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CONVERGENCE OF SOLUTIONS TO FINITE DIFFERENCE SCHEMES FOR SINGULAR LIMITS OF NONLINEAR EVOLUTIONARY PDES

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Abstract. Solutions of certain finite-difference schemes for singularly-perturbed evolutionary PDEs converge as the perturbation parameter and/or the discretization parameters tend to zero. Under suitable hypotheses a sharp convergence rate of order one-half in the time step, uniform in the perturbation parameter, is obtained.

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1. Introduction

This paper continues the development initiated in [8] of a theory for finite difference schemes analogous to the theory of singular limits for systems of PDEs containing large terms, such as

$$A(\varepsilon u)u_t = \sum_{j=1}^{d} A^j(t,x,u)u_{x_j} + \sum_{j,k=1}^{d} \partial_{x_j}\left[B^{jk}\partial_{x_k}u\right] + \frac{1}{\varepsilon}\left[\sum_{j=1}^{d} \mathcal{C}^j u_{x_j} + \mathcal{D}u\right] + F(t,x,u), \tag{1.1}$$

where ε is a small parameter [11, 12, 15, 17, 21]. Under appropriate assumptions on the matrices A^j, $B^{j,k}$, $\mathcal{C}^j$, and $\mathcal{D}$ and vector F, solutions of (1.1) with fixed smooth initial data exist and satisfy uniform bounds for a time independent of ε, and the difference between those solutions and the solutions of certain limit or profile equations tends to zero with ε. Singular limits of equations of the form (1.1) and variants thereof occur not only in the original motivating example of slightly-compressible fluid dynamics [15, 17, 22] and its variants [4, 19, 30] but also in a variety of other fields (*e.g.*, [1, 5, 24]). It is therefore of much interest to obtain convergence results for numerical methods for equations of the form (1.1) that are uniform in the parameter ε.

Here we prove the convergence of solutions of appropriate finite-difference approximations to equations like (1.1) to the solutions of limit equations as the perturbation parameter ε and/or the discretization parameters $\Delta := (\Delta t, \Delta x)$ tend to zero, provided that the solution of the difference scheme and its first time-difference are bounded uniformly in ε and Δ in discrete Sobolev spaces of sufficiently high order and the initial data of the numerical scheme converges under the same limit. The uniform bounds needed to apply this convergence result were proven for a class of finite-difference schemes in [8], and the results there are extended to certain finite-volume schemes in [23]. In contrast, as discussed in [8], neither the well-known convergence results of [25]

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for finite difference approximations to nonlinear evolution equations nor the somewhat improved results of [29] yield bounds for discretizations of (1.1) that are uniform in ε.

Three convergence results will be proven in Section 4. The first result shows convergence without a rate, under the conditions outlined above. The second result justifies that for systems without a large parameter the rate of convergence is determined by the order of accuracy of the difference scheme and its initial data. For the case when large terms are present, a uniform-in-ε $(\Delta t)^{1/2}$ rate of convergence is obtained under appropriate assumptions, and that rate is shown to be sharp. Precise statements of all results are presented in Section 2, along with statements of all the definitions and lemmas to which they refer. In particular, the numerical schemes may approximate certain evolutionary PDEs more general than (1.1). As noted in Section 2, the results of [8] imply that the convergence theorems presented here apply to the schemes considered there. The lemmas stated in Section 2 are proven in Section 3.2.

Convergence results for PDEs usually compare solutions of the differential and difference equations at the grid points of the latter (*e.g.* [13], Thm. 5.1.3; [18], Sect. 10.1.1; [26], Def. 1.4.1; [28], Def. 2.2.2). However, for PDEs like (1.1), whose natural energy estimates involve L^2-based Sobolev space norms, that approach requires that the solution of the PDE be very smooth, because projecting a function such as the solution of a PDE onto a discrete grid loses smoothness in those norms. An alternative approach is to compare a spatial interpolation of the discrete solution to the PDE solution (*e.g.* [13], Thm. 2.1.1; [26], Thm. 10.1.2). Since the latter approach potentially yields sharp estimates for the amount of smoothness required by solutions of the PDE, that is the method we use here. Moreover, although projecting onto the time grid does not lose smoothness because the estimates are assumed to be L^∞ in time, we also use time interpolation. This approach is more natural since it allows us to actually take the limit of the interpolated discrete solution rather than just showing that the difference between certain values of the discrete and PDE solutions is small. However, using interpolation for the nonlinear PDEs considered here does require us to develop certain preliminary results about interpolation, in order to show that the interpolation of a solution to a nonlinear difference scheme is almost a solution of that scheme. Although linear estimates for interpolation are well known, we need here estimates in Sobolev norms for the "nonlinear commutator" of interpolation operators with nonlinear functions. Those interpolation estimates are derived in Appendix A, and the estimate of how close the interpolation of a solution is to being a solution itself is obtained in Section 3.1.

For equations not containing large terms, our approach makes it feasible to obtain sharp results about the smoothness of numerical approximations and the norms in which convergence is obtained. In order to actually achieve those sharp results, it is necessary to approximate that initial data by numerical initial data that both maintain smoothness and achieve an arbitrary order of approximation. The required approximation of initial data for equations not containing large terms is developed in Appendix B. However, in order to obtain convergence results for equations that do contain large terms, it is necessary to approximate slow initial data for the PDE by slow initial data for the numerical scheme, *i.e.*, to approximate initial data such that the first time derivative at time zero of the solution to the PDE is bounded uniformly in ε by initial data such that the first time difference at time zero of the solution to the numerical scheme is bounded uniformly in ε. Although the fact that the numerical scheme has a finite order of accuracy means that such initial data cannot achieve arbitrary accuracy, it is shown in Section 3.3 how to make the initial data for the numerical scheme satisfy the above condition while still maintaining smoothness and achieving the same accuracy as the numerical scheme.

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2. Schemes and results

Before stating the theorems to be proven we need to specify the domains on which the PDEs and difference schemes will be defined and the norms in which we will measure the size of their solutions, the forms those PDEs and difference schemes will take, and the methods used to interpolate solutions of the difference schemes. We will also define notions of stability and order of accuracy that will be used in the theorems, state some lemmas whose hypotheses will be assumed in some of the theorems, and derive formally the equations that should be satisfied under various limits.

2.1. Domains and norms

We will let d denote the spatial dimension. For notational simplicity we let the spatial domain be periodic of length 2π in each spatial variable and assume that the grid spacing Δx is the same in all directions. The cases when some or all of the components of the spatial variable range over $\mathbb{R}$, the grid spacing is different in different spatial directions, or the lengths of the spatial domain are different may be treated similarly. In particular, changing the length of the spatial domain to $2L$ in each direction merely introduces factors of $\frac{\pi}{L}$ into various formulas. The spatial domain for the PDE is therefore $X := [-\pi, \pi)^d$ taken periodically, and the corresponding Fourier domain is $\Pi := \mathbb{Z}^d$. From now on we also assume for simplicity that the half-period π of the spatial domain is an integer multiple of Δx, so that $\Delta x \left\lfloor \frac{\pi}{\Delta x} \right\rfloor$ equals π. Then the discrete spatial and Fourier domains reduce to $X_{\Delta x} = [-\pi, \pi)^d \cap \Delta x\mathbb{Z}^d$ and $\Pi_{\Delta x} = [-\frac{\pi}{\Delta x}, \frac{\pi}{\Delta x})^d \cap \mathbb{Z}^d$, with the former also taken periodically.

Before defining discrete analogues of Sobolev norms we recall some notations: The forward and backwards shift operators are $[S_{j,\Delta x}u](x) := u(x + \Delta x e_j)$ and $[(S_{j,\Delta x})^{-1}u](x) := u(x - \Delta x e_j)$, and the forward difference operator is $D_{j,\Delta x} := \frac{S_{j,\Delta x}-1}{\Delta x}$, where as usual e_j denotes the vector whose component j equals one and other components equal zero. Note that in operator formulas a number denotes the operator of multiplication by that number; these are all scalar operators, but will be extended to operate on vectors componentwise. The time-shift operator and time-difference operator are $[S_{\Delta t}u](t, x) = u(t + \Delta t, x)$ and

$$D_{\Delta t} := \tfrac{S_{\Delta t}-1}{\Delta t}, \tag{2.1}$$

respectively. Higher-order shift and difference operators are defined by $S^\alpha_{\Delta x} := (S_{1,\Delta x})^{\alpha_1} \dots (S_{d,\Delta x})^{\alpha_d}$, where α is a multi-index vector with integer components, and $D^\alpha_{\Delta x} := (D_{1,\Delta x})^{\alpha_1} \dots (D_{d,\Delta x})^{\alpha_d}$, where α is a multi-index vector with nonnegative integer components. The length $|\alpha|$ of a multi-index is the sum $\sum_j |\alpha_j|$ of the absolute values of its components.

To define the discrete analogues of the Sobolev norms $\| \ \|_{H^s}$, we begin with the discrete spatial ℓ^2 norm defined by $\|v\|_{\ell^2} := \sqrt{\langle v, v\rangle_{\ell^2}}$, where $\langle v, w\rangle_{\ell^2} := \sum_{x\in X_{\Delta x}} v(x)\cdot w(x)\ (\Delta x)^d$ is the discrete ℓ^2 inner product. The discrete Sobolev norms are then defined for nonnegative integers s by $\|u\|_{h^s} := \sqrt{\langle u, u\rangle_{h^s}}$, where $\langle u, v\rangle_{h^s} := \sum_{|\alpha|\le s} \langle D^\alpha_{\Delta x}u, D^\alpha_{\Delta x}v\rangle_{\ell^2}$ is the discrete Sobolev inner product, and any points in the formula for $D^\alpha_{\Delta x}$ that lie outside the discrete domain $X_{\Delta x}$ are understood periodically as mentioned above. An equivalent norm

$$\|f\|_{h^s} := \left[\sum_{\xi\in\Pi_{\Delta x}} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \right]^{1/2} \tag{2.2}$$

can be defined using the discrete Fourier transform, which as noted in [8], (4.1), may be written as

$$\widehat{f}(\xi) = \sum_{x\in X_{\Delta x}} f(x)\mathrm{e}^{-i\xi\cdot x}(\Delta x)^d. \tag{2.3}$$

The use of the same notation for the two equivalent norms should not cause confusion since each calculation makes clear which version is being used. The usual Fourier inversion formula $(\widehat{f})^\vee \equiv f$ holds (*e.g.* [10], pp. 250–252), where the discrete inverse Fourier transform may be written as ([8], Eq. (4.2))

$$g^\vee(x) = \frac{1}{(2\pi)^d} \sum_{\xi\in\Pi_{\Delta x}} g(\xi)\mathrm{e}^{i\xi\cdot x}. \tag{2.4}$$

The discrete ℓ^∞ and $w^{k,p}$ norms are defined similarly as the discrete analogues of the continuous L^∞ and $W^{k,p}$ norms. For discrete functions depending on both time and spatial variables, we will use the space-time norms

$$\|u\|_{w^{k,\infty}([0,T];h^s)} := \sum_{j=0}^{k} \|(D_{\Delta t})^j u\|_{\ell^\infty([0,T];h^s)} = \sum_{j=0}^{k} \max_{t\in[0,T]\cap\Delta t\mathbb{Z}} \|(D_{\Delta t})^j u(t,\cdot)\|_{h^s}, \tag{2.5}$$

which are the discrete analogues of the continuous norms

$$\|u\|_{W^{k,\infty}([0,T];H^s)} := \sum_{j=0}^{k} \|\partial_t^j u\|_{L^\infty([0,T];H^s)} = \sum_{j=0}^{k} \sup_{t\in[0,T]} \|\partial_t^j u(t,\cdot)\|_{H^s}. \tag{2.6}$$

However, as discussed in Remark A.2, the time interpolation scheme described in Section 2.4 that we use yields functions that are only piecewise smooth. When dealing with time-interpolated functions we will therefore replace (2.6) with "piecewise-$W^{k,\infty}$" norms

$$\|u\|_{W^{k,\infty}_{\mathrm{PW}}([0,T];H^s)} := \sum_{j=0}^{k} \|\partial_t^j u\|_{L^\infty_{\mathrm{PW}}([0,T];H^s)} := \sum_{j=0}^{k} \sup_{t\in[0,T]\setminus\Delta t\mathbb{Z}} \|\partial_t^j u(t,\cdot)\|_{H^s}. \tag{2.7}$$

2.2. PDEs and numerical schemes

The general form of the PDE that will be considered here is

$$0 = \mathcal{N}_{0,\varepsilon}[u] := A(\varepsilon u)(u_t - \mu\mathcal{C}(\partial_x)u) + \tfrac{1}{\varepsilon}\mathcal{L}u + \mathcal{F}(\varepsilon, t, x, \{D^\alpha u\}_{0\le|\alpha|\le p}) \tag{2.8}$$

where μ is a constant and $\mathcal{C}$ and $\mathcal{L}$ are constant-coefficient spatial partial differential operators. In typical examples allowed by the results of [8], $\mu\mathcal{C}(\partial_x)$ is identically zero, $\mathcal{L} = \sum_j L^j \partial_{x_j}$ with the L^j being symmetric matrices, and $\mathcal{F} = \sum_{j,k} B^{jk}\partial_{x_j}\partial_{x_k} u + \sum_j A^j(t,x,u)u_{x_j} + F(t,x,u)$, with the matrices A^j and B^{jk} satisfying suitable hypotheses. However, each of those operators could include terms of higher order. Although $-\mu A(\varepsilon u)\mathcal{C}(\partial_x)$ could be absorbed into $\mathcal{F}$ it will be convenient to write that term separately because for certain schemes the presence or absence of the term $\mathcal{C}(\partial_x)$ in the PDE obtained by taking the limit of a difference scheme will depend on the relationship between Δt and Δx as both tend to zero. Keeping the parameter μ separate from $\mathcal{C}(\partial_x)$ allows us to treat the cases when that term is present or is absent simultaneously.

In order to write the corresponding difference schemes we will let $\mathcal{L}_{\Delta x} := \sum_{|\alpha|\le M} \mathcal{L}_\alpha S^\alpha_{\Delta x}$, and similarly $\mathcal{C}_{\Delta x}$, *etc.*, denote general spatial shift operators; the arguments of $\mathcal{L}_{\Delta x}$, if any, will denote the variables and parameters that the coefficients $\mathcal{L}_\alpha$ may depend on. Using these notations the difference schemes that we consider may be written as

$$\begin{aligned} 0 = \mathcal{N}_{\Delta,\varepsilon}[v] := A(\varepsilon v)\left\{D_{\Delta t}v - \tfrac{(\Delta x)^q}{\Delta t}\mathcal{C}_{\Delta x}v\right\} + \tfrac{1}{\varepsilon}\mathcal{L}_{\Delta x}(\rho S_{\Delta t}v + (1-\rho)v) \\ + \mathcal{F}_\Delta(\varepsilon, t, x, \{D^\alpha_{\Delta x}\widetilde{v}\}_{0\le|\alpha|\le p}, \{D^\alpha_{\Delta x}S_{\Delta t}\widetilde{v}\}_{0\le|\alpha|\le p}), \end{aligned} \tag{2.9}$$

where

$$\widetilde{v} := \{S^\beta_{\Delta x}v\}_{|\beta|\le M} \tag{2.10}$$

denotes a finite collection of spatial shifts of the variable v. A typical example in which the shift operator $\mathcal{C}_{\Delta x}$ is nonzero is the Lax–Friedrichs scheme, for which q in (2.9) equals two and $\mathcal{C}_{\Delta x}$ equals $\frac{1}{2d}\sum_{j=1}^{d} \frac{S_{j,\Delta x} - 2 + (S_{j,\Delta x})^{-1}}{(\Delta x)^2}$. Since the operators $\mathcal{L}$ in (2.8) and $\mathcal{L}_{\Delta x}$ in (2.9) are multiplied there by the large value $\frac{1}{\varepsilon}$, we will refer to them as the large operators.

We do not claim that all difference schemes of the form (2.9) have uniformly bounded solutions, nor that the initial-value problem for all PDEs of the form (2.8) is well posed. In this paper we demonstrate convergence of solutions to difference schemes under the assumption that solutions to the scheme are uniformly bounded, and therefore want the form of the equations to be as general as possible. The issue of showing that solutions to particular classes of schemes indeed have the uniformly bounded solutions needed to apply the convergence theory developed here has been and will be considered elsewhere (*e.g.*, [8, 23]). However, in order to understand the various results presented in this paper the reader may wish to keep in mind the numerical scheme

$$\begin{aligned} &\tfrac{1}{\rho_0+\varepsilon r}\left[D_{\Delta t}r - \frac{|\mathbf{u}|\Delta x}{2}\Delta_{\Delta x}r + \mathbf{u}\cdot\nabla_{\Delta x}r\right] + \tfrac{1}{\varepsilon}\nabla_{\Delta x}\cdot S_{\Delta t}\mathbf{u} = 0 \\ &\tfrac{\rho_0+\varepsilon r}{\mathcal{P}'(\rho_{\prime}+\varepsilon\nabla)}\left[D_{\Delta t}\mathbf{u} - \frac{|\mathbf{u}|\Delta x}{2}\Delta_{\Delta x}\mathbf{u} + \mathbf{u}\cdot\nabla_{\Delta x}\mathbf{u}\right] + \tfrac{1}{\varepsilon}\nabla_{\Delta x}S_{\Delta t}r = 0, \end{aligned} \tag{2.11}$$

where

$$\nabla_{\Delta x} := \sum_j \frac{S_{j,\Delta x} - (S_{j,\Delta x})^{-1}}{2\Delta x} \qquad \text{and} \qquad \Delta_{\Delta x} := \sum_j \frac{S_{j,\Delta x} - 2 + (S_{j,\Delta x})^{-1}}{(\Delta x)^2},$$

which approximates the Euler equations of barotropic compressible inviscid fluid dynamics

$$\begin{aligned} &\tfrac{1}{\rho_0+\varepsilon r}\left[r_t + \mathbf{u}\cdot\nabla r\right] + \tfrac{1}{\varepsilon}\nabla\cdot\mathbf{u} = 0 \\ &\tfrac{\rho_0+\varepsilon r}{\mathcal{P}'(\rho_{\prime}+\varepsilon\nabla)}\left[\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right] + \tfrac{1}{\varepsilon}\nabla r = 0, \end{aligned} \tag{2.12}$$

where $\rho_0 + \varepsilon r$ is the fluid density, $\mathcal{P}(\rho_0 + \varepsilon r)$ is the pressure, and $\mathbf{u}$ is the velocity. The numerical scheme (2.11) is the local Lax–Friedrichs scheme with the large terms treated implicitly, which was shown in ([8], Sect. 3.1) to have solutions that are bounded uniformly in ε.

2.3. Stability, consistency, and order

Although the basic convergence theorem without a rate of convergence does not require the stability of the PDE or difference scheme, the convergence theorems that include a rate of convergence do require such stability.

Definition 2.1. Let r be the largest order of any derivative in the PDE (2.8). That PDE is *stable* if there exists a finite k_0 such that for any $k \geq k_0$ and any functions $u^{(1)}$ and $u^{(2)}$ in $L^\infty([0,T];H^{k+r}) \cap W^{1,\infty}([0,T];H^k)$,

$$\|u^{(1)} - u^{(2)}\|_{L^\infty([0,T];H^k)} \leq C_k \left\{ \|\mathcal{N}_{0,\varepsilon}[u^{(1)}] - \mathcal{N}_{0,\varepsilon}[u^{(2)}]\|_{L^\infty([0,T];H^k)} + \|u^{(1)}(0,\cdot) - u^{(2)}(0,\cdot)\|_{H^k} \right\}, \tag{2.13}$$

where C_k depends on the equation, T, and the $L^\infty([0,T];H^{k+r})$ and $W^{1,\infty}([0,T];H^k)$ norms of both $u^{(1)}$ and $u^{(2)}$. If C_k is independent of ε then the equation is *stable uniformly in* ε.

Given $u \in L^\infty([0,T];H^{k+r}) \cap W^{1,\infty}([0,T];H^k)$, the operator linearized around u associated to $\mathcal{N}_{0,\varepsilon}$ is

$$\begin{aligned} \mathcal{N}'_{0,\varepsilon,u}[U] := {}& A(\varepsilon u)(U_t - \mu\mathcal{C}(\partial_x)U) + U\cdot\{\varepsilon[\nabla_u A](\varepsilon u)\}(u_t - \mu\mathcal{C}(\partial_x)u) + \tfrac{1}{\varepsilon}\mathcal{L}U \\ &+ \sum_{0\leq|\beta|\leq p} \frac{\partial\mathcal{F}(\varepsilon,t,x,\{D^\alpha u\}_{0\leq|\alpha|\leq p})}{\partial(D^\beta u)} D^\beta U \end{aligned} \tag{2.14}$$

The PDE (2.8) is *linearly stable uniformly in* ε if there is a k_0 such that for $k \geq k_0$, $0 \leq k_1 \leq k$, and $0 \leq t \leq T$,

$$\|U(t,\cdot)\|^2_{H^{k_1}} \leq C_k \left\{ \int_0^t \|\mathcal{N}'_{0,\varepsilon,u}[U(s,\cdot)]\|^2_{H^{k_1}}\,\mathrm{d}s + \|U(0,\cdot)\|^2_{H^{k_1}} \right\}, \tag{2.15}$$

where C_k depends on the equation, T, and the $L^\infty([0,T];H^{k+r})$ and $W^{1,\infty}([0,T];H^k)$ norms of u but not on ε.

Similarly, let r be the largest order of any difference in the difference scheme (2.9). That difference scheme is *stable uniformly in* ε if there exists a finite k_0 such that for any $k \geq k_0$ and any functions $u^{(1)}$ and $u^{(2)}$ in $\ell^\infty([0,T+\Delta t]\cap \Delta t\mathbb{Z}; h^{k+r}) \cap w^{1,\infty}([0,T+\Delta t]\cap \Delta t\mathbb{Z}; h^k)$,

$$\|u^{(1)}-u^{(2)}\|_{\ell^\infty([0,T]\cap\Delta t\mathbb{Z};h^k)} \leq C_k\left\{\|\mathcal{N}_{\Delta,\varepsilon}[u^{(1)}]-\mathcal{N}_{\Delta,\varepsilon}[u^{(2)}]\|_{\ell^\infty([0,T]\cap\Delta t\mathbb{Z};h^k)}+\|u^{(1)}(0,\cdot)-u^{(2)}(0,\cdot)\|_{h^k}\right\}, \tag{2.16}$$

where C_k depends on the equation and the $\ell^\infty([0,T+\Delta t]\cap\Delta t\mathbb{Z};h^{k+r})$ and $w^{1,\infty}([0,T]\cap\Delta t\mathbb{Z};h^k)$ norms of both $u^{(1)}$ and $u^{(2)}$ but not on ε.

Although the basic notions of consistency of a difference operator with a differential operator and its order of accuracy are fairly standard, we will use continuous Sobolev norms rather than the more usual discrete ℓ^2 or ℓ^∞ norms in the definitions of those concepts, and consider the difference between the discrete and continuous operators for arbitrary sufficiently-smooth functions rather than just for solutions of the PDE. That approach is common only for defining the order of individual difference operators such as $\frac{S_{\Delta x}-(S_{\Delta x})^{-1}}{2\Delta x}$ but will be used here both for such operators and for the full difference scheme (2.9).

Definition 2.2. A difference operator $\mathcal{N}_{\Delta,\varepsilon}[v]$ is *consistent* with a differential operator $\mathcal{N}_{0,\varepsilon}[u]$ provided that there exist nonnegative integers $k_{\min}$, m, and k_- such that $\|\mathcal{N}_{\Delta,\varepsilon}[u]-\mathcal{N}_{0,\varepsilon}[u]\|_{L^\infty([0,T];H^{k-k_-})}$ tends to zero as Δt and Δx tend to zero whenever $k \geq k_{\min}$ and $u \in \cap_{j=0}^m W^{j,\infty}([0,T];H^{k-jk_-})$.

Let σ_t and σ_x be positive numbers. A difference operator approximation $\mathcal{N}_{\Delta,\varepsilon}[v]$ is σ_t*-order accurate in time and* σ_x*-order accurate in space* for the differential operator $\mathcal{N}_{0,\varepsilon}[u]$ provided that there exist nonnegative integers $k_{\min}$, J, and k_- and functions C_k such that for all $k\geq k_{\min}$ and $u\in\cap_{j=0}^J W^{j,\infty}([0,T];H^{k-jk_-})$,

$$\|\mathcal{N}_{\Delta,\varepsilon}[u]-\mathcal{N}_{0,\varepsilon}[u]\|_{L^\infty([0,T];H^{k-k_--\max\{\sigma_t k_-,\sigma_x\}})} \leq C_k\left(\sum_{j=0}^J \|u\|_{W^{j,\infty}([0,T];H^{k-jk_-})}\right)\left[(\Delta t)^{\sigma_t}+(\Delta x)^{\sigma_x}\right]. \tag{2.17}$$

Remark 2.3. Since the functions C_k in Definition 2.2 depend on both $\mathcal{N}_{0,\varepsilon}$ and $\mathcal{N}_{\Delta,\varepsilon}$, when those operators include coefficients of order $\frac{1}{\varepsilon}$ then C_k will depend on $\frac{1}{\varepsilon}$, so the estimate (2.17) will be non-uniform in ε. For this reason, the general convergence-rate theorem, Theorem 2.8, will not be uniform in ε. A uniform convergence rate will be proven under additional hypotheses in Theorem 2.9.

2.4. Interpolation and projection

A spatial interpolation operator from the discrete grid $X_{\Delta x}$ of the difference scheme to the spatial domain X of the PDE can be obtained by first taking the discrete Fourier transform and then taking the inverse discrete Fourier transform of the result but allowing the spatial variable to vary over all of X rather than just $X_{\Delta x}$, *i.e.*,

$$[\mathrm{Int}_x\, f](x) := \frac{1}{(2\pi)^d}\sum_{\xi\in\Pi_{\Delta x}}\left[\sum_{y\in X_{\Delta x}} f(y)\mathrm{e}^{-i\xi\cdot y}(\Delta x)^d\right]\mathrm{e}^{i\xi\cdot x}, \qquad x\in X. \tag{2.18}$$

Applying the inverse Fourier transform in that manner is equivalent to defining the result of taking the discrete Fourier transform to be zero outside the domain of the discrete inverse Fourier transform and then taking the ordinary inverse Fourier transform. Hence the Fourier-space formulas for Sobolev norms imply that Int_x is a bounded operator from $h^s(X_{\Delta x})$ to $H^s(X)$ for any s ([8]).

Besides the spatial interpolation operator we will also need a time interpolation operator from $\ell^\infty([0,T]\cap\Delta t\mathbb{Z};H^s(X))$ to $L^\infty([0,T];H^s(X))$ for $s\geq 0$. Since only the first time derivative of solutions will be assumed to be bounded uniformly in ε at time zero, for the most part we will use the piecewise linear interpolation operator defined by

$$[\mathrm{Int}_t\, v]\,(t,x) := \left(\left\lceil\tfrac{t}{\Delta t}\right\rceil-\tfrac{t}{\Delta t}\right)v\left(\Delta t\left\lfloor\tfrac{t}{\Delta t}\right\rfloor,x\right)+\left(\tfrac{t}{\Delta t}-\left\lfloor\tfrac{t}{\Delta t}\right\rfloor\right)v\left(\Delta t\left\lceil\tfrac{t}{\Delta t}\right\rceil,x\right), \tag{2.19}$$

where $\lceil s \rceil := \min\{n \in \mathbb{Z} \mid n > s\}$ and $\lfloor s \rfloor := \max\{n \in \mathbb{Z} \mid n \le s\}$ are defined so that $\lceil s \rceil - \lfloor s \rfloor = 1$ for all real s. However, when large terms are absent and the numerical scheme has order p greater than one it will be convenient to interpolate functions in each interval $[\alpha\Delta t, (\alpha+1)\Delta t]$ by polynomial interpolation at $p+1$ points $\{(\alpha+j)\Delta t\}_{j=j_0}^{j_0+p}$. In order to avoid complications at the first few time intervals we henceforth assume that $j_0 \ge 0$. Formula (2.19) is the special case of polynomial interpolation in which $p = 1$.

The inverse operation to interpolation is pointwise projection onto the numerical grid. We will let Pr_x denote pointwise evaluation on the spatial grid and Pr_t denote pointwise evaluation on the time grid. By construction, the space and time interpolation operators satisfy

$$\mathrm{Pr}_x \,\mathrm{Int}_x = I, \qquad \mathrm{Pr}_t \,\mathrm{Int}_t = I. \tag{2.20}$$

2.5. Limit equations

We will assume that the difference equation (2.9) is consistent with the PDE (2.8). When the limit $\mathcal{C}(\partial_x)$ of the shift operator $\mathcal{C}_{\Delta x}$ is nonzero then consistency means that $\mathcal{N}_{\Delta,\varepsilon}[u]$ tends to $\mathcal{N}_{0,\varepsilon}[u]$ as $\Delta \to 0$ in such a way that $\frac{(\Delta x)^q}{\Delta t}$ tends to μ, with ε fixed. When the difference operator $\mathcal{C}_{\Delta x}$ is identically zero then we place no restriction here on the relation between Δt and Δx, although it should be noted that for some difference schemes a restriction of the form

$$\Delta t \le c(\Delta x)^k \tag{2.21}$$

is needed to ensure the boundedness of the solutions of the difference scheme assumed in the statements of the theorems. Examples of schemes that require a condition of the form (2.21) and of schemes that do not are given in ([8], Sects. 2 and 3).

We also wish to consider the case when ε tends to zero, either with Δ fixed or with Δ also tending to zero. We therefore need to determine the formal limit equations satisfied in those two cases. Taking the formal limit means calculating the limit of an equation applied to a fixed function as the coefficients of that equation vary. Although showing the formal limit of an equation does not prove that solutions of the original equation tend to solutions of the limit equation, it can be used as one step in a proof of that fact, as will be seen in the Proof of Theorem 2.7 in Section 4.1.

Since we will be assuming that both the solution v and the first time difference $D_{\Delta t}v$ of the solution of the difference equation (2.9) are bounded uniformly in ε in certain Sobolev spaces, after multiplying that equation by ε we obtain

$$\mathcal{L}_{\Delta x}(\rho S_{\Delta t} v + (1-\rho)v) = O(\varepsilon). \tag{2.22}$$

If Δ tends to zero along with ε then, after relabeling v as u for later convenience, (2.22) tends formally to

$$\mathcal{L}u = 0 \tag{2.23}$$

since the difference operator $\mathcal{L}_{\Delta x}$ will be assumed to be a consistent approximation of the differential operator $\mathcal{L}$, while $S_{\Delta t}$ tends formally to the identity operator when $\Delta t \to 0$.

The limit equation (2.23) and the corresponding equation for the difference equations can be expressed in terms of the L^2 orthogonal projection operator $\mathcal{P}$ onto the null space of $\mathcal{L}$ and the ℓ^2 orthogonal projection operator $\mathcal{P}_{\Delta x}$ onto the null space of $\mathcal{L}_{\Delta x}$. Equation (2.23) is equivalent to

$$(1-\mathcal{P})u = 0 \tag{2.24}$$

Furthermore, applying $\mathcal{P}_{\Delta x}$ to (2.9) so as to eliminate the large term, then taking the formal limit assuming that the difference equation is consistent with the PDE (2.8) and that the discrete projection $\mathcal{P}_{\Delta x}$ converges to the continuous one $\mathcal{P}$ as $\Delta x \to 0$, and using (2.24) yields the complementary limit equation

$$0 = \mathcal{P}\mathcal{N}_{0,0}[\mathcal{P}u] = \mathcal{P}A(0)(\mathcal{P}u_t - \mu\mathcal{C}(\partial_x)\mathcal{P}u) + \mathcal{P}\mathcal{F}(0,t,x,\{D^\alpha \mathcal{P}u\}_{0\le|\alpha|\le p}). \tag{2.25}$$

Since (2.25) will be taken in conjunction with (2.24), the appearance of $\mathcal{P}u$ rather than just u in (2.25) is not strictly necessary, but it makes the "coefficient" of the time derivative be the symmetric operator $\mathcal{P}A(0)\mathcal{P}$.

Taking the formal limit of (2.22) as ε tends to zero with Δ fixed yields

$$\mathcal{L}_{\Delta x}(\rho S_{\Delta t}v + (1-\rho)v) = 0. \tag{2.26}$$

The initial data will be assumed to satisfy $\mathcal{L}_{\Delta x}v_\varepsilon(0,x) = O(\varepsilon)$, so the limit initial data will satisfy $\mathcal{L}_{\Delta x}v_0(0,x) = 0$. Together these imply by finite induction that the the discrete analogue

$$(1-\mathcal{P}_{\Delta x})v = 0 \tag{2.27}$$

of (2.24) holds. Applying $\mathcal{P}_{\Delta x}$ to (2.9) to eliminate the large term and taking the formal limit as $\varepsilon \to 0$ with Δ fixed while using (2.27) yields the complementary equation.

$$\begin{aligned} 0 &= \mathcal{P}_{\Delta x}\mathcal{N}_{\Delta,0}[\mathcal{P}_{\Delta x}v] \\ &= \mathcal{P}_{\Delta x}A(0)\left\{\mathcal{P}_{\Delta x}D_{\Delta t}v - \tfrac{(\Delta x)^q}{\Delta t}\mathcal{C}_{\Delta x}\mathcal{P}_{\Delta x}v\right\} + \mathcal{P}_{\Delta x}\mathcal{F}_{\Delta}(0,t,x,\{D^\alpha_{\Delta x}\mathcal{P}_{\Delta x}\widetilde{v}\}_{0\le|\alpha|\le p},\{D^\alpha_{\Delta x}\mathcal{P}_{\Delta x}S_{\Delta t}\widetilde{v}\}_{0\le|\alpha|\le p}). \end{aligned} \tag{2.28}$$

2.6. Rate of convergence of the projection operator

The basic convergence theorem without a rate merely requires that the projection operator $\mathcal{P}_{\Delta x}$ onto the null space of the large difference operator $\mathcal{L}_{\Delta x}$ converge to the projection operator $\mathcal{P}$ onto the null space of the large differential operator $\mathcal{L}$ as $\Delta x \to 0$, However, the uniform convergence theorem requires a sufficiently high rate of convergence of that projection operator. Two lemmas giving sufficient conditions for that to occur are therefore stated here, but are proven in Section 3.2. These lemmas are also used in Lemma 3.3 to obtain slow difference-scheme initial data that approximates slow PDE initial data. Some operators $\mathcal{L}$ and $\mathcal{L}_{\Delta x}$ for which that convergence does not hold, and other cases in which the hypotheses of the two lemma either do or do not hold, will be presented in Example 3.2.

The hypotheses of both lemmas are expressed in terms of the symbols of the differential and difference operators. The symbol of a differential operator is obtained from that operator by replacing ∂_{x_j} with $i\xi_j$, while the symbol of a shift operator is obtained from that operator by replacing $S_{j,\Delta x}$ with $\mathrm{e}^{i\Delta x\xi_j}$. The symbols of constant-coefficient differential or shift operators are related to the Fourier transform by the formulas

$$\widehat{\mathcal{L}u}(\xi) = [\mathrm{Symb}(\mathcal{L})](\xi)\widehat{u}(\xi), \qquad [\mathrm{Symb}(\mathcal{L}_{\Delta x})](\xi)\widehat{u}(\xi) = \widehat{\mathcal{L}_{\Delta x}u}(\xi). \tag{2.29}$$

The first result is simpler and stronger but places a condition on the symbol of large differential operator $\mathcal{L}$. Specifically, the number of nonzero eigenvalues of that symbol must either be independent of ξ for all nonzero ξ, or more generally be independent of ξ when at least one component of ξ actually appearing in that symbol is nonzero. The former condition is satisfied by the large operator occurring in the equations (2.12) of slightly compressible fluid dynamics and various other physical systems with one spatial scale, while the more general condition remains satisfied for slightly compressible fluid dynamics with multiple spatial scales, considered in [16, 22] and references therein. In addition, the first result assumes that the difference operator is obtained from the differential operator in a specific, albeit quite natural, way. However, the assumption on the number of nonzero eigenvalues of the large operator is not satisfied in general by the equations of multi-phase geometric optics (*e.g.*, [21], Sect. 5). Hence we also present a second result that does not make that assumption and in addition allows the large difference operator to have a fairly general form, but yields an estimate that depends on the size of the nonzero eigenvalues of both large operators. Neither result requires the large differential operator to be of first order, but it must be anti-hermitian. In addition, even when the simpler lemma holds one of the hypotheses of the second lemma is needed elsewhere in the proof of the uniform convergence theorem.

Lemma 2.4. *Let $\mathcal{L} := \sum_{0\le|\alpha|\le m} \mathcal{L}_\alpha \partial_{x_j}^\alpha$ be a constant-coefficient anti-hermitian matrix-valued differential operator. Reorder the components of ξ to the form $\xi = (\xi', \xi'')$ such that every component of ξ' appears in $\mathrm{Symb}(\mathcal{L})$ but no component of ξ'' appears there, and assume that the rank of $[\mathrm{Symb}(\mathcal{L})](\xi')$ is independent of ξ' for $\xi' \neq 0$.*

Define an anti-hermitian difference operator

$$\mathcal{L}_{\Delta x} := \sum_{0\le|\alpha|\le m} \mathcal{L}_\alpha (\partial_{j,\Delta x,c})^\alpha,$$

where $\partial_{j,\Delta x,c} := \frac{1}{\Delta x}\sum_{i=-m}^{m} c_i (S_{j,\Delta x})^i$ is a central difference operator approximating ∂_x, i.e., satisfies $c_{-i} = -c_i$ as well as $\sum i c_i = 1$. Suppose that there exist a positive constant σ and a finite constant c such that each of those difference operators satisfies

$$|\mathrm{Symb}(\partial_{j,\Delta x,c}) - i\xi_j| \le c_\partial (\Delta x)^\sigma |\xi_j|^{\sigma+1}. \tag{2.30}$$

Then there is a constant c such that the L^2-orthogonal projection $\mathcal{P}$ onto the null space of $\mathcal{L}$ and the ℓ^2-orthogonal projection $\mathcal{P}_{\Delta x}$ onto the null space of $\mathcal{L}_{\Delta x}$ satisfy

$$\|(\mathcal{P}_{\Delta x} - \mathcal{P})u\|_{H^k} \le c_{\mathcal{P}}\, (\Delta x)^\sigma \|u\|_{H^{k+\sigma}}. \tag{2.31}$$

Lemma 2.5. *Let $\mathcal{L}$ be a constant-coefficient anti-hermitian matrix-valued differential operator, and let $\mathcal{L}_{\Delta x}$ be a constant-coefficient anti-hermitian finite-difference approximation to $\mathcal{L}$ whose coefficients are analytic in the spatial discretization parameter Δx for $\Delta x \neq 0$. Assume that there exists a positive constant δ such that the following hold:*

(1) *The rank of $[\mathrm{Symb}(\mathcal{L}_{\Delta x})](\xi)$ equals the rank of $[\mathrm{Symb}(\mathcal{L})](\xi)$ for $\xi \in \Pi_{\Delta x} = [-\frac{\pi}{\Delta x}, \frac{\pi}{\Delta x})^d \cap \mathbb{Z}^d$ satisfying $\Delta x|\xi| \le \delta$.*
(2) *The difference operator $\mathcal{L}_{\Delta x}$ is an approximation of some positive order σ to $\mathcal{L}$,* i.e., *for some $c < \infty$ and $m < \infty$,*
$$\|[\mathrm{Symb}(\mathcal{L}_{\Delta x})](\xi) - [\mathrm{Symb}(\mathcal{L})](\xi)\| \le c(\Delta x)^\sigma (1+|\xi|)^{\sigma+m} \qquad \text{for } \xi \in \Pi_{\Delta x}. \tag{2.32}$$
(3) *For some $M < \infty$, $c > 0$, and $\nu \in [-m, \infty)$, all nonzero eigenvalues $\lambda(\xi)$ of $[\mathrm{Symb}(\mathcal{L})](\xi)$ and nonzero eigenvalues $\lambda(\Delta x, \xi)$ of $[\mathrm{Symb}(\mathcal{L}_{\Delta x})](\xi)$ satisfy*
$$|\lambda(\xi)| \ge c|\xi|^{-\nu} \qquad \text{for } |\xi| > M, \tag{2.33}$$
$$|\lambda(\Delta x, \xi)| \ge c|\xi|^{-\nu} \qquad \text{for } |\xi| > M \text{ and } \Delta x|\xi| \le \delta. \tag{2.34}$$

Then there is a constant c such that

$$\|(\mathcal{P}_{\Delta x} - \mathcal{P})u\|_{H^k} \le c\, (\Delta x)^\sigma \|u\|_{H^{k+\sigma+m+\nu}}, \tag{2.35}$$

where $\mathcal{P}$ and $\mathcal{P}_{\Delta x}$ denote the orthogonal projection operators onto the null spaces of $\mathcal{L}$ and $\mathcal{L}_{\Delta x}$, respectively.

Remark 2.6.

(1) The nonzero eigenvalues of the large operator of the equation (2.12) of slightly compressible fluid dynamics are $O(|\xi|)$, so (2.33) holds with $\nu = -1$. An example where ν is positive is provided by the operator $\mathcal{L} := \begin{pmatrix} 0 & \partial_x & 0 \\ \partial_x & \partial_x^3 & 1 \\ 0 & -1 & 0 \end{pmatrix}$: The nonzero eigenvalues of its symbol are $-i\frac{\xi^3}{2} \pm i\frac{\sqrt{4+4\xi^2+\xi^6}}{2}$, and the eigenvalue having a smaller absolute value is $O(|\xi|^{-1})$, so (2.33) holds with $\nu = 1$.
(2) When the spatial domain of the PDE is the whole space $\mathbb{R}^d$ rather than the periodic domain considered here then the corresponding Fourier space is a continuum, and then an assumption is also needed on the behavior of the nonzero eigenvalues of $[\mathrm{Symb}(\mathcal{L})](\xi)$ and $[\mathrm{Symb}(\mathcal{L}_{\Delta x})](\xi)$ in a neighborhood of any exceptional point ξ where the number of such eigenvalues is not constant in some neighborhood.
(3) Although the assumptions of the lemma do not require that the multiplicity of the zero eigenvalue of $[\mathrm{Symb}(\mathcal{L})](\xi)$ be constant for nonzero ξ, the condition that the multiplicity of the zero eigenvalue of $[\mathrm{Symb}(\mathcal{L}_{\Delta x})](\xi)$ equal that of $[\mathrm{Symb}(\mathcal{L})](\xi)$ can be difficult to achieve when that multiplicity is not constant, as will be seen in Example 3.2 below.

2.7. Statements of theorems

We first present a variety of general convergence theorems assuming that a sequence of solutions of the difference equation satisfies uniform energy estimates in discrete Sobolev spaces of sufficiently high order and converges at time zero. After that we will show that the uniform energy estimates obtained in [8] allow the convergence results given here to be applied to the classes of difference schemes considered there. This approach of separating the problem of obtaining uniform estimates from the issue of convergence allows the results obtained here to be applied to other systems for which uniform estimates may eventually be obtained, such as the class of finite-volume schemes on rectangular grids treated in [23].

Theorem 2.7. *Let r denote the largest order of any derivative or difference appearing in the PDE $\mathcal{N}_{0,\varepsilon}$ in (2.8) or difference scheme $\mathcal{N}_{\Delta,\varepsilon}$ in (2.9), and let s be an integer at least $\lfloor d/2\rfloor+1+r$, where d is the spatial dimension. Assume that the coefficients of $\mathcal{N}_{0,\varepsilon}$ and $\mathcal{N}_{\Delta,\varepsilon}$ belong to C^s and that the difference operators $\mathcal{C}_{\Delta x}$, $\mathcal{L}_{\Delta x}$, and $\mathcal{F}_{\Delta x}$ from $\mathcal{N}_{\Delta,\varepsilon}$ are consistent approximations of the differential operators $\mathcal{C}$, $\mathcal{L}$, and $\mathcal{F}$ from $\mathcal{N}_{0,\varepsilon}$, respectively. Let $v_{\Delta,\varepsilon}$ be a sequence of solutions of the difference scheme $\mathcal{N}_{\Delta,\varepsilon}[v_{\Delta,\varepsilon}]=0$ satisfying*

$$\|v_{\Delta,\varepsilon}\|_{\ell^\infty([0,T+\Delta t]\cap\Delta t\mathbb{Z};h^s)}+\|D_{\Delta t}v_{\Delta,\varepsilon}\|_{\ell^\infty([0,T]\cap\Delta t\mathbb{Z};h^{s-r})}\le c<\infty \tag{2.36}$$

for some $T>0$. Let Int_x denote the trigonometric interpolation operator (2.18) and Int_t denote the time-interpolation operator (2.19) or its generalization described in Section 2.4.

(1) *Assume that solutions in $L^\infty([0,T];H^s)\cap W^{1,\infty}([0,T];H^{s-r})$ of the initial-value problem for the PDE (2.8) are unique, and that the spatial interpolation $\mathrm{Int}_x\, v_{\Delta,\varepsilon}(0,x)$ of the initial data for the difference scheme converges in H^{s-1} to some function u_ε^0 as $\Delta\to 0$. If the difference operator $\mathcal{C}_{\Delta x}$ is nonzero, assume in addition that the limit $\Delta\to 0$ is taken in such a way that $\frac{(\Delta x)^q}{\Delta t}$ converges to some μ. Then as $\Delta\to 0$ with ε fixed, the space-time interpolant $\mathrm{Int}_t\,\mathrm{Int}_x\, v_{\Delta x,\varepsilon}$ of the solution $v_{\Delta,\varepsilon}$ of the difference scheme (2.9) converges in the $L^\infty([0,T];H^{s-1})$ norm to the solution of the PDE (2.8) having initial data u_ε^0.*

(2) *Assume that solutions in $\ell^\infty([0,T]\cap\Delta t\mathbb{Z};h^s)\cap w^{1,\infty}([0,T]\cap\Delta t\mathbb{Z};h^{s-r})$ of the initial-value problem for the limit equations (2.27)–(2.28) are unique, and that the initial data $v_{\Delta,\varepsilon}(0,x)$ converges in h^{s-1} as ε tends to zero to a function v_Δ^0 satisfying (2.27). Then as ε converges to zero with Δ fixed the solution $v_{\Delta,\varepsilon}$ of the difference scheme (2.9) converges in the $\ell^\infty([0,T]\cap\Delta t\mathbb{Z};h^{s-1})$ norm to the solution of (2.27)–(2.28) having initial data v_Δ^0.*

(3) *Assume that solutions in $L^\infty([0,T];H^s)\cap W^{1,\infty}([0,T];H^{s-r})$ of the initial-value problem for the limit equations (2.24)–(2.25) are unique, that the projection $\mathcal{P}_{\Delta x}$ onto the null space of the difference operator $\mathcal{L}_{\Delta x}$ converges as $\Delta x\to 0$ to the projection operator $\mathcal{P}$ onto the null space of the differential operator $\mathcal{L}$, and that as both Δ and ε tend to zero the spatial interpolation $\mathrm{Int}_x\, v_{\Delta,\varepsilon}(0,x)$ of the initial data converges in H^{s-1} to a function u^0 satisfying (2.24). If the difference operator $\mathcal{C}_{\Delta x}$ is nonzero, assume in addition that the limit $\Delta\to 0$ is taken in such a way that $\frac{(\Delta x)^q}{\Delta t}$ converges to some μ. Then as both Δ and ε converge to zero the space-time interpolation $\mathrm{Int}_t\,\mathrm{Int}_x\, v_{\Delta,\varepsilon}$ of the solution of the difference scheme (2.9) converges in the $L^\infty([0,T];H^{s-1})$ norm to the solution of (2.24)–(2.25) having initial data u^0.*

In order to obtain a rate of convergence equal to the order of accuracy of the scheme we will assume that ε is a fixed parameter, because the coefficients of the discretization parameters in the estimate of the rate of convergence depend on higher time derivatives/differences of the solutions, which will not be bounded uniformly in ε. However, under appropriate assumptions it should be possible to generalize the following result to the case when ε tends to zero together with Δ provided that the initial data are specially chosen as in [2] and the proof of Lemma 4.2 below so as to make those higher time derivatives and differences be uniformly bounded.

Theorem 2.8. *Let r denote the largest order of any derivative or difference appearing in the PDE $\mathcal{N}_{0,\varepsilon}$ in (2.8) or difference scheme $\mathcal{N}_{\Delta,\varepsilon}$ in (2.9), and let ε in the PDE (2.8) and the difference scheme (2.9) be a fixed parameter. Assume that the difference operator approximation $\mathcal{N}_{\Delta,\varepsilon}$ in (2.9) is σ_t-order accurate in time and σ_x-order* *accurate in space for the differential operator* $\mathcal{N}_{0,\varepsilon}$ *in* (2.8), *with the parameters from Definition* 2.2 *having the values* $k_{\min} \le \lfloor \frac{d}{2} \rfloor + 1 + r + \max\{r\sigma_t, \sigma_x\}$, $J = \sigma_t + 1$, *and* $k_- = r$. *Assume in addition that the PDE* (2.8) *is stable in the sense of Definition* 2.1 *with the parameter* k_0 *equal to* $\lfloor \frac{d}{2} \rfloor + 1 + r$, *where* d *is the spatial dimension. Assume further that both the basic hypotheses and the additional hypotheses of the first part of Theorem* 2.7 *hold, with* s *at least* $\lfloor d/2 \rfloor + 1 + r + \max\{\sigma_t r, \sigma_x\}$.

Finally, assume that the chosen initial data $v^0_{\Delta x}$ *for the difference scheme* (2.9) *and* u^0 *for the PDE* (2.8) *satisfy* $\| \operatorname{Int}_x v^0_{\Delta x} - u^0 \|_{H^{s-r-\max\{\sigma_t r, \sigma_x\}}} \le c(\Delta x)^{\sigma_x}$. *Let* T *denote the minimum of the time of existence of the solution* u *and the uniform time of existence of* v_Δ *with those initial data. Then*

$$\| \operatorname{Int}_t \operatorname{Int}_x v_\Delta - u \|_{L^\infty([0,T]; H^{s-r-\max(r\sigma_t, \sigma_x)})} \le c((\Delta t)^{\sigma_t} + (\Delta x)^{\sigma_x}). \tag{2.37}$$

Theorem 2.9. *Let* r *be the largest order of any derivative or difference appearing in the PDE* $\mathcal{N}_{0,\varepsilon}$ *in* (2.8) *or difference scheme* $\mathcal{N}_{\Delta,\varepsilon}$ *in* (2.9). *Assume that the nonzero eigenvalues* $\lambda(\xi)$ *of* $\operatorname{Symb}(\mathcal{L})$ *satisfy* (2.33) *for some* $\nu \in [-r, \infty)$, *that all the hypotheses of either Lemma* 2.4 *or Lemma* 2.5 *hold, and that* $A(\varepsilon u)$ *is a positive-definite symmetric matrix.*

Assume also that the difference operator $\mathcal{F}_\Delta$ *from* (2.9), *after replacing* $S_{\Delta t}\widetilde{v}$ *in its last argument by* $\widetilde{v}$, *is an approximation of some order* σ_x *in* Δx *to the differential operator* $\mathcal{F}$ *in* (2.8), *with parameter values* $k_0 \le \lfloor \frac{d}{2} \rfloor + 1 + r$, $k_- \le r$, *and* $J = 0$. *If the difference operator* $\mathcal{C}_{\Delta x}$ *in* (2.9) *is nonzero then let* $\mathcal{C}_{\Delta x}$ *be an approximation of order* σ_x *in* Δx *to the differential operator* $\mathcal{C}$ *in* (2.8) *with the same parameter values as above, restrict* Δx *to satisfy* $\left| \frac{(\Delta x)^q}{\Delta t} - \mu \right| \le c(\Delta t)^{1/2}$, *where* q *and* μ *are the parameters appearing in* (2.9) *and* (2.8) *respectively, and assume that*

$$\frac{\min\{\sigma_x, \sigma\}}{q} \ge \tfrac{1}{2}, \tag{2.38}$$

where σ *is the parameter from Lemma* 2.4 *or Lemma* 2.5. *Alternatively, if* $\mathcal{C}_{\Delta x}$ *is identically zero then restrict* Δx *by* $\Delta x \le c(\Delta t)^{1/q}$, *where* q *is any positive number for which* (2.38) *holds.*

Assume that the PDE (2.8) *is stable uniformly in* ε *and linearly stable uniformly in* ε, *and the difference scheme* (2.9) *is stable uniformly in* ε, *all in the sense of Definition* 2.1 *with the parameter* k_0 *equal to* $\lfloor \frac{d}{2} \rfloor + 1 + r$, *where* d *is the spatial dimension.*

Let m *be zero if the hypotheses of Lemma* 2.4 *hold, or the parameter from Lemma* 2.5 *if the hypotheses of that lemma hold. Assume that the integer* s *satisfies*

$$s \ge \max\{\lfloor d/2 \rfloor + 1 + r + (\sigma_x + m) + 2(r + \nu), 3r + \nu\}, \tag{2.39}$$

and that the coefficients of the PDE (2.8) *are at least* C^s.

Assume that the initial data $v^0_{\Delta,\varepsilon}$ *for the difference scheme* (2.9) *and the initial data* u^0_ε *for the PDE* (2.8) *satisfy*

$$\| \operatorname{Int}_x v^0_{\Delta,\varepsilon} - u^0_\varepsilon \|_{H^{s-\sigma_x}} \le c(\Delta x)^{\frac{q}{2}}. \tag{2.40}$$

and

$$\| \mathcal{L}_{\Delta x} v^0_{\Delta,\varepsilon} \|_{h^{s-r}} \le c\varepsilon. \tag{2.41}$$

Finally, let c_{slow} *be the constant from Lemma* 4.2, *and assume that the solution* $v_{\Delta,\varepsilon}$ *of the difference scheme* (2.9) *having the initial value* $v^0_{\Delta,\varepsilon}$ *satisfies* (2.36), *the solution* u_ε *of the PDE* (2.8) *having the initial value* u^0_ε *satisfies the analogous bound*

$$\|u_\varepsilon\|_{L^\infty([0,T]; H^s)} + \|(u_\varepsilon)_t\|_{L^\infty([0,T]; H^{s-r})} \le c < \infty, \tag{2.42}$$

and that for every initial data $\widetilde{u}^0_\varepsilon$ *satisfying* $\|\widetilde{u}^0_\varepsilon - u^0_\varepsilon\|_{H^{s-r-\nu}} + \|\mathcal{L}(\widetilde{u}^0_\varepsilon - u^0_\varepsilon)\|_{H^{s-2r-\nu}} \le c_{slow}\varepsilon$ *the solution of the PDE* (2.8) *with initial data* $\widetilde{u}^0_\varepsilon$ *satisfies* (2.42) *with* s *replaced by* $s - r - \nu$, *all for some positive* T *independent of* ε. *Then*

$$\| \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon} - u_\varepsilon \|_{L^\infty([0,T]; H^{s-\max\{3r+2\nu+\sigma_x+m, 3r+\nu\}})} \le c\sqrt{\Delta t} \tag{2.43}$$

uniformly in ε.

Remark 2.10.

(1) When the operator $\mathcal{L}$ is elliptic on the orthogonal complement of its null space then $\nu = -r$, which reduces significantly the number of derivatives lost in the estimate (2.43). The slightly-compressible Euler equations of fluid dynamics (2.12) provide an example in which this condition holds. For the discretization of those equations in (2.11) in dimension three, the other parameters in the statement of Theorem 2.9 have the values $\lfloor \frac{d}{2} \rfloor = 1$, $m = 0$, $r = 1$, $\sigma_x = 1$, and hence $s \geq 4$ and the norm in estimate (2.43) is H^{s-2}.
(2) Initial data of the form (B.4) below satisfies the assumptions on the initial data in part 1 of Theorem 2.7 and in Theorem 2.8. Initial data of the form (3.13) below satisfies the assumptions on the initial data in part 2 of Theorem 2.7. Initial data of the form (3.20) below satisfies the assumptions on the initial data in part 3 of Theorem 2.7 and in Theorem 2.9.
(3) The consistency assumption (2.17) holds when $\mathcal{N}_{\Delta,\varepsilon}$ in (2.9) is obtained from $\mathcal{N}_{0,\varepsilon}$ in (2.8) by replacing each derivative operator ∂_{x_j} by a difference approximation of order σ_x and discretizing in time using a numerical scheme of order σ_t.

Lemma 2.11.

(1) *Let the PDE have the somewhat more specific form*

$$A(\varepsilon u)u_t + \tfrac{1}{\varepsilon}\mathcal{L}u + \sum_{j=1}^{d} A^j(t,x,u)u_{x_j} + \sum_{j,k=1}^{d} B^{j,k}u_{x_j x_k} + F(t,x,u) = 0, \tag{2.44}$$

where all the coefficients are sufficiently smooth, the A is strictly positive definite, A and the A^j are symmetric, the operator $\sum_{j,k} B^{j,k}\partial_{x_j}\partial_{x_k}$ is strongly elliptic or more generally the $B^{j,k}$ satisfy either ([8], Eq. (2.11) *or* [8], *Eq.* (2.12)), *and $\mathcal{L}$ is an antisymmetric constant-coefficient spatial differential operator. As before, let r denote the order of the highest derivative appearing in* (2.44). *Then the nonlinear and linearized uniform-in-ε stability assumptions* (2.13) *and* (2.15) *hold with $k_0 := \lfloor \frac{d}{2} \rfloor + 1 + r$, the uniqueness assumption holds for both the PDE and its limit equations* (2.24)–(2.25), *and every initial data satisfying $\|u^0_\varepsilon\|_{H^s} + \frac{1}{\varepsilon}\|\mathcal{L}u^0_\varepsilon\|_{H^{s-r}} \leq c$ yields a solution satisfying the uniform boundedness condition* (2.42).
(2) *As before we let r denote the order of the highest difference appearing in a difference scheme. The difference schemes from* [8] *satisfy the stability estimate* (2.16) *with $k_0 := \lfloor \frac{d}{2} \rfloor + 1 + r$ and the uniqueness condition for both the original scheme and for the limit scheme* (2.27)–(2.28).

The first part of Lemma 2.11 follows from standard estimates for singular limits of evolutionary PDEs in [12, 20]. The second part of the lemma follows from estimates in [8] plus straightforward extensions of those estimates. In particular, the differences schemes treated in [8] satisfy estimates in the continuous spaces H^s analogous to those shown there in the discrete spaces h^s since all the discrete estimates used to produce those estimates are analogues of known continuous estimates. In addition, uniqueness for the limit difference scheme (2.27)–(2.28) can be obtained *via* an ℓ^2 estimate for the difference of two solutions just like the uniqueness of the original difference scheme, because after multiplying the equation satisfied by the difference of two solutions by that difference the projection operators may be omitted since the difference lies in the range of the projection.

Since the discretization (2.11) of the equations (2.12) of slightly-compressible fluid dynamics is a particular case of the difference scheme from ([8], Sect. 3.1), that PDE satisfies the conditions of Lemma 2.4, and that discretization is a first-order approximation to the PDE, the following result shows that all the above convergence theorems apply to that example.

Corollary 2.12. *The results of Theorems 2.7–2.9 apply to systems satisfying the hypotheses of Theorem* 1 *or Theorem* 4 *of* [8], *provided that the assumptions on the consistency or order of approximation and the hypotheses from Lemma 2.4 or Lemma 2.5 hold where needed.*

3. Order of approximation

3.1. Order of accuracy of interpolation

By making repeated use of the estimate from Lemma A.5 of the "nonlinear commutator" $\operatorname{Int} F(u)-F(\operatorname{Int} u)$ of interpolation operators with nonlinear functions, along with the fact that the interpolation operators considered here commute with difference operators, we can estimate how close the interpolation of a solution of the difference equation (2.9) is to being itself a solution of that equation. No $O(\frac{1}{\varepsilon})$ terms occur in the estimate because the $O(\frac{1}{\varepsilon})$ terms in the difference equation are linear with constant coefficients and so commute with interpolations. Moreover, even though an estimate on the second time difference of the solution is needed in order to obtain the minimal time accuracy $\sigma_t=1$, the norm of that second time difference appears in the estimate (3.1) multiplied by a factor of ε, so only the first time difference of the solution needs to be uniformly bounded in order to obtain an estimate that is uniform in ε.

Lemma 3.1. *Let k be an integer satisfying $k \geq \lfloor\frac{d}{2}\rfloor+1+r$, where r is the maximal order of any difference appearing in the difference equation (2.9). Let σ_t be a positive integer, and σ_x be an integer satisfying $0 \leq \sigma_x < k-(\lfloor\frac{d}{2}\rfloor+1+r)$. Let $\mathcal{N}_{\Delta,\varepsilon}$ be the difference operator from (2.9), Int_x be the trigonometric interpolation operator (2.18), and Int_t be the time interpolation operator defined in Section 2.4 using at least σ_t+1 points. Assume that $A \in C^{k+\sigma_t+1}$ and that the other functions and coefficients appearing in $\mathcal{N}_{\Delta,\varepsilon}$ from (2.9) belong to $C^{k+\sigma_t}$. Finally, if $\mathcal{C}_{\Delta x}$ is nonzero then assume that $\frac{\varepsilon(\Delta x)^q}{\Delta t}$ is bounded. Then for any $v \in \ell^\infty([0,T]\cap\Delta t\mathbb{Z};h^k)\cap w^{\sigma_t,\infty}([0,T+\Delta t]\cap\Delta t\mathbb{Z};h^{k-\sigma_x})$ satisfying $\mathcal{N}_{\Delta,\varepsilon}[v]=0$,*

$$\begin{aligned}
&\|\mathcal{N}_{\Delta,\varepsilon}[\operatorname{Int}_t \operatorname{Int}_x v]\|_{L^\infty([0,T];H^{k-r-\sigma_x})} \leq \\
&\qquad C(\|v\|_{w^{\sigma_t,\infty}([0,T+\Delta t]\cap\Delta t\mathbb{Z};h^{k-\sigma_x})})[1+\varepsilon\|D_{\Delta t}v\|_{w^{\sigma_t,\infty}([0,T]\cap\Delta t\mathbb{Z};h^{k-r-\sigma_x})}](\Delta t)^{\sigma_t} \\
&\qquad\quad + C(\|v\|_{\ell^\infty([0,T+\Delta t]\cap\Delta t\mathbb{Z};h^k)})[1+\varepsilon\|D_{\Delta t}v\|_{\ell^\infty([0,T]\cap\Delta t\mathbb{Z};h^{k-r})}](\Delta x)^{\sigma_x}. \qquad (3.1)
\end{aligned}$$

Proof. The basic idea is to estimate $T[\operatorname{Int}_t \operatorname{Int}_x v]-\operatorname{Int}_t \operatorname{Int}_x T[v]$ for every term $T[v]$ occurring in $\mathcal{N}_{\Delta,\varepsilon}$ from (2.9); summing those estimates yields an estimate for $\mathcal{N}_{\Delta,\varepsilon}[\operatorname{Int}_t \operatorname{Int}_x v]$ since $\operatorname{Int}_t \operatorname{Int}_x \mathcal{N}_{\Delta,\varepsilon}[v]=\operatorname{Int}_t \operatorname{Int}_x 0=0$.

Note first that since the shift operator $\mathcal{L}_{\Delta x}(\rho S_{\Delta t}+(1-\rho))$ is linear and has constant coefficients, so the shift-invariance properties (A.2) and (A.10) of the interpolation operators ensure that $\frac{1}{\varepsilon}\mathcal{L}_{\Delta x}S_{\Delta t}(\operatorname{Int}_t \operatorname{Int}_x v)-\operatorname{Int}_t \operatorname{Int}_x\left(\frac{1}{\varepsilon}\mathcal{L}_{\Delta x}S_{\Delta t}v\right)=0$. Hence that term contributes nothing to the estimate, which ensures that the final estimate will be uniform in ε.

When estimating the term $\mathcal{F}_\Delta$ from (2.9) we will make the notation more compact by omitting the dependence of that function on arguments other than u. Since $r \geq p$, applying (A.18) with $k:=\sigma_t$ to $\mathcal{F}_\Delta(\boldsymbol{w})$ with $\boldsymbol{w}:=(\{D^\alpha_{\Delta x}\widetilde{u}\}_{0\leq|\alpha|\leq p},\{D^\alpha_{\Delta x}S_{\Delta t}\widetilde{u}\}_{0\leq|\alpha|\leq p})$ and using the commutativity of difference and interpolation operators and the invariance of h^k norms under spatial shifts yields

$$\begin{aligned}
&\|\mathcal{F}_\Delta(\{D^\alpha_{\Delta x}\operatorname{Int}_t \operatorname{Int}_x \widetilde{u}\}_{0\leq|\alpha|\leq p},\{D^\alpha_{\Delta x}S_{\Delta t}\operatorname{Int}_t \operatorname{Int}_x \widetilde{u}\}_{0\leq|\alpha|\leq p}) \\
&\qquad -\operatorname{Int}_t \operatorname{Int}_x \mathcal{F}_\Delta(\{D^\alpha_{\Delta x}\widetilde{u}\}_{0\leq|\alpha|\leq p},\{D^\alpha_{\Delta x}S_{\Delta t}\widetilde{u}\}_{0\leq|\alpha|\leq p})\|_{L^\infty([0,T];H^{k-r-\sigma_x})} \\
&\quad =\|\mathcal{F}_\Delta(\operatorname{Int}_t \operatorname{Int}_x \boldsymbol{w})-\operatorname{Int}_t \operatorname{Int}_x \mathcal{F}_\Delta(\boldsymbol{w})\|_{L^\infty([0,T];H^{k-r-\sigma_x})} \\
&\quad \leq C(\|\boldsymbol{w}\|_{w^{\sigma_t,\infty}([0,T]\cap\Delta t\mathbb{Z};h^{k-r-\sigma_x})})(\Delta t)^{\sigma_t}+C(\|\boldsymbol{w}\|_{\ell^\infty([0,T]\cap\Delta t\mathbb{Z};h^{k-r})})(\Delta x)^{\sigma_x} \\
&\quad \leq C(\|v\|_{w^{\sigma_t,\infty}([0,T+\Delta t]\cap\Delta t\mathbb{Z};h^{k-\sigma_x})})(\Delta t)^{\sigma_t}+C(\|\boldsymbol{v}\|_{\ell^\infty([0,T+\Delta t]\cap\Delta t\mathbb{Z};h^k)})(\Delta x)^{\sigma_x}.
\end{aligned} \qquad (3.2)$$

The remaining term in $\mathcal{N}_{\Delta,\varepsilon}$ is the term involving A. If A were constant then that term would also commute with the interpolation operators we are using. To treat the case when A is not constant, note that

$$A(\varepsilon\widetilde{v})-A(0)=\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}A(s\varepsilon\widetilde{v})\,\mathrm{d}s=\varepsilon\left[\int_0^1 \widetilde{v}\cdot\nabla_u A_u(s\varepsilon\widetilde{v})\,\mathrm{d}s\right]:=\varepsilon\widetilde{A}(\varepsilon\widetilde{v},\widetilde{v}). \qquad (3.3)$$

Using the fact that constant-coefficient linear difference operators commute with the interpolation operators we are using, (3.3), the fact that the order of $\mathcal{C}$ is at most r, and the assumption that $\frac{\varepsilon(\Delta x)^q}{\Delta t}$ is bounded, we obtain

$$\begin{aligned}
&\|A(\varepsilon \operatorname{Int}_t \operatorname{Int}_x v)(D_{\Delta t} \operatorname{Int}_t \operatorname{Int}_x v - \tfrac{(\Delta x)^q}{\Delta t}\mathcal{C}_{\Delta x} \operatorname{Int}_t \operatorname{Int}_x v) \\
&\qquad - \operatorname{Int}_t \operatorname{Int}_x \left\{A(\varepsilon \operatorname{Int}_t \operatorname{Int}_x v)(D_{\Delta t}v - \tfrac{(\Delta x)^q}{\Delta t}\mathcal{C}_{\Delta x}v)\right\} \|_{L^\infty([0,T];H^{k-r-\sigma_x})} \\
&\ = \|[A(\varepsilon \operatorname{Int}_t \operatorname{Int}_x v) - A(0)](D_{\Delta t} \operatorname{Int}_t \operatorname{Int}_x v - \tfrac{(\Delta x)^q}{\Delta t}\mathcal{C}_{\Delta x} \operatorname{Int}_t \operatorname{Int}_x v) \\
&\qquad - \operatorname{Int}_t \operatorname{Int}_x \left\{[A(\varepsilon \operatorname{Int}_t \operatorname{Int}_x v) - A(0)](D_{\Delta t}v - \tfrac{(\Delta x)^q}{\Delta t}\mathcal{C}_{\Delta x}v)\right\} \|_{L^\infty([0,T];H^{k-r-\sigma_x})} \\
&\ = \varepsilon\|\widetilde{A}(\varepsilon \operatorname{Int}_t \operatorname{Int}_x \widetilde{v}, \operatorname{Int}_t \operatorname{Int}_x \widetilde{v})(D_{\Delta t} \operatorname{Int}_t \operatorname{Int}_x v - \tfrac{(\Delta x)^q}{\Delta t}\mathcal{C}_{\Delta x} \operatorname{Int}_t \operatorname{Int}_x v) \\
&\qquad - \operatorname{Int}_t \operatorname{Int}_x \left\{\widetilde{A}(\varepsilon\widetilde{v}, \widetilde{v})(D_{\Delta t}v - \tfrac{(\Delta x)^q}{\Delta t}\mathcal{C}_{\Delta x}v)\right\} \|_{L^\infty([0,T];H^{k-r-\sigma_x})} \\
&\ \le C(\|v\|_{w^{\sigma_t,\infty}([0,T]\cap\Delta t\mathbb{Z};h^{k-r-\sigma_x})})\varepsilon\left[\|D_{\Delta t}v\|_{w^{\sigma_t,\infty}([0,T]\cap\Delta t\mathbb{Z};h^{k-r-\sigma_x})} + \|v\|_{w^{\sigma_t,\infty}([0,T]\cap\Delta t\mathbb{Z};h^{k-\sigma_x})}\right](\Delta t)^{\sigma_t} \\
&\qquad + C(\|v\|_{\ell^\infty([0,T]\cap\Delta t\mathbb{Z};h^{k-r})})\varepsilon\left[\|D_{\Delta t}v\|_{\ell^\infty([0,T]\cap\Delta t\mathbb{Z};h^{k-r})} + \|v\|_{\ell^\infty([0,T]\cap\Delta t\mathbb{Z};h^{k})}\right](\Delta x)^{\sigma_x}. \qquad (3.4)
\end{aligned}$$

Adding (3.2) and (3.4) and combining terms yields (3.1). □

3.2. Approximation of the null space of the large operator

Proof of Lemma 2.4. By the definition of the Sobolev H^k norms, in order to show that (2.35) holds it suffices to show that

$$|\operatorname{Symb}(\mathcal{P}_{\Delta x}) - \operatorname{Symb}(\mathcal{P})| \le c_{\mathcal{P}}\,|\xi|^\sigma(\Delta x)^\sigma. \qquad (3.5)$$

Since eliminating the components of ξ that do not appear in $\operatorname{Symb}(\mathcal{L})$ reduces the norm of that vector, we may ignore those components when proving (3.5). The assumption on $\operatorname{Symb}(\mathcal{L})$ then reduces to the assertion that its rank is independent of ξ for $\xi \neq 0$.

Since $\mathcal{L}$ and $\mathcal{L}_{\Delta x}$ are antisymmetric operators, the symbols of the orthogonal projections $\mathcal{P}$ and $\mathcal{P}_{\Delta x}$ onto their null spaces are matrix orthogonal projections for each value of ξ, and hence have norms bounded by one. First choose a constant positive μ such that $dc_\partial\mu < 1$, where d is the spatial dimension and c_∂ is the constant from (2.30). For all ξ satisfying $|\xi|\Delta x \ge \mu$,

$$|\operatorname{Symb}(\mathcal{P}_{\Delta x}) - \operatorname{Symb}(\mathcal{P})| \le 2 \le \tfrac{2}{\mu^\sigma}(|\xi|\Delta x)^\sigma,$$

so for this case (3.5) holds with $c_{\mathcal{P}} := \frac{2}{\mu^\sigma}$. It therefore suffices to show that (3.5) also holds when

$$|\xi|\Delta x \le \mu, \qquad (3.6)$$

possibly with a different value of $c_{\mathcal{P}}$.

Since the $\partial_{j,\Delta x,c}$ are central difference operators, their symbols are imaginary and odd, and hence $\eta_j(\Delta x,\xi) := -i[\operatorname{Symb}(\partial_{j,\Delta x,c})](\xi)$ is real. Since odd functions vanish at zero, $[\operatorname{Symb}(\mathcal{L}_{\Delta x})](0) = [\operatorname{Symb}(\mathcal{L})](0)$, which implies that (3.5) also holds for $\xi = 0$. More generally, $[\operatorname{Symb}(\mathcal{L}_{\Delta x})](\xi) = [\operatorname{Symb}(\mathcal{L})](\eta(\Delta x,\xi))$ and hence $[\operatorname{Symb}(\mathcal{P}_{\Delta x})](\xi) = [\operatorname{Symb}(\mathcal{P})](\eta(\Delta x,\xi))$, where η is the vector having components η_j.

For $\xi \neq 0$ the integral formula ([14], Eq. (II.1.16)), for the projection matrix $[\operatorname{Symb}(\mathcal{P})](\xi)$ onto the zero eigenspace of the matrix $[\operatorname{Symb}(\mathcal{L})](\xi)$ is

$$[\operatorname{Symb}(\mathcal{P})](\xi) = \frac{1}{2\pi i}\oint_{|z|=\delta(\xi)} (z - [\operatorname{Symb}(\mathcal{L})](\xi))^{-1}\,\mathrm{d}z, \qquad (3.7)$$

where $\delta(\xi)$ is chosen smaller than the absolute value of all nonzero eigenvalues of $[\operatorname{Symb}(\mathcal{L})](\xi)$. Recall that inverse of a matrix equals the transpose of its cofactor matrix divided by its determinant. Since the components of $[\operatorname{Symb}(\mathcal{L})]$ are polynomials in ξ, both cofactor matrix and the determinant of $z-[\operatorname{Symb}(\mathcal{L})](\xi)$ are polynomials in (z,ξ). Hence their ratio $(z-[\mathrm{Symb}(\mathcal{L})](\xi))^{-1}$ is a rational function of those variables. Hence (3.7) shows that the projection $[\mathrm{Symb}(\mathcal{P})](\xi)$ equals the coefficient of z^{-1} in the partial fraction decomposition of $(z-[\mathrm{Symb}(\mathcal{L})](\xi))^{-1}$, which is a rational function of ξ.

Let r be the fixed number of zero eigenvalues of $[\mathrm{Symb}(\mathcal{L})](\xi)$ for $\xi \neq 0$. Then $\det(z-[\mathrm{Symb}(\mathcal{L})](\xi)) = z^r \Lambda(z,\xi)$ for some polynomial Λ such that $\Lambda(0,\xi)$ nonzero for all nonzero ξ. Moreover, since the entries of the cofactor matrix of $z-[\mathrm{Symb}(\mathcal{L})](\xi)$ are the determinants of that matrix with one row and one column omitted, that matrix has the form $z^{r-1}C(z,\xi)$ for some polynomial ξ. Hence the coefficient of z^{-1} in the partial fraction decomposition of $(z-[\mathrm{Symb}(\mathcal{L})](\xi))^{-1}$ equals $\frac{C(0,\xi)}{\Lambda(0,\xi)}$, which shows that the denominator of the rational matrix-valued function $[\mathrm{Symb}(\mathcal{P})](\xi)$ is nonzero for nonzero ξ. Hence both $[\mathrm{Symb}(\mathcal{P})](\xi)$ and its gradient $\nabla[\mathrm{Symb}(\mathcal{P})](\xi)$ are well-defined for all nonzero ξ. Furthermore, the fact that the norm of $[\mathrm{Symb}(\mathcal{P})](\xi)$ equals one for all ξ implies that the maximum degree of any component of the numerator of that rational function with respect to any component of ξ is the same as the degree of there common denominator with respect to that component. The quotient rule therefore implies that $|\partial_{\xi_j}[\mathrm{Symb}(\mathcal{P})](\xi)| \le \frac{c}{|\xi_j|}$. Moreover, (3.6) together with (2.30) and the definition of $\eta(\Delta x,\xi)$ imply that $|\eta(\Delta x,\xi)-\xi| \le dc_\partial\mu|\xi| < c|\xi|$ for some constant $c<1$, which implies that $|s\eta+(1-s)\xi| \ge (1-cs)|\xi|$ for $0 \le s \le 1$ and hence that $\nabla[\mathrm{Symb}(\mathcal{P})](s\eta+(1-s)\xi)$ is well-defined for such s. Together with the above bounds for $\partial_{\xi_j}\mathrm{Symb}(\mathcal{P})$ this yields

$$\begin{aligned}
|[\mathrm{Symb}(\mathcal{P}](\eta(\Delta x,\xi)) - [\mathrm{Symb}(\mathcal{P})](\xi)| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}[\mathrm{Symb}(\mathcal{P})](s\eta(\Delta x,\xi)+(1-s)\xi)\,\mathrm{d}s\right| \\
&= \left|\sum_{j=1}^d (\eta_j(\Delta x,\xi)-\xi_j)\cdot\int_0^1 \partial_{\xi_j}[\,\mathrm{Symb}(\mathcal{P})](s\eta(\Delta x,\xi)+(1-s)\xi)\,\mathrm{d}s\right| \\
&\le c\sum_j \frac{|\eta_j(\Delta x,\xi)-\xi_j|}{|\xi_j|}. \qquad (3.8)
\end{aligned}$$

Hence assumption (2.30) together with the fact that $[\mathrm{Symb}(\mathcal{P})](\eta(\Delta x,\xi)))$ is the symbol of $\mathcal{P}_{\Delta x}$ yields (2.31). □

Proof of Lemma 2.5. Since $\mathcal{L}_{\Delta x}$ is a finite difference operator, $\mathrm{Symb}(\mathcal{L}_{\Delta x})$ is a sum of trigonometric polynomials of $\Delta x\,\xi$ divided by powers of Δx, times coefficients that are assumed analytic in Δx. Consider first the case when $\Delta x|\xi| \le \delta$, where δ is the constant mentioned in the lemma.

The fact that $\mathcal{L}_{\Delta x}$ approximates $\mathcal{L}$ implies that $[\mathrm{Symb}(\mathcal{L})](\xi) = \lim_{\Delta x\to 0}[\mathrm{Symb}(\mathcal{L}_{\Delta x})](\xi)$ and hence that the singularity of $[\mathrm{Symb}(\mathcal{L}_{\Delta x})](\xi)$ at $\Delta x = 0$ is removable. This means that $[\mathrm{Symb}(\mathcal{L}_{\Delta x})](\xi)$ is an analytic perturbation of $[\mathrm{Symb}(\mathcal{L})](\xi)$. Hence there is an integral formula ([14], Eq. (II.1.16), p. 77), for the projection operator onto the eigenspace of the eigenvalues that tend to zero with Δx, provided that the path of integration is chosen so as to include all those and only those eigenvalues. By assumption all those eigenvalues are identically zero, so that projection is the projection $\mathrm{Symb}(\mathcal{P}_{\Delta x})$ onto the zero eigenspace of $[\mathrm{Symb}(\mathcal{L}_{\Delta x})](\xi)$. Assumption (2.34) ensures that the path of integration can be taken to be a circle whose radius depends only on ξ, for all values of Δx under consideration. Hence

$$\mathrm{Symb}(\mathcal{P}_{\Delta x}) = \frac{1}{2\pi i}\oint_{|z|=\varepsilon(\xi)} (z-[\mathrm{Symb}(\mathcal{L}_{\Delta x})](\xi))^{-1}\,\mathrm{d}z. \qquad (3.9)$$

Moreover, since the multiplicity of the zero eigenvalue is constant in a neighborhood of $\Delta x = 0$, that point is not an exceptional point in the sense of ([14], Sect. II.1) so the limit of (3.9) as $\Delta x \to 0$ yields $\mathrm{Symb}(\mathcal{P})$.

Subtracting that limiting formula from (3.9) and using the identity $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$ yields

$$\begin{aligned}\operatorname{Symb}(\mathcal{P}_{\Delta x}) - \operatorname{Symb}(\mathcal{P}) &= \frac{1}{2\pi i}\oint_{|z|=\varepsilon(\xi)} \left[(z-[\operatorname{Symb}(\mathcal{L}_{\Delta x})](\xi))^{-1} - (z-[\operatorname{Symb}(\mathcal{L})](\xi))^{-1}\right] \mathrm{d}z \\ &= \frac{1}{2\pi i}\oint_{|z|=\varepsilon(\xi)} (z-[\operatorname{Symb}(\mathcal{L}_{\Delta x})](\xi))^{-1}\left([\operatorname{Symb}(\mathcal{L}_{\Delta x})](\xi)\right. \\ &\quad \left. -[\operatorname{Symb}(\mathcal{L})](\xi)\right)(z-[\operatorname{Symb}(\mathcal{L})](\xi))^{-1} \,\mathrm{d}z. \end{aligned} \tag{3.10}$$

By assumptions (2.34) and (2.33), $\varepsilon(\xi)$ may be taken to equal $\frac{c}{2}|\xi|^{-\nu}$ for $|\xi| > M$, and since there are only a finite number of wavenumbers satisfying $|\xi| \le M$ it may be taken to equal some constant for such ξ. Using assumption (2.32) plus the fact that both inverses appearing in the last integral of (3.10) are $O((1+|\xi|)^\nu)$ on the chosen contour of integration but the length of that contour cancels one of those factors, we obtain the estimate

$$\|\operatorname{Symb}(\mathcal{P}_{\Delta x} - \mathcal{P})\| = \|\operatorname{Symb}(\mathcal{P}_{\Delta x}) - \operatorname{Symb}(\mathcal{P})\| \le c(\Delta x)^\sigma (1+|\xi|)^{\sigma+m+\nu} \tag{3.11}$$

for $0 \le \Delta x|\xi| \le \delta$. On the other hand, for $\Delta x|\xi| \ge \delta$ (3.11) holds trivially, with a possibly larger constant c, since the left side is bounded by 2 while the right side will be at least that big provided that c is sufficiently large. Hence (3.11) holds for all Δx and all $\xi \in \Pi_{\Delta x}$, which by the definition of the H^s norms yields (2.35). $\square$

Example 3.2.

(1) The projection operator onto the null space of the operator

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & 0 & 2\partial_y \\ \partial_x & 2\partial_y & 0 \end{pmatrix}$$

has symbol

$$\operatorname{Symb}(\mathcal{P}) = \begin{pmatrix} \frac{4\xi_2^2}{\xi_1^2+4\xi_2^2} & -\frac{2\xi_1\xi_2}{\xi_1^2+4\xi_2^2} & 0 \\ -\frac{2\xi_1\xi_2}{\xi_1^2+4\xi_2^2} & \frac{\xi_1^2}{\xi_1^2+4\xi_2^2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which, in agreement with the Proof of Lemma 2.4, is a rational function of ξ. The matrix $\operatorname{Symb}(\mathcal{L})$ has constant rank for real nonzero ξ, as does the symbol of the difference operator $\mathcal{L}_{\Delta x}$ obtained by replacing ∂_x and ∂_y in $\mathcal{L}$ by the central difference operators $[\partial_{x,\Delta x,c}u](x,y) := \frac{u(x+\Delta x,y)-u(x-\Delta x,y)}{2\Delta x}$ and $[\partial_{y,\Delta x,c}u](x,y) := \frac{u(x,y+\Delta x)-u(x,y-\Delta x)}{2\Delta x}$, provided that $\Delta x \max_j |\xi_j| < \pi$. The operators $\mathcal{L}$ and $\mathcal{L}_{\Delta x}$ satisfy all the assumptions of Lemmas 2.4 and 2.5.

(2) The symbol of the operator

$$\mathcal{L} := \begin{pmatrix} \partial_x & 1 \\ -1 & \partial_x \end{pmatrix}$$

has a zero eigenvalue only when $\xi = \pm 1$. Hence Lemma 2.4 is not applicable. Replacing ∂_x by the standard central difference operator $\partial_{x,\Delta x,c} := \frac{S_{\Delta x} - (S_{\Delta x})^{-1}}{2\Delta x}$ would yield a difference operator whose symbol vanishes when ξ satisfies $\frac{\sin(\Delta x\xi)}{\Delta x} = \pm 1$ rather then when $\xi = \pm 1$ as required by Lemma 2.5. Moreover, while $\xi = \pm 1$ belongs to the Fourier domain $\mathbb{Z}$, for sufficiently small Δx the solutions of $\frac{\sin(\Delta x\xi)}{\Delta x} = \pm 1$ do not. Hence the projection operator $\mathcal{P}_{\Delta x}$ onto the null space of $\mathcal{L}_{\Delta x}$ is the zero operator, which does not converge to the nonzero projection operator $\mathcal{P}$ onto the null space of $\mathcal{L}$.

Nevertheless, Lemma 2.5. can be applied provided that we define $\mathcal{L}_{\Delta x}$ by replacing ∂_x in $\mathcal{L}$ by $\frac{\Delta x}{\sin(\Delta x)}$ times $\partial_{x,\Delta x,c}$, since the analytic coefficient multiplying the difference operator shifts the zeros of the symbol of $\mathcal{L}_{\Delta x}$ back to the location of the zeros of the symbol of $\mathcal{L}$. However, this method of moving the zeros of the symbol of the discretization is not always possible for more complicated operators.

3.3. Approximate initial data in the null space of the large operator

The results of Appendix B show how to obtain initial data for the difference scheme (2.9) satisfying the desirable conditions (B.1) and (B.2) for the case when the parameter ε is held constant. However, for the cases that include the limit as $\varepsilon \to 0$ the convergence theorems proven in this paper require that the first time difference of the numerical approximation be bounded uniformly in the small parameter ε. Theorems 1 and 4 of [8] show that this condition will hold for the schemes considered in those theorems provided that the initial data $v^0_{\Delta x,\varepsilon}$ of the numerical scheme satisfies

$$\|\mathcal{L}_{\Delta x} v^0_{\Delta x,\varepsilon}\|_{h^{s-p}} \le c\varepsilon \tag{3.12}$$

for some finite p and sufficiently large s, where as usual $\mathcal{L}_{\Delta x}$ denotes the large operator of the numerical scheme. Moreover, the proof of Theorem 1 of [8] shows that for any scheme of the form (2.9) considered here condition (3.12) ensures that the initial data for that scheme is slow, *i.e.*, is such that the first time-difference is bounded uniformly in ε at time zero.

If the initial data is not required to converge as $\Delta \to 0$ then initial data of the desired form can be obtained by letting $\widetilde{v}^0_0$ and $\widetilde{v}^0_1(\varepsilon)$ be arbitrary functions that are uniformly bounded in h^s and setting

$$v^0_{\Delta x,\varepsilon} = \mathcal{P}_{\Delta x}\widetilde{v}^0_0 + \varepsilon\widetilde{v}^0_1(\varepsilon), \tag{3.13}$$

where $\mathcal{P}_{\Delta x}$ is the projection operator onto the null space of the discrete large operator $\mathcal{L}_{\Delta x}$. However, more care is required when that initial data is required to converge to slow initial data for the PDE (2.8) as $\Delta \to 0$. Slow initial data for the PDE depending smoothly on ε has the form

$$u^0_\varepsilon(x) := u^0_0(x) + \varepsilon u^0_1(x,\varepsilon), \tag{3.14}$$

where

$$\mathcal{L}u^0_0 = 0 \tag{3.15}$$

and

$$\|u^0_0\|_{H^s} + \|u^0_1\|_{H^s} \le c \tag{3.16}$$

for some appropriate value of s. We then seek initial data for the difference scheme (2.9) of the form

$$v^0_{\Delta x,\varepsilon} = v^0_{\Delta x,0} + \varepsilon v^0_{\Delta x,1}(\varepsilon), \tag{3.17}$$

where

$$\mathcal{L}_{\Delta x} v^0_{\Delta x,0} = 0 \tag{3.18}$$

and

$$\|v^0_{\Delta x,0}\|_{h^s} + \|v^0_{\Delta x,1}(\varepsilon)\|_{h^s} \le c, \tag{3.19}$$

which will ensure that (3.12) holds with p equal to the order of the difference operator $\mathcal{L}_{\Delta x}$.

Lemma 3.3. *Suppose that the large operators $\mathcal{L}$ of the PDE and $\mathcal{L}_{\Delta x}$ of the numerical scheme satisfy the conditions of either Lemma 2.4 or 2.5 for some positive σ, and let s be an integer greater than $\frac{d}{2}+2+z$, where $z=0$ when the assumptions of Lemma 2.4 hold or $z=m+\nu$ if the assumptions of Lemma 2.5 hold.*

Let the initial data for the PDE have the form (3.14), where u^0_0 and u^0_1 satisfy (3.15)–(3.16). Define $\widetilde{v}^0_{\Delta x,0}$ and $v^0_{\Delta x,1}$ by the right side of formula (B.4) with u^0 replaced with u^0_0 and u^0_1, respectively, where in that formula ρ satisfies the hypotheses of Lemma B.1 and its Fourier transform $\widehat{\rho}(\xi)$ vanishes for $|\xi| \ge \pi$. Define $v^0_{\Delta x,0} := \mathcal{P}_{\Delta x}\widetilde{v}^0_{\Delta x,0}$, where $\mathcal{P}_{\Delta x}$ is the projection operator onto the null space of $\mathcal{L}_{\Delta x}$. Then the modified discrete initial data of the form (3.17), i.e.,

$$v^0_{\Delta x,\varepsilon} := v^0_{\Delta x,0} + \varepsilon v^0_{\Delta x,1,\varepsilon} = \mathcal{P}_{\Delta x}\widetilde{v}^0_{\Delta x,0} + \varepsilon v^0_{\Delta x,1,\varepsilon} \tag{3.20}$$

satisfies (3.18), (3.19), *and*

$$\|u^0_\varepsilon - \operatorname{Int}_x v^0_{\Delta,\varepsilon}\|_{H^{s-\sigma_1}} \le c\,(\Delta x)^{\sigma_1}(\|u^0_0\|_{H^s} + \varepsilon\|u^0_1\|_{H^s}) \qquad \textit{for } \sigma_1 \le \min\{\sigma, s\}, \tag{3.21}$$

$$\|v^0_{\Delta x,\varepsilon}\|_{h^s} \le c\,(\|u^0_0\|_{H^s} + \varepsilon\|u^0_1\|_{H^s}). \tag{3.22}$$

Proof. Condition (3.18) is satisfied on account of the projection operator $\mathcal{P}_{\Delta x}$ in the order one term of (3.20). Lemma B.1 ensures that $\widetilde{v}^0_{\Delta x,0}$ satisfies (B.2), and the projection $\mathcal{P}_{\Delta x}$ is a bounded operator, so (3.22) holds, and that estimate together with (3.16) ensures that (3.19) holds. The assumption on the spectrum of ρ ensures that there is no aliasing error, *i.e.*, $\operatorname{Int}_x \operatorname{Pr}_x(\rho_{\Delta x} \star f) = \rho_{\Delta x} \star f$, and since $\mathcal{P}_{\Delta x}$ is a multiplication operator in Fourier space it commutes with Int_x. Together with (3.15) and formula (B.4), these facts imply that

$$\operatorname{Int}_x \mathcal{P}_{\Delta x}\widetilde{v}^0_{\Delta x,0} = \operatorname{Int}_x \mathcal{P}_{\Delta x} \operatorname{Pr}_x(\rho_{\Delta x} \star u^0_0) = \mathcal{P}_{\Delta x} \operatorname{Int}_x \operatorname{Pr}_x(\rho_{\Delta x} \star u^0_0) = \mathcal{P}_{\Delta x}(\rho_{\Delta x} \star u^0_0).$$

Hence, for σ restricted as stated in the lemma,

$$\begin{aligned}\|\operatorname{Int}_x v^0_{\Delta x,0} - u^0_0\|_{H^{s-z-\sigma}} &= \|\operatorname{Int}_x \mathcal{P}_{\Delta x}\widetilde{v}^0_{\Delta x,0} - u^0_0\|_{H^{s-z-\sigma}} = \|\mathcal{P}_{\Delta x}(\rho_{\Delta x} \star u^0_0) - \mathcal{P}u^0_0\|_{H^{s-z-\sigma}}\\ &\le \|(\mathcal{P}_{\Delta x} - \mathcal{P})(\rho_{\Delta x} \star u^0_0)\|_{H^{s-z-\sigma}} + \|\mathcal{P}((\rho_{\Delta x} \star u^0_0) - u^0_0)\|_{H^{s-z-\sigma}} \le c(\Delta x)^\sigma \|u^0_0\|_{H^s},\end{aligned} \tag{3.23}$$

where z equals zero or $m+\nu$ depending which of Lemma 2.4 or 2.5 holds. Estimate (3.23) together with the corresponding estimate (B.1) for $\operatorname{Int}_x v^0_{\Delta x,1} - u^0_1$ implies that (3.21) holds ☐

4. Convergence

4.1. Convergence as Δ and/or ε tend to zero

Proof of Theorem 2.7. We present here the proof for the case when both Δ and ε tend to zero. The proof for the case when $\Delta \to 0$ with ε fixed is similar, and that case is also considered in the Proof of Theorem 2.8. The proof for the case when $\varepsilon \to 0$ with Δ fixed only requires substituting time differences for time derivatives, sums over time for integrals over time, and discrete versions developed in [8] for the continuous estimates used here.

By assumption (2.36), $v_{\Delta,\varepsilon}$ is uniformly bounded in $\ell^\infty([0, T + \Delta t] \cap \Delta t\mathbb{Z}; h^s)$ and $D_{\Delta t}v_{\Delta,\varepsilon}$ is uniformly bounded in $\ell^\infty([0, T] \cap \Delta t\mathbb{Z}; h^{s-r})$. The interpolation estimates (A.1) and (A.11) then show that $\operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}$ is uniformly bounded in $C^0([0, T]; H^s)$ and $\partial_t \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}$ is uniformly bounded in $C^0([0, T]; H^{s-r})$, in view of the continuity of the time-interpolation operator. Hence by Ascoli's theorem plus interpolation of Sobolev spaces, every sequence of solutions $v_{\Delta,\varepsilon}$ has a subsequence such that $\operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}$ converges in $L^\infty([0, T]; H^{s-1})$ to some limit u.

The standard estimate

$$\|\mathcal{F}(t, x, u) - \mathcal{F}(t, x, v)\|_{H^s} \le C(\|u\|_{H^s}, \|v\|_{H^s})\|u - v\|_{H^s} \tag{4.1}$$

for $s > \frac{d}{2}$ follows from the identity $\mathcal{F}(t, x, u) - \mathcal{F}(t, x, v) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}(t, x, su + (1-s)v)\,\mathrm{d}s = (u-v)\cdot\int_0^1 \nabla_u\mathcal{F}(t, x, su + (1-s)v)\,\mathrm{d}s$ together with (A.8) and the Sobolev embedding estimate. Using (4.1) together with the convergence of $\operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}$ on $\mathcal{P}_{\Delta x}$ applied to each term T in $\mathcal{N}_{\Delta,\varepsilon}$ except for the time-difference term shows that $\mathcal{P}_{\Delta x}T[\operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}] - \mathcal{P}_{\Delta x}T[u]$ tends to zero in $L^\infty([0, T]; H^{s-1-r})$ as Δ and ε tend to zero. Similarly, $\mathcal{L}_{\Delta x}(\rho S_{\Delta t} + (1-\rho)) \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon} - \mathcal{L}_{\Delta x}(\rho S_{\Delta t} + (1-\rho))u$ tends to zero in the same norm. To deal with the time-difference term, note that the uniform boundedness of the time difference implies that after taking a further subsequence $\partial_t \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}$ converges weak-$*$ $L^\infty([0, T]; H^{s-r})$ to some limit w. Moreover, taking the limit of the identity $\operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}(t_2, x) - \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}(t_1, x) = \int_{t=t_1}^{t_2} \partial_t \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}(t, x)$ shows that the limit w of $\partial_t \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}$ is the time derivative of the limit u of $\operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}$. Since the factor involving A converges strongly, the expression $A(\varepsilon \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon})D_{\Delta t} \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon} - A(0)\partial_t u$ converges weak-$*$ to zero Δ and ε tend to zero. To see this, let ϕ belong to the space $C^\infty_0([0, T] \times X)$ of compactly-supported smooth functions. Then

$$\begin{aligned}\langle \phi, A(\varepsilon \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon})D_{\Delta t} \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon} - A(0)u_t\rangle &= \langle \{A(\varepsilon \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}) - A(0)\}\,\phi, D_{\Delta t} \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}\rangle\\ &\quad + \langle A(0)\phi, D_{\Delta t} \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon} - u_t\rangle\,,\end{aligned}$$

and the first term on the right converges to zero since $A(\varepsilon \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}) - A(0)$ tends to zero in C^0 and $D_{\Delta t} \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}$ is uniformly bounded, while the second term on the right converges to zero on account of the weak-$*$ convergence of $D_{\Delta t} \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon} - u_t$ to zero. Since the projection $\mathcal{P}_{\Delta x}$ is bounded and preserves smoothness, applying that operator to expressions that tend weak-$*$ to zero yields results that still converge weak-$*$ to zero.

Note that by applying the operator $\mathcal{P}_{\Delta x}$ to the terms of $\mathcal{N}_{\Delta x,\varepsilon}$ and considering the operator $\mathcal{L}_{\Delta x}(\rho S_{\Delta t}+(1-\rho))$ separately without the factor $\frac{1}{\varepsilon}$ we have avoided the difficulty of trying to take the limit of a term of order $\frac{1}{\varepsilon}$, as would occur if we tried to take the limit of the term $\frac{1}{\varepsilon}\mathcal{L}_{\Delta x}(\rho S_{\Delta t} + (1-\rho)) \operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}$ directly.

Together with estimate (3.1) of $\mathcal{N}_{\Delta,\varepsilon}[\operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}]$, the above estimates imply that in order to show that the limit u satisfies the claimed limit equations it suffices to show that $\mathcal{P}_{\Delta x}\mathcal{N}_{\Delta,\varepsilon}[u]$ and $\mathcal{L}_{\Delta x}(\rho S_{\Delta t} + (1-\rho))u$ tend to those limit equations. Given the assumed consistency of the difference equation with the PDE, plus the assumed consistency of $\mathcal{P}_{\Delta x}$ with $\mathcal{P}$, those limits follow from the formal calculation of those limit equations in Section 2.5. Finally, since the initial data are assumed to converge, every subsequence of $\operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}$ has been shown to converge to a solution of the limit equations having the same initial data. Since solutions of the initial-value problem for the limit equations have been assumed to be unique, every such limit is therefore the same. By a standard result for limits, $\operatorname{Int}_t \operatorname{Int}_x v_{\Delta,\varepsilon}$ therefore converges without the need to restrict to a subsequence. □

4.2. Rate of convergence without large terms

Proof of Theorem 2.8. Since $v_{\Delta,\varepsilon}$ is assumed to be a solution of (2.9), Lemma 3.1 with $k = s+\sigma_x-\max\{\sigma_t r, \sigma_x\}$ shows that

$$\|\mathcal{N}_{\Delta,\varepsilon}[\operatorname{Int}_t \operatorname{Int}_x v_{\Delta x,\varepsilon}]\|_{L^\infty([0,T];H^{s-r-\max\{\sigma_t r,\sigma_x\}})} \le c((\Delta t)^{\sigma_t} + (\Delta x)^{\sigma_x}), \tag{4.2}$$

where c depends only on the $\ell^\infty([0, T + \Delta t] \cap \Delta t\mathbb{Z}; h^s)$ norm of v, since the $\ell^\infty([0, T] \cap \Delta t\mathbb{Z}; h^{s-jr})$ norms of $D^j_{\Delta t}v$ can be estimated in terms of the former norm by repeatedly using the difference equation plus the estimate ([8], Eq. (4.17)), for h^s norms of smooth functions $F(v)$ of an h^s function v. By the assumption that the difference scheme is an approximation of order σ_t in t and σ_x in x, with the assumed values of the parameters of Definition 2.2, we then obtain that

$$\begin{aligned}
&\|\mathcal{N}_{0,\varepsilon}[\operatorname{Int}_t \operatorname{Int}_x v_{\Delta x,\varepsilon}]\|_{L^\infty([0,T];H^{s-r-\max\{\sigma_t r,\sigma_x\}})}\\
&\quad\le \|\mathcal{N}_{\Delta,\varepsilon}[\operatorname{Int}_t \operatorname{Int}_x v_{\Delta x,\varepsilon}]\|_{L^\infty([0,T];H^{s-r-\max\{\sigma_t r,\sigma_x\}})}\\
&\qquad+ \|\mathcal{N}_{\Delta,\varepsilon}[\operatorname{Int}_t \operatorname{Int}_x v_{\Delta x,\varepsilon}] - \mathcal{N}_{0,\varepsilon}[\operatorname{Int}_t \operatorname{Int}_x v_{\Delta x,\varepsilon}]\|_{L^\infty([0,T];H^{s-r-\max\{\sigma_t r,\sigma_x\}})}\\
&\quad\le c((\Delta t)^{\sigma_t} + (\Delta x)^{\sigma_x}).
\end{aligned} \tag{4.3}$$

Applying the assumed estimate (2.13) with $k := s - r - \max\{\sigma_t r, \sigma_x\}$ to the functions $u^{(1)} := \operatorname{Int}_t \operatorname{Int}_x v$ and $u^{(2)} := u$ while taking into account the fact that $\mathcal{N}_{0,\varepsilon}[u] = 0$ and the assumed bound for $\|\operatorname{Int}_x v^0 - u^0\|_{H^k}$ then yields (2.37). □

4.3. Uniform-in-ε convergence rate

Before proving the uniform-in-ε convergence rate, we present an example to show the sharpness of that rate.

Example 4.1. Let f be a non-constant periodic function. The function $u(t,x) = \varepsilon[f(x) - f(x - \frac{t}{\varepsilon})]$ is the solution of the initial-value problem $u_t + \frac{1}{\varepsilon}u_x = f'(x)$, $u(0,x) = 0$, and its Fourier transform is $\widehat{u}(t,\xi) = \varepsilon\widehat{f}(\xi)\left[1 - \mathrm{e}^{-i\xi t/\varepsilon}\right]$. The function $v(n\Delta t, x)$ whose Fourier transform is $\widehat{v}(n\Delta t,\xi) = \varepsilon\widehat{f}(\xi)\left[1 - (1 + \frac{i\xi\Delta t}{\varepsilon})^{-n}\right]$ is the solution of the time-discretized semi-discrete approximation $D_{\Delta t}v + \frac{1}{\varepsilon}S_{\Delta t}v_x = f'(x)$ with the initial value $v(0,x) = 0$. Since $|\mathrm{e}^{is}| = 1$ and $|1 + is| = (1 + s^2)^{1/2}$ for real s, $|\widehat{v}(n\Delta t,\xi) - \widehat{u}(n\Delta t,\xi)| \ge \varepsilon|\widehat{f}(\xi)|(1 - (1 + \frac{\xi^2(\Delta t)^2}{\varepsilon^2})^{-n/2})$. For positive α and β, let $\alpha = O_S(\beta)$ denote that α is strictly of the order of β, *i.e.*, that there exist positive constants $c_\pm$ such that $c_- \le \frac{\alpha}{\beta} \le c_+$. For $|\xi| = O_S(1)$ for which $|\widehat{f}(\xi)| = O(1)$, $n = O_S(\frac{1}{\Delta t})$, and $\varepsilon = O_S(\sqrt{\Delta t})$, the expression $\left(1+\frac{\xi^2(\Delta t)^2}{\varepsilon^2}\right)^{-n/2}$ equals $(1+O_S(\Delta t))^{-O_S(\frac{1}{\Delta t})}$, which in turn equals $\mathrm{e}^{-O_S(1)}$. Since $1-\mathrm{e}^{-O_S(1)} = O_S(1)$, $|\widehat{v}(n\Delta t,\xi) - \widehat{u}(n\Delta t,\xi)| \geq O_S(\sqrt{\Delta t})$. This implies that when $\varepsilon = O_S(\sqrt{\Delta t})$ the error satisfies $\|v(t,x)-u(t,x)\|_{H^s} \geq O_S(\sqrt{\Delta t})$ for $t = O_S(1)$ and all H^s that f belongs to.

Before proving the $O(\sqrt{\Delta t})$ convergence rate for the full equations we need a technical lemma. In that lemma and the proof of the theorem we will use the notations

$$\mathcal{M}[u] := -\mu A(\varepsilon u)\mathcal{C}(\partial_x)u + \mathcal{F}(\varepsilon,t,x,\{D^\alpha u\}_{0\leq|\alpha|\leq p}) = \mathcal{N}_{0,\varepsilon}[u] - A(\varepsilon u)u_t - \tfrac{1}{\varepsilon}\mathcal{L}u \tag{4.4}$$

and

$$\begin{aligned}\mathcal{M}_\Delta[v] &:= -\tfrac{(\Delta x)^q}{\Delta t}A(\varepsilon v)\mathcal{C}_{\Delta x}v + \mathcal{F}_\Delta(\varepsilon,t,x,\{D^\alpha\widetilde{v}\}_{0\leq|\alpha|\leq p}, S_{\Delta t}\{D^\alpha\widetilde{v}\}_{0\leq|\alpha|\leq p}) \\ &= \mathcal{N}_{\Delta,\varepsilon}[v] - A(\varepsilon v)D_{\Delta t}v - \tfrac{1}{\varepsilon}\mathcal{L}_{\Delta x}(\rho S_{\Delta t}v + (1-\rho)v).\end{aligned} \tag{4.5}$$

Lemma 4.2. *Let $\mathcal{L}$ be an antisymmetric differential operator, and $A(0)$ be a symmetric positive-definite matrix. Assume that the nonzero eigenvalues $i\lambda(\xi)$ of* $\mathrm{Symb}(\mathcal{L})$ *satisfy* (2.33) *for some $\nu \in [-r,\infty)$, where r is the highest order of any derivative in* (2.8). *Let the initial data u^0_ε for* (2.8) *satisfy $\|u^0_\varepsilon\|_{H^s} \leq c$ and $\|\mathcal{L}u^0_\varepsilon\|_{H^{s-r}} \leq c\varepsilon$, where s satisfies* (2.39), *and assume that the coefficients of the PDE* (2.8) *are at least C^s.*

Then there exist a constant c_{slow} and initial data $\widetilde{u}^0_\varepsilon$ for (2.8) *satisfying*

$$\|\widetilde{u}^0_\varepsilon - u^0_\varepsilon\|_{H^{s-r-\nu}} + \|\mathcal{L}(\widetilde{u}^0_\varepsilon - u^0_\varepsilon)\|_{H^{s-2r-\nu}} \leq c_{slow}\,\varepsilon \tag{4.6}$$

such that the initial first and second time derivatives $\widetilde{u}_t(0) = -A(\varepsilon\widetilde{u}^0_\varepsilon)^{-1}(\frac{1}{\varepsilon}\mathcal{L}\widetilde{u}^0_\varepsilon + \mathcal{M}[\widetilde{u}^0_\varepsilon])$ and $\widetilde{u}_{tt}(0)$ calculated formally from the PDE (2.8) *using $\widetilde{u}^0_\varepsilon$ as the initial value of u satisfy $\|\widetilde{u}_t(0)\|_{H^{s-2r-\nu}} + \|\widetilde{u}_{tt}(0)\|_{H^{s-3r-\nu}} \leq c$.*

Proof. We will look for $\widetilde{u}^0_\varepsilon$ in the form $\widetilde{u}^0_\varepsilon = u^0_\varepsilon + \varepsilon\mathcal{U}^0_\varepsilon$, with $\|\mathcal{U}^0_\varepsilon\|_{H^{s-r-\nu}}$ bounded. Then $\|\widetilde{u}^0_\varepsilon - u^0_\varepsilon\|_{H^{s-r-\nu}} \leq c$ and $\|\mathcal{L}\widetilde{u}^0_\varepsilon\|_{H^{s-2r-\nu}} \leq c\varepsilon$, which together with the assumptions on u^0_ε will imply that (4.6) holds and that $\|\widetilde{u}_t(0)\|_{H^{s-2r-\nu}}$ is bounded uniformly in ε. We now calculate how to choose $\mathcal{U}^0_\varepsilon$ so that the remaining condition $\|\widetilde{u}_{tt}(0)\|_{H^{s-3r-\nu}} \leq c$ also holds. Differentiating (4.4) with respect to t, setting t equal to zero and $u(0)$ equal to $\widetilde{u}^0_\varepsilon$, using the above bounds, and defining $\widehat{\mathcal{L}} := A(0)^{-1/2}\mathcal{L}A(0)^{-1/2}$ yields

$$\begin{aligned}\widetilde{u}_{tt}(0) = -\tfrac{1}{\varepsilon}A(0)^{-1}\mathcal{L}\widetilde{u}_t(0) + O(1) &= \tfrac{1}{\varepsilon}A(0)^{-1}\mathcal{L}A(0)^{-1}\left\{\tfrac{1}{\varepsilon}\mathcal{L}u^0_\varepsilon + \mathcal{L}\mathcal{U}^0_\varepsilon + \mathcal{M}[u^0_\varepsilon]\right\} + O(1) \\ &= \tfrac{1}{\varepsilon}A(0)^{-1/2}\widehat{\mathcal{L}}\left\{\widehat{\mathcal{L}}A(0)^{1/2}\mathcal{U}^0_\varepsilon + A(0)^{-1/2}\left[\tfrac{1}{\varepsilon}\mathcal{L}u^0_\varepsilon + \mathcal{M}[u^0_\varepsilon]\right]\right\} + O(1),\end{aligned} \tag{4.7}$$

where $O(1)$ means a term bounded in H^{s-2r} uniformly in ε.

Recall that $\mathcal{P}$ is the orthogonal projection onto the null space of $\mathcal{L}$. Since the matrix $A(0)$ is positive definite, the operator $\mathcal{P}A(0)\mathcal{P}$ has a pseudo-inverse "$[\mathcal{P}A(0)\mathcal{P}]^{-1}$" from $\mathcal{P}H^k$ to itself for any k. Then $\widehat{\mathcal{P}} := (A(0))^{1/2}\mathcal{P}$"$[\mathcal{P}A(0)\mathcal{P}]^{-1}$"$\mathcal{P}(A(0))^{1/2}$ is the orthogonal projection onto the null space of $\widehat{\mathcal{L}}$. The assumption that (2.33) holds with $\nu \in [-r,\infty)$ ensures that $\widehat{\mathcal{L}}$ has a bounded pseudo-inverse "$\widehat{\mathcal{L}}^{-1}$" from $(I-\widehat{\mathcal{P}})H^k$ to $(I-\widehat{\mathcal{P}})H^{k-\nu}$. Since $\widehat{\mathcal{L}} = \widehat{\mathcal{L}}(I-\widehat{\mathcal{P}})$, we may insert a factor of $(I-\widehat{\mathcal{P}})$ before the term $A(0)^{-1/2}\left[\frac{1}{\varepsilon}\mathcal{L}u^0_\varepsilon + \mathcal{M}[u^0_\varepsilon]\right]$ in the last line of (4.7). Hence that equation shows that defining

$$\mathcal{U}^0_\varepsilon := -A(0)^{-1/2}\,\text{``}\widehat{\mathcal{L}}^{-1}\text{''}(I-\widehat{\mathcal{P}})A(0)^{-1/2}\left[\tfrac{1}{\varepsilon}\mathcal{L}u^0_\varepsilon + \mathcal{M}[u^0_\varepsilon]\right]$$

satisfies all the requirements of the lemma. ∎

Proof of Theorem 2.9. The assumed bounds for $\partial_t u_\varepsilon$ and $D_{\Delta t}v_{\Delta,\varepsilon}$ imply that both u and $\mathrm{Int}_t\,\mathrm{Int}_x\,v_\Delta$ differ from their values at the nearest multiple of Δt by $O(\Delta t)$, so it suffices to prove the estimate (2.43) for times that are multiples of Δt. It would suffice to prove that

$$\mathcal{N}_{\Delta,\varepsilon}[u_\varepsilon] = O(\sqrt{\Delta t}) \tag{4.8}$$

uniformly in ε, since Lemma 3.1 implies that

$$\mathcal{N}_{\Delta,\varepsilon}[\mathrm{Int}_t\,\mathrm{Int}_x\, v_{\Delta,\varepsilon}] = O(\Delta t + (\Delta x)^{\sigma_x}) = O(\sqrt{\Delta t}) \tag{4.9}$$

uniformly in ε, and hence the assumed uniform stability of the difference scheme $\mathcal{N}_{\Delta,\varepsilon}$ would then yield the desired result, in similar fashion to the proof of Theorem 2.8. Although we will not quite obtain (4.8), we will obtain a modified version of that estimate in which u_ε is replaced by the sum of u_ε and a small term, and that variant will suffice.

Hence we want to determine the remainder term R in the difference equation $\mathcal{N}_{\Delta,\varepsilon}[u_\varepsilon] = R$ satisfied by a solution of the PDE $\mathcal{N}_{0,\varepsilon}[u_\varepsilon] = 0$. For that purpose we will use Taylor's formula to write a time derivative u_t as a time difference $D_{\Delta t}u$ plus an integral remainder. However, because the large operator in the difference equation is applied to $\{\rho S_{\Delta t} + (1-\rho)\}v$ we will take ρ times the formula $S_{\Delta t}u_t = D_{\Delta t}u + \Delta t\int_0^1 s u_{tt}(t+s\Delta t, x)\,\mathrm{d}s$ plus $1-\rho$ times the formula $u_t = D_{\Delta t}u - \Delta t\int_0^1 (1-s)u_{tt}(t+s\Delta t, x)\,\mathrm{d}s$ to obtain

$$\rho S_{\Delta t}u_t + (1-\rho)u_t = D_{\Delta t}u + \Delta t I[u_{tt}], \tag{4.10}$$

where

$$I[u_{tt}] := \int_0^1 (\rho s - (1-\rho)(1-s))u_{tt}(t+s\Delta t, x)\,\mathrm{d}s. \tag{4.11}$$

Formula (4.10) with $u := u_\varepsilon$ lets us write ρ times $S_{\Delta t}$ applied to the PDE (2.8) plus $1-\rho$ times that PDE as

$$\begin{aligned}
0 &= \rho S_{\Delta t}\mathcal{N}_{0,\varepsilon}[u_\varepsilon] + (1-\rho)\mathcal{N}_{0,\varepsilon}[u_\varepsilon] \\
&= A(\varepsilon u_\varepsilon)D_{\Delta t}u_\varepsilon + \rho\left\{A(\varepsilon S_{\Delta t}u_\varepsilon) - A(\varepsilon u_\varepsilon)\right\}S_{\Delta t}(u_\varepsilon)_t + \Delta t A(\varepsilon u_\varepsilon)I[(u_\varepsilon)_{tt}] \\
&\quad + \tfrac{1}{\varepsilon}\mathcal{L}(\rho S_{\Delta t}u_\varepsilon + (1-\rho)u_\varepsilon) + \rho S_{\Delta t}\mathcal{M}[u_\varepsilon] + (1-\rho)\mathcal{M}[u_\varepsilon] \\
&= \mathcal{N}_{\Delta,\varepsilon}[u_\varepsilon] + \rho\left\{A(\varepsilon S_{\Delta t}u_\varepsilon) - A(\varepsilon u_\varepsilon)\right\}S_{\Delta t}(u_\varepsilon)_t + \rho(S_{\Delta t}-1)\mathcal{M}[u_\varepsilon] \\
&\quad + (\mathcal{M}[u_\varepsilon] - \mathcal{M}_\Delta[u_\varepsilon]) + \tfrac{1}{\varepsilon}(\mathcal{L} - \mathcal{L}_{\Delta x})(\rho S_{\Delta t}u_\varepsilon + (1-\rho)u_\varepsilon) + \Delta t A(\varepsilon u_\varepsilon)I[(u_\varepsilon)_{tt}].
\end{aligned} \tag{4.12}$$

Hence

$$N_{\Delta,\varepsilon}[u_\varepsilon] = T_1 + T_2 + T_3 + T_4 + T_5, \tag{4.13}$$

where

$$\begin{aligned}
T_1 &:= \rho\left\{A(\varepsilon u_\varepsilon) - A(\varepsilon S_{\Delta t}u_\varepsilon)\right\}S_{\Delta t}(u_\varepsilon)_t, \\
T_2 &:= \rho(1 - S_{\Delta t})\mathcal{M}[u_\varepsilon], \\
T_3 &:= \mathcal{M}_\Delta[u_\varepsilon] - \mathcal{M}[u_\varepsilon], \\
T_4 &:= \tfrac{1}{\varepsilon}(\mathcal{L}_{\Delta x} - \mathcal{L})(\rho S_{\Delta t}u_\varepsilon + (1-\rho)u_\varepsilon),
\end{aligned}$$

and

$$T_5 := -\Delta t A(\varepsilon u_\varepsilon)I[(u_\varepsilon)_{tt}].$$

The uniform bounds (2.42) on the solution u_ε and its first time derivative together with the fact the order of the highest derivative in $\mathcal{M}$ is at most r imply that

$$\|T_1\|_{L^\infty([0,T];H^{s-r})} + \|T_2\|_{L^\infty([0,T];H^{s-2r})} \le c\Delta t. \tag{4.14}$$

Since the assumptions of the theorem imply that

$$(\Delta x)^{\sigma_x} \le c(\Delta t)^{1/2}, \tag{4.15}$$

the uniform bounds (2.42) together with the fact that $\mathcal{M}_\Delta$ is an approximation of order σ_x to $\mathcal{M}$ with parameters $k_- = r$ and $J = 0$ yield

$$\|T_3\|_{L^\infty([0,T];H^{s-r-\sigma_x})} \le c(\Delta x)^{\sigma_x} \le \widetilde{c}(\Delta t)^{1/2}. \tag{4.16}$$

To deal with the term T_4, divide it into the terms

$$T_{4,a} := \tfrac{1}{\varepsilon}(\mathcal{L}_{\Delta x} - \mathcal{L})\mathcal{P}(\rho S_{\Delta t} u_\varepsilon + (1-\rho)u_\varepsilon) \quad \text{and} \quad T_{4,b} := \tfrac{1}{\varepsilon}(\mathcal{L}_{\Delta x} - \mathcal{L})(I-\mathcal{P})(\rho S_{\Delta t} u_\varepsilon + (1-\rho)u_\varepsilon), \tag{4.17}$$

where $\mathcal{P}$ is the projection operator onto the null space of $\mathcal{L}$. Begin with $T_{4,b}$. Since (2.33) holds with $\nu \in [-r, \infty)$ and the order of $\mathcal{L}$ is at most r, the operator $\mathcal{L}$ has a bounded pseudo-inverse "$\mathcal{L}^{-1}$" from $\mathcal{L}H^s \subset H^{s-r}$ to $(I-\mathcal{P})H^{s-r-\nu}$. Also, solving the PDE (2.8) for $\mathcal{L}u_\varepsilon$ and using the assumed uniform bounds (2.42) to estimate the result shows that $\|\mathcal{L}u_\varepsilon\|_{L^\infty([0,T];H^{s-r})} \le c\varepsilon$. Together these yield the estimate

$$\|(I-\mathcal{P})(\rho S_{\Delta t} u_\varepsilon + (1-\rho)u_\varepsilon)\|_{L^\infty([0,T];H^{s-r-\nu})} = \|\text{“}\mathcal{L}^{-1}\text{”}\mathcal{L}(\rho S_{\Delta t} u_\varepsilon + (1-\rho)u_\varepsilon)\|_{L^\infty([0,T];H^{s-r-\nu})} \le c\varepsilon. \tag{4.18}$$

In addition, by the hypotheses of either Lemma 2.4 or 2.5 plus (4.15),

$$\|(\mathcal{L}_{\Delta x} - \mathcal{L})u\|_{H^{k-r-\sigma_x-m}} \le c\|u\|_{H^k}(\Delta x)^{\sigma_x} \le \widetilde{c}\|u\|_{H^k}(\Delta t)^{1/2}, \tag{4.19}$$

where m equals zero if Lemma 2.4 holds or the value that parameter has in Lemma 2.5. Substituting $u := (I-\mathcal{P})(\rho S_{\Delta t} u_\varepsilon + (1-\rho)u_\varepsilon)$ into (4.19) and using (4.18) to estimate the result yields

$$\|T_{4,b}\|_{L^\infty([0,T];H^{s-2r-\nu-\sigma_x-m})} = \|\tfrac{1}{\varepsilon}(\mathcal{L}_{\Delta x} - \mathcal{L})(I-\mathcal{P})(\rho S_{\Delta t} u_\varepsilon + (1-\rho)u_\varepsilon)\|_{L^\infty([0,T];H^{s-2r-\nu-\sigma_x-m})} \le c(\Delta t)^{1/2}. \tag{4.20}$$

Next, using the identities $\mathcal{L}\mathcal{P} = 0 = \mathcal{L}_{\Delta x}\mathcal{P}_{\Delta x}$, where $\mathcal{P}_{\Delta x}$ is the projection onto the null space of $\mathcal{L}_{\Delta x}$, the term $T_{4,a}$ can be rewritten as

$$\begin{aligned} T_{4,a} = \tfrac{1}{\varepsilon}(\mathcal{L}_{\Delta x} - \mathcal{L})\mathcal{P}(\rho S_{\Delta t} u_\varepsilon + (1-\rho)u_\varepsilon) &= \tfrac{1}{\varepsilon}\mathcal{L}_{\Delta x}\mathcal{P}(\rho S_{\Delta t} u_\varepsilon + (1-\rho)u_\varepsilon) \\ &= \tfrac{1}{\varepsilon}\mathcal{L}_{\Delta x}(\mathcal{P} - \mathcal{P}_{\Delta x})(\rho S_{\Delta t} u_\varepsilon + (1-\rho)u_\varepsilon) \\ &= -\tfrac{1}{\varepsilon}\mathcal{L}_{\Delta x}(\rho S_{\Delta t} U + (1-\rho)U), \end{aligned} \tag{4.21}$$

where

$$U := (\mathcal{P}_{\Delta x} - \mathcal{P})u_\varepsilon \tag{4.22}$$

Lemma 2.4 or 2.5 plus (4.15) and the bounds (2.42) for u_ε imply that U satisfies

$$\|U\|_{L^\infty([0,T];H^{s-\sigma_x-m})} + \|D_{\Delta t}U\|_{L^\infty([0,T];H^{s-\sigma_x-m-r})} \le c(\Delta x)^{\sigma_x} \le \widetilde{c}(\Delta t)^{1/2}. \tag{4.23}$$

It does not seem possible to obtain a uniform-in-ε estimate for $T_{4,a}$. Instead, we will rewrite $\mathcal{N}_{\Delta,\varepsilon}[u_\varepsilon] - T_{4,a}$ in terms of $\mathcal{N}_{\Delta,\varepsilon}[u_\varepsilon + U]$, obtaining

$$\mathcal{N}_{\Delta,\varepsilon}[u_\varepsilon] - T_{4,a} = \mathcal{N}_{\Delta,\varepsilon}[u_\varepsilon + U] - T_6, \tag{4.24}$$

where

$$T_6 := [A(\varepsilon u_\varepsilon + U) - A(\varepsilon(u_\varepsilon))]\, D_{\Delta t} u_\varepsilon + A(\varepsilon(u_\varepsilon + U))D_{\Delta t}U + (\mathcal{M}_\Delta[u_\varepsilon + U] - \mathcal{M}_\Delta[u_\varepsilon])\,. \tag{4.25}$$

In similar fashion to the derivation of (4.14), (4.23) together with the bounds (2.42) for u_ε implies that

$$\|T_6\|_{L^\infty([0,T];H^{s-\sigma_x-m-r})} \le c(\Delta t)^{1/2}. \tag{4.26}$$

In order to treat the term T_5 we must consider two separate cases. If $\varepsilon \ge \sqrt{\Delta t}$ then the fact that $\|(u_\varepsilon)_{tt}\|_{L^\infty([0,T];H^{s-2r})} \le \frac{c}{\varepsilon}$ implies that

$$\|T_5\|_{L^\infty([0,T];H^{s-2r})} = \|\Delta t A(\varepsilon u_\varepsilon) I[(u_\varepsilon)_{tt}]\|_{L^\infty([0,T];H^{s-2r})} \le c\frac{\Delta t}{\varepsilon} \le c(\Delta t)^{1/2}. \tag{4.27}$$

Otherwise $\varepsilon \le \sqrt{\Delta t}$, in which case we replace u_ε by the solution $\widetilde{u}_\varepsilon$ having the initial data $\widetilde{u}^0_\varepsilon$ from Lemma 4.2, whose existence and boundedness on the time interval $[0,T]$ is guaranteed by the assumptions of the theorem. However, since $\widetilde{u}_\varepsilon$ only satisfies (2.42) with s replaced by $s-r-\nu$, that substitution must also be made in the norms in the estimates (4.14), (4.16), (4.20), and (4.26). Since $\varepsilon \le \sqrt{\Delta t}$, estimate (4.6) plus the assumed uniform nonlinear stability of the PDE (2.8) imply that

$$\|\widetilde{u}_\varepsilon - u_\varepsilon\|_{L^\infty([0,T];H^{s-r-\nu})} \le c\|\widetilde{u}^0_\varepsilon - u^0_\varepsilon\|_{L^\infty([0,T];H^{s-r-\nu})} \le c\varepsilon \le c(\Delta t)^{1/2}. \tag{4.28}$$

Moreover, $(\widetilde{u}_\varepsilon)_{tt}$ satisfies the modified linearized equation

$$\mathcal{N}'_{0,\varepsilon,\widetilde{u}_\varepsilon}[(\widetilde{u}_\varepsilon)_{tt}] = -2(\widetilde{u}_\varepsilon)_t \cdot \nabla_u A(\varepsilon\widetilde{u}_\varepsilon)(\widetilde{u}_\varepsilon)_{tt} + G[\widetilde{u}_\varepsilon, (\widetilde{u}_\varepsilon)_t] \tag{4.29}$$

for some spatial differential operator G of order at most r. Since $\widetilde{u}_\varepsilon$ satisfies (2.42) with s replaced by $s-r-\nu$, the assumed uniform linear stability of the PDE (2.8) implies that

$$\|(\widetilde{u}_\varepsilon)_{tt}(t,\cdot)\|^2_{H^{s-3r-\nu}} \le c\left\{\int_0^t \left[\|(\widetilde{u}_\varepsilon)_{tt}(s,\cdot)\|^2_{H^{s-3r-\nu}} + c\right] \mathrm{d}s + \|(\widetilde{u}_\varepsilon)_{tt}(0,\cdot)\|^2_{H^{s-3r-\nu}}\right\}. \tag{4.30}$$

Since Lemma 4.2 implies that $\|(\widetilde{u}_\varepsilon)_{tt}(0,\cdot)\|_{H^{s-3r-\nu}}$ is bounded uniformly in ε, Gronwall's lemma says that (4.30) implies that $\|(\widetilde{u}_\varepsilon)_{tt}\|_{L^\infty([0,T];H^{s-3r-\nu})}$ is also bounded uniformly in ε. Hence

$$\|T_5\|_{L^\infty([0,T];H^{s-3r-\nu})} = \|\Delta t A(\varepsilon u_\varepsilon) I[(u_\varepsilon)_{tt}]\|_{L^\infty([0,T];H^{s-3r-\nu})} \le c\Delta t \tag{4.31}$$

Substituting formula (4.13) into the formula (4.24) for $N_{\Delta x,\varepsilon}[u_\varepsilon + U]$, replacing u_ε with $\widetilde{u}_\varepsilon$ in the result when $\varepsilon \le \sqrt{\Delta t}$, using (4.23) and (4.28) to estimate the difference between the argument of $N_{\Delta,\varepsilon}$ and u_ε, and using the estimates (4.14), (4.16), (4.20), (4.26), (4.27), and (4.31) for the terms T_j shows that in both cases

$$\mathcal{N}_{\Delta,\varepsilon}[u_\varepsilon + O(\sqrt{\Delta t})] = O(\sqrt{\Delta t}). \tag{4.32}$$

Since $u_\varepsilon + O(\sqrt{\Delta t})$ differs at time zero from $\mathrm{Int}_t\,\mathrm{Int}_x\, v_{\Delta,\varepsilon}$ by at most $O(\sqrt{\Delta t})$, estimates (4.9) and (4.32) together with the assumed uniform stability of the difference scheme (2.9) yields (2.43) upon taking into account the norms in which the various estimates hold. $\square$

Appendix A. Estimates for interpolation operators

As noted in ([8], Lem. 4.3), the trigonometric interpolation operator (2.18) satisfies

$$\|\,\mathrm{Int}_x f\|_{H^s} \le c_s\|f\|_{h^s}. \tag{A.1}$$

Other interpolation operators could also be used as long as they satisfy (A.1) and the identities (2.20) and (A.2) and estimate (A.3) below; for problems without large terms the identities could even be relaxed by allowing the difference between the two sides to be sufficiently small rather than identically zero.

In order to obtain estimates for interpolation operators it will be convenient to use the projection operator Pr_x defined by pointwise evaluation on the spatial computational grid. Since the formula for Int_x reduces at grid points to the formula for the inverse discrete Fourier transform of the discrete Fourier transform, the first part of (2.20) holds. The definition of Int_x also implies that it commutes with shift operators, *i.e.*,

$$[\mathrm{Int}_x, (S_{\Delta t})^p S^\alpha_{\Delta x}] = 0, \tag{A.2}$$

where as usual $[A,B]$ denotes the commutator of the two operators A and B.

In order to show that interpolants of solutions of finite-difference schemes are nearly solutions of the PDEs approximated by such schemes we will need to estimate the difference between $F(\mathrm{Int}_x u)$ and $\mathrm{Int}_x F(u)$

when F is a nonlinear function. A key point of our analysis is that such estimates can be obtained by combining estimates for norms of $F(u)$ in terms of norms of u with an estimate for the linear operator $1-\operatorname{Int}_x \operatorname{Pr}_x$. The combination $\operatorname{Int}_x \operatorname{Pr}_x$ is commonly known as the Fourier pseudospectral projection operator, so the estimate

$$\|(1-\operatorname{Int}_x \operatorname{Pr}_x)f\|_{H^{s-\sigma}} \le c\,(\Delta x)^\sigma \|f\|_{H^s} \qquad \text{for } s>\tfrac{d}{2} \text{ and } 0\le\sigma\le s \tag{A.3}$$

is a particular case of ([3], Thm. 1.2). The case $\sigma=0$ of (A.3) implies that

$$\|\operatorname{Int}_x \operatorname{Pr}_x f\|_{H^s} \le \|f\|_{H^s} + \|(1-\operatorname{Int}_x \operatorname{Pr}_x)f\|_{H^s} \le c\|f\|_{H^s}. \tag{A.4}$$

Lemma A.1. *Let σ and s be integers satisfying $0\le\sigma\le s$ and $s>\frac{d}{2}$, where d is the spatial dimension, and let $\mathcal{F}$ be any function in C^s, and if the domain is infinite then assume in addition that $\|\mathcal{F}(t,x,0)\|_{H^s}$ is finite. Let Int_x be any scalar operator applied componentwise satisfying (A.1), the first part of (2.20), and (A.3). Then there exists a function C and a constant c such that for all $u\in H^s$ and v in h^s,*

$$\begin{aligned}\|\mathcal{F}(t,x,\operatorname{Int}_x \operatorname{Pr}_x u) - \operatorname{Int}_x \mathcal{F}(t,\operatorname{Pr}_x x,\operatorname{Pr}_x u)\|_{H^{s-\sigma}}&\\ \le C(\|\operatorname{Int}_x \operatorname{Pr}_x u\|_{H^s})(\Delta x)^\sigma \le c\,C(c\|u\|_{H^s})(\Delta x)^\sigma,&\end{aligned} \tag{A.5}$$

$$\|\mathcal{F}(t,x,\operatorname{Int}_x v) - \operatorname{Int}_x \mathcal{F}(t,\operatorname{Pr}_x x,v)\|_{H^{s-\sigma}} \le C(\|v\|_{h^s})(\Delta x)^\sigma. \tag{A.6}$$

Proof. By the identity $\operatorname{Pr}_x \mathcal{F}(t,x,u) = \mathcal{F}(t,\operatorname{Pr}_x x,\operatorname{Pr}_x u)$ and (2.20),

$$\operatorname{Int}_x \operatorname{Pr}_x \mathcal{F}(t,x,\operatorname{Int}_x \operatorname{Pr}_x u) = \operatorname{Int}_x \mathcal{F}(t,\operatorname{Pr}_x x,\operatorname{Pr}_x \operatorname{Int}_x \operatorname{Pr}_x u) = \operatorname{Int}_x \mathcal{F}(t,\operatorname{Pr}_x x,\operatorname{Pr}_x u). \tag{A.7}$$

By (A.7), (A.3), and the continuum version (*cf.* [17], Prop. 2.1)

$$\|\mathcal{F}(t,x,u)-\mathcal{F}(t,x,0)\|_{H^s} \le C(\|u\|_{H^s})\|u\|_{H^s} \tag{A.8}$$

of [8], (4.17), plus the assumptions on $\mathcal{F}$,

$$\begin{aligned}&\|\mathcal{F}(t,x,\operatorname{Int}_x \operatorname{Pr}_x u) - \operatorname{Int}_x \mathcal{F}(t,\operatorname{Pr}_x x,\operatorname{Pr}_x u)\|_{H^{s-\sigma}}\\ &\quad= \|(1-\operatorname{Int}_x \operatorname{Pr}_x)[\mathcal{F}(t,x,\operatorname{Int}_x \operatorname{Pr}_x u)-\mathcal{F}(t,x,0)] + (1-\operatorname{Int}_x \operatorname{Pr}_x)\mathcal{F}(t,x,0)\|_{H^{s-\sigma}}\\ &\quad\le c\,(\Delta x)^\sigma\left(\|\mathcal{F}(t,x,\operatorname{Int}_x \operatorname{Pr}_x u)-\mathcal{F}(t,x,0)\|_{H^s} + \|\mathcal{F}(t,x,0)\|_{H^s}\right)\\ &\quad\le C(\|\operatorname{Int}_x \operatorname{Pr}_x u\|_{H^s})(\Delta x)^\sigma,\end{aligned} \tag{A.9}$$

which yields the first inequality of (A.5); the second inequality there then follows by (A.4). Estimate (A.6) follows from (A.5) upon taking $u:=\operatorname{Int}_x v$ since $\operatorname{Pr}_x u$ then equals v and the H^s norm of u is bounded by the h^s norm of v by (A.1). $\square$

As discussed in Section 2.4, the time interpolation operator may be either the piecewise-linear operator (2.19) or its generalization to polynomial interpolation of order p using $p+1$ points. Any such time interpolation operator Int_t also reduces to the identity at grid points and commutes with shifts, *i.e.* satisfies (2.20) and

$$\left[\operatorname{Int}_t, (S_{\Delta t})^j S^\alpha_{\Delta x}\right] = 0. \tag{A.10}$$

The divided-difference formula for polynomial interpolation implies that

$$\|(\partial_t)^j \operatorname{Int}_t f\|_{L^\infty([m\Delta t,(m+1)\Delta t];H^s)} \le c\|(D_{\Delta t})^j f\|_{L^\infty([0,T]\cap\Delta t\mathbb{Z};H^s)} \tag{A.11}$$

for $0\le j\le p+1$, which in turn yields

$$\|(\partial_t)^j \operatorname{Int}_t \operatorname{Pr}_t f\|_{L^\infty([m\Delta t,(m+1)\Delta t];H^s)} \le c\|(\partial_t)^j f\|_{L^\infty([0,T];H^s)} \tag{A.12}$$

for such j since the formula $D_{\Delta t} f = \frac{1}{\Delta t}\int_0^{\Delta t} f'(t+s)\,\mathrm{d}s$ can be used recursively to write difference operators in terms of derivatives.

Remark A.2. The reason why the norms on the left sides of (A.11)–(A.12) involve individual time intervals of length Δt is that the derivatives of the interpolation are in general only piecewise continuous, with discontinuities at the time grid points. Hence derivatives of the interpolation of order at least two do not belong to $L^\infty([0,T])$ when considered in the distributional sense. However, interval-by-interval estimates suffice for the purpose of showing that interpolants of solutions of a difference scheme are approximate solutions of that scheme. As a more compact notation for estimates that hold in each interval $[m\Delta t,(m+1)\Delta t]$ we henceforth let $L^\infty_{\rm PW}$ denote the space of distributions that equal a bounded function on each interval $(m\Delta t,(m+1)\Delta t)$. Similarly, $W^{k,\infty}_{\rm PW}$ is the subspace of $L^\infty_{\rm PW}$ of functions whose time derivatives through order k also belong to $L^\infty_{\rm PW}$, with the norm defined in (2.7). Note that in the case of the L^∞ norm the piecewise interpretation is only needed when time derivatives of order at least two are applied to the time interpolation.

Estimates (A.11) and (A.12) are the analogues of (A.1) and (A.4) for the time interpolation operator. We now prove an analogue of estimate (A.3).

Lemma A.3. *Let* Int_t *be any polynomial time-interpolation operator using* $p+1$ *points, as described above. Then for any* $f\in W^{k,\infty}([0,T];H^s)$ *with* $s\ge 0$,

$$\|(\partial_t)^{k-\sigma}(1-\mathrm{Int}_t\,\mathrm{Pr}_t)f\|_{L^\infty_{PW}([0,T];H^s)}\le c\,(\Delta t)^\sigma\|(\partial_t)^k f\|_{L^\infty([0,T];H^s)} \tag{A.13}$$

for $0\le\sigma\le k\le p+1$.

Proof. When $\sigma=0$ then (A.13) follows from (A.11). Now assume that $\sigma\ge 1$. Since $k\le p+1$, the error of polynomial interpolation of order p certainly vanishes for polynomials for order $k-1$, so the Peano kernel theorem (*e.g.* [6], Thm. 3.7.1)) with remainder of order k is applicable to interpolation of order p. Moreover, by ([7], Thm. 2.1) that theorem remains valid for Banach-space-valued functions, with the same kernel function as in the scalar case. The Peano kernel theorem yields (*e.g.* [6], Eqs. (3.7.2) and (3.7.3), Ex. 1, p. 71)

$$[(1-\mathrm{Int}_t\,\mathrm{Pr}_t)f](m\Delta t+t)=\int_0^{p\Delta t} f^{(k)}(m\Delta t+s)K(t,s)\,\mathrm{d}s, \tag{A.14}$$

where

$$K(t,s):=\frac{1}{(k-1)!}\left[(t-s)_+^{k-1}-\sum_{i=0}^{p}(i\Delta t-s)_+^{k-1}\ell_i(t)\right]. \tag{A.15}$$

Here $\ell_i(t)$ are the polynomials of order p from the Lagrange interpolation formula, which are uniquely determined by the condition that $l_i(j\Delta t)=\delta_{ij}$. Scaling considerations or the explicit formula ([6], Eq. (2.5.1)), for the ℓ_i show that $(\partial_t)^j\ell_i=O((\Delta t)^{-j})$, so $(\partial_t)^jK(t,s)=O((\Delta t)^{k-1-j})$ for $0\le j\le k-1$ when t and s lie in the domain of integration in (A.14). Since the interval of integration in (A.14) is $O(\Delta t)$, differentiating (A.14) $k-\sigma$ times and substituting the estimate for derivatives of K into the result yields (A.13). □

Following the proof of Lemma A.1 but using (A.11), the second part of (2.20), and (A.13) in place of (A.1), the first part of (2.20), and (A.3) yields the following result for time interpolation analogous to Lemma A.1 for spatial interpolation. The smoothness requirement $s>\frac{d}{2}$ still applies to the spatial dependence because that requirement arises from (A.8); the time norm index k has no such requirement because L^∞ based norms $W^{k,\infty}$ are used for the time dependence.

Lemma A.4. *Let* σ *and* k *be integers satisfying* $0\le\sigma\le k\le p+1$, *where* p *is the order of the time interpolation operator, assume that* $s>\frac{d}{2}$, *and let* $\mathcal{F}$ *be any function in* C^{k+s}. *Let* Int_t *be any scalar operator applied componentwise satisfying* (A.11), *the second part of* (2.20), *and* (A.13).

Then there exists a function C and a constant c such that for all $u \in W^{k,\infty}([0,T];H^s)$ and v in $w^{k,\infty}([0,T]\cap \Delta t\mathbb{Z};H^s)$,

$$\begin{aligned}&\|\mathcal{F}(t,x,\operatorname{Int}_t \operatorname{Pr}_t u) - \operatorname{Int}_t \mathcal{F}(\operatorname{Pr}_t t, x, \operatorname{Pr}_t u)\|_{W^{k-\sigma,\infty}_{PW}([0,T];H^s)} \\ &\qquad \le C(\|\operatorname{Int}_t \operatorname{Pr}_t u\|_{W^{k,\infty}_{PW}([0,T];H^s)})(\Delta t)^\sigma \le cC(c\|u\|_{W^{k,\infty}([0,T];H^s)})(\Delta t)^\sigma\end{aligned} \tag{A.16}$$

$$\|\mathcal{F}(t,x,\operatorname{Int}_t v) - \operatorname{Int}_t \mathcal{F}(\operatorname{Pr}_t t,x,v)\|_{W^{k-\sigma,\infty}_{PW}([0,T];H^s)} \le C(\|v\|_{w^{k,\infty}([0,T]\cap\Delta t\mathbb{Z};H^s)})(\Delta t)^\sigma \tag{A.17}$$

Lemmas A.1 and A.4 can be combined in various ways to estimate the time and space interpolation operator $\operatorname{Int}_t \operatorname{Int}_x$, such as in the following lemma, which is obtained by writing $\mathcal{F}(t,x,\operatorname{Int}_t \operatorname{Int}_x v) - \operatorname{Int}_t \operatorname{Int}_x \mathcal{F}(\operatorname{Pr}_t t, \operatorname{Pr}_x x, v)$ as

$$[\mathcal{F}(t,x,\operatorname{Int}_t \operatorname{Int}_x v) - \operatorname{Int}_t \mathcal{F}(\operatorname{Pr}_t t,x,\operatorname{Int}_x v)] + \operatorname{Int}_t [\mathcal{F}(\operatorname{Pr}_t t,x,\operatorname{Int}_x v) - \operatorname{Int}_x \mathcal{F}(\operatorname{Pr}_t t,\operatorname{Pr}_x x,v)]\,.$$

Lemma A.5. *Let the integers k, s, σ_1 and σ_2 satisfy $0 \le \sigma_1 \le k$ and $0 \le \sigma_2 \le s - (\lfloor \frac{d}{2} \rfloor + 1)$, and let $\mathcal{F}$ be any function in $C^{k+s-\min(\sigma_1,\sigma_2)}$. Let Int_x and Int_t be any scalar operators applied componentwise satisfying (A.1), (2.20), (A.3), (A.11), and (A.13).*

Then there exists a function C and a constant c such that for all v in $w^{k,\infty}([0,T]\cap\Delta t\mathbb{Z};h^{s-\sigma_2})\cap w^{k-\sigma_1}([0,T]\cap \Delta t\mathbb{Z};h^s)$,

$$\begin{aligned}&\|\mathcal{F}(t,x,\operatorname{Int}_t \operatorname{Int}_x v) - \operatorname{Int}_t \operatorname{Int}_x \mathcal{F}(\operatorname{Pr}_t t,\operatorname{Pr}_x x,v)\|_{W^{k-\sigma_1,\infty}_{PW}([0,T];H^{s-\sigma_2})} \\ &\qquad \le C(\|v\|_{w^{k,\infty}([0,T]\cap\Delta t\mathbb{Z};h^{s-\sigma_2})})(\Delta t)^{\sigma_1} + C(\|v\|_{w^{k-\sigma_1,\infty}([0,T]\cap\Delta t\mathbb{Z};h^s)})(\Delta x)^{\sigma_2}.\end{aligned} \tag{A.18}$$

Appendix B. Approximation of initial data

It is desirable that the initial data v^0_Δ for the numerical scheme should approximate the initial data u^0 of the PDE both accurately and boundedly, *i.e.*, that

$$\|u^0 - \operatorname{Int}_x v^0_\Delta\|_{H^{s-\sigma}} \le c(s)(\Delta x)^\sigma \|u^0\|_{H^s} \tag{B.1}$$

when $s > \frac{d}{2}$ and $\sigma \le s$, and

$$\|v^0_{\Delta x}\|_{h^s} \le c(s)\|u^0\|_{H^s}. \tag{B.2}$$

Neither direct pointwise evaluation nor taking cell averages satisfy both these conditions. However, both can be obtained by taking the pointwise projection of a smoother average:

Lemma B.1. *Let $\rho : \mathbb{R}^d \mapsto \mathbb{R}$ satisfy $\int_{\mathbb{R}^d} |\rho| < \infty$, $\int_{\mathbb{R}^d} \rho = 1$,*

$$\widehat{\rho}(\xi) \equiv 1 \textit{ for } |\xi| \le \delta, \textit{ for some positive } \delta, \tag{B.3}$$

and $\sup_{x\in[-\frac12,\frac12]^d} \sum_{n\in\mathbb{Z}^d} |\rho(n+x)| < \infty$. Then, defining $\rho_{\Delta x}(x) := \frac{1}{(\Delta x)^d}\rho(\frac{x}{\Delta x})$ and $f \star g := \int_{\mathbb{R}^d} f(z)g(x-z)\,\mathrm{d}z$,

$$v^0_{\Delta x} := \operatorname{Pr}_x(\rho_{\Delta x} \star u^0) \tag{B.4}$$

satisfies (B.1)–(B.2).

Remark B.2. A function ρ satisfying the assumptions of Lemma B.1 cannot have L^1 norm equal to one, and hence cannot be nonnegative, since the fact that $\cos(s)$ is less than one almost everywhere implies that for $0 < |k| < \delta$,

$$1 = \operatorname{Re} 1 = \operatorname{Re} \widehat{\rho}(k) = \int_{\mathbb{R}^d} \rho(x)\cos(k\cdot x)\,\mathrm{d}x < \int_{\mathbb{R}^d} |\rho|.$$

Example B.3. The function $\rho(x) := \frac{2(\cos(x)-\cos(2x))}{x^2}$, whose Fourier transform is $\widehat{\rho}(\xi) = \begin{cases} 1 & |\xi|\le 1 \\ 2-|\xi| & 1\le|\xi|\le 2 \\ 0 & |\xi|\ge 2 \end{cases}$, satisfies all the hypotheses of Lemma B.1 for the case $d=1$. So does $\rho_d(x) := \prod_j \rho(x_j)$ for arbitrary d.

Proof of Lemma B.1. By a slight generalization of ([27], Eq. (1.38) of Chap. 16)

$$\|f - \rho_{\Delta x} \star f\|_{H^{s-\sigma}} \le c\,\delta^{-\sigma}\,(\Delta x)^{\sigma}\|f\|_{H^s} \tag{B.5}$$

for $0 \le \sigma \le s$, where δ is the constant appearing in (B.3) and $c := 1 + \int_{\mathbb{R}^d} |\rho|$. Taking $s = 0$ in (B.5) shows that

$$\|\rho_{\Delta x} \star f\|_{H^s} \le \|f\|_{H^s} + \|f - \rho_{\Delta x} \star f\|_{H^{s-\sigma}} \le c\|f\|_{H^s}. \tag{B.6}$$

Combining (B.5) with $f := u^0$, (A.3) with $f := \rho_{\Delta x} \star u^0$ and (B.6) with $f := u^0$ yields (B.1).

The main step towards proving (B.2) is to show that

$$\|\operatorname{Pr}_x(\rho_{\Delta x} \star u)\|_{\ell^2} \le \left[\int_{\mathbb{R}^d} |\rho|\right]^{1/2} \left[\sup_{x\in[-\frac12,\frac12]^d} \sum_{n\in\mathbb{Z}^d} |\rho(n+x)|\right]^{1/2} \|u\|_{L^2}.$$

Since the sum in the definition of the ℓ^2 norm has the form of a Riemann sum for the L^2 norm, we follow as far as possible the proof (*e.g.*, [9], Eq. 0.C) of Young's inequality: By the definitions of $\rho_{\Delta x}$ and the convolution operator $\star$, the periodicity of u, and the Cauchy–Schwartz inequality applied to $\sqrt{\sum_{m\in 2\pi\mathbb{Z}^d} |\rho_{\Delta x}(y+m)|}$ and $\sqrt{\sum_{m\in 2\pi\mathbb{Z}^d} |\rho_{\Delta x}(y+m)|}|u(y)|$,

$$\begin{aligned}
\|\operatorname{Pr}_x(\rho_{\Delta x} \star u)\|_{\ell^2}^2 &= \sum_{x\in X_{\Delta x}} \left|\int_{\mathbb{R}^d} \rho_{\Delta x}(z) u(x-z)\,\mathrm{d}z\right|^2 (\Delta x)^d \overset{y=x-z}{=} \sum_{x\in X_{\Delta x}} \left|\int_{\mathbb{R}^d} \rho_{\Delta x}(x-y) u(y)\,\mathrm{d}y\right|^2 (\Delta x)^d \\
&= \sum_{x\in X_{\Delta x}} \left|\int_{[-\pi,\pi]^d} \Big[\sum_{m\in 2\pi\mathbb{Z}^d} \rho_{\Delta x}(x-y-m)\Big] u(y)\,\mathrm{d}y\right|^2 (\Delta x)^d \\
&\le \sum_{x\in X_{\Delta x}} \int_{[-\pi,\pi]^d} \Big[\sum_{m\in 2\pi\mathbb{Z}^d} |\rho_{\Delta x}(x-y-m)|\Big]\,\mathrm{d}y \int_{[-\pi,\pi]^d} \Big[\sum_{m\in 2\pi\mathbb{Z}^d} |\rho_{\Delta x}(x-y-m)|\Big] |u(y)|^2\,\mathrm{d}y (\Delta x)^d \\
&= \sum_{x\in X_{\Delta x}} \int_{\mathbb{R}^d} |\rho_{\Delta x}(x-y-m)|\,\mathrm{d}y \int_{[-\pi,\pi]^d} \Big[\sum_{m\in 2\pi\mathbb{Z}^d} |\rho_{\Delta x}(x-y-m)|\Big] |u(y)|^2\,\mathrm{d}y (\Delta x)^d \\
&\overset{z=x-m}{=} \left[\int_{\mathbb{R}^d} |\rho|\right] \int_{[-\pi,\pi]^d} \left[\sum_{z\in(\Delta x)\mathbb{Z}^d} (\Delta x)^d |\rho_{\Delta x}(z-y)|\right] |u(y)|^2\,\mathrm{d}y \\
&\overset{z=(\Delta x)n}{=} \left[\int_{\mathbb{R}^d} |\rho|\right] \int_{[-\pi,\pi]^d} \sum_{n\in\mathbb{Z}^d} |\rho(n - \tfrac{y}{\Delta x})| |u(y)|^2\,\mathrm{d}y \le \left[\int_{\mathbb{R}^d} |\rho|\right] \left[\sup_{x\in[-\frac12,\frac12]^d} \sum_{n\in\mathbb{Z}^d} |\rho(n+x)|\right] \|u\|_{L^2([-\pi,\pi]^d)}^2.
\end{aligned}$$

The corresponding result for Sobolev norms follows from the facts that the difference operators $D_{j,\Delta x} = \frac{S_{j,\Delta x}-1}{\Delta x}$ commute with pointwise evaluation on the grid with spacing Δx and with convolution, and that $\left|\frac{\mathrm{e}^{i\Delta x \xi_j}-1}{\Delta x}\right| \le |\xi_j|$ (*e.g.*, [9], Proof of Thm. 6.21), which implies that for any difference operator $D_{\Delta x}^{\alpha} = \prod_{j=1}^{d} D_{j,\Delta x}^{\alpha_j}$, $\|D_{\Delta x}^{\alpha} u\|_{H^s} \le \|D^{\alpha} u\|_{H^s}$. ∎

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