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Singularly perturbed equations of degenerate type

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Abstract

This work is devoted to the study of nonvariational, singularly perturbed elliptic equations of degenerate type. The governing operator is anisotropic and ellipticity degenerates along the set of critical points. The singular behavior is of order $\mathrm{O}\left(\frac{1}{\epsilon}\right)$ along ϵ-level layers $\{u_\epsilon \sim \epsilon\}$, and a non-homogeneous source acts in the noncoincidence region $\{u_\epsilon > \epsilon\}$. We obtain the precise geometric behavior of solutions near ϵ-level surfaces, by means of optimal regularity and sharp geometric nondegeneracy. We further investigate Hausdorff measure properties of ϵ-level surfaces. The analysis of the asymptotic limits as the ϵ parameter goes to zero is also carried out. The results obtained are new even if restricted to the uniformly elliptic, isotropic setting.

MSC: 35B25; 35J60

Keywords: Singularly perturbed equations; Degenerate fully nonlinear operators; Geometric regularity theory

1. Introduction

In this paper we develop a systematic approach to study local geometric properties of solutions to singularly perturbed equations of the form

$$\mathscr{F}_\epsilon(X, u_\epsilon, \nabla u_\epsilon, D^2 u_\epsilon) = 0, \tag{1.1}$$

taking place within a generic domain in the d-dimensional Euclidean space, $\Omega \subset \mathbb{R}^d$. We focus our analysis to reaction-diffusion models with singular behavior near ϵ-level surfaces. The diffusion process is anisotropic and degenerate as $\nabla u_\epsilon \sim 0$. Motivations for the study of singularly perturbed problems as in (1.1) come from the several fields of applications: population dynamics, boundary transition layers, fluid mechanics, theory of combustion, certain chemical reactions, heat propagation, free boundary problems, approximating singular PDEs, segregation problems, etc.

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Singular perturbation theory refers to a rich set of methods, ideas and technics for the study of problems involving infinitesimal parameters. Such a situation is rather common in applications and this fact has fostered a massive development of the theory, whose primary ideas go back to the early 19th century. Singular perturbations are particularly important in the study of physical problems modeled by partial differential equations. Through the last decades, several methods have been invented to tackle PDE problems involving singular perturbations: matched asymptotic expansions, WKB approximations, periodic averaging, the method of multiple scales, etc. Nowadays singular perturbation theory is a well established, very active and rich field of research for mathematicians, physicists, engineers, among other scientists. For a comprehensive reference on singular perturbation theory, we refer the readers to the book [12].

For the models treated in this work, the singular behavior of the perturbed equation (1.1) is of order $\mathrm{O}\left(\epsilon^{-1}\right)$ near ϵ-level surfaces $\Gamma_\epsilon = \{u_\epsilon \sim \epsilon\}$. The diffusion of the governing operator degenerates along the a priori unknown set of critical points, $\mathfrak{C} := \{X \in \Omega \mid \nabla\phi(X)| = 0\}$. A typical example is the class of operators $\mathscr{L}_\gamma \phi = |\nabla\phi|^\gamma \Delta\phi$, for $\gamma \geq 0$. When $\gamma > 0$, the operator is non-variational and degenerate elliptic. The magnitude of the degeneracy is measured by the parameter γ; the bigger the value of γ more degenerate the operator is. This feature impels less efficient smoothing effects on the diffusion process and the regularity theory for such class of operators is more involved from the mathematical view point. In turn, our goal in this work is to establish analytic and geometric properties of nonnegative solutions to (1.1) that are uniform-in-ϵ. Such estimates will then allow us to understand asymptotic limiting problems obtained as the infinitesimal parameter ϵ goes to zero.

Singularly perturbed methods are also useful tools in the study of elliptic and parabolic equations with singular (blowing-up) potentials. Variational, isotropic singular PDEs often arise from critical point theory of nondifferentiable functionals. The study of non-variational singularly perturbed PDEs is more delicate than its variational counterpart, mainly due to lack energy estimates and monotonicity formulae. Uniform elliptic singularly perturbed problems coming from the theory of flame propagations have been treated in [5], pioneering a large literature on the subject, see also [20]. Non-variational equations with singular (blowing-up) potentials were addressed in recent works, [18,11] and [2].

General, singular perturbed equations ruled by anisotropic and degenerate elliptic operators – object of study of this paper – present several new difficulties in its mathematical treatment. Loosely speaking, solutions of such equations carry two unknown regions of singular behavior, namely the set of critical points $\mathfrak{C}_\epsilon$ and the actual *physical* free layer of transition, $\{u^\epsilon \sim \epsilon\}$. A central part of the program is then to show that near ϵ-level surfaces, Γ_ϵ, it is possible to control u^ϵ by $\sim \mathrm{dist}(X, \Gamma_\epsilon)$. The upper control concerns optimal Lipchitz continuity of solutions independent of ϵ; whereas the control by below reflects a geometric nondegeneracy property. Heuristically, such a fine geometric control on u_ϵ – independent of ϵ – implies that, *in measure*, the two free boundaries do not intersect each other. Implementing the heuristics explained above involves a number of technical steps and new tools, which ultimately produce results that are original even in the isotropic, non-degenerate case, $\gamma = 0$.

We must highlight that the analysis of anisotropic problems treated here are indeed rather more subtle than the isotropic case. For instance, even though the limiting equation of problems modeled by $|\nabla v_\epsilon|^\gamma \Delta v_\epsilon \approx \epsilon^{-1}\chi_{\{v_\epsilon \leq \epsilon\}}$ is simply the Laplace equation within the set of positivity, $\Delta v = 0$ in $\{v > 0\}$, the anisotropy of the singular perturbation leaves its signature along the limiting transition boundary. It effects the expected linear behavior of the limiting function along $\partial\{v > 0\}$ – see the comments at the beginning of Section 7.

The paper is organized as follows. In Section 2 we discuss the mathematical set-up for the singular perturbed problem to be studied. In particular the appropriate notion of solution is introduced at the end of that Section. Nondegeneracy properties of Perron's solutions are proven in Section 3 by means of useful barrier constructions. Optimal gradient bounds, uniform-in-ϵ, is derived in Section 4. In Section 6 we derive uniform Hausdorff estimates of $\{u^\epsilon \sim \epsilon\}$, in particular we show that $\mathscr{H}^{d-1}(\{u^\epsilon \sim \epsilon\} \cap B_r) \sim r^{d-1}$. The asymptotic limit as $\epsilon \to 0$ is studied in Section 7.

2. Mathematical set-up

We start off this Chapter by introducing the basic set-up we shall work on in this article. By $\mathrm{Sym}(d)$ we denote the space of $d \times d$ real, symmetric matrices. Given two positive numbers $0 < \lambda \leq \Lambda$, we denote the Pucci extremal operators by

$$\mathcal{M}^{+}_{\lambda,\Lambda}(M) := \lambda\left(\sum_{e_i<0} e_i\right) + \Lambda\left(\sum_{e_i>0} e_i\right),$$

$$\mathcal{M}^{-}_{\lambda,\Lambda}(M) := \lambda\left(\sum_{e_i>0} e_i\right) + \Lambda\left(\sum_{e_i<0} e_i\right),$$

where $e_i = e_i(M)$ are the eigenvalues of symmetric matrix M. An operator $F\colon \mathrm{Sym}(d) \to \mathbb{R}$ is said to be (λ, Λ)-uniformly elliptic provided

$$\mathcal{M}^{-}_{\lambda,\Lambda}(M) \le F(M+N) - F(M) \le \mathcal{M}^{+}_{\lambda,\Lambda}(M); \tag{2.1}$$

holds for any $M, N \in \mathrm{Sym}(d)$. Throughout this paper, F will always denote a (λ, Λ)-uniformly elliptic operator. With no loss of generality we shall always assume the normalization condition

$$F(0) = 0.$$

Existence and regularity theory for solutions to uniform elliptic equations has been a central topic of research since the late 70s, when Krylov–Safonov Harnack inequality [16,17] paved the way to such a rich subject. It is now well established that under the monotonicity assumption encoded in (2.1), the language of viscosity solutions provides an appropriate notion of weak solutions, [8]. A consequence of Krylov–Safonov Harnack inequality is that viscosity solutions to the homogeneous equation

$$F(D^2u) = 0,$$

are of class $C^{1,\alpha}$, for some $0 < \alpha < 1$ that depends only upon dimension and ellipticity constants, see [4, Chapters 4, 5]. Under concavity assumption on F, Evans and Krylov proved that solutions are classical, i.e., of class C^2, see for instance [4, Chapter 6]. Regularity theory for fully nonlinear, non-homogeneous equations, $F(D^2u) = f(X)$, is more involved, and it has been treated by Caffarelli in his epic marking article [3]. In [21], optimal modulus of continuity has been provided under appropriate conditions on the coefficients and integrability properties of f. For existence results, see [9]. The corresponding theory for fully nonlinear elliptic equations of degenerate type

$$|\nabla u|^\gamma F(D^2u) = f(X), \tag{2.2}$$

is more recent and it has received a warm attention through the last decade, see [6,10,13–15,1] among several other works on this subject. It has been shown [14,1] that viscosity solutions to (2.2) are also of class $C^{1,\beta}$, provided f is bounded. In fact [1] shows that if F is concave, then solutions are precisely of class $C^{1,\frac{1}{1+\gamma}}$, and such estimate is optimal.

In this current article, we are interested in singularly perturbed equations ruled by degenerate elliptic operators as in (2.2), i.e., our goal is to study *weak* solutions to

$$\begin{cases} |\nabla u^\epsilon|^\gamma F(D^2u^\epsilon) = \zeta_\epsilon(X, u^\epsilon) \text{ in } \Omega, \\ \qquad 0 \le u^\epsilon \le K_0, \end{cases} \tag{2.3}$$

where $\gamma \ge 0$ is a degeneracy parameter, $F\colon \mathrm{Sym}(d) \to \mathbb{R}$ is a fully nonlinear, uniformly elliptic operator. The reaction term, ζ_ϵ, represents the singular perturbation of the model. We are interested in singular behaviors of order $\mathrm{O}\left(\epsilon^{-1}\right)$ along ϵ-level layers $\{u_\epsilon \sim \epsilon\}$, hence we are led to consider singular reaction terms $\zeta_\epsilon\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$0 \le \zeta_\epsilon(X, t) \le \frac{\mathscr{B}}{\epsilon}\chi_{(0,\epsilon)}(t) + \mathscr{C}, \quad \forall (X, t) \in \Omega \times \mathbb{R}_+, \tag{2.4}$$

for nonnegative constants $\mathscr{B}, \mathscr{C} \ge 0$.

Clearly $\zeta_\epsilon \equiv 0$ satisfies (2.4), so as to assure that the reaction term is genuinely singular, we shall also consider the following non-degeneracy condition:

$$\inf_{\Omega\times[a,b]} \epsilon\zeta_\epsilon(X, \epsilon t) := \iota > 0, \tag{2.5}$$

for some $0 < a < b$, and some ι independent of ϵ. Heuristically, (2.5) says that the singular term behaves asymptotically as $\sim \epsilon^{-1}\chi_{(0,\epsilon)}$ plus a nonnegative noise that stays uniformly bounded away from infinity. Singular reaction terms built up as approximation of unity

$$\phi_\epsilon(X,t) := \frac{1}{\epsilon}\phi_1\left(\frac{t}{\epsilon}\right) + g_\epsilon(X), \tag{2.6}$$

are particular (simpler) cases covered by the analysis to be developed herein. Indeed, for such approximations, ϕ_1 is a nonnegative smooth real function with supp $\phi_1 = [0,1]$, and $0 < c_0 \le g_\epsilon(X) \le c_0^{-1}$. It is plain to check that the reaction term written in (2.6) does satisfy (2.4) and (2.5).

Throughout this paper we deal with Perron's solutions of equation (2.3). That is, given a subsolution $\underline{u}^\varepsilon$ and a supersolution $\overline{u}^\varepsilon$ of equation (2.3), for which the following inequality $\underline{u}^\varepsilon \le \overline{u}^\varepsilon$ on $\partial\Omega$ is verified, a Perron's solution is the least supersolution lying between $\overline{u}^\varepsilon$ and $\underline{u}^\varepsilon$. Note that comparison principle assures $\underline{u}^\varepsilon \le \overline{u}^\varepsilon$ pointwise in Ω.

Given a sufficiently regular boundary datum φ on $\partial\Omega$, a pair of subsolution and supersolution solutions can aways be obtained by solving the

$$|\nabla \underline{u}^\varepsilon|^\gamma F(D^2\underline{u}^\varepsilon) = \sup_{\Omega\times[0,\infty)} \zeta_\varepsilon \quad \text{and} \quad |\nabla \overline{u}^\varepsilon|^\gamma F(D^2\overline{u}^\varepsilon) = \inf_{\Omega\times[0,\infty)} \zeta_\varepsilon \quad \text{in } \Omega, \tag{2.7}$$

satisfying $\underline{u}^\varepsilon = \overline{u}^\varepsilon = \varphi$ on $\partial\Omega$. Existence of solutions of (2.7) follows for instance by [7, Proposition 2 and Proposition 3].

Once fixed a pair of subsolution and supersolution solutions of equation (2.3), the following general procedure yields existence of Perron's solution:

Theorem 2.1. *Let $g : \Omega \times [0,\infty) \to \mathbb{R}$ be a bounded function, and uniformly Lipschitz in $[0,\infty)$. Assume $\mathscr{F} : \Omega \times \mathbb{R}^d \times Sym(d) \to \mathbb{R}$ verifies the monotonicity condition*

$$\mathscr{F}(X,\vec{p},N) \le \mathscr{F}(X,\vec{p},M), \tag{2.8}$$

for any $\vec{p} \in \mathbb{R}^d$, a.e. $X \in \Omega$ and all $N, M \in Sym(d)$ verifying the $N \le M$. Assume a priori $C^{0,\alpha}$ estimates for viscosity solutions $\mathscr{F}(X,\nabla u, D^2u) = f(X) \in L^\infty$ and that the equation

$$\mathscr{F}(X,\nabla u, D^2 u) = g(X,u) \tag{2.9}$$

admits subsolution and supersolution $\underline{u}$, $\overline{u} \in C^0(\overline{\Omega})$ respectively, with $\underline{u} = \overline{u} = \varphi \in W^{2,\infty}(\partial\Omega)$, then the function

$$v(x) := \inf_{w\in\mathcal{S}} w(x) \tag{2.10}$$

is a continuous viscosity solution to (2.9), satisfying $u = \varphi$ in $\partial\Omega$, where

$$\mathcal{S} := \left\{ w \in C(\overline{\Omega}) \;\middle|\; w \text{ is a supersolution to (2.9), and } \underline{u} \le w \le \overline{u} \right\},$$

Proof. By looking at the equation (2.9) as

$$\left[\mathscr{F}(X,\nabla u, D^2u) - \lambda u\right] + (\lambda u - g(X,u)) = 0,$$

let us denote the following operator

$$\mathscr{A}_f[u] = \mathscr{A}_f(X,u,\nabla u, D^2u) := \mathscr{F}(X,\nabla u, D^2u) - \lambda u + f(X).$$

Observe that $\mathscr{A}_f$ enjoys comparison principle, see for instance [7]. Also, we define

$$h(X,z) := \lambda z - g(X,z) \tag{2.11}$$

for some number $\lambda > 0$ sufficiently large such that $\nabla_z h \ge \lambda - \nabla_z g \ge \lambda/2$.

Now, we argue by finite induction. Let us consider $u_0 := \underline{u}$ and for each integer $k \ge 0$, define u_{k+1} as the solution of

$$\begin{cases} \mathscr{A}_{f_k}(X,u,\nabla u, D^2u) &= 0 \quad \text{in} \quad \Omega \\ u &= \varphi \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.12}$$

where $f_k(X) := h(X, u_k(X))$. We claim for each $k > 0$, $u_k \le u_{k+1}$ pointwise in Ω. Indeed, by (2.12) we note that $\mathscr{A}_{f_0}[u_1] = 0 \le \mathscr{A}_{f_0}[u_0]$ in the viscosity sense and so, comparison principle implies $u_0 \le u_1$ in Ω. Now, we suppose $u_{k-1} \le u_k$ in Ω. By taking $\lambda > 0$ sufficiently large in (2.11), h becomes increasing in the variable z which guarantees

$\mathscr{A}_{f_k}[u_{k+1}] = 0 \le \mathscr{A}_{f_k}[u_k]$ in the viscosity sense. Hence, applying comparison principle once more we conclude $u_k \le u_{k+1}$ in Ω.

Similarly for each $k > 0$, we verify $u_k \le \overline{u}$ holds pointwise in Ω. In fact, for $\overline{f}(X) := h(X, \overline{u}(X))$ we have $\mathscr{A}_{\overline{f}}[u_1] \ge 0 \ge \mathscr{A}_{\overline{f}}[\,\overline{u}\,]$ in the viscosity sense, so $u_1 \le \overline{u}$ in Ω. By assuming $u_k \le \overline{u}$ in Ω and taking account that $\mathscr{A}_{\overline{f}}[u_{k+1}] \ge 0 \ge \mathscr{A}_{\overline{f}}[\,\overline{u}\,]$ in the viscosity sense, we obtain $u_{k+1} \le \overline{u}$ in Ω. Therefore, we derive the following increasing sequence

$$\underline{u} = u_0 \le u_1 \le u_2 \le \cdots \le u_k \le u_{k+1} \le \cdots \le \overline{u} \quad \text{in} \quad \Omega.$$

By a priori $C^{0,\alpha}$ estimates, up to a subsequence $\{u_k\}$ converges locally uniform to a function u_∞ defined pointwise in Ω. In addition, passing to another subsequence if necessary, $\mathscr{A}_{f_k}$ converges locally uniformly to

$$\mathscr{A}_\infty[\,u\,] = \mathscr{F}(X, \nabla u, D^2 u) - \lambda u + h(X, u_\infty),$$

and thus u_∞ is a viscosity solution of

$$\mathscr{F}(X, \nabla u, D^2 u) = g(X, u) \quad \text{in} \quad \Omega.$$

It remains to check that u_∞ satisfies (2.10). For each $v \in \mathcal{S}$ and $k > 0$, we obtain

$$\mathscr{A}_{f_k}[\,v\,] \;=\; \mathscr{F}(X, \nabla v, D^2 v) - (h(X, v) - h(X, u_k)) - g(X, v). \tag{2.13}$$

Inductively, let us analyze the case $k = 0$ in (2.13). Since $u_0 = \underline{u} \le v$ in Ω, we obtain

$$\mathscr{A}_{f_0}[u_1] = 0 \ge \mathscr{F}(X, \nabla v, D^2 v) - g(X, v) = \mathscr{A}_{f_0}[\,v\,]$$

in the viscosity sense. Thus comparison principle implies $u_1 \le v$ in Ω. Analogously, for $u_k \le v$ we obtain

$$\mathscr{A}_{f_k}[u_{k+1}] = 0 \ge \mathscr{A}_{f_k}[\,v\,]$$

and so $u_{k+1} \le v$ in Ω. Therefore for any positive integer k there holds $u_k \le v$ in Ω and by passing the limit as $k \to \infty$ we reach

$$u_\infty(x) = \inf_{v \in \mathcal{S}} v(x),$$

and the Theorem is proven. $\square$

Henceforth, it is set for this paper that for each $\varepsilon > 0$ fixed, u_ε always denotes a nonnegative Perron's solution of the singularly perturbed equation

$$|\nabla v|^\gamma F(D^2 v) = \zeta_\epsilon(X, v) \text{ in } \Omega. \tag{E_ϵ}$$

We conclude this section with a comment on the non-negativity assumption on Perron's solution. Initially we note that, if one assumes

$$\zeta_\epsilon(X, t) = 0 \text{ for } t \le 0, \tag{2.14}$$

then any solution to (E_ϵ) with nonnegative boundary value is nonnegative inside Ω, and hence entitled for our analysis. Indeed, suppose for the sake of contradiction, that v solves (E_ϵ) in the viscosity sense, $v \ge 0$ on $\partial\Omega$ and $\mathscr{N} := \{X \in \Omega \mid v(X) < 0\}$ is nonempty. Clearly $v = 0$ on $\partial\mathscr{N} \cap \Omega$ and, since $v \ge 0$ on $\partial\Omega$, we conclude $v \ge 0$ on $\partial\mathscr{N}$. Now, in view of (2.14) and [14, Lemma 6], we conclude v satisfies $F(D^2 v) = 0,$ in $\mathscr{N}$, which gives a contradiction to the maximum principle and the definition of $\mathscr{N}$.

More generally, if one assumes solely condition (2.4), it is still possible to construct a nonnegative subsolution to (E_ϵ), provided we have "enough height", i.e., the infimum of the desired boundary datum φ is big enough. Indeed, within a ball $B_R(X_0)$, for parameters $\mathrm{L} \gg 1$ and $0 < \alpha < 1$ to be chosen a posteriori, define $w(X) = w(r) := \mathrm{L}|X - X_0|^\alpha$. Direct computation shows that

$$|\nabla w|^\gamma \mathscr{M}^-_{\lambda,\Lambda}(D^2 w) = (\alpha \mathrm{L})^{1+\gamma} \left[\lambda(d-1) - \Lambda(1-\alpha)\right] r^{(\alpha-1)(1+\gamma)-1}. \tag{2.15}$$

Thus, if we choose

$$\alpha = 1 - \frac{\lambda}{2\Lambda}, \tag{2.16}$$

equation (2.15) becomes

$$|\nabla w|^{\gamma} \mathscr{M}^{-}_{\lambda,\Lambda}(D^2 w) = ([1 - \frac{\lambda}{2\Lambda}]\mathrm{L})^{1+\gamma} \lambda (d - 1 - \frac{1}{2}) r^{-(\frac{\lambda}{2\Lambda}(1+\gamma)+1)}. \tag{2.17}$$

Next, we note that

$$\{w \le \varepsilon\} = \left\{ |X| \le \left(\frac{\varepsilon}{\mathrm{L}}\right)^{\frac{2\Lambda}{2\Lambda-\lambda}} \right\},$$

and then, in $\{w \le \varepsilon\}$, there holds

$$|\nabla w|^{\gamma} \mathscr{M}^{-}_{\lambda,\Lambda}(D^2 w) \ge c\mathrm{L}^{1+\gamma} \left(\frac{\varepsilon}{\mathrm{L}}\right)^{-\frac{\lambda(1+\gamma)+2\Lambda}{2\Lambda-\lambda}} \ge \frac{\mathscr{B}}{\varepsilon} + \mathscr{C},$$

provided L is chosen properly. Another choice for L, which depends only on R and $\mathscr{C}$, assures the desired inequality outside $\{w \le \varepsilon\}$.

3. Scaling barriers and geometric nondegeneracy

As explained in the introduction, the first key difficulty in dealing with singularly perturbed degenerate equations is to prevent degeneracy along the transition layers. The appropriate tool for such a task is a geometric nondegeneracy estimate.

In this Section, we show that solutions grow *at least* in a linear fashion away from ϵ-level surfaces, inside $\{u^{\epsilon} > \epsilon\}$. This implies that *in measure* the two free boundaries do not intersect. The proof shall be based on appropriate barrier functions. We carry out a more general construction for future references. To this end, we shall look at degenerate elliptic equations of the form

$$|\nabla u|^{\gamma} \mathcal{M}^{+}_{\lambda,\Lambda}(D^2 u) = \zeta(X, u), \quad \text{in } \mathbb{R}^d,$$

where the reaction term satisfies the mild non-degeneracy assumption:

$$\inf_{\mathbb{R}^d \times [a,b]} \zeta(X, t) > 0. \tag{3.1}$$

Fixed $0 < a < b < 1$, for α and A_0 positive numbers to be chosen a posteriori, we consider the radially symmetric function $\Theta \colon \mathbb{R}^d \to \mathbb{R}$ defined as follows,

$$\Theta_L(X) := \begin{cases} a & \text{for} \quad 0 \le |X| < L; \\ A_0(|X| - L)^2 + a & \text{for} \quad L \le |X| < L + \sqrt{\frac{b-a}{A_0}}; \\ \psi(L) - \phi(L)|X|^{-\alpha} & \text{for} \quad |X| \ge L + \sqrt{\frac{b-a}{A_0}}. \end{cases} \tag{3.2}$$

Under the following choices,

$$\phi(L) := \frac{2}{\alpha}\sqrt{(b-a)A_0}\left(L + \sqrt{\frac{b-a}{A_0}}\right)^{1+\alpha} \quad \text{and} \quad \psi(L) := b + \phi(L)\left(L + \sqrt{\frac{b-a}{A_0}}\right)^{-\alpha}, \tag{3.3}$$

it is possible to verify that $\Theta_L \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$. So, we can compute the second order derivatives of Θ_L almost everywhere. Our first aim is to show, provided the appropriate parameters, that Θ_L satisfies pointwise

$$|\nabla \Theta_L(X)|^{\gamma} \mathcal{M}^{+}_{\lambda,\Lambda}(D^2 \Theta_L(X)) \le \zeta(X, \Theta_L(X)) \quad \text{in} \quad \mathbb{R}^d. \tag{3.4}$$

Indeed, clearly for $0 \le |X| < L$ the above inequality is verified. In the region $L \le |X| < L + \sqrt{\frac{b-a}{A_0}}$, we have

$$|\nabla \theta_L(X)| = 2A_0(|X| - L) \le 2A_0\sqrt{\frac{b-a}{A_0}} = 2\sqrt{A_0(b-a)}$$

and

$$D^2\Theta_L(X) = 2A_0\left[\left(\frac{1}{|X|^2} - \frac{(|X|-L)}{|X|^3}\right)X\otimes X + \frac{(|X|-L)}{|X|}\mathrm{Id}\right] \le 4A_0 \cdot Id.$$

Therefore, by using the estimates above, we obtain

$$|\nabla\Theta_L(X)|^\gamma\, \mathcal{M}^+_{\lambda,\Lambda}(D^2\Theta_L(X)) \;\le\; 4d\Lambda A_0\left(2\sqrt{A_0(b-a)}\right)^\gamma.$$

Moreover, by construction

$$a \le \Theta_L(X) \le b$$

and so, for A_0 sufficiently small, we get

$$|\nabla\Theta_L(X)|^\gamma\, \mathcal{M}^+(D^2\Theta_L(X)) \le \inf_{\mathbb{R}^d\times[a,b]} \zeta(X,t) \le \zeta(X,\Theta_L(X)).$$

Finally, let us turn our attention to the set $|X| \ge L + \sqrt{\frac{b-a}{A_0}}$. Direct computation shows

$$D^2\Theta_L(X) = \alpha\phi(L)|X|^{-(\alpha+2)}\left(-\frac{(\alpha+2)}{|X|^2}X\otimes X + Id\right)$$

hence,

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\Theta_L(X)) \le \alpha\phi(L)|X|^{-(\alpha-2)}\left(-(\alpha+1)\lambda + (d-1)\Lambda\right).$$

Finally, taking

$$\alpha \ge (d-1)\frac{\Lambda}{\lambda} - 1,$$

we get

$$|\nabla\Theta_L(X)|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2\Theta_L(X)) \le 0 \le \zeta(X,\Theta_L(X)).$$

Therefore, Θ_L satisfies (3.4).

To complete the analysis of the supersolution Θ_L, we will show that for some universal $\kappa_0 > 0$, there holds

$$\Theta_L(X) \ge \kappa_0 \cdot 4L \quad \text{for} \quad |X| \ge 4L, \tag{3.5}$$

where $L \ge L_0 := \sqrt{\frac{b-a}{A_0}}$. In fact, by (3.3)

$$|X| \ge 4L \ge 2(L+L_0) = 2\left(\frac{\phi(L)}{\psi(L)-b}\right)^{\frac{1}{\alpha}}$$

and hence,

$$\Theta_L(X) = \psi(L) - \phi(L)|X|^{-\alpha} \ge \psi(L) - 2^{-\alpha}(\psi(L)-b) \ge C_\alpha\psi(L),$$

for $\alpha > 1$, therefore,

$$\Theta_L(X) \ge \kappa_0 \cdot 4L$$

where $\kappa_0 > 0$ depends on $\alpha, d, \Lambda, \lambda$ and $(b-a)$.

As to establish lower bounds on the growth speed of solutions to (E_ϵ) inward the set $\{u^\epsilon > \epsilon\}$, the strategy now is to consider appropriate scaling versions of the universal barrier Θ_L. Hereafter, we will denote the distance of a point in the non-coincidence set $X \in \Omega \cap \{u^\epsilon > 0\}$ to the approximating transition boundary, Γ_ϵ, by

$$d_\epsilon(X_0) := \mathrm{dist}(X_0, \{u^\epsilon \le \epsilon\}).$$

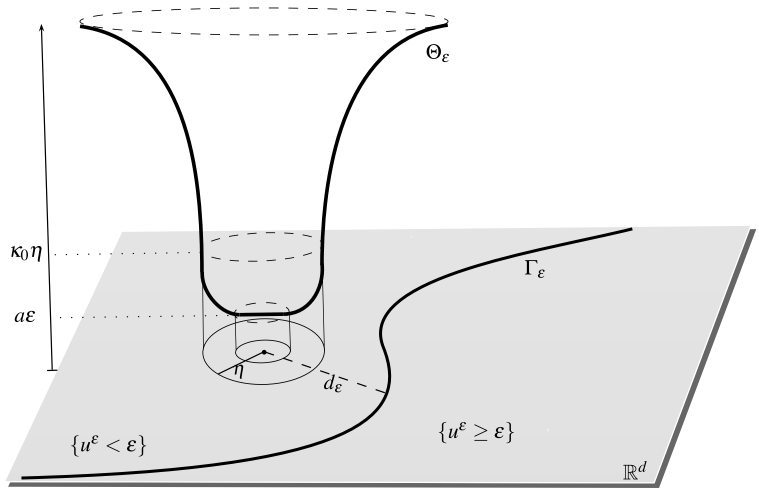

Fig. 1. Graphic illustrating the scaling barrier argument.

Theorem 3.1. *Let $\{u^\epsilon\}_{\epsilon>0}$ the Perron's solution of (E_ϵ). There exists $c > 0$ depending on universal parameters such that, for $X_0 \in \{u^\epsilon > \epsilon\}$ and $0 < \epsilon \ll d_\epsilon(X_0) \ll 1$, there holds*

$$u^\epsilon(X_0) \geq c \cdot d_\epsilon(X_0).$$

Proof. With no loss of generality, let us assume $0 \in \{u^\epsilon > \epsilon\}$. We set

$$\eta := \frac{d_\epsilon(0)}{2}$$

and consider the reaction term

$$\zeta(X,t) := \begin{cases} \epsilon\zeta_\epsilon(\epsilon X, \epsilon t), & \text{if } \epsilon X \in \Omega \\ \iota & \text{otherwise.} \end{cases}$$

Given the universal barrier Θ_L built-up above, we define

$$\Theta_\epsilon(X) := \epsilon \cdot \Theta_{\frac{\eta}{4\epsilon}}\left(\frac{X}{\epsilon}\right).$$

Easily one verifies that the scaled barrier Θ_ϵ satisfies

$$|\nabla\Theta_\epsilon|^\gamma F(D^2\Theta_\epsilon) \leq \zeta_\epsilon(X, \Theta_\epsilon),$$

and by (3.2) and (3.5), one simply checks that for $4L_0\epsilon \ll \eta$ (see Fig. 1),

$$\Theta_\epsilon(0) = a \cdot \epsilon \quad \text{and} \quad \Theta_\epsilon\big|_{\partial B_\eta} \geq \kappa_0\eta. \tag{3.6}$$

Now, we claim that there exists a $Z_0 \in \partial B_\eta$ such that

$$\Theta_\epsilon(Z_0) \leq u^\epsilon(Z_0). \tag{3.7}$$

In fact, if we assume $\Theta_\epsilon \geq u^\epsilon$ everywhere in ∂B_η, then the function

$$v^\epsilon := \min\{\Theta_\epsilon, u^\epsilon\}$$

would be a supersolution to Eq. (E_ϵ), but v^ϵ is strictly below of u^ϵ, which contradicts the minimality of u^ϵ. Therefore, by (3.6) and (3.7), we obtain

$$\kappa_0\eta \leq \Theta_\epsilon(Z_0) < u^\epsilon(Z_0) \leq \sup_{B_\eta} u^\epsilon. \tag{3.8}$$

In addition, u^ϵ solves

$$c_0 \le |\nabla u^\epsilon|^\gamma F(D^2 u^\epsilon) \le c_0^{-1} \quad \text{in} \quad B_{2\eta}.$$

So, by Harnack inequality, see [14], we get

$$\sup_{B_\eta} u^\epsilon \le C\left(u^\epsilon(0) + (2\eta)^{\frac{2+\gamma}{1+\gamma}} c_0^{\frac{1}{1+\gamma}}\right),$$

and by (3.8),

$$u^\epsilon(0) \ge \left(\kappa_0 - C c_0^{\frac{1}{1+\gamma}} \eta^{\frac{1}{1+\gamma}}\right)\eta.$$

Finally, taking $\eta > 0$ universally small, we have

$$u^\epsilon(0) \ge c\,\eta,$$

for some $0 < c \ll 1$. □

4. Lipschitz regularity

In this Section, we derive uniform gradient estimates, which in particular provides compactness in the local uniform convergence topology. In view of the results proven in Section 3, such an estimate is indeed optimal.

Theorem 4.1 *(Uniform Lipschitz Estimate). Let $\{u^\epsilon\}_{\epsilon>0}$ be a solution of (E_ϵ). Given $\Omega' \Subset \Omega$, there exists a constant C_0 depending on dimension, ellipticity constants and on Ω', but independent of $\epsilon > 0$, such that*

$$\|\nabla u^\epsilon\|_{L^\infty(\Omega')} \le C_0.$$

Proof. Initially we analyze the closed transition region $\{0 \le u^\epsilon \le \epsilon\} \cap \Omega'$. For $\epsilon \ll \operatorname{dist}(\Omega', \partial\Omega)$, fix $X_0 \in \{0 \le u^\epsilon \le \epsilon\} \cap \Omega'$ and define the auxiliary function

$$v(Y) := \frac{1}{\epsilon} u^\epsilon(X_0 + \epsilon Y) \quad \text{in} \quad B_1.$$

Direct computations show that v satisfies

$$|\nabla v(Y)|^\gamma F_\epsilon(D^2 v(Y)) = \epsilon \zeta(X_0 + \epsilon Y, u^\epsilon(X_0 + \epsilon Y)) =: f_\epsilon(Y) \quad \text{in} \quad B_1,$$

where $F_\epsilon(M) := \epsilon F(\epsilon^{-1} M)$. It follows readily from (2.4) that

$$0 \le f_\epsilon(Y) \le (\mathscr{B} + \epsilon \mathscr{C}) \le C_\star.$$

Thus, from the $C^{1,\alpha}$ regularity estimates ([14], see also [1]), we have

$$|\nabla v(0)| \le C\left\{\|v\|_{L^\infty(B_{1/2})} + C_\star^{\frac{1}{1+\gamma}}\right\}, \tag{4.1}$$

for some universal constant $C > 0$. Since,

$$v(0) = \frac{1}{\epsilon} u^\epsilon(X_0) \le 1,$$

it follows by Harnack inequality [13] (see also [15]) that

$$\|v\|_{L^\infty(B_{1/2})} \le C, \tag{4.2}$$

for a universal constant $C > 0$. Combining (4.1) and (4.2) we get

$$|\nabla u^\epsilon(X_0)| = |\nabla v(0)| \le C_0, \tag{4.3}$$

for some $C_0 > 0$ independent of ϵ.

We now proceed our analysis as to cover the open region $\{\epsilon < u^\epsilon\} \cap \Omega'$. For that, let us label

$$\Gamma_\epsilon := \{X \in \Omega' \mid u^\epsilon(X) = \epsilon\},$$

and fix a generic point X_1 inside $\{\epsilon < u^\epsilon\} \cap \Omega'$. In the sequel, we compute the distance from X_1 to Γ_ϵ and call such a number r, i.e.,

$$r := \mathrm{dist}(X_1, \Gamma_\epsilon).$$

Define the renormalized function $v_r \colon B_1 \to \mathbb{R}$ as

$$v_r(Y) := \frac{u^\epsilon(X_1 + rY) - \epsilon}{r}.$$

One easily verifies that such a function solves

$$|\nabla v_r|^\gamma F_r(D^2 v_r) = r\zeta_\epsilon(X_1 + rY, u^\epsilon(X_1 + rY)) =: \mathfrak{g}(Y),$$

in the viscosity sense, where as before, $F_r(M) := rF(r^{-1}M)$. From geometric consideration, $u^\epsilon(X_1 + rY) > \epsilon$, for all $Y \in B_1$, thus, it follows from (2.4) that $\mathfrak{g}(Y)$ is bounded, independently of ϵ, i.e.,

$$\|\mathfrak{g}\|_{L^\infty(B_1)} \le K_0$$

for a constant K_0 that depends only on $\mathscr{B}$, $\mathscr{C}$ and $\mathrm{diam}(\Omega')$. Applying once more $C^{1,\alpha}$ regularity estimates from [14] and [1], we conclude

$$|\nabla u^\epsilon(X_1)| = |\nabla v_r(0)| \le C\left(\frac{1}{r}\|u^\epsilon - \epsilon\|_{L^\infty(B_{r/2}(X_1))} + K_0^{\frac{1}{1+\gamma}}\right). \tag{4.4}$$

Now, let $Z_0 \in \Gamma_\epsilon$ be a point that realizes distance, i.e.,

$$r = |X_1 - Z_0|.$$

We will select $0 < r < r_0 \ll 1$ a posteriori to be universally small. From the Lipchitz regularity estimate early proven for points within $\{0 \le u^\epsilon \le \epsilon\}$, estimate (4.3), we know

$$|\nabla u^\epsilon(Z_0)| \le C_0. \tag{4.5}$$

Let us label

$$\mathscr{I} := \inf_{B_{r/2}(X_1)} (u^\epsilon - \epsilon),$$

and for $M \gg 1$ to be chosen a posteriori, define the auxiliary function in $B_r(X_1) \setminus B_{r/2}(X_1)$ by

$$\varrho(Z) := \frac{\mathscr{I}}{2^M - 1} r^M \left(|Z - X_1|^{-M} - r^{-M}\right). \tag{4.6}$$

Recall that according to Theorem 3.1

$$\mathscr{I} \ge cr, \tag{4.7}$$

for $\epsilon \ll r$. Direct computation yields

$$|\nabla \varrho|^\gamma \mathcal{M}^-_{\lambda,\Lambda}(D^2\varrho) \ge [\lambda(M+1) - \Lambda(n-1)]\left(\frac{\mathscr{I}}{2^M - 1} r^M \cdot M\right)^{1+\gamma} r^{-(M+1)\gamma - (M+2)} \tag{4.8}$$

in $B_r(X_1) \setminus B_{r/2}(X_1)$. We now choose M universally large so that

$$\lambda(M+1) - \Lambda(n-1) \ge \frac{\lambda}{10} M,$$

and in the sequel, taking into account (4.7), we restrict the analysis to $0 < r < r_0$, for

$$r_0 := \tilde{\delta} \frac{M^{2+\gamma}}{\mathscr{C}(2^M - 1)},$$

where $\tilde{\delta}$ is a positive, small constant that depends only on universal parameters. With such a universal choices made, we verify, in the viscosity sense,

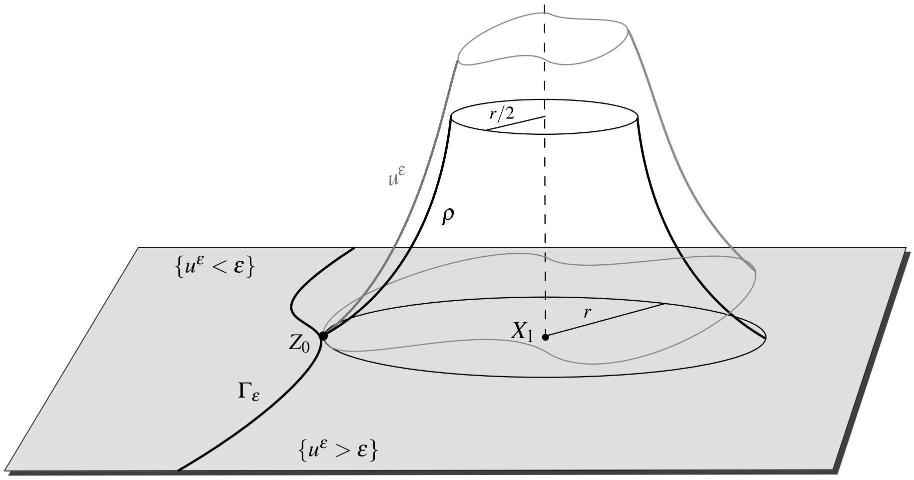

Fig. 2. Barrier for geometric upper control of the grow rate.

$$|\nabla \varrho|^\gamma F(D^2\varrho) \geq |\nabla \varrho|^\gamma \mathcal{M}^-_{\lambda,\Lambda}(D^2\varrho) \geq \mathscr{C}, \tag{4.9}$$

those, from construction and maximum principle, we have $\varrho(Z) \leq u^\epsilon(Z)$ for all points $Z \in B_d(X_1) \setminus B_{d/2}(X_1)$. (See Fig. 2.)

We estimate

$$\frac{d\varrho}{dt}\Big|_{t=r} \sim \frac{\mathscr{I}}{r}. \tag{4.10}$$

Hence, we from (4.5) and (4.10), we have

$$C_0 \geq -\partial_\nu u_\epsilon(Z_0) \geq -\frac{d\varrho}{dt}\Big|_{t=r} \geq c\frac{\mathscr{I}}{r}. \tag{4.11}$$

It also follows from the scaled Harnack inequality [14] that

$$\|u^\epsilon - \epsilon\|_{L^\infty(B_{r/2}(X_1))} \leq C\left\{\mathscr{I} + r^{2+\gamma}\mathscr{C}\right\}. \tag{4.12}$$

Finally, combining (4.4), (4.11), and (4.12), we establish gradient boundedness at interior points of $\{\epsilon < u^\epsilon\} \cap \Omega'$ and hence the proof of Theorem 4.1 is concluded. □

One should notice that for each $\epsilon > 0$, solutions u^ϵ are indeed locally of class $C^{1,\alpha}$. For instance, when F is concave, it follows from [1] that $u^\epsilon \in C^{1,\frac{1}{1+\gamma}}_{\text{loc}}$. Nonetheless, for any tiny $0 < \beta \ll 1$, near internal ϵ-layers, one verifies that

$$\lim_{\epsilon \to \infty} \|u^\epsilon\|_{C^{1,\beta}} = +\infty.$$

On the other hand, Theorem 4.1 implies that the Lipschitz norms of u^ϵ remain bounded, independently of ϵ. In such a perspective, this is an optimal estimate.

5. Geometric consequences

In this intermediary section, we discuss some geometric consequences of the sharp control of solutions, established in the previous two sections. An immediate consequence of Lipschitz regularity, Theorem 4.1, and Theorem 3.1 is the complete control of u^ϵ in terms of $d_\epsilon(X_0)$. (See Fig. 3.)

Corollary 5.1. *Given a subdomain $\Omega' \Subset \Omega$, there exists a universal constant $C = C(\Omega') > 0$ such that for $X_0 \in \Omega' \cap \{u^\epsilon > \epsilon\}$ and $\epsilon \ll d_\epsilon(X)$, there holds*

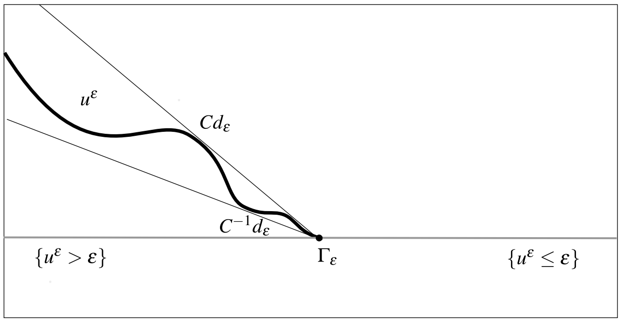

Fig. 3. Picture representing a slice of the graph of u_ϵ controlled from above and from below by $\sim d_\epsilon$ – result established in Corollary 5.1.

$$C^{-1}d_\epsilon(X_0) \leq u^\epsilon(X_0) \leq C\, d_\epsilon(X_0).$$

Proof. The inequality from below is precisely the Theorem 3.1. Now take $Y_0 \in \partial\{u^\epsilon > \epsilon\}$, such that $|Y_0 - X_0| = d_\epsilon(X_0)$. Thus, follow from Theorem 4.1,

$$u^\epsilon(X_0) \leq C\, d_\epsilon(X_0) + u^\epsilon(Y_0) \leq C\; d_\epsilon(X_0),$$

and the Corollary is proven. □

Now we prove that minimal solutions are strongly non-degenerate near ϵ-level sets. It means that the maximum of u^ϵ on the boundary of a ball B_r centered in $\{u^\epsilon > \epsilon\}$ is of the order of r. This is an additional and important information about the growth rate of u^ϵ away from ϵ-level surfaces.

Theorem 5.2. *Given $\Omega' \Subset \Omega$, there exists a universal constant $c > 0$ such that, for $X_0 \in \{u^\epsilon > \epsilon\}$, $\epsilon \ll \rho \ll 1$, there holds*

$$c\,\rho \leq \sup_{B_\rho(X_0)} u^\epsilon \leq c^{-1}(\rho + u^\epsilon(X_0)).$$

Proof. As in the proof of Theorem 3.1, taking $\Theta_\epsilon(X) = \epsilon\Theta_{\frac{\rho}{4\epsilon}}(X)$, we have

$$u^\epsilon(Z) > \Theta_\epsilon(Z),$$

for some point $Z \in \partial B_\rho(X_0)$. To conclude, we note that

$$\kappa \cdot \rho \leq \Theta_\epsilon(Z) < u^\epsilon(Z) \leq \sup_{B_\rho(X_0)} u^\epsilon.$$

The upper estimate follows directly from Lipschitz regularity. □

Remark 5.3. Given $X_0 \in \{u^\epsilon > \epsilon\}$, $\epsilon \ll \rho$ and $\rho \ll 1$ universally small, we have from strong non-degeneracy that there exists $Y_0 \in B_\rho(X_0)$ such that

$$u^\epsilon(Y_0) \geq c_0\rho.$$

By Lipschitz continuity, for $Z \in B_{\kappa\rho}(Y_0)$, we get

$$u^\epsilon(Z) - C\kappa\rho \geq u^\epsilon(Y_0).$$

Then, by estimates above, it is possible to choose $0 < \kappa \ll 1$ universally small such that

$$Z \in B_{\kappa\rho}(Y_0) \cap B_\rho(X_0) \quad \text{and} \quad u^\epsilon(Z) > \epsilon.$$

Finally, we conclude that there exists a portion of $B_\rho(X_0)$ with volume in order $\sim \rho^d$ within $\{u^\epsilon > \epsilon\}$ (see Fig. 4). By this fact, we are ready to obtain uniform positive density along level sets of minimal solutions to Eq. (E_ϵ).

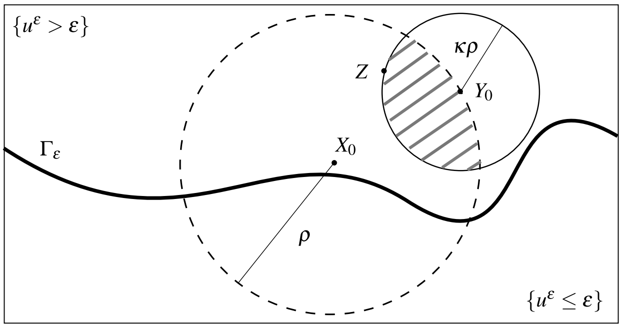

Fig. 4. Geometric idea of the proof of Corollary 5.4.

Corollary 5.4. *Given $X_0 \in \{u^\epsilon > \epsilon\}$, $\epsilon \ll \rho$ and $\rho \ll 1$ universally small, there exists a universal constant $0 < c_0 < 1$ such that*

$$\frac{\mathcal{L}^d(B_\rho(X_0) \cap \{u^\epsilon > \epsilon\})}{\mathcal{L}^d(B_\rho(X_0))} \geq c_0\,,$$

where $\mathcal{L}^d(A)$ is the Lebesgue measure of the set A.

Proof. Following the lines of Remark 5.3, we check

$$\mathcal{L}^d(B_\rho(X_0) \cap \{u^\epsilon > \mu\}) \geq \mathcal{L}^d(B_\rho(X_0) \cap B_{\kappa\rho}(Y_0)) = c_0\,\mathcal{L}^d(B_\rho(X_0)),$$

for some universal constant $0 < c_0 \ll 1$. □

Corollary 5.5. *Given $X_0 \in \{u^\epsilon > \epsilon\}$, $\epsilon \ll \rho$ and $\rho \ll 1$ universally small, then*

$$\fint_{B_\rho(X_0)} u^\epsilon dX \geq c\,\rho$$

for a universal constant $c > 0$ which does depend on ϵ.

Proof. As in Remark 5.3, there exists a universally small constant $0 < \kappa \ll 1$, such that

$$\fint_{B_\rho(X_0)} u^\epsilon dX \geq C_d \fint_{B_\rho(X_0)\cap B_{\kappa\rho}(Y_0)} u^\epsilon dX \geq c\,\rho$$

for a universal constant $0 < c \ll 1$ and some $Y_0 \in \{u^\epsilon > \epsilon\}$. □

6. Hausdoff estimates

In this section, we establish Hausdorff measure estimate of the approximating level sets. A necessary condition for the study of such an estimate is to impose the nondegeneracy of the reaction term propagates up to the transition boundary. Hence, hereafter in condition (2.5), we shall take $a = 0$, i.e.,

$$\inf_{\Omega\times[0,b]} \epsilon\zeta_\epsilon(X, \epsilon t) := \iota > 0, \tag{6.1}$$

for some $b > 0$ will be enforced. A condition at infinity on the governing operator F is also required in the Hausdorff estimate analysis. We shall discuss such a condition when time comes.

Our next result says that, in measure, the Hessian of an approximating solution blows-up near the transition boundary as $\epsilon \to 0$.

Proposition 6.1. *Fix $\Omega' \Subset \Omega$, $C \gg 1$ and $\rho < \mathrm{dist}(\Omega', \partial\Omega)$. There exists $\epsilon_0 > 0$ such that, for $\epsilon \leq \epsilon_0$ there holds*

$$\int_{B_\rho(X_\epsilon)} \big(\zeta_\epsilon(X,u^\epsilon) - C\big)\, dX \geq 0, \tag{6.2}$$

for any $X_\epsilon \in \partial\{u^\epsilon > \epsilon\} \cap \Omega'$.

Proof. Let us suppose, for sake of contradiction, that there exists $C_0 > 0$ and $\rho < \operatorname{dist}(\Omega', \partial\Omega)$, such that

$$\int_{B_\rho(X_k)} \big(\zeta_{\epsilon_k}(X,u^{\epsilon_k}) - C_0\big)\, dX < 0, \tag{6.3}$$

for points $X_{\epsilon_k} \in \partial\{u^{\epsilon_k} > \epsilon_k\} \cap \Omega'$ and a sequence $\epsilon_k \to 0$ as $k \to \infty$. Let us define

$$v_k(Y) := \frac{b}{\epsilon_k} u^{\epsilon_k}(X_{\epsilon_k} + \epsilon_k Y).$$

From uniform Lipschitz regularity, we know

$$\|v_k\|_{L^\infty(B_\rho)} \leq C_1.$$

From (6.3) we obtain

$$\int_{B_{\rho/\epsilon_k}} \Big((\epsilon_k b^{-1}) \cdot \zeta_{\epsilon_k}(X_{\epsilon_k} + \epsilon_k Y, \epsilon_k b^{-1} v_k) - C_0 \epsilon_k b^{-1}\Big)\, dX < 0. \tag{6.4}$$

Easily we verify that v_k is a Perron's solution to

$$|\nabla v_k|^\gamma F_k(D^2 v_k) = \epsilon_k b^{-1} \zeta_{\epsilon_k}(X_{\epsilon_k} + \epsilon_k Y, \epsilon_k b^{-1} v_k) =: \mathfrak{f}_\epsilon(X),$$

where $F_k(\mathscr{M}) := \epsilon_k F(\epsilon_k^{-1}\mathscr{M})$ has the same ellipticity constants as F and

$$\|\mathfrak{f}_\epsilon\|_{L^\infty(B_{\rho/\epsilon_k})} \leq \frac{\mathscr{B} + \mathscr{C}}{b},$$

independent of ϵ. From $C^{1,\alpha}$ estimates, [14,1], up to a subsequence

$$\lim_{k\to\infty} v_k =: v_\infty,$$

in the $C^1_{\mathrm{loc}}(B_\rho)$ topology. Combining (6.1) with (6.4), we deduce that

$$\text{either } v_\infty \equiv 0, \quad \text{ or else } \quad v_\infty \geq b, \text{ everywhere in } B_\rho. \tag{6.5}$$

As $v_k(0) = b$ for all $k \in \mathbb{N}$, we have that $v_\infty(0) = b > 0$, so v_∞ can not be identically zero. If $v_\infty \geq b$, we have that 0 is a minimum point, which give us a contradiction by nondegeneracy, i.e.

$$0 = |\nabla v_\infty(0)| = |\nabla u_{\epsilon_k}(0)| + \mathrm{o}(1) \geq c > 0,$$

and the proof is concluded. □

Heuristically, Proposition 6.1 implies that near the transition boundary, the governing operator F gets evaluated at very large matrices. Such an insight motivates the following structural asymptotic condition on the governing operator:

Definition 6.2. We say a uniform elliptic operator $F : \mathrm{Sym}(d) \to \mathbb{R}$ is *asymptotically concave* if there exists a positive matrix $\mathscr{F} = (f_{ij})_{d\times d}$ and a constant $C_F > 0$, such that

$$\operatorname{tr}(\mathscr{F} \cdot M) - F(M) \geq -C_F, \tag{AC}$$

for all $M \in \mathrm{Sym}(d)$.

This is a generalization on the classical concavity assumption on F, required for instance in the Evans and Krylov $C^{2,\alpha}$ regularity theorem. Improved regularity estimates for viscosity solutions of *asymptotically concave* equations were proven in [19]. Condition (AC) has shown to be a proper assumption on the governing operator F required in the Hausdorff estimate analysis. Hence, hereafter in this Section, we assume the governing operator F is asymptotically concave.

It is interesting to notice that if u^ϵ is a Perron's solution to (E_ϵ) then one verifies in the viscosity sense

$$F(D^2u^\epsilon) = \zeta_\epsilon(X, u^\epsilon)|\nabla u^\epsilon|^{-\gamma} \quad \text{in} \quad \{u^\epsilon > \epsilon\} \cap \Omega', \tag{6.6}$$

for any $\Omega' \Subset \Omega$. Hence, by non-degeneracy and Lipschitz regularity and asymptotic concavity of the operator, i.e., (AC), for $X_\epsilon \in \{u_\epsilon > \epsilon\} \cap \Omega'$ there holds

$$\int\limits_{B_\rho(X_\epsilon)} f_{ij}\, u^\epsilon_{ij}\, dX \geq \int\limits_{B_\rho(X_\epsilon)} \left(\zeta_\epsilon(X, u^\epsilon)|\nabla u^\epsilon|^{-\gamma} - C_F\right) dX \geq C_0^{-\gamma} \int\limits_{B_\rho(X_\epsilon)} \left(\zeta_\epsilon(u^\epsilon) - C_F C_0^{\gamma}\right) dX,$$

where $C_0 > 0$ comes from the universal control on the Lipschitz norm in $B_\rho(X_\epsilon)$. Here the expression $f_{ij}\, u^\epsilon_{ij}$ is understood in the viscosity sense. Thus, combining the estimate above and the Proposition 6.1, we obtain

$$\int\limits_{B_\rho(X_\epsilon)} f_{ij}\, u^\epsilon_{ij}\, dX \geq 0, \tag{6.7}$$

for $\epsilon \ll 1$.

Lemma 6.3. *There exists a constant $C > 0$ depending on $\Omega' \Subset \Omega$ and universal parameters such that, for each $X_\epsilon \in \partial\{u^\epsilon > \epsilon\} \cap \Omega'$ and $\rho \ll 1$, there holds*

$$\int\limits_{B_\rho(X_\epsilon)\cap\{\epsilon \leq u^\epsilon < \mu\}} |\nabla u^\epsilon|^2\, dX \leq C\mu\rho^{d-1}.$$

Proof. We define the following cut off function,

$$\Phi^\epsilon = \begin{cases} C_1\epsilon & \text{in} \quad \{u^\epsilon \leq C_1\epsilon\}; \\ u^\epsilon & \text{in} \quad \{C_1\epsilon < u^\epsilon \leq \mu\}; \\ \mu & \text{in} \quad \{u^\epsilon > \mu\}. \end{cases} \tag{6.8}$$

Estimate (6.7) gives

$$0 \leq \int\limits_{B_\rho(X_\epsilon)} \Phi^\epsilon f_{ij} D_{ij} u^\epsilon\, dX = \frac{1}{\rho} \int\limits_{\partial B_\rho(X_\epsilon)} f_{ij} D_i u^\epsilon \Phi^\epsilon (X^i - X^i_\epsilon) d\mathscr{H}^{d-1} - \int\limits_{B_\rho(X_\epsilon)} f_{ij} D_i \Phi^\epsilon D_j u^\epsilon\, dX$$

and hence,

$$\int\limits_{B_\rho(X_\epsilon)\cap\{\epsilon \leq u^\epsilon < \mu\}} f_{ij} D_i u^\epsilon D_j u^\epsilon\, dX \leq \frac{1}{\rho} \int\limits_{\partial B_\rho(X_\epsilon)} f_{ij} D_i u^\epsilon \Phi^\epsilon (X^i - X^i_\epsilon)\, d\mathscr{H}^{d-1}. \tag{6.9}$$

Using the regularity of u^ϵ and ellipticity, we get

$$\int\limits_{B_\rho(X_\epsilon)\cap\{\epsilon \leq u^\epsilon < \mu\}} |\nabla u^\epsilon|^2\, dX \leq C\mu\rho^{d-1},$$

for $\rho \ll 1$. $\square$

Given a measurable set $G \subset \mathbb{R}^d$ and a positive number $\delta > 0$, we denote:

$$\mathscr{N}_\delta(G) := \{X \in \mathbb{R}^d \mid \operatorname{dist}(X, G) < \delta\},$$

the δ-neighborhood of G in $\mathbb{R}^d$. As we move towards uniform bounds of the $\mathscr{H}^{d-1}$-Hausdorff measure of the level-surfaces $\partial\{u^\epsilon > \epsilon\}$, we recall a classical result from measure theory.

Lemma 6.4. *Given an open set $A \Subset \Omega$, there holds:*

a) *If there exists δ such that A has the δ-density property, then there exists a constant $C = C(\tau, d)$, where:*

$$|\mathcal{N}_\delta(\partial A) \cap B_\rho(X)| \leq \frac{1}{2^d \tau} |\mathcal{N}_\delta(\partial A) \cap B_\rho(X) \cap A| + C\delta\rho^{d-1}$$

with $X \in \partial A \cap \Omega$ and $\delta \ll \rho$.

b) *If A has uniform density in Ω along A, then $|\partial A \cap \Omega| = 0$.*

In the sequence, we obtain a d-dimensional measure estimate on ϵ-level layers, that are uniform with respect to the infinitesimal parameter ϵ.

Lemma 6.5. *Fixed $\Omega' \Subset \Omega$, there exists a positive constant $C^\star$ depending on Ω' and universal parameters, such that if $C^\star\mu \leq 2\rho \ll \operatorname{dist}(\Omega', \partial\Omega)$ then, for $\mu, \epsilon > 0$ small enough, with $3C_1\epsilon < \mu \ll \rho$, we have*

$$\mathcal{L}^d\left(\{C_1\epsilon < u^\epsilon < \mu\} \cap B_\rho(X_\epsilon)\right) \leq C^\star \mu\rho^{d-1},$$

where $X_\epsilon \in \partial\{u^\epsilon > \epsilon\} \cap \Omega'$, with $d_\epsilon(X_\epsilon) \ll \operatorname{dist}(\Omega', \partial\Omega)$ and $C_1 > 1$.

Proof. Let $\{B_j\}$ be a finite covering of $\partial\{u^\epsilon > C_1\epsilon\} \cap B_\rho(X_\epsilon)$ by balls centered at $X_j \in \partial\{u^\epsilon > C_1\epsilon\}$ with radius equals $C^\star\mu$, satisfying

$$\bigcup_j B_j \subset \left[\mathcal{N}_{\frac{\eta}{8}(\Omega')} \cap B_\rho(X_\epsilon)\right]$$

where $\eta = \operatorname{dist}(\partial\Omega', \partial\Omega)$ and $C^\star > 0$ will be chosen *a posteriori*. By Heine–Borel Theorem

$$\sum_j \chi_{B_j} \leq m.$$

We will verify that if $C^\star \gg 1$ is taken universally large, it is possible to find, another, positive universal constant C_2 such that

$$\int_{\{C_1\epsilon < u^\epsilon < \mu\} \cap B_j} |\nabla u^\epsilon|^2 dX \geq C_2 |B_j|. \tag{6.10}$$

In fact, we shall obtain two families of balls, hereby labeled $\{B_j^1\}$ and $\{B_j^2\}$, both contained in $\{B_j\}$, such that:

(A) the radii of B_j^1 and B_j^2 are proportional to μ, i.e., up to a multiplicative constant that depends only upon Ω');
(B) $\Phi^\epsilon \geq \frac{3}{4}\mu$ in B_j^1 and $\Phi^\epsilon \leq \frac{2}{3}\mu$ in B_j^2, for Φ^ϵ as in (6.8).

The existence of the above-mentioned balls goes as follows: by strong non-degeneracy, Theorem 7.5, there exists a point $X_1 \in \frac{1}{4}\overline{B}_j$ such that

$$u^\epsilon(X_1) = \sup_{\frac{1}{4}B_j} u^\epsilon \geq c \cdot \frac{C^\star\mu}{4},$$

for a universal constant $c > 0$. Now, we can select $C^\star \gg 1$ so large that

$$C^\star \cdot c > 4 \quad \text{and} \quad K := \sup_{\mathcal{N}_{\frac{\eta}{8}}(\Omega')} |\nabla u^\epsilon| > \frac{1}{C^\star}.$$

Now, if μ is small enough, we can take $r_j^1 = \frac{1}{8K}\mu$ and $r_j^2 = \frac{1}{3K}\mu$ as to verify

$$\Phi^\epsilon \geq \frac{3}{4}\mu > C_1\epsilon \quad \text{in } B_j^1 = B_{r_j^1}(X_1)$$

and

$$\Phi^{\epsilon} \le \frac{2}{3}\mu < \mu \quad \text{in } B_j^2 = B_{r_j^2}(X_j).$$

For each $j > 0$, we define

$$m_j := \fint_{B_j} \Phi^{\epsilon}.$$

We now claim that the estimate $|\Phi^{\epsilon} - m_j| > \sigma\mu$ holds in at least one of the two sub balls B_j^1, B_j^2, for some universal constant $\sigma > 0$. Indeed, if we assume, for the sake of contradiction, that this is not the case, we could find sequences $X_n \in B_j^1$, $Y_n \in B_j^2$ such that

$$\frac{|\Phi^{\epsilon}(X_n) - m_j|}{\mu} < \frac{1}{n} \quad \text{and} \quad \frac{|\Phi^{\epsilon}(Y_n) - m_j|}{\mu} < \frac{1}{n} \quad \forall n.$$

Letting $n \to \infty$ in the above estimates yields

$$\frac{|\Phi^{\epsilon}(X_n) - \Phi^{\epsilon}(Y_n)|}{\mu} \to 0,$$

which contradicts property (B). In conclusion, by Poincare's inequality, we have

$$\sigma^2\mu^2 \le \fint_{B_j} |\Phi^{\epsilon} - m_j|^2 dX \le \sigma(C^{\star}\mu)^2 \fint_{B_j} |\nabla\Phi^{\epsilon}|^2 dX,$$

and hence we deduce,

$$\int\limits_{\{C_1\epsilon < u^{\epsilon} < \mu\}\cap B_j} |\nabla u^{\epsilon}|^2 dX \ge C_2|B_j|.$$

Finally, applying nondegeneracy estimates once more, we reach

$$C_3 d_{\epsilon}(Z) \le u^{\epsilon}(Z) \le \mu,$$

for all $Z \in \{C_1\epsilon < u^{\epsilon} < \mu\} \cap B_{\rho}(X_{\epsilon})$. We have verified the following inclusion

$$\{C_1\epsilon < u^{\epsilon} < \mu\} \cap B_{\rho}(X_{\epsilon}) \subset \mathscr{N}_{\frac{1}{C_3}\mu}(\partial\{C_1\epsilon \le u^{\epsilon}\} \cap B_{2\rho}(X_{\epsilon}));$$

therefore, enlarging $C^{\star}$ if necessary and diminishing $\mu \ll \rho$ by universal proportions, we obtain

$$\{C_1\epsilon < u^{\epsilon} < \mu\} \cap B_{\rho}(X_{\epsilon}) \subset \bigcup 2B_j \subset B_{4\rho}(X_{\epsilon}).$$

From (6.10) and Lemma 6.3, we can write

$$\begin{aligned} C_4\mu\rho^{d-1} &\ge \int\limits_{B_{4\rho}(X_{\epsilon})\cap\{C_1\epsilon<u^{\epsilon}<\mu\}} |\nabla u^{\epsilon}|^2 dX \\ &\ge \frac{1}{m}\sum \int\limits_{2B_j\cap\{C_1\epsilon<u^{\epsilon}<\mu\}} |\nabla u^{\epsilon}|^2 dX \\ &\ge \frac{C_2}{m}\sum |B_j| \\ &\ge \frac{C_2}{m}\sum |\{C_1\epsilon < u^{\epsilon} < \mu\} \cap B_{\rho}(X_{\epsilon})|, \end{aligned}$$

for $C_4 > 0$, a universal constant, which concludes the proof of the Lemma. □

We are now ready to establish the $(d-1)$-Hausdorff estimate of approximating level sets, that are uniform with respect to the infinitesimal parameter ϵ. This will be performed by a combination of the optimal control we have upon solutions, together with Lemma 6.5.

Theorem 6.6. *Fixed $\Omega' \Subset \Omega$, there exists a universally positive constant $C = C(\Omega')$, such that*

$$\mathscr{L}^d\left(\mathscr{N}_\mu(\{C_1\epsilon < u^\epsilon\}) \cap B_\rho(X_\epsilon)\right) \le C\mu\rho^{d-1},$$

for $C_1 > 1$, $X_\epsilon \in \partial\{C_1\epsilon < u^\epsilon\} \cap \Omega'$, $d_\epsilon(X_\epsilon) \ll \operatorname{dist}(\Omega', \partial\Omega)$ and $C_1\epsilon \ll \rho$. In particular,

$$\mathscr{H}^{d-1}(\{u^\epsilon = C_1\epsilon\} \cap B_\rho(X_0)) \le C\rho^{d-1}, \tag{6.11}$$

for constants C and C_1 independent of ϵ.

Proof. By optimal regularity, for $Z \in \partial\{C_1\epsilon < u^\epsilon\}$ and $Y \in \mathscr{N}_\delta(\partial\{C_1\epsilon < u^\epsilon\}) \cap B_\rho(X_\epsilon) \cap \{C_1\epsilon < u^\epsilon\}$, there holds

$$u^\epsilon(Y) \le u^\epsilon(Z) + C|Z - Y| \le \mu + C\delta \le \kappa\mu,$$

for $\mu = C\,\delta$ and $\kappa > 0$ universal. Therefore, the inclusion

$$\left[\mathscr{N}_\delta(\partial\{C_1\epsilon < u^\epsilon\}) \cap B_\rho(X_\epsilon) \cap \{C_1\epsilon < u^\epsilon\}\right] \subset \left[\{C_1\epsilon < u^\epsilon < \kappa\mu\} \cap B_\rho(X_\epsilon)\right] \tag{6.12}$$

is verified. On the other hand, employing Corollary 5.4 and taking δ as above, we check that

$$\frac{\mathscr{L}^d(B_\delta(X) \cap \{u^\epsilon > C_1\epsilon\})}{\mathscr{L}^d(B_\delta(X))} \ge c \quad \text{for} \quad X \in \partial\{u^\epsilon > \epsilon\},$$

and hence we conclude that $\partial\{u^\epsilon > C_1\epsilon\}$ has the δ-density property. Lemma 6.4 sponsors the existence of the universal, positive constant M such that

$$\begin{aligned}\mathscr{L}^d(\mathscr{N}_\delta(\partial\{C_1\epsilon < u^\epsilon\}) \cap B_\rho(X_\epsilon)) \le{}& C_2\mathscr{L}^d(\mathscr{N}_\delta(\partial\{C_1\epsilon < u^\epsilon\}) \cap B_\rho(X_\epsilon) \cap \{C_1\epsilon < u^\epsilon\}) \\ &+ M\delta\rho^{d-1},\end{aligned}$$

thus, applying (6.12), we obtain

$$\mathscr{L}^d(\mathscr{N}_\delta(\partial\{C_1\epsilon < u^\epsilon\}) \cap B_\rho(X_\epsilon)) \le C_2\mathscr{L}^d(\{C_1\epsilon < u^\epsilon < \kappa\mu\} \cap B_\rho(X_\epsilon)) + M\delta\rho^{d-1},$$

for some universal constant $C_2 > 0$. Finally for $\mu \ll \rho$ Lemma 6.5 yields

$$\mathscr{L}^d\left(\mathscr{N}_\delta(\partial\{C_1\epsilon < u^\epsilon\}) \cap B_\rho(X_\epsilon)\right) \le C\delta\rho^{d-1},$$

for some $C > 0$.

In the sequel, we take a covering of $\partial\{C_1\epsilon < u^\epsilon\} \cap B_\rho(X_\epsilon)$ by balls $\{B_j\}$ centered at points along $\partial\{C_1\epsilon < u^\epsilon\} \cap B_\rho(X_\epsilon)$ with radius $\mu \ll 1$. We can write

$$\bigcup B_j \subset \mathscr{N}_\mu(\{C_1\epsilon < u^\epsilon\}) \cap B_{\rho+\mu}(X_\epsilon).$$

Thus, there exist universal constants $C_3, C_4 > 0$, such that

$$\begin{aligned}\mathscr{H}^{d-1}(\partial\{C_1\epsilon < u^\epsilon\} \cap B_\rho(X_\epsilon)) &\le C_3\sum \operatorname{Area}(\partial B_j) \\ &= \frac{C_3}{\mu}\sum \mathscr{L}^d(B_j) \\ &\le \frac{C_4}{\mu}\mathscr{L}^d(\mathscr{N}_\mu(\{C_1\epsilon < u^\epsilon\}) \cap B_{\rho+\mu}(X_\epsilon)) \\ &\le C_4C(\rho+\mu)^{d-1} = C_4C\rho^{d-1} + o(1).\end{aligned}$$

Letting $\mu \to 0$, we finish the proof of the Theorem. □

7. Asymptotic limits

In this section, we are interesting in geometric properties of a limiting function

$$u := \lim_{k\to\infty} u^{\epsilon_k},$$

for a subsequence $\epsilon_k \to 0$. From uniform Lipschitz regularity the family $\{u^\epsilon\}$ is pre-compact in $C^{0,1}_{loc}(\Omega)$. Hence, up to a subsequence, there exists a limiting function u, obtained as the uniform limit of u_ϵ, as $\epsilon \to 0$. One readily verifies that a limiting function u satisfies

(1) $0 \le u \le K_0$ in Ω;
(2) $u \in C^{0,1}_{\text{loc}}(\Omega)$;
(3) $0 \le |\nabla u|^\gamma F(D^2 u) \le \mathscr{C}$, in $\{u > 0\}$, in the viscosity sense.

Let us introduce the following notation:

$$\mathfrak{F}(u) := \partial\{u > 0\} \cap \Omega.$$

Combining (3) with the sharp regularity estimate established in [14] and [1], it follows that

$$u \in C^{1,\alpha}_{\text{loc}}(\{u > 0\}).$$

Such an estimate deteriores as we approach $\mathfrak{F}(u)$; notwithstanding from (2), the gradient remains bounded, even when $\text{dist}(X_0, \mathfrak{F}(u)) \to 0$.

In the particular case coming from the homogeneous flame propagation theory:

$$\zeta_\epsilon(t) = \frac{1}{\epsilon}\zeta\left(\frac{t}{\epsilon}\right),$$

where ζ is a continuous function supported in $[0, 1]$, then a limiting function satisfies

$$F(D^2 u) = 0, \qquad \text{in } \{u > 0\},$$

in view of [14, Lemma 6]. In this case, even though the gradient degeneracy is no longer present in the limiting equation, it does leave its *signature* on the expected linear behavior along the limiting transition boundary. For instance let us analyze one-dimensional profiles, i.e., limiting configuration of the equation

$$|u^\epsilon_x|^\gamma \cdot u^\epsilon_{xx} = \zeta_\epsilon(u^\epsilon). \tag{7.1}$$

Multiplying the above equation by $u^\epsilon_x dx$, we find the differential equality:

$$|u^\epsilon_x|^\gamma u^\epsilon_x \cdot (u^\epsilon_{xx} dx) = \zeta_\epsilon(u^\epsilon).u^\epsilon_x dx. \tag{7.2}$$

However,

$$\zeta_\epsilon(u^\epsilon).u^\epsilon_x dx = \frac{d}{dx}\mathfrak{Z}_\epsilon(u^\epsilon),$$

where $\mathfrak{Z}_\epsilon(x) := \int_0^{x/\epsilon} \zeta(s)ds \to \int \zeta(s)ds$, as $\epsilon \to 0$. Performing a change of variables

$$u^\epsilon_x(x) = v \quad \Longrightarrow \quad u^\epsilon_{xx} dx = dv,$$

we can write down:

$$\int u^\epsilon_x u^\epsilon_{xx} dx = \int |v|^\gamma v dv.$$

Thus, computing anti-derivatives in (7.2) and letting $\epsilon \to 0$, we obtain for a limiting function u that

$$|u'| = \sqrt[\gamma+2]{(\gamma+2)\int \zeta(s)ds}.$$

Taking $\gamma = 0$, we recover the classical free boundary condition in the isotropic flame propagation theory, see for instance [5].

Let us continue our discussion on the limiting geometric properties obtained as $\epsilon \to 0$ in (E_ϵ). Next we show that at each point Z_0 in the free transition boundary $\mathfrak{F}(u)$, there exists a cone like function with vertex Z_0 that traps the graph of the limiting function.

Theorem 7.1. *Let u be a limiting function, $u := \lim\limits_{k\to\infty} u^{\epsilon_k}$. Given $\Omega' \Subset \Omega$, for $X_0 \in \{u > 0\} \cap \Omega'$ with $\operatorname{dist}(X_0, \{u = 0\}) \ll \operatorname{dist}(\Omega', \partial\Omega)$, there exists a universal constant $C > 0$ such that,*

$$C^{-1}\operatorname{dist}(X_0, \mathfrak{F}(u)) \le u_0(X_0) \le C\operatorname{dist}(X_0, \mathfrak{F}(u)). \tag{7.3}$$

Proof. From Corollary 5.1, there exists $Y_\epsilon \in \{0 \le u^\epsilon \le \epsilon\} \cap \Omega'$ with $d_\epsilon(X) = |X - Y_\epsilon|$ such that

$$u^\epsilon(X) \ge c\, d_\epsilon(X) = c\,|X - Y_\epsilon|,$$

for some universal constant $c > 0$. Up to a subsequence, $Y_\epsilon \to Y_0 \in \{u = 0\}$ and hence

$$u(X) \ge c\,|X_0 - Y_0| \ge c\operatorname{dist}(X, \mathfrak{F}(u)).$$

The upper estimate follows readily from local Lipschitz continuity of u. □

Passing the limit as $\epsilon \to 0$ in Theorem 5.2, we the following sharp control on the maximum value of u within balls of universally small radii.

Theorem 7.2. *Let u be a limiting function, $u := \lim\limits_{k\to\infty} u^{\epsilon_k}$. Given $\Omega' \Subset \Omega$, there exist universal positive constants C_0 and r_0, such that*

$$C_0^{-1} r \le \sup_{B_r(X_0)} u \le C_0(r + u(X_0))$$

for any $X_0 \in \Omega' \cap \overline{\{u > 0\}}$ with $\operatorname{dist}(X_0, \partial\{u > 0\}) \ll \operatorname{dist}(X_0, \partial\Omega')$ and $r \le r_0$.

In the sequel, we show that the set $\{u > 0\}$ is the limit, in the Hausdorff distance, of $\{u^\epsilon > \epsilon\}$ as $\epsilon \to 0$. More precisely,

Theorem 7.3. *Let u be a limiting function, $u := \lim\limits_{k\to\infty} u^{\epsilon_k}$. Given $C_1 > 1$, the following inclusions,*

$$\{u > 0\} \cap \Omega' \subset \mathcal{N}_\delta(\{u^{\epsilon_k} > C_1\epsilon_k\}) \cap \Omega' \quad \text{and} \quad \{u^{\epsilon_k} > C_1\epsilon_k\} \cap \Omega' \subset \mathcal{N}_\delta(\{u > 0\}) \cap \Omega',$$

hold for $\delta \ll 1$ and $\epsilon_k \ll \delta$.

Proof. We will prove the first inclusion. Let assume for purpose of contradiction that there exist a subsequence $\epsilon_k \to 0$ and points $X_k \in \{u > 0\} \cap \Omega'$ such that

$$\operatorname{dist}(X_k, \{u^{\epsilon_k} > C_1\epsilon_k\}) > \delta. \tag{7.4}$$

By Theorem 7.2, and taking $k \gg 1$, we get

$$u^{\epsilon_k}(Y_k) = \sup_{B_{\frac{\delta}{2}}(X_k)} u^{\epsilon_k}(X_k) \ge \frac{1}{2} \cdot \sup_{B_{\frac{\delta}{2}}(X_k)} u(X_k) \ge c\delta \ge C_1\epsilon_k$$

for some $Y_k \in \overline{B_{\frac{\delta}{2}}(X_k)} \cap \{u^{\epsilon_k} > C_1\epsilon_k\}$, which contradicts (7.4). Similarly, we obtain the other inclusion. □

Theorem 7.4. *Given a subdomain $\Omega' \Subset \Omega$, there exists a universal constants $C > 0$ and $\rho_0 > 0$ depending on Ω' and universal parameters such that, for any $X_0 \in \mathfrak{F}(u)$ and $\rho \le \rho_0$, there holds*

$$C^{-1}\rho \le \fint_{\partial B_\rho(X_0)} u\, d\mathscr{H}^{d-1} \le C\rho. \tag{7.5}$$

As u satisfies the condition (7.5), we say that u is *locally uniformly non-degenerate* in $\mathfrak{F}(u)$. Such property is another way to rephrase Lipschitz continuity and nondegeneracy of u.

Proof of Theorem 7.4. By Lipschitz continuity, it is easily to check that upper estimate is valid. To prove the lower inequality, we consider $Y_\epsilon \in \partial\{u_\epsilon > 0\}$, satisfying

$$|Y_\epsilon - X_0| = \operatorname{dist}(X_0, \partial\{u_\epsilon > 0\}).$$

From Theorem 7.3, $Y_\epsilon \to X_0$. We now can pass the limit as $\epsilon_k \to 0$ in the thesis of Corollary 5.5 and Theorem is proven. □

Now, we show that the positive set $\{u > 0\}$ has uniform density along the free transition boundary $\mathfrak{F}(u)$.

Theorem 7.5. *Given $\Omega' \Subset \Omega$, there exists a universal constants $c_0 > 0$, such that for $X_0 \in \mathfrak{F}(u) \cap \Omega'$ there holds*

$$\frac{\mathscr{L}^d(B_\rho(X_0) \cap \{u > 0\})}{\mathscr{L}^d(B_\rho(X_0))} \geq c_0, \tag{7.6}$$

for $\rho \ll 1$. In particular, $\mathcal{L}^d(\mathfrak{F}(u)) = 0$.

Proof. Estimate (7.6) follows as in the proof of Theorem 5.4 and Remark 5.3. By Lebesgue differentiation theorem and simple covering arguments we conclude the proof of Theorem above. □

From this point on, we restrict our analysis to the class of operators satisfying asymptotically concavity, i.e. the condition (AC). The ultimate goal is to prove that the limiting free transition boundary $\mathfrak{F}(u)$ has local finite $\mathscr{H}^{d-1}$-Hausdorff measure.

Theorem 7.6. *Given $\Omega' \Subset \Omega$, there exists a constant $C > 0$, depending on Ω' and universal parameters such that, for $X_0 \in \mathfrak{F}(u) \cap \Omega'$,*

$$\mathscr{H}^{d-1}(\mathfrak{F}(u) \cap B_\rho(X_0)) \leq C\rho^{d-1}.$$

Proof. From Theorem 7.3, for $k \gg 1$ large enough, we have

$$\left[\mathscr{N}_\delta(\mathfrak{F}(u)) \cap B_\rho(X_0)\right] \subset \left[\mathscr{N}_{4\delta}(\partial\{u^{\epsilon_k} > C_1\epsilon_k\}) \cap B_{2\rho}(X_0)\right].$$

Assuming, $\epsilon_k \ll \delta \ll \rho \ll \operatorname{dist}(\Omega', \partial\Omega)$, the assumptions of Theorem 6.6 are satisfied, providing the following estimate on the Lebesgue measure of the δ-neighborhood,

$$\mathscr{L}^d(\mathscr{N}_\delta(\mathfrak{F}(u)) \cap B_\rho(X_0)) \leq \overline{C} \cdot \delta\rho^{d-1}.$$

Let $\{B_j\}$ be a covering of $\mathfrak{F}(u) \cap B_\rho(X_0)$ by balls centered at free boundary point on $\mathfrak{F}(u) \cap B_\rho(X_0)$ and radius $\delta > 0$. Clearly

$$\bigcup B_j \subset \mathscr{N}_\delta(\mathfrak{F}(u)) \cap B_{\rho+\delta}(X_0).$$

In conclusion, there exists a universal constant $\overline{C} > 0$ such that

$$\begin{aligned}
\mathscr{H}_\delta^{d-1}(\mathfrak{F}(u) \cap B_\rho(X_0)) &\leq \overline{C} \sum \operatorname{Area}(\partial B_j) = \frac{\overline{C}}{\delta}\mathscr{L}(B_j) \\
&\leq \frac{\overline{C}}{\delta}\mathscr{L}(\mathscr{N}_\delta(\mathfrak{F}(u)) \cap B_{\rho+\delta}(X_0)) \\
&\leq \overline{C}C(\rho+\delta)^{d-1} = \overline{C}C\rho^{d-1} + o(1).
\end{aligned}$$

The proof of the Theorem follows by letting $\delta \to 0$. □

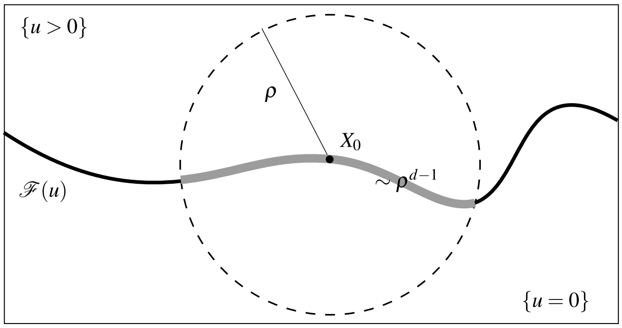

Fig. 5. Geometry of the $\mathscr{H}^{d-1}$-Hausdorff estimate.

The final result we deliver here states that the reduced transition boundary $\partial_{\text{red}}\{u > 0\} := \mathfrak{F}^\star(u)$ has total $\mathscr{H}^{d-1}$-measure (see Fig. 5). In fact we shall obtain an even stronger geometric measure estimate on such a set.

Theorem 7.7. *Given a subdomain $\Omega' \Subset \Omega$, there exists a positive universal constant $C = C(\Omega')$, such that for $\rho \ll 1$ and $X_0 \in \mathfrak{F}(u)$, there holds*

$$C^{-1}\rho^{d-1} \le \mathscr{H}^{d-1}(\mathfrak{F}^\star(u) \cap B_\rho(X_0)) \le C\,\rho^{d-1}.$$

In particular,

$$\mathscr{H}^{d-1}(\mathfrak{F}(u) \setminus \mathfrak{F}^\star(u)) = 0.$$

Proof. The upper estimate is a direct consequence of Theorem 7.6. Let us focus on the lower control. Initially we note that for $X_0 \in \{u > 0\}$, there holds

$$\int_{B_\rho(X_0)} f_{ij}\, D_{ij} u^{\epsilon_k}\, dX \ge 0, \tag{7.7}$$

for $\rho \ll 1$ and $k > 0$ large enough. Now, let us define the normalized function $v^k : B_1 \to \mathbb{R}$ by

$$v^k(X) := \frac{u^{\epsilon_k}(X_0 - \rho X)}{\rho}.$$

By Lipschitz estimates, up to a subsequence, $v^k \to v$ uniformly over compact subsets. We will furnish a special barrier. For that, let ψ be a nonnegative smooth function in B_1, with $\psi \equiv 1$ in $B_{1/5}$ and $\psi \equiv 0$ outside $B_{1/4}$. Let Φ be the solution to the Dirichlet problem

$$\begin{cases} L\Phi = -\psi, & \text{in} \quad B_1 \\ \Phi = 0, & \text{on} \quad \partial B_1, \end{cases}$$

where $Lv := \operatorname{tr}(f_{ij} D_{ij} v)$. It follows from classical elliptic regularity theory that

$$\|\Phi\|_{C^{1,\alpha}(B_{1/2})} \le C, \tag{7.8}$$

by some universal constant $C > 0$. By the maximum principle, $\Phi > 0$ in B_1, and so, Hopf maximum principle yields

$$f_{ij}\, \partial_i \Phi\, \eta_j \ge c > 0 \quad \text{along} \quad \partial B_1, \tag{7.9}$$

where η_j is the j-th coordinate of the outward normal vector to ∂B_1. Applying the generalized Gauss–Green formula, we derive

$$\begin{aligned} \int_{\{v>0\}\cap B_1} \left\{ \Phi L(v^k) - v^k L\Phi \right\} dX = & \int_{\partial_{\text{red}}\{v>0\}\cap B_1} \left\{ \Phi f_{ij} \partial_i v^k - v^k f_{ij} \partial_i \Phi \right\} \eta_j\, d\mathscr{H}^{d-1} \\ & - \int_{\{v>0\}\cap \partial B_1} v^k f_{ij}\, \partial_i \Phi\, \eta_j\, d\mathscr{H}^{d-1}. \end{aligned} \tag{7.10}$$

By (7.7), there holds

$$\liminf_{k\to\infty} \int_{\{v>0\}\cap B_1} \left\{\Phi L(v^k) - v^k L(\Phi)\right\} dX \geq \int_{B_1} \psi v^k dX \geq \int_{B_{1/5}} v^k dX. \tag{7.11}$$

Moreover, from uniform gradient bounds of v^k, ellipticity and (7.8), we estimate

$$\left| \int_{\partial_{\mathrm{red}}\{v>0\}\cap B_1} \Phi f_{ij}\, \partial_i v^k\, \eta_j\, d\mathscr{H}^{d-1} \right| \leq C_\star \mathscr{H}^{d-1}(\partial_{\mathrm{red}}\{v>0\}\cap B_1), \tag{7.12}$$

for a universal constant $C_\star > 0$. On the other hand,

$$\int_{\partial_{\mathrm{red}}\{v>0\}\cap B_1} v^k f_{ij}\, \partial_i \Phi\, \eta_j\, d\mathscr{H}^{d-1} = o(1), \tag{7.13}$$

as $k \to \infty$ and so, by (7.9), we obtain

$$\begin{aligned} \liminf_{k\to\infty} \int_{\{v>0\}\cap\partial B_1} v^k f_{ij}\, \partial_i \Phi\, \eta_j\, d\mathscr{H}^{d-1} &\geq \int_{\{v>0\}\cap\partial B_1} v\, f_{ij}\, \partial_i \Phi\, \eta_j\, d\mathscr{H}^{d-1} \\ &\geq 0. \end{aligned} \tag{7.14}$$

Combining (7.10)–(7.14), we deduce

$$\int_{B_{1/5}} v\, dX \leq C_\star \mathscr{H}^{d-1}(\partial_{\mathrm{red}}\{v>0\}\cap B_1). \tag{7.15}$$

Finally, by non-degeneracy, as in the proof of Theorem 7.4, there holds

$$\fint_{B_{1/5}} v\, dX \geq C, \tag{7.16}$$

for a positive universal constant C. Finally, from (7.15) and (7.16) we conclude

$$\mathscr{H}^{d-1}(\partial_{\mathrm{red}}\{v>0\}\cap B_1) \geq c_0,$$

for a universal constant c_0, and the estimate by below is established. The total measure of the reduced transition boundary follows by classical considerations. □

Conflict of interest statement

There is no conflict of interest.

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