Letters in Nonlinear Analysis and its Applications **1** (2023) No. 3, 96–100
https://doi.org/10.5281/zenodo.8362009
Available online at Lett. Nonlinear Anal. Appl.
Research Article

Letters in Nonlinear Analysis and its Application

Peer Reviewed Scientific Journal

On fractal cubes in dimension 3 and their components.

Dmitry Drozdov[a,b], Andrei Tetenov[b], Marina Tulina[c]

[a] *Novosibirsk State Technical University, Novosibirsk 630073, Russia*
[b] *Novosibirsk State University, Novosibirsk 630090, Russia.*
[c] *Gorno-Altaisk State University, Gorno-Altaisk 649000, Russia.*

Abstract

We show that a fractal cube F in $\mathbb{R}^3$ may have an uncountable set Q of connected components which are not contained in any plane, and the set Q is a totally disconnected self-similar subset of the hyperspace $C(\mathbb{R}^3)$, isomorphic to a Cantor set.

Keywords: fractal cube; fractal square; Hausdorff dimension; hyperspace.
2010 MSC: 28A80

1. Introduction.

Let $n \geq 2$ and let $\mathcal{D}$ be some subset $\{0, 1, \dots, n-1\}^k$, $2 \leq \#\mathcal{D} < n^k$. We call $\mathcal{D}$ a *digit set.* The unique non-empty compact set $F \subset \mathbb{R}^k$ satisfying

$$F = \frac{F + \mathcal{D}}{n}$$

is called a *fractal k-cube* (or fractal square if $k = 2$).

Let $H = F + \mathbb{Z}^k$ and $H^c = \mathbb{R}^k \setminus H$. We also define $I = [0, 1]^k$. The Hutchinson operator T for the cube F is defined by $T(A) := \dfrac{A + \mathcal{D}}{n}$ and we denote $F_1 = T(I)$.

For the case $k = 2$ when F is a fractal square, K. S. Lau, J. J. Luo and H. Rao [3] proved the following theorem.

URL: `d.drozdov1@g.nsu.ru` (Dmitry Drozdov), `a.tetenov@gmail.com` (Andrei Tetenov), `aniram.ru@googlemail.com` (Marina Tulina)

Received January 23, 2023, Accepted: May 01, 2022 , Online: May 14, 2023

Theorem 0.1. *Let F be a fractal square. Then F satisfies either*
(i) H^c has a bounded component, which is also equivalent to: F contains a non-trivial component that is not a line segment; or
(ii) H^c has an unbounded component, then F is either totally disconnected or all non-trivial components of F are parallel line segments.

We show that in the case of fractal cubes in $\mathbb{R}^3$, the situation is completely different. In our short note, we prove the following theorem

Theorem 0.2. *There is a fractal cube $F\subset\mathbb{R}^3$ such that the set H^c is connected and the set H is an uncountable union of unbounded components H_α, $\alpha \in \{0,1\}\infty$, each H_α being invariant with respect to $\mathbb{Z}^3$-translations.*

We construct F as a fractal cube for which $n = 5$ and whose digit set $\mathcal{D}\subset\{0, 1, ..., 4\}^3$ is a disjoint union of its subsets $\mathcal{D}_0$ and $\mathcal{D}_1$. These digit sets define disjoint connected fractal cubes K_0 and K_1.

The Hutchinson operators $T_i(A) := \dfrac{\mathcal{D}_i + A}{5}$ for $\mathcal{D}_i, i = 0, 1$ can be considered as contraction maps $\widetilde{T_i}$ of the hyperspace $C(\mathbb{R}^3)$ to itself whose Lipschitz constant is 1/5. For any finite $\bar{\alpha} = \alpha_1 \ldots \alpha_k \in \{0, 1\}^k, k \in \mathbb{N}$, we define $T_{\bar{\alpha}} = T_{\alpha_1}\circ\ldots\circ T_{\alpha_k}$ and $F_{\bar{\alpha}} = T_{\bar{\alpha}}\circ I$. Let $\mathcal{K}_{\bar{\alpha}}$ be the non-empty compact set satisfying $\mathcal{K}_{\bar{\alpha}} = T_{\bar{\alpha}}(\mathcal{K}_{\bar{\alpha}})$. For any infinite string $\alpha = \alpha_1\alpha_2.... \in \{0, 1\}^\infty$ we write $|\bar{\alpha}| = k$ and define

$$K_\alpha = \bigcap_{k=1}^{\infty} T_{\alpha_1\ldots\alpha_k}(I)$$

In our case, each K_α is a connected component of F and

$$F = \bigcup_{\alpha\in\{0,1\}^\infty} K_\alpha$$

.

ince all components K_α of F are not line segments, but H^c has no bounded components, so the statement (i) of Theorem0.1 does not hold. From the other side, the set H^c is connected and unbounded, but all components of F are not line segments, so (ii) does not work too.
Let Q be the set of connected components $\mathcal{K}_\alpha$ of F. We consider it as a subset of the set $C(\mathbb{R}^3)$ of compact subsets of $\mathbb{R}^3$ supplied with Hausdorff metrics. The set Q is uncountable and totally disconnected and isomorphic to a Cantor set.

Theorem 0.3. *The set $Q\subset C(\mathbb{R}^3)$ of connected components $\mathcal{K}_\alpha$ of F is a self-similar set generated by two contractions $\tilde{T_0}$ and $\tilde{T_1}$ of the hyperspace $C(\mathbb{R}^3)$. There is a Hölder homeomorphism $\varphi : Q \to C_{1/3}$ of the set Q to the middle-third Cantor set $C_{1/3}$ which induces the isomorphism of self-similar structures on these sets.*

Let $\alpha = \alpha_1\alpha_2 \ldots \in \{0, 1\}^\infty$. For any $m \in \mathbb{N}$ let λ_m be the density of zeros in the word $\alpha_1 \ldots \alpha_m$. Put $\bar{\lambda}_\alpha = \limsup_{m\to\infty} \lambda_m$, $\underline{\lambda}_\alpha = \liminf_{m\to\infty} \lambda_m$ and if $\underline{\lambda}_\alpha = \bar{\lambda}_\alpha$, we write $\lambda_\alpha = \lim_{m\to\infty} \lambda_m$. If α is preperiodic with period length p, the λ_α is equal to the density of zeros in any period of α.

We prove the following estimates for the dimension of the components K_α:

Theorem 0.4. *For any $\alpha \in \{0, 1\}^\infty$*

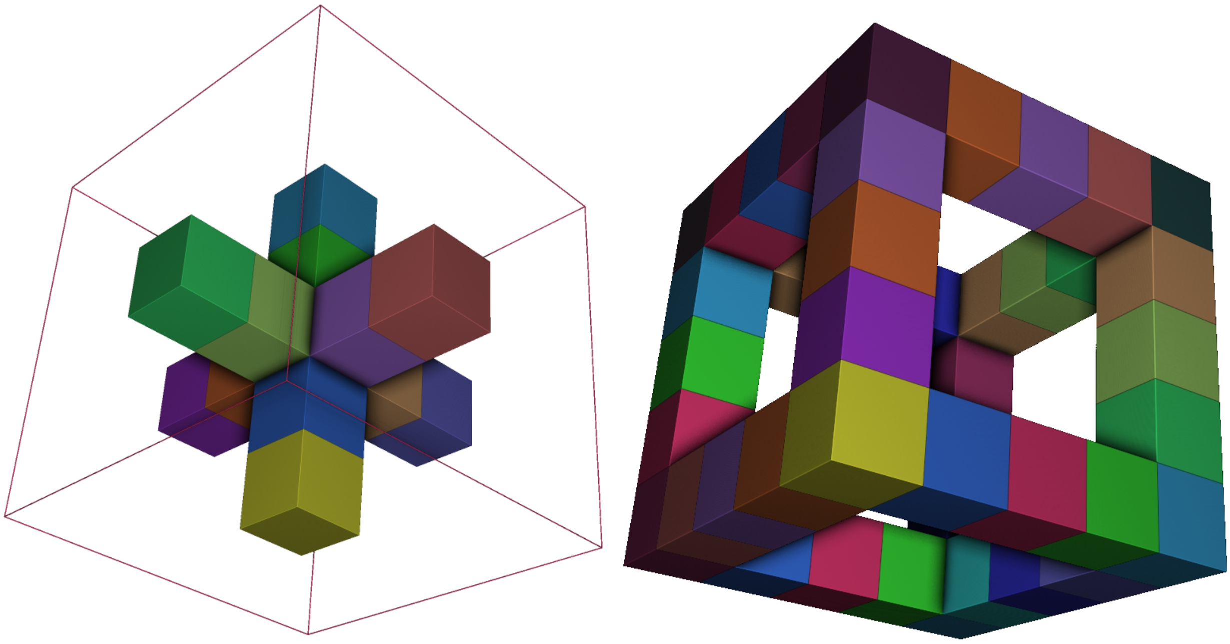
$$\underline{\dim}_B(\mathcal{K}_\alpha) = \bar{\lambda}_\alpha \log_5 13 + (1 - \bar{\lambda}_\alpha) \log_5 44,$$

$$\bar{\dim}_B(\mathcal{K}_\alpha) = \underline{\lambda}_\alpha \log_5 13 + (1 - \underline{\lambda}_\alpha) \log_5 44.$$

If α is preperiodic, then

$$\dim_H(\mathcal{K}_\alpha) = \dim_B(\mathcal{K}_\alpha) = \lambda_\alpha \log_5 13 + (1 - \lambda_\alpha) \log_5 44.$$

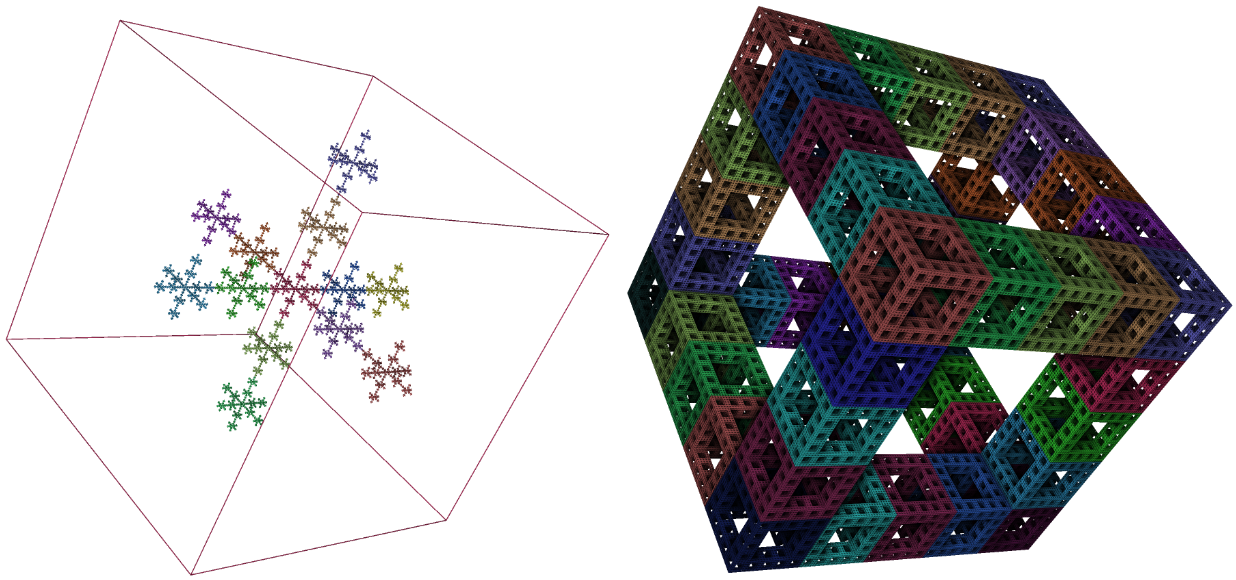
2. Construction of the set F



Cross(left) and Frame (right).

Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be the sets of the coordinates defining the cubes $\bar{x} + I$ forming subsets "Cross" and "Frame" of a cube $5 \cdot I$, so that $\mathcal{D}_0 + I$ is a "Cross" and $\mathcal{D}_1 + I$ is a "Frame".

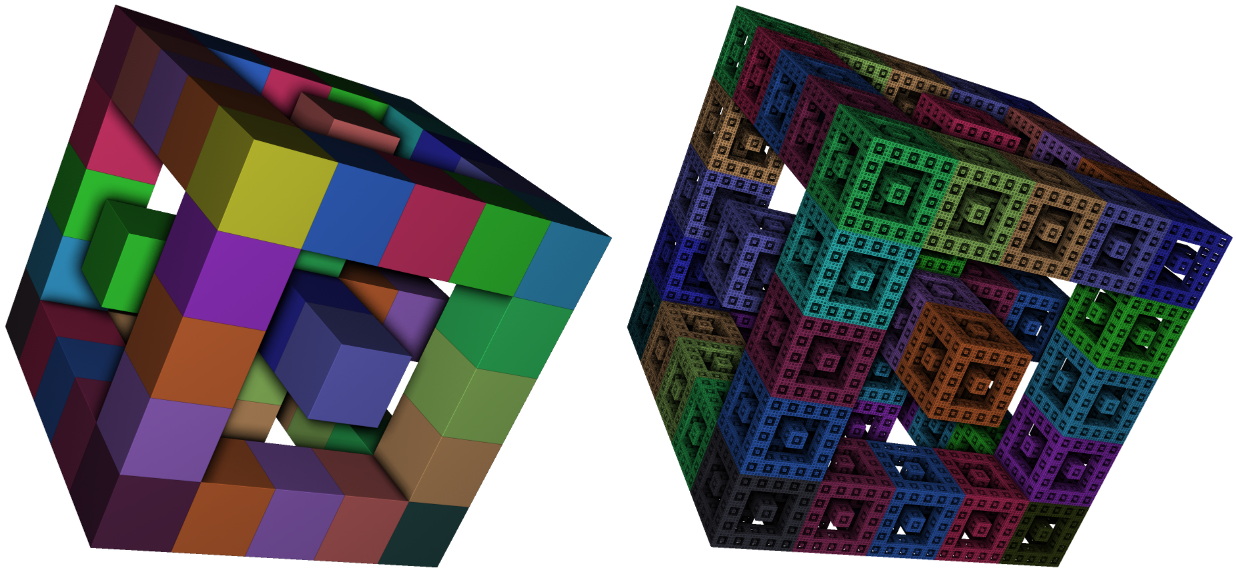
Let $T_0(A) := \dfrac{\mathcal{D}_0 + A}{5}$ and $T_1(A) := \dfrac{\mathcal{D}_1 + A}{5}$ be Hutchinson operators defined by $\mathcal{D}_0$ and $\mathcal{D}_1$. Let $\mathcal{D} = \mathcal{D}_0 \bigcup \mathcal{D}_1$ and $T(A) := \dfrac{\mathcal{D} + A}{5}$. Let $\mathcal{K}_0$ and $\mathcal{K}_1$ be the fractal cubes corresponding to $\mathcal{D}_0$ and $\mathcal{D}_1$.



Fractal cubes $\mathcal{K}_0$(left) and $\mathcal{K}_1$ (right).

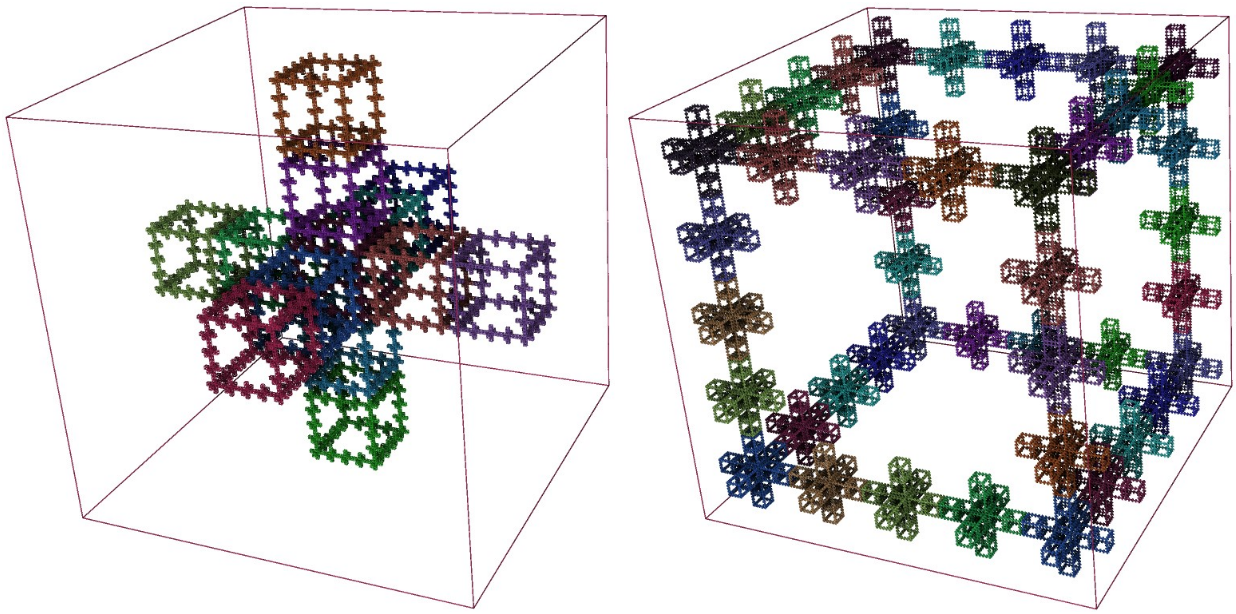
Applying the approach, developed by L.Cristea and B.Steinsky in [1], we see that since the set $\mathcal{D}_0 + I$ has exactly one pair of entrance and exit points on each pair of opposite faces of the cube $5I$ and has empty intersection with its edges and the intersection graph of $\mathcal{D}_0 + I$ is a tree, the fractal cube $\mathcal{K}_0$ is a dendrite, all of whose ramification points have order 6.

The fractal cube $\mathcal{K}_1$ is a $5 \times 5 \times 5$ version of Menger sponge.



The set $T(I)$ (left) and the fractal cube F (right).

We see from the construction that $\#\mathcal{D}_0 = 13$, $\#\mathcal{D}_1 = 44$. Moreover, the Hausdorff distance between $\mathcal{D}_0 + I$ and $\mathcal{D}_1 + I$ is $2\sqrt{2}$, while the minimal distance $d(\mathcal{D}_0 + I, \mathcal{D}_1 + I)$ between the points of those sets is equal to 1.



The components K_{01} and K_{10}

Lemma 0.5. *For any $\bar{\alpha}$, the sets $F_{\bar{\alpha}}$ and $\mathcal{K}_{\bar{\alpha}}$ are connected.*

Proof. For $i = 0, 1$, the intersections of F_i with any two opposite faces of the cube I are congruent with respect to the translation moving one face to the other.

For any two cubes $\dfrac{a+I}{5}, \dfrac{b+I}{5} \subset F_i$ with side $1/5$ in F_i there is a path from one cube to the other which intersects transversely the faces of pairs of adjacent cubes and does not intersect any of their edges.

It follows then, that for any i, j the sets $T_i(F_j)$ are connected and possess the same opposite face property.

Proceeding by induction, we get that the sets $T_{\alpha_1 \dots \alpha_k}(F_i)$ are connected and inherit the same property.

Since $\mathcal{K}_{\bar{\alpha}}$ is the intersection of a nested family of compact connected sets $(T_{\bar{\alpha}})^m(I), \mathcal{K}_{\bar{\alpha}}$ is connected too.

Lemma 0.6. *If* $\bar{\alpha}, \bar{\beta} \in \{0,1\}^k$ *and* $\bar{\alpha} \neq \bar{\beta}$, *then*
$F_{\bar{\alpha}} \bigcap F_{\bar{\beta}} = \varnothing$, $d_H(F_{\bar{\alpha}}, F_{\bar{\beta}}) < \dfrac{3\sqrt{5}}{5^{|\bar{\alpha}\wedge\bar{\beta}|}}$, *and* $d(F_{\bar{\alpha}}, F_{\bar{\beta}}) \geq \dfrac{1}{5^{|\bar{\alpha}\wedge\bar{\beta}|+1}}$

Proof. If $\alpha_1 \neq \beta_1$ then $F_{\bar{\alpha}} \subset F_{\alpha_1}, F_{\bar{\beta}} \subset F_{\beta_1}$, therefore

$$\begin{aligned} d_H(F_{\bar{\alpha}}, F_{\bar{\beta}}) &\leq d_H(F_{\bar{\alpha}}, F_{\alpha_1}) + d_H(F_{\alpha_1}, F_{\beta_1}) + d_H(F_{\bar{\beta}}, F_{\beta_1}) \\ &\leq \frac{2\sqrt{2}}{25} + \frac{2\sqrt{2}}{5} + \frac{2\sqrt{2}}{25} < \frac{3\sqrt{2}}{5} \end{aligned} \tag{0.1}$$

and $d(F_{\bar{\alpha}}, F_{\bar{\beta}}) \geq d(F_{\alpha_1}, F_{\beta_1}) > 1/5$. Notice, that $d_H(F_{\sigma\alpha}, F_{\sigma\beta}) \leq \dfrac{d_H(F_\alpha, F_\beta)}{5^{|\sigma|}}$ to get the desired statement.

Let $\bar{\alpha} = \alpha_1\alpha_2\alpha_3 \ldots \in \{0,1\}^\infty$. We define $\mathcal{K}_{\bar{\alpha}} = \bigcap\limits_{k=1}^{\infty} F_{\alpha_1 \ldots \alpha_k}$.

Corollary 0.7. *For any* $\bar{\alpha}, \bar{\beta} \in \{0,1\}^\infty$

$$\frac{1}{5^{|\bar{\alpha}\wedge\bar{\beta}|+1}} \leq d_H(F_{\bar{\alpha}}, F_{\bar{\beta}}) < \frac{3\sqrt{5}}{5^{|\bar{\alpha}\wedge\bar{\beta}|+1}},$$

and

$$d(F_{\bar{\alpha}}, F_{\bar{\beta}}) \geq \frac{1}{5^{|\bar{\alpha}\wedge\bar{\beta}|+1}}$$

Consider $\bar{\alpha} \in \{0,1\}^k, k \in \mathbb{N}$. The number $N_\delta(F_{\bar{\alpha}})$ of δ-mesh cubes for $(F_{\bar{\alpha}} = (T_{\bar{\alpha}}(I)$ and $\delta = 5^{-k}$ is equal to $13^m \cdot 44^{k-m}$, where $m = \#\{i \leq k : \alpha_i = 0\}$. Thus $-\dfrac{\log N_\delta}{\log \delta} = \dfrac{m}{k} \log_5 13 + (1 - \dfrac{m}{k}) \log_5 44$.

For any $\alpha \in \{0,1\}^\infty$, the minimal number of δ-mesh cube containing F_α is equal to $N_\delta(F_{\bar{\alpha}})$ if $\delta = 5^{-k}$ and $\bar{\alpha} = \alpha_1 \ldots \alpha_k$ is the k-th initial segment of α. Taking upper and lower limits as $k \to \infty$ we get desired estimates for $\dim_B(\mathcal{K}_\alpha)$.

References

[1] Cristea L. L., Steinsky B., Curves of infinite length in 4×4-labyrinth fractals,Geom. Dedicata, 141(2009), 1–17.
[2] Kuratowski K., Topology, Volumes 1 and 2, Academic Press and PWN,New York, 1966.
[3] K. S. Lau, J. J. Luo and H. Rao, Topological structure of fractal squares, Math. Proc.Camb.Philos.Soc. 155 (2013) 73-86.
[4] J.J. Luo, J.-C. Liu, On the classification of fractal squares, Fractals 24 (2016) 1650008.