CELLULAR AUTOMATA AND POWERS OF $p/q^{\stackrel{\wedge}{\sim}}$

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Abstract. We consider one-dimensional cellular automata $F_{p,q}$ which multiply numbers by p/q in base pq for relatively prime integers p and q. By studying the structure of traces with respect to $F_{p,q}$ we show that for $p \geq 2q-1$ (and then as a simple corollary for p > q > 1) there are arbitrarily small finite unions of intervals which contain the fractional parts of the sequence $\xi(p/q)^n$, $(n=0,1,2,\ldots)$ for some $\xi > 0$. To the other direction, by studying the measure theoretical properties of $F_{p,q}$, we show that for p > q > 1 there are finite unions of intervals approximating the unit interval arbitrarily well which don't contain the fractional parts of the whole sequence $\xi(p/q)^n$ for any $\xi > 0$.

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1. Introduction

In [11] Weyl proved that for any $\alpha > 1$ the sequence of numbers $\{\xi \alpha^i\}$, $i \in \mathbb{N}$ is uniformly distributed in the interval [0,1) for almost every choice of $\xi > 0$, where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. In particular, $\{\{\xi \alpha^i\} \mid i \in \mathbb{N}\}$ is dense in [0,1) for almost every $\xi > 0$. However, this doesn't hold for every $\xi > 0$, and it would be interesting to know what other types of distribution the set $\{\{\xi \alpha^i\} \mid i \in \mathbb{N}\}$ can exhibit for different choices of ξ .

As a special case of this problem, in [8] Mahler posed the question of whether there exist so called Z-numbers, i.e. real numbers $\xi > 0$ such that

$$\left\{\xi\left(\frac{3}{2}\right)^i\right\} \in [0,1/2)$$

for every $i \in \mathbb{N}$. We will work with the following generalization of the notion of Z-numbers: let p > q > 1 be relatively prime integers and let $S \subseteq [0,1)$ be a finite union of intervals. Then if we denote by $Z_{p/q}(S)$ the set of real numbers $\xi > 0$ such that

$$\left\{ \xi \left(\frac{p}{q}\right)^i \right\} \in S$$

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for every $i \in \mathbb{N}$, Z-numbers are the elements of the set $Z_{3/2}([0,1/2))$ and Mahler's question can be reformulated as whether $Z_{3/2}([0,1/2)) = \emptyset$ or not.

A natural approach to the emptiness problem of $Z_{3/2}([0,1/2))$ is to seek sets S as small as possible such that $Z_{p/q}(S) \neq \emptyset$ and sets S as large as possible such that $Z_{p/q}(S) = \emptyset$ (for previous results, see e.g. [1, 2, 3, 4]). In this paper we prove that for $p \geq 2q-1$ and k>0 there exists a union of q^{2k} intervals $I_{p,q,k}$ of total length at most $(q/p)^k$ such that $Z_{p/q}(I_{p,q,k})$ is non-empty. From this it follows as a simple corollary that for p>q and $\epsilon>0$ there exists a finite union of intervals $J_{p,q,\epsilon}$ of total length at most ϵ such that $Z_{p/q}(J_{p,q,\epsilon})$ is non-empty. On the other hand, for p>q and $\epsilon>0$ we prove that there exists a finite union of intervals $K_{p,q,\epsilon}$ of total length at least $1-\epsilon$ such that $Z_{p/q}(K_{p,q,\epsilon})$ is empty. The proofs of emptiness and non-emptiness are based on the study of the cellular automaton $F_{p,q}$ that implements multiplication by p/q in base pq. This cellular automaton was introduced in [7] in relation with the problem of universal pattern generation and the connection to Mahler's problem was pointed out in [6].

2. Preliminaries

For a finite set A (an alphabet) the set $A^{\mathbb{Z}}$ is called a configuration space and its elements are called configurations. An element $c \in A^{\mathbb{Z}}$ is a bi-infinite sequence and the element at position i in the sequence is denoted by c(i). A factor of c is any finite sequence $c(i)c(i+1)\ldots c(j)$ where $i,j\in\mathbb{Z}$, and we interpret the sequence to be empty if j < i. Any finite sequence $a(1)a(2)\ldots a(n)$ (also the empty sequence, which is denoted by λ) where $a(i) \in A$ is a word over A. The set of all words over A is denoted by A^* , and the set of non-empty words is $A^+ = A^* \setminus \{\lambda\}$. The set of words of length n is denoted by A^n . For a word $w \in A^*$, |w| denotes its length, i.e. $|w| = n \iff w \in A^n$.

Definition 2.1. Any $w \in A^+$ and $i \in \mathbb{Z}$ determine a *cylinder*

$${\rm Cyl}_A(w,i) = \{c \in A^{\mathbb{Z}} \mid c(i)c(i+1)\dots c(i+|w|-1) = w\}.$$

The collection of all cylinders over A is

$$C_A = \{ \operatorname{Cyl}_A(w, i) \mid w \in A^+, i \in \mathbb{Z} \}.$$

The subscript A is omitted when the used alphabet is clear from the context.

The configuration space $A^{\mathbb{Z}}$ becomes a topological space when endowed with the topology \mathcal{T} generated by \mathcal{C} . It can be shown that this topology is metrizable, and that a set $S \subseteq A^{\mathbb{Z}}$ is compact if and only if it is closed. $A^{\mathbb{Z}}$ can also be endowed with measure theoretical structure: it is known that there is a unique probability space $(A^{\mathbb{Z}}, \Sigma(\mathcal{C}), \mu)$, where $\Sigma(\mathcal{C})$ is the sigma-algebra generated by \mathcal{C} and $\mu: \Sigma(\mathcal{C}) \to \mathbb{R}$ is a measure such that $\mu(\mathrm{Cyl}(w,i)) = |A|^{-|w|}$ for every $\mathrm{Cyl}(w,i) \in \mathcal{C}$. Note that $\mathcal{T} \subseteq \Sigma(\mathcal{C})$ because \mathcal{C} is a countable basis of \mathcal{T} , so $\Sigma(\mathcal{C})$ is actually the collection of Borel sets of $A^{\mathbb{Z}}$.

Definition 2.2. A one-dimensional cellular automaton (CA) is a 3-tuple (A, N, f), where A is a finite state set, $N = (n_1, \ldots, n_m) \in \mathbb{Z}^m$ is a neighborhood vector and $f: A^m \to A$ is a local rule. A given CA (A, N, f) is customarily identified with a corresponding CA function $F: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by

$$F(c)(i) = f(c(i + n_1), \dots, c(i + n_m))$$

for every $c \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$.

To every configuration space $A^{\mathbb{Z}}$ is associated a (left) shift CA $(A, (1), \iota)$, where $\iota : A \to A$ is the identity function. Put in terms of the CA-function determined by this 3-tuple, the left shift is $\sigma_A : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $\sigma_A(c)(i) = c(i+1)$ for every $c \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$.

c	• • •	0	1	1	0	1	0	0	1	
$\sigma_A(c)$	• • • •	1	1	0	1	0	0	1		
$\sigma_A^2(c)$	• • • •	1	0	1	0	0	1			

FIGURE 1. An example of a space-time diagram.

For a given CA $F: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ and a configuration $c \in A^{\mathbb{Z}}$ it is often helpful to consider a space-time diagram of c with respect to F. A space-time diagram is a picture which depicts elements of the sequence $(F^i(c))_{i \in \mathbb{N}}$ (or possibly $(F^i(c))_{i \in \mathbb{Z}}$ in the case when F is reversible) in such a way that $F^{i+1}(c)$ is drawn below $F^i(c)$ for every i. As an example, Figure 1 contains a space-time diagram of $c = \dots 01101001\dots$ with respect to the left shift on $A = \{0, 1\}$.

All CA-functions are continuous with respect to \mathcal{T} and commute with the shift.

3. The cellular automata $G_{p,q}$ and $F_{p,q}$

In this section we define auxiliary CA $G_{p,q}$ for relatively prime $p,q \geq 2$ and show that they multiply numbers by p in base pq. Then we use $G_{p,q}$ in constructing the CA $F_{p,q}$ which multiply numbers by p/q in base pq, and cover some basic properties of $F_{p,q}$.

Let us denote by A_n the set of digits $\{0, 1, 2, \ldots, n-1\}$ for $n \in \mathbb{N}$, n > 1. To perform multiplication using a CA we need be able to represent a nonnegative real number as a configuration in $A_n^{\mathbb{Z}}$. If $\xi \geq 0$ is a real number and $\xi = \sum_{i=-\infty}^{\infty} \xi_i n^i$ is the unique base n expansion of ξ such that $\xi_i \neq n-1$ for infinitely many i < 0, we define $\operatorname{config}_n(\xi) \in A_n^{\mathbb{Z}}$ by

$$\operatorname{config}_n(\xi)(i) = \xi_{-i}$$

for all $i \in \mathbb{Z}$. In reverse, whenever $c \in A_n^{\mathbb{Z}}$ is such that c(i) = 0 for all sufficiently small i, we define

$$\operatorname{real}_n(c) = \sum_{i=-\infty}^{\infty} c(-i)n^i.$$

For words $w = w(1)w(2)\dots w(k) \in A_n^k$ we define analogously

$$\operatorname{real}_n(w) = \sum_{i=1}^k w(i)n^{-i}.$$

Clearly $\operatorname{real}_n(\operatorname{config}_n(\xi)) = \xi$ and $\operatorname{config}_n(\operatorname{real}_n(c)) = c$ for every $\xi \geq 0$ and every $c \in A_n^{\mathbb{Z}}$ such that c(i) = 0 for all sufficiently small i and $c(i) \neq n-1$ for infinitely many i > 0.

For relatively prime integers $p,q \geq 2$ let $g_{p,q}: A_{pq} \times A_{pq} \to A_{pq}$ be defined as follows. Digits $x,y \in A_{pq}$ are represented as $x = x_1q + x_0$ and $y = y_1q + y_0$, where $x_0, y_0 \in A_q$ and $x_1, y_1 \in A_p$: such representations always exist and they are unique. Then

$$g_{p,q}(x,y) = g_{p,q}(x_1q + x_0, y_1q + y_0) = x_0p + y_1.$$

An example in the particular case (p,q)=(3,2) is given in Figure 2.

The CA function $G_{p,q}: A_{pq}^{\mathbb{Z}} \to A_{pq}^{\mathbb{Z}}, G_{p,q}(c)(i) = g_{p,q}(c(i), c(i+1))$ determined by $(A_{pq}, (0,1), g_{p,q})$ implements multiplication by p in base pq in the sense of the following lemma.

$ \begin{array}{c c} x \setminus y \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} $	0	1	2	3	4	5
0	0	0	1	1	2	2
1	3	3	4	4	5	5
2	0	0	1	1	2	2
3	3	3	4	4	5	5
$_4$	0	0	1	1	2	2
5	3	3	4	4	5	5

FIGURE 2. The values of $g_{p,q}(x,y)$ in the case (p,q)=(3,2).

Lemma 3.1. real_{pq} $(G_{p,q}(\text{config}_{pq}(\xi))) = p\xi \text{ for all } \xi \geq 0.$

Proof. Let $c = \text{config}_{pq}(\xi)$. For every $i \in \mathbb{Z}$, denote by $c(i)_0$ and $c(i)_1$ the natural numbers such that $0 \le c(i)_0 < q$, $0 \le c(i)_1 < p$ and $c(i) = c(i)_1 q + c(i)_0$. Then

$$\operatorname{real}_{pq}(G_{p,q}(\operatorname{config}_{pq}(\xi))) = \operatorname{real}_{pq}(G_{p,q}(c)) = \sum_{i=-\infty}^{\infty} G_{p,q}(c)(-i)(pq)^{i}$$

$$= \sum_{i=-\infty}^{\infty} g_{p,q}(c(-i), c(-i+1))(pq)^{i} = \sum_{i=-\infty}^{\infty} (c(-i)_{0}p + c(-i+1)_{1})(pq)^{i}$$

$$= \sum_{i=-\infty}^{\infty} (c(-i)_{0}p(pq)^{i} + c(-i+1)_{1}pq(pq)^{i-1})$$

$$= \sum_{i=-\infty}^{\infty} (c(-i)_{0}p(pq)^{i} + c(-i)_{1}pq(pq)^{i})$$

$$= p \sum_{i=-\infty}^{\infty} (c(-i)_{1}q + c(-i)_{0})(pq)^{i} = p \operatorname{real}_{pq}(c) = p \operatorname{real}_{pq}(\operatorname{config}_{pq}(\xi)) = p\xi.$$

We also define $G_{p,q}(w)$ for words $w = w(1)w(2) \dots w(|w|)$ such that $|w| \geq 2$:

$$G_{p,q}(w) = u = u(1) \dots u(|w| - 1) \in A_{pq}^{|w|-1},$$

where $u(i) = g_{p,q}(w(i), w(i+1))$ for $1 \le i \le |w| - 1$. Inductively it is possible to define $G_{p,q}^t(w)$ for every t > 0 and word w such that $|w| \ge t + 1$:

$$G_{p,q}^t(w) = G_{p,q}(G_{p,q}^{t-1}(w)) \in A_{pq}^{|w|-t}.$$

Clearly the shift CA $\sigma_{A_{pq}}$ multiplies by pq in base pq and its inverse divides by pq. This combined with Lemma 3.1 shows that the composition $F_{p,q} = \sigma_{A_{pq}}^{-1} \circ G_{p,q} \circ G_{p,q}$ implements multiplication by p/q in base pq. The value of $F_{p,q}(c)(i)$ is given by the local rule $f_{p,q}$ defined as follows:

$$\begin{split} F_{p,q}(c)(i) = & f_{p,q}(c(i-1),c(i),c(i+1)) \\ = & g_{p,q}(g_{p,q}(c(i-1),c(i)),g_{p,q}(c(i),c(i+1))). \end{split}$$

FIGURE 3. Iterated application of $F_{p,q}$ on w for (p,q)=(3,2) and w=3434205.

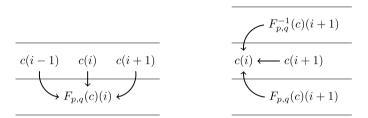


Figure 4. Determination of digits in the space-time diagram of c with respect to $F_{p,q}$.

The CA function $F_{p,q}$ is reversible: if $c \in A_{pq}^{\mathbb{Z}}$ is a configuration with a finite number of non-zero coordinates, then

$$\begin{split} F_{p,q}(F_{q,p}(c)) &= F_{p,q}(F_{q,p}(\text{config}_{pq}(\text{real}_{pq}(c)))) \\ \stackrel{L3.1}{=} \text{config}_{pq}((p/q)(q/p) \, \text{real}_{pq}(c)) &= c. \end{split}$$

Since $F_{p,q} \circ F_{q,p}$ is continuous and agrees with the identity function on a dense set, it follows that $F_{p,q}(F_{q,p}(c)) = c$ for all configurations $c \in A_{pq}^{\mathbb{Z}}$. We will denote the inverse of $F_{p,q}$ interchangeably by $F_{q,p}$ and $F_{p,q}^{-1}$. As for $G_{p,q}$, we define $F_{p,q}(w)$ for words $w = w(1)w(2)\dots w(|w|)$ such that $|w| \geq 3$:

$$F_{p,q}(w) = u = u(1) \dots u(|w| - 2) \in A_{pq}^{|w|-2},$$

where $u(i) = f_{p,q}(w(i), w(i+1), w(i+2))$ for $1 \le i \le |w| - 2$, and $F_{p,q}^t(w)$ for every t > 0 and word w such that $|w| \ge 2t + 1$:

$$F_{p,q}^t(w) = F_{p,q}(F_{p,q}^{t-1}(w)) \in A_{pq}^{|w|-2t}$$

(see an example in Fig. 3).

By the definition of $F_{p,q}$, for every $c \in A_{pq}^{\mathbb{Z}}$ and every $i \in \mathbb{Z}$ the value of $F_{p,q}(c)(i)$ is uniquely determined by c(i-1), c(i) and c(i+1), the three nearest digits above in the space-time diagram. Proposition 3.5 gives similarly that each digit in the space-time diagram is determined by the three nearest digits to the right (see Fig. 4). Its proof is broken down into the following sequence of lemmas.

Lemma 3.2. If $g_{p,q}(x,z) = g_{p,q}(y,w)$, then $x \equiv y \pmod{q}$.

Proof. Let $x = x_1q + x_0$, $y = y_1q + y_0$, $z = z_1q + z_0$ and $w = w_1q + w_0$. Then

$$g_{p,q}(x,z) = g_{p,q}(y,w) \implies x_0p + z_1 = y_0p + w_1$$

 $\implies x_0 = y_0 \implies x \equiv y \pmod{q}.$

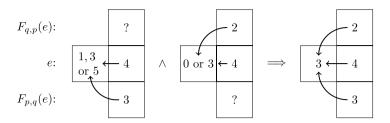


FIGURE 5. The proof of Proposition 3.5 (here (p,q)=(3,2)).

Lemma 3.3. $g_{p,q}(x,a) \equiv g_{p,q}(y,a) \pmod{q} \iff x \equiv y \pmod{q}$.

Proof. Let $x = x_1q + x_0$, $y = y_1q + y_0$ and $a = a_1q + a_0$. Then

$$g_{p,q}(x,a) \equiv g_{p,q}(y,a) \pmod{q} \iff x_0p + a_1 \equiv y_0p + a_1 \pmod{q}$$

 $\iff x_0 = y_0 \iff x \equiv y \pmod{q}$

Lemma 3.4. If $f_{p,q}(x, a, y) = f_{p,q}(z, a, w)$, then $x \equiv z \pmod{q}$.

Proof.

$$f_{p,q}(x, a, y) = f_{p,q}(z, a, w)$$

$$\Longrightarrow g_{p,q}(g_{p,q}(x, a), g_{p,q}(a, y)) = g_{p,q}(g_{p,q}(z, a), g_{p,q}(a, w))$$

$$\stackrel{L3.2}{\Longrightarrow} g_{p,q}(x, a) \equiv g_{p,q}(z, a) \pmod{q} \stackrel{L3.3}{\Longrightarrow} x \equiv z \pmod{q}.$$

Proposition 3.5. For every $c \in A_{pq}^{\mathbb{Z}}$ and for all $k, i \in \mathbb{Z}$, the value of $F_{p,q}^k(c)(i)$ is uniquely determined by the values of $F_{p,q}^{k-1}(c)(i+1)$, $F_{p,q}^k(c)(i+1)$ and $F_{p,q}^{k+1}(c)(i+1)$.

Proof. Denote $e = \sigma_{A_{pq}}^i(F_{p,q}^k(c))$. It suffices to show that e(0) is uniquely determined by $F_{q,p}(e)(1)$, e(1) and $F_{p,q}(e)(1)$. Because $F_{p,q}(e)(1) = f_{p,q}(e(0), e(1), e(2))$, by Lemma 3.4 e(1) and $F_{p,q}(e)(1)$ determine the value of e(0) modulo q (see Fig. 5, left). Similarly, because $F_{q,p}(e)(1) = f_{q,p}(e(0), e(1), e(2))$, by the same lemma e(1) and $F_{q,p}(e)(1)$ determine the value of e(0) modulo p (Fig. 5, middle). In total, $F_{q,p}(e)(1)$, e(1) and $F_{p,q}(e)(1)$ determine the value of e(0) both modulo q and modulo p (Fig. 5, right). Because $e(0) \in A_{pq}$, the value of e(0) is uniquely determined.

4. Traces of configurations

For $\xi \geq 0$ we are interested in the values of $\{\xi(p/q)^i\}$ as i ranges over \mathbb{N} . In terms of the configuration $\operatorname{config}_{pq}(\xi)$ these correspond to the tails of the configurations $F_{p,q}^i(\operatorname{config}_{pq}(\xi))$, i.e. to the digits $F_{p,q}^i(\operatorname{config}_{pq}(\xi))(j)$ for j>0. Partial information on the tails is preserved in the traces of a configuration. In this section we study traces with respect to $F_{p,q}$ to prove in the case $p\geq 2q-1$ the existence of small sets S such that $Z_{p/q}(S)$ is non-empty, and then as a corollary for all p>q>1.

				1 1				
$F_{3,2}^{-2}(c)$	 5	4	0	1	5	3	4	
$F_{3,2}^{-1}(c)$	 2						3	
c	 3	4	3	4	2	0	5	
$F_{3,2}(c)$	 5			3			2	
$F_{3,2}^{2}(c)$	 5	2	5	2	1	5	1	

FIGURE 6. A trace of a configuration.

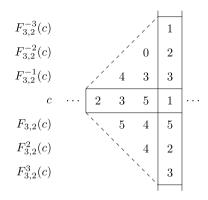


FIGURE 7. A trace determining part of the configuration.

Definition 4.1. For any $k \in \mathbb{Z}$, the k-trace of a configuration $c \in A_{pq}^{\mathbb{Z}}$ (with respect to $F_{p,q}$) is the sequence

$$\operatorname{Tr}_{p,q}(c,k) = (F_{p,q}^n(c)(k))_{n \in \mathbb{Z}}.$$

In the special case k = 1, we denote $\operatorname{Tr}_{p,q}(c,1) = \operatorname{Tr}_{p,q}(c)$.

A k-trace of c is simply the sequence of digits in the k-th column of the space-time diagram of c with respect to $F_{p,q}$ (see Fig. 6).

Definition 4.2. The set of allowed words of $Tr_{p,q}$ is

$$L(p,q) = \{ w \in A_{pq}^* \mid w \text{ is a factor of } \operatorname{Tr}_{p,q}(c) \text{ for some } c \in A_{pq}^{\mathbb{Z}} \},$$

i.e. the set of words that can appear in the columns of space-time diagrams with respect to $F_{p,q}$.

The following is a reformulation of Proposition 3.5 in terms of traces (see Fig. 7).

Corollary 4.3. For every $c \in A_{pq}^{\mathbb{Z}}$ and k > 0, the values of $\operatorname{Tr}_{p,q}(c,k)(i)$ for $-(k-1) \leq i \leq (k-1)$ uniquely determine the values of c(j) for $1 \leq j \leq k$.

Proof. The proof is by induction. The case k=1 follows from the fact that $\operatorname{Tr}_{p,q}(c,1)(0)=c(1)$. Next assume that the claim holds for some k>0 and consider the values of $\operatorname{Tr}_{p,q}(c,k+1)(i)$ for $-k\leq i\leq k$. By Proposition 3.5 these determine $\operatorname{Tr}_{p,q}(c,k)(i)$ for $-(k-1)\leq i\leq (k-1)$, which in turn determine c(j) for $1\leq j\leq k$ by the induction hypothesis. The value of c(k+1) is determined by $\operatorname{Tr}_{p,q}(c,k+1)(0)=c(k+1)$.

Next we prove an important restriction on the words in the set L(q, p) when $p \ge 2q - 1$. Note that the words in L(q, p) are mirror images of the words in L(p, q) (traces with respect to $F_{p,q}$ are read "from bottom to top").

Lemma 4.4. Let $p > q \ge 2$ be relatively prime such that $p \ge 2q - 1$, and for every $d \in A_q$ let $k_d \in A_p$ and $j_d \in A_q$ be the unique elements such that $k_d q \equiv d \pmod{p}$ and $k_d q = j_d p + d$. If $wab \in L(q, p)$ for some $w \in A_{pq}^*$, $a, b \in A_{pq}$ and $a \equiv k_d \pmod{p}$, then $b \equiv j_d \pmod{q}$.

Proof. From $wab \in L(q, p)$ it follows that $b = f_{q,p}(x, a, y)$ for some $x, y \in A_{pq}$. Let us write $a = a_1p + a_0, y = y_1p + y_0, g_{q,p}(x, a) = z = z_1p + z_0$ and $g_{q,p}(a, y) = w = w_1p + w_0$, where $a_0, y_0, z_0, w_0 \in A_p$ and $a_1, y_1, z_1, w_1 \in A_q$. Here $a_0 = k_d$ because $a \equiv k_d \pmod{p}$ and $w_1 = j_d$ because $g_{q,p}(a, y) = k_d q + y_1 = j_d p + (d + y_1)$ and $d + y_1 \le (q - 1) + (q - 1) < p$. Now

$$f_{q,p}(x,a,y) = g_{q,p}(g_{q,p}(x,a), g_{q,p}(a,y)) = g_{q,p}(z,w) = z_0q + j_d,$$

and thus $b \equiv j_d \pmod{q}$.

Based on the previous lemma, we define a special set of digits

$$D_{p,q} = \{ a \in A_{pq} \mid a \equiv k_d \pmod{p} \text{ for some } d \in A_q \},$$

where the digits k_d are as above.

Example 4.5. Consider the case p=3 and q=2. Then $A_q=\{0,1\}$ and $D_{3,2}=\{0,2,3,5\}$ consists of the elements of A_6 which are congruent to either $k_0=0$ or $k_1=2\pmod{3}$.

Lemma 4.6. If $p \ge 2q - 1$, then $|L(p,q) \cap D_{p,q}^n| \le q^{n+1}$ for every n > 0.

Proof. The proof is by induction. The case n=1 is clear because $|D_{p,q}|=q^2$. Next assume that the claim holds for some n>0. It is sufficient to compute an upper bound for $|L(q,p)\cap D^{n+1}_{p,q}|$, because the words in L(p,q) are mirror images of the words in L(q,p). If $v\in L(q,p)\cap D^{n+1}_{p,q}$, by the previous lemma it can be written in the form v=wab, where $a\equiv k_d\pmod p$ and $b\equiv j_d\pmod q$ for some $d\in A_q$. Because $wa\in L(q,p)\cap D^n_{p,q}$, by the induction hypothesis there are at most q^{n+1} different choices for the word wa. Let us fix wa and $d\in A_q$ such that $a\equiv k_d\pmod p$. To prove the claim, it is enough to show that there are at most q choices for the digit b.

Let us assume to the contrary that there are distinct digits $b_1, b_2, \dots b_{q+1} \in D_{p,q}$ such that $wab_i \in L(q,p) \cap D_{p,q}^{n+1}$ whenever $1 \le i \le q+1$. For every i the congruence $b_i \equiv k_{d_i} \pmod{p}$ holds for some $d_i \in A_q$. By pigeonhole principle we may assume that $d_1 = d_2$ and therefore $b_1 \equiv k_{d_1} \equiv b_2 \pmod{p}$. Because $wab_1, wab_2 \in L(q,p) \cap D_{p,q}^{n+1}$, we also have $b_1 \equiv j_d \equiv b_2 \pmod{q}$. Because $b_1, b_2 \in A_{pq}$ are congruent both modulo p and modulo p, they are equal, contradicting the distinctness of $b_1, b_2, \dots b_{q+1}$.

As in the introduction, for relatively prime p > q > 1 and any $S \subseteq [0,1)$ we denote

$$Z_{p/q}(S) = \left\{ \xi > 0 \mid \left\{ \xi \left(\frac{p}{q} \right)^i \right\} \in S \text{ for every } i \in \mathbb{N} \right\}.$$

In [1] it was proved that if p, q > 1 are relatively prime integers such that $p > q^2$, then for every $\epsilon > 0$ there exists a finite union of intervals $J_{p,q,\epsilon}$ of total length at most ϵ such that $Z_{p/q}(J_{p,q,\epsilon}) \neq \emptyset$. We extend this result to the case p > q > 1, which in particular covers p/q = 3/2. The following theorem by Akiyama, Froughy and Sakarovitch is needed.

Theorem 4.7 (Akiyama, Frougny, Sakarovitch [2]). If $p \geq 2q - 1$, then $Z_{p/q}(Y_{p,q}) \neq \emptyset$, where

$$Y_{p,q} = \bigcup_{d \in A_a} \left[\frac{1}{p} k_d, \frac{1}{p} (k_d + 1) \right)$$

and $k_d \in A_p$ are as in Lemma 4.4.

Corollary 4.8. If $p \geq 2q - 1$, then $Z_{p/q}(X_{p,q}) \neq \emptyset$, where

$$X_{p,q} = \bigcup_{a \in D_{p,q}} \left[\frac{1}{pq} a, \frac{1}{pq} (a+1) \right).$$

Proof. If $\xi \in Z_{p/q}(Y_{p,q})$, then $\xi/q \in Z_{p/q}(X_{p,q})$.

Theorem 4.9. If $p \ge 2q - 1$ and k > 0, then there exists a finite union of intervals $I_{p,q,k}$ of total length at $most (q/p)^k$ such that $Z_{p/q}(I_{p,q,k}) \ne \emptyset$.

Proof. Let k > 0 be fixed and choose any $\xi' \in Z_{p/q}(X_{p,q})$, where $X_{p,q}$ is the set in the previous corollary. Let $\xi = \xi'(pq)^{-(k-1)}(p/q)^{k-1}$ and denote $c = \text{config}_{pq}(\xi)$. Based on c we define a collection of words

$$W = \{ w = e(1)e(2) \dots e(k) \mid e = F_{p,q}^n(c) \text{ for some } n \in \mathbb{N} \}.$$

The set W determines a finite union of intervals

$$I_{p,q,k} = \bigcup_{w \in W} \left[\operatorname{real}_{pq}(w), \operatorname{real}_{pq}(w) + (pq)^{-k} \right)$$

and $\xi \in Z_{p/q}(I_{p,q,k})$ by the definition of W. Each interval in $I_{p,q,k}$ has length $(pq)^{-k}$, so to prove that the total length of $I_{p,q,k}$ is at most $(q/p)^k$ it is sufficient to show that $|W| \leq q^{2k}$.

By the definition of $X_{p,q}$, $\text{Tr}_{p,q}(\text{config}_{pq}(\xi'))(i) \in D_{p,q}$ for every $i \geq 0$. For the k-trace of c

$$\begin{split} & \operatorname{Tr}_{p,q}(c,k)(i) = \operatorname{Tr}_{p,q}(\operatorname{config}_{pq}(\xi'(pq)^{-(k-1)}(p/q)^{k-1}), k)(i) \\ & = \operatorname{Tr}_{p,q}(\sigma_{A_{pq}}^{-(k-1)}(F_{p,q}^{k-1}(\operatorname{config}_{pq}(\xi'))), k)(i) = \operatorname{Tr}_{p,q}(F_{p,q}^{k-1}(\operatorname{config}_{pq}(\xi')), 1)(i) \\ & = \operatorname{Tr}_{p,q}(\operatorname{config}_{pq}(\xi'))(i+(k-1)) \text{ for every } i \in \mathbb{N}, \end{split}$$

from which it follows that $\operatorname{Tr}_{p,q}(c,k)(i) \in D_{p,q}$ for every $i \geq -(k-1)$. Thus, the words in the set

$$V = \{ \operatorname{Tr}_{p,q}(F_{p,q}^{n}(c), k)(-(k-1)) \dots \operatorname{Tr}_{p,q}(F_{p,q}^{n}(c), k)(k-1) \mid n \in \mathbb{N} \}$$

also belong to $L(p,q)\cap D^{2k-1}_{p,q},$ and by Corollary 4.3 and Lemma 4.6

$$|W| \le |V| \le |L(p,q) \cap D_{p,q}^{2k-1}| \le q^{2k}$$
.

Remark 4.10. The set $I_{p,q,k}$ constructed in the proof of the previous theorem is a union of q^{2k} intervals, each of which is of length $(pq)^{-k}$.

Corollary 4.11. If p > q > 1 and $\epsilon > 0$, then there exists a finite union of intervals $J_{p,q,\epsilon}$ of total length at most ϵ such that $Z_{p/q}(J_{p,q,\epsilon}) \neq \emptyset$.

Proof. Choose some n > 0 such that $p^n \ge 2q^n - 1$. Then by the previous theorem there exists a finite union of intervals I_0 of total length at most $\eta = \epsilon(p-1)/(p^n-1)$ such that $Z_{p^n/q^n}(I_0) \ne \emptyset$. For 0 < i < n define inductively

$$I_i = \left\{ \left\{ \xi \frac{p}{q} \right\} \in [0, 1) \mid \xi \ge 0 \text{ and } \{\xi\} \in I_{i-1} \right\},$$

each of which is a finite union of intervals of total length at most $p^i\eta$. Then $J_{p,q,\epsilon} = \bigcup_{i=0}^{n-1} I_i$ is a finite union of intervals of total length at most

$$\sum_{i=0}^{n-1} (p^i) \eta = \frac{p^n - 1}{p - 1} \eta = \epsilon$$

and $Z_{p/q}(J_{p,q,\epsilon}) \supseteq Z_{p^k/q^k}(I_0) \neq \emptyset$.

5. Ergodicity of $F_{p,q}$

In this section we study the measure theoretical properties of $F_{p,q}$ to prove the existence of large sets S such that $Z_{p/q}(S)$ is empty.

Definition 5.1. A CA function $F: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is measure preserving if $\mu(F^{-1}(S)) = \mu(S)$ for every $S \in \Sigma(\mathcal{C})$.

Definition 5.2. A measure preserving CA function $F: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is *ergodic* if for every $S \in \Sigma(\mathcal{C})$ with $F^{-1}(S) = S$ either $\mu(S) = 0$ or $\mu(S) = 1$.

The next lemma is a special case of a well known measure theoretical result (see e.g. Thm. 2.18 in [9]):

Lemma 5.3. For every $S \in \Sigma(\mathcal{C})$ and $\epsilon > 0$ there is an open set $U \subseteq A^{\mathbb{Z}}$ such that $S \subseteq U$ and $\mu(U \setminus S) < \epsilon$.

Lemma 5.4. If $F: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is an ergodic CA, then for every $\epsilon > 0$ there is a finite collection of cylinders $\{U_i\}_{i \in I}$ such that $\mu(\bigcup_{i \in I} U_i) < \epsilon$ and

$$\left\{c \in A^{\mathbb{Z}} \mid F^{t}(c) \in \bigcup_{i \in I} U_{i} \text{ for some } t \in \mathbb{N}\right\} = A^{\mathbb{Z}}.$$

Proof. Let $C \in \mathcal{C}$ be such that $0 < \mu(C) < \epsilon/2$. By continuity of F, $B = \bigcup_{t \in \mathbb{N}} F^{-t}(C)$ is open and $\mu(B) = 1$ by ergodicity of F (see Thm. 1.5 in [10]). Equivalently, $B' = A^{\mathbb{Z}} \setminus B$ is closed (and compact) and $\mu(B') = 0$. Let V be an open set such that $B' \subseteq V$ and $\mu(V) < \epsilon/2$: such a set exists by Lemma 5.3. Because \mathcal{C} is a basis of \mathcal{T} , there is a collection of cylinders $\{V_i\}_{i \in I}$ such that $V = \bigcup_{i \in I} V_i$. By compactness of B' there is a finite set $I' \subseteq J$ such that $B' \subseteq \bigcup_{i \in I'} V_i$. Now $\{U_i\}_{i \in I} = \{C\} \cup \{V_i\}_{i \in I'}$ is a finite collection of cylinders such that $\mu(\bigcup_{i \in I} U_i) < \epsilon$ and

$$\left\{c \in A^{\mathbb{Z}} \mid F^t(c) \in \bigcup_{i \in I} U_i \text{ for some } t \in \mathbb{N}\right\} \supseteq B \cup \bigcup_{i \in I'} V_i \supseteq B \cup B' = A^{\mathbb{Z}}.$$

To apply this lemma in our setup, we need to show that $F_{p,q}$ is ergodic for p > q > 1. In fact, it turns out that a stronger result holds.

Definition 5.5. A measure preserving CA function $F: A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is strongly mixing if

$$\lim_{t \to \infty} \mu(F^{-t}(U) \cap V) = \mu(U)\mu(V)$$

for every $U, V \in \Sigma(\mathcal{C})$.

We will prove that $F_{p,q}$ is strongly mixing. For the statement of the following lemmas, we define a function $\operatorname{int}: A_{pq}^+ \to \mathbb{N}$ by

$$\operatorname{int}(w(1)w(2)\dots w(k)) = \sum_{i=0}^{k-1} w(k-i)(pq)^i,$$

i.e. int(w) is the integer having w as a base pq representation.

Lemma 5.6. Let $w_1, w_2 \in A_{pq}^k$ for some $k \geq 2$ and let t > 0 be a natural number. Then

1.
$$\operatorname{int}(w_1) < q^t \implies \operatorname{int}(G_{p,q}(w_1)) < q^{t-1} \text{ and}$$

2. $\operatorname{int}(w_2) \equiv \operatorname{int}(w_1) + q^t \pmod{(pq)^k} \implies \operatorname{int}(G_{p,q}(w_2)) \equiv \operatorname{int}(G_{p,q}(w_1)) + q^{t-1} \pmod{(pq)^{k-1}}.$

Proof. Let $c_i \in A_{pq}^{\mathbb{Z}}$ (i = 1, 2) be such that $c_i(-(k-1))c_i(-(k-1)+1)\ldots c_i(0) = w_i$ and $c_i(j) = 0$ for j < -(k-1) and j > 0. From this definition of c_i it follows that $\operatorname{int}(w_i) = \operatorname{real}_{pq}(c_i)$. Denote $e_i = G_{p,q}(c_i)$. We have

$$\sum_{j=-\infty}^{\infty} e_i(-j)(pq)^j = \operatorname{real}_{pq}(e_i) = p \operatorname{real}_{pq}(c_i) = p \operatorname{int}(w_i)$$

and

$$\inf(G_{p,q}(w_i)) = \inf(e_i(-(k-1)) \dots e_i(-1))$$

$$= \sum_{j=1}^{k-1} e_i(-j)(pq)^{j-1} \equiv [\inf(w_i)/q] \pmod{(pq)^{k-1}}.$$

Also note that $int(G_{p,q}(w_i)) < (pq)^{k-1}$.

For the proof of the first part, assume that $int(w_1) < q^t$. Combining this with the observations above yields $\operatorname{int}(G_{p,q}(w_i)) \leq |\operatorname{int}(w_i)/q| < q^{t-1}.$

For the proof of the second part, assume that $int(w_2) \equiv int(w_1) + q^t \pmod{(pq)^k}$. Then there exists $n \in \mathbb{Z}$ such that $int(w_2) = int(w_1) + q^t + n(pq)^k$ and

$$\inf(G_{p,q}(w_2)) \equiv \left[\inf(w_2)/q \right] \equiv \left[\inf(w_1)/q \right] + q^{t-1} + np(pq)^{k-1}$$
$$\equiv \left[\inf(w_1)/q \right] + q^{t-1} \equiv \inf(G_{p,q}(w_1)) + q^{t-1} \pmod{(pq)^{k-1}}.$$

Lemma 5.7. Let t > 0 and $w_1, w_2 \in A_{pq}^k$ for some $k \ge 2t + 1$. Then

1.
$$\operatorname{int}(w_1) < q^{2t} \implies \operatorname{int}(F_{p,q}^t(w_1)) = 0$$
 and
2. $\operatorname{int}(w_2) \equiv \operatorname{int}(w_1) + q^{2t} \pmod{(pq)^k} \implies \operatorname{int}(F_{p,q}^t(w_2)) \equiv \operatorname{int}(F_{p,q}^t(w_1)) + 1 \pmod{(pq)^{k-2t}}.$

Proof. First note that $F_{p,q}(w) = G_{p,q}^2(w)$ for every $w \in A_{pq}^*$ such that $|w| \ge 3$, because $F_{p,q} = \sigma_{A_{pq}}^{-1} \circ G_{p,q} \circ G_{p,q}$. Then both claims follow by repeated application of the previous lemma.

The content of Lemma 5.7 is as follows. Assume that $\{w_i\}_{i=0}^{(pq)^k-1}$ is the enumeration of all the words in A_{pq}^k in the lexicographical order, meaning that $w_0 = 00 \dots 00, w_1 = 00 \dots 01, w_2 = 00 \dots 02$ and so on. Then let i run through all the integers between 0 and $(pq)^k - 1$. For the first q^{2t} values of i we have $F_{p,q}^t(w_i) = 00...00$, for the next q^{2t} values of i we have $F_{p,q}^t(w_i) = 00...01$, and for the following q^{2t} values of i we have $F_{p,q}^t(w_i) = 00...02$. Eventually, as i is incremented from $q^{2t}(pq)^{k-2t} - 1$ to $q^{2t}(pq)^{k-2t}$, the word $F_{p,q}^t(w_i)$ loops from $(pq-1)(pq-1)\dots(pq-1)(pq-1)$ back to $00\dots00$.

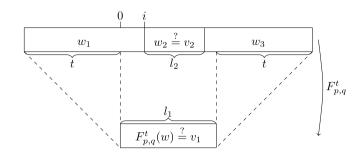


FIGURE 8. Relations between the words v_1 , v_2 and $w_1w_2w_3$.

Theorem 5.8. If p > q > 1, then $F_{p,q}$ is strongly mixing.

Proof. Firstly, because $F_{p,q}$ is surjective, the fact that $F_{p,q}$ is measure preserving follows from Theorem 5.4 in [5]. Then, by Theorem 1.17 in [10] it is sufficient to verify the condition

$$\lim_{t \to \infty} \mu(F_{p,q}^{-t}(C_1) \cap C_2) = \mu(C_1)\mu(C_2)$$

for every $C_1, C_2 \in \mathcal{C}$. Without loss of generality we may consider cylinders $C_1 = \text{Cyl}(v_1, 0)$ and $C_2 = \text{Cyl}(v_2, i)$.

Denote $l_1 = |v_1|$, $l_2 = |v_2|$ and let $t \ge i + l_2$ be a natural number. Consider an arbitrary word $w \in A_{pq}^{2t+l_1}$ and its decomposition $w = w_1 w_2 w_3$, where $w_1 \in A_{pq}^{t+i}$, $w_2 \in A_{pq}^{l_2}$ and $w_3 \in A_{pq}^{t+l_1-i-l_2}$. The following conditions may or may not be satisfied by w (see Fig. 8):

- 1. $F_{p,q}^t(w) = v_1$ 2. $w_2 = v_2$.

Note that if w satisfies condition (1), then $F_{p,q}^t(\mathrm{Cyl}(w,-t)) \subseteq C_1$, and otherwise $F_{p,q}^t(\mathrm{Cyl}(w,-t)) \cap C_1 = \emptyset$. Also, if w satisfies condition (2), then $\mathrm{Cyl}(w,-t)\subseteq C_2$, and otherwise $\mathrm{Cyl}(w,-t)\cap C_2=\emptyset$. Let $W_t\subseteq A_{pq}^{2t+l_1}$ be the collection of those words w that satisfy both conditions. It follows that

$$\mu(F_{p,q}^{-t}(C_1) \cap C_2) = \mu\left(\bigcup_{w \in W_t} \text{Cyl}(w, -t)\right) = |W_t|(pq)^{-(2t+l_1)}.$$

Next, we estimate the number of words $w = w_1 w_2 w_3$ in W_t . In any case, to satisfy condition (2), w_2 must equal v_2 . Then, for any of the $(pq)^{t+i}$ choices of w_1 , the number of choices for w_3 that satisfy condition (1) is between $(pq)^{t+l_1-i-l_2}/(pq)^{l_1}-q^{2t}$ and $(pq)^{t+l_1-i-l_2}/(pq)^{l_1}+q^{2t}$ by Lemma 5.7 (and the paragraph following it). Thus,

$$((pq)^{t-i-l_2} - q^{2t}) (pq)^{t+i} (pq)^{-(2t+l_1)} \le \mu(F_{p,q}^{-t}(C_1) \cap C_2)$$

$$\le ((pq)^{t-i-l_2} + q^{2t}) (pq)^{t+i} (pq)^{-(2t+l_1)},$$

and as t tends to infinity,

$$\lim_{t \to \infty} \mu(F_{p,q}^{-t}(C_1) \cap C_2) = (pq)^{-l_1 - l_2} = \mu(C_1)\mu(C_2).$$

Theorem 5.9. If p > q > 1 and $\epsilon > 0$, then there exists a finite union of intervals $K_{p,q,\epsilon}$ of total length at least $1 - \epsilon$ such that $Z_{p/q}(K_{p,q,\epsilon}) = \emptyset$.

Proof. The previous theorem implies that $F_{p,q}$ is ergodic: if $S \in \Sigma(\mathcal{C})$ is such that $F_{p,q}^{-1}(S) = S$, then

$$\mu(S) = \lim_{t \to \infty} \mu(F_{p,q}^{-t}(S) \cap S) = \mu(S)\mu(S),$$

which means that $\mu(S) = 0$ or $\mu(S) = 1$.

Since $F_{p,q}$ is ergodic, by Lemma 5.4 there is a finite collection of cylinders $\{U_i\}_{i\in I}$ such that $\mu(\bigcup_{i\in I} U_i) < \epsilon$ and

$$\left\{c \in A_{pq}^{\mathbb{Z}} \mid F_{p,q}^t(c) \in \bigcup_{i \in I} U_i \text{ for some } t \in \mathbb{N}\right\} = A_{pq}^{\mathbb{Z}}.$$

Without loss of generality we may assume that for every $i \in I$, $U_i = \text{Cyl}(w_i, 1)$ and $w_i \in A_{pq}^k$ for a fixed k > 0. Consider the collection of words $W = A_{pq}^k \setminus \{w_i\}_{i \in I}$ and define

$$K_{p,q,\epsilon} = \bigcup_{v \in W} \left[\operatorname{real}_{pq}(v), \operatorname{real}_{pq}(v) + (pq)^{-k} \right).$$

The set $K_{p,q,\epsilon}$ has total length

$$\frac{|W|}{(pq)^k} = 1 - \frac{|I|}{(pq)^k} = 1 - \mu\left(\bigcup_{i \in I} U_i\right) \ge 1 - \epsilon.$$

Now let $\xi > 0$ be arbitrary and denote $c = \text{config}_{pq}(\xi)$. There exists a $t \in \mathbb{N}$ such that $F_{p,q}^t(c) \in \bigcup_{i \in I} U_i$, and equivalently, $F_{p,q}^t(c) \notin \bigcup_{v \in W} (\text{Cyl}(v,1))$. This means that $\{\xi(p/q)^t\} \notin K_{p,q,\epsilon}$, and therefore $Z_{p/q}(K_{p,q,\epsilon}) = \emptyset$. \square

6. Conclusions

We have shown in Theorem 4.9 and Corollary 4.11 that for p>q>1 and $\epsilon>0$ there exists a finite union of intervals $J_{p,q,\epsilon}$ of total length at most ϵ such that $Z_{p/q}(J_{p,q,\epsilon})\neq\emptyset$. Moreover, by following the proof of this result, it is possible (at least in principle) to explicitly construct the set $J_{p,q,\epsilon}$ for any given ϵ . We have also shown in Theorem 5.9 that for p>q>1 and $\epsilon>0$ there exists a finite union of intervals $K_{p,q,\epsilon}$ of total length at least $1-\epsilon$ such that $Z_{p/q}(K_{p,q,\epsilon})=\emptyset$. The proof of this theorem is non-constructive.

Problem 6.1. Assume that p > q > 1. Is it possible to construct explicitly for every $\epsilon > 0$ a finite union of intervals S such that the total length of S is at least $1 - \epsilon$ and $Z_{p/q}(S) = \emptyset$?

References

- [1] S. Akiyama, Mahler's Z-number and 3/2 number system. Unif. Distrib. Theory 3 (2008) 91–99.
- [2] S. Akiyama, Ch. Frougny and J. Sakarovitch, Powers of rationals modulo 1 and rational base number systems. *Isr. J. Math.* **168** (2008) 53–91.
- [3] A. Dubickas, On the powers of 3/2 and other rational numbers. Math. Nachr. 281 (2008) 951–958.
- [4] L. Flatto, J.C. Lagarias and A.D. Pollington, On the range of fractional parts $\{\xi(p/q)^n\}$. Acta Arith. 70 (1995) 125–147.
- [5] G. Hedlund, Endomorphisms and automorphisms of shift dynamical systems. Math. Syst. Theory 3 (1969) 320–375.
- [6] J. Kari, Cellular automata, the Collatz conjecture and powers of 3/2, in Developments in Language Theory. Vol. 7410 of Lecture Notes in Computer Science (2012) 40–49.
- [7] J. Kari, Universal pattern generation by cellular automata. Theor. Comput. Sci. 429 (2012) 180-184.
- [8] K. Mahler, An unsolved problem on the powers of 3/2. J. Aust. Math. Soc. 8 (1968) 313-321.
- [9] W. Rudin, Real and Complex Analysis. McGraw-Hill Book Company, New York (1966).

- [10] P. Walters, An Introduction to Ergodic Theory. Springer-Verlag, New York (1982).
- [11] H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins. Math. Ann. 77 (1916) 313–352.