Deformations of half-canonical Gorenstein curves in codimension four

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ABSTRACT

Recent work of Ablett (2021) and Kapustka, Kapustka, Ranestad, Schenck, Stillman and Yuan (2021) outlines a number of constructions for singular Gorenstein codimension four varieties. Earlier work of Coughlan, Gołębiowski, Kapustka and Kapustka (2016) details a series of nonsingular Gorenstein codimension four constructions with different Betti tables. In this paper we exhibit a number of flat deformations between Gorenstein codimension four varieties in the same Hilbert scheme, realising many of the singular varieties as specialisations of the earlier nonsingular varieties.

1. Introduction

Gorenstein varieties in low codimension The explicit construction of Gorenstein varieties is well understood in codimension three or less. Reid (2015) gave a general structure theorem for codimension four, but constructing Gorenstein ideals of codimension four is still not fully understood. Thus the case of codimension four is still an area of active study. By understanding concrete examples of codimension four Gorenstein varieties and the relationships between these varieties, we can hope to understand more about the general case.

Table 1

A stratification of the Betti tables of Schenck et al. (2022).

Classifying curves by genus and degree		
Degree	Genus	Corresponding Betti table
14	15	CGKK 1
15	16	CGKK 2, SSY 7, SSY 8
16	17	CGKK 3, SSY 3, SSY 4, SSY 6
17	18	CGKK 4, CGKK 5,6, SSY 2, SSY 5
18	19	CGKK 7,8, SSY 1
19	20	CGKK 9,10
20	21	CGKK 11

Definition 1.1. For a projective variety $X \subset \mathbb{P}^n$, we say X is projectively Gorenstein, or alternatively arithmetically Gorenstein, if the coordinate ring R of X is Cohen–Macaulay and the canonical module $\omega_R \cong R(a)$, where a represents a shift in the grading.

We henceforth refer to such varieties as Gorenstein varieties. The condition on the canonical module leads to interesting symmetry in the free resolution for such varieties. In codimension two it is shown by Serre (1960) that a variety is Gorenstein if and only if it is a complete intersection. In codimension three, the Buchsbaum–Eisenbud structure theorem Buchsbaum and Eisenbud (1977) shows that all Gorenstein ideals of height 3 are given by the $2n \times 2n$ Pfaffians of a $(2n + 1) \times (2n + 1)$ skew-symmetric matrix, for $n \in \mathbb{Z}_{\geq 1}$.

Attempts to classify codimension 4 Gorenstein varieties Recent work on Gorenstein varieties has focused in particular on Gorenstein Calabi–Yau threefolds, henceforth GoCY threefolds. Studying GoCY threefolds in $\mathbb{P}^7$ is an interesting subcase of the Gorenstein codimension four problem. This was first studied by Bertin (2009), and later by Coughlan, Gołębiowski, Kapustka and Kapustka, who published a list of nonsingular GoCY threefolds in $\mathbb{P}^7$ Coughlan et al. (2016) and gave evidence to suggest this list may be complete. Schenck, Stillman and Yuan made further progress, listing every possible Betti table for Artinian Gorenstein algebras of codimension and regularity four Schenck et al. (2022). The regularity four condition is a direct result of the Calabi–Yau condition. Note that since the coordinate ring R for a Gorenstein variety X is Cohen–Macaulay, cutting by a regular sequence leaves an Artinian ring with the same Betti table. Thus any higher dimensional codimension four Gorenstein variety whose coordinate ring has regularity four will have a Betti table given in Schenck et al. (2022). For each Betti table of Schenck et al. (2022), there are explicit descriptions of one or more families of varieties with that Betti table (see Coughlan et al. (2016), Schenck et al. (2022), Ablett (2021), Kapustka et al. (2021)). Further, the Betti tables of Schenck et al. (2022) may be split into two parts. Those appearing in Table 1 of Schenck et al. (2022) correspond to at least one family of nonsingular threefolds in $\mathbb{P}^7$ (see Coughlan et al. (2016)), and are therefore referred to as the CGKK Betti tables. Those appearing in Table 2 of Schenck et al. (2022) do not occur for nonsingular threefolds (see Schenck et al. (2022)), and we refer to these as the SSY Betti tables.

The results of this paper It was shown in Schenck et al. (2022) that the SSY Betti tables cannot correspond to smooth GoCY threefolds. Moreover, almost all the families of curves given in Ablett (2021) are families of singular stable nodal curves, in contrast to the families of nonsingular threefolds of Coughlan et al. (2016). See Table 1 for a description of the families of curves in terms of their genus and degree. It thus becomes an interesting question of whether the singular varieties in Ablett (2021), Kapustka et al. (2021) can be smoothed to one of the nonsingular varieties of Coughlan et al. (2016) in the same Hilbert scheme. We summarise our results in the following theorem, which answers this question in the affirmative.

Theorem 1.2. *Let β be an SSY Betti table of Schenck et al. (2022). Then*

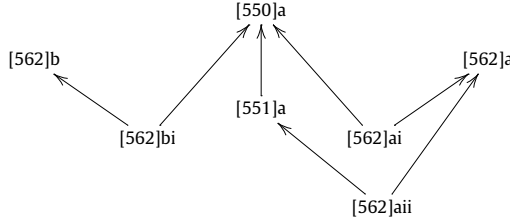(i) *There is at least one family of curves with Betti table β.*

Fig. 1. Degree 15 strata and incidences.

(ii) *Each family in part (i) contains a subfamily whose general element admits a flat deformation to a curve with a CGKK Betti table.*

Refined statements listing all the families and subfamilies that we consider are in Figs. 1, 3, 4 and §5.

By a result of Hartshorne (1966), the Hilbert scheme parametrising subschemes of $\mathbb{P}^n_k$ with fixed Hilbert function is connected. Hence Theorem 1.2 is unsurprising. On the other hand, our varieties and subsequent deformations are outlined concretely in terms of ideal generators, which does not follow from Hartshorne's result.

We can often express these generators using Pfaffians of matrices and other similar techniques, giving a simple description of both the ideal and the deformation. We repeatedly utilise the "Cramer's rule" format originally seen in Kustin and Miller (1980) and explained in (Papadakis and Reid, 2000, §2.8). Consider a 3×4 matrix M and column vector v of length 4 with entries in some polynomial ring, and a further parameter s. Then the equations obtained from

$$Mv = 0, \quad \bigwedge^3 M = sv$$

define a variety. By $\bigwedge^3 M = sv$ we mean that $sv_i = (-1)^i \det M_{\hat{i}}$, where $M_{\hat{i}}$ is the 3×3 matrix obtained by deleting the ith column of M. Several of our families of varieties may be described by imposing certain conditions on M, v and s. For example, when M and v have generic linear entries, and s is also general and quadratic, we obtain a family of varieties with Betti table CGKK 4.

We explain our naming convention for the families described in Coughlan et al. (2016), Schenck et al. (2022), Ablett (2021) and Kapustka et al. (2021). For simplicity, in this paragraph we only consider families of curves. As observed in Kapustka et al. (2021), each Betti table is uniquely identified by its first row $b_{12}b_{23}b_{34}$. The symbol $[b_{12}b_{23}b_{34}]$ refers to the locus $\mathcal{F}_{[b_{12}b_{23}b_{34}]}$ of quaternary quartics whose apolar ring has that Betti table. An additional letter inside the brackets (for example [300a]), specifies an irreducible subset (of the locus $\mathcal{F}_{[300]}$) as identified in Kapustka et al. (2021). We add decorations outside the brackets to denote the families discussed in this paper. For example, consider the family of curves [441b]a. The first row of the Betti table of a general curve in this family is 441. The b inside the brackets indicates that the family corresponds to a certain stratum of quartics, $\mathcal{F}_{[441b]}$. The a outside of the brackets is our primary identifier for this family of varieties. Different families corresponding to the same stratum of quartics are differentiated by our primary identifier. For example, the two families corresponding to $\mathcal{F}_{[562]}$ are called [562]a and [562]b. A specialisation of a family is denoted by appending roman numerals to the name e.g. [441b]a specialises to [441b]ai. We sometimes write [562]a to refer to the family of curves [562]a. An element of (the family of curves) [562]a is called a curve in [562]a or sometimes a curve of Type [562]a.

Fig. 1 gives an overview of the results for degree 15. Similarly, Figs. 3 and 4 describe the results for degrees 16 and 17 respectively. An arrow $A \to B$ means that we can exhibit a flat deformation whose central fibre is a general element of the stratum A and whose general fibre is a general element of the stratum B.

It is interesting to compare and contrast our strata and their incidences with the stratification of the space of quartics in four variables in Kapustka et al. (2021). We do not claim to have a stratifica-

tion of the Hilbert scheme of curves in $\mathbb{P}^5$. Indeed, we know of other strata which are not included in the scope of this paper. Moreover, we have not counted parameters or moduli.

Lifting to higher dimensional varieties   Using equations to describe our families of curves means that we can consider them more generally as families of codimension four varieties in a given n-dimensional projective space. These are referred to as liftings in Kapustka et al. (2021). In many situations, we can lift our smoothings of curves to smoothings of higher dimensional varieties too. However, it may happen that the lifted variety does not have a smoothing, because the deformation is obstructed in some lower dimension.

We make this more precise in degree 15. Family [550]a lifts to dimension 6, where it is the intersection of a cubic hypersurface with the 7-dimensional projective cone over $\mathrm{Gr}(2, 5)$. Any further lifting results in singularities where the cubic intersects the vertex of the cone.

Family [551]a lifts to at least dimension 5 if we assume that a_i, b_i are coordinates along with x_i and y. Similarly, the special family [562]bi lifts to dimension 5 if we assume that L_0, L_1, L_2, L_3 are coordinates. The smoothings of [551]a and [562]bi to [550]a both lift to dimension 5 without obstruction.

On the other hand, family [562]a lifts to dimension 9 if we assume that a_{ij}, b_{ij} are coordinates along with x_i, y_j. Since there is a deformation from [562]ai to [551]a and the maximal dimension of a lifting of [551]a is 6, it follows that the smoothing of [562]ai must be obstructed in dimensions ≥ 7. Indeed, the deformation to [550]a collects all the terms involving b_{ij} into the equation for the cubic hypersurface. The resulting 9-dimensional variety of type [550]a is singular along the intersection of this cubic hypersurface with the 3-dimensional vertex of the cone over $\mathrm{Gr}(2, 5)$ with coordinates b_{ij}.

How we prove our results   We prove Theorem 1.2 by constructing explicit flat families, typically over $\mathbb{A}^1$ but in one case over $\mathbb{A}^4$, whose special fibre is a singular variety with an SSY Betti table and whose general fibre is a variety in one of the families of Coughlan, Gołębiowski, Kapustka and Kapustka. Explicit details of the deformations are outlined in sections 2, 3, 4 and 5. Note that the deformations for the curves of Type [551]a and [562]a are due to Jan Stevens.

To show that the deformations are indeed flat we utilise the fact that for a Noetherian integral scheme T and a family $X \subset \mathbb{P}_T^n$, X is flat over T if the Hilbert polynomial is independent of $t \in T$ at every fibre X_t (see Hartshorne (1977), III.9, page 261). Thus in our case, where $t \in \mathbb{A}^1$ or occasionally $\mathbb{A}^4$, it is enough to check the dimension and corresponding Betti table of every fibre X_t, since the Hilbert polynomial can be directly calculated from this information. Table 1 organises the Betti tables of Schenck et al. (2022) into strata according to the degree and genus of the stable curves in Ablett (2021). Equidimensional varieties with Betti tables in the same strata will have the same Hilbert polynomial. We use Magma Bosma et al. (1997) to check that our families have the correct dimension and Betti table.

In sections 2, 3, 4 and 5 we outline details of each family of varieties and its deformations. The different sections correspond to different degrees, with section 2 describing the degree 15 varieties up to section 5 describing the degree 18 varieties. Further each section is split into subsections, with each subsection describing a different construction in that degree and its possible deformations.

This work builds on the talk "Gorenstein curves of codimension four" given by Ablett at the MEGA 2022 conference, which discussed the results of Ablett (2021). The original constructions in Ablett (2021) and the flat families in this paper make frequent use of the Magma Bosma et al. (1997) computer algebra software. The code is available at

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github.com/PatienceAblett/GorensteinCodim4Deformations
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and runs in the Magma online calculator.

Many of the varieties in the paper are defined using skew-symmetric matrices. We utilise the convention of only specifying the upper right triangular entries of the skew-symmetric matrix, since the rest of the entries are given by the skew-symmetry. We use $\mathrm{Pf}_{\hat{i}} M$ to denote the maximal Pfaffian of the submatrix of M obtained by removing the ith row and column. When there is only one matrix used in a construction we drop the M here for brevity.

2. Deformations in degree 15

In this section we outline a series of deformations of degree 15 curves, exhibiting each of our singular curves in a flat family, whose general fibre is a nonsingular curve section of one of the constructions of Coughlan, Gołębiowski, Kapustka and Kapustka. In our naming convention, family no. 2 of Coughlan, Gołębiowski, Kapustka and Kapustka is called family [550]a. Note that we do not claim this is the only possible family of varieties with this Betti table. The family in Ablett (2021) with Betti table SSY 7 is called family [551]a. The two constructions with Betti table SSY 8, as seen in Ablett (2021) and Kapustka et al. (2021), are referred to as families [562]a and [562]b. Here the letter distinguishing between the two families is outside the brackets. Both families are associated to the same irreducible locus of quaternary quartics from Kapustka et al. (2021). Note that the smoothing of a general curve in the Type [551]a family is outlined in Ablett (2021), as well as further details on all the families described in this section.

2.1. Family [550]a

The family [550]a is originally outlined in Coughlan et al. (2016), as $\mathrm{Gr}(2,5) \cap X_3 \cap \mathbb{P}^7$, where X_3 is a general cubic hypersurface. In other words, we take a linear $\mathbb{P}^7$ section of $\mathrm{Gr}(2,5)$ in its Plücker embedding and intersect with a cubic hypersurface. If we instead intersect $\mathrm{Gr}(2,5)$ with $\mathbb{P}^5$ to obtain a surface, this is the degree 5 del Pezzo surface. Subsequently intersecting with a cubic hypersurface defines a nonsingular curve with Betti table CGKK 2.

	0	1	2	3	4
0	1	–	–	–	–
1	–	5	5	–	–
2	–	1	–	1	–
3	–	–	5	5	–
4	–	–	–	–	1

Betti table CGKK 2 Schenck et al. (2022)

2.2. Family [551]a

We start by recasting family [551]a, which was originally constructed in Ablett (2021), as an example of the "Cramer's rule" format.

Lemma 2.1. *Consider the following matrix M, vector v, and parameter s:*

$$M = \begin{pmatrix} 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \\ 0 & c_0 & c_1 & c_2 \end{pmatrix}, \quad v = \begin{pmatrix} F_2 \\ x_0 \\ x_1 \\ x_2 \end{pmatrix}, \quad s = y.$$

Here the a_i and b_i are linear forms, the c_i are quadratic forms and F_2 is a cubic form in $x_0, x_1, x_2, x_3, x_4, y$. Let C be the variety in $\mathbb{P}^5$ defined by equations $Mv = 0$, $\bigwedge^3 M = sv$. Then $C = C_1 + C_2$ is a curve with Betti table SSY 7. We call this family [551]a.

	0	1	2	3	4
0	1	–	–	–	–
1	–	5	5	1	–
2	–	1	2	1	–
3	–	1	5	5	–
4	–	–	–	–	1

Betti table SSY 7 Schenck et al. (2022)

We use the terminology $C_1 + C_2$ to denote the union of two curves which intersect transversely.

We describe the geometry of the Type [551]a curve constructed in the above Lemma more precisely. Three of the five quadrics defining C are

$$Q_0 = x_0 y, \quad Q_1 = x_1 y, \quad Q_2 = x_2 y.$$

It follows that C breaks into two pieces, $C_1 \subset \mathbb{P}^4_{\langle x_0, \ldots, x_4 \rangle}$ and $C_2 \subset \mathbb{P}^2_{\langle x_3, x_4, y \rangle}$.

The three equations coming from the product $Mv = 0$ are

$$Q_3 = a_0 x_0 + a_1 x_1 + a_2 x_2, \quad Q_4 = b_0 x_0 + b_1 x_1 + b_2 x_2, \quad F_1 = c_0 x_0 + c_1 x_1 + c_2 x_2.$$

Let Γ be the $(2, 2, 3)$ complete intersection defined by Q_3, Q_4, F_1. Then Γ contains the line $l = \mathbb{P}^1_{\langle x_3, x_4 \rangle}$ and C_1 is the residual curve to l in Γ, which is defined by Q_3, Q_4, F_1 and a further quartic H, which is the determinant of the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}.$$

The plane curve C_2 is then defined by the more general quartic $H' = H + y F_2$, where F_2 is a cubic form. The ideal defining C is $I_C = (Q_0, \ldots, Q_4, F_1, H')$.

Thus $C = C_1 + C_2 \subset \mathbb{P}^5$, where $C_1 \subset \mathbb{P}^4$ is residual to a line in a $(2, 2, 3)$ complete intersection and $C_2 \subset \mathbb{P}^2$ is a plane quartic "ear", meeting C_1 in four points on a line.

Jan Stevens showed that there is a flat family over $\mathbb{A}^1$ whose general fibre is a curve of Type [550]a and whose fibre over $t = 0$ is a general curve of Type [551]a, see Ablett (2021). Writing the equations using Cramer's rule allows us to describe this deformation to Type [550]a with ease. Replacing M with

$$M_t = \begin{pmatrix} 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \\ t & c_0 & c_1 & c_2 \end{pmatrix},$$

the equations $M_t v = 0$, $\bigwedge^3 M_t = sv$ now define a Type [550]a variety if t is invertible, as shown in Ablett (2021).

2.3. Family [562]a

We present two different families of curves with Betti table SSY 8 and outline a deformation for each construction.

	0	1	2	3	4
0	1	–	–	–	–
1	–	5	6	2	–
2	–	2	4	2	–
3	–	2	6	5	–
4	–	–	–	–	1

Betti table SSY 8 Schenck et al. (2022)

We construct the first family, which is called [562]a. Let $x_1, \ldots, x_4, y_1, y_2$ be coordinates on $\mathbb{P}^5$ and choose cubics

$$F_1 = a_{13} x_3 x_1 + a_{14} x_4 x_1 + a_{23} x_3 x_2 + a_{24} x_4 x_2,$$

$$F_2 = b_{13} x_3 x_1 + b_{14} x_4 x_1 + b_{23} x_3 x_2 + b_{24} x_4 x_2$$

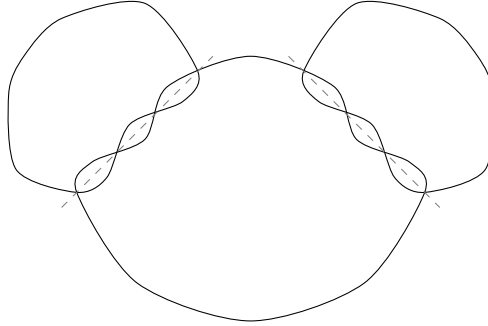where a_{ij}, b_{ij} are linear forms. Now we can state:

Fig. 2. Curve [562]a – Big Ears.

Lemma 2.2. *Let C be the curve in $\mathbb{P}^5$ defined by the 4×4 Pfaffians of the following two skew-symmetric matrices, and one additional equation:*

$$M_1 = \begin{pmatrix} y_2 & 0 & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \\ & 0 & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ & & x_2 & -x_1 \\ & & & H_{34} \end{pmatrix}, \quad M_2 = \begin{pmatrix} y_1 & 0 & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} \\ & 0 & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} \\ & & x_4 & -x_3 \\ & & & H_{12} \end{pmatrix}, \quad y_1 y_2 = 0. \tag{2.1}$$

Here H_{12} and H_{34} are general cubics. Then C has Betti table SSY 8. We call this family of curves [562]a.

Note that two of the Pfaffians of M_1 are also repeated as Pfaffians of M_2. These are the cubics $F_1 = \mathrm{Pf}_{\hat{1}} M_1 = \mathrm{Pf}_{\hat{1}} M_2$, and $F_2 = \mathrm{Pf}_{\hat{2}} M_1 = \mathrm{Pf}_{\hat{2}} M_2$. Thus there are nine ideal generators.

Writing out the quadric equations from the Lemma, we get

$$Q_1 = x_3 y_1, \quad Q_2 = x_4 y_1, \quad Q_3 = x_1 y_2, \quad Q_4 = x_2 y_2, \quad Q_5 = y_1 y_2.$$

Hence $C = C_1 + C_2 + C_3$ where $C_1 \subset \mathbb{P}^3_{\langle x_1, \dots, x_4 \rangle}$, $C_2 \subset \mathbb{P}^2_{\langle x_3, x_4, y_2 \rangle}$, and $C_3 \subset \mathbb{P}^2_{\langle y_1, x_1, x_2 \rangle}$.

First consider C_1. The two repeated Pfaffians F_1 and F_2 define a $(3, 3)$ complete intersection which is residual to the lines $l_{34} = \mathbb{P}^1_{\langle x_3, x_4 \rangle}$ and $l_{12} = \mathbb{P}^1_{\langle x_1, x_2 \rangle}$. The residual curve is C_1, with ideal $(F_1, F_2, q_{12}, q_{34})$, where the quartics q_{12} and q_{34} are given by the determinants of Jacobian matrices as follows:

$$q_{34} = \frac{\partial(F_1, F_2)}{\partial(x_1, x_2)} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix}, \quad q_{12} = \frac{\partial(F_1, F_2)}{\partial(x_3, x_4)} = \begin{pmatrix} \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} \\ \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} \end{pmatrix}.$$

The quartic $G_{34} = q_{34} + y_2 H_{34}$ defines $C_2 \subset \mathbb{P}^2_{\langle x_3, x_4, y_2 \rangle}$, and $G_{12} = q_{12} + y_1 H_{12}$ defines $C_3 \subset \mathbb{P}^2_{\langle y_1, x_1, x_2 \rangle}$. Here H_{12} and H_{34} are general cubics.

Thus $C \subset \mathbb{P}^5$ is a union of three curves, $C_1 \subset \mathbb{P}^3$ residual to a line pair in a $(3, 3)$ complete intersection, and two plane quartic "ears" C_2 and C_3. Each quartic ear meets C_1 in four nodes lying on a residual line. We therefore refer to a curve in family [562]a colloquially as "Big Ears" (see Fig. 2 for a schematic diagram).

Proposition 2.3. *There is a specialisation [562]ai of the family [562]a such that a general curve in [562]ai admits a deformation to a curve in family [550]a. We construct a flat family over $\mathbb{A}^1$ whose special fibre is a curve in family [562]ai and whose general fibre is a curve in family [550]a.*

Proof. The following deformation was first explained to us by Jan Stevens. Let [562]ai denote the special subfamily of family [562]a where $H = H_{12} = H_{23}$. Let $t \in \mathbb{A}^1$ be a degree 0 deformation parameter. We perturb the matrices (2.1) to the following:

$$M_1' = \begin{pmatrix} y_2 & t & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \\ & 0 & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ & & x_2 & -x_1 \\ & & & H \end{pmatrix}, \quad M_2' = \begin{pmatrix} y_1 & t & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} \\ & 0 & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} \\ & & x_4 & -x_3 \\ & & & H \end{pmatrix}.$$

The quadric Q_5 also becomes $Q_5' = y_1 y_2 - t^2(a_{14}a_{23} - a_{13}a_{24})$. Since we assumed that $H_{12} = H_{34}$, we maintain the coincidences $\mathrm{Pf}_{\hat{i}} M_1' = \mathrm{Pf}_{\hat{i}} M_2'$ for $i = 1, 2$. We write out the perturbed Pfaffians in full. The Pfaffian quadrics deform to

$$Q_3' = \mathrm{Pf}_{\hat{4}} M_1' = x_1 y_2 + t \frac{\partial F_1}{\partial x_2}, \qquad Q_4' = \mathrm{Pf}_{\hat{5}} M_1' = x_2 y_2 - t \frac{\partial F_1}{\partial x_1},$$
$$Q_1' = \mathrm{Pf}_{\hat{4}} M_2' = x_3 y_1 + t \frac{\partial F_1}{\partial x_4}, \qquad Q_2' = \mathrm{Pf}_{\hat{5}} M_2' = x_4 y_1 - t \frac{\partial F_1}{\partial x_3},$$

one of the repeated cubic Pfaffians deforms to

$$F_2' = \mathrm{Pf}_{\hat{2}} M_1' = \mathrm{Pf}_{\hat{2}} M_2' = F_2 + tH,$$

and the three remaining Pfaffians (one repeated) are unchanged:

$$F_1' = \mathrm{Pf}_{\hat{1}} M_1' = \mathrm{Pf}_{\hat{1}} M_2' = F_1,$$
$$G_{12}' = \mathrm{Pf}_{\hat{3}} M_1' = y_1 H + \det \frac{\partial(F_1, F_2)}{\partial(x_3, x_4)} = G_{12},$$
$$G_{34}' = \mathrm{Pf}_{\hat{3}} M_2' = y_2 H + \det \frac{\partial(F_1, F_2)}{\partial(x_1, x_2)} = G_{34}.$$

If t is invertible, then these three unchanged Pfaffians are redundant. That is, they are not needed as ideal generators. Indeed, using Q_i' to eliminate the partial derivatives of F_1, we have

$$tF_1 = t\frac{\partial F_1}{\partial x_1} x_1 + t\frac{\partial F_1}{\partial x_2} x_2 = (x_2 y_2)x_1 + (-x_1 y_2)x_2 = 0,$$

and

$$tG_{12} = ty_1 H + t \det \begin{pmatrix} \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} \\ \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} \end{pmatrix} = ty_1 H + \det \begin{pmatrix} x_4 y_1 & -x_3 y_1 \\ \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} \end{pmatrix},$$
$$= ty_1 H + y_1(\frac{\partial F_2}{\partial x_3} x_3 + \frac{\partial F_2}{\partial x_4} x_4) = y_1 F_2'.$$

Similarly, G_{34}' reduces to $y_2 F_2'$ and thus, F_1', G_{12}' and G_{34}' are redundant when t is invertible. The remaining Pfaffian equations Q_1', Q_2', Q_3', Q_4', F_2' and the extra equation $y_1 y_2 - t^2(a_{14}a_{23} - a_{13}a_{24})$ define a curve in family [550]a as follows:

$$\mathrm{Pf} \begin{pmatrix} y_1 & x_2 & ta_{14} & ta_{13} \\ & -x_1 & ta_{24} & ta_{23} \\ & & -x_3 & x_4 \\ & & & y_2 \end{pmatrix}, \quad F_2 + tH.$$

Consider the ideal $I_t = (Q_1', \ldots, Q_4', Q_5, F_1, F_2', G_{12}, G_{34})$. By the above, when t is invertible I_t defines a curve of Type [550]a. Clearly I_0 defines a curve of Type [562]ai. Every fibre has the same dimension and degree, and consequently the same Hilbert polynomial. Since the Hilbert polynomial at each fibre is independent of t this is indeed a flat family over $\mathbb{A}^1$. $\quad\square$

There is a second, symmetric deformation to [550]a if we put t in entry 23 of the perturbed matrices M_i', instead of entry 13.

Proposition 2.4. *Let [562]aii be the specialisation of the family of curves [562]a obtained by requiring $H_{34} = 0$. Then a general curve in [562]aii can be deformed to a curve in [551]a. We exhibit a flat family over $\mathbb{A}^1$ whose fibre at $t = 0$ is a general curve in [562]aii, and whose general fibre is a curve in [551]a.*

Proof. The following deformation was first explained to us by Jan Stevens. Let [562]aii denote the special subfamily of [562]a where $H_{34} = 0$. Replace M_1 with the matrix

$$M_1' = \begin{pmatrix} y_2 & t & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \\ & 0 & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ & & x_2 & -x_1 \\ & & & 0 \end{pmatrix},$$

and leave M_2 as before. Two Pfaffians are perturbed and the rest remain unchanged. Specifically, we now have

$$Q_3' = \mathrm{Pf}_{\hat{4}}\, M_1' = x_1 y_2 + t \frac{\partial F_1}{\partial x_2}, \quad Q_4' = \mathrm{Pf}_{\hat{5}}\, M_1' = x_2 y_2 - t \frac{\partial F_1}{\partial x_1},$$

and we can use these to show that G_{01} and F_1 are not needed as ideal generators when t is invertible:

$$t F_1 = t \frac{\partial F_1}{\partial x_1} x_1 + t \frac{\partial F_1}{\partial x_2} x_2 = (x_2 y_2) x_1 + (-x_1 y_2) x_2 = 0,$$

$$t G_{34} = t \det \frac{\partial (F_1, F_2)}{\partial (x_1, x_2)} = (x_2 y_2) \frac{\partial F_2}{\partial x_2} - (-x_1 y_2) \frac{\partial F_2}{\partial x_1} = y_2 F_2.$$

Moreover, for t invertible, the remaining seven generators define a curve in [551]a as follows:

$$M = \begin{pmatrix} 0 & a_{23} & a_{24} & x_1 \\ 0 & a_{13} & a_{14} & -x_2 \\ 0 & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} & 0 \end{pmatrix}, \quad v = \begin{pmatrix} H_{12} \\ t x_3 \\ t x_4 \\ y_2 \end{pmatrix}, \quad s = y_1.$$

Consequently the ideal $I_t = (Q_1, Q_2, Q_3', Q_4', Q_5, F_1, F_2, G_{12}, G_{23})$ defines a Type [562]aii curve when $t = 0$ and a Type [551]a curve otherwise, and this is again a flat family over $\mathbb{A}^1$. $\square$

The specialisation $H_{34} = 0$ imposes a nodal ear on the general curve of Type [562]aii. The deformation smooths the central curve and the nodal ear together into a curve of degree 11 which is residual to a line in a $(2, 2, 3)$ complete intersection, just as in Type [551]a. The other ear is deformed trivially.

2.4. Family [562]b

The second family with Betti table SSY 8 is outlined in Kapustka et al. (2021), and is similar to family [441b]a described in 3.5.

Lemma 2.5. *Let $x_0, x_1, y_0, y_1, z_0, z_1$ be coordinates on $\mathbb{P}^5$ and let C be a curve in $\mathbb{P}^5$ defined by the following nine equations:*

$$Q_1 = x_0 y_0, \quad Q_2 = x_0 y_1, \quad Q_3 = x_1 y_0, \quad Q_4 = x_1 y_1,$$

$$F_1 = A_0 y_0 - A_1 y_1, \quad F_2 = B_0 y_0 - B_1 y_1,$$

$$G_1 = L_0 x_0 - L_1 x_1, \quad G_2 = D_0 x_0 - D_1 x_1, \quad H.$$

Here A_0, A_1, B_0, B_1 are quadratic forms, L_0, L_1 are linear forms, D_0, D_1 are cubic forms and H is a quartic which agrees with $A_0 B_1 - B_0 A_1$ on $\mathbb{P}^3_{\langle y_0, y_1, z_0, z_1 \rangle}$ and which agrees with $L_0 D_1 - L_1 D_0$ on $\mathbb{P}^3_{\langle x_0, x_1, z_0, z_1 \rangle}$. Then C has Betti table SSY 8. We call this family of curves [562]b.

The four quadrics Q_1, Q_2, Q_3, Q_4 are reducible so $C = C_1 + C_2 \subset \mathbb{P}^5$ where $C_1 \subset \mathbb{P}^3_{\langle y_0, y_1, z_0, z_1 \rangle}$, $C_2 \subset \mathbb{P}^3_{\langle x_0, x_1, z_0, z_1 \rangle}$. Moreover, C_1 is residual to the line $y_0 = y_1 = 0$ in the complete intersection defined by (F_1, F_2). Similarly C_2 is residual to the line $x_0 = x_1 = 0$ in the complete intersection defined by (G_1, G_2). The curves C_1 and C_2 are each defined by an additional determinantal quartic, given by

$A_0 B_1 - B_0 A_1$ for C_1 and $L_0 D_1 - L_1 D_0$ for C_2. The condition on H implies that these quartics agree with the restriction of H to the line $\mathbb{P}^1_{\langle z_0, z_1 \rangle}$.

This determinantal condition on H is tricky, because the entries of the two 2×2 matrices have different degrees:

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

To show that family [562]b exists, we focus on the case that the cubics D_0 and D_1 break up into a linear and quadric form or three linear forms. Let [562]bi denote the special case

$$L_1 = x_1, \quad A_0 = L_3 x_1, \quad B_0 = L_0 L_1, \quad B_1 = L_0 L_2,$$
$$D_0 = L_0 L_2 L_3, \quad D_1 = L_1 A_1$$

where L_0, L_1, L_2, L_3 are general linear forms. Then

$$F_1 = L_3 x_1 y_0 - A_1 y_1, \quad F_2 = L_0 L_1 y_0 - L_0 L_2 y_1,$$
$$G_1 = L_0 x_0 - x_1^2, \quad G_2 = L_0 L_2 L_3 x_0 - L_1 A_1 x_1, \quad H = L_0 L_2 L_3 x_1 - L_0 L_1 A_1.$$

Together with the quadrics Q_i, these define a reduced curve with several irreducible components.

Proposition 2.6. *The general curve in the specialised family [562]bi as described above can be deformed to a curve of Type [550]a. We exhibit a flat family over $\mathbb{A}^1$ whose special fibre at $t = 0$ is of Type [562]bi and whose general fibre is of Type [550]a.*

Proof. Let t in $\mathbb{A}^1$ be a degree 0 deformation parameter and consider the deformed quadrics

$$Q_1' = x_0 y_0 - t L_2 x_1, \qquad Q_2' = x_0 y_1 - t L_1 x_1,$$
$$Q_3' = x_1 y_0 - t L_2 L_0, \qquad Q_4' = x_1 y_1 - t L_1 L_0.$$

When t is invertible the ideal of the five quadrics $(Q_1', Q_2', Q_3', Q_4', G_1)$ can be generated by the 4×4 Pfaffians of the skew-symmetric matrix

$$M = \begin{pmatrix} x_0 & x_1 & L_2 & 0 \\ & 0 & t L_1 & t x_1 \\ & & y_1 & t L_0 \\ & & & y_0 \end{pmatrix}.$$

Consequently when t is invertible $(Q_1', Q_2', Q_3', Q_4', G_1)$ defines a degree 5 del Pezzo surface. Note that this is untrue when $t = 0$, since it is necessary to be able to cancel t. Furthermore, we can show that when t is invertible, F_2, G_2 and H are not needed as ideal generators. Indeed, we have

$$t F_2 = y_1 Q_3' - y_0 Q_4',$$
$$t G_2 = A_1 Q_2' + x_0 F_1 - L_3 x_1 Q_1' + t L_2 L_3 G_1,$$
$$t H = A_1 Q_4' - L_3 x_1 Q_3' + x_1 F_1.$$

It follows that F_1 defines a cubic hypersurface in the del Pezzo surface cut out by the five quadrics. The ideals $I_t = (Q_1', Q_2', Q_3', Q_4', G_1, G_2, F_1, F_2, H)$ thus define a flat family of curves C_t, where C_0 is in family [562]bi and C_t is in [550]a for t invertible. Since the Hilbert polynomial is independent of t, this is indeed a flat family. $\square$
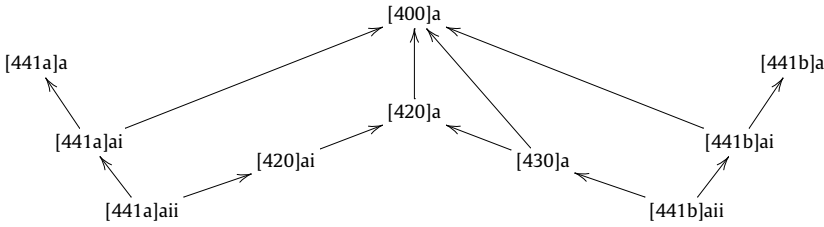
Fig. 3. Degree 16 strata and incidences.

3. Deformations in degree 16

We now turn our attention to the degree 16 varieties. There are two different irreducible subsets of the locus $\mathcal{F}_{[441]}$ corresponding to Betti table SSY 6, denoted $\mathcal{F}_{[441a]}$ and $\mathcal{F}_{[441b]}$. The a and b are included inside the bracket to indicate the different components of the space of quarternary quartics in Kapustka et al. (2021). The corresponding families of varieties are denoted by [441a]a and [441b]a, with both admitting specialisations which smooth to a nonsingular variety of Coughlan et al. (2016). There are further families with Betti tables SSY 3 and SSY 4, which are denoted by [430]a and [420]a respectively. We connect each of these families to family [400]a, which is the family of complete intersections of four quadrics. The full results for this section are seen in Fig. 3.

3.1. Family [400]a

The family [400]a consists of $(2,2,2,2)$ complete intersections. This family has Betti table CGKK 3. We construct a number of smoothings of varieties in the [420]a, [430]a, [441a]a and [441b]a families to varieties of this type.

	0	1	2	3	4
0	1	—	—	—	—
1	—	4	—	—	—
2	—	—	6	—	—
3	—	—	—	4	—
4	—	—	—	—	1

Betti table CGKK 3 Schenck et al. (2022)

3.2. Family [420]a

The original construction of the [420]a family is found in Schenck et al. (2022). This family corresponds to Betti table SSY 4, and unlike the other "SSY" constructions is a family of nonsingular, irreducible curves. Note that when embedded as a threefold in $\mathbb{P}^7$, Schenck, Stillman and Yuan show that this threefold is singular.

	0	1	2	3	4
0	1	—	—	—	—
1	—	4	2	—	—
2	—	2	6	2	—
3	—	—	2	4	—
4	—	—	—	—	1

Betti table SSY 4 Schenck et al. (2022)

Let $x_0, \ldots, x_5$ be coordinates on $\mathbb{P}^5$ and consider the following matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ & q_1 & q_2 & x_3 \\ & & q_3 & x_4 \\ & & & x_5 \end{pmatrix},$$

where the q_i are general quadric forms. The 4×4 Pfaffians of this matrix define a codimension three variety, which when intersected with a fourth general quadric Q is a Gorenstein codimension four curve of Type [420]a. The Pfaffians are given explicitly as

$$\mathrm{Pf}_{\hat{1}} = x_5 q_1 - x_4 q_2 + x_3 q_3,$$

$$\mathrm{Pf}_{\hat{2}} = x_1 x_5 - x_2 x_4,$$

$$\mathrm{Pf}_{\hat{3}} = x_0 x_5 - x_2 x_3,$$

$$\mathrm{Pf}_{\hat{4}} = x_0 x_4 - x_1 x_3,$$

$$\mathrm{Pf}_{\hat{5}} = x_2 q_1 - x_1 q_2 + x_0 q_3.$$

Proposition 3.1. *A curve of Type [420]a may be smoothed to the complete intersection of four quadrics. Explicitly, we construct a flat family of curves over $\mathbb{A}^1$ where the special fibre is a curve of the [420]a family and the general fibre is in the [400]a family.*

Proof. The deformation is fairly simple and proceeds as follows. We introduce the deformation parameter $t \in \mathbb{A}^1$. Replacing the zero entry of the above matrix with the parameter t we observe that $\mathrm{Pf}_{\hat{2}}, \mathrm{Pf}_{\hat{3}}$ and $\mathrm{Pf}_{\hat{4}}$ each gain a term, becoming

$$\mathrm{Pf}_{\hat{2}}' = x_1 x_5 - x_2 x_4 + t q_2, \quad \mathrm{Pf}_{\hat{3}}' = x_0 x_5 - x_2 x_3 + t q_3, \quad \mathrm{Pf}_{\hat{4}}' = x_0 x_4 - x_1 x_3 + t q_1.$$

From the syzygies between the three original quadric Pfaffians we obtain the following relations on the cubic Pfaffians:

$$t \mathrm{Pf}_{\hat{1}} = x_0 \mathrm{Pf}_{\hat{3}}' - x_1 \mathrm{Pf}_{\hat{2}}' + x_2 \mathrm{Pf}_{\hat{4}}',$$

$$t \mathrm{Pf}_{\hat{5}} = x_3 \mathrm{Pf}_{\hat{3}}' - x_4 \mathrm{Pf}_{\hat{2}}' + x_5 \mathrm{Pf}_{\hat{4}}'.$$

It follows that when t is invertible the cubic Pfaffians are not needed as ideal generators, and the ideal $(\mathrm{Pf}_{\hat{2}}', \mathrm{Pf}_{\hat{3}}', \mathrm{Pf}_{\hat{4}}', Q)$ defines a $(2,2,2,2)$ complete intersection. Again since the Hilbert polynomial is independent of t, this is a flat family over $\mathbb{A}^1$. □

3.3. Family [430]a

We now describe family [430]a, again utilising the "Cramer's rule" format.

	0	1	2	3	4
0	1	—	—	—	—
1	—	4	3	—	—
2	—	3	6	3	—
3	—	—	3	4	—
4	—	—	—	—	1

Betti table SSY 3 Schenck et al. (2022)

Let $x_0, x_1, x_2, y_1, y_2, z$ be coordinates on $\mathbb{P}^5$, let A, C be linear forms and B, D, P, Q quadratic forms.

Lemma 3.2. *Consider the following matrix M, vector v and parameter s:*

$$M = \begin{pmatrix} Q & P & A & C \\ -x_0 & x_1 & 0 & 0 \\ x_1 & -x_2 & 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} y_2 \\ y_1 \\ D \\ -B \end{pmatrix}, \quad s = z.$$

Then the seven equations obtained from $Mv = 0$, $\bigwedge^3 M = sv$ define a curve $C = C_1 + C_2 \subset \mathbb{P}^5$ with Betti table SSY 3. This is the [430]a family.

We describe the geometry of C in more detail. Two of the four quadric ideal generators, $y_1 z$ and $y_2 z$ are reducible. Hence $C = C_1 + C_2$, with $C_1 \subset \mathbb{P}^3_{\langle x_0, x_1, x_2, z \rangle}$ and $C_2 \subset \mathbb{P}^4_{\langle x_0, x_1, x_2, y_1, y_2 \rangle}$. Consider the cubic surface scroll in $\mathbb{P}^4_{\langle x_0, x_1, x_2, y_1, y_2 \rangle}$ defined by the vanishing of the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & y_1 \\ x_1 & x_2 & y_2 \end{pmatrix}.$$

The first two minors of the above matrix give the remaining two quadrics in the ideal of C. Consider the third minor $x_0 x_2 - x_1^2$. Let

$$F = A(x_0 x_2 - x_1^2) - Bz,$$
$$G = C(x_0 x_2 - x_1^2) - Dz.$$

Then C_1 is the curve residual to $x_0 x_2 - x_1^2$ in the complete intersection $\Gamma \subset \mathbb{P}^3_{\langle x_0, x_1, x_2, z \rangle}$ defined by the vanishing of (F, G). The cubic $AD - BC + Py_1 + Qy_2$, cuts out C_2 inside the cubic scroll in $\mathbb{P}^4$.

Proposition 3.3. *A general curve of Type [430]a can be deformed to a curve of Type [420]a, and to the complete intersection of four quadrics.*

Proof. We consider the flat family C_t over $\mathbb{A}^4$ with coordinates $t = (t_1, t_2, t_3, t_4)$. Denote by N the 2×2 block of zeroes in M, which we replace with the 2×2 matrix

$$N_t = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}.$$

Then C_t is defined using the "Cramer's rule" format, with N_t replacing N in M. Depending on the rank of N_t, we observe three different possibilities for the homological behaviour of C_t. When N_t has rank 0, i.e. $N_t = N$, C_t is a curve of type [430]a. When N_t has rank 1, C_t is a curve of type [420]a. When N_t has rank 2, C_t is a curve of type [400]a.

When N_t has rank 1 we may write

$$N_t = \begin{pmatrix} t_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where t_1 is invertible. Consequently the polynomials defining C_t are given by

$$Q_1 = \tfrac{1}{t_1} z y_1 - C x_1,$$
$$Q_2 = \tfrac{1}{t_1} z y_2 - C x_2,$$
$$Q_3 = x_0 y_2 - x_1 y_1 - t_1 D,$$
$$Q_4 = x_1 y_2 - x_2 y_1,$$
$$F_1 = A(x_0 x_2 - x_1^2) - Bz + t_1(Q x_2 + P x_1),$$
$$F_2 = C(x_0 x_2 - x_1^2) - Dz,$$
$$F_3 = AD - BC + Py_1 + Qy_2.$$

Note that we can write $\frac{1}{t_1}$ since t_1 is invertible. We can rearrange $Q_1 = 0$ and $Q_2 = 0$ to obtain

$$Cx_1 = \tfrac{1}{t_1}zy_1, \quad Cx_2 = \tfrac{1}{t_1}zy_2.$$

Substituting these into F_2 we observe that $F_2 = \frac{z}{t_1}Q_3$, thus this polynomial is not required as an ideal generator. We show that the remaining polynomials define a curve of Type [420]a. Five of the polynomials are given (modulo Q_3) by the 4×4 Pfaffians of the matrix

$$\begin{pmatrix} z & t_1C & -t_1Q - x_0A & t_1P - x_1A \\ & 0 & x_1 & x_2 \\ & & y_1 & y_2 \\ & & & -B \end{pmatrix},$$

and the remaining polynomial Q_3 is the general quadric.

Finally if N_t has rank 2, then we assume the matrix is diagonal with

$$N_t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}.$$

The equations become

$$Q_1 = zy_1 + Qt_1t_2 + Ax_0t_2 - Cx_1t_1, \quad Q_2 = zy_2 - Pt_1t_2 + Ax_1t_2 - Cx_2t_1,$$
$$Q_3 = x_0y_2 - x_1y_1 - t_1D, \quad Q_4 = x_1y_2 - x_2y_1 - t_2B,$$
$$F_1 = A(x_0x_2 - x_1^2) - Bz + t_1(Qx_2 + Px_1),$$
$$F_2 = C(x_0x_2 - x_1^2) - Dz + t_2(Qx_1 + Px_0),$$
$$F_3 = AD - BC + Py_1 + Qy_2.$$

We can eliminate P and Q, since

$$Q = \tfrac{1}{t_1t_2}(-zy_1 - Ax_0t_2 + Cx_1t_1),$$
$$P = \tfrac{1}{t_1t_2}(zy_2 + Ax_1t_2 - Cx_2t_1),$$

and use this to show F_1, F_2 and F_3 are not necessary to generate the ideal. Thus when N has rank two C_t is a curve of Type [400]a. Since the Hilbert polynomial at each fibre is independent of t we have a flat family of curves over $\mathbb{A}^4$. $\square$

3.4. Family [441a]a

In Ablett (2021) we present two different families with Betti table SSY 6.

	0	1	2	3	4
0	1	–	–	–	–
1	–	4	4	1	–
2	–	4	8	4	–
3	–	1	4	4	–
4	–	–	–	–	1

Betti table SSY 6 Schenck et al. (2022)

We outline a deformation for each family, starting with [441a]a.

Lemma 3.4. *Let $C \subset \mathbb{P}^4_{\langle x_0,\dots,x_4,y \rangle}$ be a curve defined by the quadrics*

$$Q_0 = x_0y, \quad Q_1 = x_1y, \quad Q_2 = x_2y,$$

the quartic

$$q = (d_0 d + d_1 b + d_2 c)x_3 - c^2 + b^2 + Py,$$

where P is a cubic form in (x_3, x_4, y) and d_0, d_1, d_2 are linear forms, and the five Pfaffians of

$$M = \begin{pmatrix} b & c & d & 0 \\ & x_3 & x_1 & x_0 \\ & & x_0 & x_1 \\ & & & x_2 \end{pmatrix}, \tag{3.1}$$

where b, c and d are quadratic forms, with $d = d_0 x_0 + d_1 x_1 + d_2 x_2$. Then C has Betti table SSY 6. We denote this family of curves by [441a]a.

Brown, Kerber and Reid (2012) discuss conditions that can be imposed on the entries of a matrix so that its Pfaffians lie in a given ideal. Two of the known solutions to this problem are called Tom and Jerry. The matrix M is in the Jerry format, which appears frequently in this paper. Thus the Pfaffians of M lie in the ideal (x_0, x_1, x_2). It is an open problem as to whether we can construct similar deformations using the Tom matrix format instead.

We discuss the geometry of C. Let $\Gamma \subset \mathbb{P}^4_{\langle x_0, \dots, x_4 \rangle}$ be the codimension three variety cut out by the Pfaffians of M, and let l be the line defined by (x_0, x_1, x_2) in this copy of $\mathbb{P}^4$. Note that the Pfaffians of M and the quartic q are a set of minimal ideal generators for the colon ideal $(I_\Gamma : I_l)$. By looking at the quadrics defining C, we observe that $C = C_1 + C_2$, where $C_1 \subset \mathbb{P}^4_{\langle x_0, \dots, x_4 \rangle}$ is the curve residual to l in Γ, defined by the five Pfaffians of M and the quartic q. On the other hand, $C_2 \subset \mathbb{P}^2_{\langle x_0, x_1, x_2 \rangle}$ is the plane quartic defined by q. The Pfaffians of M are

$$\text{Pf}_{\hat{1}} = x_0^2 - x_1^2 + x_2 x_3,$$
$$\text{Pf}_{\hat{2}} = x_2 c - x_1 d,$$
$$\text{Pf}_{\hat{3}} = x_2 b - x_0 d,$$
$$\text{Pf}_{\hat{4}} = x_1 b - x_0 c,$$
$$\text{Pf}_{\hat{5}} = x_0 b - x_1 c + x_3 d.$$

Notice that Pfaffians $\text{Pf}_{\hat{2}}$, $\text{Pf}_{\hat{3}}$ and $\text{Pf}_{\hat{4}}$ may be written as three of the 2×2 minors of the matrix

$$N = \begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ b & c & d & y \end{pmatrix},$$

with the other three minors giving the quadrics Q_0, Q_1, Q_2. Further, the remaining Pfaffians

$$Q_3 = x_0^2 - x_1^2 + x_2 x_3,$$
$$F = b x_0 - c x_1 + d x_3,$$

are in the "rolling factors" format of Duncan Dicks Reid (1989) with $x_0 \to b$, $x_1 \to c$ and $x_2 \to d$. Moreover, since d is in the ideal (x_0, x_1, x_2) we may write $d = d_0 x_0 + d_1 x_1 + d_2 x_2$. The quartic q may be obtained from the cubic $F = (d_0 x_0 + d_1 x_1 + d_2 x_2)x_3 - c x_1 + b x_0$ by rolling factors and adding an additional term in y, which gives

$$q = (d_0 d + d_1 b + d_2 c)x_3 - c^2 + b^2 + Py.$$

Proposition 3.5. Let [441a]ai be the specialisation of the [441a]a family, where $P = yA$ for some quadric $A \in (x_3, x_4, y)$. Then a curve in the [441a]ai family may be deformed to the complete intersection of four quadrics. We show this explicitly by constructing a flat family over $\mathbb{A}^1$.

Proof. Let $t \in \mathbb{A}^1$ be a deformation parameter. We deform the matrix N to N_t by replacing 0 with t in the top right entry:

$$N_t = \begin{pmatrix} x_0 & x_1 & x_2 & t \\ b & c & d & y \end{pmatrix}.$$

Then the quadrics are now given by

$$Q_0' = x_0 y - bt,$$
$$Q_1' = x_1 y - ct,$$
$$Q_2' = x_2 y - dt.$$

If t is invertible then the other three minors are not needed to generate the ideal, since

$$t(x_2 b - x_0 d) = x_0 Q_2' - x_2 Q_0',$$
$$t(x_2 c - x_1 d) = x_1 Q_2' - x_2 Q_1',$$
$$t(x_0 c - x_1 b) = x_1 Q_0' - x_0 Q_1'.$$

We further deform the remaining Pfaffians to

$$Q_3' = x_0^2 - x_1^2 + x_2 x_3 + t^2 A,$$
$$F' = b x_0 - x_1 c + x_3 d + t y A.$$

The quartic remains unchanged. Again if t is invertible the cubic Pfaffian is now a tautology since

$$t(b x_0 - x_1 c + x_3 d + t y A) = x_1 Q_1' - x_0 Q_0' - x_3 Q_2' + y Q_3'.$$

Similarly, we may write the quartic q as

$$tq = (c - x_3 d_1) Q_1' - (b + x_3 d_0) Q_0' - x_3 d_2 Q_2' + y F',$$

so this is also not needed to generate the ideal when t is invertible. Consider the family of curves over $\mathbb{A}^1$ defined by the vanishing of

$$I_t = (Q_0', Q_1', Q_2', Q_3', \mathrm{Pf}_{\hat{2}}, \mathrm{Pf}_{\hat{3}}, \mathrm{Pf}_{\hat{4}}, F', q).$$

It follows from the above that the general fibre is a $(2, 2, 2, 2)$ complete intersection defined by the vanishing of

$$Q_0' = x_0 y - bt,$$
$$Q_1' = x_1 y - ct,$$
$$Q_2' = x_2 y - dt,$$
$$Q_3' = x_0^2 - x_1^2 + x_2 x_3 + t^2 A.$$

On the other hand if $t = 0$ the curve defined by I_t is in the [441a]ai family. Since the Hilbert polynomial at each fibre is independent of t this is a flat family over $\mathbb{A}^1$. $\square$

We now consider a different specialisation of the [441a]a family, which admits a deformation to [420]a. We replace the matrix M (3.1) used in the construction of the [441a]a family with the following degenerate version, where we have assumed certain linear dependencies between some entries in the Jer$_{45}$ matrix:

$$M' = \begin{pmatrix} b & c & d & 0 \\ & x_3 & 0 & x_1 \\ & & x_0 & 0 \\ & & & x_2 \end{pmatrix}.$$

The three quadrics $Q_0 = x_0 y$, $Q_1 = x_1 y$ and $Q_2 = x_2 y$ are unchanged. The additional quartic is now given by $bc + x_3 cd_0 + Ay^2$, and we further set $A = c$, so that $q = bc + x_3 cd_0 + cy^2$. We refer to this specialisation as [441a]aii. This is a degeneration of [441a]ai which no longer fits into the rolling factors format.

Proposition 3.6. *A general curve in the family [441a]aii admits a deformation to a curve of Type [420]ai, which is a specialisation of [420]a with one extra zero in the Pfaffian matrix.*

Proof. The Pfaffians of M' are now

$$\mathrm{Pf}_{\hat{1}} = x_0 x_1 + x_2 x_3, \quad \mathrm{Pf}_{\hat{2}} = x_2 c, \quad \mathrm{Pf}_{\hat{3}} = x_2 b - x_1 d,$$
$$\mathrm{Pf}_{\hat{4}} = -x_1 c, \quad \mathrm{Pf}_{\hat{5}} = x_0 b + x_3 d.$$

Let $t \in \mathbb{A}^1$ be a degree 0 deformation parameter. We deform Q_0 and $\mathrm{Pf}_{\hat{5}}$ to

$$Q_0' = x_0 y + tc, \quad \mathrm{Pf}_{\hat{5}}' = x_0 b + x_3 d - tyc.$$

When t is invertible, we can eliminate $\mathrm{Pf}_{\hat{2}}$, $\mathrm{Pf}_{\hat{4}}$ and the additional quartic q. For example,

$$tq = (b + d_0 x_3) Q_0' - y \mathrm{Pf}_{\hat{5}}' + d_1 x_3 Q_1 + d_2 x_3 Q_2.$$

The remaining three Pfaffians along with Q_1 and Q_2 fit into the 4×4 Pfaffians of the 5×5 skew-symmetric matrix

$$\begin{pmatrix} x_2 & x_1 & 0 & 0 \\ & tc & d & x_0 \\ & & b & -x_3 \\ & & & -y \end{pmatrix}. \tag{3.2}$$

These Pfaffians and the last quadric Q_0' define a curve in the [420]ai family. This is a degeneration of [420]a because the matrix (3.2) has an extra zero entry. Thus the ideal $I_t = (Q_0', Q_1, Q_2, \mathrm{Pf}_{\hat{1}}, \mathrm{Pf}_{\hat{2}}, \mathrm{Pf}_{\hat{3}}, \mathrm{Pf}_{\hat{4}}, \mathrm{Pf}_{\hat{5}}', q)$ defines a curve of Type [441a]aii when $t = 0$, and a curve of Type [420]ai when t is invertible. $\square$

3.5. Family [441b]a

We now discuss the family of curves associated to the stratum $\mathcal{F}_{[441b]}$ with Betti table SSY 6.

Lemma 3.7. *Let $x_0, x_1, y_0, y_1, z_0, z_1$ be coordinates on $\mathbb{P}^5$, and further let A, B, C, D, P, M, N, Q be quadratic forms. Consider the matrices*

$$N_1 = \begin{pmatrix} x_0 & D' & B' \\ x_1 & C' & A' \end{pmatrix}, \quad N_2 = \begin{pmatrix} y_0 & Q' & N' \\ y_1 & P' & M' \end{pmatrix},$$

where A', B', C', D' are given by the restriction of A, B, C, D to $y_0 = y_1 = 0$ and M', N', P', Q' are given by the restriction of M, N, P, Q to $x_0 = x_1 = 0$. Let $C_1 \subset \mathbb{P}^3_{\langle x_0, x_1, z_0, z_1 \rangle}$ be defined by the vanishing of the 2×2 minors of N_1 and $C_2 \subset \mathbb{P}^3_{\langle y_0, y_1, z_0, z_1 \rangle}$ be defined by the vanishing of the 2×2 minors of N_2. Suppose further that the quartics $A'D' - B'C'$ and $P'N' - Q'M'$ agree on $\mathbb{P}^1_{\langle z_0, z_1 \rangle}$. Then $C = C_1 + C_2 \subset \mathbb{P}^5$ is a Gorenstein codimension four curve with Betti table SSY 6, corresponding to the stratum $\mathcal{F}_{[441b]}$. We call this family of curves [441b]a.

The curves C_1 and C_2 are both degree 8 and genus 7, and are each residual to a line in a complete intersection. We consider the specialisation [441b]ai where $P = A$, $Q = C$, $M = B$ and $N = D$, so that

$$H = AD - BC = PN - QM$$

is the quartic in the ideal of C. The equations defining C are thus given by the 2×2 minors of the matrix

$$N = \begin{pmatrix} 0 & x_0 & x_1 \\ y_0 & D & C \\ y_1 & B & A \end{pmatrix}.$$

Proposition 3.8. *Any curve in the [441b]ai family may be deformed to the complete intersection of four quadrics. We explicitly construct a flat family over $\mathbb{A}^1$ whose general fibre is in the [400]a family and whose special fibre is in the [441b]ai family.*

Proof. Introduce a degree 0 deformation parameter $t \in \mathbb{A}^1$ and consider the deformed matrix

$$N_t = \begin{pmatrix} -t & x_0 & x_1 \\ y_0 & D & C \\ y_1 & B & A \end{pmatrix}.$$

The 2×2 Pfaffians of this matrix define a $(2, 2, 2, 2)$ complete intersection. The quadrics become

$$Q_1' = x_0 y_0 + tD,$$
$$Q_2' = x_0 y_1 + tB,$$
$$Q_3' = x_1 y_0 + tC,$$
$$Q_4' = x_1 y_1 + tA.$$

If t is invertible then the four cubics and quartic are tautologies. Indeed, we have

$$tF_1 = x_0 Q_4' - x_1 Q_2',$$
$$tF_2 = x_0 Q_3' - x_1 Q_1',$$
$$tG_1 = y_0 Q_4' - y_1 Q_3',$$
$$tG_2 = y_0 Q_2' - y_1 Q_1',$$
$$tH = D Q_4' - B Q_3' + x_1 G_2.$$

We see that (Q_1', Q_2', Q_3', Q_4') defines a complete intersection. We again consider the family of curves defined by ideals of the form

$$I_t = (Q_1', Q_2', Q_3', Q_4', F_1, F_2, G_1, G_2, H).$$

As shown above, the general fibre is the complete intersection of the four quadrics (Q_1', Q_2', Q_3', Q_4'), whereas the special fibre is in the [441b]ai family. □

We now exhibit a specialisation [441b]aii of the [441b]ai family which admits a deformation to [430]a. Let [441b]aii be the specialisation of [441b]ai with $A = x_0 a + y_0 d$ and $C = x_0 c + y_1 b$, so that the matrix N becomes:

$$N = \begin{pmatrix} 0 & x_0 & x_1 \\ y_0 & D & x_0 c + y_1 b \\ y_1 & B & x_0 a + y_0 d \end{pmatrix}.$$

Proposition 3.9. *There is a flat family over $\mathbb{A}^1$ whose special fibre at $t = 0$ is in the [441b]aii family and whose general fibre when t is invertible is in the [430]a family.*

Proof. The 2×2 minors of N can be simplified using the quadrics Q_i to give the following equations for the fibre over $t = 0$:

$$Q_1 = x_1 y_1, \quad Q_2 = x_1 y_0, \quad Q_3 = x_0 y_0, \quad Q_4 = x_0 y_1,$$
$$F_1 = Bx_1 - ax_0^2, \quad F_2 = Dx_1 - cx_0^2, \quad F_3 = Dy_1 - By_0, \quad F_4 = y_0^2 d - y_1^2 b,$$
$$H = D(x_0 a + y_0 d) - B(x_0 c + y_1 b).$$

We deform two of the quadrics and three of the cubics to:

$$Q_3' = x_0 y_0 - tby_1, \quad Q_4' = x_0 y_1 - tdy_0, \quad F_1' = Bx_1 - a(x_0^2 - t^2 bd),$$
$$F_2' = Dx_1 - c(x_0^2 - t^2 bd), \quad F_3' = D(y_1 + ta) - B(y_0 + tc).$$

Then we have

$$tF_4 = y_1 Q_3' - y_0 Q_4', \quad tH = -x_0 F_3' - B Q_3' + D Q_4',$$

so that when t is invertible, F_4 and H are not needed as ideal generators.

We show that the remaining seven ideal generators fit into family [430]a. Recall the Cramer's rule format for the [430]a family, defined using a matrix M, vector v and parameter s. Choosing M, v, s to be

$$M = \begin{pmatrix} \frac{1}{t}D & -\frac{1}{t}B & a & c \\ -tb & x_0 & 0 & 0 \\ x_0 & -td & 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} y_1 \\ y_0 \\ D \\ -B \end{pmatrix}, \quad s = x_1,$$

expresses the seven remaining ideal generators as an element of [430]a Note that the quadratic forms D, B appear both in M and v.

In conclusion the ideal $I_t = (Q_1, Q_2, Q_3', Q_4', F_1', F_2', F_3', F_4, H)$ defines a family of curves C_t, where the special fibre at $t = 0$ is in the [441b]aii family, and is in the [430]a family when t is invertible.　□

The degeneration from [441b]ai to [441b]aii breaks the two degree 8 components C_i of a curve in the [441b]ai family into $C_i' + \mathbb{P}^1_{\langle z_0, z_1 \rangle}$. Thus a curve of Type [441b]aii contains the residual line $\mathbb{P}^1_{\langle z_0, z_1 \rangle}$ as a nonreduced component. The above deformation to [430]a smooths this double line into the residual conic in the construction of the general curve of Type [430]a curve. We do not know if there is a reduced curve of Type [441b]ai which deforms to [430]a.

4. Deformations in degree 17

We describe three families in degree 17, along with their specialisations. The results for this section can be seen in Fig. 4.

4.1. Family [300a]a

Family [300a]a is one of the nonsingular families of Coughlan et al. (2016) (see also (Papadakis and Reid, 2000, §2.8)).

Lemma 4.1. *Let M be a 3×4 matrix and v a 4×1 column vector with linear entries, with a further degree 2 parameter s. Then the variety defined by the equations*
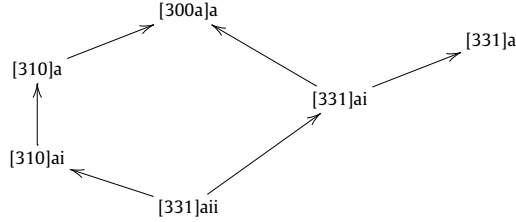
Fig. 4. Degree 17 strata and incidences.

$$Mv = 0, \quad \bigwedge^{3} M = sv$$

has Betti table CGKK 4. This is family [300a]a, and it is a lift of the stratum $\mathcal{F}_{[300a]}$ in the space of quartics in four variables.

	0	1	2	3	4
0	1	–	–	–	–
1	–	3	–	–	–
2	–	4	12	4	–
3	–	–	–	3	–
4	–	–	–	–	1

Betti table CGKK 4 Schenck et al. (2022)

4.2. Family [310]a

We next describe family [310]a.

	0	1	2	3	4
0	1	–	–	–	–
1	–	3	1	–	–
2	–	5	12	5	–
3	–	–	1	3	–
4	–	–	–	–	1

Betti table SSY 2 Schenck et al. (2022)

Lemma 4.2. *Let $x_0, \ldots, x_7, y$ be coordinates on $\mathbb{P}^8$ and let M be the 5×5 skew-symmetric matrix*

$$M = \begin{pmatrix} A & B & C & D \\ & x_2 & x_3 & x_4 \\ & & x_5 & x_6 \\ & & & x_7 \end{pmatrix}$$

where the entries of the first row are

$$A = a_0 x_0 + a_1 x_1, \quad B = b_0 x_0 + b_1 x_1, \quad C = c_0 x_0 + c_1 x_1, \quad D = d_0 x_0 + d_1 x_1,$$

with a_i, b_i, c_i, d_i linear forms. The 4×4 Pfaffians of M are four cubics and one quadric, $\mathrm{Pf}_{\hat{1}}$. Let

$$Q_0 = x_0 y, \quad Q_1 = x_1 y, \quad Q_2 = \mathrm{Pf}_{\hat{1}} + yL,$$

where L is a linear form. Next define the cubic

$$H = (c_0 d_1 - c_1 d_0)x_2 - (b_0 d_1 - b_1 d_0)x_3 + (a_0 d_1 - a_1 d_0)x_4$$
$$+ (b_0 c_1 - b_1 c_0)x_5 - (a_0 c_1 - a_1 c_0)x_6 + (a_0 b_1 - a_1 b_0)x_7, \quad (4.1)$$

and let E be a general quadratic form. Then the cubic Pfaffians of M, along with Q_0, Q_1, Q_2 and $H + yE$ define a fourfold $X \subset \mathbb{P}^8$ with Betti table SSY 2. Intersecting X with a 5-dimensional linear subspace gives a family of curves in $\mathbb{P}^5$ which we call [310]a.

Since the quadrics Q_0 and Q_1 are reducible, we observe that $X = X_1 + X_2$, with $X_1 \subset \mathbb{P}^7_{\langle x_0, \dots, x_7 \rangle}$ and $X_2 \subset \mathbb{P}^6_{\langle x_2, \dots, x_7, y \rangle}$. Let Γ be the variety defined by the Pfaffians of M in $\mathbb{P}^7$, and Y the variety cut out by $(x_0, x_1, \mathrm{Pf}_{\hat{1}})$. Note that Y is a component of Γ, and the Pfaffians of M together with the cubic H generate the colon ideal $(I_\Gamma : I_Y)$. Thus X_1 is residual to Y in Γ. On the other hand, X_2 is cut out by the ideal $(Q_2, H + yE)$ in $\mathbb{P}^6$, and is thus a $(2, 3)$ complete intersection. Intersecting this construction with a general $\mathbb{P}^5$ produces a Gorenstein curve of codimension four, as in Ablett (2021).

Proposition 4.3. *There is a flat family over $\mathbb{A}^1$ whose special fibre is a variety of Type [310]a and whose general fibre is a variety of Type [300a]a.*

Proof. First we collect the terms of $\mathrm{Pf}_{\hat{5}}$ involving x_0, x_1 as follows:

$$\mathrm{Pf}_{\hat{5}} = (x_5 a_0 - x_3 b_0 + x_2 c_0)x_0 + (x_5 a_1 - x_3 b_1 + x_2 c_1)x_1.$$

Let $t \in \mathbb{A}^1$ be a deformation parameter. We deform the quadrics Q_0 and Q_1 to

$$Q_0' = x_0 y + t(x_5 a_1 - x_3 b_1 + x_2 c_1), \quad Q_1' = x_1 y + t(x_5 a_0 - x_3 b_0 + x_2 c_0).$$

Then $t\,\mathrm{Pf}_{\hat{5}} = x_1 Q_0' - x_0 Q_1'$. Hence if t is invertible, $\mathrm{Pf}_{\hat{5}}$ is no longer needed as an ideal generator. The remaining cubic Pfaffians deform to

$$\mathrm{Pf}_{\hat{2}}' = \mathrm{Pf}_{\hat{2}} + t(x_5 E + (b_1 c_0 - b_0 c_1)L),$$
$$\mathrm{Pf}_{\hat{3}}' = \mathrm{Pf}_{\hat{3}} + t(x_3 E + (a_1 c_0 - a_0 c_1)L),$$
$$\mathrm{Pf}_{\hat{4}}' = \mathrm{Pf}_{\hat{4}} + t(x_2 E + (a_1 b_0 - a_0 b_1)L).$$

When t is invertible we may also replace the ideal generator $H + yE$ with the polynomial

$$(H + yE) - \frac{1}{t}(d_0 Q_0' + d_1 Q_1').$$

Hence when t is invertible, the remaining three quadrics and four cubic generators fit into the "Cramer's rule" format in the following way:

$$N = \begin{pmatrix} x_1 & a_0 & -b_0 & c_0 \\ x_0 & -a_1 & b_1 & -c_1 \\ tL & x_4 & -x_6 & x_7 \end{pmatrix}, \quad v = \begin{pmatrix} \frac{1}{t}y \\ x_5 \\ x_3 \\ x_2 \end{pmatrix}, \quad s = D + tE.$$

For invertible t, the three quadrics are obtained from $Nv = 0$ and the remaining four cubics are given by $\bigwedge^3 N = sv$.

It follows that $(Q_0', Q_1', Q_2, \mathrm{Pf}_{\hat{2}}', \mathrm{Pf}_{\hat{3}}', \mathrm{Pf}_{\hat{4}}', \mathrm{Pf}_{\hat{5}}, H + yE)$ defines a family of codimension four varieties whose general fibre is of Type [300a]a, and whose special fibre at $t = 0$ is of Type [310]a. Since the Hilbert polynomial at every fibre does not depend on t, this family is flat. $\square$

4.3. Family [331]a

We now describe a family of varieties with Betti table SSY 5.

	0	1	2	3	4
0	1	–	–	–	–
1	–	3	3	1	–
2	–	7	14	7	–
3	–	1	3	3	–
4	–	–	–	–	1

Betti table SSY 5 Schenck et al. (2022)

Lemma 4.4. *Let $x_0, \ldots, x_{12}, y$ be coordinates on $\mathbb{P}^{13}$ and let M be the skew-symmetric matrix*

$$M = \begin{pmatrix} x_3 & x_4 & x_6 & x_9 & a_0 x_0 & 0 \\ & x_5 & x_7 & x_{10} & b_1 x_1 & 0 \\ & & x_8 & x_{11} & 0 & 0 \\ & & & x_{12} & 0 & x_0 \\ & & & & 0 & x_1 \\ & & & & & x_2 \end{pmatrix} \tag{4.2}$$

where a_0, b_1 are scalars. Let $X \subset \mathbb{P}^{13}$ be defined by the seven 6×6 Pfaffians of M, the quadrics

$$Q_0 = x_0 y, \quad Q_1 = x_1 y, \quad Q_2 = x_2 y,$$

and the quartic $H = q + yF$ where q is the following combination of the 4×4 Pfaffians of M:

$$q = a_0 \, \mathrm{Pf}_{1234} \, \mathrm{Pf}_{2345} - b_1 \, \mathrm{Pf}_{1345} \, \mathrm{Pf}_{1235},$$

and F is a general cubic form.

Then X is a 9-dimensional Gorenstein codimension four variety with Betti table SSY 5. The intersection of X with a general linear subspace of dimension 5 is a curve C and we call this family [331]a.

In the language of Brown, Kerber and Reid (2012) M is a Jer_{67}, with all the 6×6 cubic Pfaffians of M lying in the ideal (x_0, x_1, x_2). Indeed M is a normal form for the following Jer_{67} matrix:

$$M' = \begin{pmatrix} x_3 & x_4 & x_6 & x_9 & A & S \\ & x_5 & x_7 & x_{10} & B & T \\ & & x_8 & x_{11} & C & U \\ & & & x_{12} & D & x_0 \\ & & & & E & x_1 \\ & & & & & x_2 \end{pmatrix}$$

where A, B, C, D, E, S, T, U are general in (x_0, x_1, x_2).

In the statement of the lemma, we write Pf_{ijkl} for the 4×4 Pfaffian of the skew-symmetric submatrix of M which is obtained by removing those row-columns whose index does not appear in $\{i, j, k, l\}$. Hence,

$$\mathrm{Pf}_{1234} = x_3 x_8 - x_4 x_7 + x_5 x_6, \qquad \mathrm{Pf}_{1235} = x_3 x_{11} - x_4 x_{10} + x_5 x_9,$$

$$\mathrm{Pf}_{1236} = b_1 x_1 x_4 - a_0 x_0 x_5, \qquad \mathrm{Pf}_{2345} = x_5 x_{12} - x_7 x_{11} + x_8 x_{10},$$

$$\mathrm{Pf}_{1345} = x_4 x_{12} - x_6 x_{11} + x_8 x_9, \qquad \mathrm{Pf}_{1245} = x_3 x_{12} - x_6 x_{10} + x_7 x_9.$$

Then three of the 6×6 Pfaffians are

$$\mathrm{Pf}_{\hat{6}} = x_1 \mathrm{Pf}_{1234} - x_0 \mathrm{Pf}_{1235},$$

$$\mathrm{Pf}_{\hat{5}} = x_0 \mathrm{Pf}_{1236} + x_2 \mathrm{Pf}_{1234},$$

$$\mathrm{Pf}_{\hat{4}} = x_1 \mathrm{Pf}_{1236} + x_2 \mathrm{Pf}_{1235}.$$

Since the quadric generators are reducible, $X = X_1 + X_2$, where $X_1 \subset \mathbb{P}^{12}_{\langle x_0, \ldots, x_{12} \rangle}$ and $X_2 \subset \mathbb{P}^{10}_{\langle x_3 \ldots, x_{12}, y \rangle}$. Further, let $\Gamma \subset \mathbb{P}^{12}$ be the variety cut out by the seven cubic Pfaffians of M, and l be the linear subspace given by the vanishing of (x_0, x_1, x_2). Then the quartic q and seven cubic Pfaffians generate the colon ideal $(I_\Gamma : I_l)$. Thus X_1 is residual to l in Γ. On the other hand, $X_2 \subset \mathbb{P}^{10}$ is a quartic hypersurface cut out by the polynomial $H = q + yF$.

Proposition 4.5. *Let [331]ai denote the special case of [331]a where the cubic F is zero so that $H = q$. There is a flat family over $\mathbb{A}^1$ whose special fibre is a variety of Type [331]ai and general fibre is a variety of Type [300a]a.*

Proof. Let $t \in \mathbb{A}^1$ be a deformation parameter. We first deform the quadrics using Pf_{1234}, Pf_{1235} and Pf_{1236}:

$$Q_0' = x_0 y + t \, \mathrm{Pf}_{1234}, \quad Q_1' = x_1 y + t \, \mathrm{Pf}_{1235}, \quad Q_2' = x_2 y - t \, \mathrm{Pf}_{1236} .$$

If t is invertible, then

$$t \, \mathrm{Pf}_{\hat{6}} = x_1 Q_0' - x_0 Q_1', \quad t \, \mathrm{Pf}_{\hat{5}} = x_2 Q_0' - x_0 Q_2', \quad t \, \mathrm{Pf}_{\hat{4}} = x_2 Q_1' - x_1 Q_2'.$$

It follows that $\mathrm{Pf}_{\hat{4}}, \mathrm{Pf}_{\hat{5}}$ and $\mathrm{Pf}_{\hat{6}}$ are redundant as ideal generators when t is invertible. Moreover,

$$tH = a_0 \mathrm{Pf}_{2345} \, Q_0' - b_1 \, \mathrm{Pf}_{1345} \, Q_1' - y \, \mathrm{Pf}_{\hat{7}},$$

and consequently if t is invertible H is also redundant. The remaining four cubic Pfaffians and three quadrics fit into the "Cramer's rule" format as follows:

$$N = \begin{pmatrix} x_0 & x_8 & -x_7 & x_6 \\ x_1 & x_{11} & -x_{10} & x_9 \\ x_2 & 0 & -b_1 x_1 & a_0 x_0 \end{pmatrix}, \quad v = \begin{pmatrix} \frac{1}{t} y \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad s = -x_2 x_{12}.$$

Then the three quadrics Q_i' are given by $Nv = 0$, and the remaining four cubics $\mathrm{Pf}_{\hat{4}}, \mathrm{Pf}_{\hat{5}}, \mathrm{Pf}_{\hat{6}}, \mathrm{Pf}_{\hat{7}}$ come from $\bigwedge^3 N = sv$.

Thus, the ideal $I = (Q_0', Q_1', Q_2', \mathrm{Pf}_{\hat{1}}, \ldots, \mathrm{Pf}_{\hat{7}}, H)$ defines a flat family of varieties, whose special fibre at $t = 0$ is of Type [331]ai and whose general fibre is of Type [300a]a. □

Next we study the varieties which lie between the families [310]a and [331]a. We consider the degeneration of [331]ai to [331]aii, which is obtained by setting the x_0 entry in the last column of (4.2) to zero and again setting the cubic $F = 0$ in H. Thus the matrix M becomes

$$M = \begin{pmatrix} x_3 & x_4 & x_6 & x_9 & a_0 x_0 & 0 \\ & x_5 & x_7 & x_{10} & b_1 x_1 & 0 \\ & & x_8 & x_{11} & 0 & 0 \\ & & & x_{12} & 0 & 0 \\ & & & & 0 & x_1 \\ & & & & & x_2 \end{pmatrix}.$$

The construction proceeds as before, with X being defined by the quadrics

$$Q_0 = x_0 y, \quad Q_1 = x_1 y, \quad Q_2 = x_2 y,$$

the 7 cubic Pfaffians of M

$$\mathrm{Pf}_{\hat{1}} = x_2\,\mathrm{Pf}_{2345} - b_1 x_1^2 x_8,$$

$$\mathrm{Pf}_{\hat{2}} = x_2\,\mathrm{Pf}_{1345} - a_0 x_0 x_1 x_8,$$

$$\mathrm{Pf}_{\hat{3}} = x_2\,\mathrm{Pf}_{1245} - a_0 x_0 x_1 x_7 + b_1 x_1^2 x_6,$$

$$\mathrm{Pf}_{\hat{4}} = x_2\,\mathrm{Pf}_{1235} - a_0 x_0 x_1 x_5 + b_1 x_1^2 x_4,$$

$$\mathrm{Pf}_{\hat{5}} = x_2\,\mathrm{Pf}_{1234},$$

$$\mathrm{Pf}_{\hat{6}} = x_1\,\mathrm{Pf}_{1234},$$

$$\mathrm{Pf}_{\hat{7}} = a_0 x_0\,\mathrm{Pf}_{2345} - b_1 x_1\,\mathrm{Pf}_{1345},$$

and the additional quartic

$$H = \mathrm{Pf}_{1234}\,\mathrm{Pf}_{2345},$$

where all Pf_{ijkl} are as before.

Proposition 4.6. *There exists a flat deformation whose central fibre is of Type [331]aii and whose general fibre is of Type [310]ai (see the proof for a description of family [310]ai).*

Proof. Introducing a deformation parameter $t \in \mathbb{A}^1$, we first deform Q_0 to $Q_0' = x_0 y + t\,\mathrm{Pf}_{1234}$. Thus Pfaffians $\mathrm{Pf}_{\hat{5}}$ and $\mathrm{Pf}_{\hat{6}}$ are not required as ideal generators when t is invertible and

$$t a_0 H = a_0\,\mathrm{Pf}_{2345}\,Q_0' - b_1\,\mathrm{Pf}_{1345}\,Q_1 - y\,\mathrm{Pf}_{\hat{7}}$$

as before, so that H is also not required.

Moreover, Pfaffians $\mathrm{Pf}_{\hat{1}}, \mathrm{Pf}_{\hat{2}}, \mathrm{Pf}_{\hat{3}}, \mathrm{Pf}_{\hat{4}}$ and the quadric Pf_{1234} are the 4×4 Pfaffians of the 5×5 matrix

$$N = \begin{pmatrix} x_2 x_9 - a_0 x_0 x_1 & x_2 x_{10} - b_1 x_1^2 & x_2 x_{11} & x_2 x_{12} \\ & x_3 & x_4 & x_6 \\ & & x_5 & x_7 \\ & & & x_8 \end{pmatrix}.$$

Comparing this with §4.2, with Q_1, Q_2, Q_0' as the three quadrics, and $\mathrm{Pf}_{\hat{7}}$ as the extra cubic (4.1), we see that the general fibre is in the [310]a family. Since N is quite special, we call the resulting subfamily [310]ai.

Thus the ideal $I_t = (Q_0', Q_1, Q_2, \mathrm{Pf}_{\hat{1}}, \mathrm{Pf}_{\hat{2}}, \mathrm{Pf}_{\hat{3}}, \mathrm{Pf}_{\hat{4}}, \mathrm{Pf}_{\hat{5}}, \mathrm{Pf}_{\hat{6}}, \mathrm{Pf}_{\hat{7}}, H)$ defines a variety X_0 of Type [331]aii when $t = 0$ and a variety X_t of Type [310]ai when t is invertible. Dimension is preserved, so this is a flat family over $\mathbb{A}^1$. □

Let us call the above deformation $\mathcal{X} \to \mathbb{A}^1$. Then the total space $\mathcal{X}$ has three components, each of which dominates $\mathbb{A}^1$ by flatness. The first component $\mathcal{X}_1 \to \mathbb{A}^1$ is a trivial deformation whose fibre X_1 is the degree 11 component of a general variety in the [310]ai family. The second component $\mathcal{X}_2 \to \mathbb{A}^1$ is again trivial, and its fibre X_2 is the quadric hypersurface ($\mathrm{Pf}_{1234} = 0$) in the linear subspace $x_0 = x_1 = x_2 = 0$. The last component $\mathcal{X}_3 \to \mathbb{A}^1$ is a nontrivial deformation, defined by $(x_0 y - t\,\mathrm{Pf}_{1234} = \mathrm{Pf}_{2345} = 0)$ in the linear subspace $x_1 = x_2 = 0$. The general fibre X_3 of $\mathcal{X}_3$ is a complete intersection of two quadrics and $X_2 + X_3$ is the $(2,3)$ complete intersection $(x_0 y - t\,\mathrm{Pf}_{1234} = x_0\,\mathrm{Pf}_{2345} = 0)$. The central fibre $Y_3 = Y_3' + Y_3''$ breaks into two hyperplane sections $Y_3': (y = 0)$ respectively $Y_3'': (x_0 = 0)$ of the quadric ($\mathrm{Pf}_{2345} = 0$).

Returning to the original deformation $\mathcal{X} \to \mathbb{A}^1$, we see that the general fibre $\mathcal{X}_t = X_1 + (X_2 + X_3)$ is in a degeneration of the [310]a family, which we called [310]ai. Moreover, $\mathcal{X}_0 = (X_1 + Y_3') + (X_2 + Y_3'')$ where $(X_1 + Y_3')$ is the degenerate degree 13 component of a general variety of Type [331]aii, and $(X_2 + Y_3'')$ is a degenerate quartic ear.

5. Deformations in degree 18

In degree 18, we discuss the singular family [210]a with Betti table [210]. We show that a variety in the specialisation [210]ai admits a smoothing to a variety in the [200]a family described in Coughlan et al. (2016) involving a linear section of the fourfold $\mathbb{P}^2 \times \mathbb{P}^2$.

5.1. Families [200]a and [200]b

Two families [200]a and [200]b are outlined in Coughlan et al. (2016). These higher degree varieties are more complicated than those previously outlined, so their descriptions are not as complete. One of the families from Coughlan et al. (2016) is bilinked to a linear section of $\mathbb{P}^2 \times \mathbb{P}^2$ in a 4-dimensional $(2, 2, 3)$ complete intersection. We call this family [200]a. The other family is bilinked to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and we call this [200]b. Both constructions have the CGKK 7/8 Betti table. For curves, there is no difference between [200]a and [200]b. Both families are bilinked to the normal elliptic curve of degree 6. Thus we may refer to this family of curves as [200].

	0	1	2	3	4
0	1	–	–	–	–
1	–	2	–	–	–
2	–	8	18	8	–
3	–	–	–	2	–
4	–	–	–	–	1

Betti table CGKK 7/8 Schenck et al. (2022)

5.2. Family [210]a

The curve C of Type [210]a was first described in Ablett (2021), and has Betti table SSY 1.

	0	1	2	3	4
0	1	–	–	–	–
1	–	2	1	–	–
2	–	9	18	9	–
3	–	–	1	2	–
4	–	–	–	–	1

Betti table SSY 1 Schenck et al. (2022)

Lemma 5.1. *Let $x_0, \ldots, x_5, y_0, y_1, z_0, z_1, z_2, w$ be coordinates on $\mathbb{P}^{11}$ and let M be the skew-symmetric matrix*

$$
M = \begin{pmatrix}
x_0 & x_1 & x_2 & y_0 & y_1 & 0 \\
 & x_3 & x_4 & y_1 & ay_0 & 0 \\
 & & x_5 & 0 & by_1 & y_0 \\
 & & & 0 & y_0 & y_1 \\
 & & & & z_0 & z_1 \\
 & & & & & z_2
\end{pmatrix}
$$

where a, b are scalars. We define quadrics

$$Q_0 = y_0 w, \quad Q_1 = y_1 w, \quad Q_2 = x_0 x_5 - x_1 x_4 + x_2 x_3,$$

where Q_2 is the Pfaffian of the upper left 4×4 block of M. Finally, let F_0 be the following cubic:

$$F_0 = ab(y_0 y_1 z_1 - x_2 z_1^2) + a(y_0^2 z_0 - x_1 z_0 z_1 + x_5 z_1^2) + b(y_1^2 z_2 + x_0 z_1^2 - x_4 z_1 z_2)$$
$$- x_0 z_0^2 + x_4 z_0 z_1 + x_3 z_1^2 + x_2 z_0 z_2 - x_3 z_0 z_2 + x_1 z_1 z_2 - x_5 z_2^2$$
$$- y_0^2 z_2 - y_0 y_1 z_1 - y_1^2 z_0.$$

Suppose that X be defined by the quadrics Q_0, Q_1, the seven cubic 6×6 Pfaffians of M, the cubic $F_1 = F_0 + wG$ where G is a quadratic form, and the further cubic $F_2 = w Q_2$.

Then X is a Gorenstein codimension 4 variety with Betti table SSY 1. The intersection of X with a 5-dimensional linear subspace is a curve C and we call this family of curves [210]a.

We start from the following skew-symmetric matrix M'

$$M' = \begin{pmatrix} x_0 & x_1 & x_2 & a_{15} & a_{16} & a_{17} \\ & x_3 & x_4 & a_{25} & a_{26} & a_{27} \\ & & x_5 & a_{35} & a_{36} & a_{37} \\ & & & a_{45} & a_{46} & a_{47} \\ & & & & z_0 & z_1 \\ & & & & & z_2 \end{pmatrix}$$

where the a_{ij} are generic linear combinations of y_0 and y_1 so that the 6×6 cubic Pfaffians of M' lie in the ideal (y_0, y_1, Q_2). Since we assumed that M' is generic, by performing row-column operations, rescaling and relabelling variables in a prudent manner, we may reduce M' to the normal form M displayed in the lemma.

Since the quadrics Q_0, Q_1 are reducible, it follows that $X = X_1 + X_2$ where X_1 lies in the copy of $\mathbb{P}^{10}$ defined by $w = 0$, while X_2 lies in $\mathbb{P}^9$ defined by $y_0 = y_1 = 0$.

By construction, X_1 is residual to the quadric (y_0, y_1, Q_2) in the variety Γ defined by the 6×6 Pfaffians of M. Thus $X_1 \in \mathbb{P}^{10}$ is defined by the seven cubic Pfaffians, along with the cubic F_0, which we obtained by computing the colon ideal $(I_\Gamma : (y_0, y_1, Q_2))$.

The other component X_2 is the $(2, 3)$ complete intersection defined by Q_2 and F_1. In Ablett (2021), the quadric Q_2 had to be of rank 2, but here we can have quadrics of rank 6.

To construct a smoothing to Type [200], we consider the specialisation to family [210]ai where we assume that $G = 0$.

Proposition 5.2. *A curve of Type [210]ai can be deformed to a curve of Type [200].*

Proof. We now construct a flat family over $\mathbb{A}^1$, introducing $t \in \mathbb{A}^1$ as our deformation parameter. Note that the final four Pfaffians, $\mathrm{Pf}_{\hat{4}}, \ldots, \mathrm{Pf}_{\hat{7}}$ are all in the ideal (y_0, y_1), so each Pfaffian may therefore be written as $A_i y_0 + B_i y_1$ for appropriate quadrics A_i, B_i.

For instance, $\mathrm{Pf}_{\hat{4}} = Ay_0 + By_1$ where

$$A = -ay_0^2 + ax_1 z_1 + x_0 z_0 + x_3 z_2, \tag{5.1}$$
$$B = -bx_0 z_1 + y_0 y_1 - x_3 z_1 - x_1 z_2. \tag{5.2}$$

If we deform the quadrics to $Q_0' = y_0 w + tB$, $Q_1' = y_1 w - tA$, then

$$t\mathrm{Pf}_{\hat{4}} = y_0 Q_1' - y_1 Q_0'.$$

Thus, when t is invertible $\mathrm{Pf}_{\hat{4}}$ becomes redundant. We further deform the cubic $F_2 = w Q_2$ to

$$F_2' = w Q_2 - t(ax_1 x_2 y_0 - ax_1^2 y_1 - bx_0 x_4 y_0 + bx_0 x_2 y_1 + bx_0 x_3 y_1$$
$$- x_0 x_1 y_0 - x_3 x_4 y_0 + x_3^2 y_1 - x_0 x_5 y_1). \tag{5.3}$$

This extra term is chosen so that the syzygies $y_i F_2 = Q_2 Q_i$ for $i = 0, 1$ extend to the deformed ideal. For completeness, the extensions are:

$$y_0 F_2' = Q_2 Q_0' + t((bx_0 + x_3) \operatorname{Pf}_{\hat{6}} + x_1 \operatorname{Pf}_{\hat{7}}),$$

$$y_1 F_2' = Q_2 Q_1' + t(ax_1 \operatorname{Pf}_{\hat{6}} + x_0 \operatorname{Pf}_{\hat{1}} + x_3 \operatorname{Pf}_{\hat{7}}).$$

Now consider the ideal

$$I_t = (Q_1', Q_2', F_1, F_2', \operatorname{Pf}_{\hat{1}}, \ldots, \operatorname{Pf}_{\hat{7}}),$$

defining a variety X_t. The fibre over $t = 0$ is of Type [210]i. Moreover, the general fibre is irreducible, reduced, and its ideal is generated by eight cubics and two quadrics, and has Betti table CGKK 7/8.

Let C_t be a general fibre of the family. We use Magma Bosma et al. (1997) to work out the bilinkage of C_t in a $(2, 2, 3)$ complete intersection. We find that C_t is indeed bilinked to a normal elliptic curve of degree 6. We therefore have a one parameter deformation of curves with general fibre of Type [200] and special fibre at $t = 0$ of Type [210]ai. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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