \tag{5.52}$$

Using the inequality due to (5.42),

$$\|u_n(\delta_n(t)) - u_n(t)\|\,\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \leq 2\nu\big(\,]\delta_n(t), t]\,\big)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \quad \nu\text{-a.e. } t \in I,$$

and using (2.10), we also see that $\|u_n(\delta_n(t)) - u_n(t)\| \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \longrightarrow 0$ as $n \longrightarrow \infty$ for ν-a.e. $t \in I$. By (5.52) and according to the Lipschitz property, for each $t \in I$, of $f(t, \cdot)$ on $\alpha\mathbb{B}$ and to the inequality $\|u_n(t)\| \leq \alpha$, for ν-almost every $t \in I$, we have

$$\lim_{n \to +\infty} \Big(B_n(t) - f\big(t, u_n(t)\big)\Big)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = 0. \tag{5.53}$$

By the Lebesgue dominated convergence theorem, it ensues by (5.44) and (5.49) that

$$\int_{]T_0,T]} \varphi_{n,m}(t)\mathrm{d}\nu(t) \longrightarrow 0 \text{ as } n, m \longrightarrow +\infty.$$

For all $n, m \in \mathbb{N}$, setting

$$\begin{aligned}
A_{n,m} = \frac{1}{2}\int_{]T_0,T]} \Bigg\{ & \varphi_{n,m}(t) + \varphi_{m,n}(t) + 2\nu\big(\,]t, \theta_n(t)]\,\big) + 2\nu\big(\,]t, \theta_m(t)]\,\big) \\
& + \frac{1}{2r}\left[\big(\gamma_n(t) + 2\gamma_m(t)\big)^2 + \big(\gamma_m(t) + 2\gamma_n(t)\big)^2\right] \\
& + \frac{1}{2r}\big(4\gamma_n^2(t) + 8\alpha\gamma_n(t) + 4\gamma_m(t) + 8\alpha\gamma_m(t)\big) \\
& + \left[\frac{1}{r}\big(2\gamma_n(t) + 2\alpha\big) + 1\right]\big(\gamma_n(t) + 2\gamma_m(t)\big) \\
& + \left[\frac{1}{r}\big(2\gamma_m(t) + 2\alpha\big) + 1\right]\big(\gamma_m(t) + 2\gamma_n(t)\big) \Bigg\}\, \mathrm{d}\nu(t)
\end{aligned}$$

we see that $A_{n,m} \longrightarrow 0$ as $n, m \longrightarrow +\infty$. On the other hand, Proposition 3.2 says that

$$d(\|u_n(\cdot) - u_m(\cdot)\|^2) \leq 2\left\langle \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(\cdot) - \frac{\mathrm{d}u_m}{\mathrm{d}\nu}(\cdot), u_n(\cdot) - u_m(\cdot) \right\rangle \mathrm{d}\nu \quad \text{for all } n, m \in \mathbb{N}. \tag{5.54}$$

Fix for a moment $n, m \in \mathbb{N}$ with $n, m \geq N$. Putting for all $t \in I$, $\psi_{n,m}(t) = \|u_n(t) - u_m(t)\|^2$ and noting that $u_n(T_0) = u_m(T_0)$, we deduce from (5.51) that, for all $t \in I$

$$\psi_{n,m}(t) \leq \int_{]T_0,t]} 2\left(L_\alpha(s)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(s) + \frac{1}{r}\right)\psi_{n,m}(s)\mathrm{d}\nu(s) + A_{n,m}.$$

According to (2.10), we have $L_\alpha(s)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(s)\nu(\{s\}) = 0$ for ν-almost every $s \in I$. It follows, for ν-almost every $t \in]T_0, T]$

$$2\left(L_\alpha(t)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) + \frac{1}{r}\right)\nu(\{t\}) = \frac{2}{r}\nu(\{t\}) = \frac{2}{r}\mu(\{t\}) \leq \frac{2}{r}\sup_{s\in]T_0,T]}\mu(\{s\}) < 1,$$

where the last inequality is due to the assumption $\sup_{s\in]T_0,T]}\mu(\{s\}) < \frac{r}{2}$. We can apply Lemma 3.1, and this yields, for all $t \in]T_0, T]$

$$\begin{aligned}\psi_{n,m}(t) &\leq A_{n,m}\exp\left(\frac{1}{1-\theta}\int_{]T_0,t]} 2\left(L_\alpha(s)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(s) + \frac{1}{r}\right)\mathrm{d}\nu(s)\right)\\ &\leq A_{n,m}\exp\left(\frac{1}{1-\theta}\left(\int_{]T_0,T]} 2L_\alpha(s)\mathrm{d}\nu(s) + \frac{2}{r}\nu(]T_0,T]\right)\right)\end{aligned}$$

where $\theta = \frac{2}{r}\sup_{s\in]T_0,T]}\mu(\{s\})$. This ensures that the sequence $(u_n(\cdot))_n$ satisfies the Cauchy property with respect to the norm of uniform convergence on the space of all bounded mappings from I into $\mathcal{H}$. Consequently, this sequence $(u_n(\cdot))_n$ converges uniformly on I to some mapping $u(\cdot)$. Furthermore, by (5.29), extracting a subsequence if necessary, we may suppose that $(\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(\cdot))_n$ converges weakly in $L^2(I, \mathcal{H}, \nu)$ to some mapping $h(\cdot) \in L^2(I, \mathcal{H}, \nu)$, so, for every $t \in I$,

$$\int_{]T_0,t]}\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(s)\mathrm{d}\nu(s) \underset{n\to+\infty}{\longrightarrow} \int_{]T_0,t]} h(s)\mathrm{d}\nu(s) \text{ weakly in } \mathcal{H}.$$

Since $\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(\cdot)$ is a density of $\mathrm{d}u_n$ relative to ν for all $n \in \mathbb{N}$, we also have for all $n \in \mathbb{N}$, for all $t \in I$, $u_n(t) = u_0 + \int_{]T_0,t]}\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(s)\mathrm{d}\nu(s)$. Thus it ensues that, for all $t \in I$, $u(t) = u_0 + \int_{]T_0,t]} h(s)\mathrm{d}\nu(s)$ and this tells us that $u(\cdot)$ is right continuous with bounded variation on I and the vector measure $\mathrm{d}u$ has $h(\cdot) \in L^2(I, \mathcal{H}, \nu)$ as a density relative to ν and $\frac{\mathrm{d}u}{\mathrm{d}\nu}(\cdot) = h(\cdot)$ ν-almost everywhere. We also deduce that

$$\frac{\mathrm{d}u_n}{\mathrm{d}\nu}(\cdot) \underset{n\to+\infty}{\longrightarrow} \frac{\mathrm{d}u}{\mathrm{d}\nu}(\cdot) \text{ weakly in } L^2(I, \mathcal{H}, \nu).$$

Step 3. Let us prove that $u(\cdot)$ is a solution and that (5.2) holds.

First, from (5.43) and the uniform convergence of $(u_n(\cdot))_n$ to $u(\cdot)$ we get

$$\left\|u(t) - u(t^-)\right\| \leq \mu(\{t\}) \quad \text{for all } t \in]T_0, T], \tag{5.55}$$

which is the property (5.2).

Let us notice that by (5.11), we have for all $n \in \mathbb{N}$ and all $t \in I$, $0 \leq \theta_n(t) - t \leq \varepsilon_n$. Using (5.42), we can write $\|u_n(\theta_n(t)) - u(t)\| \leq \|u_n(t) - u(t)\| + 2\nu\big(]t, \theta_n(t)]\big)$. By letting $n \to +\infty$, we obtain

$$\theta_n(t) \downarrow t \text{ and } u_n(\theta_n(t)) \longrightarrow u(t) \quad \text{for all } t \in I. \tag{5.56}$$

Further, from (5.1), (5.33) and (5.10), we have for all $t \in I$, all $n \in \mathbb{N}$

$$d_{C(t)}(u_n(\theta_n(t))) = \left|d_{C(t)}(u_n(\theta_n(t))) - d_{C(\theta_n(t))}(u_n(\theta_n(t)))\right| \leq \mu(]t, \theta_n(t)]) \leq \varepsilon_n.$$

We then see according to (5.56) and the closedness of $C(t)$ that,

$$u(t) \in C(t) \quad \text{for all } t \in I. \tag{5.57}$$

Now, let us show that

$$\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \in -N(C(t);u(t)) \quad \nu\text{-a.e. } t \in I.$$

From (5.53) we first notice that, for ν-almost every $t \in I$

$$e_n(t) := f\big(t, u_n(\delta_n(t))\big)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \underset{n\to+\infty}{\longrightarrow} f\big(t,u(t)\big)\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) =: e(t).$$

By this and (5.28) the Lebesgue dominated convergence theorem yields that $(e_n(\cdot))_n$ converges strongly to $e(\cdot)$ in $L^2(I,\mathcal{H},\nu)$. Following a technique due to Castaing [5] and putting for ν-a.e. $t \in I$, for all $n \in \mathbb{N}$, $\zeta_n(t) = \frac{\mathrm{d}u_n}{\mathrm{d}\nu}(t) + e_n(t)$ and for ν-a.e. $t \in I$, $\zeta(t) = \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + e(t)$, the sequence $(\zeta_n(\cdot))_n$ converges weakly in $L^2(I,\mathcal{H},\nu)$ to $\zeta(\cdot)$ and by Mazur's lemma there is a sequence $(\xi_n(\cdot))_n$ converging strongly in $L^2(I,\mathcal{H},\nu)$ to $\zeta(\cdot)$ with

$$\xi_n(\cdot) \in \operatorname{co}\{\zeta_k(\cdot) : k \geq n\}.$$

This sequence $(\xi_n(\cdot))_n$ has a subsequence (that we do not relabel) converging ν-almost everywhere to $\zeta(\cdot)$, hence, there is some Borel set $I_0 \subset I$ with $\nu(I \setminus I_0) = 0$ such that, for all $t \in I \setminus I_0$,

$$\zeta(t) \in \bigcap_{n\in\mathbb{N}} \overline{\operatorname{co}}\{\zeta_k(t) : k \geq n\}.$$

We may also suppose that the inclusion (5.32) is satisfied for all $t \in I \setminus I_0$ and all $n \in \mathbb{N}$. Then, fixing any $t \in I \setminus I_0$, we obtain for any fixed $w \in \mathcal{H}$, $\langle w, \zeta(t)\rangle \leq \inf_{n\in\mathbb{N}} \sup_{k\geq n} \langle w, \zeta_k(t)\rangle$, which entails by (5.32), $\langle w, \zeta(t)\rangle \leq \limsup_{n\to+\infty} \sigma(w, -\partial_P d_{C(\theta_n(t))}(u_n(\theta_n(t))))$, so Proposition 3.4 tells us that

$$\langle w, \zeta(t)\rangle \leq \sigma(w, -\partial_P d_{C(t)}(u(t))).$$

As the Clarke subdifferential is always closed and convex, this last inequality yields (using (2.1)), for all $t \in I\setminus I_0$,

$$\zeta(t) \in -\partial_C d_{C(t)}(u(t)) \subset -N^C(C(t);u(t)).$$

As a consequence, $u(\cdot)$ is a solution satisfying (5.2).

It remains to show that (5.3) holds true. Clearly, the inequality is invariant with respect to absolutely continuously equivalent measures, hence it suffices to show it with the measure ν involved in the development above. Consider first any $t \in [T_0, T]$ with $\nu(\{t\}) > 0$. For such an element t, we know that $\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = 0$ and by (5.55)

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}\nu}(t)\right\| = \left\|\frac{u(t) - u(t^-)}{\nu(\{t\})}\right\| \leq \frac{\mu(\{t\})}{\nu(\{t\})} = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(t),$$

which guarantees the inequality

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right\| \leq \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(t) + \|f(t,u(t))\|\,\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t).$$

On the other hand, take $A \subset I$ with $\nu(A) = 0$ such that for all $t \in I \setminus A$ the inclusion

$$\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \in -N\big(C(t);u(t)\big)$$

is fulfilled (so, in particular both $\frac{\mathrm{d}u}{\mathrm{d}\nu}(t)$ and $\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)$ exist). Fix any such t satisfying $\nu(\{t\}) = 0$. By definition of proximal normal, there is some real $a > 0$ such that

$$u(t) \in \operatorname{Proj}_{C(t)}\left(u(t) - a\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) - af(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right)\cdot$$

Since $\nu(]s,t]) \to \nu(\{t\}) = 0$ as $s \uparrow t$, we can choose some $s_0 \in I$ with $s_0 < t$ such that $0 < \nu(]s,t]) < a$ for all $s \in [s_0, t[$. Thus, for every $s \in [s_0, t[$, with $a_s := \nu(]s,t])$ we derive from the latter inclusion that

$$\begin{aligned} a_s \left\| \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right\| &= d_{C(t)}\left(u(t) - a_s\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) - a_s f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right) \\ &\le d_{C(s)}(u(s)) + \mu(]s,t]) + \left\| u(t) - u(s) - a_s\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) - a_s f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right\|, \end{aligned}$$

where the inequality follows from the variation assumption of $C(\cdot)$. Fix any $s \in [s_0, t[$. Since $d_{C(s)}(u(s)) = 0$, dividing by $a_s > 0$ we obtain

$$\left\| \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right\| \le \frac{\mu(]s,t])}{\nu(]s,t])} + \left\| \frac{u(t)-u(s)}{\nu(]s,t])} - \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) - f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right\|\cdot$$

Making $s \uparrow t$ yields

$$\begin{aligned} \left\| \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right\| &\le \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(t) + \left\| \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) - \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) - f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right\| \\ &= \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(t) + \|f(t,u(t))\|\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\cdot \end{aligned}$$

Consequently, as desired, (5.3) holds true for all $t \in I \setminus A$. This finishes the proof of existence of a solution satisfying (5.2) and (5.3), in the case when the inequality

$$\int_{[T_0,T]} (\beta(s)+1)\mathrm{d}\lambda(s) \le \frac{1}{4}$$

is fulfilled.

Case 2. Assume that

$$\int_{[T_0,T]} (\beta(s)+1)\mathrm{d}\lambda(s) > \frac{1}{4}\cdot$$

Let us follow [1,12]. First, we note that the mapping $u(\cdot)$ in the above case is also a solution with measure $\mu+\lambda$ in place of ν therein, since the measure $\mu+\lambda$ is absolutely continuous with respect to ν and *vice versa*. Let $T_0, \dots, T_p$ (with $p \in \mathbb{N}$) be such that, for each $i \in \{1, \dots, p\}$,

$$\int_{[T_{i-1},T_i]} (\beta(s)+1)\mathrm{d}\lambda(s) \le \frac{1}{4}\cdot$$

For each $i \in \{1, \dots, p\}$, denote by μ_i (resp. λ_i) the Radon measure induced on $[T_{i-1}, T_i]$ by μ (resp. λ) and set $\nu_i := \mu_i + \lambda_i$. Then, the Case 1 provides a mapping $u_1 : [T_0, T_1] \longrightarrow \mathcal{H}$ right continuous on $[T_0, T_1]$ with bounded variation, such that $u_1(t) \in C(t)$ for all $t \in [T_0, T_1]$, $u_1(T_0) = u_0$,

$$\left\|u_1(t) - u_1(t^-)\right\| \le \mu_1(\{t\}) = \mu(\{t\}) \quad \text{for all } t \in]T_0, T_1],$$

$$\left\| \frac{\mathrm{d}u_1}{\mathrm{d}\nu_1}(t) + f(t,u_1(t))\frac{\mathrm{d}\lambda_1}{\mathrm{d}\nu_1}(t)\right\| \le \frac{\mathrm{d}\mu_1}{\mathrm{d}\nu_1}(t) + \|f(t,u_1(t))\|\frac{\mathrm{d}\lambda_1}{\mathrm{d}\nu_1}(t) \quad \nu_1\text{-a.e. } t \in [T_0, T_1],$$

$\mathrm{d}u_1$ has $\frac{\mathrm{d}u_1}{\mathrm{d}\nu_1}$ in $L^1([T_0,T_1],\mathcal{H},\nu_1)$ as density relative to ν_1 and

$$\frac{\mathrm{d}u_1}{\mathrm{d}\nu_1}(t)+f(t,u_1(t))\frac{\mathrm{d}\lambda_1}{\mathrm{d}\nu_1}(t)\in -N(C(t);u_1(t))\ \ \nu_1\text{-a.e. } t\in[T_0,T_1].$$

Similarly, there is a right continuous with bounded variation mapping $u_2:[T_1,T_2]\longrightarrow\mathcal{H}$ such that $u_2(t)\in C(t)$ for all $t\in[T_1,T_2]$, $u_2(T_1)=u_1(T_1)$,

$$\left\|u_2(t)-u_2(t^-)\right\|\le\mu_2(\{t\})=\mu(\{t\})\quad\text{for all } t\in]T_1,T_2],$$

$$\left\|\frac{\mathrm{d}u_2}{\mathrm{d}\nu_2}(t)+f(t,u_2(t))\frac{\mathrm{d}\lambda_2}{\mathrm{d}\nu_2}(t)\right\|\le\frac{\mathrm{d}\mu_2}{\mathrm{d}\nu_2}(t)+\|f(t,u_2(t))\|\frac{\mathrm{d}\lambda_2}{\mathrm{d}\nu_2}(t)\quad\nu_2\text{-a.e. } t\in[T_1,T_2],$$

$\mathrm{d}u_2$ has $\frac{\mathrm{d}u_2}{\mathrm{d}\nu_2}$ as a density in $L^1([T_1,T_2],\mathcal{H},\nu_2)$ relative to ν_2 and

$$\frac{\mathrm{d}u_2}{\mathrm{d}\nu_2}(t)+f(t,u_2(t))\frac{\mathrm{d}\lambda_2}{\mathrm{d}\nu_2}(t)\in -N(C(t);u_2(t))\ \ \nu_2\text{-a.e. } t\in[T_1,T_2].$$

So, by induction, we obtain for each $i\in\{1,\dots,p\}$ a mapping $u_i:[T_{i-1},T_i]\longrightarrow\mathcal{H}$ with bounded variation and right continuous such that, $u_i(t)\in C(t)$ for all $t\in[T_{i-1},T_i]$, $u_i(T_{i-1})=u_{i-1}(T_{i-1})$,

$$\left\|u_i(t)-u_i(t^-)\right\|\le\mu_i(\{t\})=\mu(\{t\})\quad\text{for all } t\in]T_{i-1},T_i],$$

$$\left\|\frac{\mathrm{d}u_i}{\mathrm{d}\nu_i}(t)+f(t,u_i(t))\frac{\mathrm{d}\lambda_i}{\mathrm{d}\nu_i}(t)\right\|\le\frac{\mathrm{d}\mu_i}{\mathrm{d}\nu_i}(t)+\|f(t,u_i(t))\|\frac{\mathrm{d}\lambda_i}{\mathrm{d}\nu_i}(t)\quad\nu_i\text{-a.e. } t\in[T_{i-1},T_i],$$

the vector measure $\mathrm{d}u_i$ has $\frac{\mathrm{d}u_i}{\mathrm{d}\nu_i}$ as a density in $L^1([T_{i-1},T_i],\mathcal{H},\nu_i)$ relative to ν_i, and there exists a Borel set of $[T_{i-1},T_i]$ with $\nu_i(B_i)=0$ such that

$$\frac{\mathrm{d}u_i}{\mathrm{d}\nu_i}(t)+f(t,u_i(t))\frac{\mathrm{d}\lambda_i}{\mathrm{d}\nu_i}(t)\in -N(C(t);u_i(t))\quad\text{for all } t\in[T_{i-1},T_i]\setminus B_i. \tag{5.58}$$

Then, the mapping $u:[T_0,T]\longrightarrow\mathcal{H}$ with $u(t):=u_i(t)$ if $t\in[T_{i-1},T_i]$ $(i\in\{1,\dots,p\})$ is well defined and right continuous with bounded variation, and the inclusions $u(t)\in C(t)$, for all $t\in[T_0,T]$, along with the equality $u(T_0)=u_0$ hold true. Further, with the measure $\nu_0:=\mu+\lambda$ on $[T_0,T]$, the inequalities

$$\left\|u(t)-u(t^-)\right\|\le\mu(\{t\})\quad\text{for all } t\in]T_0,T]$$

and

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}\nu_0}(t)+f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu_0}(t)\right\|\le\frac{\mathrm{d}\mu}{\mathrm{d}\nu_0}(t)+\|f(t,u(t))\|\frac{\mathrm{d}\lambda}{\mathrm{d}\nu_0}(t)\quad\nu\text{-a.e. } t\in[T_0,T],$$

are obviously fulfilled. On the other hand, considering the function g defined for ν_0-almost every $t\in[T_0,T]$ by

$$g(t):=1_{[T_0,T_1]}(t)\frac{\mathrm{d}u_1}{\mathrm{d}\nu_1}(t)+\sum_{i=2}^{p}1_{]T_{i-1},T_i]}(t)\frac{\mathrm{d}u_i}{\mathrm{d}\nu_i}(t), \tag{5.59}$$

we easily see that $u(t)=u(T_0)+\int_{]T_0,t]}g(s)\mathrm{d}\nu_0(s)$ for all $t\in[T_0,T]$, so the vector measure $\mathrm{d}u$ has $g(\cdot)\in L^1([T_0,T],\mathcal{H},\nu_0)$ as a density relative to ν_0 and $\frac{\mathrm{d}u}{\mathrm{d}\nu_0}(\cdot)=g(\cdot)$ ν_0-a.e., that is, there is some Borel set $B'\subset I$ with $\nu_0(B')=0$ such that

$$\frac{\mathrm{d}u}{\mathrm{d}\nu_0}(t)=g(t)\quad\text{for all } t\in[T_0,T]\setminus B'. \tag{5.60}$$

Let $B := B' \cup \bigcup_{i=1}^{p} B_i$. Fix any $\tau \in [T_0, T] \setminus B$. Either $\tau \in [T_0, T_1]$ or $\tau \in]T_{i_0}, T_{i_0+1}]$ for some $i_0 \geq 1$. On the one hand, considering the case where τ is an interior or endpoint of the corresponding interval and using (2.11) we see that $\frac{\mathrm{d}\lambda_{i_0}}{\mathrm{d}\nu_{i_0}}(\tau) = \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(\tau)$. On the other hand, from (5.59) and (5.60), we have

$$\frac{\mathrm{d}u_{i_0}}{\mathrm{d}\nu_{i_0}}(\tau) = g(\tau) = \frac{\mathrm{d}u}{\mathrm{d}\nu_0}(\tau).$$

For all $t \in I \setminus B$, it results that

$$\frac{\mathrm{d}u}{\mathrm{d}\nu_0}(t) + f(t, u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu_0}(t) \in -N(C(t); u(t)),$$

so $u(\cdot)$ is a solution on the whole interval $[T_0, T]$ satisfying both conditions (5.2) and (5.3). □

The next result can be viewed as an extension of the equality (2.13) of Moreau [23] to Lipschitz perturbed BV sweeping process with nonconvex prox-regular sets.

Corollary 5.2. *Under the assumptions of Theorem 5.1, for each $u_0 \in C(T_0)$, the solution obtained in this theorem for the measure differential perturbed sweeping process*

$$(\mathcal{P}) \begin{cases} -\mathrm{d}u \in N(C(t); u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \end{cases}$$

also satisfies the equality

$$u(t) = \mathrm{P}_{C(t)}(u(t^-)) \quad \textit{for all}\, t \in]T_0, T].$$

Proof. Fix any $u_0 \in C(T_0)$. We know by (5.2) that $(\mathcal{P})$ has a solution satisfying,

$$\left\| u(t) - u(t^-) \right\| \leq \mu(\{t\}) \quad \text{for all}\, t \in]T_0, T].$$

Fix any $t \in]T_0, T]$.

Case 1. $\mu(\{t\}) = 0$. Under this assumption the above inequality ensures that

$$u(t^-) = u(t) \in C(t),$$

hence $u(t) = \mathrm{P}_{C(t)}(u(t^-))$.

Case 2. $\mu(\{t\}) > 0$. In this second case, we have

$$\left\| u(t) - u(t^-) \right\| \leq \mu(\{t\}) \leq \sup_{s \in]T_0, T]} \mu(\{s\}) < r. \tag{5.61}$$

With $\nu := \mu + \lambda$, the inequality $\mu(\{t\}) > 0$ also entails $\nu(\{t\}) > 0$, thus by definition of a solution

$$\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t, u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \in -N(C(t); u(t)).$$

By (2.10), we have $\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = 0$, so the last inclusion gives us

$$\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) \in -N(C(t); u(t)).$$

Using this and (2.12), we obtain

$$-N\big(C(t); u(t)\big) \ni \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) = \lim_{s\uparrow t} \frac{\mathrm{d}u(]s,t])}{\nu(]s,t])} = \lim_{s\uparrow t} \frac{u(t)-u(s)}{\nu(]s,t])} = \frac{u(t)-u(t^-)}{\nu(\{t\})},$$

which is equivalent to

$$u(t^-) - u(t) \in N(C(t); u(t)), \tag{5.62}$$

since $N(C(t); u(t))$ is a cone. Using (5.61) and (5.62), we can apply Proposition 3.5 to get

$$u(t) = \mathrm{P}_{C(t)}(u(t^-)). \qquad \square$$

6. Uniqueness

Requiring a control on the jumps, we have the following result of uniqueness of solution for our measure differential inclusion.

Theorem 6.1. *Under the assumptions of Theorem 5.1, for each $u_0 \in C(T_0)$, the perturbed sweeping process*

$$(\mathcal{P}) \begin{cases} -\mathrm{d}u \in N(C(t); u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \end{cases}$$

has one and only one solution satisfying

$$\sup_{s\in]T_0,T]} \big\|u(s) - u(s^-)\big\| < \frac{r}{2}\cdot$$

Further, this solution has the properties:

$$u(t) = P_{C(t)}\big(u(t^-)\big) \quad \textit{for all } t \in]T_0, T],$$

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right\| \le \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(t) + \|f(t,u(t))\|\,\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \quad \nu\textit{-a.e. } t \in [T_0, T].$$

Proof. Fix any $u_0 \in C(T_0)$.

Existence. Using Theorem 5.1 and Corollary 5.2, $(\mathcal{P})$ has a solution $u(\cdot)$ satisfying

$$\big\|u(t) - u(t^-)\big\| \le \mu(\{t\}) \quad \text{for all } t \in]T_0, T],$$

as well as the property $u(t) = P_{C(t)}\big(u(t^-)\big)$ for all $t \in]T_0, T]$ and the inequality

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right\| \le \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(t) + \|f(t,u(t))\|\,\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \quad \text{for } \nu\text{-a.e. } t \in I,$$

where $\nu = \mu + \lambda$. Combining the inequality above and $\sup_{s\in]T_0,T]} \mu(\{s\}) < \frac{r}{2}$, we get

$$\sup_{s\in]T_0,T]} \big\|u(s) - u(s^-)\big\| < \frac{r}{2}.$$

Uniqueness. To prove the uniqueness, consider two solutions $u_1(\cdot), u_2(\cdot) : I \longrightarrow \mathcal{H}$ of $(\mathcal{P})$ (with the same initial condition u_0) such that for each $i \in \{1, 2\}$,

$$\sup_{s\in]T_0,T]} \big\|u_i(s) - u_i(s^-)\big\| < \frac{r}{2}\cdot$$

Let $\nu := \mu + \lambda$. Since the concept of solution does not depend on the Radon measure absolutely continuously equivalent to $\mu + \lambda$, one has for each $i \in \{1, 2\}$,

$$A_i(t) := \frac{\mathrm{d}u_i}{\mathrm{d}\nu}(t) + f(t, u_i(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \in -N(C(t); u_i(t)) \quad \nu\text{-a.e. } t \in I.$$

By the r-prox-regularity of the sets $C(t)$, the latter inclusion and Theorem 2.2, it ensues that for ν-almost every $t \in I$,

$$\langle A_1(t) - A_2(t), u_1(t) - u_2(t)\rangle \leq \frac{1}{2r}\|u_1(t) - u_2(t)\|^2 \left(\sum_{i=1}^{2}\left(\left\|\frac{\mathrm{d}u_i}{\mathrm{d}\nu}(t)\right\| + \|f(t, u_i(t))\|\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right)\right)\cdot$$

Since the BV mappings $u_1(\cdot)$ and $u_2(\cdot)$ are in particular bounded on $I = [T_0, T]$, we can choose some real $\alpha > 0$ such that, for each $i \in \{1, 2\}$, $\|u_i(t)\| \leq \alpha$ for all $t \in I$. We then obtain, for ν-almost every $t \in I$,

$$\begin{aligned}
&\left\langle \frac{\mathrm{d}u_1}{\mathrm{d}\nu}(t) - \frac{\mathrm{d}u_2}{\mathrm{d}\nu}(t), u_1(t) - u_2(t)\right\rangle \\
&\leq \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\,\langle f(t, u_2(t)) - f(t, u_1(t)), u_1(t) - u_2(t)\rangle \\
&\quad + \frac{1}{2r}\|u_1(t) - u_2(t)\|^2 \left(\sum_{i=1}^{2}\left(\left\|\frac{\mathrm{d}u_i}{\mathrm{d}\nu}(t)\right\| + \|f(t, u_i(t))\|\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right)\right) \\
&\leq \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)L_\alpha(t)\|u_1(t) - u_2(t)\|^2 + \frac{1}{2r}\|u_1(t) - u_2(t)\|^2 \left(\sum_{i=1}^{2}\left(\left\|\frac{\mathrm{d}u_i}{\mathrm{d}\nu}(t)\right\| + \|f(t, u_i(t))\|\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right)\right)\cdot
\end{aligned}$$

Using Proposition 3.2, we deduce that, for all $t \in I$,

$$\|u_1(t) - u_2(t)\|^2 \leq \int_{]T_0, t]} g(s)\|u_1(s) - u_2(s)\|^2 \,\mathrm{d}\nu(s),$$

where $g(t) := 2[\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)L_\alpha(t) + \frac{1}{2r}(\sum_{i=1}^{2}(\|\frac{\mathrm{d}u_i}{\mathrm{d}\nu}(t)\| + \|f(t, u_i(t))\|\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)))]$ for ν-almost every $t \in I$. Observe also that (see (2.10)), for ν-almost every $t \in I$, $\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\nu(\{t\}) = 0$. Furthermore, for each $i \in \{1, 2\}$, since $\frac{\mathrm{d}u}{\mathrm{d}\nu}$ is a density of $\mathrm{d}u$ relative to ν, with $\gamma := 2\max\limits_{1\leq i\leq 2}\sup\limits_{s\in]T_0,T]}\|u_i(s) - u_i(s^-)\| < r$ one also has, for ν-almost every $t \in I$,

$$\left\|\frac{\mathrm{d}u_i}{\mathrm{d}\nu}(t)\right\|\nu(\{t\}) = \left\|u_i(t) - u_i(t^-)\right\| \leq \sup_{s\in]T_0,T]}\left\|u_i(s) - u_i(s^-)\right\| \leq \frac{\gamma}{2}\cdot$$

It ensues that

$$0 \leq g(t)\nu(\{t\}) = \frac{1}{r}\sum_{i=1}^{2}\left\|\frac{\mathrm{d}u_i}{\mathrm{d}\nu}(t)\right\|\nu(\{t\}) \leq \frac{\gamma}{r} < 1,$$

and this allows us to use Lemma 3.1 to obtain for all $t \in I$, $\|u_1(t) - u_2(t)\|^2 \leq 0$. This proves the uniqueness. □

7. Existence and uniqueness on a non-compact interval

Now, we investigate the case where I is a non-compact interval $[T_0, \tau[$ of $\mathbb{R}$ with $\tau \in]T_0, +\infty]$.

Theorem 7.1. *Let $\tau \in \mathbb{R} \cup \{+\infty\}$ and let $T_0 \in \mathbb{R}$ with $T_0 < \tau$. Let $r \in]0, +\infty]$ and let $C(\cdot) : I := [T_0, \tau[\rightrightarrows \mathcal{H}$ be a set-valued mapping from I into the r-prox-regular subsets of the real Hilbert space $\mathcal{H}$ for which there exists a positive Radon measure μ on I such that,*

$$|d(y, C(t)) - d(y, C(s))| \le \mu\left(]s, t]\right) \quad \textit{for all } y \in \mathcal{H}, \textit{ for all } s, t \in I \textit{ with } s \le t.$$

Assume that, one has $\sup\limits_{s \in]T_0, \tau[} \mu(\{s\}) < \frac{r}{2}$. Let $f : I \times \mathcal{H} \longrightarrow \mathcal{H}$ be a mapping such that:

(i) *the mapping $f(\cdot, x)$ is Lebesgue measurable for each $x \in \mathcal{H}$ and there exists a nonnegative function $\beta : I \longrightarrow \mathbb{R}$ with $\beta \in L^1_{\text{loc}}(I, \mathbb{R}, \lambda)$ such that, for all $t \in I$, $x \in \bigcup\limits_{\tau \in I} C(\tau)$,*

$$\|f(t, x)\| \le \beta(t)(1 + \|x\|);$$

(ii) *for each real $\alpha \ge 0$, there exists some nonnegative function $L_\alpha : I \longrightarrow \mathbb{R}$ with $L_\alpha \in L^1_{\text{loc}}(I, \mathbb{R}, \lambda)$ such that, for all $t \in I$, for all $x, y \in \alpha\mathbb{B}_{\mathcal{H}}$,*

$$\|f(t, x) - f(t, y)\| \le L_\alpha(t) \, \|x - y\| \, .$$

Then, for each $u_0 \in C(T_0)$, the following measure differential inclusion sweeping process on $[T_0, \tau[$

$$(\mathcal{P}) \begin{cases} -\mathrm{d}u \in N(C(t); u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \end{cases}$$

has a unique solution satisfying

$$\sup_{s \in]T_0, \tau[} \left\|u(s) - u(s^-)\right\| < \frac{r}{2}\cdot$$

Further, one has $u(t) = P_{C(t)}\big(u(t^-)\big)$ for all $t \in]T_0, \tau[$ and with $\nu := \mu + \lambda$

$$\left\| \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t, u(t)) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \right\| \le \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(t) + \|f(t, u(t))\| \, \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \ \nu\textit{-a.e. } t \in I.$$

Proof. Fix any $u_0 \in C(T_0)$.

Existence. Let us adjoin to T_0 an increasing sequence $(T_i)_{i \ge 1}$ in $]T_0, \tau[$ tending to τ. For each integer $i \ge 0$ denote by μ_i (resp. λ_i) the measure induced by μ (resp. λ) on $[T_i, T_{i+1}]$. Put also $\nu_i = \mu_i + \lambda_i$ and notice that, for all $y \in \mathcal{H}$,

$$|d(y, C(t)) - d(y, C(s))| \le \mu_i(]s, t]) \quad \text{for all } s, t \in [T_i, T_{i+1}] \text{ with } s < t.$$

Using Theorem 5.1, there exists a solution $U_0 : [T_0, T_1] \longrightarrow \mathcal{H}$ of the sweeping process

$$(\mathcal{P}_0) \begin{cases} -\mathrm{d}U_0 \in N(C(t); U_0(t)) + f(t, U_0(t)) \\ U_0(T_0) = u_0 \end{cases}$$

which satisfies

$$\left\|U_0(t) - U_0(t^-)\right\| \le \mu_0(\{t\}) = \mu(\{t\}) \text{ and } U_0(t) = P_{C(t)}\big(U_0(t^-)\big) \text{ for all } t \in]T_0, T_1],$$

and also

$$\left\| \frac{\mathrm{d}U_0}{\mathrm{d}\nu_0}(t) + f(t, U_0(t)) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu_0}(t) \right\| \le \frac{\mathrm{d}\mu}{\mathrm{d}\nu_0}(t) + \|f(t, U_0(t))\| \, \frac{\mathrm{d}\lambda}{\mathrm{d}\nu_0}(t) \quad \text{for } \nu_0\text{-a.e. } t \in [T_0, T_1].$$

Besides U_0 we construct by induction (thanks to Thm. 5.1) a sequence $(U_j)_{j\geq 1}$ of mappings such that, for all integer $j \geq 1$, $U_j : [T_j, T_{j+1}] \longrightarrow \mathcal{H}$ is a solution of

$$(\mathcal{P}_j)\begin{cases} -\mathrm{d}U_j \in N(C(t); U_j(t)) + f(t, U_j(t)) \\ U_j(T_j) = U_{j-1}(T_j) \end{cases}$$

and satisfies

$$\big\|U_j(t) - U_j(t^-)\big\| < \mu_j(\{t\}) = \mu(\{t\}) \ \text{ and } \ U_j(t) = P_{C(t)}\big(U_j(t^-)\big) \ \text{ for all } t \in]T_j, T_{j+1}]$$

as well as

$$\left\|\frac{\mathrm{d}U_j}{\mathrm{d}\nu_j}(t) + f(t, U_j(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu_j}(t)\right\| \leq \frac{\mathrm{d}\mu}{\mathrm{d}\nu_j}(t) + \|f(t, U_j(t))\| \frac{\mathrm{d}\lambda}{\mathrm{d}\nu_j}(t) \quad \text{for } \nu_j\text{-a.e. } t \in [T_j, T_{j+1}].$$

Let us define $u : [T_0, \tau[\longrightarrow \mathcal{H}$ with $u(t) := U_j(t)$ if $t \in [T_j, T_{j+1}]$ for some $j \geq 0$. Proceeding as in the proof of Theorem 5.1 Case 2 with the function g defined for ν-almost every $t \in I$ (where $\nu := \mu + \lambda$) by

$$g(t) := \mathbf{1}_{[T_0,T_1]}\frac{\mathrm{d}U_0}{\mathrm{d}\nu_0}(t) + \sum_{j=1}^{+\infty} \mathbf{1}_{]T_j,T_{j+1}]}(t)\frac{\mathrm{d}U_j}{\mathrm{d}\nu_j}(t),$$

we see that u is a solution of $(\mathcal{P})$. Moreover, we have

$$\big\|u(t) - u(t^-)\big\| \leq \mu(\{t\}) \quad \text{and} \quad u(t) = P_{C(t)}\big(u(t^-)\big) \qquad \text{for all } t \in]T_0, \tau[,$$

hence in particular $\sup\limits_{s \in]T_0,\tau[} \|u(s) - u(s^-)\| < \frac{r}{2}$. Further, we also have

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t, u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t)\right\| \leq \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(t) + \|f(t, u(t))\| \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \quad \text{for } \nu\text{-a.e. } t \in [T_0, T].$$

This finishes the proof of the existence of a solution with the desired properties.

Uniqueness. Let u_1, u_2 be two solutions of $(\mathcal{P})$ such that, for each $i \in \{1, 2\}$

$$\sup_{s \in]T_0,\tau[} \big\|u_i(s) - u_i(s^-)\big\| < \frac{r}{2}.$$

For each $i \in \{1, 2\}$, for all integer $j \geq 0$, we have

$$\sup_{s \in]T_j,T_{j+1}]} \big\|u_{i|[T_j,T_{j+1}]}(s) - u_{i|[T_j,T_{j+1}]}(s^-)\big\| < \frac{r}{2}.$$

Applying Theorem 6.1, we get by induction on j that, for every integer $j \geq 0$,

$$u_{1|[T_j,T_{j+1}]}(t) = u_{2|[T_j,T_{j+1}]}(t), \quad \text{for all } t \in]T_j, T_{j+1}].$$

It results that $u_1 = u_2$, which finishes the proof. □

In the case when the sets $C(t)$ are convex for all $t \in I$, the condition $\sup \mu(\{s\}) < \frac{r}{2}$ holds automatically because in such a case, for all $t \in I$, $C(t)$ is r-prox-regular with $r = +\infty$. So, we retreive the existence part of ([1], Thm. 4.1).

Even for $f \equiv 0$, the problem $(\mathcal{P})$ may have more than one solution. For this fact, we refer to Remark 3.1(2) of [12].

8. Absolutely continuous sweeping process

In this section, we deal with the case where the measure μ is absolutely continuous relative to λ.

Proposition 8.1. *Let $T \in \mathbb{R}$ (resp. $\tau \in \mathbb{R} \cup \{+\infty\}$) and $I = [T_0, T]$ with $T_0 \in \mathbb{R}$ and $T_0 < T$ (resp. $I = [T_0, \tau[$ with $T_0 \in \mathbb{R}$ and $T_0 < \tau$). Let $C : I \rightrightarrows \mathcal{H}$ be a set-valued mapping from I into the real Hilbert space $\mathcal{H}$ such that, for some $r \in]0, +\infty]$, $C(t)$ is r-prox-regular for every $t \in I$. Assume that there exists a nondecreasing locally absolutely continuous function $v(\cdot)$ on I such that*

$$|d(y, C(s)) - d(y, C(t))| \le v(t) - v(s) \quad \textit{for all } y \in \mathcal{H}, \textit{ for all } s, t \in I \textit{ with } s \le t.$$

Assume also that the mapping f satisfies conditions (i) and (ii) in Theorem 5.1.

Then, considering the Radon measure μ with $\mu(]s,t]) = v(t) - v(s)$ for $s < t$, a solution of the measure differential sweeping process

$$(\mathcal{P}) \begin{cases} -\mathrm{d}u \in N(C(t); u(t)) + f(t, u(t)) \\ u(T_0) = u_0 \in C(T_0) \end{cases}$$

is a solution in the classical sense, that is:

(a) *u is absolutely continuous on I (resp. locally absolutely continuous on I);*
(b) *$-\frac{\mathrm{d}u}{\mathrm{d}t}(t) \in N(C(t); u(t)) + f(t, u(t))$ λ-a.e. $t \in I$;*
(c) *$u(T_0) = u_0$ and $u(t) \in C(t)$ for all $t \in I$.*

So, $(\mathcal{P})$ admits one and only one absolutely (resp. locally absolutely) continuous solution $u(\cdot)$ on I, and further

$$\left\| \frac{\mathrm{d}u}{\mathrm{d}t}(t) + f(t, u(t)) \right\| \le \frac{\mathrm{d}v}{\mathrm{d}t}(t) + \|f(t, u(t))\| \quad \lambda\textit{-a.e. } t \in I.$$

Proof. Let $u(\cdot) : I \longrightarrow \mathcal{H}$ be a solution of $(\mathcal{P})$ in the measure differential sense. Set $\nu = \mu + \lambda$ and observe that the restriction of the measure ν to any compact interval of I is absolutely continuously equivalent to the restriction of the Lebesgue measure λ to that compact interval. Then, there exists a mapping $h : I \longrightarrow [0, +\infty[$ λ-integrable (resp. locally λ-integrable) on I such that $\nu = h(\cdot)\lambda$. Thanks to the equalities (λ-a.e.) $h(\cdot) = \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(\cdot)$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(\cdot)\frac{\mathrm{d}u}{\mathrm{d}\nu}(\cdot) = \frac{\mathrm{d}u}{\mathrm{d}\lambda}(\cdot)$, we have

$$u(t) = u_0 + \int_{]T_0,t]} h(s) \frac{\mathrm{d}u}{\mathrm{d}\nu}(s) \mathrm{d}\lambda(s) \quad \text{for all } t \in I.$$

As a consequence, the mapping $u(\cdot)$ is absolutely (resp. locally absolutely) continuous on I and there exists a Borel set B_1 of I with $\lambda(B_1) = 0$ such that

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = h(t) \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) \quad \text{for all } t \in I \setminus B_1.$$

Since $u(\cdot)$ is a solution of $(\mathcal{P})$ in the measure differential sense, there exists a Borel set B_2 in I with $\nu(B_2) = 0$ such that

$$\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t, u(t)) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \in -N(C(t); u(t)) \quad \text{for all } t \in I \setminus B_2.$$

Setting $B = B_1 \cup B_2$, we see that $\lambda(B) = 0$ and, for all $t \in I \setminus B$,

$$h(t) \frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t, u(t)) h(t) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) \in -N(C(t); u(t)).$$

On the other hand, for all $s, t \in I$ with $s < t$,

$$\int_{]s,t]} h(\theta) \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(\theta) \mathrm{d}\lambda(\theta) = \int_{]s,t]} \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(\theta) h(\theta) \mathrm{d}\lambda(\theta) = \int_{]s,t]} \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(\theta) \mathrm{d}\nu(\theta) = \int_{]s,t]} \mathrm{d}\lambda(\theta).$$

It follows that

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + f(t,u(t)) \in -N(C(t);u(t)) \quad \lambda\text{-a.e. } t \in I,$$

and this finishes the proof. $\square$

9. Application to nonlinear differential complementarity systems

In view of application to nonlinear differential complementarity systems, we will start with a subsection devoted to the prox-regularity of sublevel sets.

9.1. Prox-regularity of the moving set described by inequality constraints

The theorem of this subsection providing sufficient conditions for the uniform prox-regularity of a set of sublevel constraints is the following:

Theorem 9.1. *Let I be a nonempty set, $\mathcal{H}$ be a real Hilbert space, and $g_k : I \times \mathcal{H} \to \mathbb{R}$ with $k = 1,\dots,m$ be functions such that, for each $t \in I$, the set*

$$C(t) = \{x \in \mathcal{H} : g_1(t,x) \le 0, \dots, g_m(t,x) \le 0\}$$

is nonempty. Assume that there exists some $\rho \in]0,+\infty]$ such that:

(i) *for all $t \in I$, for all $k \in \{1,\dots,m\}$, $g_k(t,\cdot)$ is of class C^1 on $U_\rho(C(t))$;*

(ii) *there exists a real $\gamma > 0$ such that, for all $t \in I$, for all $x \in \operatorname{bdry} C(t)$, for all $y \in U_\rho(C(t))$, for all $k \in \{1,\dots,m\}$ with $g_k(t,x) = 0$,*

$$\langle \nabla g_k(t,\cdot)(y) - \nabla g_k(t,\cdot)(x), y - x\rangle \ge -\gamma \|y-x\|^2 .$$

Assume also that there is a real $\delta > 0$ such that, for any $(t,x) \in I \times \mathcal{H}$ with $x \in \operatorname{bdry} C(t)$ and any $\zeta \in \operatorname{co}\{\nabla g_k(t,\cdot)(x) : k \in K(t,x)\}$ where $K(t,x) := \{k \in \{1,\dots,m\} : g_k(t,x) = 0\}$, there exists $v(t,x,\zeta) \in \mathbb{B}_{\mathcal{H}}$ satisfying $\langle \zeta, v(t,x,\zeta)\rangle \le -\delta$.

Then, for all $t \in I$, the set $C(t)$ is r-prox-regular with $r = \min\left\{\rho, \frac{\delta}{\gamma}\right\}$.

Proof. All the sets $C(t)$ are clearly closed according to the continuity of the functions $g_k(t,\cdot)$ over $U_\rho(C(t))$. Set $K := \{1,\dots,m\}$ and

$$g(t,x) := \max_{k\in K} g_k(t,x) \quad \text{for all } (t,x) \in I \times \mathcal{H}.$$

Put also, for all $x \in \mathcal{H}$, for all $t \in I$, $K(t,x) = \{k \in K : g_k(t,x) = 0\}$. Fix now any $t \in I$. One observes that $C(t) = \{x \in \mathcal{H} : g(t,x) \le 0\}$. By assumption (i), for all $k \in K$, the function $g_k(t,\cdot)$ is locally Lipschitz continuous on $U_\rho(C(t))$. Using (ii) and ([8], Cor. 10.23), one has

$$\partial_C g(t,\cdot)(x) \subset \operatorname{co}\{\nabla g_k(t,\cdot)(x) : k \in K(t,x)\} \tag{9.1}$$

for all $x \in \operatorname{bdry} C(t)$. It is straighforward by the assumption involving $v(t,x,\zeta)$ and (9.1) that $0 \notin \partial_C g(t,\cdot)(x)$ for all $x \in \operatorname{bdry} C(t)$. So, by ([8], Thm. 10.42) we obtain that, for all $x \in \operatorname{bdry} C(t)$

$$N^C(C(t);x) = \bigcup_{\alpha\ge 0} \alpha \partial_C g(t,\cdot)(x). \tag{9.2}$$

Fix any $x, y \in C(t)$ with $\|y-x\| < 2\rho$ and $x \in \operatorname{bdry} C(t)$ (hence $g(t,x)=0$). For all $s \in [0,1]$, one has

$$\begin{aligned} d_{C(t)}(x+s(y-x)) &\le \min\{\|x+s(y-x)-x\|, \|x+s(y-x)-y\|\} \\ &= \min\{s, 1-s\}\|y-x\| \\ &\le \frac{1}{2}\|y-x\| < \rho, \end{aligned}$$

i.e., $x+s(y-x) \in U_\rho(C(t))$. For all $s \in]0,1]$, for all $k \in K(t,x)$, one has

$$\begin{aligned} \langle \nabla g_k(t,\cdot)(x+s(y-x)) - \nabla g_k(t,\cdot)(x), y-x\rangle &= \frac{1}{s}\langle \nabla g_k(t,\cdot)(x+s(y-x)) - \nabla g_k(t,\cdot)(x), s(y-x)\rangle \\ &\ge -\frac{1}{s}\gamma\|s(y-x)\|^2 = -\gamma s\|x-y\|^2. \end{aligned}$$

Then, for every $k \in K(t,x)$

$$\begin{aligned} 0 \ge g_k(t,y) - g_k(t,x) &= \int_0^1 \langle \nabla g_k(t,\cdot)(x+s(y-x)), y-x\rangle \, \mathrm{d}s \\ &= \langle \nabla g_k(t,\cdot)(x), y-x\rangle + \int_0^1 \langle \nabla g_k(t,\cdot)(x+s(y-x)) - \nabla(g_k(t,\cdot)(x), y-x\rangle \, \mathrm{d}s \\ &\ge \langle \nabla g_k(t,\cdot)(x), y-x\rangle - \gamma\|y-x\|^2 \int_0^1 s \, \mathrm{d}s, \end{aligned}$$

hence $\langle \nabla g_k(t,\cdot)(x), y-x\rangle \le \frac{\gamma}{2}\|y-x\|^2$. This and the equality (9.1) imply

$$\langle \zeta, y-x\rangle \le \frac{\gamma}{2}\|y-x\|^2 \quad \text{for all } \zeta \in \partial_C g(t,\cdot)(x).$$

Further, for any $\zeta \in \partial g(x)$, the relation (9.1) again ensure that $\langle \zeta, -v(t,x,\zeta)\rangle \ge \delta$. Since $\|-v(t,x,\zeta)\| \le 1$, for every $\zeta \in \partial_C g(t,\cdot)(x)$ it follows that $\|\zeta\| \ge \delta$, thus

$$\langle \zeta, y-x\rangle \le \frac{\gamma}{2\delta}\|\zeta\|\,\|y-x\|^2.$$

It ensues that, for any $x, y \in C(t)$ with $x \in \operatorname{bdry} C(t)$ and $\|y-x\| < 2r$,

$$\langle \zeta, y-x\rangle \le \frac{1}{2r}\|\zeta\|\,\|y-x\|^2 \quad \text{for all } \zeta \in \partial_C g(t,\cdot)(x),$$

or equivalently according to (9.2)

$$\langle \zeta, y-x\rangle \le \frac{1}{2r}\|\zeta\|\,\|y-x\|^2 \quad \text{for all } \zeta \in N^C(C(t);x).$$

This and Proposition 2.4 justify the r-prox-regularity of the set $C(t)$. □

Given a nonempty open convex subset U of a real Hilbert space $\mathcal{H}$, a C^1-function $g : U \longrightarrow \mathbb{R}$ is known to be γ-*prox-regular* on U for some real $\gamma \ge 0$ if and only if

$$\langle \nabla g(x), y-x\rangle \le \gamma\|y-x\|^2 \quad \text{for all } x, y \in U,$$

that is, the function $g + \frac{\gamma}{2}\|\cdot\|^2$ is convex on U.

As a consequence, we get the following result.

Corollary 9.2. *Let I be a nonempty set, $\mathcal{H}$ be a real Hilbert space, and $g_k : I \times \mathcal{H} \to \mathbb{R}$ with $k = 1, \dots, m$ be functions such that, for each $t \in I$, the set*

$$C(t) = \{x \in \mathcal{H} : g_1(t,x) \le 0, \dots, g_m(t,x) \le 0\}$$

is nonempty. Assume that there exists some $\rho \in]0, +\infty]$ such that:

(i) *for all $t \in I$, for all $k \in \{1, \dots, m\}$, $g_k(t, \cdot)$ is C^1 on $U_\rho(C(t))$;*
(ii) *for some real $\gamma \ge 0$, the functions $g_k(t, \cdot)$ are γ-prox-regular on an open convex set containing $U_\rho(C(t))$ for all $t \in I$ and all $k \in \{1, \dots, m\}$.*

Assume also that there is a real $\delta > 0$ such that, for any $(t,x) \in I \times \mathcal{H}$ with $x \in \operatorname{bdry} C(t)$ and any $\zeta \in \operatorname{co}\{\nabla g_k(t,\cdot)(x) : k \in K(t,x)\}$ where $K(t,x) := \{k \in \{1, \dots, m\} : g_k(t,x) = 0\}$, there exists $v(t,x,\zeta) \in \mathbb{B}_{\mathcal{H}}$ satisfying $\langle \zeta, v(t,x,\zeta) \rangle \le -\delta$.

Then, for all $t \in I$, the set $C(t)$ is r-prox-regular with $r = \min\left\{\rho, \frac{\delta}{2\gamma}\right\}$.

A previous result has been established by Vial ([35], Prop. 4.10) for the prox-regularity (called therein weak convexity) of a set in the form $\{x \in \mathbb{R}^n : \varphi(x) \le 0\}$, with a single function $\varphi : \mathbb{R}^n \to \mathbb{R}$ which is weakly convex on $\mathbb{R}^n$; that result is encompassed by Corollary 9.2 in the context of differentiable functions. More general results on operations with (uniform) prox-regular sets will appear in a forthcoming paper. We also cite Venel ([34], Prop. 2.9) for a result with $\mathcal{H} = \mathbb{R}^n$ under some boundedness assumptions related to the first and second derivatives of the constraints functions $g_k(t, \cdot)$. Our statement, approach and proof of Theorem 9.1 are general and different from those of the aforementioned results in [34, 35].

9.2. Nonlinear differential complementarity systems

For a matrix $M \in \mathbb{R}^m \times \mathbb{R}^n$, M^T stands for the transpose matrix of M.

Nonlinear Complementarity Systems (NCS) is an important class of nonsmooth dynamical systems with a wide range of applications in mechanical and electrical engineering. It consists of an ordinary differential equation coupled with a nonlinear complementarity problem in the constraint. The novelty is that time-varying inequality constraints are allowed to take into account the constraints evolution with respect to time. NDCS belongs to the large class of hybrid dynamical system defined generally by a finite number of smooth modes described by an ordinary differential inclusion with transition between the modes through a switching surface. NCS plays a fundamental role in nonsmooth mechanics (multibody dynamics with contact, friction and impact), in nonregular electrical circuits (switched electrical networks, relay systems, circuit breakers), in control systems as well as in dynamical games. In this paragraph, we will show how to transform a NDCS involving inequality constraints to a sweeping process of the form (1.1).

Let $T > 0$ be a real, $I = [0,T]$, $n, m \in \mathbb{N}$, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : I \times \mathbb{R}^n \to \mathbb{R}^m$ two given mappings. Assuming that $g(t, \cdot)$ is differentiable for each $t \in I$, the NDCS (associated with f and g) can be described as

$$(NDCS) \begin{cases} -\mathrm{d}u = f(t, u(t)) + \nabla g(t,\cdot)(u(t))^T z(t) \\ 0 \le z(t) \perp g(t,x) \le 0, \end{cases}$$

where $z : I \to \mathbb{R}^m$ is unknown mapping. The term $\nabla g(t,\cdot)(u(t))^T z(t)$ can be seen as the generalized reactions due to the constraints in mechanics.

Of course, the behaviour of a solution with respect to t is connected to the variation with respect to t of the set constraint

$$C(t) := \{x \in \mathbb{R}^n : g_1(t,x) \le 0, \dots, g_m(t,x) \le 0\}$$

where we set $g(t,\cdot) = (g_1(t,\cdot),\dots,g_m(t,\cdot))$ for each $t \in I$. Generally, absolute continuity is required for the solution. Here jumps will be allowed. So, we assume that:

(H1) There exists a positive Radon measure μ on I such that

$$|d(y, C(t)) - d(y, C(s))| \le \mu\,(]s,t]) \quad \text{for all } y \in \mathbb{R}^n, \text{ for all } s,t \in I \text{ with } s \le t.$$

In this context, a mapping $u : I \to \mathbb{R}^n$ is a solution of (NDCS) whenever:

(a) $u(\cdot)$ is of bounded variation and right-continuous on I, and there is a Borel-measurable mapping $z : I \to \mathbb{R}^m$ with $z(I) \subset \mathbb{R}^m_+$ (where $\mathbb{R}_+ := [0,+\infty[$) and

$$\langle z(t), g(t,u(t))\rangle = 0 \quad \text{for all } t \in I;$$

(b) with $\nu := \mu + \lambda$, the differential measure $\mathrm{d}u$ is absolutely continuous with respect to ν, and for ν-almost every $t \in I$,

$$\frac{\mathrm{d}u}{\mathrm{d}\nu}(t) + f(t,u(t))\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(t) = -\nabla g(t,\cdot)(u(t))^T z(t).$$

If the variation of the set constraint is absolutely continuous, then the measure $\nu = \mu + \lambda$ is absolutely continuously equivalent to the Lebesgue measure λ, so the above definition is reduced to the classical concept of absolutely continuous solution for (NDCS).

For each $t \in I$, assume that $g(t,\cdot)$ is differentiable on the open enlargement $U_\rho(C(t)) := \{x \in \mathbb{R}^n : d(x, C(t)) < \rho\}$ and that $\nabla g(t,\cdot)$ is $\gamma-$Lipschitz continuous on $U_\rho(C(t))$, for some reals $\rho, \gamma > 0$. For any $x \in \mathbb{R}^n$, assume that $g(\cdot,x) : I \to \mathbb{R}^m$ is a Borel function. Assume also that:

(H2) There is a real $\delta > 0$ such that, for any $(t,x) \in [0,T] \times \mathbb{R}^n$ with $x \in \operatorname{bdry} C(t)$, there exists $\overline{v} \in \mathbb{B}$ satisfying for all $k \in \{1,\dots,m\}$

$$\nabla g_k(t,\cdot)(x)\,\overline{v}^T \le -\delta;$$

(H3) For each $t \in I$, $x \in C(t)$, $\nabla g(t,\cdot)(x)$ is of rank m.

It is straightforward that Theorem 9.1 ensures the r-prox-regularity of the set $C(t)$ with $r := \min\{\rho, \frac{\delta}{\gamma}\}$. Clearly, recalling that ψ_S denotes the indicator function of a set S (see Sect. 2.2), the following equality $\psi_{C(t)} = \psi_{\mathbb{R}^m_-} \circ g(t,\cdot)$ holds true, where $\mathbb{R}_- :=] - \infty, 0]$. Let $u : I \longrightarrow \mathbb{R}^n$ be a mapping.
Note that for any mapping $z : I \longrightarrow \mathbb{R}^m$, we have for each $t \in I$,

$$z(t) \in \mathbb{R}^m_+ \quad \text{and} \quad z(t)\,g(t,u(t))^T = 0 \Longleftrightarrow z(t) \in N(\mathbb{R}^m_-; g(t,u(t))). \tag{9.3}$$

Therefore, (NDCS) reduces to

$$-\mathrm{d}u \in f(t,u(t)) + \nabla g_t(u(t))^T\big(N(\mathbb{R}^m_-; g(t,u(t)))\big). \tag{9.4}$$

Further, invoking a chain rule of Clarke subdifferential (see [29], p. 428), we have

$$\partial_C \psi_{C(t)}(x) = \nabla g(t,\cdot)(x)^T\big(N(\mathbb{R}^m_-; g(t,u(t)))\big) \quad \text{for all } t \in I,\ x \in C(t). \tag{9.5}$$

Hence, u is a solution to the following sweeping process

$$-\mathrm{d}u \in f(t,u(t)) + N\big(C(t); u(t)\big). \tag{9.6}$$

whenever it is a solution of (NDCS).

Now, we show the converse implication. Assume that $u(\cdot)$ is a solution of (9.6). Let us note that $g(\cdot,u(\cdot))$ is Borel measurable. Hence, from ([29], Thm. 14.26) we deduce that $N(\mathbb{R}^m_-; g(\cdot,u(\cdot)))$ is a Borel-measurable closed-valued set-valued mapping. As a consequence (see, *e.g.*, [29], Cor. 14.6), there is a Borel-measurable mapping $z : [0,T] \longrightarrow \mathbb{R}^m$ such that $z(t) \in N\big(\mathbb{R}^m_-; g(t,u(t))\big)$ for all $t \in I$. Using (9.5) and (9.3), $u(\cdot)$ is a solution of (NDCS).

All together say, according to Theorems 5.1 and 6.1, that we have proved the following theorem:

Theorem 9.3. *Assume that* **H1**−**H3** *and conditions* (i) *and* (ii)*, for the mapping* f*, in Theorem* 5.1 *are satisfied, and assume also that* $\sup_{t\in]0,T]} \mu(\{t\}) < \frac{r}{2}$*. Then, for every initial data* u_0 *with* $g(0, u_0) \leq 0$*, problem (NDCS) has one and only one solution* $u(\cdot)$ *such that*

$$\sup_{s\in]0,T]} \|u(t) - u(t^-)\| < \frac{r}{2}.$$

10. Concluding remarks

In this paper, we studied the existence and the uniqueness of solution to a discontinuous sweeping process where the state trajectories are constrained to evolve in a prox-regular moving set having a variation given by a positive Radon measure. Various properties and estimates of jumps of the solution are also provided. This kind of problem arises in unilateral mechanics, in elastoplasticity, in mathematical economics as well as in the simulation of crowd motion. A sufficient condition ensuring the prox-regularity of the moving set, when it is described by inequality constraints, is given. An application to nonlinear differential complementarity system is also discussed in detail. The cornerstone of the existence proof is Moreau's catching-up algorithm adapted to prox-regular moving sets. This leads naturally to the numerical treatment of discontinuous nonconvex sweeping processes. It will be interesting to perform some numerical experiments on concrete examples. In nonsmooth mechanics, some constraints could be nondifferentiable. A natural question would be to generalize the conditions in Theorem 9.1. We contented ourselves with studying the single-valued perturbation f. Adding a set-valued one F is of a great interest in economical problems (see, *e.g.*, [10, 13, 15]). This and the study of preservation of prox-regularity under various operations are out of the scope of this manuscript. Both studies will be the subject of forthcoming research projects.

References

[1] S. Adly, T. Haddad and L. Thibault, Convex sweeping process in the framework of measure differential inclusions and evolution variational inequalities. *Math. Program. Ser. B* **148** (2014) 5–47.

[2] H. Benabdellah, Existence of solutions to the nonconvex sweeping process. *J. Differ. Eqs.* **164** (2000) 286–295.

[3] M. Bounkhel, L. Thibault, On various notions of regularity of sets in nonsmooth analysis. *Nonlinear Anal. Ser. A: Theory Methods* **48** (2002) 223–246.

[4] M. Bounkhel and L. Thibault, Nonconvex sweeping process and prox-regularity in Hilbert space. *J. Nonlinear Convex Anal.* **6** (2005) 359–374.

[5] C. Castaing, Equation différentielle multivoque avec contrainte sur l'état dans les espaces de Banach. Travaux Sém. Anal. Convexe. Montpellier (1978) Exposé 13.

[6] C. Castaing and M.D.P. Monteiro Marques, BV periodic solutions of an evolution problem associated with continuous moving convex sets. *Set-Valued Anal.* **3** (1995) 381–399.

[7] C. Castaing, M.D.P. Monteiro Marques, Evolution problems associated with non-convex closed moving sets with bounded variation. *Portugal. Math.* **53** (1996) 73–87.

[8] F.H. Clarke, Functional Analysis, Calculus of Variations and Optimal Control. Springer, London (2013).

[9] G. Colombo and V.V. Goncharov, The sweeping processes without convexity. *Set-Valued Anal.* **7** (1999) 357–374.

[10] B. Cornet, Existence of slow solutions for a class of differential inclusions. *J. Math. Anal. Appl.* **96** (1983) 130–147.

[11] N. Dinculeanu, *Vector Measures, Pergamon*, Oxford (1967).

[12] J.F. Edmond and L. Thibault, BV solutions of nonconvex sweeping process differential inclusions with perturbation. *J. Differ. Eqs.* **226** (2006) 135–179.

[13] M. Falcone, P. Saint-Pierre, Slow and quasi-slow solutions of differential inclusions. *Nonlinear Anal.* **11** (1987) 367–377.

[14] H. Federer, Curvature measures. *Trans. Amer. Math. Soc.* **93** (1959) 418–491.

[15] C. Henry, An existence theorem for a class of differential equations with multivalued right-hand side. *J. Math. Anal. Appl.* **41** (1973) 179–186.

[16] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces. Cambridge Univ. Press, London (1995).

[17] M.D.P. Monteiro Marques, Perturbations convexes semi-continues supérieurement de problèmes d'évolution dans les espaces de Hilbert. Travaux Sém. Anal. Convexe. Montpellier (1984) Exposé 2.

[18] B. Maury and J. Venel, A mathematical framework for a crowd motion model, *C. R. Math. Acad. Sci.* Paris **346** (2008) 1245–1250.

[19] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation I. Vol. 330 *Grundlehren Series*. Springer (2006).

[20] J.J. Moreau, Rafle par un convexe variable I. Travaux Sém. Anal. Convexe. Montpellier (1971) Exposé 15.

[21] J.J. Moreau, On unilateral constraints, friction and plasticity. *New Variational Techniques in Mathematical Physics* (C.I.M.E., II Ciclo 1973). Edizioni Cremonese, Rome (1974) 171–322.

[22] J.J. Moreau, Sur les mesures différentielles des fonctions vectorielles à variation bornée. Travaux Sém. Anal. Convexe. Montpellier (1975) Exposé 17.

[23] J.J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space. *J. Differ. Eqs.* **26** (1977) 347–374.

[24] J.J. Moreau, Bounded variation in time. Topics in nonsmooth mechanics, Vol. 174. Birkhäuser, Basel (1988).

[25] J.J. Moreau, Numerical aspects of the sweeping process. *Comput. Methods Appl. Mech. Engrg.* **177** (1999) 329–349.

[26] J.J. Moreau, An introduction to unilateral dynamics. Novel Approaches in Civil Engineering. Edited by M. Frémond and F. Maceri. Springer, Berlin (2002).

[27] J.J. Moreau and M. Valadier, A chain rule involving vector functions of bounded variation. *J. Funct. Anal.* **74** (1987) 333–345.

[28] R.A. Poliquin, R.T. Rockafellar, L. Thibault, Local differentiability of distance functions. *Trans. Amer. Math. Soc.* **352** (2000) 5231–5249.

[29] R.T. Rockafellar and R.J.-B. Wets, Variational Analysis. *Grundlehren der Mathematischen Wissenschaften*, vol. 317. Springer, New York (1998).

[30] A. Tanwani, B. Brogliato and C. Prieur, Stability and observer design for Lur'e systems with multivalued, nonmonotone, time-varying nonlinearities and state jumps. *SIAM J. Control Optim.* **52** (2014) 3639–3672.

[31] L. Thibault, Sweeping process with regular and nonregular sets. *J. Differ. Eqs.* **193** (2003) 1–26.

[32] M. Valadier, Quelques problèmes d'entraînement unilatéral en dimension finie, Travaux Sém. Anal. Convexe. Montpellier (1988) Exposé 8.

[33] M. Valadier, Rafle et viabilité. Travaux Sém. Anal. Convexe. Montpellier (1992) Exposé 17.

[34] J. Venel, A numerical scheme for a class of sweeping processes. *Numer. Math.* **118** (2011) 367–400.

[35] J.-P. Vial, Strong and weak convexity of sets and functions. *Math. Oper. Res.* **8** (1983) 231–259.