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TERM RATES, MULTICURVE TERM STRUCTURES AND OVERNIGHT RATE BENCHMARKS: A ROLL–OVER RISK APPROACH

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Abstract. In the current LIBOR transition to overnight–rate benchmarks, it is important to understand theoretically and empirically what distinguishes actual term rates from overnight benchmarks or "synthetic" term rates based on such benchmarks. The well–known "multi–curve" phenomenon of tenor basis spreads between term structures associated with different payment frequencies provides key information on this distinction. This information can be extracted using a modelling framework based on the concept of "roll–over risk", i.e., the risk a borrower faces of not being able to refinance a loan at (or at a known spread to) a market benchmark rate. Separating the roll–over risk priced by tenor basis spreads into a credit–downgrade and a funding–liquidity component, the theoretical modelling and the empirical evidence show that proper term rates based on the new benchmarks remain elusive and that a multi–curve environment will persist even for rates secured by repurchase agreements.

1. Introduction.

The ongoing transition from the London Interbank Offered Rate (LIBOR) and other, similar benchmarks in major jurisdictions (IBORs) to new benchmarks based on overnight rates is starting to move term rates to the center of attention. Term is an important feature of borrowing transactions. This is not simply because of interest rate risk or market expectations of rising or falling interest rates reflected in higher or lower long-term versus short-term rates: Consider a borrower who takes out a floating–rate loan, agreeing to pay a floating interest rate updated every three months on a one–year loan, as compared to a borrower who also needs to borrow for one year, but chooses to do so by borrowing at a fixed rate for three months and then refinancing this loan every three months for a total of one year. Both borrowers are equally exposed to interest rate risk, but the second borrower faces the additional risk that they may not be able to refinance at

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the same market rate. This additional source of risk is "refinancing risk" or *roll–over risk*[1] (since it appears when seeking to roll over a three–month loan into a new three–month loan). It can manifest itself due to the borrower experiencing a credit downgrade or due to not being able to borrow at the market rate because of dwindling market liquidity—as it happened when money markets dried up in the wake of the default of Lehman Brothers in 2008. In fact, it is since 2008 that the market has been putting a clearly discernible price on roll–over risk: Floating–for–floating interest rate swap (i.e., basis swap) spreads added to the higher payment frequency rate increased sharply at the time and then decreased post-crisis, but never returned to the negligible levels seen pre-crisis. This price on roll–over risk is a form of term premium reflected in term rates—the longer the term, the higher the premium.

Clearly, an overnight rate such as SOFR, SONIA, EONIA or €STR[2] will not reflect this premium, creating a substantial, and substantially risky, mismatch between the new risk–free rate (RFR) benchmarks and any borrowing or lending for periods longer than overnight. Compounded–in–arrears indices based on these benchmarks[3] do not address this problem. Moreover, they introduce the additional shortcoming that they are backward–looking, thus not known until the end of the accrual period. The forward–looking "synthetic term rates" which rely on derivative financial instruments to convert the floating overnight rate into a fixed rate over a given term do not reflect this premium either. These swaps are derivative financial instruments which do not involve an exchange of notional principal between counterparties, so involve no borrowing/lending transactions, and therefore only serve to transfer interest rate risk, not roll–over risk. Simply put, a hypothetical market participant able to borrow at an overnight benchmark rate would still pay a premium above the OIS–implied term rate when borrowing for a fixed term (longer than overnight) to avoid roll–over risk.

Thus it is important to understand theoretically and empirically what distinguishes term rates (at which market participants can actually borrow) from overnight benchmarks or "synthetic" term rates based on these benchmarks. The now well–recognised "multicurve" phenomenon in interest rate markets provides key clues to explaining these differences: if we know what drives the basis spreads between term structures based on genuine (benchmark) term rates of different tenors, then we understand what is missing in (a) the new benchmarks based on overnight rates and (b) the possible ways being discussed at present to fill the term rate gap. This then leads to the conclusion that in jurisdictions where IBORs are discontinued,[4] proper term rates remain elusive—and we are able to formally (both theoretically and empirically) demonstrate what is missing. Though the discontinuation of LIBOR might be seen by some as an opportunity to ridden fixed-income markets of term risk (basis spreads), we argue the contrary: The end of LIBOR will not result

[1]The term "rollover risk" first appears in the literature in connection with the financial crisis of 2007/8 in Acharya, Gale and Yorulmazer (2011).

[2]Secured Overnight Funding Rate (in US dollars), Sterling Overnight Index Average, Euro Overnight Index Average and Euro Short–Term Rate, respectively.

[3]For example, the Bank of England favours compounded–in–arrears SONIA in many cases where proper term rates would have been used previously, see Bailey (2021).

[4]This is now more or less locked in for USD and GBP. For EUR, on the other hand, EURIBOR still may or may not survive, and for some currencies no discontinuation is currently on the table, e.g., there is no indication that the Australian IBOR equivalent, the Bank Bill Swap Rate (BBSW), will be discontinued.

in a return to the “single–curve” interest rate environment that was prevalent prior to the 2008 financial crisis.

In this paper, we propose a framework that links the existence of the tenor basis spreads to roll–over risk, including IBOR/OIS spreads as a limit case of a tenor basis spread. To our knowledge, in prior literature only Filipović and Trolle (2013), Gallitschke, Seifried and Seifried (2017) and Alfeus, Grasselli and Schlögl (2020) have taken a similar approach (see also a discussion of the relevant literature in Section 2 below). The first two papers predate the LIBOR transition, so they do not consider the implications of roll–over risk (which they call “interbank risk”) for the distinction between term rates and rates implied by the new RFR benchmarks. Furthermore, Gallitschke et al. (2017) only model the funding liquidity component (in the form of a “liquidity freeze”) of roll–over risk, not the credit downgrade component.[5] While our modelling of the credit downgrade component is similar to Filipović and Trolle (2013) in the specific instance of our framework, which we use for econometric estimation, we are explicit in the modelling of the funding liquidity component. We also find that their assumption around the overnight–market component of the default intensity/credit spread is untenable on our data set. That is, setting this market component to a small constant is too restrictive, resulting in an over–estimation of downgrade risk and often in over–pricing the EURIBOR–OIS spread, even in the absence of any contribution from funding–liquidity risk. Our model performs substantially better on the data and gives a more meaningful split between the credit downgrade and funding–liquidity components of the tenor basis spreads. Our treatment of overnight–market credit risk suggests that on our data set, about 30% is the empirically observed upper bound for the contribution of downgrade risk to tenor basis spreads. This implies that roll–over risk is also central for secured term rates, and overnight rates such as SOFR and associated synthetic term rates remain inadequate benchmarks for term borrowing. Thus our modelling and empirical analysis provides a rigorous basis for the view also recently expressed in practitioners’ articles (see, e.g., Nelson (2020) and Albanese, Iabichino and Mammola (2021)) that an appropriate benchmark for term borrowing should reflect this type of funding liquidity risk.

The rest of the paper is organised as follows. After giving some context in terms of the recent literature on multicurve term structure modelling and on the development of new benchmarks to replace LIBOR in Section 2, Section 3 presents our modelling framework and derives the dynamics of a quoted benchmark rate such as IBOR. Section 4 provides a specific model instance of the framework introduced in Section 3, making the dynamics of the “credit” and “liquidity” components of roll–over risk concrete in a manner which allows the prices of key market instruments to be computed. An econometric analysis of the model applied to Euro data (OIS, vanilla interest rate swaps, basis swaps and credit default swaps) is conducted in Section 5. Since roll–over risk is not solely due to credit risk, it remains crucial even if credit risk is entirely mitigated, e.g., if going forward one were to consider a purely SOFR–based world: This would be a special case in our model, which is discussed in Section 6. Section 7 concludes.

[5] In Gallitschke et al. (2017), credit risk enters only in the form of default risk, not via the risk of facing an increased credit spread when attempting to refinance roll–over borrowing. Thus it is unsurprising that they find, “Tenor basis spreads are almost exclusively due to the presence of funding liquidity risk.”

2. **Background.** In the literature, direct modelling of basis spreads (without any recourse to the underlying financial risks driving these spreads) can be traced back to Boenkost and Schmidt (2004), who constructed a model for cross–currency swap valuation in the presence of a basis spread. Kijima, Tanaka and Wong (2009) applied this approach to single–currency (frequency) basis spreads, which were first highlighted in an article by Henrard (2007), initiating an extensive literature on "multicurve" models. Fujii, Shimada and Takahashi (2009) considered how the absence of arbitrage opportunities can be ensured in models of this type. Mercurio (2009, 2010) proposed multicurve versions of the LIBOR Market Model, Kenyon (2010) constructed a short–rate modelling framework, Mercurio and Xie (2012) based their model on stochastic additive basis spreads, while Henrard (2010, 2013) considered, respectively, deterministic and stochastic multiplicative basis spreads. Focusing on a stochastic, multiplicative spot spread, Cuchiero, Fontana and Gnoatto (2019) developed a general framework nesting all existing affine multicurve models. The model of Moreni and Pallavicini (2014) consists of two curves, for risk–free instantaneous forward rates and forward LIBORs, which are Markovian in a common set of state variables. Grasselli and Miglietta (2016) show how the widespread practice of calibrating interest rate term structure models to market data via a deterministic time–shift can be extended to a multicurve framework. Macrina and Mahomed (2018) embed multicurve models (for discount curves in different currencies, or real vs nominal interest rates, or for different tenors) in a general pricing kernel framework. These works do not model the economic structural links between different term structures.

Elsewhere, the literature incorporating potential causes of frequency basis spreads revolves around either credit risk, or funding liquidity risk, or both. Papers by Morini (2009) and by Bianchetti (2010) focus on counterparty credit risk as the driver of the basis. Crépey and Douady (2013) construct a stylised equilibrium model of credit risk and funding liquidity risk to explain the LIBOR/OIS spread. Taking a different perspective, Crépey (2015) integrates funding cost and counterparty credit risk into a model for credit valuation adjustment (CVA), but does not explicitly consider spreads between different tenor frequencies arising from roll–over risk.

Filipović and Trolle (2013) consider the risk of loss resulting from lending in the interbank money market, which they call "interbank risk". They separate this risk into two parts, a default and a non–default component, and study the associated risk premia based on time series data of overnight index swaps (OIS), the IBOR–style (e.g. LIBOR, EURIBOR, etc.) money market, the vanilla interest rate swap (IRS) market, and the basis swap market. Their "default" component is interpreted in terms of the risk of a deterioration of creditworthiness of a LIBOR reference panel bank, resulting in it dropping out of the LIBOR panel.[6] Such a bank would no longer be able to roll debt over at the overnight reference rate, while the rate on any LIBOR borrowing would remain fixed until the end of the accrual period (i.e., typically for several months). They show that this differential impact of a credit downgrade on rolling debt explains part of the LIBOR/OIS spread; the residual is labelled the "liquidity" component.[7] Both components of this "interbank risk"

[6]This is also known as the "renewal effect," see Collin-Dufresne and Solnik (2001) and Grinblatt (2001).

[7]Past empirical studies, in particular of the Global Financial Crisis, also found that credit risk alone is insufficient to explain the LIBOR/OIS spread; see e.g. Eisenschmidt and Tapking (2009).

manifest themselves as additional cost when rolling debt over. Thus, although Filipović and Trolle (2013) do not use this terminology, this can be seen as roll–over risk, consisting of "downgrade risk" (the credit component) and "funding–liquidity risk"[8] (the non–credit component).[9]

Starting from the observation that the frequency basis gives rise to naive "arbitrage" strategies involving lending at a longer tenor and borrowing at a shorter tenor, Alfeus et al. (2020) argue that the phenomenon of the frequency basis can only persist (as empirically observed) if the putative "arbitrage" channel is closed off by the presence of roll–over risk. They model both components as spreads applied to the overnight borrowing cost, showing how these block potential trading strategies to take advantage of the LIBOR/OIS spread and/or the frequency basis. That paper, however, takes a "cross–sectional" perspective on market data, calibrating the model to market data observed at a particular point in time. Specifically, although roll–over risk has a term structure in Alfeus et al. (2020), that term structure is static in process time, therefore the model is not amenable to econometric estimation. Lifting this restriction is a key point of distinction of our approach, as explained in Section 3 below.

As noted by Alfeus et al. (2020), the IBOR/OIS spread can be interpreted as a premium paid by the borrower at IBOR to avoid roll–over risk over the length of the IBOR loan. This links roll–over risk with what the literature commonly calls "term premia" or "term funding risk". In other words, the presence of roll–over risk gives rise to a term premium in addition to the premium due to the market price of interest rate risk.[10] In view of the current transition away from IBOR to benchmarks such as the Secured Overnight Funding Rate (SOFR), this points to a critical difference between IBOR and the proposed replacements, in that even when interest rate risk is eliminated (e.g., through an OIS–type derivative financial instrument), one obtains only a pseudo–term rate not reflecting roll–over risk. This was long under–appreciated in the debate on the IBOR transition, but it is becoming more widely recognised as an important issue.[11]

By explicitly modelling roll–over risk as the driver of the basis between overnight rates and term rates, and between rates of different terms (a.k.a. tenors), our framework provides important insights into the implications of phasing out IBORs and for introducing proposed replacements (all of which, at the time of writing,

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[8]"Funding–liquidity risk" has also been considered explicitly in a separate strand of the literature. For example, Acharya and Skeie (2011) model liquidity hoarding by participants in the interbank market. In their model, there is a positive feedback effect between roll–over risk and liquidity hoarding (via term premia on interbank lending rates), which in the extreme case can lead to a freeze of interbank lending. Brunnermeier and Pedersen (2009) model a similar adverse feedback effect between market liquidity and funding liquidity.

[9]As noted above, roll–over risk drives the spread between reference rates for term borrowing versus reference rates based on overnight borrowing. Thus modelling this risk by a credit and a liquidity component is consistent with the findings of the econometric literature on term premia, which finds contributions from both of these components, see e.g. Michaud and Upper (2008) and Gefang, Koop and Potter (2011).

[10]For a review of traditional concepts of term premia, see e.g. Kim and Orphanides (2007).

[11]Schrimpf and Sushko (2019) highlight this point repeatedly in an article in the *BIS Quarterly Review*, noting for example, "A crucial, yet challenging, area of the reform process is the extension of the reference curve from O/N to term rates." In a paper examining alternative benchmarks to replace LIBOR, Klingler and Syrstad (2021) remark that these alternatives "lack a term premium, which detaches these rates from banks' costs of term funding and introduces problems for loan issuance."

reference overnight rates). This goes beyond what can be considered the current state of the LIBOR transition debate:

- The move to replace IBORs with the RFR benchmarks is motivated by moving towards a primary benchmark that is both closely related to actual transactions and less susceptible to manipulation. It is clear that replacing IBOR rates with transaction–based RFRs lives up to that objective. In the USA, the current plan is to replace LIBOR with SOFR, which is a weighted average of rates on overnight repurchase transactions. Several challenges remain though, including whether SOFR is an appropriate proxy for actual interbank (or, for that matter, corporate) funding cost, as well as in relation to converting existing LIBOR derivatives to SOFR equivalents. While the latter problem is of considerable magnitude,[12] some of the implications of the former (the resolution of which is a pre–requisite for satisfactorily addressing the latter) are only beginning to be discussed in earnest.
- In response to a letter from a group of bank representatives,[13] the Board of Governors of the Federal Reserve System, the US Office of the Comptroller of the Currency, and the Federal Deposit Insurance Corporation established a LIBOR Transition Credit Sensitivity Group (CSG), "to focus on the issues surrounding a credit sensitive rate/spread that could be added to SOFR". Here, the issue raised by the group of bank representatives, and thus the issue which the CSG seeks to resolve, is one of a disconnect *due to credit risk* between the cost of funding of private–sector banks (subject to credit risk) and a SOFR benchmark based on repo transactions (subject to negligible credit risk). Our analysis in the present paper suggests that this disconnect is not solely due to credit spreads in the traditional sense, but also due to premia for avoiding roll–over risk by borrowing at term.
- In the broader debate, problems arising from a mismatch in tenor between the new candidate benchmarks (i.e., overnight) and the old IBOR benchmarks (e.g., three months, six months, etc.) are only addressed in passing. This may be because of the hope that "term rates" based on SOFR will become available once SOFR is well–established as the market benchmark, based on OIS referencing SOFR as the overnight rate (in a manner similar to how current OIS contracts reference the overnight Fed funds rate), see for example the discussion in Henrard (2019). Alternatively, a consultation by the ISDA (ISDA (2018), where rates derived from a SOFR–OIS market were not included as an option) concludes that a backward–looking approach, where a (constant) term– and currency–dependent spread is added to a compounded average of SOFR (often called SAFR), is the preferred option for replacing LIBOR in existing derivatives. From the analysis laid out in the present paper, we conclude that neither of these approaches leads to benchmarks with the term–rate properties of the old IBOR benchmarks. In particular, SOFR–based OIS–implied term rates would be more suitable to replace the current Fed–Funds–based OIS–implied term rates, not the benchmark IBORs. Intuitively, this is both unsurprising (though largely unrecognised) and economically significant when

[12] In the US dollar fixed income markets alone, it is estimated that 36 of the 200 trillion USD gross notional value of LIBOR exposed derivatives and loans extend beyond the 2021 deadline, see ARRC (2018), Table 1.

[13] See `https://www.newyorkfed.org/medialibrary/media/newsevents/events/markets/2020/credit-sensitivity-letters.pdf`.

one considers that tenor basis spreads (between one–month, three–month and six–month tenors, and between these tenors and OIS) are known to be substantial and volatile. From a perspective of sound mathematical modelling and the associated econometric analysis, this is due to "roll–over risk", consisting of downgrade risk (an aspect of credit risk quite distinct from that considered by the CSG) and funding liquidity risk.

It is worth noting that there are efforts to develop indices of rates at which market participants can actually borrow. In a recent paper, Berndt, Duffie and Zhu (2020) propose an "across–the–curve" credit spread index to address the issues under consideration by the CSG, by developing a robust benchmark spread which could then be added to SOFR to better reflect the actual cost of funding. This proposed index is a single number (i.e., a weighted average across maturities), and intended to reflect the average spread (over SOFR) of bank borrowings, based on their recent composition. It thus avoids dealing with the tenor basis (or term premia) except in the most aggregate manner. In contrast, the ICE Bank Yield Index (IBYI), developed by the Intercontinental Exchange (ICE), seeks to provide a benchmark term structure, which is calculated from OIS term rates plus credit/funding spreads derived from actual transactions. This makes this index essentially a transactions–based version of LIBOR, see the white paper ICE (2019). However, Berndt et al. (2020) remark on the IBYI, "The underlying pool of transactions, while much broader than that used to fix LIBOR, is not sufficiently deep for heavy use in derivatives market applications."

3. **Construction of the interest rate system.** Let the filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{Q})$ be the model for a financial market, where $(\mathcal{G}_t)_{t\geq 0}$ is the market filtration, and $\mathbb{Q}$ is a probability measure equivalent to the physical (or real–world or objective) probability measure $\mathbb{P}$. We introduce a money–market account with value process

$$B(t) = B(0)\,\exp\left(\int_0^t r(s)\,\mathrm{d}s\right), \tag{3.1}$$

for $t \geq 0$. The short rate of interest process $r(t)_{t\geq 0}$ is assumed risk–free; in particular it is free of credit risk. It is further assumed that the money–market account is the numéraire asset associated with the measure $\mathbb{Q}$, which is thus identified as the (spot) risk–neutral measure. The price process of a (non–dividend–paying) financial instrument that is discounted by $B(t)$ is a $(\mathcal{G}_t, \mathbb{Q})$–martingale. The measure $\mathbb{Q}$ is used to price any traded asset in the considered financial market. The risk–free zero–coupon bond is an example of a traded asset with a price process satisfying

$$P(t,T) = B(t)\,\mathbb{E}^{\mathbb{Q}}\left[1/B(T)\,|\,\mathcal{G}_t\right],$$

for $0 \leq t \leq T$. Before we proceed any further, we shall briefly discuss various interest rates (and interest rate benchmarks) mentioned or used in this paper. We also take the opportunity to recall the pricing of a canonical OIS contract allowing us to derive the *OIS rate*.

1. We have already introduced the *risk–free rate* $r(t)_{t\geq 0}$, which is recognised as the secured (i.e. credit–risk–free) overnight benchmark rate. We emphasise that we afford[14] assuming that any overnight interest rate may be considered

[14]This is a stylistic choice common in interest rate term structure modelling. Strictly speaking, in the context of the present paper this approximation ignores any basis spread between refinancing

approximately as a continuously compounded rate, i.e.,

$$e^{\int_t^T r(s)\mathrm{d}s} \approx \prod_{i=0}^{N-1} \left[1 + \delta_i \rho(t_i, t_{i+1})\right], \tag{3.2}$$

where $\rho(t_i, t_{i+1})$ is the overnight rate for $t \leq t_0 \leq \ldots \leq t_N = T$ and $\delta_i = t_{i+1} - t_i$.

2. The *unsecured overnight benchmark rate* $r^c(t)_{t\geq 0}$ is subject to credit risk. We set $r^c(t) := r(t) + \Lambda(t)$, where $\Lambda(t)_{t\geq 0}$ is the overnight credit spread process for entities able to borrow at the unsecured overnight benchmark rate. Examples of such overnight benchmark rates are the Effective Fed Funds Rate (EFFR) in the USA and the EONIA rate in the EU. It then follows that $r(t) = r^c(t) - \Lambda(t)$ and in the absence of market imperfections this would correspond to a *secured overnight benchmark rate*. Examples of secured overnight benchmark rates are SOFR in the USA and SARON in Switzerland. However, it is worth noting that SOFR is often at a spread *above* EFFR, thus—perhaps for reasons of market illiquidity—it is not a viable proposition to use $\Lambda(t)$ to link the markets for unsecured and secured borrowing. In our empirical analysis in Section 5, we focus on the market for unsecured borrowing only, estimating dynamics for $r^c(t)$ and $\Lambda(t)$. This would then imply a "shadow" risk–free rate $r(t)$, but this rate is not used in the empirical analysis.
3. Collateralised financial instruments require capital (e.g. cash, treasuries, etc.) to be deposited as collateral. Such deposits accrue interest at the *collateral rate*. An important example is the OIS, which swaps a fixed rate for an accumulated overnight rate (typically unsecured, such as Fed Funds). We now consider a single–period OIS; for a contract maturing at time T, the payoff (of the party receiving the floating leg) is

$$H(T) = \exp\left(\int_t^T r^c(u)\,\mathrm{d}u\right) - (1 + \delta\,\mathrm{OIS}(t,T)),$$

where $\delta := T - t$ and $\mathrm{OIS}(t,T)$ is the fixed OIS rate prevailing at time t. OIS are collateralised derivative contracts, so to price this payoff at a point in time prior to maturity T, the collateral rate should be used for discounting.[15] We assume this collateral rate to be given by the unsecured overnight rate r^c on which the OIS is written.[16] In practice, this is typically the case, and is the case for the OIS and other data we consider later in the paper; see Section 5.1 for further discussions. At time $t \in [0,T]$, the price process of the OIS is therefore given by

$$H(t,T) = B^c(t)\mathbb{E}^{\mathbb{Q}}\left[H(T)/B^c(T)\,|\,\mathcal{G}_t\right],$$

where $B^c(t) = B^c(0)\exp\left(\int_0^t r^c(s)\,\mathrm{d}s\right)$. Hence, we have

$$H(t,T) = 1 - P^c(t,T)\,(1 + \delta\,\mathrm{OIS}(t,T)), \tag{3.3}$$

every business day vs. continuously. See Appendix C.2 for evidence that this distinction is indeed negligible.

[15]That is, the cash flows of a fully collateralised contract must be discounted at the collateral rate in order to obtain risk–neutral martingales. We refer to, e.g., Piterbarg (2010), Fujii, Shimada and Takahashi (2011), Henrard (2014) and references therein, for material on the pricing of collateralised contracts.

[16]Thus the credit–risky rate r^c corresponds to the collateral rate and we do not need to introduce any additional notation for the latter.

where

$$P^c(t,T) := B^c(t)\,\mathbb{E}^{\mathbb{Q}}\left[1/B^c(T)\,|\,\mathcal{G}_t\right]. \tag{3.4}$$

The market convention is that an OIS has a present value of zero at inception, i.e. $H(t,T)=0$, for all $T \geq t$, at time of inception t. This implies that for the market *OIS rate* we have

$$\mathrm{OIS}(t,T) = \frac{1}{\delta}\left(\frac{1}{P^c(t,T)} - 1\right). \tag{3.5}$$

Multi–period OIS involve cash flows, swapping fixed for an accumulated overnight rate, at the end of multiple accrual periods. See Appendix A.1 for details.

3.1. **Construction of IBOR term rates.** An IBOR term rate is a benchmark meant to reflect the cost of borrowing for a fixed term. It is typically constructed as a trimmed average of rates reported by members of a panel. The rates themselves can originate from actual transactions, but are usually best estimates of the rates at which the panel members themselves can borrow, which is the case for LIBOR. Alternatively, EURIBOR is calculated by asking a panel what they believe is the borrowing rate faced by a representative (first–tier) bank within the panel. The numbers are reported to a datagathering institution, which calculates the average and publishes the benchmark on a daily basis. If we imagine m panel members contributing the term offer rates $\{L^k(t,T)\}_{0\leq t\leq T}^{k=1,\ldots,m}$ at time $t \in \mathbb{R}^+$ for the tenor $\delta = T - t$, then an IBOR can be roughly thought of as

$$L(t,T) := \Upsilon\left(L^1(t,T),\ldots,L^m(t,T)\right). \tag{3.6}$$

The map Υ is the (jurisdiction–specific) averaging mechanism producing the IBOR quote. While the contributed individual quotes are made public, proposing a model for each rate $L^k(t,T)$ is impractical, not least because observable market data is insufficient to identify the idiosyncratic model parameters associated with each individual panel member $k \in \{1,\ldots,m\}$. In the case of EURIBOR this might even be misleading, as the individual panel members are not asked to assess at what rate they themselves can borrow, but merely that of a representative entity.

In our model, the IBOR/OIS spreads are attributed to roll–over risk, which we decompose into downgrade risk and funding–liquidity risk. Credit default swaps are instruments by which the default risk of individual entities is traded, so calibrating the downgrade risk component to individual credit default swap spreads is possible in principle. However, there are no such instruments by which funding–liquidity risk of individual entities is traded, rendering a bottom–up approach to the modelling of $L(t,T)$, understood as modelling each $L^k(t,T)$ individually, infeasible. Therefore, we will calibrate our model of downgrade and funding–liquidity risk as it is reflected *in aggregate* by $L(t,T)$ or, more specifically, by the frequency basis between $L(t,T)$ for different tenors, as well as by the spread between $L(t,T)$ and $\mathrm{OIS}(t,T)$. In this, we focus on the roll–over risk faced by an arbitrary entity, which at time t is able to borrow at the benchmark rates (i.e., $r^c(t)$ overnight and $L(t,T)$ for a fixed term ending at T).

One should note that this also lends itself to an alternative interpretation, where the term rate benchmark $L(t,T)$ is based on the lending/borrowing transactions at time t. In this interpretation, the mapping Υ is the mechanism by which transacted lending/borrowing rates $L^k(t,T)$ are mapped to obtain a market benchmark $L(t,T)$. By remaining agnostic about the exact nature of Υ, our approach is therefore equally applicable regardless of whether the benchmark term rate is transaction–based or survey–based.

3.2. **Short rate model for IBOR.** Throughout this section we consider an arbitrary entity (be it a bank or highly rated corporate entity) that is assumed representative of (highly creditworthy) corporate borrowers in the market at time t, in the sense that it is able to borrow at the (credit–risky) overnight market benchmark rate $r^c(t)$ at time t, and also in the sense that it faces the level of roll–over risk priced by the tenor basis observed in the market (in a way formalised below). Since we only observe the market–aggregated pricing of roll–over risk (aggregated by way of price discovery in the swap markets), such an entity is sufficient for our modelling purposes. We assume that, at time $u > t$, this entity faces the instantaneous funding cost

$$\bar{r}_t(u) := r^c(u) + \gamma_t(u), \tag{3.7}$$

where $\gamma_t(u)_{u\geq t}^{t\geq 0}$ is the stochastic *roll–over–risk spread*. Note that the subscript t makes explicit that these values are specific to an arbitrary (but fixed from time t onwards) entity which is representative of the market at time t. Since, at time t, the entity can borrow at the prevailing overnight benchmark rate $r^c(t)$, the initial condition $\gamma_t(t) = 0$ must hold. However, reflective of the level of roll–over risk priced by the tenor basis, any entity able to borrow at $r^c(t)$ at time t faces the risk of no longer being able to borrow at $r^c(u)$ at $u > t$. Thus $\gamma_t(u) > 0$ at time $u > t$ means that the entity has been impacted by roll–over risk and is unable to roll over its borrowing at the market benchmark rate.

Note that the introduction of the subscript time index t is necessary in order to make the model amenable to econometric estimation and thus a crucial extension beyond the static roll–over risk term structure featured in Alfeus et al. (2020), which one could nest in our framework by fixing $t = 0$ while allowing $u \in [0, T^*]$ for some model time horizon T^*. In other words, Alfeus et al. (2020) only consider the pricing of roll–over risk for a borrower which is representative of the market at time $t = 0$ in the sense that it can borrow at the benchmark rate at that time. At future times $t > 0$, market rates price in the future roll–over risk faced by a borrower which is able to borrow at the benchmark rate at that t—this may be a different borrower compared to $t = 0$ (which is why we need this additional dimension). As we formalise the relationship between (3.7) and market observables below, it will become clear that the dynamics in the dimension indexed by u (i.e., the dynamics of the instantaneous funding cost for a fixed borrower) matter only in terms of the risk–neutral expectations embedded in market prices—the dynamics in this dimension under the physical probability measure are not empirically observed. What is observed are the dynamics of market instruments in the process time indexed by the subscript t.

The roll–over risk spread process $\gamma_t(u)_{u\geq t}^{t\geq 0}$ consists of two components: We introduce $\phi_t(u)_{u\geq t}^{t\geq 0}$, the *funding–liquidity spread* process, and $\lambda_t(u)_{u\geq t}^{t\geq 0}$, the *credit–downgrade spread* process.[17] The latter represents the additional credit spread arising from a deterioration of the credit quality of an entity relative to the market. It is thus an idiosyncratic component of the total credit spread, with the total credit

[17]Conceivably, one could extend this to allow also for *credit upgrades*, i.e. $\lambda_t(u) < 0$. However, since the level of credit risk reflected by the overnight rate benchmark $r^c(t)$, $\Lambda(t)$, in practice is for highly creditworthy entities, the risk of credit downgrades vs. the potential of upgrades is very asymmetric, so we can ignore the possibility of upgrades in the current modelling context.

spread at time $u \geq t$ given by $\Lambda(u) + \lambda_t(u)$. In the sequel, for notational simplicity, we will assume that there is zero recovery in default, and therefore the total credit spread is also the default intensity. This assumption does not represent any substantial loss of generality, since the relationships we derive below hold either exactly or in close approximation even when this assumption is lifted:[18] To verify the robustness of the empirical results obtained in Section 5 under the zero–recovery assumption, we reimplemented our model assuming a recovery rate of 40% and the results were for all intents and purposes unchanged under this departure from the zero–recovery assumption.

The funding–liquidity spread process reflects the fact that an entity, which is able to access funding at the benchmark rate at time t, may not be able to do so at some time $u > t$ even though its credit quality has *not* deteriorated relative to the market. Typically, one would expect this to happen during times of insufficient liquidity in the money market, as when the interbank lending market tightened following the bankruptcy of Lehman Brothers in 2008.[19] Thus we have a roll–over risk spread process of

$$\gamma_t(u) = \phi_t(u) + \lambda_t(u), \tag{3.8}$$

and the total funding rate $\bar{r}_t(u)$, at time $u \geq t$, is the risk–free rate plus the funding–liquidity spread and the total credit spread. That is,

$$\bar{r}_t(u) = r(u) + \phi_t(u) + \Lambda(u) + \lambda_t(u). \tag{3.9}$$

In steady and mature markets, one would expect that roll–over risk events be rare in a short time interval $(t, T]$, and hence $\gamma_t(u) = 0$ to prevail, except on rare occasions, for $u \in (t, T]$. This feature of the funding–liquidity and credit–downgrade spreads will be exhibited by the chosen stochastic model. In Section 4, an explicit model is proposed that is driven by compound Poisson processes, devised in such a way as to only jump infrequently. The stochastic jump intensity offers the necessary parametric freedom to calibrate the expected jump frequency in line with market data.

A random default time τ_t can be associated with the default intensity given by $(\Lambda(u) + \lambda_t(u))_{u \geq t}$, which represents the default time of an entity fixed at t, able to borrow at the market benchmark rate at that time. Thus we always have $\tau_t > t$, since this must be an entity that has not defaulted by time t. For $u > t$, however, the possibility of default is controlled by the intensity. Following the standard intensity–based approach, we consider a market filtration $(\mathcal{G}_t)_{t \geq 0}$ that consists of two components: $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for all $t \geq 0$, where $(\mathcal{H}_t)_{t \geq 0}$ is generated by the default indicator process $(\mathbf{1}\{\tau_s \leq t\})_{0 \leq s \leq u < t}$, see Appendix A.2 for details.

We are now in the position to construct the process $A(t, T)_{t \in [0,T]}$ of the present value of the *unsecured roll–over–risk–adjusted borrowing account*. That is, the account value at time t—given that no defualt has occurred until time t—is given by the expected discounted value of the repayment at time T of the continuously rolled borrowing over the period $[t, T]$, for a borrower which at time t can borrow at the

[18] For example, one could adopt a "fractional recovery in default", a.k.a. "recovery of market value" model as in Duffie and Singleton (1999), in which case a total credit spread of $\Lambda(u) + \lambda_t(u)$ and a loss fraction in default q would imply a default intensity $(\Lambda(u) + \lambda_t(u))/q$—but the final lines of (3.10) and (3.11) below would still apply.

[19] Arguably it was during the 2008 financial crisis that the market "learned" that such funding–liquidity squeezes were possible, reflected in the funding–liquidity component of roll–over risk priced by the market ever since. This component of roll–over risk can be expected to persist even if there is a move entirely to secured rate benchmarks like SOFR—we treat this case in Section 6.

market benchmark rate $\bar{r}_t(t) = r^c(t)$ (i.e., $\gamma_t(t) = 0$). Applying the decomposition (3.9), we obtain

$$\begin{aligned} A(t,T) &:= B(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T)}\exp\left(\int_t^T \bar{r}_t(u)\,\mathrm{d}u\right)\mathbf{1}\{\tau_t > T\}\,\middle|\,\mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T \phi_t(u) + \Lambda(u) + \lambda_t(u)\,\mathrm{d}u\right)\mathbf{1}\{\tau_t > T\}\,\middle|\,\mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T \phi_t(u)\,\mathrm{d}u\right)\,\middle|\,\mathcal{F}_t\right], \qquad (3.10) \end{aligned}$$

where the last step relies on Lemma A.3 given in Appendix A.2, and is based on a sub–filtration $\mathcal{F}_t \subset \mathcal{G}_t$. Here, we assume that all processes (other than the default indicator process), and especially the default intensities, are adapted to $(\mathcal{F}_t)_{t\geq 0}$. Note how the two indexing dimensions t and u from (3.7) enter into this calculation: The dynamics in the dimension indexed by u only enter in expectation, thus $A(t,T)$ does not depend on u. On the other hand, t determines the information on which we condition (via the filtration $\mathcal{G}_t$) and the borrower which we are considering, characterised by $\phi_t(t) = 0$, $\lambda_t(t) = 0$. In other words, when considering observable market prices at time t, these market prices reflect (by definition) the borrowing cost of a borrower able to borrow at the market benchmark rate, so the departure of the instantaneous funding cost from the market benchmark must be reset to zero.

The credit spread component $\Lambda(u) + \lambda_t(u)$ of the roll–over–risky borrowing rate $\bar{r}_t(u)$ cancels with the hazard–rate term emerging from the default indicator. Thus, the present value of the cost of borrowing from time t to T only depends on the funding–liquidity risk spread in the said time interval.

Again using Lemma A.3, and assuming that no default has occurred until time t, the price process $Q(t,T)_{t\in[0,T]}$ of the *defaultable zero–coupon bond* is given by

$$\begin{aligned} Q(t,T) &= B(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T)}\mathbf{1}\{\tau_t > T\}\,\middle|\,\mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T [r(u) + \Lambda(u) + \lambda_t(u)]\,\mathrm{d}u\right)\,\middle|\,\mathcal{F}_t\right]. \qquad (3.11) \end{aligned}$$

This is the expected discounted value, at time t, of a (promised) payment of one unit of currency at time T by an entity able to borrow at the market benchmark rate at time t.

In order to arrive at the "fair" term rate available to a market–average entity in the presence of credit risk, we apply the following argument:

1. Borrow one unit of currency at time t and continuously roll over this loan until time T. The present value at time t of the repayment at time T is given by $A(t,T)$.
2. Suppose now that, alternatively, the entity is in a position to borrow at the term rate $L(t,T)$. The value at time t of the repayment at time T is

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}(1 + \delta L(t,T))\mathbf{1}\{\tau_t > T\}\,\middle|\,\mathcal{G}_t\right] = [1 + \delta L(t,T)]Q(t,T).$$

3. In equilibrium, the values of the continuously rolled–over loan must be equal to that of the term loan.[20] One must therefore have $A(t,T) = [1+\delta L(t,T)]Q(t,T)$, and thus
$$L(t,T) = \frac{1}{\delta}\left(\frac{A(t,T)}{Q(t,T)} - 1\right). \tag{3.12}$$

The multiplicative relation for the IBOR–OIS spread can thus be given by
$$\mathrm{Sp}(t,T) = \frac{1+\delta L(t,T)}{1+\delta \mathrm{OIS}(t,T)} = \frac{A(t,T)}{Q(t,T)}P^c(t,T). \tag{3.13}$$

In our approach, the relations $Q(t,T) \leq P^c(t,T)$ and $A(t,T) \geq 1$ hold by construction, so $\mathrm{Sp}(t,T) \geq 1$. Furthermore the spread itself is a stochastic process, depending on the joint dynamics of all three components. This spread is the premium paid by the borrower to avoid the refinancing risk that borrowing on a roll–over basis entails.

Remark 3.1. It is worth noting the distinction between our expression (3.12) for IBOR and the expression in Filipović and Trolle (2013),
$$L(t,T) = \frac{1}{(T-t)}\left(\frac{1}{B(t,T)} - 1\right)\Xi(t,T) \tag{3.14}$$
where $B(t,T)$ corresponds to $Q(t,T)$ in our notation. In (3.14), $\Xi(t,T)$ is a multiplicative residual term needed to fit the market data, labelled "funding liquidity risk" without explanation of any underlying mechanism by which funding liquidity risk would translate into such a term. The expression (3.12), on the other hand, is derived from modelling the risk of any individual entity not being able to refinance roll–over borrowing at the prevailing market rate. It is a consequence of the model (not an ad hoc assumption) that this risk is one of the determinants of a term rate via $A(t,T)$, the present value of the roll–over risk adjusted borrowing account. The relations (3.12) and (3.14) are neither mathematically nor economically equivalent.

Remark 3.2. The term rate benchmark $L(t,T)$ and the infinitesimal benchmark $r^c(t)$ are in fact consistent, in the following sense. Assuming that $L(t,T)$ is differentiable with respect to T, we have:
$$\begin{aligned}
\lim_{T\to t} L(t,T) &= \lim_{T\to t}\frac{1}{T-t}\left(\frac{A(t,T)}{Q(t,T)} - 1\right) = \frac{\partial}{\partial T}\left(\frac{A(t,T)}{Q(t,T)} - 1\right)\Bigg|_{T=t} \\
&= \frac{\frac{\partial}{\partial T}A(t,T)\big|_{T=t} - \frac{\partial}{\partial T}Q(t,T)\big|_{T=t}}{\left(Q(t,T)\right)^2\big|_{T=t}} \\
&= \frac{\partial}{\partial T}A(t,T)\big|_{T=t} - \frac{\partial}{\partial T}Q(t,T)\big|_{T=t} \\
&= \phi_t(t) + r^c(t) + \lambda_t(t) \\
&= r^c(t),
\end{aligned} \tag{3.15}$$

[20] This ensures that the model is arbitrage–free, internally and with respect to any instruments to which the model is calibrated. It does not necessarily imply that a departure from model–implied prices results in exploitable arbitrage. This is because the market is not complete with respect to all risk sources, implying that $\mathbb{Q}$ is not the unique risk–neutral measure. Nevertheless, we follow the standard approach of pricing with a particular (market–chosen) measure, and identifying the stochastic dynamics under this measure by calibrating to liquid instruments.

where we recall that $\phi_t(t) = 0$ and $\lambda_t(t) = 0$. The result (3.15) states that the roll–over–risk premium, embedded in $L(t, T)$, is a premium that is offered only through term rates.

Remark 3.3. One could argue that consistency between the short– and the long–term benchmarks is not required, since the short–term benchmarks are calculated differently than their longer–term counterparts. The dominating short–term benchmark in the Eurozone during our data period was EONIA, which was calculated as the volume–weighted average of actual interbank overnight borrowing transactions. Although the EONIA panel was very similar to the EURIBOR panel, the EURIBOR term rates were not necessarily linked to actual transactions, but merely best guesses of a term rate at which a representative entity can borrow. Similarly, in the US the short–term unsecured benchmark is EFFR, which is calculated as the weighted average of overnight borrowing rates among all institutions eligible to make deposits at the Federal Reserve. This arguably makes the potential structural difference between long–term and short–term rates even greater than in the Eurozone. As no true short–term IBOR is observable in either markets, we leave such extensions for future work, but one could easily extend the model by including an additional spread process in (3.7).

We conclude the section by noting that IBOR–dependent instruments can be priced in the above framework, for instance an *interest–rate swap*, which involves swapping IBOR–linked payments for fixed payments. For a swap with payment dates T_1, T_2, ..., T_n, the rate R_t^L fixed at time t must satisfy

$$\sum_{i=1}^{n} R_t^L (T_i - T_{i-1}) B^c(t) \mathbb{E}^{\mathbb{Q}} \left[1/B^c(T_i) \,|\, \mathcal{F}_t \right] = \sum_{i=1}^{n} (T_i - T_{i-1}) B^c(t) \mathbb{E}^{\mathbb{Q}} \left[L(T_{i-1}, T_i)/B^c(T_i) \,|\, \mathcal{F}_t \right]. \quad (3.16)$$

This ensures equal value for both parties, where the rates $\{L(T_{i-1}, T_i)\}_{i=1,2,\ldots,n}$ determining the IBOR–linked payment leg are given by (3.12). The swap is assumed to be collateralised, so the cash flows are discounted with the unsecured overnight rate (which, as in (3.3), we assume to be the collateral rate). It is worth noting the distinction between the condition (3.16) for interest rate swaps and the condition (A.1) for multi–period OIS in Appendix A.1. The spread between the fixed–leg rates of the two instruments is a function of roll–over risk; like the spot IBOR/OIS spread, it vanishes to zero if we set the components $\phi_t(u)_{u \geq t}^{t \geq 0}$ and $\lambda_t(u)_{u \geq t}^{t \geq 0}$ of the roll–over risk process to zero. Similarly, Appendix A.1 shows how a non–zero *basis spread* added to the shorter tenor leg of a single–currency floating–for–floating basis swap (a.k.a. tenor swap) arises from roll–over risk.

Finally, *credit–default swaps*, where the reference entity can borrow at the market benchmark rate at time t, can be considered. These do not depend on liquidity risk, but do depend on the total credit spread (consisting of $\Lambda(t)_{t \geq 0}$ and $\lambda_t(u)_{u \geq t}$). See Appendix A.2 for details.

4. **Modelling with specific roll–over risk dynamics.**

A specific example of the framework in Section 3 is obtained by specifying the following four processes: The risk–free short rate $r(t)_{t \geq 0}$, the benchmark overnight credit spread $\Lambda(t)_{t \geq 0}$, and, for a particular entity fixed at time t, the credit–downgrade spread process

$\lambda_t(u)_{u\geq t}$ and the funding–liquidity spread $\phi_t(u)_{u\geq t}$. The overnight benchmark and collateral rate are then implied by the specification, where $r^c(t) = r(t) + \Lambda(t)$.

Before introducing our full model, we treat the special case that does not involve correlation. At the end of the section, we explain how the affine nature of our model allows for relevant quantities to be calculated.

4.1. **Uncorrelated model.** We specify the model initially under the risk–neutral measure $\mathbb{Q}$, so that expressions in Section 3 can be computed. The specification uses several scalar $\mathbb{Q}$–Brownian motions, all mutually independent, and all denoted $W(t)_{t\geq 0}$ with some additional superscript and/or subscript, e.g., $W^c(t)$ or $W^c_*(t)$. All quantities not explicitly dependent on t, e.g., θ^c_* or σ^Λ, are constants.

It is convenient to model $r^c(t)_{t\geq 0}$ directly, i.e., we specify $r^c(t)$ and $\Lambda(t)$, rather than $r(t)$ and $\Lambda(t)$. To accommodate non–trivial changes to the OIS term structure, and also the possibility of negative OIS rates, we model $r^c(t)_{t\geq 0}$ with a two–factor Gaussian process:

$$\mathrm{d}r^c(t) = \kappa^c\left(\theta^c(t) - r^c(t)\right)\mathrm{d}t + \sigma^c\,\mathrm{d}W^c(t), \tag{4.1}$$

$$\mathrm{d}\theta^c(t) = \kappa^c_*(\theta^c_* - \theta^c(t))\,\mathrm{d}t + \sigma^c_*\left(\rho^c\,\mathrm{d}W^c(t) + \sqrt{1-(\rho^c)^2}\,\mathrm{d}W^c_*(t)\right). \tag{4.2}$$

Next, for the benchmark overnight credit spread, we set

$$\mathrm{d}\Lambda(t) = \kappa^\Lambda(\theta^\Lambda - \Lambda(t))\,\mathrm{d}t + \sigma^\Lambda\sqrt{\Lambda(t)}\,\mathrm{d}W^\Lambda(t). \tag{4.3}$$

For $\lambda_t(u)_{u\geq t}$, the downgrade risk aspect of the model, we specify the following, which applies for each fixed t and uses u as the time variable:

$$\mathrm{d}\lambda_t(u) = -\beta^\lambda\lambda_t(u)\,\mathrm{d}u + \mathrm{d}J^\lambda_t(u), \quad \lambda_t(t) = 0, \tag{4.4}$$

where $J^\lambda_t(u)_{u\geq t}$ is a pure jump process that models downgrade events for an entity fixed at time t.

For each time at which an entity can be fixed, we allow for different drivers of downgrade risk, e.g., $J^\lambda_s(u)_{u\geq s}$ versus $J^\lambda_t(u)_{u\geq t}$. However, we use one (scalar) stochastic process, denoted $\xi^\lambda(u)_{u\geq 0}$, to model the jump intensity of $J^\lambda_t(u)_{u\geq t}$ at all fixing points t. That is, letting $N^\lambda_t(u)_{u\geq t}$ count the jumps of $J^\lambda_t(u)_{u\geq t}$, we have

$$\xi^\lambda(u) = \lim_{h\downarrow 0}\frac{\mathbb{Q}\left[N^\lambda_t(u+h) > N^\lambda_t(u)\,|\,\mathcal{F}_u\right]}{h}, \tag{4.5}$$

for all $t \geq 0$ and $u \geq t$.[21] All jump sizes are exponentially distributed, with a fixed mean of 2%.[22] Before specifying the stochastic intensity process $\xi^\lambda(u)_{u\geq 0}$, we emphasise the fact that all downgrade risk (and, extended below, all roll–over risk) has a certain stationarity property in our model. Because all jump processes refer to the same stochastic intensity, conditional on it having a certain value, downgrade risk is modelled equally in distribution across fixing times.

The stochastic intensity becomes a key state variable, controlling the downgrade risk outlook over time. We emphasise that the downgrade risk process $\lambda_t(u)_{u\geq t}$ is not a state variable of the model; it is initialised at zero by definition, and it is only

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[21] This specifies the distribution (conditional on the stochastic intensity) of the jump times for the processes initialised at all fixing times t. It does not specify how the processes are related, i.e., it does not specify the joint distribution of the processes across t.

[22] As in Filipović and Trolle (2013), the jump distribution becomes entangled with the effect of the jump intensity, and it is convenient to fix the former and focus on the latter. Adjusting the jump size mean, for instance, has a similar effect (on the instruments we consider) to adjusting the intensity, causing a redundancy and potential identification issues.

the expectation of the growth from zero that is relevant to, e.g., interbank rates. We model the intensity with

$$\mathrm{d}\xi^{\lambda}(t) = \kappa^{\lambda}(\theta^{\lambda} - \xi^{\lambda}(t))\,\mathrm{d}t + \sigma^{\lambda}\sqrt{\xi^{\lambda}(t)}\,\mathrm{d}W^{\lambda}(t). \tag{4.6}$$

For the liquidity aspect of the model, our specification is similar to that of downgrade risk. At each fixed time t, we have

$$\mathrm{d}\phi_t(u) = -\beta^{\phi}\phi_t(u)\,\mathrm{d}u + \mathrm{d}J_t^{\phi}(u), \quad \phi_t(t) = 0, \tag{4.7}$$

where $J_t^{\phi}(u)_{u\geq t}$ is a jump process. As formalised in (4.5), jump times (for all the jump processes) are distributed according to a stochastic intensity process $\xi^{\phi}(u)_{u\geq 0}$. Reverting to t as the time variable, we specify that

$$\mathrm{d}\xi^{\phi}(t) = \kappa^{\phi}(\theta^{\phi}(t) - \xi^{\phi}(t))\,\mathrm{d}t + \sigma^{\phi}\sqrt{\xi^{\phi}(t)}\,\mathrm{d}W^{\phi}(t). \tag{4.8}$$

The mean–reversion level above is stochastic, and we find it necessary to implement a two–factor structure for the liquidity aspect of the model. The downgrade–risk aspect above could easily be generalised in this way, by making θ^{λ} a state variable. Nevertheless, in Section 5 we show that the one–factor downgrade–risk model gives a surprisingly good fit. The stochastic mean process satisfies

$$\mathrm{d}\theta^{\phi}(t) = \kappa_*^{\phi}(\theta_*^{\phi} - \xi^{\phi}(t))\,\mathrm{d}t + \sigma_*^{\phi}\sqrt{\theta^{\phi}(t)}\,\mathrm{d}W_*^{\phi}(t). \tag{4.9}$$

The jump sizes are again all exponentially distributed with mean of 2%.

To model the time–series behaviour of the quantities in Section 3, we specify a market–price–of–risk process that relates the six–dimensional Brownian motion

$$W^{\mathbb{Q}}(t) = \Big[W^c(t), W_*^c(t), W^{\Lambda}(t), W^{\lambda}(t), W^{\phi}(t), W_*^{\phi}(t)\Big]^{\top} \tag{4.10}$$

used above to a corresponding Brownian motion $W^{\mathbb{P}}(t)_{t\geq 0}$ under the physical measure $\mathbb{P}$. That is,

$$\mathrm{d}W^{\mathbb{Q}}(t) = \mathrm{d}W^{\mathbb{P}}(t) + \mu(t)\,\mathrm{d}t, \tag{4.11}$$

where

$$\mu(t) = \Big[\mu^c, \mu_*^c, \mu^{\Lambda}\sqrt{\Lambda(t)}, \mu^{\lambda}\sqrt{\xi^{\lambda}(t)}, \mu^{\phi}\sqrt{\xi^{\phi}(t)}, \mu_*^{\phi}\sqrt{\theta^{\phi}(t)}\Big]^{\top}. \tag{4.12}$$

4.2. **Correlated model.** The model in Section 4.1 has sufficient cross–sectional flexibility, but does not include correlation between the various components of the model. Correlations cannot be added directly to the Brownian motions, because this prevents the model from being affine, which is essential for the computation of pricing equations. We propose the following two–step extension to allow for correlation but preserve the affine nature of the model.

First, instead of specifying $r^c(t)_{t\geq 0}$ directly, we specify

$$\mathrm{d}X^c(t) = \kappa^c(\theta^c(t) - X^c(t))\,\mathrm{d}t + \sigma^c\,\mathrm{d}W^c(t), \tag{4.13}$$

$$\mathrm{d}\theta^c(t) = \kappa_*^c(\theta_*^c - \theta^c(t))\,\mathrm{d}t + \sigma_*^c\,(\rho^c\,\mathrm{d}W^c(t) + \sqrt{1-(\rho^c)^2}\,\mathrm{d}W_*^c(t)). \tag{4.14}$$

Then we let $r^c(t) = X^c(t) + a^{\lambda}\xi^{\lambda}(t) + a^{\phi}\xi^{\phi}(t)$. The uncorrelated model is recovered if the correlation parameters a^{λ} and a^{ϕ} are zero, while non–zero values imply a correlation between $r^c(t)_{t\geq 0}$ and other state variables.

Second, we let the jump intensity of $J_t^\lambda(u)_{u\geq t}$ be given by

$$\xi^\lambda(u) + \rho^\phi \xi^\phi(u). \tag{4.15}$$

That is, in the correlated model, $\xi^\lambda(u)$ no longer represents the total intensity of the downgrade jumps, but only the part of the intensity that is uncorrelated from the liquidity aspect of the model. Again, if $\rho^\phi = 0$, the uncorrelated model is recovered. On the other hand, a non–zero value induces correlation between the downgrade and liquidity risk aspects of the model.

4.3. **Implementing the model.** At any fixed time t, using u to denote time thereafter, the *enlarged state process* is given by

$$X_t(u) = \Big[X^c(u), \theta^c(u), \Lambda(u), \lambda_t(u), \xi^\lambda(u), \phi_t(u), \xi^\phi(u), \theta^\phi(u) \Big]^\top. \tag{4.16}$$

Without the correlation introduced in Section 4.2, $X^c(u)$ reduces to $r^c(u)$. The dynamics specified above ensure that $X_t(u)_{u\geq t}$ is an affine jump–diffusion, making the well–known affine numerical techniques applicable; see Duffie, Pan and Singleton (2000). For instance, the quoted interbank rate in (3.12) can be written as

$$L(t,T) = \frac{1}{T-t}\left(\exp\left(\alpha^L(T-t) + \beta^L(T-t)^\top X_t(t)\right) - 1\right), \tag{4.17}$$

where $\alpha^L(\cdot)$ and $\beta^L(\cdot)$ are, respectively, $\mathbb{R}$– and $\mathbb{R}^8$–valued deterministic functions that satisfy the Riccati equations in Duffie et al. (2000) and elsewhere. The instruments covered in Appendix A (overnight–index swaps, interest–rate swaps, basis swaps and credit–default swaps) can be priced in this manner; see the Internet Appendix for details.

We recall, however, that $\lambda_t(t) = \phi_t(t) = 0$. These processes must be incorporated into the state space for cross–sectional calculations, but do not have values that need to be tracked over time; only the anticipation of their change, always from an initialisation of zero, is relevant. They are therefore excluded from the *reduced state process*, which is independent of the fixing times and is given by

$$X(t) = \Big[X^c(t), \theta^c(t), \Lambda(t), \xi^\lambda(t), \xi^\phi(t), \theta^\phi(t) \Big]^\top. \tag{4.18}$$

The dynamics earlier in the section, along with the measure change given by (4.11), imply the physical–measure dynamics for the reduced state process $X(t)_{t\geq 0}$. At any particular time, it must be enlarged with $\lambda_t(u)_{u\geq t}$ and $\phi_t(u)_{u\geq t}$ to obtain $X_t(u)_{u\geq t}$, for cross–sectional pricing purposes.

5. **Empirical implementation.**

5.1. **Data.** Our data set covers the period 2 January 2014 to 31 December 2021, with weekly observations therein. We consider data on Eurozone benchmarks EONIA and 3–month and 6–month EURIBOR. EONIA (Euro Overnight Index Average) is a volume weighted average of unsecured overnight lending in the Eurozone.[23] The EURIBOR (Euro Interbank Offer Rate) is calculated as a trimmed average of daily submissions by the panel banks. The banks are asked what they believe a

[23]On 2 October 2019, EONIA was reformed to be the new transactions based index €STR plus 8.5 bps. The major clearing houses switched from EONIA to €STR discounting on 27 July 2020. In our empirical analysis we account for this change by simply replacing EONIA based data with €STR based data from 27 July 2020. This did not induce any apparent break in the analysis or the empirical results.

prime bank would charge another prime bank in an unsecured loan for a given maturity. It thus measures the same as EONIA at a different term, but the calculation methodology and panel are slightly different. For a full description of the different EONIA, EURIBOR and €STR we refer to The European Money Markets Institute (2021). Our study focuses on the following:

- OIS rates for maturities of 1, 2, 3, 4, 5, 7 and 10 years. The rates are derived from collateralized contracts with EONIA index as the underlying and as collateral rate. We use the standard approximation method described in Appendix A.1 to relate OIS rates to discount factors. The discount factors can then be directly inferred using equation (A.1).
- Interest–rate swap rates, swapping EURIBOR for a fixed rate, and basis swap spreads, swapping 3–month for 6–month EURIBOR. The market convention is that standard vanilla swaps are quoted with a 3–month EURIBOR for the 1–year maturity, and 6–month EURIBOR for maturities above 1 year. The fixed leg is paid annually in both vanilla and basis swaps. Using the OIS discounting curve, we can easily combine the vanilla swap data with basis–swap spreads to calculate a synthetic 1–year swap rate for a swap written on 6–month EURIBOR, and synthetic swap rates for maturities above 1 year but with 3–month EURIBOR underlying. This leaves us with swap rates for both 3–month and 6–month tenors, for maturities of 1, 2, 3, 4, 5, 7 and 10 years.[24]
- The credit spread of a representative prime bank in the Eurozone is calculated following Filipović and Trolle (2013) by using the median of the CDS spreads of EURIBOR panel member banks.[25] This number is calculated for maturities 0.5, 1, 2, 3, 4, 5, 7 and 10 years every day in the sample. The results are plotted in Figure 1 below.

The data set containing swap rates comes from Bloomberg Financial Systems (2021). The CDS data come from IHS Markit (2021), and the EURIBOR panel membership data from The European Money Markets Institute (2021).

5.2. **Model identification.** The main identification issue revolves around disentangling the market–wide level of overnight credit risk from idiosyncratic credit–downgrade risk. This is a challenge, because the CDS spreads are driven by the sum of the two (as CDS contracts refer to a specific entity that can downgrade relative to the panel/market). We now describe this issue, and our resolution, in more detail. This has a significant bearing on how idiosyncratic credit spread risk, i.e., downgrade risk, is disentangled from liquidity risk, as we discuss below. The only other identification issue is more minor, concerning the means of the jumps that drive $\phi_t(u)_{u\geq t}$ and $\lambda_t(u)_{u\geq t}$, which cannot be identified separately from the jump intensity. Recall that these mean values were fixed in Section 4.

The OIS rates, across dates and maturities, are sufficient to identify the model for $r^c(t)_{t\geq 0}$ (i.e., to determine the value of this state variable over time, as well as its associated risk–neutral and physical–measure parameters). Given the OIS model, the

[24]From informal discussions with market participants we have learned that 3–month and 6–month tenors carry by far the dominant share of the liquidity in the market. While basis swaps against other tenors such a 1–month and 12–month do exist we have found the data quality to be low, with many missing values and stale quotes - especially in the earlier part of the data

[25]The number of banks in the panel ranges from 32 in the beginning of the sample to 19 in the end of the sample. We have not been able to find CDS data on CECE, BCEE and Bank of Ireland.

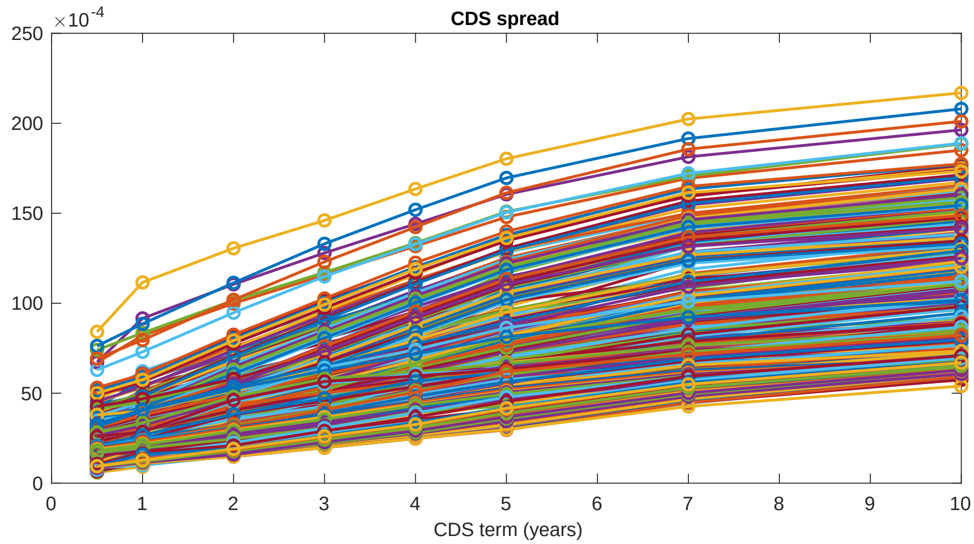

FIGURE 1. CDS term structures for each week in the sample.

CDS data is a function of both overnight–market credit risk, with spread and intensity process $\Lambda(t)_{t\geq 0}$, and idiosyncratic downgrade relative to the market benchmark, i.e., additional intensity $\lambda_t(u)_{u\geq t}$. See Appendix A.2 for the specific expressions. Strictly speaking, there is insufficient information to separate overnight–market and idiosyncratic credit spread risk, as they both contribute to the observed spreads.

Previous papers, such as Filipović and Trolle (2013) and Alfeus et al. (2020), have fixed the overnight–market default intensity at a low, constant value of five basis points. In those papers, this is justified by comparing the average overnight repo rate with an average unsecured overnight benchmark rate. Setting the market component to a small constant means that downgrade risk will account for the vast majority of the default risk observed in CDS, and for all of the observed variation. Consider, however, Figure 1, which shows the CDS term structure prevailing at each date covered by our data sample. Note that the shorter the CDS term, the less dependent the CDS spread is on downgrade risk; as the term is taken to zero, there is less and less potential for downgrade, and at the limit, the initial, overnight intensity completely determines average–panel CDS spreads. Thus, as the CDS term is shortened, the average–panel CDS spread becomes exactly equal to the overnight default intensity $\Lambda(t)$.[26]

Figure 1 suggests that the short–end of the CDS term structure, and therefore the value of $\Lambda(t)$, varies considerably over time, *and* tends to exceed the five basis points. In particular, it varies from virtually zero to sixty-five basis points during the sample, with an average of roughly twenty. We thus find the previous approach

[26]This is similar to how short–term credit spreads coincide with the default intensity, but note that these identities rely on our assumption of zero recovery. Under a "fractional recovery in default", a.k.a. "recovery of market value" model as in Duffie and Singleton (1999), if the CDS protection leg recognised a recovery of, e.g., 60%, then short–term CDS spreads would be given by 40% of the default intensity. To first order, under this and other constant recovery assumptions, a non-zero recovery in default amounts to a scaling between credit spreads and default intensities, and this scaling is irrelevant to our empirical analysis, which requires only credit spreads, not defaults.

to be untenable on our dataset, and instead propose to identify the overnight–market credit spread with an extrapolation of the CDS term structure to a term of zero at each observation date. Such an extrapolation gives a direct measurement of overnight–market credit spread, allowing it to be identified separately from the downgrade risk aspect of the model.[27] The regularity of the cross–sectional term structures, apparent in Figure 1, encourages us that this extrapolation is a reasonable inference from the data. Indeed, we find that the results are virtually unaffected by changes to the extrapolation method.[28]

With measurements of $\Lambda(t)$ over time, as per the above, the physical–measure dynamics of this process can be estimated. The market price of risk associated with changed to $\Lambda(t)$ cannot, however, be identified. This is because the CDS spreads depend on a risk–neutral expectation of both the market default intensity and idiosyncratic downgrades (i.e., on both $\Lambda(t)_{t\geq 0}$ and $\lambda_t(u)_{u\geq t}$; again see Appendix A.2 for details).

We therefore assume that the market price of risk associated with $\Lambda(t)$ to be zero (i.e., we assume $\mu^{\Lambda} = 0$).[29] This resolves the identification problem. Furthermore, the real–world dynamics of $\Lambda(t)$, empirically, involve a high degree of mean reversion. Section 5.4 below confirms this for our data, but this is consistent with other findings, e.g., Filipović and Trolle (2013). This is a reflection of the so–called "renewal" effect: panel members that have downgraded sufficiently are excluded, as the panel is renewed, so there is a continuous selection effect in favour of high credit–quality banks. Market participants incorporate this into their expectation, and it limits the effect of the market price of risk (at the limit, higher and higher mean reversion results in a constant process with no market price of risk).

Finally, note that, given the OIS model, the interest–rate and basis swaps are a function of total roll–over risk, i.e., liquidity $\phi_t(u)_{u\geq t}$ and downgrade $\lambda_t(u)_{u\geq t}$ (this stems from the IBOR in (3.12); see Appendix A.1 for full details). The amount of downgrade risk that is ascertained from the CDS data is therefore a crucial determinant of how the model decomposes the total roll–over risk that is observed in interest–rate and basis swaps.

5.3. **Estimation method.** We apply (quasi) maximum likelihood in conjunction with a Kalman filter. The likelihood function is constructed as follows. The physical–measure dynamics from Section 4 are given an Euler discretisation, where the change over each time step is approximated as Gaussian, giving rise to a *transition equation*. A *measurement equation* links the observed data to corresponding model–implied quantities, introducing a Gaussian, zero–mean error term between

[27]We emphasise the importance of the measurement corresponding to a term of zero. Even the six–month average–panel CDS spread (the shortest CDS term we observe) is a function of both market and idiosyncratic contributions to default risk, and there will always be an identification issue relating to this decomposition. For example, the CDS data in isolation would be consistent, strictly speaking, with zero default risk in the overnight market, and all default risk driven by idiosyncratic downgrades.

[28]For example, a simple linear extrapolation of the 6–month and 1–year spreads hardly differs from a cubic–spline extrapolation based on the whole term structure. We opt for the former, using the intuition that the comparison between the 6–month and 1–year spreads reflects the potential for downgrade between these two horizons, and therefore gives a proxy for how much downgrade potential is inherent in the 6–month spread.

[29]For the avoidance of doubt: This is the market price of the risk that the benchmark level of the market overnight credit spread changes. It is not the market price of default risk—the market price of default risk is not required in our analysis.

the two (model–implied OIS rates, CDS spreads etc., are computed as per Appendix B.1). The Kalman filter is the mean–variance optimal algorithm for estimating the discretised model's state variable paths. The observed error terms, for a particular parameter set, imply a likelihood value, which is maximised. The likely parameter sets are ones that result in small errors, clustered around zero with a small standard deviations. The errors have a standard deviation of σ_{rates} for OIS rates and CDS spreads (which are of a similar magnitude), and of σ_{spreads} for swap–OIS spreads.

The standard Kalman filter assumes a measurement equation that is linear in the state variables; we use the unscented Kalman filter (see, e.g., Christoffersen, Dorion, Jacobs and Karoui (2014)) to handle non–linear pricing. This involves approximating the distribution of estimated state variables with a discrete distribution, and transforming this distribution with the measurement equation, after which one can estimate the variances and covariances necessary to proceed (i.e., to optimally update the state variable estimate, given the week's prevailing data). Also, a truncation is occasionally needed, for the CIR–type processes, to prevent the Gaussian–approximated transition from becoming negative. See Appendix B.2 for details.

As discussed in Section 5.2 above, we first estimate the real–world dynamics of $\Lambda(t)$, based on the short–end of the CDS term structure over time. With the assumption of a zero market price of risk for $\Lambda(t)$, these parameters are fixed before the remaining parameters are varied to numerically maximise the likelihood function. Various initialisations are used to verify a global maximum is obtained. We estimate asymptotic standard errors based on the Fisher information.

5.4. **Parameter estimates.** Table 1 shows the parameters obtained for the uncorrelated model of Section 4.1 along with asymptotic standard errors. The pre–estimated overnight–market credit parameters show strong mean reversion to a level of sixteen basis points, with mild volatility only, as discussed in Section 5.2 above. The standard errors are relatively high, because these parameters are ascertained from the time series only.

The risk–neutral parameters are generally in typical ranges given their roles. One exception is the negative value for θ^c_*; note, however, that this is paired with a very low, negative mean reversion rate κ^c_*.[30] The other exception is the very low value for θ^ϕ_*; this is bounded below at zero, and although the state variable $\theta^\phi(t)$ is substantially away from zero for a majority of the sample, the optimal fit is obtained when the risk–neutral mean is very low.

As expected in estimations of this kind, the market–price–of–risk parameters have large standard errors. However, μ^ϕ, associated with downgrade risk, is negative. This indicates a (positive) risk premium for bearing downgrade risk, a consistent finding in prior literature (see the discussion in Filipović and Trolle (2013, §5.1)). In addition, μ^λ is significantly negative—despite the difficulties in precisely estimated market–price–of–risk parameters, we can conclude that the market awards a significant risk premium for liquidity risk.

The uncorrelated model gives a close fit to the market data—this is examined in the next subsection, but is apparent here from the low noise parameters σ_{rates} and σ_{spreads}.

[30]Instead of specifying a drift of $\kappa^c_*(\theta^c_* - \theta^c(t))$, we could have set $a + b\theta^c(t)$, in which case a would have a moderate value and b would be small.

Model aspect		OIS	O/N–market credit	Downgrade	Liquidity
State variables		r_t^c, θ_t^c	Λ_t	ξ_t^λ, λ_t	ξ_t^ϕ, θ_t^ϕ, ϕ_t
Superscript		c	Λ	λ	ϕ
Model parameters	κ	0.0971	4.3104	0.0525	0.2668
		(0.0026)	(0.7740)	(0.0005)	(0.0061)
	θ	-	0.0016	0.3966	-
			(0.0002)	(0.0034)	
	σ	0.0012	0.0819	0.8648	0.7929
		(0.0001)	(0.0030)	(0.0062)	(0.0534)
	κ_*	-0.0172	-	-	0.2022
		(0.0002)			(0.0049)
	θ_*	-0.2706	-	-	0.0002
		(0.0041)			(0.0002)
	σ_*	0.0120	-	-	0.5526
		(0.0006)			(0.0403)
	β	-	-	0.0493	4.7920
				(0.0021)	(0.0912)
	ρ	0.2502	-	-	-
		(0.0967)			
	μ	0.0192	-	-1.2081	-1.9761
		(0.1354)		(1.0706)	(0.7443)
	μ_*	-0.4035	-	-	-0.0166
		(0.3611)			(0.1167)
Filter parameters		$\sigma_{\text{rates}} \times 10^4$	$\sigma_{\text{spreads}} \times 10^4$	$\mathcal{L} \times 10^{-4}$	
		4.4240	1.5794	7.4718	
		(0.0448)	(0.0163)		

TABLE 1. Maximum likelihood results for the uncorrelated model, including asymptotic standard errors in parentheses. The log-likelihood value is denoted $\mathcal{L}$.

Table 2 shows the results for the full, correlated model from Section 4.2. This involves estimating three additional parameters: a^λ, a^ϕ and ρ^ϕ. These take on non–zero values, much larger in magnitude than their standard errors, and a significantly improved likelihood is obtained compared to the uncorrelated case. The improvement is easily sufficient for the uncorrelated model to be rejected in any standard statistical test, e.g., a likelihood–ratio test. The parameters a^λ and a^ϕ are negative, indicating a negative correlation between OIS and both downgrade and liquidity risk. A positive association between downgrade and liquidity risk is reflected by positive ρ^ϕ. This is consistent with the expected mechanism that an increasing liquidity spread naturally increases the overall credit risk, and in this model by impacting the credit–downgrade spread. The data thus strongly supports the inclusion of correlation in the model.

Model aspect		OIS	O/N–market credit	Downgrade	Liquidity
State variables		r^c_t, θ^c_t	Λ_t	ξ^λ_t, λ_t	ξ^ϕ_t, θ^ϕ_t, ϕ_t
Superscript		c	Λ	λ	ϕ
Model parameters	κ	0.1069	4.3104	0.0278	0.2250
		(0.0040)	(0.7740)	(0.0008)	(0.0047)
	θ	-	0.0016	0.2472	-
			(0.0002)	(0.0058)	
	σ	0.0012	0.0819	0.6977	0.7983
		(0.0001)	(0.0030)	(0.0080)	(0.0584)
	κ_*	-0.0183	-	-	0.2098
		(0.0003)			(0.0045)
	θ_*	-0.1953	-	-	0.0009
		(0.0033)			(0.0038)
	σ_*	0.0110	-	-	0.5830
		(0.0005)			(0.0423)
	β	-	-	0.0907	4.7959
				(0.0018)	(0.0696)
	ρ	0.2511	-	-	0.0830
		(0.0917)			(0.0010)
	μ	0.0182	-	-1.4948	-1.7886
		(0.1254)		(1.1729)	(0.7560)
	μ_*	-0.3373	-	-	-0.0872
		(0.3527)			(0.2186)
	a	-	-	-0.0028	-0.0024
				(0.0006)	(0.0002)
Filter parameters		$\sigma_{\text{rates}} \times 10^4$	$\sigma_{\text{spreads}} \times 10^4$	$\mathcal{L} \times 10^{-4}$	
		4.2031	1.5867	7.5021	
		(0.0432)	(0.0167)		

TABLE 2. Maximum likelihood results for the full, correlated model, including asymptotic standard errors in parentheses. The log–likelihood value is denoted $\mathcal{L}$.

The remaining parameters in the correlated model are estimated at similar values as in the uncorrelated case, including the market price of risk pattern discussed above.

5.5. **Model fit.** We hereafter focus on the correlated case. Figure 2 illustrates the fit to the data obtained by the full, correlated model, plotting time–series of market data and their model–fitted counterparts. The uncorrelated case is shown in Appendix C.1, as are the state variable paths that correspond to the fitted rates and spreads.

The top–left panel considers a short–, medium– and long–term OIS rate, showing that the model accommodates the changing level and shape of the OIS term

			EURIBOR–OIS	
	OIS	CDS	6m	3m
Uncorrelated model	3 .39	5.12	1.33	1.69
Full model	3.29	4.66	1.32	1.71

TABLE 3. Root mean squared fitting errors, in basis points, for our two model specifications.

structure, and obtains a close fit. Negative rates are observed and fitted well. The top–right panel plots CDS spreads, for similarly representative times to maturity. Recall that the overnight–market credit model is fitted in advance, given the short–end of the CDS term structure only, with the downgrade risk model capturing the term structure at non–zero times to maturity. This structure results in a close fit, with fitting errors no more than a few basis points for a vast majority of the data (formal fitting measures are follow below). The bottom panels pertain to the EURIBOR–OIS spread; the two–factor liquidity model (which combines with the CDS–informed downgrade risk model) achieves a close fit to the six–month spread, shown in the bottom–left panel, with changes to both the level of the spreads in general, and the shape across the spread's term structure being matched by the model. The fit to the three–month spread is not as close; with the short–term spread dropping very low relative to the six–month spread during 2016/7. Outside of this, however, the fitting error is around one basis point in magnitude on average.

The model fit is summarised in Table 3, which gives the root mean square error between the market data and corresponding model–implied quantities. Like with the likelihood function, the additional parameters in the correlated model result in a substantial improvement. The next subsection specifically considers the relative contributions of downgrade and liquidity risk.

5.6. **Downgrade versus liquidity risk.** With the EURIBOR–OIS spread being a function of total roll–over risk, consisting of both downgrade and liquidity risk, our model allows us to decompose the contributions of the two. Figure 3 shows, for the full correlated model, the fitted spread (with the six– and three–month tenor in either panel) as well as the spread our model implies without any contribution from liquidity.[31] On our dataset, downgrade risk never explains a dominant part of the observed spreads. It tends to constitute a significant part of the total roll–over risk, but well less than half, and the contribution is small for short–term spreads when spreads are low (in late 2017 and early 2018). Table 4 summarises the downgrade–liquidity decomposition we obtain, giving the average percentage contribution of downgrade risk to total roll–over risk at each term. On average, we estimate the downgrade component to be roughly 30%.

Finally, we note that we find very little correlation between typical liquidity measures and our model's liquidity risk variables, but we do find a positive correlation with our model's downgrade risk variable. Computing the correlation coefficients between weekly changes in the "K–measure" of Schwarz (2019), and weekly changes in the cross–sectional–average EURIBOR–OIS spread implied by our model, we find a correlation of about 35%. The K–measure measures the spread between German

[31]Specifically, the downgrade–risk–only spread is calculated assuming $\phi_t(u) = 0$ for all $u \geq t$, but with all other aspects and parameters of the model unchanged. That is, we do not re–estimate the model, but simply set the liquidity component of the estimated model to zero.

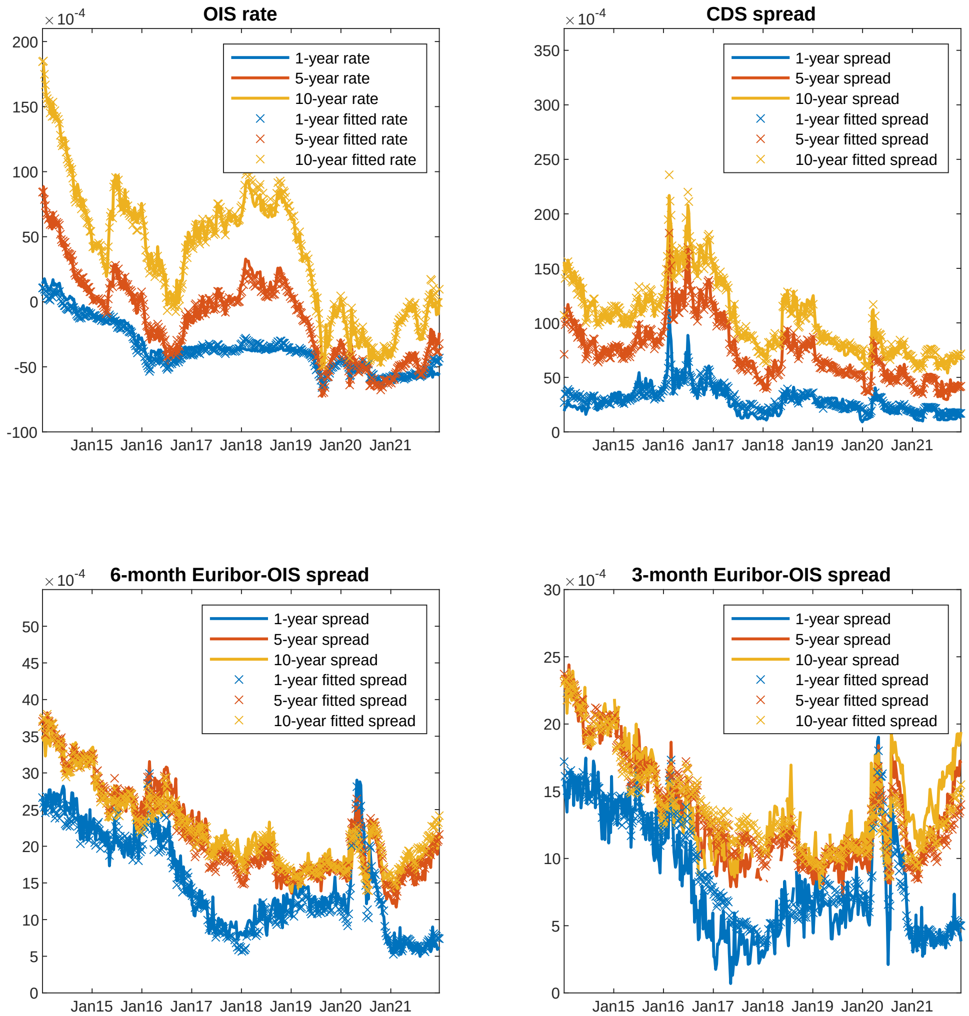

Figure 2. Market data compared to corresponding quantities, fitted by the full, correlated model.

Term (years)	1	2	3	4	5	7	10	Ave.
Uncorrelated model								
6–month spread percentage	38.22	34.08	32.56	32.21	32.51	34.21	38.10	34.56
3–month spread percentage	32.43	28.51	27.07	26.70	26.94	28.41	31.90	28.85
Full model								
6–month spread percentage	39.53	35.63	33.87	33.05	32.74	33.00	34.56	34.63
3–month spread percentage	33.75	29.98	28.30	27.50	27.20	27.40	28.78	28.99

Table 4. Percentage contribution of downgrade risk to fitted EURIBOR–OIS spread.

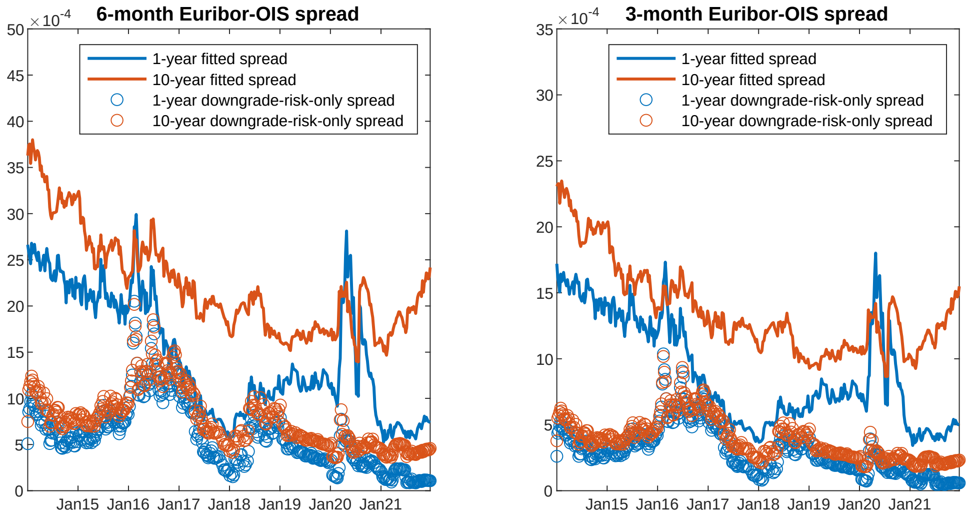

FIGURE 3. Fitted EURIBOR–OIS spreads, based on the full correlated model, as well as a version of the model that lacks liquidity risk.

government bonds and less liquid, but still government guaranteed agency bonds (KfW bonds). However, when we use the spread that is driven by liquidity risk only (i.e., with $\lambda_t(u) = 0$ for all $u \geq t$; see Footnote 31) the correlation drops to less than 5%; when the downgrade-risk-only spread is used instead, the correlation returns to nearly 35%, showing that this drives the vast majority of the total correlation. This is likely due to the fact that the K–measure, as explained in Schwarz (2019), is a non–specific measure of liquidity, whereas our model very specifically measures the expected funding–liquidity risk of prime banks.

6. **Repurchase agreement term rate benchmark.** While anticipating future market developments and in view of the fact that replacement benchmarks such as SOFR tend to reference collateralised rates (i.e., repurchase agreement (repo) rates), it is worth considering a specific instance of the framework introduced in Section 3, in which credit risk (and with it, downgrade risk) is “turned off,” leaving only the funding–liquidity component of roll–over risk. Thus our framework is equally applicable in an interest rate market in which credit risk is completely mitigated by collateralisation.[32] This will highlight how the presence of roll–over risk confounds efforts to create proper term rate benchmarks based solely on contracts referencing SOFR:

1. If one replaces the (unsecured) overnight Fed funds rate underlying an OIS by SOFR, then a family of such OIS for a set of maturities will imply a term structure of rates and discount factors. However, even in the absence of any

[32] In theory, one might be tempted to use this to extend the econometric analysis of the previous section to include repo rates. In practice, however, at the present stage of development of the market, there still seems to be a structural break between the repo market and the market for unsecured borrowing, as evidenced for example by the positive spread of SOFR over the effective Fed funds rate (EFFR) — credit risk mitigation alone would imply that this spread should go the other way. Thus including repo rates in the econometric analysis remains beyond the scope of the present paper.

market imperfections, these rates will differ from rates for secured borrowing at term (i.e., repo term rates), due to roll–over risk (now based solely on funding-liquidity, not credit, risk). That is, we would expect an OIS/repo term rate tenor basis.

2. Neither will this problem be resolved by averaging SOFR over some longer accrual period, nor by compounding the SOFR on a unit of investment over time, which are additional benchmarks published by the Federal Reserve Bank of New York under the names *SOFR Average* and *SOFR Index*, see Federal Reserve Bank of New York (2020). Aside from not being proper term rates, these rates are not known until the end of the accrual period, rather than at the beginning. While the technical issues involved in pricing contracts which reference backward–looking rates are surmountable,[33] irrespective of roll–over risk, the backward–looking averaging means that SOFR Average and SOFR Index will have a much lower volatility than fixed–in–advance spot term rates, be they unsecured (LIBOR) or secured (repo term rates). This issue is removed when one considers the forward rates implied by futures on SOFR Averages,[34] at least as long as the beginning of the averaging period still lies in the future. However, even after the appropriate convexity adjustments, we would still expect a tenor basis between such forward rates and the corresponding repo forward term rates due to roll–over risk.

Thus let us now consider the risk–free rate $r(t)_{t\geq 0}$ to be equivalent to the continuous–time approximation of SOFR, and therefore a rate that is representative of the cost of overnight repurchase transactions in the USA. In our example we will assume this is a rate at which any entity who has not been impacted by roll–over risk at time t can lend and borrow. (We recall that such an entity *is exposed* to being impacted by roll–over risk in the future.) We consider a repurchase agreement with an underlying security being sold by one counterparty to another. The agreement requires the asset be repurchased later at a higher value. We assume the repo, which is of sufficiently low risk such that in the event of default of the counterparty the market risk of the underlying Treasury is negligible,[35] fully mitigates credit risk. As in previous sections, the borrowing entity may face the risk that while it is able to borrow at SOFR at time t, it may not continue to be so at a future time $u > t$. This means the rate at which it can "repo out" the asset at time $u > t$ may change at a future point in time. Therefore, the continuous repo rate at time $u > t$, available to a borrower that at time t could borrow at the SOFR, is given by

$$\bar{r}_t(u) = r(u) + \phi_t(u). \tag{6.1}$$

We recall that $\phi_t(u)_{u\geq t}^{t\geq 0}$ is the funding-liquidity spread process with initial condition $\phi_t(t) = 0$, for all $t \geq 0$.

6.1. **Term rates versus OIS–implied rates in the absence of credit risk.** In order to arrive at the "fair" term rate available to a market–average entity when credit risk is eliminated (by virtue of collateralisation in the repo), we can repeat

[33] See, e.g., Lyashenko and Mercurio (2019) and Macrina and Skovmand (2020).

[34] Futures contracts referencing compounded daily SOFR interest during a specified accrual period (available lengths one and three months) are traded on CME Globex and CME ClearPort, see CME Group (2018).

[35] This is a common assumption if the underlying asset is a US Treasury.

the argument from Section 3 when roll–over risk consists of funding liquidity risk only:

1. Borrow one unit of currency at time t and continuously roll over this loan until time T. The present value at time t of the repayment at time T is given by $A(t,T)$, though the derivation differs from (3.10) by the absence of credit risk, i.e.,
$$\begin{aligned} A(t,T) &:= B(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T)}\exp\left(\int_t^T \bar{r}_t(u)\,\mathrm{d}u\right)\,\middle|\,\mathcal{F}_t\right] \\ &= B(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T)}\exp\left(\int_t^T r(u)+\phi_t(u)\,\mathrm{d}u\right)\,\middle|\,\mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^T \phi_t(u)\,\mathrm{d}u\right)\,\middle|\,\mathcal{F}_t\right]. \end{aligned} \tag{6.2}$$
2. Suppose now that, alternatively, the entity is in a position to borrow at the repo term rate $R(t,T)$. The value at time t of the repayment at time T is
$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}(1+\delta R(t,T))\,\middle|\,\mathcal{F}_t\right] = [1+\delta R(t,T)]P(t,T).$$
3. In equilibrium, the values of the continuously rolled–over loan must be equal to that of the term loan, and consequently
$$R(t,T) = \frac{1}{\delta}\left(\frac{A(t,T)}{P(t,T)} - 1\right). \tag{6.3}$$

As noted in the introduction, it is likely that a liquid OIS market with SOFR as the floating leg benchmark (as opposed to an unsecured rate) will appear in the near future, meaning that any exposure to interest rate risk in point 1 above can be eliminated via an OIS. Furthermore, if SOFR is also the collateral rate on such an OIS, the argument in Appendix A still holds and the discount factors implied by such an OIS are the $P(t,T)$. The multiplicative relation for the spread between a repo term rate and the rate on an OIS referencing SOFR is therefore
$$\mathrm{Sp}(t,T) = \frac{1+\delta R(t,T)}{1+\delta \mathrm{OIS}(t,T)} = A(t,T). \tag{6.4}$$

6.2. **Term rates versus backward–looking rate indices.** The SOFR Index defined in Federal Reserve Bank of New York (2020) is an example of a term risk–free reference rate, or "term–RFR", benchmark. It is standard to apply the approximation (3.2) to the term–RFR in mathematical models, see for example Lyashenko and Mercurio (2019). For a daily compounded term–RFR, denoted $\bar{R}(T,U)$, covering the accrual period $[T,U]$, we thus write
$$\bar{R}(T,U) \approx R^*(T,U) = \frac{1}{\delta}\left(\frac{B_U}{B_T} - 1\right), \tag{6.5}$$
where $R^*(T,U)$ is the continuously compounded term–RFR. The bank account process $B(t)$ is defined by (3.1), and $\delta = U - T$.

Macrina and Skovmand (2020) highlight several consequences of the fact that $\bar{R}(T,U)$ and $R^*(T,U)$ are not T–measurable, but rather U–measurable: Comparing the "backward–looking" $R^*(T,U)$ with a traditional "forward–looking" risk–free rate implied by the discount factor over $[T,U]$, i.e.,

$$F(T,U)=\frac{1}{\delta}\left(\frac{1}{P(T,U)}-1\right), \tag{6.6}$$

they note that a single fixed–for–floating payment on a swap (a "swaplet") covering the accrual period $[T,U]$ has the same present value at time $t\leq T$ regardless of whether the floating leg references $R^*(T,U)$ or $F(T,U)$, but this is not true for $T<t\leq U$, because at such a time $F(T,U)$ is already known while $R^*(T,U)$ is still uncertain. This additional uncertainty also means that caplets referencing $R^*(T,U)$ are more valuable than caplets referencing $F(T,U)$, all other things being equal.

Furthermore, it is worth noting that the spot rate dynamics of fixed *time-to-maturity* rates $F(t,t+\delta)$ and $\bar{R}(t-\delta,t)$ (written here so that both expressions are t–measurable) will have markedly different volatilities, with the volatility of $\bar{R}(t-\delta,t)$ being much lower due to the fact that $\bar{R}(t-\delta,t)$ is an average of realised overnight rates. For *forward rates*, this problem disappears, since the relevant expectations under the forward measure to the end of the accrual period (denoted here by the superscript U) are the same, i.e., for $t\leq T$,

$$\mathbb{E}^U[F(T,U)\,|\,\mathcal{F}_t]=\mathbb{E}^U[R^*(T,U)\,|\,\mathcal{F}_t],$$

from which it follows that the associated forward rates are the same.

Thus, in the absence of roll–over risk, one can argue that $\bar{R}(T,U)$ is an appropriate benchmark replacement for $F(T,U)$ when considering *linear* instruments (not caplets) at times $t\leq T$. This is already a lot more restrictive than a blanket claim that $\bar{R}(T,U)$ is an appropriate replacement for $F(T,U)$.

However, even this restricted assertion fails in the presence of roll–over risk. This is because the term rate $R(t,T)$ given by (6.3) differs from $F(t,T)$ by the presence of the present value $A(t,T)$ of the roll–over–risk–adjusted borrowing account, and therefore replacing a reference term rate $R(T,U)$ by a term–RFR benchmark like the SOFR Index, i.e. by $\bar{R}(T,U)$ (or its continuous–time approximation $R^*(T,U)$), will change the present value of the floating side of a swaplet at *any* time $t<U$, including $t\leq T$.

The correlated model estimated on European data in Section 5 allows us to construct the theoretical repo term rates defined in (6.3), and compare them to the theoretical OIS implied rates defined by (6.6). Both rates are completely devoid of credit risk, but the funding liquidity risk component is present in the repo term rate. In the Eurozone these rates are not empirically observable at present, as the primary overnight rate (EONIA/€STR) is unsecured. Nevertheless, the comparison allows us to gauge the economic significance and dynamics of the impact of the funding liquidity component on term rates. In Figure 4 in the left panel, these rates are plotted for a six–month term and for comparison we have also included the model implied backward-looking rate. Removing the credit component puts the rates in negative territory for almost the entire sample period. The presence of the roll–over risk spread is particularly apparent in the earlier, higher rate regime of the sample period. Furthermore, we also see that while expectations of forward–looking and backward–looking rates may be the same, their actual realisations also differ quite substantially, with the backward–looking rate floating both above and below the forward–looking rates. The right hand side of Figure 4 demonstrates that the roll–over risk premium as measured by the spread between the OIS–implied and repo term rate also scales in a level–dependent fashion with the length of the term. Thus

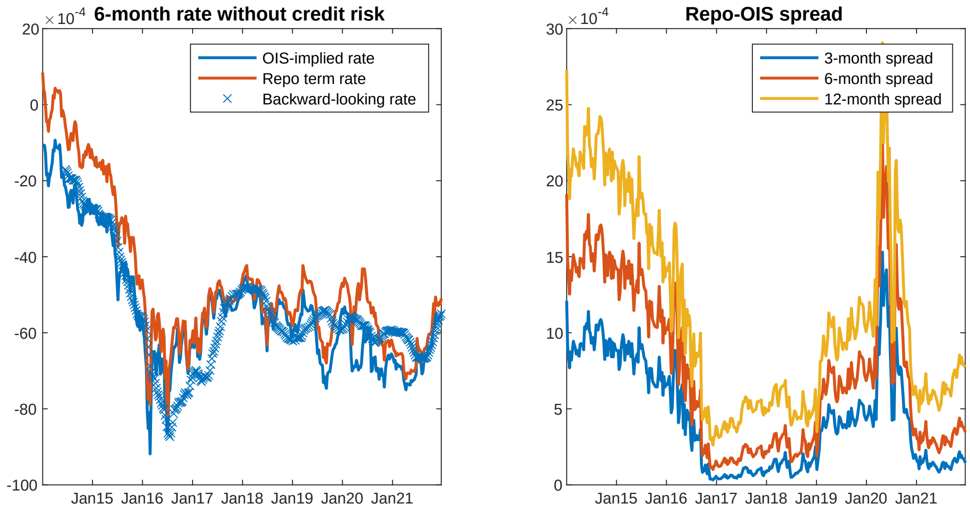

FIGURE 4. Six–month term rates (left panel) and the spread between repo term rates and OIS–implied term rates (right panel), all in the absence of credit risk, based on the model parameters and state variables estimated in the correlated version of the model in Section 5.

this (at present theoretical) basis spread implied by the funding–liquidity component as estimated empirically in Section 5 is both economically significant (up to about 30 bps at the twelve–month term) and volatile.

6.3. **SOFR futures.** In the absence of OIS on overnight RFR benchmarks such as SOFR, futures contracts can be used to glean information about the term structure of risk–free interest rates. This involves a model–dependent convexity correction: Lyashenko and Mercurio (2019) provide an approximation for this in their model, while Macrina and Skovmand (2020) obtain a closed–form expression in their context. Note that the convexity correction in the case of a SOFR futures, i.e., a futures on $\bar{R}(T,U)$ or $R^*(T,U)$, differs from the convexity correction for a futures on $F(T,U)$. In the absence of market imperfections and assuming that the model used to derive the convexity correction is a sufficiently accurate reflection of reality, RFR discount factors thus can be extracted from SOFR futures. However, these discount factors correspond to the OIS–implied $P(t,T)$, which by themselves are insufficient to determine the term rates $R(t,T)$, as noted in (6.3) above. This reinforces the argument that the "term–RFRs" $\bar{R}(T,U)$ or $R^*(T,U)$ are not appropriate replacements for IBOR term rates, even when credit risk is set aside.

7. **Conclusion.** In the framework for multicurve interest rate modelling developed in this paper, the presence of roll–over risk blocks the naive arbitrage between term structures for different payment frequencies (tenors). Unlike Alfeus et al. (2020), our set–up explicitly models the dynamics of the term structure of roll–over risk, and therefore of term rates such as LIBOR. Thus a concrete model instance in our framework lends itself to econometric estimation. For such an instance, one that is based on affine dynamics, we have performed an empirical analysis on EUR data for OIS, interest–rate swaps, basis swaps and credit default swaps. Unlike

a key prior empirical study on similar data (Filipović and Trolle (2013)), we find that credit risk typically contributes only about 30% of the IBOR/OIS spread, where the extrapolated short–term credit spread faced by entities able to borrow at the market benchmark rates is greater than the five basis points assumed in that study. The balance of the IBOR/OIS spread is due to the funding liquidity component of roll–over risk, which we have modelled more explicitly than the prior literature. Our results provide empirical support for the view, expressed for example in Albanese et al. (2021), that "the [LIBOR] transition's most problematic aspect revolves around *Funding Risk* transfer, which current proposals do not robustly tackle."[36] In our framework, this funding liquidity component of roll–over risk is reflected in the present value of a roll–over risk–adjusted borrowing account. Furthermore, we allow for correlation between the state variables, and we obtain a substantially better econometric fit than Filipović and Trolle (2013). The model appears to be robust, i.e., we observe a similar good quality of fit over various sub–sets of the data.

The framework is "reduced–form" in the sense that it takes a "top–down" approach by modelling the dynamics of market benchmark rates (of which the specific instances EONIA and EURIBOR are used in the empirical section) and the risk that a borrower might not be able to roll–over debt at (a constant spread to) the benchmark rates in the future. Consequently, the framework is unaffected by the specific mechanism by which such benchmark rates are determined, be it panel–based or transaction–based. Thus, as various jurisdictions transition away from panel–based benchmarks towards transaction–based ones (such as SOFR in the United States), the framework presented in this paper remains applicable. Although our empirical analysis is conducted on EUR data, the issue it identifies is of global relevance for all jurisdictions in which the future of IBOR–type benchmarks is being debated or (as in the US and the UK) has been decided.

In fact, our framework allows for a unified treatment of overnight benchmarks based on unsecured rates (e.g., EONIA) or secured rates (e.g., SOFR). As we have shown, roll–over risk leads to a term premium beyond the traditional term premia associated with the market price of interest rate risk. When roll–over risk is decomposed into a credit and a non-credit component, the latter is substantial, therefore even if credit risk is entirely mitigated, the presence of roll–over risk means that mooted term rate replacements such as SOFR compounded over three months are not proper term rates at all, nor are proper forward term rates implied by futures on compounded SOFR. If the market continues to incorporate the risk of not being able to roll over borrowing at (a constant spread to) the market reference rate into a basis spread between tenors, then even if there is a move entirely to secured rate benchmarks, a "multicurve" term structure environment can be expected to persist. For the Eurozone, the policy implication is that there is certainly utility in EURIBOR continuing to exist alongside €STR.

Appendix A. Interest–rate instruments.

A.1. **Swaps.** We first consider multi–period OIS. For the ith of n payments, let $[T_{i-1}, T_i]$ denote the accrual period over which the reference rate is compounded.

[36]See Albanese et al. (2021), p. 2. In that paper, Albanese et al. (2021) propose that in the absence of LIBOR, banks be immunaised from funding risks by periodic exchanges of funding valuation adjustment (FVA) payments, calculated by bespoke independent legal entities created for this purpose.

Letting R^c_t denote the fair fixed rate at time t, we must have

$$\sum_{i=1}^{n} B^c(t)\mathbb{E}^{\mathbb{Q}}\left[\delta_i R^c_t / B^c(T_i)\,\middle|\,\mathcal{F}_t\right] = \sum_{i=1}^{n} B^c(t)\mathbb{E}^{\mathbb{Q}}\left[\left(\mathrm{e}^{\int_{T_{i-1}}^{T_i} r^c(u)\,\mathrm{d}u} - 1\right)/B^c(T_i)\,\middle|\,\mathcal{F}_t\right],$$

where $\delta_i = T_i - T_{i-1}$. This ensures that the two legs of the swap have equal value at time t. Note the overnight–continuous compounding approximation, as in (3.2). Both sides can be simplified to obtain

$$R^c_t \sum_{i=1}^{n} \delta_i P^c(t, T_i) = \sum_{i=1}^{n} P^c(t, T_{i-1}) - P^c(t, T_i),$$

and therefore

$$R^c_t = \frac{\sum_{i=1}^{n} P^c(t, T_{i-1}) - P^c(t, T_i)}{\sum_{i=1}^{n} \delta_i P^c(t, T_i)} = \frac{1 - P^c(t, T_n)}{\sum_{i=1}^{n} \delta_i P^c(t, T_i)}. \tag{A.1}$$

Considering now an interest–rate swap with the same payment dates, (3.16) implies a fixed rate of

$$R^L_t = \frac{\sum_{i=1}^{n} \delta_i B^c(t)\mathbb{E}^{\mathbb{Q}}\left[L(T_{i-1}, T_i)/B^c(T_i)\,|\,\mathcal{F}_t\right]}{\sum_{i=1}^{n} \delta_i P^c(t, T_i)}.$$

Note that in the absence of roll–over risk, one has $L(T_{i-1}, T_i) = \mathrm{OIS}(T_{i-1}, T_i)$; to see this, compare (3.5) and (3.12) when $\phi_t(u)^{t\geq 0}_{u\geq t}$ and $\lambda_t(u)^{t\geq 0}_{u\geq t}$ are zero. In this case, the numerator can be simplified to show that $R^c_t = R^L_t$.

It is possible for the accrual and payment structure to differ between the floating and fixed legs. To accommodate this, let T^L_1, T^L_2, ..., T^L_n denote the dates of the n floating rate payments, and T^{fix}_1, T^{fix}_2, ..., T^{fix}_m the dates of the m fixed payments. Almost always one has $T^L_n = T^c_m$. Then (3.16) must be modified as follows:

$$\sum_{i=1}^{m} R^L_t (T^{\mathrm{fix}}_i - T^{\mathrm{fix}}_{i-1}) B^c(t)\mathbb{E}^{\mathbb{Q}}\left[1/B^c(T^{\mathrm{fix}}_i)\,|\,\mathcal{F}_t\right] = \sum_{i=1}^{n} (T^L_i - T^L_{i-1}) B^c(t)\mathbb{E}^{\mathbb{Q}}\left[L(T^L_{i-1}, T^L_i)/B^c(T^L_i)\,|\,\mathcal{F}_t\right],$$

giving

$$R^L_t = \frac{\sum_{i=1}^{n} (T^L_i - T^L_{i-1}) B^c(t)\mathbb{E}^{\mathbb{Q}}\left[L(T^L_{i-1}, T^L_i)/B^c(T^L_i)\,|\,\mathcal{F}_t\right]}{\sum_{i=1}^{m} (T_i - T^{\mathrm{fix}}_{i-1}) P^c(t, T^{\mathrm{fix}}_i)}.$$

Finally, basis swaps involve the swapping of IBOR–based floating payment streams of different frequency. We let $T^{(j)}_1$, $T^{(j)}_2$, ..., $T^{(j)}_{n_j}$ denote the payment times of the jth stream, for $j = 1$ or $j = 2$ (as above, $T^{(1)}_{n_1} = T^{(2)}_{n_2}$). The value of either leg at time t is

$$\sum_{i=1}^{n_j} (T^{(j)}_i - T^{(j)}_{i-1}) B^c(t)\mathbb{E}^{\mathbb{Q}}\left[L(T^{(j)}_{i-1}, T^{(j)}_i)/B^c(T^{(j)}_i)\,|\,\mathcal{F}_t\right]. \tag{A.2}$$

With roll–over risk priced into the IBOR rates, the leg with the lower payment frequency will be more valuable. The party receiving this leg must therefore pay the *basis swap spread* to their counterparty, to ensure the swap is fair. Let $T^{(3)}_1$, $T^{(3)}_2$, ..., $T^{(3)}_{n_3}$ denote the spread payment times (which do not necessarily coincide with the payment times of either IBOR leg), and let R_t^{BS} denote the spread at time t. The value at time t of the stream of spread payments is

$$\sum_{i=1}^{n_3}(T_i^{(3)} - T_{i-1}^{(3)})B^c(t)\mathbb{E}^{\mathbb{Q}}\left[R_t^{\mathrm{BS}}/B^c(T_i^{(3)})\,|\,\mathcal{F}_t\right] = R_t^{\mathrm{BS}}\sum_{i=1}^{n_3}(T_i^{(3)} - T_{i-1}^{(3)})P^c(t,T_i^{(3)}). \tag{A.3}$$

The spread R_t^{BS} must be set so that the value of the spread payments, in (A.3), is equal to the difference between the two IBOR–based legs, valued in (A.2).

A.2. **Credit–risky instruments.** We use the so–called reduced–form approach to model credit risk in this paper, and follow the setup presented in Bielecki, Jeanblanc and Rutkowski (2009). On the probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, we introduce the right-continuous default indicator process $H(t) := \mathbf{1}\{\tau \leq t\}_{t>0}$ generating the filtration $\mathcal{H}_t := \sigma(H(s)_{0\leq s\leq t})$, where τ is the random default time. The market filtration is given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, and while τ is a $\mathcal{G}_t$-stopping time, it is not necessarily an $(\mathcal{F}_t)$-stopping time. Next, we introduce the conditional probability $F(t) = \mathbb{Q}[\tau \leq t\,|\,\mathcal{F}_t]$, and define the $(\mathcal{F}_t)$-adapted hazard process $\Gamma(t)_{t>0}$ by $\Gamma(t) = -\ln(1 - F(t))$. The so-called "key lemma", can now be recalled, see, e.g., Bielecki et al. (2009):

Lemma A.1. *Let X be $\mathcal{F}_T$-measurable and $\mathbb{Q}$-integrable. Then for every $t \leq T$, one has*

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[X\,\mathbf{1}\{\tau > T\}\,|\,\mathcal{G}_t\right] &= \mathbf{1}\{\tau > t\}\frac{\mathbb{E}^{\mathbb{Q}}\left[X\,\mathbf{1}\{\tau > T\}\,|\,\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[X\,\mathbf{1}\{\tau > t\}\,|\,\mathcal{F}_t\right]} \\ &= \mathbf{1}\{\tau > t\}\mathbb{E}^{\mathbb{Q}}\left[X\exp\left[\Gamma(t) - \Gamma(T)\right]\,|\,\mathcal{F}_t\right].\end{aligned}$$

Assumption A.2. The hazard process used to model the credit-risk component of the roll-over-risk-adjusted borrowing account (3.10) is given by

$$\Gamma_s(s,t) = \int_s^t \psi_s(u)\mathrm{d}u,$$

where $s \in [0,t]$ and $\psi_s(u) = \Lambda(u) + \lambda_s(u)$, so that $\mathbb{Q}[\tau_s \in [u, u+\mathrm{d}u]\,|\,\tau_s \geq u, \mathcal{F}_u] = \psi_s(u)\mathrm{d}u$, $u \geq s$. In the context of Lemma A.1, one then has

$$\Gamma(t) - \Gamma(T) = \Gamma_0(0,t) - \Gamma_0(0,T) = \Gamma_t(t,T) = \int_t^T \psi_t(u)\mathrm{d}u.$$

We interpret $\Gamma_t(t,T)_{t\in[0,T]}$ as the *total credit* hazard process for the time period $[t,T]$, and $\psi_t(u)_{u\geq t}$ as the *total hazard rate* process.

Then, the "key lemma" takes the following form in our intensity-based setting, c.f., relation (3.10):

Lemma A.3. *For an integrable random variable X and the default time $\tau_t > t$ described above, one has*

$$\mathbb{E}^{\mathbb{Q}}\left[X\,\mathbf{1}\{\tau_t > T\}\,|\,\mathcal{G}_t\right] = \mathbf{1}\{\tau_t > t\}\mathbb{E}^{\mathbb{Q}}\left[X\exp\left(-\int_t^T \left[\Lambda(u) + \lambda_t(u)\right]\mathrm{d}u\right)\,\middle|\,\mathcal{F}_t\right].$$

One could drop the indicator $\mathbf{1}\{\tau_t > t\}$ on the right–hand side of the above equation if one specified that $\mathbb{Q}[\tau_t \leq t] = 0$. We also note that we have stated the lemma in terms of sigma-algebras at time t; in general, one may need to evaluate the above at any time point until T, but this is not required for our purposes.

Consider now a CDS written at time t, referring to the entity with default time τ_t. Let $[T_0, T_n]$ denote the period covered by the CDS (where $T_0 \geq t$), and let T_1, T_2, ..., T_n denote the payment dates. The protection leg of the CDS, at time t and with a unit notional, has value

$$B^c(t)\mathbb{E}^{\mathbb{Q}}\left[1/B^c(\tau_t)\mathbf{1}\{\tau_t \leq T_n\}\,|\,\mathcal{G}_t\right]. \tag{A.4}$$

Note that here again we are using the assumption of zero recovery in default. As argued in Section 3.2, this assumption has no material effect on our results.[37]

In the event of default during the CDS period (i.e., if $\tau_t \in [T_0, T_n]$), the protection seller makes a payment equal to the notional amount (recall that zero recovery is assumed throughout the paper). Letting C_t denote the annual spread at time t, the payment leg of the unit–notional CDS has value at time t of

$$\sum_{i=1}^{n} B^c(t)C_t(T_i - T_{i-1})\mathbb{E}^{\mathbb{Q}}\left[1/B^c(T_i)\mathbf{1}\{\tau_t > T_i\}\,|\,\mathcal{G}_t\right] \tag{A.5}$$

$$+\sum_{i=1}^{n} B^c(t)C_t\mathbb{E}^{\mathbb{Q}}\left[1/B^c(\tau_t)(\tau_t - T_{i-1})\mathbf{1}\{T_{i-1} < \tau_t \leq T_i\}\,|\,\mathcal{G}_t\right]. \tag{A.6}$$

The first sum gives the value of future spread payments, in the event that the reference entity does not default. The second sum gives the value of the partial spread payment that is made in the event of default. The CDS spread C_t must ensure that the payment and protection legs are of equal value.

Appendix B. Model implementation.

B.1. Instrument pricing.

It is convenient to collate the (risk–neutral) dynamics of the individual state variables, and write the dynamics of the enlarged state process in (4.16):

$$\mathrm{d}X_t(u) = \kappa(\theta - X_t(u))\,\mathrm{d}u + \sigma(u)\,\mathrm{d}W^{\mathbb{Q}}(u) + \mathrm{d}J_t(u), \tag{B.1}$$

where

$$\kappa = \begin{bmatrix} \kappa^c & -\kappa^c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa^c_* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa^\Lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta^\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa^\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta^\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \kappa^\phi & -\kappa^\phi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa^\phi_* \end{bmatrix}, \qquad \theta = \begin{bmatrix} \theta^c_* \\ \theta^c_* \\ \theta^\Lambda \\ 0 \\ \theta^\lambda \\ 0 \\ \theta^\phi_* \\ \theta^\phi_* \end{bmatrix},$$

[37] As noted in Section 3.2, a robustness check was carried out by repeating the analysis assuming a 40% recovery rate. In that departure from the zero–recovery assumption, all affected formulas (including CDS pricing) were appropriately modified.

$$\sigma(u) = \begin{bmatrix} \sigma^c & 0 & 0 & 0 & 0 & 0 \\ \sigma^c_* \rho^c & \sigma^c_* \sqrt{1-(\rho^c)^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^\Lambda \sqrt{\Lambda(u)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^\lambda \sqrt{\xi^\lambda(u)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^\phi \sqrt{\xi^\phi(u)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^\phi_* \sqrt{\theta^\phi(u)} \end{bmatrix},$$

and where $J_t(u)_{u \geq t}$ is an $\mathbb{R}^8$–valued process, with $J^\lambda_t(u)$ in its fourth element, $J^\phi_t(u)$ in its sixth, and zeros elsewhere. Using $(\cdot)^\top$ to denote a transpose, let

$$\sigma(u)\sigma(u)^\top = v_0 + \sum_{i=1}^{n} [X_t(u)]_i v_i$$

where $v_0, v_1, ..., v_8$ are each eight–by–eight constant matrices, and $[\cdot]_i$ extracts the ith component of a multi–dimensional vector.

The enlarged state process is an affine jump–diffusion, making the techniques in Duffie et al. (2000) applicable. Beginning with the OIS bond in (3.4), one has

$$P^c(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r^c(u)\,\mathrm{d}u} \,\middle|\, \mathcal{G}_t \right] = \exp\left(\alpha^c(T-t) + \beta^c(T-t)^\top X_t(t) \right), \quad \text{(B.2)}$$

where $\alpha^c(\cdot)$ and $\beta^c(\cdot)$ are, respectively, $\mathbb{R}$– and $\mathbb{R}^8$–valued deterministic functions satisfying

$$\frac{\partial \alpha^c(u)}{\partial u} = (\kappa\theta)^\top \beta^c(u) + \tfrac{1}{2}\beta^c(u)^\top v_0\, \beta^c(u),$$

$$\frac{\partial \beta^c_1(u)}{\partial u} = -\kappa^c \beta^c_1(u) - [R]_1,$$

$$\frac{\partial \beta^c_2(u)}{\partial u} = \kappa^c \beta^c_1(u) - \kappa^c_* \beta^c_2(u) - [R]_2,$$

$$\frac{\partial \beta^c_3(u)}{\partial u} = -\kappa^\Lambda \beta^c_3(u) + \tfrac{1}{2}\left(\beta^c_3(u)\sigma^\Lambda\right)^2 - [R]_3,$$

$$\frac{\partial \beta^c_4(u)}{\partial u} = -\beta^\lambda \beta^c_4(u) - [R]_4,$$

$$\frac{\partial \beta^c_5(u)}{\partial u} = -\kappa^\lambda \beta^c_5(u) + \tfrac{1}{2}\left(\beta^c_5(u)\sigma^\lambda\right)^2 + \frac{\beta^c_4(u)}{(0.02)^{-1} - \beta^c_4(u)} - [R]_5,$$

$$\frac{\partial \beta^c_6(u)}{\partial u} = -\beta^\phi \beta^c_6(u) - [R]_6,$$

$$\frac{\partial \beta^c_7(u)}{\partial u} = -\kappa^\phi \beta^c_7(u) + \tfrac{1}{2}\left(\beta^c_7(u)\sigma^\phi\right)^2 + \frac{\beta^c_6(u)}{(0.02)^{-1} - \beta^c_6(u)} + \frac{\rho^\phi \beta^c_4(u)}{(0.02)^{-1} - \beta^c_4(u)} - [R]_7,$$

$$\frac{\partial \beta^c_8(u)}{\partial u} = \kappa^\phi \beta^c_7(u) - \kappa^\phi_* \beta^c_8(u) + \tfrac{1}{2}\left(\beta^c_8(u)\sigma^\phi_*\right)^2 - [R]_8,$$

and the initial conditions $\alpha^c(0) = 0$ and $\beta^c_i(0) = 0$ for $i = 1, ..., 8$. Note that $\beta^c_i(\cdot)$ refers to the ith scalar component of the eight–dimensional function $\beta^c(\cdot)$. Here, $R = [1, 0, a^\Lambda, 0, a^\lambda, 0, a^\phi, 0]^\top$, i.e., the linear coefficients used to define the short rate. Recall also that 0.02 was fixed as the mean of the jump sizes.

Standard Runge–Kutta methods can be used to numerically solve the above system of equation, and accurately approximate the coefficient functions in (B.2). We note that this calculation, and several of the ones below, can be made more efficient by excluding certain non–necessary components of the state process; for instance, for the above calculation, the processes $\lambda_t(u)_{u\geq t}$ and $\phi_t(u)_{u\geq t}$ are not necessary to include, and one can reduce the dimension of the equation system. Further exclusions can be made in the uncorrelated model.

To compute IBOR in (3.12), the quantities in (3.10) and (3.11), respectively, can be given by

$$\begin{aligned} A(t,T) &= \exp\left(\alpha^A(T-t) + \beta^A(T-t)^\top X_t(t)\right), \\ Q(t,T) &= \exp\left(\alpha^Q(T-t) + \beta^Q(T-t)^\top X_t(t)\right), \end{aligned} \tag{B.3}$$

where, in both cases, the coefficient functions are computed in the same way as those in (B.2), except that the linear coefficient vector R must be adjusted. For $A(t,T)$, one sets $R = [0,0,0,0,0,-1,0,0]^\top$; for $Q(t,T)$, one sets $R = [1,0,a^\Lambda,0,1+a^\lambda,0,a^\phi,0]^\top$. The coefficient functions in (4.17) are then given by

$$\alpha^L(u) = \alpha^A(u) - \alpha^Q(u) \quad \text{ and } \quad \beta^L(u) = \beta^A(u) - \beta^Q(u).$$

In order to price the IBOR–dependent instruments detailed in Appendix A.1, one needs to evaluate expressions of the form

$$B^c(t)\mathbb{E}^{\mathbb{Q}}\left[L(T,U)/B^c(U)\,|\,\mathcal{F}_t\right]$$

for $t \leq T < U$. Consider the following representation:

$$\begin{aligned} &B^c(t)\mathbb{E}^{\mathbb{Q}}\left[L(T,U)/B^c(U)\,|\,\mathcal{F}_t\right] \\ &= B^c(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{U-T}\left(\frac{A(T,U)}{Q(T,U)}-1\right)/B^c(U)\,|\,\mathcal{F}_t\right] \\ &= \frac{B^c(t)}{U-T}\mathbb{E}^{\mathbb{Q}}\left[\frac{A(T,U)}{Q(T,U)}/B^c(U)\,|\,\mathcal{F}_t\right] - \frac{P^c(t,U)}{U-T} \\ &= \frac{1}{U-T}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^U r^c(u)\,\mathrm{d}u}\frac{A(T,U)}{Q(T,U)}\,|\,\mathcal{F}_t\right] - \frac{P^c(t,U)}{U-T} \\ &= \frac{1}{U-T}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^U r^c(u)\,\mathrm{d}u}\frac{A(T,U)}{Q(T,U)}\,|\,\mathcal{F}_T\right]\,|\,\mathcal{F}_t\right] - \frac{P^c(t,U)}{U-T} \\ &= \frac{1}{U-T}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r^c(u)\,\mathrm{d}u}\frac{A(T,U)}{Q(T,U)}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_T^U r^c(u)\,\mathrm{d}u}\,|\,\mathcal{F}_T\right]\,|\,\mathcal{F}_t\right] - \frac{P^c(t,U)}{U-T} \\ &= \frac{1}{U-T}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r^c(u)\,\mathrm{d}u}\frac{A(T,U)}{Q(T,U)}P^c(T,U)\,|\,\mathcal{F}_t\right] - \frac{P^c(t,U)}{U-T}. \end{aligned}$$

The right–hand term can be computed as per (B.2); the expectation in the left–hand term can be written, using (B.2) and (B.3), as

$$\begin{aligned} &\mathrm{e}^{\alpha^c(U-T)+\alpha^A(U-T)-\alpha^Q(U-T)} \\ &\times \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r^c(u)\,\mathrm{d}u}\mathrm{e}^{(\beta^c(U-T)+\beta^A(U-T)-\beta^Q(U-T))^\top X_T(T)}\,|\,\mathcal{F}_t\right] \\ &= \mathrm{e}^{\alpha^c(U-T)+\alpha^A(U-T)-\alpha^Q(U-T)}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r^c(u)\,\mathrm{d}u}\mathrm{e}^{\hat\beta(U-T)^\top X_t(T)}\,|\,\mathcal{F}_t\right], \end{aligned}$$

where $\hat\beta(U-T)$ is defined as the $\mathbb{R}^8$–valued vector equal to $\beta^c(U-T)+\beta^A(U-T)-\beta^Q(U-T)$ except with the fourth and the sixth elements set to zero. These two elements correspond to $\lambda_T(T)$ and $\phi_T(T)$, residing in $X_T(T)$, which are both equal to zero. The above quantity can then be given as

$$\exp\left(\alpha^c(U-T)+\alpha^A(U-T)-\alpha^Q(U-T)+\alpha^*(T-t)+\beta^*(T-t)^\top X_t(t)\right),$$

where $\alpha^*(\cdot)$ and $\beta^*(\cdot)$ are calculated exactly as $\alpha^c(\cdot)$ and $\beta^c(\cdot)$ above, except that the initial condition must be given by $\beta^*(0) = \hat{\beta}(U-T)$ and $\alpha^*(0) = 0$. The value for R does not change, i.e., $R = [1, 0, a^\Lambda, 0, a^\lambda, 0, a^\phi, 0]^\top$.

Consider now the CDS detailed in Appendix A.2. The following representation is essential for computing the protection leg:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[B^c(t)/B^c(\tau_t)\mathbf{1}\{\tau_t \leq T_n\}\,|\,\mathcal{G}_t\right] &= \lim_{m\to\infty}\sum_{i=1}^{m}\mathbb{E}^{\mathbb{Q}}\left[B^c(t)/B^c(\tau_t)\mathbf{1}\{t_{i-1} < \tau_t \leq t_i\}\,|\,\mathcal{G}_t\right] \\ &= \lim_{m\to\infty}\sum_{i=1}^{m}\mathbb{E}^{\mathbb{Q}}\left[B^c(t)/B^c(t_{i-1})\mathbf{1}\{t_{i-1} < \tau_t \leq t_i\}\,|\,\mathcal{G}_t\right],\end{aligned}$$

where $t_i = t + i\frac{T_n - t}{m}$ for $i = 0, 1, ..., m$, i.e., $\{t_i\}$ is a even mesh over the interval $[t, T_n]$. The above quantity can then be written, using Lemma A.3, as

$$\begin{aligned}&\lim_{m\to\infty}\sum_{i=1}^{m}\mathbb{E}^{\mathbb{Q}}\left[B^c(t)/B^c(t_{i-1})(\mathbf{1}\{t_{i-1} < \tau_t\} - \mathbf{1}\{t_i \leq \tau_t\})\,|\,\mathcal{G}_t\right] \\ &= \lim_{m\to\infty}\sum_{i=1}^{m}\mathbb{E}^{\mathbb{Q}}\left[B^c(t)/B^c(t_{i-1})\left(\mathrm{e}^{-\int_t^{t_{i-1}}[\Lambda(s)+\lambda_t(s)]\,\mathrm{d}s} - \mathrm{e}^{-\int_t^{t_i}[\Lambda(s)+\lambda_t(s)]\,\mathrm{d}s}\right)\,|\,\mathcal{F}_t\right] \\ &= \lim_{m\to\infty}\sum_{i=1}^{m}\mathbb{E}^{\mathbb{Q}}\left[B^c(t)/B^c(t_{i-1})\left[\Lambda(t_{i-1}) + \lambda_t(t_{i-1})\right]\mathrm{e}^{-\int_t^{t_{i-1}}[\Lambda(s)+\lambda_t(s)]\,\mathrm{d}s}\frac{T_n - t}{m}\,|\,\mathcal{F}_t\right],\end{aligned} \tag{B.4}$$

where the final line expresses the increment of the function $f(u) = \mathrm{e}^{-\int_t^u[\Lambda(s)+\lambda_t(s)]\,\mathrm{d}s}$ as

$$\begin{aligned}f(t_{i-1}) - f(t_i) = -f'(t_{i-1})(t_i - t_{i-1}) = \\ [\Lambda(t_{i-1}) + \lambda_t(t_{i-1})]\,\mathrm{e}^{-\int_t^{t_{i-1}}[\Lambda(s)+\lambda_t(s)]\,\mathrm{d}s}(t_i - t_{i-1}).\end{aligned}$$

Then (B.4), and therefore the CDS protection leg value, can be written as

$$\begin{aligned}&\int_t^{T_n}\mathbb{E}^{\mathbb{Q}}\left[B^c(t)/B^c(u)\left[\Lambda(u) + \lambda_t(u)\right]\mathrm{e}^{-\int_t^u[\Lambda(s)+\lambda_t(s)]\,\mathrm{d}s}\,|\,\mathcal{F}_t\right]\mathrm{d}u \\ &= \int_t^{T_n}\mathbb{E}^{\mathbb{Q}}\left[\left[\Lambda(u) + \lambda_t(u)\right]\mathrm{e}^{-\int_t^u[r^c(s)+\Lambda(s)+\lambda_t(s)]\,\mathrm{d}s}\,|\,\mathcal{F}_t\right]\mathrm{d}u.\end{aligned} \tag{B.5}$$

Then, the integrand on the right–hand side of (B.5) can be computed with the so–called extended transform of Duffie et al. (2000), and integrated numerically. That is, the integrand can be given as

$$\left(\bar{\alpha}^{\mathrm{CDS}}(u-t) + \bar{\beta}^{\mathrm{CDS}}(u-t)^\top X_t(t)\right)\exp\left(\alpha^{\mathrm{CDS}}(u-t) + \beta^{\mathrm{CDS}}(u-t)^\top X_t(t)\right), \tag{B.6}$$

where $\alpha^{\mathrm{CDS}}(\cdot)$ and $\beta^{\mathrm{CDS}}(\cdot)$ are calculated exactly as $\alpha^c(\cdot)$ and $\beta^c(\cdot)$ above, except with $R = [1, 0, 1 + a^\Lambda, 0, 1 + a^\lambda, 0, a^\phi, 0]^\top$. Then $\bar{\alpha}^{\mathrm{CDS}}(\cdot)$ and $\bar{\beta}^{\mathrm{CDS}}(\cdot)$ must satisfy

$$\begin{aligned}\frac{\partial\bar{\alpha}^{\mathrm{CDS}}(u)}{\partial u} &= (\kappa\theta)^\top\bar{\beta}^{\mathrm{CDS}}(u) + \beta^{\mathrm{CDS}}(u)^\top v_0\,\bar{\beta}^{\mathrm{CDS}}(u), \\ \frac{\partial\bar{\beta}_1^{\mathrm{CDS}}(u)}{\partial u} &= -\kappa^c\bar{\beta}_1^{\mathrm{CDS}}(u), \\ \frac{\partial\bar{\beta}_2^{\mathrm{CDS}}(u)}{\partial u} &= \kappa^c\bar{\beta}_1^{\mathrm{CDS}}(u) - \kappa_*^c\bar{\beta}_2^{\mathrm{CDS}}(u),\end{aligned}$$

$$\frac{\partial \bar{\beta}_3^{\mathrm{CDS}}(u)}{\partial u} = -\kappa^{\Lambda}\bar{\beta}_3^{\mathrm{CDS}}(u) + \beta_3^{\mathrm{CDS}}(u)\bar{\beta}_3^{\mathrm{CDS}}(u)\left(\sigma^{\Lambda}\right)^2,$$
$$\frac{\partial \bar{\beta}_4^{\mathrm{CDS}}(u)}{\partial u} = -\beta^{\lambda}\bar{\beta}_4^{\mathrm{CDS}}(u),$$
$$\frac{\partial \bar{\beta}_5^{\mathrm{CDS}}(u)}{\partial u} = -\kappa^{\lambda}\bar{\beta}_5^{\mathrm{CDS}}(u) + \beta_5^{\mathrm{CDS}}(u)\bar{\beta}_5^{\mathrm{CDS}}(u)\left(\sigma^{\lambda}\right)^2 + \left(\frac{\bar{\beta}_4^{\mathrm{CDS}}(u)}{(0.02)^{-1} - \beta_4^{\mathrm{CDS}}(u)}\right)^2,$$
$$\frac{\partial \bar{\beta}_6^{\mathrm{CDS}}(u)}{\partial u} = -\beta^{\phi}\bar{\beta}_6^{\mathrm{CDS}}(u),$$
$$\frac{\partial \bar{\beta}_7^{\mathrm{CDS}}(u)}{\partial u} = -\kappa^{\phi}\bar{\beta}_7^{\mathrm{CDS}}(u) + \beta_7^{\mathrm{CDS}}(u)\bar{\beta}_7^{\mathrm{CDS}}(u)\left(\sigma^{\phi}\right)^2 + \left(\frac{\bar{\beta}_6^{\mathrm{CDS}}(u)}{(0.02)^{-1} - \beta_6^{\mathrm{CDS}}(u)}\right)^2 + \left(\frac{\rho^{\phi}\bar{\beta}_4^{\mathrm{CDS}}(u)}{(0.02)^{-1} - \beta_4^{\mathrm{CDS}}(u)}\right)^2,$$
$$\frac{\partial \bar{\beta}_8^{\mathrm{CDS}}(u)}{\partial u} = \kappa^{\phi}\bar{\beta}_7^{\mathrm{CDS}}(u) - \kappa^{\phi}_{*}\bar{\beta}_8^{\mathrm{CDS}}(u) + \beta_8^{\mathrm{CDS}}(u)\bar{\beta}_8^{\mathrm{CDS}}(u)\left(\sigma^{\phi}_{*}\right)^2,$$

and the initial conditions $\bar{\alpha}^{\mathrm{CDS}}(0) = 0$ and $\bar{\beta}^{\mathrm{CDS}}(0) = [0,0,1,0,1,0,0,0]^{\top}$.

To compute the value of the payment leg (for a given spread value C_t), the first summation, (A.5), can be evaluated by first applying Lemma A.3 to each item in the sum. In particular,

$$B^c(t)\mathbb{E}^{\mathbb{Q}}\left[1/B^c(T_i)\mathbf{1}\{\tau_t > T_i\}\,|\,\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T [r^c(u) + \Lambda(u) + \lambda_t(u)]\,\mathrm{d}u\right)\,|\,\mathcal{F}_t\right]$$
$$= \exp\left(\alpha^{\mathrm{CDS}}(T-t) + \beta^{\mathrm{CDS}}(T-t)^{\top}X_t(t)\right),$$

where $\alpha^{\mathrm{CDS}}(\cdot)$ and $\beta^{\mathrm{CDS}}(\cdot)$ are the same functions used to compute (B.5). The second summation, (A.6), requires the disintegration technique used to arrive at (B.5), i.e., one can write

$$\mathbb{E}^{\mathbb{Q}}\left[B^c(t)/B^c(\tau_t)(\tau_t - T_{i-1})\mathbf{1}\{T_{i-1} < \tau_t \le T_i\}\,|\,\mathcal{G}_t\right]$$
$$= \int_{T_{i-1}}^{T_t} (u - T_{i-1})\mathbb{E}^{\mathbb{Q}}\left[[\Lambda(u) + \lambda_t(u)]\,\mathrm{e}^{-\int_t^u [r^c(s)+\Lambda(s)+\lambda_t(s)]\,\mathrm{d}s}\,|\,\mathcal{F}_t\right]\mathrm{d}u,$$

which can be numerically evaluated in the same manner as (B.5). The representation (B.6) remains applicable, but the deterministic part of the integrand above must be included as well.

B.2. **Filtering.** For a given set of model parameters, the filtering algorithm constructs estimates of the reduced state process in (4.18). We let $\hat{X}(t)$ denote the estimate of $X(t)$, which will be obtained at a number of discrete values of t, corresponding to the observation dates in our data sample described in Section 5.1. We set $\Delta t = 1/50$ as the time period between observation dates, to approximate the weekly observation frequency.

The physical–measure dynamics of the reduced state process are of the form

$$\mathrm{d}X(t) = \kappa^{\mathbb{P}}(\theta^{\mathbb{P}} - X(t))\,\mathrm{d}t + \sigma^{\mathbb{P}}(X(t))\,\mathrm{d}W^{\mathbb{P}}(u),$$

where $\kappa^{\mathbb{P}}$ and $\theta^{\mathbb{P}}$ ($\mathbb{R}^{6\times 6}$– and $\mathbb{R}^6$–valued respectively) can be deduced from the specification (B.1) as well as the measure–change (4.11). Note that $\sigma^{\mathbb{P}}(X(t))$ is identical to $\sigma(t)$, except that the fourth and sixth rows are deleted, and that the quantity is now given as an explicit function of the reduced state process.

The *transition equation* is based on an Euler discretisation of these dynamics. On this basis, the following quantities are defined at each discrete point t:

$$\hat{X}^+(t) = \hat{X}(t) + \kappa^{\mathbb{P}}(\theta^{\mathbb{P}} - \hat{X}(t))\,\Delta t, \text{ and}$$
$$P^+(t) = -\kappa^{\mathbb{P}} P(t)(-\kappa^{\mathbb{P}})^\top + \sigma^{\mathbb{P}}(\hat{X}(t))\sigma^{\mathbb{P}}(\hat{X}(t))^\top \Delta t,$$

representing an estimate for the state process at the next time point, i.e., the expectation of $X(t+\Delta t)$ conditional on information available before the data at time $t+\Delta t$ is incorporated, and the associated conditional variance.

Because the observations have a non–linear dependence on the state process, the conditional distribution of $X(t+\Delta t)$ is approximated with a discrete distribution over the following thirteen *sigma points*, constructed symmetrically around the mean:

$$X^{(1)}(t+\Delta t) = \hat{X}^+(t),$$
$$X^{(i)}(t+\Delta t) = \hat{X}^+(t) + \tfrac{1}{2}\gamma_i(t) \text{ for } i = 2, ..., 7,$$
$$X^{(i)}(t+\Delta t) = \hat{X}^+(t) - \tfrac{1}{2}\gamma_i(t) \text{ for } i = 8, ..., 13,$$

where $\gamma_i(t)$ is the ith column of the Cholesky decomposition of $P^+(t)$, so that $\gamma(t)\gamma(t)^\top = P^+(t)$, where $\gamma(t)$ is the full lower–triangular Cholesky matrix. The coefficient of one half is due to a particular parameterisation of the discrete distribution approximation. This value of a half appears several times below, and could be set differently; some authors favour a low value, such as the 10^{-3} value mentioned in Wan and Van Der Merwe (2000). Now define

$$\begin{aligned} Y^+(t+\Delta t) &= \sum_{i=1}^{13} z\left(X^{(i)}(t+\Delta t)\right) w_i \\ &= z\left(X^{(1)}(t+\Delta t)\right)\left(1 - \frac{6}{(1/2)^2}\right) + \sum_{i=2}^{13} z\left(X^{(i)}(t+\Delta t)\right)\frac{1}{2(1/2)^2}, \end{aligned}$$

where $z(\cdot)$ is the pricing function, i.e., a function that computes model–implied OIS rates, CDS spreads, etc. (see Appendix B.1) corresponding to the market–observed quantities at each date (see Section 5.1). The weights $\{w_i\}_{i=1,...,13}$ are specified in the far right–hand expression; they depend on the one–half value mentioned above, and the dimension of the state process $X(t)_{t\geq 0}$ (six). The quantity $Y^+(t+\Delta t)$ aggregates over the various model–implied quantities, giving an estimate for what will in fact be observed. We let $Y(t+\Delta t)$ denote these market observations. We will see $Y^+(t+\Delta t)$ and $Y(t+\Delta t)$ compared below.

Next define

$$\begin{aligned} S(t+\Delta t) = \sum_{i=1}^{13} &\left(z\left(X^{(i)}(t+\Delta t)\right) - Y^+(t+\Delta t)\right) \\ &\cdot \left(z\left(X^{(i)}(t+\Delta t)\right) - Y^+(t+\Delta t)\right)^\top w_i^* + R, \end{aligned}$$

where R is a diagonal matrix with σ^2_{rates} in the rows corresponding to OIS or CDS spreads, and $\sigma^2_{\text{spreads}}$ in rows corresponding to EURIBOR–OIS or basis–swap spread observations, and where the "second–order" weights $\{w_i^*\}$ are equivalent to the "first–order" ones $\{w_i\}$, except that $w_1^* = w_1 + 1 - (\frac{1}{2})^2 + 2$. This adjustment is a further free aspect of the unscented transform implementation, and this addition of two is typical. Then define

$$K(t+\Delta t) = \left(\sum_{i=1}^{13} \left(X^{(i)}(t+\Delta t) - \hat{X}^{+}(t) \right) \left(z\left(X^{(i)}(t+\Delta t) \right) - Y^{+}(t+\Delta t) \right)^{\top} w_i^{*} \right) S(t{+}\Delta t)^{-1}.$$

Applying a truncation to the non–negative state variables, we have

$$[\hat{X}(t+\Delta t)]_i = \left[\hat{X}^{+}(t) + K(t+\Delta t) \left(Y(t+\Delta t) - Y^{+}(t+\Delta t) \right) \right]_i$$

for $i = 1, 2$, and

$$\begin{aligned} [\hat{X}(t+\Delta t)]_i &= \max\left(\left[\hat{X}^{+}(t) + K(t+\Delta t) \left(Y(t+\Delta t) - Y^{+}(t+\Delta t) \right) \right]_i , 0 \right), \\ P(t+\Delta t) &= P^{+}(t) - K(t+\Delta t) S(t+\Delta t) K(t+\Delta t)^{\top}, \end{aligned}$$

for $i = 3, 4, 5, 6$. This concludes an iteration of the algorithm. Starting from $\hat{X}(t)$ and $P(t)$, the information inherent in $Y(t{+}\Delta t)$ has been used to construct $\hat{X}(t{+}\Delta t)$ and $P(t+\Delta t)$. The log–likelihood is then given by

$$-\frac{1}{2} \left(\sum_t n_t \log(2\pi) + \log(\det S(t)) + \left(Y(t) - Y^{+}(t) \right) S(t)^{-1} \left(Y(t) - Y^{+}(t) \right)^{\top} \right),$$

where n_t is the number of observations at each observation date t (for us, $n_t = 29$; see Section 5.1). The model parameters are then varied in order to maximise this log–likelihood value.

Appendix C. Further results.

C.1. **Fit using the uncorrelated model.** Figure 5 illustrates the fit to the data obtained by the uncorrelated model. Figure 6 shows the state variables paths obtained from the Kalman filter at the maximum likelihood parameters. The correlated (uncorrelated) model is considered in the top (bottom) panels.

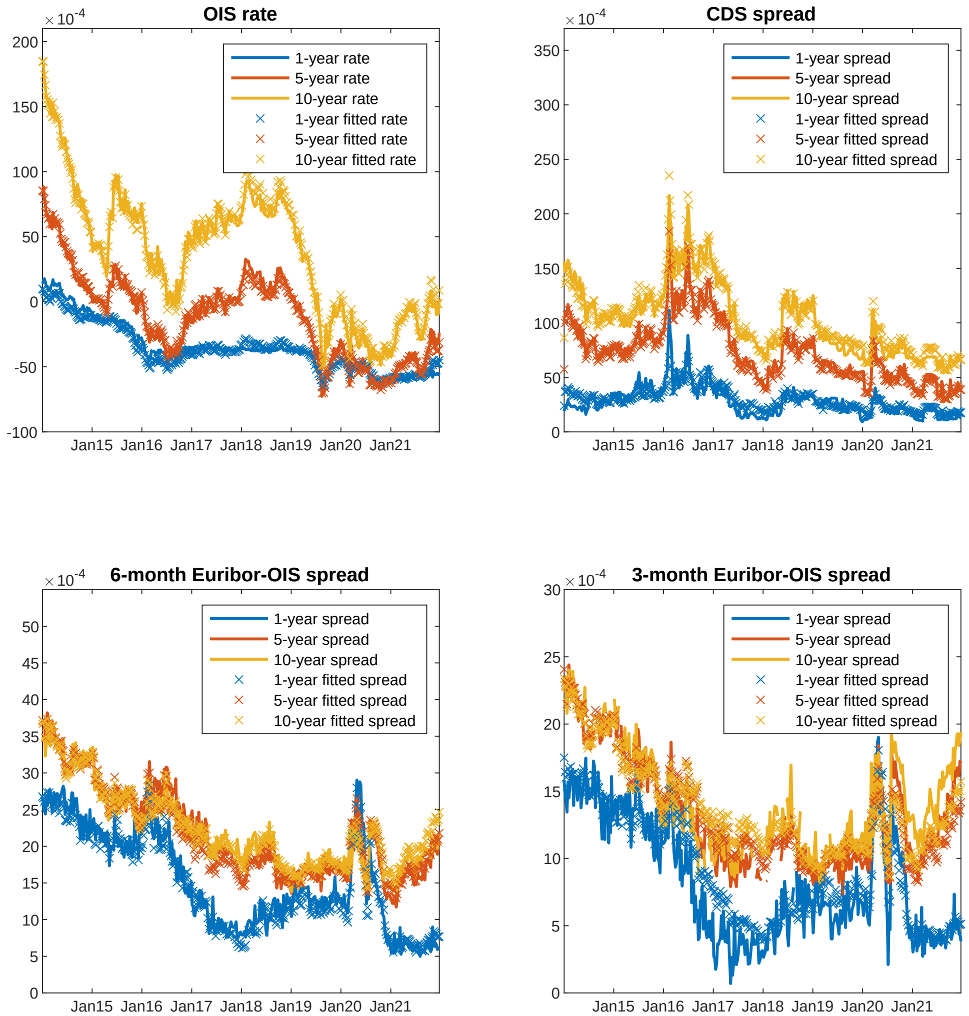

FIGURE 5. Market data compared to corresponding quantities, fitted by the uncorrelated model.

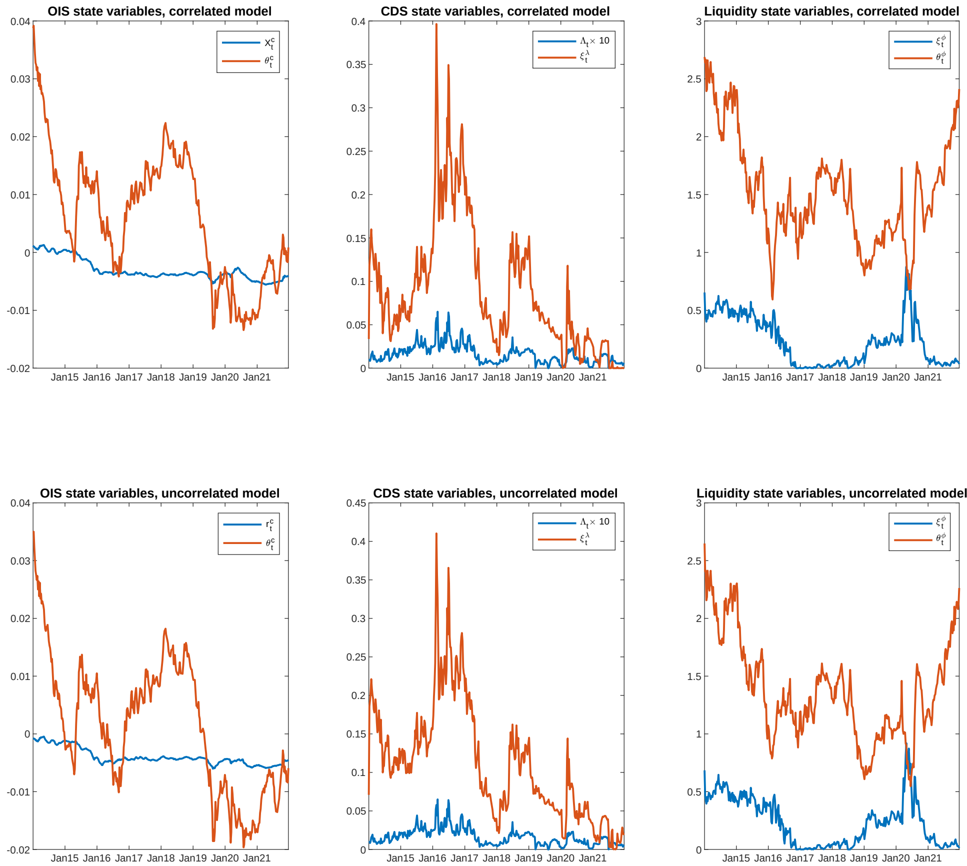

FIGURE 6. State variable paths determined by the unscented Kalman filter. The top (bottom) panels pertain to the correlated (uncorrelated) model.

C.2. **Overnight vs. continuous refinancing in the money–market account.** Our approximation of conflating overnight with continuous roll–over in the money–market account is a stylistic choice common in interest rate term structure modelling. It is mainly done for reading comprehension and mathematical tractability. Strictly speaking, this approximation ignores any basis spread between refinancing continuously vs overnight. However, such basis spreads are indeed negligible: With our model, using the full model parameters for roll–over risk that we have estimated from the data, we can calculate the "term rates" for short terms, as well as their basis spreads to the instantaneous rate $r^c(u)$:

Term	Term rate (bps)	Term premium (bps)
0	-39.57	0
0.001	-39.53	0.04
0.005	-39.38	0.19
0.01	-39.19	0.38
0.05	-37.74	1.83
0.1	-36.02	3.54

TABLE 5. Term premia (basis spreads to $r^c(u)$) for very short borrowing terms, based on the fitted model parameters.

A term of "overnight" (i.e., next business day) corresponds to a year fraction of less than 0.005 (or less than 0.01 if over a weekend). As Table 5 shows, the term premia (basis spreads to $r^c(u)$) corresponding to these terms are well under half a basis point.

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