ESAIM: M2AN 51 (2017) 1637–1661
DOI: 10.1051/m2an/2016078

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

NUMERICAL ANALYSIS OF THE OSEEN-TYPE PETERLIN VISCOELASTIC MODEL BY THE STABILIZED LAGRANGE–GALERKIN METHOD PART I: A NONLINEAR SCHEME

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Abstract. We present a nonlinear stabilized Lagrange–Galerkin scheme for the Oseen-type Peterlin viscoelastic model. Our scheme is a combination of the method of characteristics and Brezzi–Pitkäranta's stabilization method for the conforming linear elements, which yields an efficient computation with a small number of degrees of freedom. We prove error estimates with the optimal convergence order without any relation between the time increment and the mesh size. The result is valid for both the diffusive and non-diffusive models for the conformation tensor in two space dimensions. We introduce an additional term that yields a suitable structural property and allows us to obtain required energy estimate. The theoretical convergence orders are confirmed by numerical experiments. In a forthcoming paper, Part II, a linear scheme is proposed and the corresponding error estimates are proved in two and three space dimensions for the diffusive model.

Mathematics Subject Classification. 65M12, 65M25, 65M60, 76A10.

Received April 22, 2016. Revised October 23, 2016. Accepted December 5, 2016.

1. Introduction

In the daily life we encounter many biological, industrial or geological fluids that do not satisfy the Newtonian assumption, *i.e.*, the linear dependence between the stress tensor and the deformation tensor. These fluids belong to the class of the non-Newtonian fluids. In order to describe such complex fluids the stress tensor is represented as a sum of the viscous (Newtonian) part and the extra stress due to the polymer contribution.

In literature we can find several models that are employed to describe various aspects of complex viscoelastic fluids. One of the well-known viscoelastic models is the Oldroyd-B model, which is derived from the Hookean dumbbell model with a linear spring force law. The model is a system of equations for the velocity, the pressure and the extra stress tensor, *cf.*, *e.g.*, [31,32].

Numerical schemes for the Oldroyd-B type models have been studied by many authors. For example, we can find a finite difference scheme based on the reformulation of the equation for the extra stress tensor by using

Keywords and phrases. Error estimates, Peterlin viscoelastic model, Lagrange–Galerkin method, Pressure-stabilization.

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the log-conformation representation in Fattal and Kupferman [12,13], free energy dissipative Lagrange–Galerkin schemes with or without the log-conformation representation in Boyaval *et al.* [5], finite element schemes using the idea of the generalized Lie derivative in Lee and Xu [15] and Lee *et al.* [16], and further related numerical schemes and computations in [1, 4, 11, 14, 20, 22, 24, 39] and references therein. To the best of our knowledge, however, there are no results on error estimates of numerical schemes for the Oldroyd-B model. As for the simplified Oldroyd-B model with no convection terms Picasso and Rappaz [30] and Bonito *et al.* [3] have given error estimates for stationary and non-stationary problems, respectively. The development of stable and convergent numerical methods for the Oldroyd-B type models, especially in the elasticity-dominated case, is still an active research area.

In this paper, Part I, and the forthcoming paper [18], Part II, we consider the so-called Peterlin viscoelastic model, which is a system of the flow equations and an equation for the conformation tensor, *cf.* [31,32]. In [29] Peterlin proposed a mean-field closure model according to which the average of the elastic force over thermal fluctuations is replaced by the value of the force at the mean-squared polymer extension. More precisely, instead of the nonlinear spring force law $F(R) = \gamma(|R|^2)R$ that acts in polymer dumbbells the Peterlin approximation $F(R) \approx \gamma(\langle|R|^2\rangle)R$ is applied, where R is the vector connecting the dumbbell beads and γ is the spring constant. That means, that the length of the spring in the spring constant γ is replaced by the average length of the spring $\langle|R|^2\rangle \equiv \operatorname{tr}\mathbf{C}$. Consequently, we can derive an evolution equation for the conformation tensor $\mathbf{C}$, which is in a closed form, *cf.* [19, 23, 31, 32, 34]. Note that in literature one can also find the Peterlin approximation in the context of finitely extensible nonlinear elastic (FENE) dumbbell model, which was subsequently termed the FENE-P model, *cf.* [2]. In this model the denominator of the FENE force of the corresponding kinetic model is replaced by the mean value of the elongation yielding the macroscopic FENE-P model. On the other hand, Renardy recently proposed a general macroscopic constitutive model, that is motivated by Peterlin dumbbell theories with a nonlinear spring law for an infinitely extensible spring, see Renardy [33,34] and a recent paper by Lukáčová−Medvid'ová *et al.* [21], where the global existence of weak solutions has been obtained. The diffusive Peterlin viscoelastic model has been obtained by a particular choice of these general constitutive functions. This model has been studied analytically by Lukáčová−Medvid'ová *et al.* [19], where the global existence of weak solutions and the uniqueness of regular solutions have been proved. Let us mention that, even when the velocity field is given, the equation for the conformation tensor in the Peterlin model is still nonlinear, while the Oldroyd-B model is linear with respect to the extra stress tensor. Hence, we can say that the nonlinearity of the Peterlin model is stronger than that of the Oldroyd-B model. As a starting point of the numerical analysis of the Peterlin model, we consider the Oseen-type model, where the velocity of the material derivative is replaced by a known one, in order to concentrate on the treatment of nonlinear terms arising from the elastic stress.

Our aim is to develop a stabilized Lagrange–Galerkin method for the Peterlin viscoelastic model. It consists of the method of characteristics and Brezzi–Pitkäranta's stabilization method [8] for the conforming linear elements. The method of characteristics yields the robustness in convection-dominated flow problems, and the stabilization method reduces the number of degrees of freedom in computation. In our recent works by Notsu and Tabata [26–28] the stabilized Lagrange–Galerkin method has been applied successfully for the Oseen, Navier–Stokes and natural convection problems and optimal error estimates have been proved.

We establish the numerical analysis of the stabilized Lagrange–Galerkin method for the Oseen-type Peterlin model in this paper, Part I, and the forthcoming paper [18], Part II. The results of the two papers are summarized in Tables 1 and 2, where ε is the diffusion coefficient in the equation for the conformation tensor, d is the spatial dimension, h is the representative mesh size and Δt is the time increment.

In Part I, a nonlinear stabilized Lagrange–Galerkin scheme for the diffusive ($\varepsilon > 0$) and the non-diffusive ($\varepsilon = 0$) Peterlin model is presented and error estimates with the optimal convergence order are proved without any relation between discretization parameters Δt and h in two dimensions. For the proof we rely on a key lemma, *cf.* Lemma 5.5, in which a special structural property using an additional term $(\operatorname{div}\mathbf{u}_h^n(\mathbf{C}_h^n)^\#, \mathbf{D}_h)$ is shown. However, this property does not hold in three-dimensional case. This is the reason why the convergence result is shown only in two space dimensions. The theoretical convergence orders are confirmed by numerical experiments. Since the scheme is nonlinear, the existence and uniqueness of the scheme are studied additionally,

TABLE 1. Summary of our results in Part I and Part II. (ε is the diffusion coefficient for the conformation tensor and d is the spatial dimension).

	Part I	Part II
Scheme	Nonlinear	Linear
ε	≥ 0	> 0
d	2	2 and 3

TABLE 2. Conditions on the time increment Δt with respect to the mesh size h. ($\varnothing$ means that no condition is required).

	Part I, $d = 2$		Part II, $\varepsilon > 0$	
Existence	$\varnothing$		$\varnothing$	
	$\varepsilon > 0$	$\varepsilon = 0$		
Uniqueness	$O\left(\dfrac{1}{(1+\lvert\log h\rvert)^2}\right)$	$O(h)$	$\varnothing$	
			$d = 2$	$d = 3$
Optimal error estimates	$\varnothing$		$O\left(\dfrac{1}{\sqrt{1+\lvert\log h\rvert}}\right)$	$O(\sqrt{h}\,)$

and we show that the scheme has a solution without any relation between h and Δt and that the solution is unique for the diffusive and the non-diffusive cases under the conditions $\Delta t = O(1/(1+|\log h|)^2)$ and $\Delta t = O(h)$, respectively, in two dimensions.

In Part II a linear scheme for the diffusive model is presented and optimal error estimates are proved under mild stability conditions, $\Delta t = O(1/\sqrt{1+|\log h|}\,)$ and $\Delta t = O(\sqrt{h}\,)$, in two and three dimensions, respectively. Moreover, the existence and uniqueness of its numerical solution is shown as well. The theoretical convergence orders are again confirmed by numerical experiments.

Let us summarize that in both papers, Part I (nonlinear scheme) and Part II (linear scheme), we present the results for optimal error estimates (i) for the non-diffusive case ($\varepsilon = 0$) in two space dimensions and (ii) for the diffusive case ($\varepsilon > 0$) in three space dimensions, respectively.

As mentioned in Boyaval *et al.* [5], the positive definiteness of the conformation tensor is important in the analysis of numerical schemes for the Oldroyd-B model and has been overcome by using, *e.g.*, the log-conformation representation in Fattal and Kupferman [12, 13]. While some schemes preserving the positive definiteness have been developed, there are, as far as we know, no convergence results of such schemes. In our papers, Part I and Part II, we have obtained the convergence results without any assumption on the positive definiteness. This is an additional feature of our proof.

The paper is organized as follows. In Section 2 the mathematical formulation of the Oseen-type Peterlin viscoelastic model is described. In Section 3 a nonlinear stabilized Lagrange–Galerkin scheme is presented. The main result on the convergence with optimal error estimates is stated in Section 4, and proved in Section 5. In Section 6 uniqueness of the numerical solution is shown. Theoretical order of convergence is confirmed by numerical experiments in Section 7.

2. The Oseen-type Peterlin viscoelastic model

The function spaces and the notation to be used throughout the paper are as follows. Let Ω be a bounded domain in $\mathbb{R}^2$, $\Gamma \coloneqq \partial\Omega$ the boundary of Ω, and T a positive constant. For $m \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$ we use the Sobolev spaces $W^{m,p}(\Omega)$, $W^{1,\infty}_0(\Omega)$, $H^m(\Omega)\,(= W^{m,2}(\Omega))$, $H^1_0(\Omega)$ and $L^2_0(\Omega) \coloneqq \{q \in L^2(\Omega);\, \int_\Omega q \,\mathrm{d}x = 0\}$. Furthermore, we employ function spaces $H^m_{sym}(\Omega) \coloneqq \{\mathbf{D} \in H^m(\Omega)^{2\times 2};\ \mathbf{D} = \mathbf{D}^T\}$ and $C^m_{sym}(\bar\Omega) \coloneqq C^m(\bar\Omega)^{2\times 2} \cap H^m_{sym}(\Omega)$, where the superscript T stands for the transposition. For any normed space S with norm $\|\cdot\|_S$, we define function spaces $H^m(0, T; S)$ and $C([0, T]; S)$ consisting of S-valued functions in $H^m(0, T)$ and $C([0, T])$, respectively. We use the same notation $(\cdot, \cdot)$ to represent the $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions. The dual pairing between S and the dual space S' is denoted by $\langle\cdot, \cdot\rangle$. The norms on $W^{m,p}(\Omega)$ and $H^m(\Omega)$ and their seminorms are simply denoted by $\|\cdot\|_{m,p}$ and $\|\cdot\|_m\,(= \|\cdot\|_{m,2})$ and by $|\cdot|_{m,p}$ and $|\cdot|_m\,(= |\cdot|_{m,2})$, respectively. The notations $\|\cdot\|_{m,p}$, $|\cdot|_{m,p}$, $\|\cdot\|_m$ and $|\cdot|_m$ are employed not only for scalar-valued functions but also for vector- and matrix-valued ones. We also denote the norm on $H^{-1}(\Omega)^2$ by $\|\cdot\|_{-1}$. For t_0 and $t_1 \in \mathbb{R}$ we introduce the function space,

$$Z^m(t_0, t_1) \coloneqq \left\{\psi \in H^j(t_0, t_1; H^{m-j}(\Omega));\ j = 0, \ldots, m,\ \|\psi\|_{Z^m(t_0,t_1)} < \infty\right\}$$

with the norm

$$\|\psi\|_{Z^m(t_0,t_1)} \coloneqq \left\{\sum_{j=0}^{m} \|\psi\|^2_{H^j(t_0,t_1;H^{m-j}(\Omega))}\right\}^{1/2},$$

and set $Z^m \coloneqq Z^m(0, T)$. We often omit $[0, T]$, Ω, and the superscripts 2 and 2×2 for the vector and the matrix if there is no confusion, *e.g.*, we shall write $C(L^\infty)$ in place of $C([0, T]; L^\infty(\Omega)^{2\times 2})$. For square matrices $\mathbf{A}$ and $\mathbf{B} \in \mathbb{R}^{2\times 2}$ we use the notation $\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}$.

We consider the system of equations describing the unsteady motion of an incompressible viscoelastic fluid,

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \operatorname{div}\big(2\nu \mathrm{D}(\mathbf{u})\big) + \nabla p = \operatorname{div}[(\operatorname{tr}\mathbf{C})\mathbf{C}] + \mathbf{f} \qquad \text{in} \quad \Omega \times (0, T), \tag{2.1a}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in} \quad \Omega \times (0, T), \tag{2.1b}$$

$$\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t} - \varepsilon\Delta\mathbf{C} = (\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u})^T - (\operatorname{tr}\mathbf{C})^2\,\mathbf{C} + (\operatorname{tr}\mathbf{C})\mathbf{I} + \mathbf{F} \qquad \text{in} \quad \Omega \times (0, T), \tag{2.1c}$$

$$\mathbf{u} = \mathbf{0}, \quad \varepsilon\frac{\partial\mathbf{C}}{\partial\mathbf{n}} = \mathbf{0}, \qquad \text{on} \quad \Gamma \times (0, T), \tag{2.1d}$$

$$\mathbf{u} = \mathbf{u}^0, \quad \mathbf{C} = \mathbf{C}^0, \qquad \text{in} \quad \Omega, \text{ at} \quad t = 0, \tag{2.1e}$$

where $(\mathbf{u}, p, \mathbf{C}) : \Omega \times (0, T) \to \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{2\times 2}_{sym}$ are the unknown velocity, pressure and conformation tensor, $\nu > 0$ is a fluid viscosity, $\varepsilon \in [0, 1]$ is an elastic stress viscosity, $(\mathbf{f}, \mathbf{F}) : \Omega \times (0, T) \to \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ is a pair of given external forces, $\nabla\mathbf{u}$ is the (matrix-valued) velocity gradient defined by $(\nabla\mathbf{u})_{ij} \coloneqq \partial u_i/\partial x_j$ $(i, j = 1, 2)$, $\mathrm{D}(\mathbf{u}) \coloneqq (1/2)[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$ is the symmetric part of the velocity gradient, $\mathbf{I}$ is the identity matrix, $\mathbf{n} : \Gamma \to \mathbb{R}^2$ is the outward unit normal, $(\mathbf{u}^0, \mathbf{C}^0) : \Omega \to \mathbb{R}^2 \times \mathbb{R}^{2\times 2}_{sym}$ is a pair of given initial functions, and $\mathrm{D}/\mathrm{D}t$ is the material derivative defined by

$$\frac{\mathrm{D}}{\mathrm{D}t} \coloneqq \frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla,$$

where $\mathbf{w} : \Omega \times (0, T) \to \mathbb{R}^2$ is a given velocity.

Remark 2.1.

(i) In this paper we pay attention to the dependency on ε to include the degenerate case $\varepsilon = 0$. The upper bound 1 of ε is not essential but replaced by any positive constant ε_0, *i.e.*, $\varepsilon \in [0, \varepsilon_0]$. The upper bound is needed in choosing the constants h_0, Δt_0 and $c_\dagger$ independent of ε in Theorem 4.5 below, where it is used for the estimate (5.8g) in Lemma 5.10.

(ii) When $\varepsilon > 0$, under regularity condition on $\mathbf{w}$ the global existence of a weak solution of (2.2) below can be proved in a similar way to the fully nonlinear case [19].

(iii) When $\varepsilon = 0$, there is neither the diffusion term in (2.1c) nor the boundary condition on $\mathbf{C}$ in (2.1d). Because of the loss of the ellipticity, $\mathbf{C}(t)$ does not belong to $H^1(\Omega)^{2\times 2}$ in general. If there exists a solution satisfying Hypothesis 4.4 below, then we can show the convergence of the finite element solution to the exact one in Theorem 4.5.

We formulate an assumption for the given velocity $\mathbf{w}$.

Hypothesis 2.2. *The function* $\mathbf{w}$ *satisfies* $\mathbf{w} \in C([0,T]; W_0^{1,\infty}(\Omega)^2)$.

Let $V := H_0^1(\Omega)^2$, $Q := L_0^2(\Omega)$ and $W := H^1_{sym}(\Omega)$. We define the bilinear forms a_u on $V \times V$, b on $V \times Q$, $\mathcal{A}$ on $(V \times Q) \times (V \times Q)$ and a_c on $W \times W$ by

$$a_u(\mathbf{u}, \mathbf{v}) := 2\big(\mathrm{D}(\mathbf{u}), \mathrm{D}(\mathbf{v})\big), \qquad b(\mathbf{u}, q) := -(\operatorname{div}\mathbf{u}, q), \qquad \mathcal{A}\big((\mathbf{u},p),(\mathbf{v},q)\big) := \nu a_u(\mathbf{u},\mathbf{v}) + b(\mathbf{u},q) + b(\mathbf{v},p),$$
$$a_c(\mathbf{C}, \mathbf{D}) := (\nabla \mathbf{C}, \nabla \mathbf{D}),$$

respectively. We present the weak formulation of the problem (2.1); find $(\mathbf{u}, p, \mathbf{C}) : (0,T) \to V \times Q \times W$ such that for $t \in (0,T)$

$$\left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}(t), \mathbf{v}\right) + \mathcal{A}\big((\mathbf{u},p)(t),(\mathbf{v},q)\big) = -\left(\operatorname{tr}\mathbf{C}(t)\,\mathbf{C}(t), \nabla\mathbf{v}\right) + \left(\mathbf{f}(t), \mathbf{v}\right), \tag{2.2a}$$

$$\left(\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t}(t), \mathbf{D}\right) + \varepsilon a_c\big(\mathbf{C}(t), \mathbf{D}\big) = 2\big((\nabla\mathbf{u}(t))\mathbf{C}(t), \mathbf{D}\big) - \big((\operatorname{tr}\mathbf{C}(t))^2\mathbf{C}(t), \mathbf{D}\big) + (\operatorname{tr}\mathbf{C}(t)\mathbf{I}, \mathbf{D}) + (\mathbf{F}(t), \mathbf{D}), \tag{2.2b}$$

$$\forall(\mathbf{v}, q, \mathbf{D}) \in V \times Q \times W,$$

with $(\mathbf{u}(0), \mathbf{C}(0)) = (\mathbf{u}^0, \mathbf{C}^0)$.

3. A nonlinear stabilized Lagrange–Galerkin scheme

The aim of this section is to present a nonlinear stabilized Lagrange–Galerkin scheme for (2.1).

Let Δt be a time increment, $N_T := \lfloor T/\Delta t \rfloor$ the total number of time steps and $t^n := n\Delta t$ for $n = 0, \ldots, N_T$. Let $\mathbf{g}$ be a function defined in $\Omega \times (0,T)$ and $\mathbf{g}^n := \mathbf{g}(\cdot, t^n)$. For the approximation of the material derivative we employ the first-order characteristics method,

$$\frac{\mathrm{D}\mathbf{g}}{\mathrm{D}t}(x, t^n) = \frac{\mathbf{g}^n(x) - \big(\mathbf{g}^{n-1} \circ X_1^n\big)(x)}{\Delta t} + O(\Delta t), \tag{3.1}$$

where $X_1^n : \Omega \to \mathbb{R}^2$ is a mapping defined by

$$X_1^n(x) := x - \mathbf{w}^n(x)\Delta t,$$

and the symbol $\circ$ means the composition of functions,

$$(\mathbf{g}^{n-1} \circ X_1^n)(x) := \mathbf{g}^{n-1}(X_1^n(x)).$$

For the details on deriving the approximation (3.1) of $\mathrm{D}\mathbf{g}/\mathrm{D}t$, see, *e.g.*, [27]. The point $X_1^n(x)$ is called the upwind point of x with respect to $\mathbf{w}^n$. The next proposition, which is a direct consequence of [35,37], presents sufficient conditions to ensure that all upwind points defined by X_1^n are in Ω and that its Jacobian $J^n := \det(\partial X_1^n/\partial x)$ is around 1.

Proposition 3.1. *Suppose Hypothesis 2.2 holds. Then, we have the following for $n \in \{0, \ldots, N_T\}$.*

(i) *Under the condition $\Delta t|\mathbf{w}|_{C(W^{1,\infty})} < 1$, $X_1^n : \Omega \to \Omega$ is bijective.*
(ii) *Furthermore, under the condition*

$$\Delta t|\mathbf{w}|_{C(W^{1,\infty})} \le 1/4, \tag{3.2}$$

the estimate $1/2 \le J^n \le 3/2$ holds.

For the sake of simplicity we suppose that Ω is a polygonal domain. Let $\mathcal{T}_h = \{K\}$ be a triangulation of $\bar{\Omega}$ $(= \bigcup_{K\in\mathcal{T}_h} K)$, h_K the diameter of $K \in \mathcal{T}_h$ and $h \coloneqq \max_{K\in\mathcal{T}_h} h_K$ the maximum element size. We consider a regular family of subdivisions $\{\mathcal{T}_h\}_{h\downarrow 0}$ satisfying the inverse assumption [9], *i.e.*, there exists a positive constant α_0 independent of h such that

$$\frac{h}{h_K} \le \alpha_0, \quad \forall K \in \mathcal{T}_h,\ \forall h.$$

We define the discrete function spaces X_h, V_h, M_h, Q_h and W_h by

$$\begin{aligned}
X_h &\coloneqq \left\{\mathbf{v}_h \in C(\bar{\Omega})^2;\ \mathbf{v}_{h|K} \in P_1(K)^2, \forall K \in \mathcal{T}_h\right\}, & V_h &\coloneqq X_h \cap V,\\
M_h &\coloneqq \left\{q_h \in C(\bar{\Omega});\ q_{h|K} \in P_1(K), \forall K \in \mathcal{T}_h\right\}, & Q_h &\coloneqq M_h \cap Q,\\
W_h &\coloneqq \left\{\mathbf{D}_h \in C_{sym}(\bar{\Omega});\ \mathbf{D}_{h|K} \in P_1(K)^{2\times 2}, \forall K \in \mathcal{T}_h\right\},
\end{aligned}$$

respectively, where $P_1(K)$ is the polynomial space of linear functions on $K \in \mathcal{T}_h$.

Let δ_0 be a small positive constant fixed arbitrarily and $(\cdot,\cdot)_K$ the $L^2(K)^2$ inner product. We define the bilinear forms $\mathcal{A}_h$ on $(V \times H^1(\Omega)) \times (V \times H^1(\Omega))$ and $\mathcal{S}_h$ on $H^1(\Omega) \times H^1(\Omega)$ by

$$\mathcal{A}_h\left((\mathbf{u}, p), (\mathbf{v}, q)\right) \coloneqq \nu a_u\left(\mathbf{u}, \mathbf{v}\right) + b(\mathbf{u}, q) + b(\mathbf{v}, p) - \mathcal{S}_h(p, q), \qquad \mathcal{S}_h(p, q) \coloneqq \delta_0 \sum_{K\in\mathcal{T}_h} h_K^2 (\nabla p, \nabla q)_K.$$

For $\mathbf{D} \in \mathbb{R}^{2\times 2}_{sym}$ let $\mathbf{D}^\# \in \mathbb{R}^{2\times 2}_{sym}$ be the adjugate matrix of $\mathbf{D}$ defined by

$$\mathbf{D}^\# \coloneqq \begin{pmatrix} D_{22} & -D_{12} \\ -D_{12} & D_{11} \end{pmatrix}.$$

Let $(\mathbf{f}_h, \mathbf{F}_h) \coloneqq (\{\mathbf{f}_h^n\}_{n=1}^{N_T}, \{\mathbf{F}_h^n\}_{n=1}^{N_T}) \subset L^2(\Omega)^2 \times L^2(\Omega)^{2\times 2}$ and $(\mathbf{u}_h^0, \mathbf{C}_h^0) \in V_h \times W_h$ be given. A nonlinear stabilized Lagrange–Galerkin scheme for (2.1) is to find $(\mathbf{u}_h, p_h, \mathbf{C}_h) \coloneqq \{(\mathbf{u}_h^n, p_h^n, \mathbf{C}_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h \times W_h$ such that, for $n = 1, \ldots, N_T$,

$$\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\big((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)\big) = -\big((\operatorname{tr} \mathbf{C}_h^n)\mathbf{C}_h^n, \nabla \mathbf{v}_h\big) + (\mathbf{f}_h^n, \mathbf{v}_h), \tag{3.3a}$$

$$\begin{aligned}
\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \varepsilon a_c\left(\mathbf{C}_h^n, \mathbf{D}_h\right) &= 2\big((\nabla \mathbf{u}_h^n)\mathbf{C}_h^n, \mathbf{D}_h\big) + \big(\operatorname{div} \mathbf{u}_h^n (\mathbf{C}_h^n)^\#, \mathbf{D}_h\big) - \big((\operatorname{tr} \mathbf{C}_h^n)^2 \mathbf{C}_h^n, \mathbf{D}_h\big)\\
&\quad + \big((\operatorname{tr} \mathbf{C}_h^n)\mathbf{I}, \mathbf{D}_h\big) + (\mathbf{F}_h^n, \mathbf{D}_h), \qquad (3.3b)\\
&\qquad \forall (\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h.
\end{aligned}$$

In Remark 5.6 below we show that an additional term, the second term on the right-hand side of (3.3b), is added in order to derive a desired energy inequality.

4. The main result

In this section we present the main result on error estimates with the optimal convergence order of scheme (3.3).

We use c to represent a generic positive constant independent of the discretization parameters h and Δt. We also use constants c_w and c_s independent of h and Δt but dependent on $\mathbf{w}$ and the solution $(\mathbf{u}, p, \mathbf{C})$ of (2.2), respectively, and c_s often depends on $\mathbf{w}$ additionally. c, c_w and c_s may be dependent on ν but are independent of ε. The symbol "$\prime$ (prime)" is sometimes used in order to distinguish two constants, *e.g.*, c_s and c_s', from each other. We use the following notation for the norms and seminorms, $\|\cdot\|_V = \|\cdot\|_{V_h} \coloneqq \|\cdot\|_1$, $\|\cdot\|_Q = \|\cdot\|_{Q_h} \coloneqq \|\cdot\|_0$,

$$\|(\mathbf{u}, \mathbf{C})\|_{Z^2(t_0,t_1)} \coloneqq \left\{ \|\mathbf{u}\|^2_{Z^2(t_0,t_1)} + \|\mathbf{C}\|^2_{Z^2(t_0,t_1)} \right\}^{1/2}, \qquad \|\mathbf{u}\|_{\ell^\infty(X)} \coloneqq \max_{n=0,\ldots,N_T} \|\mathbf{u}^n\|_X,$$

$$\|\mathbf{u}\|_{\ell^2(X)} \coloneqq \left\{ \Delta t \sum_{n=1}^{N_T} \|\mathbf{u}^n\|_X^2 \right\}^{1/2}, \qquad |\mathbf{u}|_{\ell^2(X)} \coloneqq \left\{ \Delta t \sum_{n=1}^{N_T} |\mathbf{u}^n|_X^2 \right\}^{1/2},$$

$$|p|_h \coloneqq \left\{ \sum_{K \in \mathcal{T}_h} h_K^2 (\nabla p, \nabla p)_K \right\}^{1/2}, \qquad |p|_{\ell^2(|\cdot|_h)} \coloneqq \left\{ \Delta t \sum_{n=1}^{N_T} |p^n|_h^2 \right\}^{1/2},$$

for $X = L^2(\Omega)$ or $H^1(\Omega)$. $\overline{D}_{\Delta t}$ is the backward difference operator defined by $\overline{D}_{\Delta t} u^n \coloneqq (u^n - u^{n-1})/\Delta t$.

The existence of the solution of scheme (3.3) is guaranteed by the next proposition whose proof is given in the next section.

Proposition 4.1 (existence)**.** *Suppose Hypothesis 2.2 holds. Then for any $h > 0$ and $\Delta t \in (0, 1/2)$ satisfying (3.2), there exists a solution $(\mathbf{u}_h, p_h, \mathbf{C}_h) \subset V_h \times Q_h \times W_h$ of scheme (3.3).*

We state the main result after preparing a projection and a hypothesis.

Definition 4.2 (Stokes projection)**.** For $(\mathbf{u}, p) \in V \times Q$ we define the Stokes projection $(\hat{\mathbf{u}}_h, \hat{p}_h) \in V_h \times Q_h$ of $(\mathbf{u}, p)$ by

$$\mathcal{A}_h \left((\hat{\mathbf{u}}_h, \hat{p}_h), (\mathbf{v}_h, q_h) \right) = \mathcal{A} \left((\mathbf{u}, p), (\mathbf{v}_h, q_h) \right), \quad \forall (\mathbf{v}_h, q_h) \in V_h \times Q_h. \tag{4.1}$$

The Stokes projection derives an operator $\Pi_h^{\mathrm{S}} : V \times Q \to V_h \times Q_h$ defined by $\Pi_h^{\mathrm{S}}(\mathbf{u}, p) := (\hat{\mathbf{u}}_h, \hat{p}_h)$. The first component $\hat{\mathbf{u}}_h$ of $\Pi_h^{\mathrm{S}}(\mathbf{u}, p)$ is denoted by $[\Pi_h^{\mathrm{S}}(\mathbf{u}, p)]_1$. Let $\Pi_h : L^2(\Omega) \to M_h$ be the Clément interpolation operator [10]. The Clément operators on $L^2(\Omega)^2$ and $L^2(\Omega)^{2\times 2}$ are denoted by the same symbol Π_h.

Remark 4.3. The Clément operator is defined for functions from $L^2(\Omega)$. When a function belongs to $C(\bar{\Omega})$, we can replace the Clément operator by the Lagrange operator $\Pi_h^L : C(\bar{\Omega}) \to M_h$.

Hypothesis 4.4. *The solution $(\mathbf{u}, p, \mathbf{C})$ of (2.2) satisfies $\mathbf{u} \in Z^2(0,T)^2 \cap H^1(0,T; V \cap H^2(\Omega)^2) \cap C([0,T]; W^{1,\infty}(\Omega)^2)$, $p \in H^1(0,T; Q \cap H^1(\Omega))$ and*

$$\mathbf{C} \in \begin{cases} Z^2(0,T)^{2\times 2} \cap L^2(0,T; W) \cap C([0,T]; H^2(\Omega)^{2\times 2}) & (\varepsilon > 0), \\ Z^2(0,T)^{2\times 2} \cap L^2(0,T; W) \cap C([0,T]; L^\infty(\Omega)^{2\times 2}) & (\varepsilon = 0). \end{cases}$$

We now impose the conditions

$$(\mathbf{u}_h^0, \mathbf{C}_h^0) = ([\Pi_h^{\mathrm{S}}(\mathbf{u}^0, 0)]_1, \Pi_h \mathbf{C}^0), \quad (\mathbf{f}_h, \mathbf{F}_h) = (\mathbf{f}, \mathbf{F}). \tag{4.2}$$

Theorem 4.5 (error estimates)**.** *Suppose Hypotheses 2.2 and 4.4 hold. Then, there exist positive constants h_0, Δt_0 and $c_\dagger$ independent of ε such that, for any pair $(h, \Delta t)$ satisfying*

$$h \in (0, h_0], \quad \Delta t \in (0, \Delta t_0], \tag{4.3}$$

and any solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ of scheme (3.3) with (4.2), it holds that

$$\begin{aligned}&\|\mathbf{u}_h - \mathbf{u}\|_{\ell^\infty(L^2)},\ \sqrt{\nu}\|\mathbf{u}_h - \mathbf{u}\|_{\ell^2(H^1)},\ |p_h - p|_{\ell^2(|.|_h)},\\ &\|\mathbf{C}_h - \mathbf{C}\|_{\ell^\infty(L^2)},\ \sqrt{\varepsilon}|\mathbf{C}_h - \mathbf{C}|_{\ell^2(H^1)},\ \big\|\mathrm{tr}\,(\mathbf{C}_h - \mathbf{C})(\mathbf{C}_h - \mathbf{C})\big\|_{\ell^2(L^2)} \le c_\dagger(h + \Delta t).\end{aligned} \tag{4.4}$$

Remark 4.6.

(i) The estimates (4.4) hold even for $\varepsilon = 0$. Then, of course, the fifth term of the left-hand side of (4.4) vanishes.
(ii) Here we do not need uniqueness of the solution of scheme (3.3). Uniqueness of the numerical solution will be discussed later in Proposition 6.1.
(iii) The positive definiteness of the exact and numerical solutions is not required for the above error estimates.

5. Proofs

In what follows we prove Proposition 4.1 and Theorem 4.5.

5.1. Preliminaries

Let us list lemmas directly employed below in the proofs. In the lemmas, α_i, $i = 1, \ldots, 4$, are numerical constants. They are independent of h, Δt, ν and ε but may depend on Ω.

Lemma 5.1 [25]**.** *Let Ω be a bounded domain with a* Lipschitz*-continuous boundary. Then, the following inequalities hold.*

$$\|\mathrm{D}(\mathbf{v})\|_0 \le \|\mathbf{v}\|_1 \le \alpha_1 \|\mathrm{D}(\mathbf{v})\|_0, \qquad \forall \mathbf{v} \in H^1_0(\Omega)^2.$$

We introduce the function

$$D(h) := (1 + |\log h|)^{1/2}, \tag{5.1}$$

which is used in the sequel.

Lemma 5.2 [6, 9, 10]**.** *The following inequalities hold.*

$$\begin{aligned}
\|\Pi_h \mathbf{g}\|_{0,\infty} &\le \|\mathbf{g}\|_{0,\infty}, && \forall \mathbf{g} \in L^\infty(\Omega)^s,\\
\|\Pi_h \mathbf{g}\|_{1,\infty} &\le \alpha_{20} \|\mathbf{g}\|_{1,\infty}, && \forall \mathbf{g} \in W^{1,\infty}(\Omega)^s,\\
\|\Pi_h \mathbf{g} - \mathbf{g}\|_0 &\le \alpha_{21} h \|\mathbf{g}\|_1, && \forall \mathbf{g} \in H^1(\Omega)^s \cap L^\infty(\Omega)^s,\\
\|\Pi_h \mathbf{g} - \mathbf{g}\|_1 &\le \alpha_{22} h \|\mathbf{g}\|_2, && \forall \mathbf{g} \in H^2(\Omega)^s,\\
\|\mathbf{g}_h\|_{0,\infty} &\le \alpha_{23} h^{-1} \|\mathbf{g}_h\|_0, && \forall \mathbf{g}_h \in S_h,\\
\|\mathbf{g}_h\|_{0,\infty} &\le \alpha_{24} D(h) \|\mathbf{g}_h\|_1, && \forall \mathbf{g}_h \in S_h,\\
\|\mathbf{g}_h\|_{1,\infty} &\le \alpha_{25} h^{-1} \|\mathbf{g}_h\|_1, && \forall \mathbf{g}_h \in S_h,\\
\|\mathbf{g}_h\|_1 &\le \alpha_{26} h^{-1} \|\mathbf{g}_h\|_0, && \forall \mathbf{g}_h \in S_h,
\end{aligned} \tag{5.2}$$

where $s = 2$ or 2×2 and $S_h = V_h$ or W_h.

Lemma 5.3 [7]**.** *Assume* $(\mathbf{u}, p) \in (V \cap H^2(\Omega)^2) \times (Q \cap H^1(\Omega))$*. Let* $(\hat{\mathbf{u}}_h, \hat{p}_h) \in V_h \times Q_h$ *be the* Stokes *projection of* $(\mathbf{u}, p)$ *by* (4.1)*. Then, the following inequalities hold,*

$$\|\hat{\mathbf{u}}_h - \mathbf{u}\|_1, \quad \|\hat{p}_h - p\|_0, \quad |\hat{p}_h - p|_h \le \alpha_3 h \|(\mathbf{u}, p)\|_{H^2 \times H^1}.$$

Lemma 5.4 [35]**.** *Under Hypothesis* 2.2 *and the condition* (3.2) *the following inequality holds for any* $n \in \{0, \ldots, N_T\}$

$$\|\mathbf{g} \circ X_1^n\|_0 \le (1 + \alpha_4 |\mathbf{w}^n|_{1,\infty} \Delta t) \|\mathbf{g}\|_0, \qquad \forall \mathbf{g} \in L^2(\Omega)^s,$$

where $s = 2$ *or* 2×2*.*

We present a key lemma in order to deal with the nonlinear terms.

Lemma 5.5. *For* $\mathbf{E} \in \mathbb{R}^{2\times 2}$ *and* $\mathbf{D} \in \mathbb{R}^{2\times 2}_{sym}$ *it holds that*

$$(\mathrm{tr}\,\mathbf{D})\mathbf{D} : \mathbf{E} - \mathbf{E}\mathbf{D} : \mathbf{D} - \frac{1}{2}(\mathrm{tr}\,\mathbf{E})\mathbf{D}^{\#} : \mathbf{D} = 0. \tag{5.3}$$

Proof. The direct calculation yields the result, see also Remark 5.6. □

Remark 5.6. Let $(\mathbf{u}, p, \mathbf{C})$ be a solution of (2.1). Multiplying (2.1a) and (2.1c) by $\mathbf{u}$ and $\mathbf{C}/2$, respectively, and adding them, we can obtain an energy inequality on $(\mathbf{u}, \mathbf{C})$ since the term derived from the nonlinear terms of (2.1a) and (2.1c) vanishes,

$$(\mathrm{div}\,[(\mathrm{tr}\,\mathbf{C})\mathbf{C}], \mathbf{u}) + \frac{1}{2}((\nabla\mathbf{u})\mathbf{C} + \mathbf{C}(\nabla\mathbf{u}), \mathbf{C}) = 0. \tag{5.4}$$

Identity (5.4) is proved as follows. The left-hand side is equal to

$$\begin{aligned}
&-((\mathrm{tr}\,\mathbf{C})\mathbf{C}, \nabla\mathbf{u}) + ((\nabla\mathbf{u})\mathbf{C}, \mathbf{C}) = (\nabla\mathbf{u}, \mathbf{C}\mathbf{C}^T - (\mathrm{tr}\,\mathbf{C})\mathbf{C}) \\
&= \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \sum_{k=1}^{2} (C_{ik}C_{jk} - C_{kk}C_{ij})\, \mathrm{d}x \\
&= \int_\Omega \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)(C_{12}C_{12} - C_{11}C_{22})\, \mathrm{d}x = -\frac{1}{2}\big((\mathrm{div}\,\mathbf{u})\mathbf{C}^{\#}, \mathbf{C}\big)
\end{aligned} \tag{5.5}$$

Since $\mathrm{div}\, u = 0$, (5.5) implies (5.4). In the approximate solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ the exact incompressibility $\mathrm{div}\,\mathbf{u}_h = 0$ does not hold. Hence, (5.4) is not true, in general, for $(\mathbf{u}_h, \mathbf{C}_h)$. On the other hand, (5.5) is always valid regardless of the property of u. Therefore, by adding the second term of the right-hand side in (5b), $(\mathrm{div}\,\mathbf{u}_h^n(\mathbf{C}_h^n)^{\#}, \mathbf{D}_h)$, we can obtain the corresponding equation to (5.4) for $(\mathbf{u}_h^n, \mathbf{C}_h^n)$,

$$-((\mathrm{tr}\,\mathbf{C}_h^n)\mathbf{C}_h^n, \nabla\mathbf{u}_h^n) + ((\nabla\mathbf{u}_h^n)\mathbf{C}_h^n, \mathbf{C}_h^n) + \frac{1}{2}(\mathrm{div}\,\mathbf{u}_h^n(\mathbf{C}_h^n)^{\#}, \mathbf{C}_h^n) = 0,$$

which plays a key role in the following stability analysis. Identity (5.3) is proved similarly to (5.5) by replacing $\mathbf{C}$ and $\nabla\mathbf{u}$ by $\mathbf{D}$ and $\mathbf{E}$, respectively.

Remark 5.7.

(i) Lemma 5.5 does not hold in three-dimensional case. This is the reason why we consider two-dimensional case in this paper.

(ii) By virtue of the term $(\operatorname{div} \mathbf{u}_h^n (\mathbf{C}_h^n)^\#, \mathbf{D}_h)$ in scheme (3.3), we can prove the error estimates for $\varepsilon = 0$, which is an advantage of the nonlinear scheme. In Part II, we propose a linear scheme for the model (2.1) and prove error estimates for $\varepsilon > 0$, where the presence of $\Delta\mathbf{C}$ in (2.1c) is essentially employed. It is, therefore, not easy to show error estimates of the linear scheme in a similar way for $\varepsilon = 0$. On the other hand, the linear scheme has an advantage that the proof of the error estimates can be extended to three-dimensional problems.

Lemma 5.8 [36]**.** *Let a_i, $i = 1, 2$, be non-negative numbers, Δt a positive number, and $\{x^n\}_{n\geq 0}$, $\{y^n\}_{n\geq 1}$ and $\{b^n\}_{n\geq 1}$ non-negative sequences. Assume $\Delta t \in (0, 1/(2a_0)]$ for $a_0 \neq 0$. Suppose*

$$\overline{D}_{\Delta t} x^n + y^n \leq a_0 x^n + a_1 x^{n-1} + b^n, \quad \forall n \geq 1.$$

Then, it holds that

$$x^n + \Delta t \sum_{i=1}^{n} y^i \leq \exp[(2a_0 + a_1) n \Delta t] \left(x^0 + \Delta t \sum_{i=1}^{n} b^i \right), \quad \forall n \geq 1.$$

Lemma 5.9 ([38], Chapt. II, Lem. 1.4, [17], Chapt. I, Lem. 4.3)**.** *Let X be a finite dimensional* Hilbert *space with inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$ and let $\mathcal{P}$ be a continuous mapping from X into itself such that $(\mathcal{P}(\xi), \xi)_X > 0$ for $\|\xi\|_X = \rho_0 > 0$. Then, there exists $\xi \in X$, $\|\xi\|_X \leq \rho_0$, such that $\mathcal{P}(\xi) = 0$.*

5.2. Proof of Proposition 4.1

We apply Lemma 5.9 for the proof. Let $n \in \{1, \ldots, N_T\}$ be a fixed number and $(\mathbf{u}_h^{n-1}, \mathbf{C}_h^{n-1}) \in V_h \times W_h$ a pair of given functions. We set $\mu_0 := (1 - 2\Delta t)/2 > 0$. We define a finite dimensional inner product space $X := V_h \times Q_h \times W_h$ equipped with the inner product,

$$\begin{aligned}\big((\mathbf{u}_h, p_h, \mathbf{C}_h), (\mathbf{v}_h, q_h, \mathbf{D}_h)\big)_X &:= \frac{1}{\Delta t}(\mathbf{u}_h, \mathbf{v}_h) + 4\nu\big(\mathrm{D}(\mathbf{u}_h), \mathrm{D}(\mathbf{v}_h)\big) \\ &\quad + 2\delta_0 \sum_{K\in\mathcal{T}_h} h_K^2 (p_h, q_h)_K + \frac{\mu_0}{\Delta t}(\mathbf{C}_h, \mathbf{D}_h) + \varepsilon(\nabla \mathbf{C}_h, \nabla \mathbf{D}_h),\end{aligned}$$

which induces the norm $\|\cdot\|_X$ for any $\varepsilon \geq 0$. Let $\mathcal{P} : V_h \times Q_h \times W_h \to V_h \times Q_h \times W_h$ be a mapping defined by

$$\begin{aligned}\big(\mathcal{P}(\mathbf{u}_h, p_h, \mathbf{C}_h), (\mathbf{v}_h, q_h, \mathbf{D}_h)\big)_X &= \left(\frac{\mathbf{u}_h - \mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\big((\mathbf{u}_h, p_h), (\mathbf{v}_h, -q_h)\big) + \big((\operatorname{tr}\mathbf{C}_h)\mathbf{C}_h, \nabla \mathbf{v}_h\big) \\ &\quad - (\mathbf{f}_h^n, \mathbf{v}_h) + \frac{1}{2}\left(\frac{\mathbf{C}_h - \mathbf{C}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \frac{\varepsilon}{2} a_c(\mathbf{C}_h, \mathbf{D}_h) - \big((\nabla \mathbf{u}_h)\mathbf{C}_h, \mathbf{D}_h\big) \\ &\quad - \frac{1}{2}\big((\operatorname{div}\mathbf{u}_h)\mathbf{C}_h^\#, \mathbf{D}_h\big) + \frac{1}{2}\big((\operatorname{tr}\mathbf{C}_h)^2\mathbf{C}_h, \mathbf{D}_h\big) - \frac{1}{2}\big((\operatorname{tr}\mathbf{C}_h)\mathbf{I}, \mathbf{D}_h\big) \\ &\quad - \frac{1}{2}(\mathbf{F}_h^n, \mathbf{D}_h), \qquad \forall (\mathbf{u}_h, p_h, \mathbf{C}_h), (\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h. \end{aligned} \tag{5.6}$$

Obviously $\mathcal{P}$ is continuous. Substituting $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ into $(\mathbf{v}_h, q_h, \mathbf{D}_h)$ in (5.6) and using the inequality $\|\operatorname{tr}\mathbf{C}_h\|_0 \leq \sqrt{2}\|\mathbf{C}_h\|_0$, we have

$$\begin{aligned}&\big(\mathcal{P}(\mathbf{u}_h, p_h, \mathbf{C}_h), (\mathbf{u}_h, p_h, \mathbf{C}_h)\big)_X \\ &= \left(\frac{\mathbf{u}_h - \mathbf{u}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{u}_h\right) + 2\nu\|\mathrm{D}(\mathbf{u}_h)\|_0^2 + \delta_0|p_h|_h^2 - (\mathbf{f}_h^n, \mathbf{u}_h) \\ &\quad + \frac{1}{2}\left(\frac{\mathbf{C}_h - \mathbf{C}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{C}_h\right) + \frac{\varepsilon}{2}|\mathbf{C}_h|_1^2 + \frac{1}{2}\|(\operatorname{tr}\mathbf{C}_h)\mathbf{C}_h\|_0^2 - \frac{1}{2}\|\operatorname{tr}\mathbf{C}_h\|_0^2 - \frac{1}{2}(\mathbf{F}_h^n, \mathbf{C}_h) \\ &\geq \frac{1}{\Delta t}\big(\|\mathbf{u}_h\|_0^2 - \|\mathbf{u}_h^{n-1}\circ X_1^n\|_0\|\mathbf{u}_h\|_0\big) + 2\nu\|\mathrm{D}(\mathbf{u}_h)\|_0^2 + \delta_0|p_h|_h^2 - \|\mathbf{f}_h^n\|_0\|\mathbf{u}_h\|_0\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2\Delta t}\big(\|\mathbf{C}_h\|_0^2 - \|\mathbf{C}_h^{n-1}\circ X_1^n\|_0\|\mathbf{C}_h\|_0\big) + \frac{\varepsilon}{2}|\mathbf{C}_h|_1^2 - \|\mathbf{C}_h\|_0^2 - \frac{1}{2}\|\mathbf{F}_h^n\|_0\|\mathbf{C}_h\|_0 \qquad \text{(by Schwarz' inequality)}\\
&\ge \frac{1}{2\Delta t}\Big\{2\|\mathbf{u}_h\|_0^2 - \beta_0\|\mathbf{u}_h\|_0^2 - \frac{1}{\beta_0}\|\mathbf{u}_h^{n-1}\circ X_1^n\|_0^2 + \|\mathbf{C}_h\|_0^2 - \beta_1\|\mathbf{C}_h\|_0^2 - \frac{1}{4\beta_1}\|\mathbf{C}_h^{n-1}\circ X_1^n\|_0^2\Big\}\\
&\quad + 2\nu\|\mathrm{D}(\mathbf{u}_h)\|_0^2 + \delta_0|p_h|_h^2 - \frac{\beta_2}{2\Delta t}\|\mathbf{u}_h\|_0^2 - \frac{\Delta t}{2\beta_2}\|\mathbf{f}_h^n\|_0^2 + \frac{\varepsilon}{2}|\mathbf{C}_h|_1^2 - \|\mathbf{C}_h\|_0^2 - \frac{\beta_3}{2\Delta t}\|\mathbf{C}_h\|_0^2 - \frac{\Delta t}{8\beta_3}\|\mathbf{F}_h^n\|_0^2\\
&\qquad\qquad \text{(by } ab \le \tfrac{\beta}{2}a^2 + \tfrac{1}{2\beta}b^2\text{)}\\
&\ge \frac{1}{2\Delta t}\Big\{(2-\beta_0-\beta_2)\|\mathbf{u}_h\|_0^2 + (1-\beta_1-2\Delta t-\beta_3)\|\mathbf{C}_h\|_0^2\Big\} + 2\nu\|\mathrm{D}(\mathbf{u}_h)\|_0^2 + \delta_0|p_h|_h^2\\
&\quad + \frac{\varepsilon}{2}|\mathbf{C}_h|_1^2 - \frac{1}{2\beta_0\Delta t}\|\mathbf{u}_h^{n-1}\circ X_1^n\|_0^2 - \frac{1}{8\beta_1\Delta t}\|\mathbf{C}_h^{n-1}\circ X_1^n\|_0^2 - \frac{\Delta t}{2\beta_2}\|\mathbf{f}_h^n\|_0^2 - \frac{\Delta t}{8\beta_3}\|\mathbf{F}_h^n\|_0^2 \qquad \text{(by Lem. 5.4)}
\end{aligned}
$$

for any $\beta_i > 0$. Choosing $\beta_0 = \beta_2 = 1/2$ and $\beta_1 = \beta_3 = \mu_0/2$, we get

$$
\begin{aligned}
\big(\mathcal{P}(\mathbf{u}_h,p_h,\mathbf{C}_h),(\mathbf{u}_h,p_h,\mathbf{C}_h)\big)_X &\ge \frac{1}{2}\Bigg[\Big\{\frac{1}{\Delta t}\|\mathbf{u}_h\|_0^2 + 4\nu\|\mathrm{D}(\mathbf{u}_h)\|_0^2 + 2\delta_0|p_h|_h^2 + \frac{\mu_0}{\Delta t}\|\mathbf{C}_h\|_0^2 + \varepsilon|\mathbf{C}_h|_1^2\Big\}\\
&\qquad - \Big\{\frac{2\|\mathbf{u}_h^{n-1}\circ X_1^n\|_0^2}{\Delta t} + \frac{\|\mathbf{C}_h^{n-1}\circ X_1^n\|_0^2}{2\mu_0\Delta t} + 2\Delta t\|\mathbf{f}_h^n\|_0^2 + \frac{\Delta t\|\mathbf{F}_h^n\|_0^2}{2\mu_0}\Big\}\Bigg]\\
&= \frac{1}{2}\Big[\|(\mathbf{u}_h,p_h,\mathbf{C}_h)\|_X^2 - \beta_*^2\Big],
\end{aligned}
$$

where

$$
\beta_* := \Big\{\frac{2\|\mathbf{u}_h^{n-1}\circ X_1^n\|_0^2}{\Delta t} + \frac{\|\mathbf{C}_h^{n-1}\circ X_1^n\|_0^2}{2\mu_0\Delta t} + 2\Delta t\|\mathbf{f}_h^n\|_0^2 + \frac{\Delta t\|\mathbf{F}_h^n\|_0^2}{2\mu_0}\Big\}^{1/2}.
$$

The right-hand side is, therefore, positive on the sphere of radius $\rho_0 = \beta_* + 1$. From Lemma 5.9 there exists an element $(\mathbf{u}_h, p_h, \mathbf{C}_h) \in V_h \times Q_h \times W_h$ such that $\mathcal{P}(\mathbf{u}_h, p_h, \mathbf{C}_h) = 0$, which is nothing but a solution of equations (3.3).

5.3. A system of equations for the error and the estimate of remainder terms

In this subsection we prepare a system of equations for the error and a lemma for the estimate of remainder terms in the system before starting the proof of Theorem 4.5.

Let $(\hat{\mathbf{u}}_h, \hat{p}_h)(t) := \Pi_h^{\mathrm{S}}(\mathbf{u}, p)(t) \in V_h \times Q_h$ and $\check{\mathbf{C}}_h(t) := \Pi_h \mathbf{C}(t) \in W_h$ for $t \in [0, T]$ and let

$$
\mathbf{e}_h^n := \mathbf{u}_h^n - \hat{\mathbf{u}}_h^n, \qquad \epsilon_h^n := p_h^n - \hat{p}_h^n, \qquad \mathbf{E}_h^n := \mathbf{C}_h^n - \check{\mathbf{C}}_h^n, \qquad \boldsymbol{\eta}(t) := (\mathbf{u} - \hat{\mathbf{u}}_h)(t), \qquad \boldsymbol{\Xi}(t) := (\mathbf{C} - \check{\mathbf{C}}_h)(t).
$$

Then, from (3.3), (4.1) and (2.2), we have for $n \ge 1$

$$
\Big(\frac{\mathbf{e}_h^n - \mathbf{e}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{v}_h\Big) + \mathcal{A}_h\big((\mathbf{e}_h^n,\epsilon_h^n),(\mathbf{v}_h,q_h)\big) = -\big((\operatorname{tr}\mathbf{E}_h^n)\mathbf{E}_h^n, \nabla\mathbf{v}_h\big) + {}_{V_h'}\langle \mathbf{r}_h^n, \mathbf{v}_h\rangle_{V_h}, \tag{5.7a}
$$

$$
\Big(\frac{\mathbf{E}_h^n - \mathbf{E}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{D}_h\Big) + \varepsilon a_c(\mathbf{E}_h^n, \mathbf{D}_h) = 2\big((\nabla\mathbf{e}_h^n)\mathbf{E}_h^n, \mathbf{D}_h\big) + \big((\operatorname{div}\mathbf{e}_h^n)(\mathbf{E}_h^n)^{\#}, \mathbf{D}_h\big) + {}_{W_h'}\langle \mathbf{R}_h^n, \mathbf{D}_h\rangle_{W_h}, \tag{5.7b}
$$

$$
\forall(\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h,
$$

where

$$\mathbf{r}_h^n := \sum_{i=1}^{4} \mathbf{r}_{hi}^n \in V_h', \qquad \mathbf{R}_h^n := \sum_{i=1}^{11} \mathbf{R}_{hi}^n \in W_h',$$

$$(\mathbf{r}_{h1}^n, \mathbf{v}_h) := \left(\frac{\mathrm{D}\mathbf{u}^n}{\mathrm{D}t} - \frac{\mathbf{u}^n - \mathbf{u}^{n-1} \circ X_1^n}{\Delta t}, \mathbf{v}_h \right),$$

$$(\mathbf{r}_{h2}^n, \mathbf{v}_h) := \frac{1}{\Delta t} \left(\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1} \circ X_1^n, \mathbf{v}_h \right),$$

$${}_{V_h'}\langle \mathbf{r}_{h3}^n, \mathbf{v}_h \rangle_{V_h} := -\big((\mathrm{tr}\, \check{\mathbf{C}}_h^n) \mathbf{E}_h^n + (\mathrm{tr}\, \mathbf{E}_h^n) \check{\mathbf{C}}_h^n, \nabla \mathbf{v}_h \big),$$

$${}_{V_h'}\langle \mathbf{r}_{h4}^n, \mathbf{v}_h \rangle_{V_h} := \big((\mathrm{tr}\, \check{\mathbf{C}}_h^n) \boldsymbol{\Xi}^n + (\mathrm{tr}\, \boldsymbol{\Xi}^n) \mathbf{C}^n, \nabla \mathbf{v}_h \big),$$

$$(\mathbf{R}_{h1}^n, \mathbf{D}_h) := \left(\frac{\mathrm{D}\mathbf{C}^n}{\mathrm{D}t} - \frac{\mathbf{C}^n - \mathbf{C}^{n-1} \circ X_1^n}{\Delta t}, \mathbf{D}_h \right),$$

$$(\mathbf{R}_{h2}^n, \mathbf{D}_h) := \frac{1}{\Delta t} \left(\boldsymbol{\Xi}^n - \boldsymbol{\Xi}^{n-1} \circ X_1^n, \mathbf{D}_h \right),$$

$${}_{W_h'}\langle \mathbf{R}_{h3}^n, \mathbf{D}_h \rangle_{W_h} := \varepsilon a_c(\boldsymbol{\Xi}^n, \mathbf{D}_h),$$

$$(\mathbf{R}_{h4}^n, \mathbf{D}_h) := 2\big((\nabla \hat{\mathbf{u}}_h^n) \mathbf{E}_h^n + (\nabla \mathbf{e}_h^n) \check{\mathbf{C}}_h^n, \mathbf{D}_h \big),$$

$$(\mathbf{R}_{h5}^n, \mathbf{D}_h) := -2\big((\nabla \hat{\mathbf{u}}_h^n) \boldsymbol{\Xi}^n + (\nabla \boldsymbol{\eta}^n) \mathbf{C}^n, \mathbf{D}_h \big),$$

$$(\mathbf{R}_{h6}^n, \mathbf{D}_h) := \big((\mathrm{div}\, \hat{\mathbf{u}}_h^n) (\mathbf{E}_h^n)^\# + (\mathrm{div}\, \mathbf{e}_h^n) (\check{\mathbf{C}}_h^n)^\#, \mathbf{D}_h \big),$$

$$(\mathbf{R}_{h7}^n, \mathbf{D}_h) := -\big((\mathrm{div}\, \hat{\mathbf{u}}_h^n) (\boldsymbol{\Xi}^n)^\# + (\mathrm{div}\, \boldsymbol{\eta}^n) (\mathbf{C}^n)^\#, \mathbf{D}_h \big),$$

$$(\mathbf{R}_{h8}^n, \mathbf{D}_h) := -\big([\mathrm{tr}\, (\mathbf{E}_h^n + \check{\mathbf{C}}_h^n)]^2 \mathbf{E}_h^n, \mathbf{D}_h \big),$$

$$(\mathbf{R}_{h9}^n, \mathbf{D}_h) := -\big([\mathrm{tr}\, (\mathbf{E}_h^n + 2\check{\mathbf{C}}_h^n)] (\mathrm{tr}\, \mathbf{E}_h^n) \check{\mathbf{C}}_h^n, \mathbf{D}_h \big),$$

$$(\mathbf{R}_{h10}^n, \mathbf{D}_h) := \big((\mathrm{tr}\, \check{\mathbf{C}}_h^n)^2 \boldsymbol{\Xi}^n + [\mathrm{tr}\, (\mathbf{C}^n + \check{\mathbf{C}}_h^n)] (\mathrm{tr}\, \boldsymbol{\Xi}^n) \mathbf{C}^n, \mathbf{D}_h \big),$$

$$(\mathbf{R}_{h11}^n, \mathbf{D}_h) := \left([\mathrm{tr}\, (\mathbf{E}_h^n - \boldsymbol{\Xi}^n)] \mathbf{I}, \mathbf{D}_h \right).$$

The remainder terms are evaluated by the next lemma.

Lemma 5.10. *Suppose Hypotheses* 2.2 *and* 4.4 *hold. Let* $n \in \{1, \ldots, N_T\}$ *be any fixed number. Then, under the condition* (3.2) *it holds that*

$$\|\mathbf{r}_{h1}^n\|_0 \le c_w \sqrt{\Delta t} \|\mathbf{u}\|_{Z^2(t^{n-1}, t^n)}, \tag{5.8a}$$

$$\|\mathbf{r}_{h2}^n\|_0 \le \frac{c_w h}{\sqrt{\Delta t}} \|(\mathbf{u}, p)\|_{H^1(t^{n-1}, t^n; H^2 \times H^1)}, \tag{5.8b}$$

$$\|\mathbf{r}_{h3}^n\|_{-1} \le c_s \|\mathbf{E}_h^n\|_0, \tag{5.8c}$$

$$\|\mathbf{r}_{h4}^n\|_{-1} \le c_s h, \tag{5.8d}$$

$$\|\mathbf{R}_{h1}^n\|_0 \le c_w \sqrt{\Delta t} \|\mathbf{C}\|_{Z^2(t^{n-1}, t^n)}, \tag{5.8e}$$

$$\|\mathbf{R}_{h2}^n\|_0 \le \frac{c_w h}{\sqrt{\Delta t}} \|\mathbf{C}\|_{H^1(t^{n-1}, t^n; H^1) \cap L^2(t^{n-1}, t^n; H^2)}, \tag{5.8f}$$

$${}_{W_h'}\left\langle \mathbf{R}_{h3}^n, \frac{1}{2} \mathbf{E}_h^n \right\rangle_{W_h} \le \frac{\varepsilon}{4} |\mathbf{E}_h^n|_1^2 + c_s h^2, \tag{5.8g}$$

$$\|\mathbf{R}_{h4}^n\|_0 \le c_s (\|\mathbf{e}_h^n\|_1 + \|\mathbf{E}_h^n\|_0), \tag{5.8h}$$

$$\|\mathbf{R}_{h5}^n\|_0 \le c_s h, \tag{5.8i}$$

$$\|\mathbf{R}_{h6}^n\|_0 \le c_s\big(\|\mathbf{e}_h^n\|_1 + \|\mathbf{E}_h^n\|_0\big), \tag{5.9a}$$
$$\|\mathbf{R}_{h7}^n\|_0 \le c_s h, \tag{5.9b}$$
$$\left(\mathbf{R}_{h8}^n, \frac{1}{2}\mathbf{E}_h^n\right) \le -\frac{3}{8}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2 + c_s\|\mathbf{E}_h^n\|_0^2, \tag{5.9c}$$
$$\left(\mathbf{R}_{h9}^n, \frac{1}{2}\mathbf{E}_h^n\right) \le \frac{1}{8}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2 + c_s\|\mathbf{E}_h^n\|_0^2, \tag{5.9d}$$
$$\|\mathbf{R}_{h10}^n\|_0 \le c_s h, \tag{5.9e}$$
$$\|\mathbf{R}_{h11}^n\|_0 \le c_s(\|\mathbf{E}_h^n\|_0 + h), \tag{5.9f}$$

where c_w and c_s are the constants given in the beginning of Section 4.

Proof. Let $t(s) \coloneqq t^{n-1} + s\Delta t$ ($s \in [0,1]$) and $y(x,s) \coloneqq x - (1-s)\mathbf{w}^n(x)\Delta t$.

We prove (5.8a). We have that

$$\begin{aligned}
\mathbf{r}_{h1}^n(x) &= \left\{\left(\frac{\partial}{\partial t} + \mathbf{w}^n(x)\cdot\nabla\right)\mathbf{u}\right\}(x,t^n) - \frac{1}{\Delta t}\left[\mathbf{u}\big(y(x,s),t(s)\big)\right]_{s=0}^1 \\
&= \left\{\left(\frac{\partial}{\partial t} + \mathbf{w}^n(x)\cdot\nabla\right)\mathbf{u}\right\}(x,t^n) - \int_0^1 \left\{\left(\frac{\partial}{\partial t} + \mathbf{w}^n(x)\cdot\nabla\right)\mathbf{u}\right\}\big(y(x,s),t(s)\big)\mathrm{d}s \\
&= \Delta t\int_0^1 \mathrm{d}s\int_s^1 \left\{\left(\frac{\partial}{\partial t} + \mathbf{w}^n(x)\cdot\nabla\right)^2\mathbf{u}\right\}(y(x,s_1),t(s_1))\,\mathrm{d}s_1 \\
&= \Delta t\int_0^1 s_1\left\{\left(\frac{\partial}{\partial t} + \mathbf{w}^n(x)\cdot\nabla\right)^2\mathbf{u}\right\}(y(x,s_1),t(s_1))\,\mathrm{d}s_1,
\end{aligned}$$

which implies

$$\|\mathbf{r}_{h1}^n\|_0 \le \Delta t\int_0^1 s_1\left\|\left\{\left(\frac{\partial}{\partial t} + \mathbf{w}^n(\cdot)\cdot\nabla\right)^2\mathbf{u}\right\}(y(\cdot,s_1),t(s_1))\right\|_0 \mathrm{d}s_1 \le c_w\sqrt{\Delta t}\|\mathbf{u}\|_{Z^2(t^{n-1},t^n)},$$

where for the last inequality we have changed the variable from x to y and used the evaluation $\det(\partial y(x,s_1)/\partial x) \ge 1/2$ ($\forall s_1 \in [0,1]$) from Proposition 3.1-(ii).

We prove (5.8b). From the equalities,

$$\mathbf{r}_{h2}^n = \frac{1}{\Delta t}\left[\boldsymbol{\eta}\left(y(\cdot,s),t(s)\right)\right]_{s=0}^1 = \int_0^1 \left\{\left(\frac{\partial}{\partial t} + \mathbf{w}^n(\cdot)\cdot\nabla\right)\boldsymbol{\eta}\right\}(y(\cdot,s),t(s))\,\mathrm{d}s,$$

we have

$$\begin{aligned}
\|\mathbf{r}_{h2}^n\|_0 &\le \int_0^1 \Big\|\Big\{\Big(\frac{\partial}{\partial t} + \mathbf{w}^n(\cdot)\cdot\nabla\Big)\boldsymbol{\eta}\Big\}\big(y(\cdot,s),t(s)\big)\Big\|_0 \mathrm{d}s \le \int_0^1\Big(\Big\|\frac{\partial\boldsymbol{\eta}}{\partial t}\big(y(\cdot,s),t(s)\big)\Big\|_0 + c_w\big\|\nabla\boldsymbol{\eta}\big(y(\cdot,s),t(s)\big)\big\|_0\Big)\mathrm{d}s \\
&\le \sqrt{2}\int_0^1\Big\{\Big\|\frac{\partial\boldsymbol{\eta}}{\partial t}\big(\cdot,t(s)\big)\Big\|_0 + c_w\big\|\nabla\boldsymbol{\eta}\big(\cdot,t(s)\big)\big\|_0\Big\}\mathrm{d}s \le \sqrt{\frac{2}{\Delta t}}\Big(\Big\|\frac{\partial\boldsymbol{\eta}}{\partial t}\Big\|_{L^2(t^{n-1},t^n;L^2)} + c_w\big\|\nabla\boldsymbol{\eta}\big\|_{L^2(t^{n-1},t^n;L^2)}\Big) \\
&\le \sqrt{\frac{2}{\Delta t}}\alpha_{31}h(1+c_w)\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} \le \frac{c_w' h}{\sqrt{\Delta t}}\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)},
\end{aligned}$$

which leads to (5.8b), where Proposition 3.1-(ii) has been used for the third inequality.

From Lemmas 5.2 and 5.3, (5.8c) and (5.8d) are obtained as follows:

$$\|\mathbf{r}_{h3}^n\|_{-1} \le \|(\operatorname{tr}\check{\mathbf{C}}_h^n)\mathbf{E}_h^n + (\operatorname{tr}\mathbf{E}_h^n)\check{\mathbf{C}}_h^n\|_0 \le c\|\check{\mathbf{C}}_h^n\|_{0,\infty}\|\mathbf{E}_h^n\|_0 \le c\|\mathbf{C}\|_{C(L^\infty)}\|\mathbf{E}_h^n\|_0 \le c_s\|\mathbf{E}_h^n\|_0,$$

$$\|\mathbf{r}_{h4}^n\|_{-1} \le \|(\operatorname{tr}\check{\mathbf{C}}_h^n)\boldsymbol{\Xi}^n + (\operatorname{tr}\boldsymbol{\Xi}^n)\mathbf{C}^n\|_0 \le c\|\check{\mathbf{C}}_h^n\|_{0,\infty}\|\boldsymbol{\Xi}_h^n\|_0 \le c\|\mathbf{C}\|_{C(L^\infty)}\alpha_{21}h\|\mathbf{C}\|_{C(H^1)} \le c_s h.$$

The estimate (5.8e) is obtained by replacing $\mathbf{u}$ with $\mathbf{C}$ in the proof of (5.8a).

We prove (5.8f). Replacing $\boldsymbol{\eta}$ with $\boldsymbol{\Xi}$ in the estimate of $\|\mathbf{r}_{h2}^n\|_0$ above, we have

$$\begin{aligned}\|\mathbf{R}_{h2}^n\|_0 &\le \sqrt{\frac{2}{\Delta t}}\left(\left\|\frac{\partial\boldsymbol{\Xi}}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^2)} + c_w\left\|\nabla\boldsymbol{\Xi}\right\|_{L^2(t^{n-1},t^n;L^2)}\right)\\ &\le \sqrt{\frac{2}{\Delta t}}h\Big(\alpha_{21}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^1)} + c_w\alpha_{22}\|\mathbf{C}\|_{L^2(t^{n-1},t^n;H^2)}\Big)\\ &\le \frac{c_w' h}{\sqrt{\Delta t}}\|\mathbf{C}\|_{H^1(t^{n-1},t^n;H^1)\cap L^2(t^{n-1},t^n;H^2)},\end{aligned}$$

which implies (5.8f).

The estimate (5.8g) is obtained from

$$\begin{aligned}{}_{W_h'}\left\langle \mathbf{R}_{h3}^n, \frac{1}{2}\mathbf{E}_h^n\right\rangle_{W_h} &\le \frac{\varepsilon}{2}|\boldsymbol{\Xi}^n|_1|\mathbf{E}_h^n|_1 \le \frac{\varepsilon}{4}(|\mathbf{E}_h^n|_1^2 + |\boldsymbol{\Xi}^n|_1^2) \qquad \text{(by } ab \le (a^2+b^2)/2)\\ &\le \frac{\varepsilon}{4}(|\mathbf{E}_h^n|_1^2 + \alpha_3^2 h^2\|\mathbf{C}\|_{C(H^2)}^2) \le \frac{\varepsilon}{4}|\mathbf{E}_h^n|_1^2 + c_s h^2.\end{aligned}$$

In order to prove estimates (5.8h)–(5.9b) we prepare the boundedness of $\|\nabla\hat{\mathbf{u}}_h^n\|_{0,\infty}$. Let $\check{\mathbf{u}}_h(t) := (\Pi_h\mathbf{u})(t)$ for $t\in[0,T]$. We have

$$\begin{aligned}\|\nabla\hat{\mathbf{u}}_h^n\|_{0,\infty} &\le \|\hat{\mathbf{u}}_h^n\|_{1,\infty} \le \|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_{1,\infty} + \|\check{\mathbf{u}}_h^n\|_{1,\infty} \le \alpha_{25}h^{-1}\|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_1 + \alpha_{20}\|\mathbf{u}^n\|_{1,\infty}\\ &\le \alpha_{25}h^{-1}\big(\|\hat{\mathbf{u}}_h^n - \mathbf{u}^n\|_1 + \|\mathbf{u}^n - \check{\mathbf{u}}_h^n\|_1\big) + \alpha_{20}\|\mathbf{u}^n\|_{1,\infty}\\ &\le \alpha_{25}h^{-1}\big(\alpha_3 h\|(\mathbf{u},p)^n\|_{H^2\times H^1} + \alpha_{22}h\|\mathbf{u}^n\|_2\big) + \alpha_{20}\|\mathbf{u}^n\|_{1,\infty}\\ &\le \alpha_{25}(\alpha_{22}+\alpha_3)\|(\mathbf{u},p)\|_{C(H^2\times H^1)} + \alpha_{20}\|\mathbf{u}\|_{C(W^{1,\infty})} \le c_s.\end{aligned} \tag{5.10}$$

We prove (5.8h)–(5.9b) by using (5.10) and (5.2) as follows.

$$\begin{aligned}\|\mathbf{R}_{h4}^n\|_0 &\le 2(\|(\nabla\hat{\mathbf{u}}_h^n)\mathbf{E}_h^n\|_0 + \|(\nabla\mathbf{e}_h^n)\check{\mathbf{C}}_h^n\|_0) \le c(c_s\|\mathbf{E}_h^n\|_0 + \|\mathbf{C}\|_{C(L^\infty)}\|\nabla\mathbf{e}_h^n\|_0) \le c_s'(\|\mathbf{e}_h^n\|_1 + \|\mathbf{E}_h^n\|_0),\\ \|\mathbf{R}_{h5}^n\|_0 &\le 2(\|(\nabla\hat{\mathbf{u}}_h^n)\boldsymbol{\Xi}^n\|_0 + \|(\nabla\boldsymbol{\eta}^n)\mathbf{C}^n\|_0) \le c(\|\nabla\hat{\mathbf{u}}_h^n\|_{0,\infty}\|\boldsymbol{\Xi}^n\|_0 + \|\mathbf{C}\|_{C(L^\infty)}\|\nabla\boldsymbol{\eta}^n\|_0)\\ &\le c_s(\|\boldsymbol{\Xi}^n\|_0 + \|\boldsymbol{\eta}^n\|_1) \le c_s h(\alpha_{21}\|\mathbf{C}\|_{C(H^1)} + \alpha_3\|(\mathbf{u},p)\|_{C(H^2\times H^1)}) \le c_s' h,\\ \|\mathbf{R}_{h6}^n\|_0 &\le \|\nabla\hat{\mathbf{u}}_h^n\|_{0,\infty}\|\mathbf{E}_h^n\|_0 + \|\check{\mathbf{C}}_h^n\|_{0,\infty}\|\mathbf{e}_h^n\|_1 \le c_s\|\mathbf{E}_h^n\|_0 + \|\mathbf{C}\|_{C(L^\infty)}\|\mathbf{e}_h^n\|_1 \le c_s'(\|\mathbf{E}_h^n\|_0 + \|\mathbf{e}_h^n\|_1),\\ \|\mathbf{R}_{h7}^n\|_0 &\le \|\nabla\hat{\mathbf{u}}_h^n\|_{0,\infty}\|\boldsymbol{\Xi}^n\|_0 + \|\mathbf{C}^n\|_{0,\infty}\|\boldsymbol{\eta}^n\|_1 \le c_s(\|\boldsymbol{\Xi}^n\|_0 + \|\boldsymbol{\eta}^n\|_1)\\ &\le c_s h(\alpha_{21}\|\mathbf{C}\|_{C(H^1)} + \alpha_3\|(\mathbf{u},p)\|_{C(H^2\times H^1)}) \le c_s' h.\end{aligned}$$

The remainder estimates (5.9c)–(5.9f) are obtained from

$$\begin{aligned}\left(\mathbf{R}^n_{h8}, \frac{1}{2}\mathbf{E}^n_h\right) &= -\frac{1}{2}\big([(\mathrm{tr}\,\mathbf{E}^n_h)^2 + 2(\mathrm{tr}\,\mathbf{E}^n_h)(\mathrm{tr}\,\check{\mathbf{C}}^n_h) + (\mathrm{tr}\,\check{\mathbf{C}}^n_h)^2]\mathbf{E}^n_h, \mathbf{E}^n_h\big)\\
&\le -\frac{1}{2}\|(\mathrm{tr}\,\mathbf{E}^n_h)\mathbf{E}^n_h\|_0^2 - \big((\mathrm{tr}\,\mathbf{E}^n_h)\mathbf{E}^n_h, (\mathrm{tr}\,\check{\mathbf{C}}^n_h)\mathbf{E}^n_h\big)\\
&\le -\frac{1}{2}\|(\mathrm{tr}\,\mathbf{E}^n_h)\mathbf{E}^n_h\|_0^2 + \frac{1}{8}\|(\mathrm{tr}\,\mathbf{E}^n_h)\mathbf{E}^n_h\|_0^2 + 2\|(\mathrm{tr}\,\check{\mathbf{C}}^n_h)\mathbf{E}^n_h\|_0^2\\
&\le -\frac{3}{8}\|(\mathrm{tr}\,\mathbf{E}^n_h)\mathbf{E}^n_h\|_0^2 + c\|\mathbf{C}\|^2_{C(L^\infty)}\|\mathbf{E}^n_h\|_0^2 \le -\frac{3}{8}\|(\mathrm{tr}\,\mathbf{E}^n_h)\mathbf{E}^n_h\|_0^2 + c_s\|\mathbf{E}^n_h\|_0^2 \qquad \text{(by (5.2))},\\
\left(\mathbf{R}^n_{h9}, \frac{1}{2}\mathbf{E}^n_h\right) &= -\frac{1}{2}\big((\mathrm{tr}\,\mathbf{E}^n_h)\check{\mathbf{C}}^n_h, (\mathrm{tr}\,\mathbf{E}^n_h)\mathbf{E}^n_h\big) - \big((\mathrm{tr}\,\check{\mathbf{C}}^n_h)(\mathrm{tr}\,\mathbf{E}^n_h)\check{\mathbf{C}}^n_h, \mathbf{E}^n_h\big)\\
&\le \frac{1}{8}\|(\mathrm{tr}\,\mathbf{E}^n_h)\mathbf{E}^n_h\|_0^2 + c\|\mathbf{C}\|^2_{C(L^\infty)}\|\mathbf{E}^n_h\|_0^2 \le \frac{1}{8}\|(\mathrm{tr}\,\mathbf{E}^n_h)\mathbf{E}^n_h\|_0^2 + c_s\|\mathbf{E}^n_h\|_0^2,\\
\|\mathbf{R}^n_{h10}\|_0 &\le c\big[\|\check{\mathbf{C}}^n_h\|^2_{0,\infty} + \|\mathbf{C}^n\|_{0,\infty}(\|\mathbf{C}^n\|_{0,\infty} + \|\check{\mathbf{C}}^n_h\|_{0,\infty})\big]\|\boldsymbol{\Xi}^n\|_0\\
&\le c'\|\mathbf{C}\|_{C(L^\infty)}\big(1 + \|\mathbf{C}\|_{C(L^\infty)}\big)\|\boldsymbol{\Xi}^n\|_0 \qquad \text{(by (5.2))}\\
&\le c_s\|\boldsymbol{\Xi}^n\|_0 \le c_s\alpha_{21}h\|\mathbf{C}^n\|_1 \le c'_s h,\\
\|\mathbf{R}^n_{h11}\|_0 &\le c(\|\mathbf{E}^n_h\|_0 + \|\boldsymbol{\Xi}^n\|_0) \le c(\|\mathbf{E}^n_h\|_0 + \alpha_{21}h\|\mathbf{C}\|_{C(H^1)}) \le c_s(\|\mathbf{E}^n_h\|_0 + h).\end{aligned}$$

□

5.4. Proof of Theorem 4.5

The constant h_0 can be chosen arbitrarily, say, $h_0 = 1$. We fix Δt_0 by

$$\Delta t_0 = \min\left\{\frac{1}{4|\mathbf{w}|_{C(W^{1,\infty})}}, \frac{1}{2c_s}\right\}, \tag{5.11}$$

where c_s is the constant appearing in (5.15) below. We consider any pair $(h, \Delta t)$ satisfying (4.3) and any solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ of scheme (3.3) with (4.2). We return to the argument in the previous subsection. Substituting $(\mathbf{e}^n_h, -\epsilon^n_h, \frac{1}{2}\mathbf{E}^n_h)$ into $(\mathbf{v}_h, q_h, \mathbf{D}_h)$ in (5.7) and noting that

$$\left(\frac{\mathbf{e}^n_h - \mathbf{e}^{n-1}_h\circ X^n_1}{\Delta t}, \mathbf{e}^n_h\right) \ge \frac{1}{2\Delta t}\left[\|\mathbf{e}^n_h\|_0^2 - (1+\alpha_4|\mathbf{w}^n|_{1,\infty}\Delta t)^2\|\mathbf{e}^{n-1}_h\|_0^2\right] \ge \overline{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{e}^n_h\|_0^2\right) - c_w\|\mathbf{e}^{n-1}_h\|_0^2 \tag{5.12}$$

(by $(b-a)b \ge (b^2-a^2)/2$ and Lem. 5.4),

$$\mathcal{A}_h\big((\mathbf{e}^n_h, \epsilon^n_h), (\mathbf{e}^n_h, -\epsilon^n_h)\big) = 2\nu\|\mathrm{D}(\mathbf{e}^n_h)\|_0^2 + \delta_0|\epsilon^n_h|_h^2 \ge \frac{2\nu}{\alpha_1^2}\|\mathbf{e}^n_h\|_1^2 + \delta_0|\epsilon^n_h|_h^2 \qquad \text{(by Lem. 5.1)},$$

$${}_{V'_h}\langle \mathbf{r}^n_h, \mathbf{e}^n_h\rangle_{V_h} \le \|\mathbf{r}^n_h\|_{-1}\|\mathbf{e}^n_h\|_1 \le \frac{\alpha_1^2}{4\nu}\|\mathbf{r}^n_h\|_{-1}^2 + \frac{\nu}{\alpha_1^2}\|\mathbf{e}^n_h\|_1^2 \qquad \text{(by } ab \le (\beta/4)a^2 + (1/\beta)b^2),$$

$$\left(\frac{\mathbf{E}^n_h - \mathbf{E}^{n-1}_h\circ X^n_1}{\Delta t}, \frac{1}{2}\mathbf{E}^n_h\right) \ge \overline{D}_{\Delta t}\left(\frac{1}{4}\|\mathbf{E}^n_h\|_0^2\right) - c_w\|\mathbf{E}^{n-1}_h\|_0^2 \qquad (cf.\,(5.12)),$$

$$\varepsilon a_c\left(\mathbf{E}^n_h, \frac{1}{2}\mathbf{E}^n_h\right) = \frac{\varepsilon}{2}|\mathbf{E}^n_h|_1^2,$$

and Lemma 5.5, we have

$$\begin{aligned}&\overline{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{e}^n_h\|_0^2 + \frac{1}{4}\|\mathbf{E}^n_h\|_0^2\right) + \frac{\nu}{\alpha_1^2}\|\mathbf{e}^n_h\|_1^2 + \delta_0|\epsilon^n_h|_h^2 + \frac{\varepsilon}{2}|\mathbf{E}^n_h|_1^2\\
&\qquad \le c_w(\|\mathbf{e}^{n-1}_h\|_0^2 + \|\mathbf{E}^{n-1}_h\|_0^2) + \frac{\alpha_1^2}{4\nu}\|\mathbf{r}^n_h\|_{-1}^2 + {}_{W'_h}\left\langle \mathbf{R}^n_h, \frac{1}{2}\mathbf{E}^n_h\right\rangle_{W_h}.\end{aligned} \tag{5.13}$$

Since the condition (3.2) is satisfied, Lemma 5.10 implies that

$$\|\mathbf{r}_h^n\|_{-1}^2 \le c_s\|\mathbf{E}_h^n\|_0^2 + c_s'\left[\Delta t\|\mathbf{u}\|_{Z^2(t^{n-1},t^n)}^2 + h^2\left(\frac{1}{\Delta t}\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}^2 + 1\right)\right], \tag{5.14a}$$

$$\begin{aligned}{}_{W_h'}\left\langle \mathbf{R}_h^n, \frac{1}{2}\mathbf{E}_h^n\right\rangle_{W_h} &\le c_s\|\mathbf{E}_h^n\|_0^2 + \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h^n\|_1^2 + \frac{\varepsilon}{4}|\mathbf{E}_h^n|_1^2 - \frac{1}{4}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2 \\ &\quad + c_s'\left[\Delta t\|\mathbf{C}\|_{Z^2(t^{n-1},t^n)}^2 + h^2\left(\frac{1}{\Delta t}\|\mathbf{C}\|_{Z^2(t^{n-1},t^n)}^2 + 1\right)\right]. \end{aligned}\tag{5.14b}$$

Combining (5.14) with (5.13), we obtain

$$\begin{aligned}&\overline{D}_{\Delta t}\left(\frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{1}{4}\|\mathbf{E}_h^n\|_0^2\right) + \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h^n\|_1^2 + \delta_0|\epsilon_h^n|_h^2 + \frac{\varepsilon}{4}|\mathbf{E}_h^n|_1^2 + \frac{1}{4}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2 \\ &\le c_s\left(\frac{1}{2}\|\mathbf{e}_h^{n-1}\|_0^2 + \frac{1}{4}\|\mathbf{E}_h^{n-1}\|_0^2 + \frac{1}{4}\|\mathbf{E}_h^n\|_0^2\right) \\ &\quad + c_s'\left[\Delta t\|(\mathbf{u},\mathbf{C})\|_{Z^2(t^{n-1},t^n)}^2 + h^2\left\{\frac{1}{\Delta t}\left(\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}^2 + \|\mathbf{C}\|_{Z^2(t^{n-1},t^n)}^2\right) + 1\right\}\right]. \end{aligned}\tag{5.15}$$

From (4.3) and (5.11) it holds that $\Delta t \in (0, 1/(2c_s)]$. As for the initial value we have

$$(\mathbf{e}_h^0, \mathbf{E}_h^0) = (\mathbf{u}_h^0, \mathbf{C}_h^0) - (\hat{\mathbf{u}}_h^0, \check{\mathbf{C}}_h^0) = ([\Pi_h^{\mathrm{S}}(\mathbf{0}, -p^0)]_1, \mathbf{0}) = ([(I - \Pi_h^{\mathrm{S}})(\mathbf{0}, p^0)]_1, \mathbf{0}),$$

which derives the estimates,

$$\|\mathbf{e}_h^0\|_0 \le \alpha_3 h\|(0,p^0)\|_{H^2\times H^1} = \alpha_3 h\|p\|_{C(H^1)}, \quad \|\mathbf{E}_h^0\|_0 = 0. \tag{5.16}$$

By applying Lemma 5.8 to (5.15) with

$$x^n = \frac{1}{2}\|\mathbf{e}_h^n\|_0^2 + \frac{1}{4}\|\mathbf{E}_h^n\|_0^2, \quad y^n = \frac{\nu}{2\alpha_1^2}\|\mathbf{e}_h^n\|_1^2 + \delta_0|\epsilon_h^n|_h^2 + \frac{\varepsilon}{4}|\mathbf{E}_h^n|_1^2 + \frac{1}{4}\|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0^2, \quad a_0 = a_1 = c_s,$$

$$b^n = c_s'\left[\Delta t\|(\mathbf{u},\mathbf{C})\|_{Z^2(t^{n-1},t^n)}^2 + h^2\left\{\frac{1}{\Delta t}\left(\|(\mathbf{u},p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)}^2 + \|\mathbf{C}\|_{Z^2(t^{n-1},t^n)}^2\right) + 1\right\}\right],$$

and (5.16), there exists a positive constant

$$\tilde{c}_\dagger = c\exp(3c_sT/2)\left[\|p\|_{C(H^1)} + \sqrt{c_s'}\left(\|(\mathbf{u},\mathbf{C})\|_{Z^2} + \|(\mathbf{u},p)\|_{H^1(H^2\times H^1)} + \sqrt{T}\right)\right]$$

independent of ε such that

$$\|\mathbf{e}_h\|_{\ell^\infty(L^2)},\ \sqrt{\nu}\|\mathbf{e}_h\|_{\ell^2(H^1)},\ |\epsilon_h|_{\ell^2(|\cdot|_h)},\ \|\mathbf{E}_h\|_{\ell^\infty(L^2)},\ \sqrt{\varepsilon}|\mathbf{E}_h|_{\ell^2(H^1)},\ \left\|(\mathrm{tr}\,\mathbf{E}_h)\mathbf{E}_h\right\|_{\ell^2(L^2)} \le \tilde{c}_\dagger(h + \Delta t). \tag{5.17}$$

Hence, we obtain (4.4) from (5.17) and the estimates,

$$\begin{aligned}\|\mathbf{u}_h^n - \mathbf{u}^n\|_k &\le \|\mathbf{e}_h^n\|_k + \|\boldsymbol{\eta}^n\|_1 \le \|\mathbf{e}_h^n\|_k + \alpha_3 h\|(\mathbf{u},p)\|_{C(H^2\times H^1)}, \\ |p_h^n - p^n|_h &\le |\epsilon_h^n|_h + |\hat{p}_h^n - p^n|_h \le |\epsilon_h^n|_h + \alpha_3 h\|(\mathbf{u},p)\|_{C(H^2\times H^1)}, \\ \|\mathbf{C}_h^n - \mathbf{C}^n\|_k &\le \|\mathbf{E}_h^n\|_k + \|\boldsymbol{\Xi}^n\|_k \le \|\mathbf{E}_h^n\|_k + \alpha_{2(k+1)}h\|\mathbf{C}\|_{C(H^{k+1})}, \\ \|\mathrm{tr}\,(\mathbf{C}_h^n - \mathbf{C}^n)(\mathbf{C}_h^n - \mathbf{C}^n)\|_0 &= \|\mathrm{tr}\,(\mathbf{E}_h^n - \boldsymbol{\Xi}^n)(\mathbf{E}_h^n - \boldsymbol{\Xi}^n)\|_0 \\ &\le \|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0 + \|(\mathrm{tr}\,\boldsymbol{\Xi}^n)\mathbf{E}_h^n\|_0 + \|(\mathrm{tr}\,\mathbf{E}_h^n)\boldsymbol{\Xi}^n\|_0 + \|(\mathrm{tr}\,\boldsymbol{\Xi}^n)\boldsymbol{\Xi}^n\|_0 \\ &\le \|(\mathrm{tr}\,\mathbf{E}_h^n)\mathbf{E}_h^n\|_0 + c_s h(\|\mathbf{E}_h^n\|_0 + 1),\end{aligned}$$

for $k = 0$ and 1.

When $\varepsilon = 0$, (4.4) is still valid, since $\mathbf{R}_{h3}^n$ vanishes and $c_\dagger$ is independent of ε.

6. Uniqueness of the solution

In this section we present and prove the result on the uniqueness of the solution of scheme (3.3). Let us remind that the function $D(h)$ has been defined in (5.1).

Proposition 6.1. *Suppose Hypotheses 2.2 and 4.4 hold. Then, for any pair $(h, \Delta t)$ satisfying the following condition (6.1) or (6.2), the solution of scheme (3.3) with (4.2) is unique.*

(i) *When $\varepsilon > 0$,*

$$h \in (0, h_\star], \quad \Delta t \le D(h)^{-2}, \tag{6.1}$$

where the constant $h_\star$ is defined by (6.14) below.

(ii) *When $\varepsilon = 0$,*

$$h \in (0, \bar{h}_\star], \quad \Delta t \le \bar{c}_\star h, \tag{6.2}$$

where the constants $\bar{h}_\star$ and $\bar{c}_\star$ are defined by (6.15) and (6.18) below.

The proof is given after preparing the next lemma.

Lemma 6.2. *Suppose Hypotheses 2.2 and 4.4 hold. Then, for any pair $(h, \Delta t)$ satisfying the following condition (6.4) or (6.5), any solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ of scheme (3.3) with (4.2) satisfies*

$$\|\mathbf{C}_h\|_{\ell^\infty(L^\infty)} \le c_c, \qquad \|\mathbf{u}_h\|_{\ell^\infty(L^\infty)} \le c_u, \tag{6.3}$$

where c_c and c_u are positive constants independent of h and Δt defined just below.

(i) *When $\varepsilon > 0$,*

$$h \in (0, h_\dagger], \quad \Delta t \le D(h)^{-2}, \tag{6.4}$$

where $h_\dagger$ is defined by (6.6d) below. Furthermore, $c_c = c_{\dagger c}$ and $c_u = c_{\dagger u}$, which are defined by (6.6e) and (6.6f).

(ii) *When $\varepsilon = 0$,*

$$h \in (0, \bar{h}_\dagger], \quad \Delta t \le h, \tag{6.5}$$

where $\bar{h}_\dagger$ is defined by (6.6a) below. Furthermore, $c_c = \bar{c}_{\dagger c}$ and $c_u = \bar{c}_{\dagger u}$, which are defined by (6.6b) and (6.6c).

Proof. Let $n \in \{0, \dots, N_T\}$ be fixed arbitrarily, and let h_0, Δt_0 and $\tilde{c}_\dagger$ be the positive constants in the statement of Theorem 4.5 and in (5.17). We fix a positive constant $h_1 \in (0, 1]$ such that

$$h_1 \le D(h_1)^{-2} \le \Delta t_0.$$

We prepare the following constants to be used in the proof:

$$\bar{h}_\dagger := \min\{h_0, \Delta t_0\}, \tag{6.6a}$$

$$\bar{c}_{\dagger c} := 2\alpha_{23}\tilde{c}_\dagger + \|\mathbf{C}\|_{C(L^\infty)}, \tag{6.6b}$$

$$\bar{c}_{\dagger u} := \alpha_{23}\big[2\tilde{c}_\dagger + (\alpha_{21} + \alpha_3)\|(\mathbf{u}, p)\|_{C(H^2 \times H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)}, \tag{6.6c}$$

$$c_1 := \tilde{c}_\dagger \max\{1, (T + \varepsilon^{-1})^{1/2}, \nu^{-1/2}\},$$

$$h_\dagger := \min\{\bar{h}_\dagger, h_1\}, \tag{6.6d}$$

$$c_{\dagger c} := \max\{2\alpha_{24}c_1 + \|\mathbf{C}\|_{C(L^\infty)}, \bar{c}_{\dagger c}\}, \tag{6.6e}$$

$$c_{\dagger u} := \max\{\alpha_{24}\big[2c_1 + (\alpha_{22} + \alpha_3)\|(\mathbf{u}, p)\|_{C(H^2 \times H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)}, \bar{c}_{\dagger u}\}. \tag{6.6f}$$

Firstly, we prove (6.3) in case (ii). Since condition (6.5) implies (4.3), Theorem 4.5 ensures (5.17). Then, the boundedness of $\|\mathbf{C}_h^n\|_{0,\infty}$ is obtained as follows:

$$\begin{aligned}\|\mathbf{C}_h^n\|_{0,\infty} &\le \|\mathbf{E}_h^n\|_{0,\infty} + \|\check{\mathbf{C}}_h^n\|_{0,\infty} \le \alpha_{23} h^{-1}\|\mathbf{E}_h^n\|_0 + \|\mathbf{C}\|_{C(L^\infty)}\\ &\le \alpha_{23} h^{-1}\tilde{c}_\dagger(\Delta t + h) + \|\mathbf{C}\|_{C(L^\infty)} \le 2\alpha_{23}\tilde{c}_\dagger + \|\mathbf{C}\|_{C(L^\infty)}\\ &= \bar{c}_{\dagger c}.\end{aligned}$$

Let $\check{\mathbf{u}}_h(t) \coloneqq (\Pi_h \mathbf{u})(t)$ for $t \in [0,T]$. The boundedness of $\|\mathbf{u}_h^n\|_{0,\infty}$ is obtained as follows:

$$\begin{aligned}\|\mathbf{u}_h^n\|_{0,\infty} &\le \|\mathbf{e}_h^n\|_{0,\infty} + \|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_{0,\infty} + \|\check{\mathbf{u}}_h^n\|_{0,\infty} \le \alpha_{23}h^{-1}\big[\|\mathbf{e}_h^n\|_0 + \|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_0\big] + \|\mathbf{u}\|_{C(L^\infty)}\\ &\le \alpha_{23}h^{-1}\big[\|\mathbf{e}_h^n\|_0 + \|\hat{\mathbf{u}}_h^n - \mathbf{u}^n\|_0 + \|\mathbf{u}^n - \check{\mathbf{u}}_h^n\|_0\big] + \|\mathbf{u}\|_{C(L^\infty)}\\ &\le \alpha_{23}h^{-1}\big[\tilde{c}_\dagger(\Delta t + h) + \alpha_3 h\|(\mathbf{u},p)\|_{C(H^2\times H^1)} + \alpha_{21}h\|\mathbf{u}\|_{C(H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)}\\ &\le \alpha_{23}\big[2\tilde{c}_\dagger + (\alpha_{21}+\alpha_3)\|(\mathbf{u},p)\|_{C(H^2\times H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)}\\ &= \bar{c}_{\dagger u}.\end{aligned}$$

Secondly, we prove (6.3) in case (i). Since condition (6.4) implies (4.3), the estimates (5.17) and the definition of c_1 lead to

$$\|\mathbf{e}_h\|_{\ell^\infty(L^2)},\ \|\mathbf{e}_h\|_{\ell^2(H^1)},\ \|\mathbf{E}_h\|_{\ell^\infty(L^2)},\ \|\mathbf{E}_h\|_{\ell^2(H^1)} \le c_1(\Delta t + h).$$

When $\Delta t \le h$, we have $\|\mathbf{C}_h^n\|_{0,\infty} \le \bar{c}_{\dagger c} \le c_{\dagger c}$ and $\|\mathbf{u}_h^n\|_{0,\infty} \le \bar{c}_{\dagger u} \le c_{\dagger u}$ from the proof in case (ii) above. When $(D(h)^2h^2 \le)\ h \le \Delta t \le D(h)^{-2}$, we have

$$\begin{aligned}\|\mathbf{C}_h^n\|_{0,\infty} &\le \|\mathbf{E}_h^n\|_{0,\infty} + \|\mathbf{C}\|_{C(L^\infty)} \le \alpha_{24}D(h)\|\mathbf{E}_h^n\|_1 + \|\mathbf{C}\|_{C(L^\infty)} \le \alpha_{24}D(h)\Delta t^{-1/2}\|\mathbf{E}_h\|_{\ell^2(H^1)} + \|\mathbf{C}\|_{C(L^\infty)}\\ &\le \alpha_{24}c_1 D(h)(\Delta t^{1/2} + \Delta t^{-1/2}h) + \|\mathbf{C}\|_{C(L^\infty)} \le 2\alpha_{24}c_1 + \|\mathbf{C}\|_{C(L^\infty)}\\ &\le c_{\dagger c},\\ \|\mathbf{u}_h^n\|_{0,\infty} &\le \|\mathbf{e}_h^n\|_{0,\infty} + \|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_{0,\infty} + \|\check{\mathbf{u}}_h^n\|_{0,\infty} \le \alpha_{24}D(h)\big[\|\mathbf{e}_h^n\|_1 + \|\hat{\mathbf{u}}_h^n - \check{\mathbf{u}}_h^n\|_1\big] + \|\mathbf{u}\|_{C(L^\infty)}\\ &\le \alpha_{24}D(h)\big[\Delta t^{-1/2}\|\mathbf{e}_h\|_{\ell^2(H^1)} + \|\hat{\mathbf{u}}_h^n - \mathbf{u}^n\|_1 + \|\mathbf{u}^n - \check{\mathbf{u}}_h^n\|_1\big] + \|\mathbf{u}\|_{C(L^\infty)}\\ &\le \alpha_{24}D(h)\big[c_1(\Delta t^{1/2} + \Delta t^{-1/2}h) + (\alpha_{22}+\alpha_3)h\|(\mathbf{u},p)\|_{C(H^2\times H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)}\\ &\le \alpha_{24}\big[2c_1 + (\alpha_{22}+\alpha_3)\|(\mathbf{u},p)\|_{C(H^2\times H^1)}\big] + \|\mathbf{u}\|_{C(L^\infty)}\\ &\le c_{\dagger u}.\end{aligned}$$

Thus, we obtain (6.3). □

Proof of Proposition 6.1. The definitions (6.14), (6.15) and (6.18) below of the constants $h_\star$, $\bar{h}_\star$ and $c_\star$ imply $h_\star \le h_\dagger$, $\bar{h}_\star \le \bar{h}_\dagger$ and $\bar{c}_\star \le 1$. Hence any pair of $(h, \Delta t)$ in Proposition 6.1 satisfies the assumptions of Lemma 6.2 for $\varepsilon \ \ge\ 0$.

Suppose $(\tilde{\mathbf{u}}_h, \tilde{p}_h, \tilde{\mathbf{C}}_h)$ and $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ are any two solutions of scheme (3.3) with (4.2). Let $(\tilde{\mathbf{e}}_h, \tilde{\epsilon}_h, \tilde{\mathbf{E}}_h) \coloneqq (\tilde{\mathbf{u}}_h, \tilde{p}_h, \tilde{\mathbf{C}}_h) - (\mathbf{u}_h, p_h, \mathbf{C}_h)$ be the difference. Since both of $(\tilde{\mathbf{u}}_h, \tilde{p}_h, \tilde{\mathbf{C}}_h)$ and $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ satisfy scheme (3.3) with (4.2), we have

$$\left(\frac{\tilde{\mathbf{e}}_h^n - \tilde{\mathbf{e}}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{v}_h\right) + \mathcal{A}_h\big((\tilde{\mathbf{e}}_h^n, \tilde{\epsilon}_h^n), (\mathbf{v}_h, q_h)\big) = -\big((\operatorname{tr}\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n, \nabla\mathbf{v}_h\big) + {}_{V_h'}\langle\tilde{\mathbf{r}}_h^n, \mathbf{v}_h\rangle_{V_h}, \tag{6.7a}$$

$$\left(\frac{\tilde{\mathbf{E}}_h^n - \tilde{\mathbf{E}}_h^{n-1}\circ X_1^n}{\Delta t}, \mathbf{D}_h\right) + \varepsilon a_c(\tilde{\mathbf{E}}_h^n, \mathbf{D}_h) = 2\big((\nabla\tilde{\mathbf{e}}_h^n)\tilde{\mathbf{E}}_h^n, \mathbf{D}_h\big) + \big((\operatorname{div}\tilde{\mathbf{e}}_h^n)(\tilde{\mathbf{E}}_h^n)^\#, \mathbf{D}_h\big) + {}_{W_h'}\langle\tilde{\mathbf{R}}_h^n, \mathbf{D}_h\rangle_{W_h}, \tag{6.7b}$$

$$\forall(\mathbf{v}_h, q_h, \mathbf{D}_h) \in V_h \times Q_h \times W_h,$$

where

$$
\begin{aligned}
&\tilde{\mathbf{r}}_h^n \in V_h', \qquad \tilde{\mathbf{R}}_h^n := \sum_{i=1}^{5} \tilde{\mathbf{R}}_{hi}^n \in W_h',\\
&{}_{V_h'}\langle \tilde{\mathbf{r}}_h^n, \mathbf{v}_h \rangle_{V_h} := -\big((\operatorname{tr}\mathbf{C}_h^n)\tilde{\mathbf{E}}_h^n + (\operatorname{tr}\tilde{\mathbf{E}}_h^n)\mathbf{C}_h^n, \nabla \mathbf{v}_h\big),\\
&(\tilde{\mathbf{R}}_{h1}^n, \mathbf{D}_h) := 2\big((\nabla \mathbf{u}_h^n)\tilde{\mathbf{E}}_h^n + (\nabla \tilde{\mathbf{e}}_h^n)\mathbf{C}_h^n, \mathbf{D}_h\big),\\
&(\tilde{\mathbf{R}}_{h2}^n, \mathbf{D}_h) := \big((\operatorname{div}\mathbf{u}_h^n)(\tilde{\mathbf{E}}_h^n)^{\#} + (\operatorname{div}\tilde{\mathbf{e}}_h^n)(\mathbf{C}_h^n)^{\#}, \mathbf{D}_h\big),\\
&(\tilde{\mathbf{R}}_{h3}^n, \mathbf{D}_h) := -\big([\operatorname{tr}(\tilde{\mathbf{E}}_h^n + \mathbf{C}_h^n)]^2 \tilde{\mathbf{E}}_h^n, \mathbf{D}_h\big),\\
&(\tilde{\mathbf{R}}_{h4}^n, \mathbf{D}_h) := -\big([\operatorname{tr}(\tilde{\mathbf{E}}_h^n + 2\mathbf{C}_h^n)](\operatorname{tr}\tilde{\mathbf{E}}_h^n)\mathbf{C}_h^n, \mathbf{D}_h\big),\\
&(\tilde{\mathbf{R}}_{h5}^n, \mathbf{D}_h) := \big((\operatorname{tr}\tilde{\mathbf{E}}_h^n)\mathbf{I}, \mathbf{D}_h\big),
\end{aligned}
$$

and $(\tilde{\mathbf{e}}_h^0, \tilde{\mathbf{E}}_h^0) = (\mathbf{0}, \mathbf{0})$. Substituting $(\tilde{\mathbf{e}}_h^n, -\tilde{\epsilon}_h^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n)$ into $(\mathbf{v}_h, q_h, \mathbf{D}_h)$ in (6.7) and using Lemma 5.5 and similar estimates in the derivation of (5.13), we have

$$
\begin{aligned}
\overline{D}_{\Delta t}\left(\frac{1}{2}\|\tilde{\mathbf{e}}_h^n\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2\right) &+ \frac{\nu}{\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 + \delta_0|\tilde{\epsilon}_h^n|_h^2 + \frac{\varepsilon}{2}|\tilde{\mathbf{E}}_h^n|_1^2\\
&\le c_w(\|\tilde{\mathbf{e}}_h^{n-1}\|_0^2 + \|\tilde{\mathbf{E}}_h^{n-1}\|_0^2) + \frac{\alpha_1^2}{4\nu}\|\tilde{\mathbf{r}}_h^n\|_{-1}^2 + \left(\tilde{\mathbf{R}}_h^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n\right).
\end{aligned} \tag{6.8}
$$

The following estimates are obtained for the functionals $\tilde{\mathbf{r}}_h^n$ and $\tilde{\mathbf{R}}_h^n$:

$$
\|\tilde{\mathbf{r}}_h^n\|_{-1} \le c\|\mathbf{C}_h^n\|_{0,\infty}\|\tilde{\mathbf{E}}_h^n\|_0, \tag{6.9}
$$

$$
\left(\tilde{\mathbf{R}}_{h1}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n\right),\ \left(\tilde{\mathbf{R}}_{h2}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n\right) \le c\|\tilde{\mathbf{E}}_h^n\|_0\big(\|\mathbf{u}_h^n\|_{0,\infty}|\tilde{\mathbf{E}}_h^n|_1 + \|\mathbf{C}_h^n\|_{0,\infty}|\tilde{\mathbf{e}}_h^n|_1\big), \tag{6.10a}
$$

$$
\left(\tilde{\mathbf{R}}_{h3}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n\right) \le -\frac{3}{8}\|(\operatorname{tr}\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2 + c\|\mathbf{C}_h^n\|_{0,\infty}^2\|\tilde{\mathbf{E}}_h^n\|_0^2, \tag{6.10b}
$$

$$
\left(\tilde{\mathbf{R}}_{h4}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n\right) \le \frac{1}{8}\|(\operatorname{tr}\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2 + c\|\mathbf{C}_h^n\|_{0,\infty}^2\|\tilde{\mathbf{E}}_h^n\|_0^2, \tag{6.10c}
$$

$$
\|\tilde{\mathbf{R}}_{h5}^n\|_0 \le c\|\tilde{\mathbf{E}}_h^n\|_0. \tag{6.10d}
$$

We note that the estimates (6.10a) are proved by the integration by parts,

$$
\begin{aligned}
\left(\tilde{\mathbf{R}}_{h1}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n\right) &= \big((\nabla \mathbf{u}_h^n)\tilde{\mathbf{E}}_h^n, \tilde{\mathbf{E}}_h^n\big) + \big((\nabla \tilde{\mathbf{e}}_h^n)\mathbf{C}_h^n, \tilde{\mathbf{E}}_h^n\big) = -\big(\mathbf{u}_h^n, \nabla(\tilde{\mathbf{E}}_h^n\tilde{\mathbf{E}}_h^n)\big) + \big((\nabla \tilde{\mathbf{e}}_h^n)\mathbf{C}_h^n, \tilde{\mathbf{E}}_h^n\big)\\
&\le c\big(\|\mathbf{u}_h^n\|_{0,\infty}\|\tilde{\mathbf{E}}_h^n\|_0|\tilde{\mathbf{E}}_h^n|_1 + \|\mathbf{C}_h^n\|_{0,\infty}|\tilde{\mathbf{e}}_h^n|_1\|\tilde{\mathbf{E}}_h^n\|_0\big),\\
\left(\tilde{\mathbf{R}}_{h2}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n\right) &= \frac{1}{2}\big((\operatorname{div}\mathbf{u}_h^n)(\tilde{\mathbf{E}}_h^n)^{\#}, \tilde{\mathbf{E}}_h^n\big) + \frac{1}{2}\big((\operatorname{div}\tilde{\mathbf{e}}_h^n)(\mathbf{C}_h^n)^{\#}, \tilde{\mathbf{E}}_h^n\big)\\
&= -\frac{1}{2}\big(\mathbf{u}_h^n\nabla(\tilde{\mathbf{E}}_h^n)^{\#}, \tilde{\mathbf{E}}_h^n\big) - \frac{1}{2}\big((\tilde{\mathbf{E}}_h^n)^{\#}, \mathbf{u}_h^n\nabla\tilde{\mathbf{E}}_h^n\big) + \frac{1}{2}\big((\operatorname{div}\tilde{\mathbf{e}}_h^n)(\mathbf{C}_h^n)^{\#}, \tilde{\mathbf{E}}_h^n\big)\\
&\le c\big(\|\mathbf{u}_h^n\|_{0,\infty}|\tilde{\mathbf{E}}_h^n|_1\|\tilde{\mathbf{E}}_h^n\|_0 + \|\mathbf{C}_h^n\|_{0,\infty}|\tilde{\mathbf{e}}_h^n|_1\|\tilde{\mathbf{E}}_h^n\|_0\big),
\end{aligned}
$$

and that the other estimates (6.9), (6.10b), (6.10c) and (6.10d) are obtained similarly to (5.8c), (5.9c), (5.9d) and (5.9f), respectively. Applying Lemma 6.2 to (6.9), we have

$$
\|\tilde{\mathbf{r}}_h^n\|_{-1} \le cc_c\|\tilde{\mathbf{E}}_h^n\|_0. \tag{6.11}
$$

We consider case (i). The estimates (6.10) and Lemma 6.2 lead to

$$\left(\tilde{\mathbf{R}}_h^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n\right) \le \frac{c}{\varepsilon}(c_c^2 + c_u^2 + 1)\|\tilde{\mathbf{E}}_h^n\|_0^2 + \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 + \frac{\varepsilon}{4}|\tilde{\mathbf{E}}_h^n|_1^2 - \frac{1}{4}\|(\mathrm{tr}\,\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2. \tag{6.12}$$

Combining (6.11) and (6.12) with (6.8), we have

$$\overline{D}_{\Delta t}\left(\frac{1}{2}\|\tilde{\mathbf{e}}_h^n\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2\right) + \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 + \delta_0|\tilde{\epsilon}_h^n|_h^2 + \frac{\varepsilon}{4}|\tilde{\mathbf{E}}_h^n|_1^2 + \frac{1}{4}\|(\mathrm{tr}\,\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2$$
$$\le \frac{c}{\varepsilon}(c_c^2 + c_u^2 + 1)\left(\frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2\right) + c_w\left(\frac{1}{2}\|\tilde{\mathbf{e}}_h^{n-1}\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^{n-1}\|_0^2\right). \tag{6.13}$$

Let $\Delta t_\star := \varepsilon/[2c(c_c^2+c_u^2+1)]$, and we fix a positive constant $h_2 \in (0,1]$ such that $D(h_2)^{-2} \le \Delta t_\star$. We define $h_\star$ by

$$h_\star := \min\{h_\dagger, h_2\}. \tag{6.14}$$

Condition (6.1) implies $\Delta t \le D(h_2)^{-2} \le \varepsilon/[2c(c_c^2 + c_u^2 + 1)]$ $(= \Delta t_\star)$. Applying Lemma 5.8 to (6.13) with

$$x^n = \frac{1}{2}\|\tilde{\mathbf{e}}_h^n\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2, \qquad y^n = \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 + \delta_0|\tilde{\epsilon}_h^n|_h^2 + \frac{\varepsilon}{4}|\tilde{\mathbf{E}}_h^n|_1^2 + \frac{1}{4}\|(\mathrm{tr}\,\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2,$$
$$a_0 = \frac{c}{\varepsilon}(c_c^2 + c_u^2 + 1), \quad a_1 = 0, \qquad b^n = c_w\left(\frac{1}{2}\|\tilde{\mathbf{e}}_h^{n-1}\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^{n-1}\|_0^2\right),$$

and using the fact $(\tilde{\mathbf{e}}_h^0, \tilde{\mathbf{E}}_h^0) = (\mathbf{0}, \mathbf{0})$, we get $(\tilde{\mathbf{e}}_h, \tilde{\epsilon}_h, \tilde{\mathbf{E}}_h) = (\mathbf{0}, 0, \mathbf{0})$.

We prove (ii). In place of (6.10a) we use the estimates,

$$\left(\tilde{\mathbf{R}}_{h1}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n\right), \ \left(\tilde{\mathbf{R}}_{h2}^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n\right) \le c\|\tilde{\mathbf{E}}_h^n\|_0\big(\alpha_{26}h^{-1}\|\mathbf{u}_h^n\|_{0,\infty}\|\tilde{\mathbf{E}}_h^n\|_0 + \|\mathbf{C}_h^n\|_{0,\infty}|\tilde{\mathbf{e}}_h^n|_1\big). \tag{6.10a$'$}$$

We define $\bar{h}_\star$ by

$$\bar{h}_\star := \min\left\{\bar{h}_\dagger, 1/c_u, c_u/c_c^2\right\}. \tag{6.15}$$

For any $h \in (0, \bar{h}_\star]$ the estimates (6.10), Lemma 6.2 and (6.15) lead to

$$\left(\tilde{\mathbf{R}}_h^n, \frac{1}{2}\tilde{\mathbf{E}}_h^n\right) \le c\left(\frac{c_u}{h} + c_c^2 + 1\right)\|\tilde{\mathbf{E}}_h^n\|_0^2 + \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 - \frac{1}{4}\|(\mathrm{tr}\,\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2$$
$$\le \frac{c'c_u}{h}\|\tilde{\mathbf{E}}_h^n\|_0^2 + \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 - \frac{1}{4}\|(\mathrm{tr}\,\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2. \tag{6.16}$$

Combining (6.11) and (6.16) with (6.8), we have

$$\overline{D}_{\Delta t}\left(\frac{1}{2}\|\tilde{\mathbf{e}}_h^n\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2\right) + \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 + \delta_0|\tilde{\epsilon}_h^n|_h^2 + \frac{1}{4}\|(\mathrm{tr}\,\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2$$
$$\le \frac{cc_u}{h}\left(\frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2\right) + c_w\left(\frac{1}{2}\|\tilde{\mathbf{e}}_h^{n-1}\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^{n-1}\|_0^2\right). \tag{6.17}$$

We define $\bar{c}_\star$ by

$$\bar{c}_\star := \min\big\{1, 1/(2cc_u)\big\}. \tag{6.18}$$

Since condition (6.2) implies $\Delta t \le h/(2cc_u)$, applying Lemma 5.8 to (6.17) with

$$x^n = \frac{1}{2}\|\tilde{\mathbf{e}}_h^n\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^n\|_0^2, \qquad y^n = \frac{\nu}{2\alpha_1^2}\|\tilde{\mathbf{e}}_h^n\|_1^2 + \delta_0|\tilde{\epsilon}_h^n|_h^2 + \frac{1}{4}\|(\mathrm{tr}\,\tilde{\mathbf{E}}_h^n)\tilde{\mathbf{E}}_h^n\|_0^2,$$
$$a_0 = \frac{cc_u}{h}, \qquad a_1 = 0, \qquad b^n = c_w\left(\frac{1}{2}\|\tilde{\mathbf{e}}_h^{n-1}\|_0^2 + \frac{1}{4}\|\tilde{\mathbf{E}}_h^{n-1}\|_0^2\right),$$

and using the fact $(\tilde{\mathbf{e}}_h^0, \tilde{\mathbf{E}}_h^0) = (\mathbf{0}, \mathbf{0})$, we obtain $(\tilde{\mathbf{e}}_h, \tilde{\epsilon}_h, \tilde{\mathbf{E}}_h) = (\mathbf{0}, 0, \mathbf{0})$, which completes the proof of (ii).

7. Numerical experiments

In this section we present numerical results by scheme (3.3) in order to confirm the theoretical convergence order. For the detailed description of the algorithm we refer to [23].

Example 7.1. In problem (2.1) we set $\Omega = (0,1)^2$ and $T = 0.5$, and we consider three cases for the pair of ν and ε,

$$(\nu, \varepsilon) = (10^{-1}, 10^{-1}),\ (10^{-1}, 10^{-3}),\ (1, 0).$$

The functions $\mathbf{f}$, $\mathbf{F}$, $\mathbf{u}^0$ and $\mathbf{C}^0$ are given such that the exact solution to (2.1) is as follows:

$$\begin{aligned}
\mathbf{u}(x,t) &= \left(\frac{\partial\psi}{\partial x_2}(x,t), -\frac{\partial\psi}{\partial x_1}(x,t)\right), \quad p(x,t) = \sin\{\pi(x_1 + 2x_2 + t)\},\\
C_{11}(x,t) &= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+t)\} + 1,\\
C_{22}(x,t) &= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_2+t)\} + 1,\\
C_{12}(x,t) &= \frac{1}{2}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+x_2+t)\}\ (= C_{21}(x,t)),\\
\psi(x,t) &\coloneqq \frac{\sqrt{3}}{2\pi}\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1+x_2+t)\}.
\end{aligned} \tag{7.1}$$

Note that we set $\mathbf{w} \equiv \mathbf{u}$ in the material derivative $\mathrm{D}/\mathrm{D}t$.

Since Theorem 4.5 holds for any fixed positive constant δ_0, we simply fix $\delta_0 = 1$. Let N be the division number of each side of the square domain. We set $N = 32, 64, 128$ and 256, and (re)define $h \coloneqq 1/N$. The time increment is set as $\Delta t = h/2$.

Let us recall that $\Pi_h^L : C(\bar{\Omega}) \to M_h$ is the Lagrange interpolation operator. We use the same symbol Π_h^L to represent the Lagrange operators on $C(\bar{\Omega})^2$ and $C(\bar{\Omega})^{2\times 2}$. We apply the scheme (3.3) with the initial conditions (4.2), where Π_h^L is employed in place of Π_h for the choice of the initial value $\mathbf{C}_h^0$ in (4.2). Let us note that when the exact conformation tensor $\mathbf{C}(t)$ belongs to $C(\bar{\Omega})^2$, the error estimates (4.4) in Theorem 4.5 hold true also for the choice of initial value with Π_h^L. For the solution $(\mathbf{u}_h, p_h, \mathbf{C}_h)$ of scheme (3.3) and the exact solution $(\mathbf{u}, p, \mathbf{C})$ given by (7.1) we define the relative errors $Er\,i$, $i = 1, \ldots, 6$, by

$$Er\,1 = \frac{\|\mathbf{u}_h - \Pi_h^L\mathbf{u}\|_{\ell^\infty(L^2)}}{\|\Pi_h^L\mathbf{u}\|_{\ell^\infty(L^2)}}, \qquad Er\,2 = \frac{\|\mathbf{u}_h - \Pi_h^L\mathbf{u}\|_{\ell^2(H^1)}}{\|\Pi_h^L\mathbf{u}\|_{\ell^2(H^1)}},$$

$$Er\,3 = \frac{\|p_h - \Pi_h^L p\|_{\ell^2(L^2)}}{\|\Pi_h^L p\|_{\ell^2(L^2)}}, \qquad Er\,4 = \frac{|p_h - \Pi_h^L p|_{\ell^2(|\cdot|_h)}}{\|\Pi_h^L p\|_{\ell^2(L^2)}},$$

$$Er\,5 = \frac{\|\mathbf{C}_h - \Pi_h^L\mathbf{C}\|_{\ell^\infty(L^2)}}{\|\Pi_h^L\mathbf{C}\|_{\ell^\infty(L^2)}}, \qquad Er\,6 = \frac{\|\mathbf{C}_h - \Pi_h^L\mathbf{C}\|_{\ell^2(H^1)}}{\|\Pi_h^L\mathbf{C}\|_{\ell^2(H^1)}}.$$

In the following we show three pairs of table and figure. Table 3 summarizes the symbols used in the figures. Tables & Figures 1, 2 and 3 present the results for the cases $(\nu, \varepsilon) = (10^{-1}, 10^{-1})$, $(10^{-1}, 10^{-3})$ and $(1, 0)$, respectively. In the tables the values of the errors and the slopes are presented, and in the figures the graphs of the errors versus h in logarithmic scale are shown. In each figure the slope of the triangle is equal to 1, which shows the convergence order $O(h)$.

We can see that all the errors except $Er\,6$ for $(\nu, \varepsilon) = (1, 0)$ are almost of the first order in h for all the cases. These results support Theorem 4.5. In the case of $(\nu, \varepsilon) = (1, 0)$ there is no diffusion for $\mathbf{C}$ in equation (2.1c) and the error estimate of the conformation tensor in $\ell^2(H^1)$-seminorm disappear from (4.4). It is, therefore, natural that the slope of $Er\,6$ does not attain 1. Although we do not have any theoretical result for $Er\,3$ at present, scheme (3.3) has produced convergence results also in this norm.

Table 3. Symbols used in the figures.

$\mathbf{u}_h$		p_h		$\mathbf{C}_h$	
○	●	△	▲	□	■
$Er\,1$	$Er\,2$	$Er\,3$	$Er\,4$	$Er\,5$	$Er\,6$

h	$Er\,1$	slope	$Er\,2$	slope
1/32	2.07×10^{-2}	–	2.91×10^{-2}	–
1/64	8.29×10^{-3}	1.32	1.21×10^{-2}	1.27
1/128	3.72×10^{-3}	1.16	5.85×10^{-3}	1.05
1/256	1.77×10^{-3}	1.07	2.60×10^{-3}	1.17
h	$Er\,3$	slope	$Er\,4$	slope
1/32	6.73×10^{-2}	–	5.08×10^{-2}	–
1/64	2.06×10^{-2}	1.71	1.86×10^{-2}	1.45
1/128	6.80×10^{-3}	1.60	8.38×10^{-3}	1.15
1/256	2.59×10^{-3}	1.39	3.68×10^{-3}	1.19
h	$Er\,5$	slope	$Er\,6$	slope
1/32	1.12×10^{-2}	–	4.80×10^{-1}	–
1/64	4.33×10^{-3}	1.37	1.66×10^{-2}	1.54
1/128	1.92×10^{-3}	1.18	6.56×10^{-3}	1.34
1/256	9.09×10^{-4}	1.08	2.90×10^{-3}	1.18

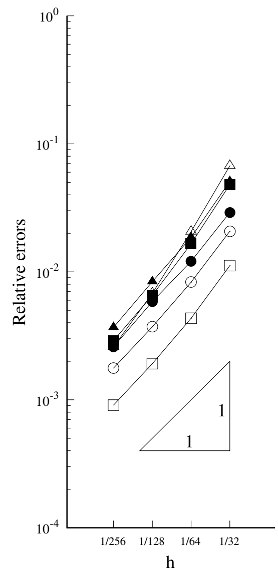

Table & Figure 1. Errors and slopes for $(\nu, \varepsilon) = (10^{-1}, 10^{-1})$.

h	$Er\,1$	slope	$Er\,2$	slope
1/32	1.75×10^{-2}	–	2.71×10^{-2}	–
1/64	6.74×10^{-3}	1.37	1.12×10^{-2}	1.28
1/128	2.91×10^{-3}	1.21	5.49×10^{-3}	1.03
1/256	1.37×10^{-3}	1.09	2.44×10^{-3}	1.17
h	$Er\,3$	slope	$Er\,4$	slope
1/32	9.77×10^{-2}	–	6.56×10^{-2}	–
1/64	3.17×10^{-2}	1.62	2.22×10^{-2}	1.56
1/128	1.02×10^{-2}	1.63	9.01×10^{-3}	1.30
1/256	3.62×10^{-3}	1.50	3.78×10^{-3}	1.25
h	$Er\,5$	slope	$Er\,6$	slope
1/32	2.06×10^{-2}	–	2.76×10^{-1}	–
1/64	7.36×10^{-3}	1.49	1.16×10^{-1}	1.25
1/128	2.93×10^{-3}	1.33	4.40×10^{-2}	1.40
1/256	1.31×10^{-3}	1.17	1.51×10^{-2}	1.54

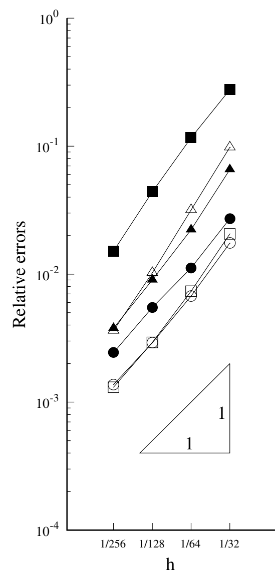

Table & Figure 2. Errors and slopes for $(\nu, \varepsilon) = (10^{-1}, 10^{-3})$.

h	$Er\,1$	slope	$Er\,2$	slope
1/32	1.36×10^{-2}	–	2.30×10^{-2}	–
1/64	4.26×10^{-3}	1.67	9.68×10^{-3}	1.25
1/128	1.40×10^{-3}	1.60	4.84×10^{-3}	1.00
1/256	5.15×10^{-4}	1.44	2.08×10^{-3}	1.22
h	$Er\,3$	slope	$Er\,4$	slope
1/32	2.03×10^{-1}	–	9.39×10^{-2}	–
1/64	6.98×10^{-2}	1.54	3.00×10^{-2}	1.65
1/128	2.16×10^{-2}	1.69	1.19×10^{-2}	1.34
1/256	6.86×10^{-3}	1.66	5.05×10^{-3}	1.23
h	$Er\,5$	slope	$Er\,6$	slope
1/32	2.13×10^{-2}	–	6.71×10^{-1}	–
1/64	7.64×10^{-3}	1.48	5.89×10^{-1}	0.19
1/128	2.81×10^{-3}	1.44	4.51×10^{-1}	0.38
1/256	1.11×10^{-3}	1.37	3.08×10^{-1}	0.55

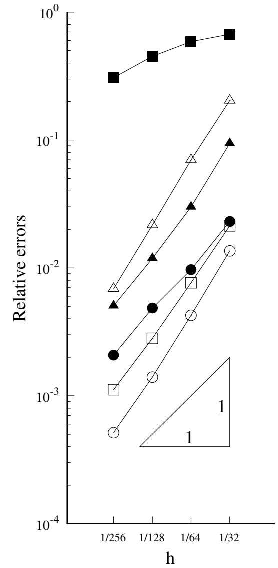

TABLE & FIGURE 3. Errors and slopes for $(\nu, \varepsilon) = (1, 0)$.

8. CONCLUSIONS

We have presented a nonlinear stabilized Lagrange–Galerkin scheme (3.3) for the Oseen-type Peterlin viscoelastic model. The scheme employs the conforming linear finite elements for all unknowns, velocity, pressure and conformation tensor, together with Brezzi–Pitkäranta's stabilization method. In Theorem 4.5 we have established error estimates with the optimal convergence order, which remain true even for $\varepsilon = 0$. We have also presented the result on the uniqueness of the solution of the scheme in Proposition 6.1. It is noted that any solution of the scheme converges to the exact solution without any relation between h and Δt, while the condition (6.1) or (6.2) is needed for the uniqueness of the solution. Theoretical convergence order has been confirmed by two-dimensional numerical experiments.

Although we have dealt with the stabilized scheme to reduce the number of degrees of freedom, the extension of the results to the combination of stable pairs for the velocity and the pressure, and conventional elements for the conformation tensor, *e.g.*, P2/P1/P2 element, is straightforward. Note that our analysis of the stabilized Lagrange-Galerkin method does not require to deal with the dissipation of the discrete free energy and positive definiteness of the conformation tensor $\mathbf{C}_h$, as it was the case of the characteristic-based scheme of Boyaval *et al.* [5] applied to the dissipative Oldroyd-B viscoelastic model. Since the strong solution of the Peterlin model (2.1) indeed satisfies these properties, *cf.* [23], they may be a useful tool in order to extend our numerical analysis to the Peterlin viscoelastic model with the nonlinear convective terms in future.

The extension of the presented scheme to the three-dimensional case is not straightforward due to Lemma 5.5. Three-dimensional problems are fully treated in a forthcoming paper, Part II, by a linear scheme, where the convergence with the best possible order is proved for any of $\varepsilon > 0$.

Acknowledgements. This research was supported by the German Science Agency (DFG) under the grants IRTG 1529 "Mathematical Fluid Dynamics" and TRR 146 "Multiscale Simulation Methods for Soft Matter Systems", and by the Japan Society for the Promotion of Science (JSPS) under the Japanese-German Graduate Externship "Mathematical Fluid Dynamics". H.M. was partially supported by the German Academic Exchange Service. M.L.-M. and H.M. wish to thank B. She (Czech Academy of Science, Prague) for fruitful discussion on the topic. H.N. and M.T. are indebted to JSPS also for Grants-in-Aid for Young Scientists (B), No. 26800091 and for Scientific Research (C), No. 25400212 and Scientific Research (S), No. 24224004, respectively. H.N. is supported by Japan Science and Technology Agency (JST), PRESTO.

References

[1] M. Aboubacar, H. Matallah and M.F. Webster, Highly elastic solutions for Oldroyd-B and Phan-Thien/Tanner fluids with a finite volume/element method: planar contraction flows. *J. Non-Newtonian Fluid Mech.* **103** (2002) 65–103.

[2] R.B. Bird, P.J. Dotson and N.L. Johnson, Polymer-solution rheology based on a finitely extensible bead-spring chain model. *J. Non-Newtonian Fluid Mech.* **7** (1980) 213–235.

[3] A. Bonito, P. Clément and M. Picasso, Mathematical and numerical analysis of a simplified time-dependent viscoelastic flow. *Numer. Math.* **107** (2007) 213–255.

[4] A. Bonito, M. Picasso and M. Laso, Numerical simulation of 3D viscoelastic flows with free surfaces. *J. Comput. Phys.* **215** (2006) 691–716.

[5] S. Boyaval, T. Lelièvre and C. Mangoubi, Free-energy-dissipative schemes for the Oldroyd-B model. *ESAIM: M2AN* **43** (2009) 523–561.

[6] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods. Springer, New York, 3rd edition (2008).

[7] F. Brezzi and J. Douglas Jr. Stabilized mixed methods for the Stokes problem. *Numer. Math.* **53** (1988) 225–235.

[8] F. Brezzi and J. Pitkäranta, On the stabilization of finite element approximations of the Stokes equations. In Efficient Solutions of Elliptic Systems, edited by W. Hackbusch. Wiesbaden. Vieweg (1984) 11–19.

[9] P.G. Ciarlet, The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978).

[10] P. Clément, Approximation by finite element functions using local regularization. *RAIRO Anal. Numér.* **9** (1975) 77–84.

[11] M.J. Crochet and R. Keunings, Finite element analysis of die swell of a highly elastic fluid. *J. Non-Newtonian Fluid Mech.* **10** (1982) 339–356.

[12] R. Fattal and R. Kupferman, Constitutive laws for the matrix-logarithm of the conformation tensor. *J. Non-Newtonian Fluid Mech.* **123** (2004) 281–285.

[13] R. Fattal and R. Kupferman, Time-dependent simulation of viscoelastic flows at high Weissenberg number using the log-conformation representation. *J. Non-Newtonian Fluid Mech.* **126** (2005) 23–37.

[14] R. Keunings, On the high Weissenberg number problem. *J. Non-Newtonian Fluid Mech.* **20** (1986) 209–226.

[15] Y.-J. Lee and J. Xu, New formulations, positivity preserving discretizations and stability analysis for non-Newtonian flow models. *Comput. Methods Appl. Mech. Engrg.* **195** (2006) 1180–1206.

[16] Y.-J. Lee, J. Xu and C.-S. Zhang, Global existence, uniqueness and optimal solvers of discretized viscoelastic flow models. *Math. Models Methods Appl. Sci.* **21** (2011) 1713–1732.

[17] J.L. Lions, Quelques Méthodes de Résolutiondes Problèmes aux Limites Non Linéaires. Dunod et Gauthier-Villars, Paris (1969).

[18] M. Lukáčová−Medvid'ová, H. Mizerová, H. Notsu and M. Tabata, Numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange–Galerkin method, Part II: A linear scheme. *ESAIM: M2AN* **51** (2017) 1663–1689.

[19] M. Lukáčová−Medvid'ová, H. Mizerová and Š. Nečasová, Global existence and uniqueness result for the diffusive Peterlin viscoelastic model. *Nonlin. Anal.: Theory, Methods Appl.* **120** (2015) 154–170.

[20] M. Lukáčová−Medvid'ová, H. Notsu and B. She, Energy dissipative characteristic schemes for the diffusive Oldroyd-B viscoelastic fluid. *Int. J. Numer. Meth. Fluids* **81** (2016) 523–55.

[21] M. Lukáčová−Medvid'ová, H. Mizerová, Š. Nečasová and M. Renardy, Global existence result for the generalized Peterlin viscoelastic model. *SIAM J. Math. Anal.* **49** (2017) 2950–2964.

[22] J.M. Marchal and M.J. Crochet, A new mixed finite element for calculating viscoelastic flow. *J. Non-Newtonian Fluid Mech.* **26** (1987) 77–114.

[23] H. Mizerová, *Analysis and numerical solution of the Peterlin viscoelastic model*. Ph.D. thesis, University of Mainz, Germany (2015).

[24] L. Nadau and A. Sequeira, Numerical simulations of shear-dependent viscoelastic flows with a combined finite element-finite volume method. *Comput. Math. Appl.* **53** (2007) 547–568.

[25] J. Nečas, Les Méthodes Directes en Théories des Équations Elliptiques. Masson, Paris (1967).

[26] H. Notsu and M. Tabata, Error estimates of stable and stabilized Lagrange–Galerkin schemes for natural convection problems. Preprint arXiv:1511.01234 [math.NA] (2015).

[27] H. Notsu and M. Tabata, Error estimates of a pressure-stabilized characteristics finite element scheme for the Oseen equations. *J. Sci. Comput.* **65** (2015) 940–955.

[28] H. Notsu and M. Tabata, Error estimates of a stabilized Lagrange–Galerkin scheme for the Navier–Stokes equations. *ESAIM: M2AN* **50** (2016) 361–380.

[29] A. Peterlin, Hydrodynamics of macromolecules in a velocity field with longitudinal gradient. *J. Polymer Sci. Part B: Polymer Lett.* **4** (1966) 287–291.

[30] M. Picasso and J. Rappaz, Existence, *a priori* and *a posteriori* error estimates for a nonlinear three-field problem arising from Oldroyd-B viscoelastic flows. *ESAIM: M2AN* **35** (2001) 879–897.

[31] M. Renardy, Mathematical Analysis of Viscoelastic Flows. *CBMS-NSF Conference Series in Applied Mathematics*. SIAM, New York **73** (2000).

[32] M. Renardy, Mathematical analysis of viscoelastic fluids. In Vol. 4 of Handbook of Differential Equations: Evolutionary Equations, Amsterdam, North-Holland (2008) 229–265.

[33] M. Renardy, The mathematics of myth: Yield stress behaviour as a limit of non-monotone constitutive theories. *J. Non-Newtonian Fluid Mech.* **165** (2010) 519–526.

[34] M. Renardy and T. Wang, Large amplitude oscillatory shear flows for a model of a thixotropic yield stress fluid. *J. Non-Newtonian Fluid Mech.* **222** (2015) 1–17.

[35] H. Rui and M. Tabata, A second order characteristic finite element scheme for convection-diffusion problems. *Numer. Math.* **92** (2002) 161–177.

[36] M. Tabata and D. Tagami, Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients. *Numer. Math.* **100** (2005) 351–372.

[37] M. Tabata and S. Uchiumi, An exactly computable Lagrange–Galerkin scheme for the Navier–Stokes equations and its error estimates. To appear in *Math. comp.* (2017). DOI: 10.1090/mcom/3222.

[38] R. Temam, Navier–Stokes Equations. North-Holland, Amsterdam (1984).

[39] P. Wapperom, R. Keunings and V. Legat, The backward-tracking Lagrangian particle method for transient viscoelastic flows. *J. Non-Newtonian Fluid Mech.* **91** (2000) 273–295.