Research Article

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Lipschitz regularity for degenerate elliptic integrals with *p*, *q*-growth

https://doi.org/10.1515/acv-2020-0120
Received December 16, 2020; revised July 22, 2021; accepted July 29, 2021

Abstract: We establish the local Lipschitz continuity and the higher differentiability of vector-valued local minimizers of a class of energy integrals of the Calculus of Variations. The main novelty is that we deal with possibly degenerate energy densities with respect to the x-variable.

Keywords: Nonstandard growth conditions, p, q-growth, degenerate ellipticity, Lipschitz continuity

MSC 2010: 49N60, 35J50

Communicated by: Frank Duzaar

1 Introduction

The paper deals with the regularity of minimizers of integral functionals of the Calculus of Variations of the form

$$F(u) = \int_{\Omega} f(x, Du)\, dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded open set, $u\colon \Omega \to \mathbb{R}^N$, $N \geq 1$, is a Sobolev map. The main feature of (1.1) is the possible degeneracy of the lagrangian $f(x, \xi)$ with respect to the x-variable. We assume that the Carathéodory function $f = f(x, \xi)$ is convex and of class C^2 with respect to $\xi \in \mathbb{R}^{N\times n}$, with $f_{\xi\xi}(x, \xi)$, $f_{\xi x}(x, \xi)$ also Carathéodory functions and $f(\,\cdot\,, 0) \in L^1(\Omega)$. We emphasize that the $N \times n$ matrix of the second derivatives $f_{\xi\xi}(x, \xi)$ not necessarily is uniformly elliptic and it may degenerate at some $x \in \Omega$.

In the vector-valued case $N > 1$, minimizers of functionals with general structure may lack regularity, see [19, 43, 49], and it is natural to assume a modulus-gradient dependence for the energy density, i.e. that there exists $g = g(x, t)\colon \Omega \times [0, +\infty) \to [0, +\infty)$ such that

$$f(x, \xi) = g(x, |\xi|). \tag{1.2}$$

Without loss of generality, we can assume $g(x, 0) = 0$; indeed, the minimizers of F are minimizers of

$$u \mapsto \int_{\Omega} (f(x, Du) - f(x, 0))\, dx$$

too. Moreover, by (1.2) and the convexity of f, $g(x, t)$ is a nonnegative, convex and increasing function of $t \in [0, +\infty)$.

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As far as the growth and the ellipticity assumptions are concerned, we assume that there exist exponents p, q, nonnegative measurable functions $a(x)$, $k(x)$ and a constant $L > 0$ such that

$$\begin{cases} a(x)(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \le \langle f_{\xi\xi}(x,\xi)\lambda,\lambda\rangle \le L(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2, \quad 2 \le p \le q, \\ |f_{\xi x}(x,\xi)| \le k(x)(1+|\xi|^2)^{\frac{q-1}{2}} \end{cases} \tag{1.3}$$

for a.e. $x \in \Omega$ and for every $\xi, \lambda \in \mathbb{R}^{N\times n}$. We allow the coefficient $a(x)$ to be zero so that $(1.3)_1$ is a not uniform ellipticity condition. As proved in Lemma 2.3, $(1.3)_1$ implies the following possibly degenerate p, q-growth conditions for f:

$$\frac{a(x)}{p(p-1)}(1+|\xi|^2)^{\frac{p-2}{2}}|\xi|^2 \le f(x,\xi) \le \frac{L}{2}(1+|\xi|^2)^{\frac{q-2}{2}}|\xi|^2, \quad \text{a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^{N\times n}. \tag{1.4}$$

Our main result concerns the local Lipschitz regularity and the higher differentiability of the local minimizers of F.

Theorem 1.1. *Let the functional F in* (1.1) *satisfy* (1.2) *and* (1.3). *Assume moreover that*

$$\frac{1}{a} \in L^s_{\mathrm{loc}}(\Omega), \quad k \in L^r_{\mathrm{loc}}(\Omega), \tag{1.5}$$

with $r, s > n$ and

$$\frac{q}{p} < \frac{s}{s+1}\Big(1 + \frac{1}{n} - \frac{1}{r}\Big). \tag{1.6}$$

If $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of F, then for every ball $B_{R_0} \Subset \Omega$, the estimates

$$\|Du\|_{L^\infty(B_{R_0/2})} \le C\mathcal{K}^{\vartheta}_{R_0}\Big(\int_{B_{R_0}} (1 + f(x,Du))\,dx\Big)^{\vartheta}, \tag{1.7}$$

$$\int_{B_{R_0/2}} a(x)(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx \le C\mathcal{K}^{\vartheta}_{R_0}\Big(\int_{B_{R_0}} (1 + f(x,Du))\,dx\Big)^{\vartheta}, \tag{1.8}$$

hold with the exponent ϑ depending on the data, the constant C also depending on R_0 and where

$$\mathcal{K}_{R_0} = 1 + \|a^{-1}\|_{L^s(B_{R_0})}\|k\|^2_{L^r(B_{R_0})} + \|a\|_{L^\infty(B_{R_0})}.$$

It is well known that, to get regularity under p, q-growth, the exponents q and p cannot be too far apart; usually, the gap between p and q is described by a condition relating p, q and the dimension n. In our case, we take into account the possible degeneracy of $a(x)$ and condition $(1.3)_2$ on the mixed derivatives $f_{\xi x}$ in terms of a possibly unbounded coefficient $k(x)$; then we deduce that the gap depends on s, the summability exponent of a^{-1} that "measures" how much a is degenerate, and the exponent r that tells us how far $k(x)$ is from being bounded. If $s = r = \infty$, then (1.6) reduces to $\frac{q}{p} < 1 + \frac{1}{n}$ that is what one expects; see [12] and for instance [40]. Moreover, if $s = \infty$ and $n < r \le +\infty$, then (1.6) reduces to $\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}$, and we recover the result of [23].

Motivated by applications to the theory of elasticity, recently, Colombo and Mingione [8, 9] (see also [2, 17, 18, 24]) studied the so-called *double phase integrals*

$$\int_\Omega |Du|^p + b(x)|Du|^q\,dx, \quad 1 < p < q.$$

When applied to this case, Theorem 1.1 gives the local Lipschitz continuity of the minimizers if $b(x) \in W^{1,r}_{\mathrm{loc}}$, for some $r > n$, and $\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}$ ([23]).

As it is well known, weak solutions to the elliptic equation in divergence form of the type

$$-\operatorname{div}(A(x,Du)) = 0 \quad \text{in } \Omega$$

are locally Lipschitz continuous provided the vector field $A\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable with respect to ξ and satisfies the uniform ellipticity conditions

$$\Lambda_1(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \langle A_\xi(x,\xi)\lambda,\lambda\rangle \leq \Lambda_2(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2.$$

Trudinger [50] started the study of the interior regularity of solutions to linear elliptic equations of the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\Big) = 0, \quad x \in \Omega \subseteq \mathbb{R}^n, \tag{1.9}$$

where the measurable coefficients a_{ij} satisfy the non-uniform condition

$$\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq n^2\mu(x)|\xi|^2$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. Here $\lambda(x)$ is the minimum eigenvalue of the symmetric matrix $A(x) = (a_{ij}(x))$ and $\mu(x) := \sup_{ij}|a_{ij}|$. Trudinger proved that any weak solution of (1.9) is locally bounded in Ω, under the following integrability assumptions on λ and μ:

$$\lambda^{-1} \in L^r_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \mu_1 = \lambda^{-1}\mu^2 \in L^\sigma_{\mathrm{loc}}(\Omega) \quad \text{with } \frac{1}{r} + \frac{1}{\sigma} < \frac{2}{n}.$$

Equation (1.9) is usually called *degenerate* when $\lambda^{-1} \notin L^\infty(\Omega)$, whereas it is called *singular* when $\mu \notin L^\infty(\Omega)$. These names in this case refer to the *degenerate* and the *singular* cases with respect to the x-variable, but in the mathematical literature, these names often refer to the gradient variable; this happens for instance with the *p-Laplacian operator* $-\mathrm{div}(|Du|^{p-2}Du)$. We do not study in this paper the degenerate case with respect to the gradient variable, but we refer for instance to the analysis made by Duzaar and Mingione [21], who studied an L^∞-gradient bound for solutions to non-homogeneous p-Laplacian type systems and equations; see also Cianchi and Maz'ya [7] and the references therein for the rich literature on the subject.

The result by Trudinger was extended in many settings and directions: firstly, by Trudinger himself in [51] and later by Fabes, Kenig and Serapioni in [29]; Pingen in [48] dealt with systems. More recently, for the regularity of solutions and minimizers, we refer to [3, 4, 10, 15, 16, 32]. For the higher integrability of the gradient, we refer to [33] (see also [6]). Very recently, Calderon–Zygmund's estimates for the p-Laplace operator with degenerate weights have been established in [1]. The literature concerning non-uniformly elliptic problems is extensive, and we refer the interested reader to the references therein.

The study of the Lipschitz regularity in the p,q-growth context started with the papers by Marcellini [35, 36], and since then, many and various contributions to the subject have been provided; see the references in [40, 42]. The vectorial homogeneous framework was considered in [37, 41] and by Esposito, Leonetti and Mingione [27, 28]. Condition $(1.3)_2$ for general non-autonomous integrands $f = f(x, Du)$ has been first introduced in [22–24]. It is worth to highlight that, due to the x-dependence, the study of regularity is significantly harder and the techniques more complex. The research on this subject is intense, as confirmed by the many articles recently published; see e.g. [11, 13, 14, 20, 25, 30, 38–40, 45–47].

Let us briefly sketch the tools to get our regularity result. First, for Lipschitz and higher differentiable minimizers, we prove a weighted summability result for the second-order derivatives of minimizers of functionals with possibly degenerate energy densities; see Proposition 3.2. Next, in Theorem 3.3, we get an *a priori estimate* for the L^∞-norm of the gradient. To establish the *a priori estimate*, we use Moser's iteration method [44] for the gradient and the ideas of Trudinger [50]. An approximation procedure allows us to conclude. Actually, if u is a local minimizer of (1.1), we construct a sequence of suitable variational problems in a ball $B_R \subset\subset \Omega$ with boundary value data u. In order to apply the a priori estimate to the minimizers of the approximating functionals, we prove a higher differentiability result, see Theorem 3.3, for minimizers of the class of functionals with p,q-growth studied in [23], where only the Lipschitz continuity was proved. By applying the previous a priori estimate to the sequence of the solutions, we obtain a uniform control in L^∞ of the gradient which allows to transfer the local Lipschitz continuity property to the original minimizer u.

Another difficulty due to the x-dependence of the energy density is that the *Lavrentiev phenomenon* may occur. A local minimizer of F is a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that $f(x, Du) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega f(x, Du)\,dx \leq \int_\Omega f(x, Du + D\varphi)\,dx$$

for every $\varphi \in C^1_0(\Omega)$. If u is a local minimizer of the functional F, by virtue of (1.4), we have $a(x)|Du|^p \in L^1_{\mathrm{loc}}(\Omega)$ and, by (1.5), $u \in W^{1,\frac{ps}{s+1}}_{\mathrm{loc}}(\Omega)$ since

$$\int_{B_R} |Du|^{\frac{ps}{s+1}}\,dx \leq \Big(\int_{B_R} a|Du|^p\,dx\Big)^{\frac{s}{s+1}}\Big(\int_{B_R} \frac{1}{a^s}\,dx\Big)^{\frac{1}{s+1}} < +\infty \tag{1.10}$$

for every ball $B_R \subset \Omega$. Therefore, in our context, a priori, the presence of the Lavrentiev phenomenon cannot be excluded. Indeed, due to the growth assumptions on the energy density, the integral in (1.1) is well defined if $u \in W^{1,q}$, but a priori, this is not the case if $u \in W^{1,\frac{ps}{s+1}}(\Omega) \setminus W^{1,q}_{\mathrm{loc}}(\Omega)$. However, as a consequence of Theorem 1.1, under the stated assumptions (1.2), (1.3), (1.5), (1.6), the Lavrentiev phenomenon for the integral functional F in (1.1) cannot occur. For the gap in the Lavrentiev phenomenon, we refer to [5, 26, 28, 52].

We conclude this introduction by observing that, even in the one-dimensional case, the Lipschitz continuity of minimizers for non-uniformly elliptic integrals is not obvious. Indeed, if we consider a local minimizer u to the one-dimensional integral

$$F(u) = \int_{-1}^{1} a(x)(1 + |u'(x)|^2)^{\frac{p}{2}}\,dx, \quad p > 1, \tag{1.11}$$

then Euler's first variation takes the form

$$\int_{-1}^{1} a(x)p(1 + |u'(x)|^2)^{\frac{p-2}{2}} u'(x)\varphi'(x)\,dx = 0 \quad \text{for all } \varphi \in C^1_0(-1, 1).$$

This implies that the quantity $a(x)(1 + |u'(x)|^2)^{\frac{p-2}{2}} u'(x)$ is constant in $(-1, 1)$; it is a nonzero constant, unless $u(x)$ itself is constant in $(-1, 1)$, a trivial case that we do not consider here. In particular, the sign of $u'(x)$ is constant, and we get

$$(1 + |u'(x)|^2)^{\frac{p-2}{2}} |u'(x)| = \frac{c}{a(x)}, \quad \text{a.e. } x \in (-1, 1).$$

Therefore, if $a(x)$ vanishes somewhere in $(-1, 1)$, then $|u'(x)|$ is unbounded (and vice versa), independently of the exponent $p > 1$. Thus, for $n = 1$, the local Lipschitz regularity of the minimizers does not hold in general if the coefficient $a(x)$ vanishes somewhere.

We can compare this one-dimensional fact with the general conditions considered in the Theorem 1.1. In the case $a(x) = |x|^\alpha$ for some $\alpha \in (0, 1)$, then, taking into account the assumptions in (1.5), for the integral in (1.11), we have $k(x) = a'(x) = \alpha|x|^{\alpha-2}x$ and

$$\begin{cases} \dfrac{1}{a} \in L^s_{\mathrm{loc}}(-1, 1) \iff 1 - \alpha s > 0 \iff \alpha < \dfrac{1}{s}, \\ k = a' \in L^r_{\mathrm{loc}}(-1, 1) \iff r(\alpha - 1) > -1 \iff \alpha > 1 - \dfrac{1}{r}. \end{cases}$$

These conditions are compatible if and only if $1 - \frac{1}{r} < \frac{1}{s}$. Therefore, also in the one-dimensional case, we have a counterexample to the L^∞-gradient bound in (1.7) if

$$\frac{1}{r} + \frac{1}{s} > 1. \tag{1.12}$$

This is a condition that can be easily compared with assumption (1.6) for the validity of L^∞-gradient bound (1.7) in the general n-dimensional case. In fact, since $1 \leq \frac{q}{p}$, (1.6) implies

$$1 < \frac{s}{s+1}\Big(1 + \frac{1}{n} - \frac{1}{r}\Big) \iff \frac{1}{r} + \frac{1}{s} < \frac{1}{n},$$

which essentially is the complementary condition to (1.12) when $n = 1$.

The plan of the paper is the following. In Section 2, we list some definitions and preliminary results. In Section 3, we prove a priori estimates of the L^∞-norm of the gradient of local minimizers and a higher differentiability result; see Theorem 3.3. In the last section, we complete the proof of Theorem 1.1.

2 Preliminary results

We shall denote by C or c a general positive constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies will be suitably emphasized using parentheses or subscripts. In what follows, $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$ will denote the ball centered at x of radius r. We shall omit the dependence on the center and on the radius when no confusion arises.

To prove our higher differentiability result (see Theorem 3.3 below), we use the finite difference operator. For a function $u\colon \Omega \to \mathbb{R}^k$, Ω open subset of $\mathbb{R}^n$, given $\ell \in \{1,\dots,n\}$, we define

$$\tau_{\ell,h}u(x) := u(x+he_\ell) - u(x), \quad x \in \Omega_{|h|},$$

where e_ℓ is the unit vector in the x_ℓ direction, $h \in \mathbb{R}$ and

$$\Omega_{|h|} := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > |h|\}.$$

We now list the main properties of this operator.

(i) If $u \in W^{1,t}(\Omega)$, $1 \le t \le \infty$, then $\tau_{\ell,h}u \in W^{1,t}(\Omega_{|h|})$ and

$$D_i(\tau_{\ell,h}u) = \tau_{\ell,h}(D_i u).$$

(ii) If f or g has support in $\Omega_{|h|}$, then

$$\int_\Omega f\tau_{\ell,h}g\,dx = \int_\Omega g\tau_{\ell,-h}f\,dx.$$

(iii) If $u, u_{x_\ell} \in L^t(B_R)$, $1 \le t < \infty$, and $0 < \rho < R$, then for every h, $|h| \le R-\rho$,

$$\int_{B_\rho} |\tau_{\ell,h}u(x)|^t\,dx \le |h|^t \int_{B_R} |u_{x_\ell}(x)|^t\,dx.$$

(iv) If $u \in L^t(B_R)$, $1 < t < \infty$, and for $0 < \rho < R$, there exists $K > 0$ such that, for every h, $|h| < R-\rho$,

$$\sum_{\ell=1}^n \int_{B_\rho} |\tau_{\ell,h}u(x)|^t\,dx \le K|h|^t,$$

then letting h go to 0, there exists $Du \in L^t(B_\rho)$ and $\|u_{x_\ell}\|_{L^t(B_\rho)} \le K$ for every $\ell \in \{1,\dots,n\}$.

We recall the following estimate for the auxiliary function:

$$V_p(\xi) := (1+|\xi|^2)^{\frac{p-2}{4}}\xi,$$

which is a convex function since $p \ge 2$ (see [34, Step 2] and the proof of [31, Lemma 8.3]).

Lemma 2.1. *Let $1 < p < \infty$. There exists a constant $c = c(n,p) > 0$ such that*

$$c^{-1}(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} \le \frac{|V_p(\xi)-V_p(\eta)|^2}{|\xi-\eta|^2} \le c(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}$$

for any $\xi, \eta \in \mathbb{R}^n$, $\xi \ne \eta$.

Remark 2.2. Note that if $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is such that $V_p(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, then there exists a constant $c(p)$ such that

$$c(p)^{-1}|D^2v|^2(\mu^2+|Dv|^2)^{\frac{p-2}{2}} \le |DV_p(Dv)|^2 \le c(p)|D^2v|^2(\mu^2+|Dv|^2)^{\frac{p-2}{2}}.$$

In the next lemma, we prove that $(1.3)_1$ implies the, possibly degenerate, p, q-growth condition stated in (1.4).

Lemma 2.3. *Let $f = f(x,\xi) = g(x,|\xi|)$ be of class C^2 with respect to the ξ-variable. Let us assume that the quadratic form of the second derivatives $D_{\xi\xi}f(x,\xi)$ of f satisfies the conditions*

$$a(x)(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \langle D_{\xi\xi}f(x,\xi)\lambda,\lambda\rangle \leq b(x)(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2 \tag{2.1}$$

for some exponents $1 < p \leq q$ and nonnegative functions a, b and for all $\xi, \lambda \in \mathbb{R}^n$. Then f satisfies the following p, q-growth condition:

$$c_1 a(x)(1+|\xi|^2)^{\frac{p-2}{2}}|\xi|^2 \leq f(x,\xi) - f(x,0) \leq c_2 b(x)(1+|\xi|^2)^{\frac{q-2}{2}}|\xi|^2, \tag{2.2}$$

where $c_1 = \min\{\frac{1}{(p-1)p}; \frac{1}{2}\}$ and $c_2 = \max\{\frac{1}{(q-1)q}; \frac{1}{2}\}$.

Remark 2.4. As noticed by an anonymous referee, that we thank for her/his careful reading of the manuscript, in Lemma 2.3, it is sufficient to assume the validity of (2.1) only for every $\lambda, \xi \in \mathbb{R}^n$ with λ proportional to ξ (see formulas (2.5), (2.6) below in the proof); in this case, (2.1) simplifies into

$$a(x)(1+t^2)^{\frac{p-2}{2}} \leq g_{tt}(x,t) \leq b(x)(1+t^2)^{\frac{q-2}{2}}. \tag{2.3}$$

Therefore, alternatively, in Lemma 2.3, it is sufficient to assume directly (2.3) instead of (2.1).

Proof. For $x \in \Omega$ and $s \in \mathbb{R}$, let us set $\varphi(s) = g(x, st)$, where we recall that g is linked to f by (1.2). The assumptions on f imply that $\varphi \in C^2(\mathbb{R})$ with $g_t(x,0) = 0$. Since

$$\varphi'(s) = g_t(x,st)\cdot t, \quad \varphi''(s) = g_{tt}(x,st)\cdot t^2,$$

the Taylor expansion formula in integral form yields

$$\varphi(1) = \varphi(0) + \varphi'(0) + \int_0^1 (1-r)\varphi''(r)\,dr.$$

Recalling the definition of φ, since $\varphi(1) = g(x,t)$ and $\varphi'(0) = g_t(x,0)\cdot t = 0$, we get

$$g(x,t) = g(x,0) + \int_0^1 (1-r)g_{tt}(x,rt)\cdot t^2\,dr. \tag{2.4}$$

We recall [41, formula (3.3)],

$$\min\Big\{g_{tt}(x,|\xi|); \frac{g_t(x,|\xi|)}{|\xi|}\Big\} \leq \frac{\langle D_{\xi\xi}f(x,\xi)\lambda,\lambda\rangle}{|\lambda|^2} \leq \max\Big\{g_{tt}(x,|\xi|); \frac{g_t(x,|\xi|)}{|\xi|}\Big\}, \tag{2.5}$$

which holds with equality when λ is proportional to ξ, when (2.5) simplifies to

$$\frac{\langle D_{\xi\xi}f(x,\xi)\lambda,\lambda\rangle}{|\lambda|^2}\bigg|_{\lambda \text{ proportional to } \xi} = g_{tt}(x,|\xi|).$$

Therefore, in the particular case with $|\lambda| = |\xi| = t$, assumption (2.1) translates into

$$a(x)(1+t^2)^{\frac{p-2}{2}} \leq g_{tt}(x,t) \leq b(x)(1+t^2)^{\frac{q-2}{2}}. \tag{2.6}$$

Inserting (2.6) in (2.4), we obtain

$$a(x)t^2\int_0^1 (1-r)[1+(rt)^2]^{\frac{p-2}{2}}\,dr \leq g(x,t) - g(x,0) \leq b(x)t^2\int_0^1 (1-r)[1+(rt)^2]^{\frac{q-2}{2}}\,dr. \tag{2.7}$$

Assume $p \geq 2$. For the integral in the left-hand side of (2.7), since $r \in [0, 1]$ and $p \geq 2$, we obtain

$$\int_0^1 (1-r)[1+(rt)^2]^{\frac{p-2}{2}}\,dr \geq \int_0^1 (1-r)[r^2+(rt)^2]^{\frac{p-2}{2}}\,dr = (1+t^2)^{\frac{p-2}{2}} \int_0^1 (1-r)r^{p-2}\,dr$$
$$= (1+t^2)^{\frac{p-2}{2}} \left[\frac{r^{p-1}}{p-1} - \frac{r^p}{p}\right]_{r=0}^{r=1} = \frac{(1+t^2)^{\frac{p-2}{2}}}{(p-1)p}. \tag{2.8}$$

Similarly (and simpler) for the integral in the right-hand side of (2.7),

$$\int_0^1 (1-r)[1+(rt)^2]^{\frac{q-2}{2}}\,dr \leq (1+t^2)^{\frac{q-2}{2}} \int_0^1 (1-r)\,dr = \frac{1}{2}(1+t^2)^{\frac{q-2}{2}}. \tag{2.9}$$

Combining the last two estimates and recalling in (2.7) that $f(x, \xi) = g(x, |\xi|) = g(x, t)$, we get

$$\frac{a(x)}{(p-1)p}(1+|\xi|^2)^{\frac{p-2}{2}}|\xi|^2 \leq f(x,\xi) - f(x,0) \leq \frac{b(x)}{2}(1+|\xi|^2)^{\frac{q-2}{2}}|\xi|^2 \quad (\text{when } q \geq p \geq 2).$$

If $p < 2$, then we modify (2.8) into

$$\int_0^1 (1-r)[1+(rt)^2]^{\frac{p-2}{2}}\,dr \geq \int_0^1 (1-r)(1+t^2)^{\frac{p-2}{2}}\,dr = \frac{1}{2}(1+t^2)^{\frac{p-2}{2}};$$

then, if also $1 < q < 2$, we modify (2.9) into

$$\int_0^1 (1-r)[1+(rt)^2]^{\frac{q-2}{2}}\,dr \leq \int_0^1 (1-r)[r^2+(rt)^2]^{\frac{q-2}{2}}\,dr = \frac{(1+t^2)^{\frac{q-2}{2}}}{(q-1)q}.$$

In this case, we obtain

$$\frac{a(x)}{2}(1+|\xi|^2)^{\frac{p-2}{2}}|\xi|^2 \leq f(x,\xi) - f(x,0) \leq \frac{b(x)}{(q-1)q}(1+|\xi|^2)^{\frac{q-2}{2}}|\xi|^2 \quad (\text{when } 1 < p \leq q \leq 2).$$

The conclusion (2.2) follows by combining all these cases. □

We end this preliminary section with two well-known properties. The first lemma has important applications in the so-called *hole-filling method*. Its proof can be found for example in [31, Lemma 6.1].

Lemma 2.5. *Let $h\colon [r, R_0] \to \mathbb{R}$ be a nonnegative bounded function and $0 < \vartheta < 1$, $A, B \geq 0$ and $\beta > 0$. Assume that*

$$h(s) \leq \vartheta h(t) + \frac{A}{(t-s)^\beta} + B$$

for all $r \leq s < t \leq R_0$. Then

$$h(r) \leq \frac{cA}{(R_0-r)^\beta} + cB,$$

where $c = c(\vartheta, \beta) > 0$.

We will also use the following (see e.g. [31]).

Lemma 2.6. *Let $\xi, \eta \in \mathbb{R}^n$. For every $s > -1$ and $r > 0$, there exist positive constants $c_1(r, s)$ and $c_2(r, s)$ such that*

$$c_1(r,s)(1+|\xi|^2+|\eta|^2)^{\frac{s}{2}} \leq \int_0^1 (1-t)^r(1+|(1-t)\xi + t\eta|^{\frac{s}{2}}\,dt \leq c_2(r,s)(1+|\xi|^2+|\eta|^2)^{\frac{s}{2}}.$$

We end this section with a remark that we will use when considering the summability of k.

Lemma 2.7. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$. If $r, s > n$ and*

$$1 < \frac{s}{s+1}\Big(1 + \frac{1}{n} - \frac{1}{r}\Big),$$

then $k \in L^r_{\mathrm{loc}}(\Omega)$ implies $k \in L^{\frac{2s}{s-1}}_{\mathrm{loc}}(\Omega)$.

Proof. Since $k \in L^r_{\mathrm{loc}}(\Omega)$, we need to prove that $\frac{2s}{s-1} \leq r$ that is equivalent to $\frac{2}{r} + \frac{1}{s} \leq 1$. This holds true because

$$\frac{s}{s+1}\Big(1 + \frac{1}{n} - \frac{1}{r}\Big) > 1 \iff \frac{n}{r} + \frac{n}{s} < 1, \tag{2.10}$$

and we conclude because $n \geq 2$. □

3 The a priori estimate

The main result in this section is an a priori estimate of the L^∞-norm of the gradient, and a higher differentiability result, of local minimizers of the functional F in (1.1) satisfying assumptions (1.2), (1.3), (1.5) and (1.6), with exponents p and q satisfying $1 < p \leq q$ and not the more restrictive condition $2 \leq p \leq q$, assumed in Theorem 1.1.

We use the following weighted Sobolev type inequality, whose proof relies on the Hölder's inequality; see e.g. [16].

Lemma 3.1. *Let $p > 1$, $s \geq 1$ and $w \in W_0^{1,\frac{ps}{s+1}}(\Omega;\mathbb{R}^N)$ ($w \in W_0^{1,p}(\Omega;\mathbb{R}^N)$ if $s = \infty$), with $1 < \frac{ps}{s+1} < n$, and let $\lambda\colon \Omega \to [0, +\infty)$ be a measurable function such that $\lambda^{-1} \in L^s(\Omega)$. Then there exists a constant $c = c(n)$ such that*

$$\Big(\int_\Omega |w|^{\sigma^*}\,dx\Big)^{\frac{p}{\sigma^*}} \leq c(n)\|\lambda^{-1}\|_{L^s(\Omega)} \int_\Omega \lambda |Dw|^p\,dx,$$

where $\sigma = \frac{ps}{s+1}$ ($\sigma = p$ if $s = +\infty$).

In establishing the a priori estimate, we need to deal with quantities that involve the L^2-norm of the second derivatives of the minimizer weighted with the function $a(x)$. The next result tells that local Lipschitz continuous minimizers u of F possess weak second derivatives such that

$$V_p(Du) \in W^{1,\frac{2s}{s+1}}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad a(x)|D(V_p(Du))|^2 \in L^1_{\mathrm{loc}}(\Omega).$$

More precisely, we have the following proposition.

Proposition 3.2. *Consider the functional F in (1.1) satisfying assumptions (1.3) with $1 < p \leq q$, and*

$$\frac{1}{a} \in L^s_{\mathrm{loc}}(\Omega), \quad k \in L^{\frac{2s}{s-1}}_{\mathrm{loc}}(\Omega), \tag{3.1}$$

with $s > 1$. If $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a local minimizer of F, then

$$V_p(Du) \in W^{1,\frac{2s}{s+1}}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad a(x)|D(V_p(Du))|^2 \in L^1_{\mathrm{loc}}(\Omega).$$

Proof. Since u is a local minimizer of the functional F, then u satisfies the Euler system

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(x, Du)\varphi^\alpha_{x_i}(x)\,dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega;\mathbb{R}^N). \tag{3.2}$$

Fix $x_0 \in \Omega$, $B_{R_0}(x_0) \Subset \Omega$, $0 < \rho < R < R_0$. Consider a cut-off function $\eta \in C_0^\infty(\Omega)$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ in B_ρ, $\operatorname{supp}\eta \subseteq B_R$ and $|h| \ll 1$. Assume $|D\eta| \leq \frac{2}{R-\rho}$.

Fix $\ell = 1, \dots, n$, and consider the function

$$\varphi^\alpha := \tau_{\ell,-h}(\eta^2 \tau_{\ell,h} u^\alpha), \quad \alpha = 1, \dots, N.$$

Thanks to the assumption $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, we can use φ as test function in equation (3.2), thus getting

$$\int_\Omega \tau_{\ell,h}\Big(\sum_{i,\alpha} f_{\xi_i^\alpha}(x, Du)\Big)(\eta^2 \tau_{\ell,h} u^\alpha_{x_i} + 2\eta\eta_{x_i}\tau_{\ell,h}u^\alpha)\,dx = 0, \tag{3.3}$$

where we used property (ii) of the finite difference operator. Now, we observe that

$$\begin{aligned}\tau_{\ell,h} f_{\xi_i^\alpha}(x, Du) &= f_{\xi_i^\alpha}(x + he_\ell, Du(x + he_\ell)) - f_{\xi_i^\alpha}(x + he_\ell, Du(x)) + f_{\xi_i^\alpha}(x + he_\ell, Du(x)) - f_{\xi_i^\alpha}(x, Du(x))\\ &= \int_0^1 \sum_{j,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x + he_\ell, Du(x) + t\tau_{\ell,h}Du)\,dt\,\tau_{\ell,h}u^\beta_{x_j} + \int_0^1 f_{\xi_i^\alpha x_\ell}(x + the_\ell, Du(x))\,dt \cdot h,\end{aligned}$$

and so (3.3) can be written as follows:

$$\begin{aligned}0 &= \int_\Omega 2\eta\Big(\int_0^1 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x + he_\ell, Du(x) + t\tau_{\ell,h}Du)\,dt\Big)\eta_{x_i}\tau_{\ell,h}u^\alpha \tau_{\ell,h}u^\beta_{x_j}\,dx\\ &\quad + \int_\Omega \eta^2\Big(\int_0^1 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x + he_\ell, Du(x) + t\tau_{\ell,h}Du)\,dt\Big)\tau_{\ell,h}u^\alpha_{x_i}\tau_{\ell,h}u^\beta_{x_j}\,dx\\ &\quad + \int_\Omega 2\eta\Big(\int_0^1 \sum_{i,\alpha} f_{\xi_i^\alpha x_\ell}(x + the_\ell, Du(x))\,dt\Big)\cdot h \cdot \eta_{x_i}\tau_{\ell,h}u^\alpha\,dx\\ &\quad + \int_\Omega \eta^2\Big(\int_0^1 \sum_{i,\alpha} f_{\xi_i^\alpha x_\ell}(x + the_\ell, Du(x))\,dt\Big)\cdot h \cdot \tau_{\ell,h}u^\alpha_{x_i}\,dx =: J_1 + J_2 + J_3 + J_4.\end{aligned}$$

By the use of Cauchy–Schwarz and Young's inequalities and by virtue of the second inequality of (1.3), we can estimate the integral J_1. If $q > 2$, we have

$$\begin{aligned}|J_1| &\le 2\int_\Omega \Big\{\Big(\int_0^1 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x + he_\ell, Du(x) + t\tau_{\ell,h}Du)\,dt\Big)\eta^2_{x_i}\tau_{\ell,h}u^\alpha\eta_{x_j}\tau_{\ell,h}u^\beta\Big\}^{\frac12}\\ &\qquad\cdot\Big\{\eta^2\Big(\int_0^1 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x + he_\ell, Du(x) + t\tau_{\ell,h}Du)\,dt\Big)\tau_{\ell,h}u^\alpha_{x_i}\tau_{\ell,h}u^\beta_{x_j}\Big\}^{\frac12}\,dx\\ &\le C\int_\Omega |D\eta|^2(1 + |Du(x)|^2 + |Du(x + he_\ell)|^2)^{\frac{q-2}{2}}|\tau_{\ell,h}u|^2\,dx\\ &\quad + \frac12\int_\Omega \eta^2\Big(\int_0^1 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x + he_\ell, Du(x) + t\tau_{\ell,h}Du)\,dt\Big)\tau_{\ell,h}u^\alpha_{x_i}\tau_{\ell,h}u^\beta_{x_j}\,dx\\ &\le C\Big(\int_\Omega |D\eta|^{\frac{2q}{q-2}}(1 + |Du(x)|^2 + |Du(x + he_\ell)|^2)^{\frac q2}\,dx\Big)^{\frac{q-2}{q}}\Big(\int_{\operatorname{supp}\eta} |\tau_{\ell,h}u|^q\,dx\Big)^{\frac2q}\\ &\quad + \frac12\int_\Omega \eta^2\Big(\int_0^1 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x + he_\ell, Du(x) + t\tau_{\ell,h}Du)\,dt\Big)\tau_{\ell,h}u^\alpha_{x_i}\tau_{\ell,h}u^\beta_{x_j}\,dx\\ &\le \frac{C}{(R-\rho)^2}\|1 + |Du|\|^q_{L^\infty(B_{R_0})}|h|^2\\ &\quad + \frac12\int_\Omega \eta^2\Big(\int_0^1 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x + he_\ell, Du(x) + t\tau_{\ell,h}Du)\,dt\Big)\tau_{\ell,h}u^\alpha_{x_i}\tau_{\ell,h}u^\beta_{x_j}\,dx,\end{aligned}$$

where we used in turn Lemma 2.6, property (iii) of the finite difference operator and the assumption $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

If $q \le 2$, a similar estimate of J_1 holds true, and it can be obtained in a simpler way. Indeed, in this case,

$$\int_\Omega |D\eta|^2(1+|Du(x)|^2+|Du(x+he_\ell)|^2)^{\frac{q-2}{2}}|\tau_{\ell,h}u|^2\,dx$$

can be estimated as follows:

$$\begin{aligned}\int_\Omega |D\eta|^2(1+|Du(x)|^2+|Du(x+he_\ell)|^2)^{\frac{q-2}{2}}|\tau_{\ell,h}u|^2\,dx\\ \le \int_\Omega |D\eta|^2|\tau_{\ell,h}u|^2\,dx \le \frac{C}{(R-\rho)^2}\int_{\operatorname{supp}\eta}|\tau_{\ell,h}u|^2\,dx \le \frac{C}{(R-\rho)^2}|h|^2\|Du\|^2_{L^\infty(B_{R_0})}.\end{aligned}$$

At the end, we obtain

$$|J_1| \le \frac{C}{(R-\rho)^2}\|1+|Du|\|^{\max\{2,q\}}_{L^\infty(B_{R_0})}|h|^2 + \frac{1}{2}\int_\Omega \eta^2\Big(\int_0^1 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x+he_\ell, Du(x)+t\tau_{\ell,h}Du)\,dt\Big)\tau_{\ell,h}u^\alpha_{x_i}\tau_{\ell,h}u^\beta_{x_j}\,dx.$$

By the last inequality in (1.3), Hölder's inequality, again by property (iii) of the finite difference operator and the assumption $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, we obtain

$$\begin{aligned}|J_3| &\le 2|h|\int_\Omega \eta\Big(\int_0^1 k(x+the_\ell)\,dt\Big)(1+|Du|^2)^{\frac{q-1}{2}}\sum_{i,\alpha}|\eta_{x_i}\tau_{\ell,h}u^\alpha|\,dx\\ &\le C\|1+|Du|\|^{q-1}_{L^\infty(B_{R_0})}|h|\Big(\int_\Omega |D\eta|^{\frac{2s}{s-1}}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^{\frac{2s}{s-1}}dx\Big)^{\frac{s-1}{2s}}\sum_{i,\alpha}\Big(\int_\Omega \eta|\tau_{\ell,h}u^\alpha|^{\frac{2s}{s+1}}\Big)^{\frac{s+1}{2s}}\\ &\le C\|1+|Du|\|^{q}_{L^\infty(B_{R_0})}|h|^2\Big(\int_\Omega |D\eta|^{\frac{2s}{s-1}}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^{\frac{2s}{s-1}}dx\Big)^{\frac{s-1}{2s}},\end{aligned}$$

and also, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}|J_4| &\le c|h|\int_\Omega \eta^2\int_0^1 k(x+the_\ell)\,dt\,(1+|Du|^2)^{\frac{q-1}{2}}|\tau_{\ell,h}Du|\,dx\\ &\le \varepsilon\int_\Omega \eta^2 a(x)(1+|Du|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_{\ell,h}Du|^2\,dx\\ &\quad + C_\varepsilon|h|^2\int_\Omega \eta^2\frac{1}{a(x)}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^2(1+|Du|^2)^{\frac{2q-p}{2}}\,dx\\ &\le \varepsilon\int_\Omega \eta^2 a(x)(1+|Du|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_{\ell,h}Du|^2\,dx\\ &\quad + C_\varepsilon\|1+|Du|\|^{2q-p}_{L^\infty(B_{R_0})}|h|^2\int_\Omega \eta^2\frac{1}{a(x)}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^2dx.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\int_\Omega \eta^2\Big(\int_0^1\sum_{i,j,\ell,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x+he_\ell, Du(x)+t\tau_{\ell,h}Du)\,dt\Big)\tau_{\ell,h}u^\alpha_{x_i}\tau_{\ell,h}u^\beta_{x_j}\,dx\\ &\le \frac{1}{2}\int_\Omega \eta^2\Big(\int_0^1\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x+he_\ell, Du(x)+t\tau_{\ell,h}Du)\,dt\Big)\tau_{\ell,h}u^\alpha_{x_i}\tau_{\ell,h}u^\beta_{x_j}\,dx\\ &\quad + \frac{C}{(R-\rho)^2}|h|^2 + C|h|^2\Big(\int_\Omega |D\eta|^{\frac{2s}{s-1}}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^{\frac{2s}{s-1}}dx\Big)^{\frac{s-1}{2s}}\\ &\quad + \varepsilon\int_\Omega \eta^2 a(x)(1+|Du|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_{\ell,h}Du|^2\,dx\\ &\quad + C_\varepsilon|h|^2\int_\Omega \eta^2\frac{1}{a(x)}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^2dx,\end{aligned}$$

where all the constants in the previous estimate depend also on $\|Du\|_{L^\infty(B_{R_0})}$. Reabsorbing the first integral in the right-hand side by the left-hand side, we obtain

$$\begin{aligned}\int_\Omega \eta^2\Big(\int_0^1 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x+he_\ell, Du(x)+t\tau_{\ell,h}Du)\,dt\Big)\tau_{\ell,h}u^\alpha_{x_i}\tau_{\ell,h}u^\beta_{x_j}\,dx \\ \le 2\varepsilon\int_\Omega \eta^2 a(x)(1+|Du|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_{\ell,h}Du|^2\,dx \\ + \frac{C}{(R-\rho)^2}|h|^2 + c|h|^2\Big(\int_\Omega |D\eta|^{\frac{2s}{s-1}}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^{\frac{2s}{s-1}}dx\Big)^{\frac{s-1}{2s}} \\ + C_\varepsilon|h|^2\int_\Omega \eta^2\frac{1}{a(x)}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^2 dx. \end{aligned} \tag{3.4}$$

By the ellipticity assumption in (1.3) and by Lemma 2.6, we get that, for some $c_1(p) > 0$,

$$\begin{aligned}c_1(p)\int_\Omega \eta^2 a(x)(1+|Du|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_{\ell,h}Du|^2\,dx \\ \le 2\varepsilon\int_\Omega \eta^2 a(x)(1+|Du|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_{\ell,h}Du|^2\,dx \\ + \frac{C}{(R-\rho)^2}|h|^2 + c|h|^2\Big(\int_\Omega |D\eta|^{\frac{2s}{s-1}}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^{\frac{2s}{s-1}}dx\Big)^{\frac{s-1}{2s}} \\ + C_\varepsilon|h|^2\int_\Omega \eta^2\frac{1}{a(x)}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^2 dx. \end{aligned}$$

Choosing $\varepsilon = \frac{c_1(p)}{4}$, we can reabsorb the first integral in the right-hand side by the left-hand side, thus getting

$$\begin{aligned}\int_\Omega \eta^2 a(x)(1+|Du|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_{\ell,h}Du|^2\,dx \\ \le \frac{C}{(R-\rho)^2}|h|^2 + c|h|^2\Big(\int_\Omega |D\eta|^{\frac{2s}{s-1}}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^{\frac{2s}{s-1}}dx\Big)^{\frac{s-1}{2s}} \\ + c_\varepsilon|h|^2\int_\Omega \eta^2\frac{1}{a(x)}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^2 dx. \end{aligned}$$

Using Lemma 2.1 to control the left-hand side of the previous estimate from below, we get

$$\begin{aligned}\int_\Omega \eta^2 a(x)|\tau_{\ell,h}V_p(Du)|^2\,dx \le \frac{C}{(R-\rho)^2}|h|^2 + c|h|^2\Big(\int_\Omega |D\eta|^{\frac{2s}{s-1}}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^{\frac{2s}{s-1}}dx\Big)^{\frac{s-1}{2s}} \\ + c_\varepsilon|h|^2\int_\Omega \eta^2\frac{1}{a(x)}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^2 dx, \end{aligned}$$

and so, by the assumption $\frac{1}{a} \in L^s_{\mathrm{loc}}(\Omega)$ and Hölder's inequality,

$$\begin{aligned}\Big(\int_\Omega \eta^2|\tau_{\ell,h}V_p(Du)|^{\frac{2s}{s+1}}\,dx\Big)^{\frac{s+1}{s}} &\le \Big(\int_\Omega \eta^2\frac{1}{a^s(x)}\,dx\Big)^{\frac{1}{s}}\int_\Omega \eta^2 a(x)|\tau_{\ell,h}V_p(Du)|^2\,dx \\ &\le \Big\{\frac{C}{(R-\rho)^2}|h|^2 + c|h|^2\Big(\int_\Omega |D\eta|^{\frac{2s}{s-1}}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^{\frac{2s}{s-1}}dx\Big)^{\frac{s-1}{2s}} \\ &\quad + c_\varepsilon|h|^2\int_\Omega \eta^2\frac{1}{a(x)}\Big(\int_0^1 k(x+the_\ell)\,dt\Big)^2 dx\Big\}\|a^{-1}\|_{L^s(B_{R_0})}. \end{aligned}$$

Dividing both sides of the previous inequality by $|h|^2$ and letting $h \to 0$, and using (3.1), we obtain

$$
\begin{aligned}
\lim_{|h|\to 0}\Big(\int_\Omega \eta^2\Big(\frac{|\tau_{\ell,h}V_p(Du)|}{|h|}\Big)^{\frac{2s}{s+1}}\,dx\Big)^{\frac{s+1}{s}} &\leq \lim_{|h|\to 0}\frac{1}{|h|^2}\int_\Omega \eta^2 a(x)|\tau_{\ell,h}V_p(Du)|^2\,dx\,\|a^{-1}\|_{L^s(B_{R_0})}\\
&\leq \Big\{\frac{C}{(R-\rho)^2}+c\Big(\int_\Omega |D\eta|^{\frac{2s}{s-1}}k^{\frac{2s}{s-1}}(x)\,dx\Big)^{\frac{s-1}{2s}}+c_\varepsilon\int_\Omega \eta^2\frac{k^2(x)}{a(x)}\,dx\Big\}\|a^{-1}\|_{L^s(B_{R_0})}\\
&\leq \|a^{-1}\|_{L^s(B_{R_0})}\Big\{\frac{C}{(R-\rho)^2}+c\Big(\int_\Omega |D\eta|^{\frac{2s}{s-1}}k^{\frac{2s}{s-1}}(x)\,dx\Big)^{\frac{s-1}{2s}}\\
&\qquad + c\Big(\int_\Omega \eta^2\frac{1}{a^s(x)}\,dx\Big)^{\frac{1}{s}}\Big(\int_\Omega \eta^2 k^{\frac{2s}{s-1}}(x)\,dx\Big)^{\frac{s-1}{s}}\Big\}\leq K,
\end{aligned}
$$

with K positive constant depending on $\|k\|_{L^{\frac{2s}{s-1}}(B_{R_0})}$ and $\|a^{-1}\|_{L^s(B_{R_0})}$. Therefore, by property (iv) of the finite difference operator, we get

$$V_p(Du)\in W^{1,\frac{2s}{s+1}}_{\mathrm{loc}}(\Omega),$$

and for a sequence h_n converging to 0, the sequence $|\tau_{\ell,h_n}V_p(Du)|/|h_n|$ strongly converges to $|D(V_p(Du))|$ in $L^{\frac{2s}{s+1}}_{\mathrm{loc}}(\Omega)$ and also a.e. up to a subsequence. Hence, by Fatou's lemma, we also have

$$\int_\Omega \eta^2 a(x)|DV_p(Du)|^2\,dx \leq \lim_{|h|\to 0}\frac{1}{|h|^2}\int_\Omega \eta^2 a(x)|\tau_{\ell,h}V_p(Du)|^2\,dx \leq K\cdot\|a^{-1}\|_{L^s(B_{R_0})}. \qquad \square$$

We are now ready to establish the main result of this section, that holds true with the condition $1 < p \leq q$ and not only for $2 \leq p \leq q$.

Theorem 3.3. *Consider the functional F in* (1.1) *satisfying assumptions* (1.2), (1.3), (1.5) *and* (1.6), *with* $1 < p \leq q$. *If $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a local minimizer of F, then for every ball $B_{R_0} \Subset \Omega$,*

$$\|Du\|_{L^\infty(B_{R_0/2})} \leq C\mathcal{K}^{\vartheta}_{R_0}\Big(\int_{B_{R_0}}(1+f(x,Du))\,dx\Big)^{\vartheta}, \tag{3.5}$$

$$\int_{B_\rho} a(x)(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx \leq c\Big(\int_{B_{R_0}}(1+f(x,Du))\,dx\Big)^{\vartheta} \tag{3.6}$$

hold for any $\rho < \frac{R}{2}$. Here

$$\mathcal{K}_{R_0} = 1+\|a^{-1}\|_{L^s(B_{R_0})}\|k\|^2_{L^r(B_{R_0})}+\|a\|_{L^{\frac{rs}{2s+r}}(B_{R_0})},$$

$\vartheta > 0$ depends on the data, C depends also on R_0 and c depends also on ρ and $\mathcal{K}_{R_0}$.

Proof. We begin by noting that, by Lemma 2.7 and Proposition 3.2, if $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a local minimizer of F, then

$$V_p(Du)\in W^{1,\frac{2s}{s+1}}_{\mathrm{loc}}(\Omega) \quad\text{and}\quad a(x)|D(V_p(Du))|^2 \in L^1_{\mathrm{loc}}(\Omega).$$

Moreover, u satisfies the Euler system

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(x,Du)\psi^\alpha_{x_i}(x)\,dx = 0 \quad \text{for all } \psi \in C^\infty_0(\Omega;\mathbb{R}^N).$$

Fix $\ell = 1,\dots,n$. By considering in the equality above $\psi = \varphi_{x_\ell}$ with $\varphi \in C^\infty_0(\Omega;\mathbb{R}^N)$, we get

$$\int_\Omega\Big\{\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)\varphi^\alpha_{x_i}u^\beta_{x_\ell x_j}+\sum_{i,\alpha} f_{\xi_i^\alpha x_\ell}(x,Du)\varphi^\alpha_{x_i}\Big\}\,dx = 0 \quad \text{for all } \varphi \in C^\infty_0(\Omega;\mathbb{R}^N). \tag{3.7}$$

Fix a cut-off function $\eta \in C^\infty_0(\Omega)$, and define for any $\gamma \geq 0$ the function

$$\varphi^\alpha := \eta^4 u^\alpha_{x_\ell}(1+|Du|^2)^{\frac{\gamma}{2}}, \quad \alpha = 1,\dots,N.$$

One can easily check that

$$\varphi^\alpha_{x_i} = 4\eta^3\eta_{x_i}u^\alpha_{x_\ell}(1+|Du|^2)^{\frac{\gamma}{2}} + \eta^4 u^\alpha_{x_\ell x_i}(1+|Du|^2)^{\frac{\gamma}{2}} + \gamma\eta^4 u^\alpha_{x_\ell}(1+|Du|^2)^{\frac{\gamma-2}{2}}|Du|(|Du|)_{x_i}.$$

By the a priori regularity properties of u, due to Proposition 3.2, and by observing that, by inequality (3.4), it can be easily deduced that

$$x \mapsto \sum_{i,j,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)u^\beta_{x_\ell x_j}u^\alpha_{x_\ell x_i}$$

is in $L^1_{\mathrm{loc}}(\Omega)$, through a density argument, we can use φ as test function in equation (3.7), thus getting

$$\begin{aligned}
0 &= \int_\Omega 4\eta^3(1+|Du|^2)^{\frac{\gamma}{2}} \sum_{i,j,\ell,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)\eta_{x_i}u^\alpha_{x_\ell}u^\beta_{x_\ell x_j}\,dx \\
&\quad + \int_\Omega \eta^4(1+|Du|^2)^{\frac{\gamma}{2}} \sum_{i,j,\ell,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)u^\alpha_{x_\ell x_i}u^\beta_{x_\ell x_j}\,dx \\
&\quad + \gamma\int_\Omega \eta^4(1+|Du|^2)^{\frac{\gamma-2}{2}}|Du| \sum_{i,j,\ell,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)u^\alpha_{x_\ell}u^\beta_{x_\ell x_j}(|Du|)_{x_i}\,dx \\
&\quad + \int_\Omega 4\eta^3(1+|Du|^2)^{\frac{\gamma}{2}} \sum_{i,\ell,\alpha} f_{\xi^\alpha_i x_\ell}(x,Du)\eta_{x_i}u^\alpha_{x_\ell}\,dx \\
&\quad + \int_\Omega \eta^4(1+|Du|^2)^{\frac{\gamma}{2}} \sum_{i,\ell,\alpha} f_{\xi^\alpha_i x_\ell}(x,Du)u^\alpha_{x_\ell x_i}\,dx \\
&\quad + \gamma\int_\Omega \eta^4(1+|Du|^2)^{\frac{\gamma-2}{2}}|Du| \sum_{i,\ell,\alpha} f_{\xi^\alpha_i x_\ell}(x,Du)u^\alpha_{x_\ell}(|Du|)_{x_i}\,dx \\
&=: I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned} \tag{3.8}$$

Estimate of I_1. By the use of Cauchy–Schwarz and Young's inequalities and by virtue of the second inequality in (1.3), we can estimate the integral I_1 as follows:

$$\begin{aligned}
|I_1| &\le 4\int_\Omega (1+|Du|^2)^{\frac{\gamma}{2}}\Big\{\eta^2 \sum_{i,j,\ell,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)\eta_{x_i}u^\alpha_{x_\ell}\eta_{x_j}u^\beta_{x_\ell}\Big\}^{\frac12}\Big\{\eta^4 \sum_{i,j,\ell,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)u^\alpha_{x_\ell x_i}u^\beta_{x_\ell x_j}\Big\}^{\frac12} \\
&\le C(\varepsilon,L)\int_\Omega \eta^2|D\eta|^2(1+|Du|^2)^{\frac{q+\gamma}{2}}\,dx + \varepsilon\int_\Omega \eta^4(1+|Du|^2)^{\frac{\gamma}{2}} \sum_{i,j,\ell,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)u^\alpha_{x_\ell x_i}u^\beta_{x_\ell x_j}\,dx,
\end{aligned} \tag{3.9}$$

where $\varepsilon > 0$ will be chosen later.

Estimate of I_3. Since

$$f_{\xi^\alpha_i\xi^\beta_j}(x,\xi) = \Big(\frac{g_{tt}(x,|\xi|)}{|\xi|^2} - \frac{g_t(x,|\xi|)}{|\xi|^3}\Big)\xi^\alpha_i\xi^\beta_j + \frac{g_t(x,|\xi|)}{|\xi|}\delta_{ij}\delta^{\alpha\beta}$$

and

$$(|Du|)_{x_i} = \frac{1}{|Du|}\sum_{\alpha,\ell} u^\alpha_{x_i x_\ell}u^\alpha_{x_\ell}, \tag{3.10}$$

then

$$\begin{aligned}
&\sum_{i,j,\ell,\alpha,\beta} f_{\xi^\alpha_i\xi^\beta_j}(x,Du)u^\alpha_{x_\ell}u^\beta_{x_\ell x_j}(|Du|)_{x_i} \\
&\quad = \Big(\frac{g_{tt}(x,|Du|)}{|Du|^2} - \frac{g_t(x,|Du|)}{|Du|^3}\Big)\sum_{i,j,\ell,\alpha,\beta} u^\alpha_{x_\ell}u^\beta_{x_\ell x_j}u^\alpha_{x_i}u^\beta_{x_j}(|Du|)_{x_i} + \frac{g_t(x,|Du|)}{|Du|}\sum_{i,\ell,\alpha} u^\alpha_{x_\ell}u^\alpha_{x_\ell x_i}(|Du|)_{x_i} \\
&\quad = \Big(\frac{g_{tt}(x,|Du|)}{|Du|} - \frac{g_t(x,|Du|)}{|Du|^2}\Big)\sum_\alpha\Big(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\Big)^2 + g_t(x,|Du|)|D(|Du|)|^2.
\end{aligned}$$

Thus,

$$I_3 = \gamma\int_\Omega \eta^4(1+|Du|^2)^{\frac{\gamma-2}{2}}|Du|\Big\{\Big(\frac{g_{tt}(x,|Du|)}{|Du|} - \frac{g_t(x,|Du|)}{|Du|^2}\Big)\sum_\alpha\Big(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\Big)^2 + g_t(x,|Du|)|D(|Du|)|^2\Big\}\,dx.$$

Using the Cauchy–Schwarz inequality, i.e.

$$\sum_\alpha \Big(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\Big)^2 \le |Du|^2 |D(|Du|)|^2$$

and observing that $g_t(x, |Du|) \ge 0$, we conclude

$$I_3 \ge \gamma \int_\Omega \eta^4 (1+|Du|^2)^{\frac{\gamma-2}{2}} |Du| \frac{g_{tt}(x,|Du|)}{|Du|} \sum_\alpha \Big(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\Big)^2 \, dx \ge 0. \tag{3.11}$$

Estimate of I_4. By using the last inequality in (1.3), we obtain

$$|I_4| \le 4 \int_\Omega \eta^3 k(x)(1+|Du|^2)^{\frac{q-1+\gamma}{2}} \sum_{i,\ell,\alpha} |\eta_{x_i} u^\alpha_{x_\ell}| \, dx \le 4 \int_\Omega \eta^3 |D\eta| k(x)(1+|Du|^2)^{\frac{q+\gamma}{2}} \, dx. \tag{3.12}$$

Estimate of I_5. Using the last inequality in (1.3) and Young's inequality, we have that

$$\begin{aligned} |I_5| &\le \int_\Omega \eta^4 k(x)(1+|Du|^2)^{\frac{q-1+\gamma}{2}} |D^2u| \, dx \\ &\le \sigma \int_\Omega \eta^4 a(x)(1+|Du|^2)^{\frac{p-2+\gamma}{2}} |D^2u|^2 \, dx + C_\sigma \int_\Omega \eta^4 \frac{k^2(x)}{a(x)} (1+|Du|^2)^{\frac{2q-p+\gamma}{2}} \, dx, \end{aligned} \tag{3.13}$$

where $\sigma \in (0,1)$ will be chosen later and a is the function appearing in (1.3).

Estimate of I_6. Using the last inequality in (1.3) and (3.10), we get

$$\begin{aligned} |I_6| &\le \gamma \int_\Omega \eta^4 k(x)(1+|Du|^2)^{\frac{q-1+\gamma}{2}} |D(|Du|)| \, dx \le \gamma \int_\Omega \eta^4 (1+|Du|^2)^{\frac{q-1+\gamma}{2}} k(x)|D^2u| \, dx \\ &\le \sigma \int_\Omega \eta^4 a(x)(1+|Du|^2)^{\frac{p-2+\gamma}{2}} |D^2u|^2 \, dx + C_\sigma \gamma^2 \int_\Omega \eta^4 \frac{k^2(x)}{a(x)} (1+|Du|^2)^{\frac{2q-p+\gamma}{2}} \, dx, \end{aligned} \tag{3.14}$$

where we used Young's inequality again. Since equality (3.8) can be written as

$$I_2 + I_3 = -I_1 - I_4 - I_5 - I_6,$$

by virtue of (3.11), we get

$$I_2 \le |I_1| + |I_4| + |I_5| + |I_6|,$$

and therefore, recalling estimates (3.9), (3.12), (3.13) and (3.14), we obtain

$$\begin{aligned} &\int_\Omega \eta^4 (1+|Du|^2)^{\frac{\gamma}{2}} \sum_{i,j,\ell,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du) u^\alpha_{x_\ell x_i} u^\beta_{x_\ell x_j} \, dx \\ &\quad \le \varepsilon \int_\Omega \eta^4 (1+|Du|^2)^{\frac{\gamma}{2}} \sum_{i,j,\ell,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du) u^\alpha_{x_\ell x_i} u^\beta_{x_\ell x_j} \, dx \\ &\qquad + 4 \int_\Omega \eta^3 |D\eta| k(x)(1+|Du|^2)^{\frac{q+\gamma}{2}} \, dx + 2\sigma \int_\Omega \eta^4 a(x)(1+|Du|^2)^{\frac{p-2+\gamma}{2}} |D^2u|^2 \, dx \\ &\qquad + C_\sigma(1+\gamma^2) \int_\Omega \eta^4 \frac{k^2(x)}{a(x)} (1+|Du|^2)^{\frac{2q-p+\gamma}{2}} \, dx + C_\varepsilon \int_\Omega \eta^2 |D\eta|^2 (1+|Du|^2)^{\frac{q+\gamma}{2}} \, dx. \end{aligned}$$

Choosing $\varepsilon = \frac{1}{2}$, we can reabsorb the first integral in the right-hand side by the left-hand side, thus getting

$$\begin{aligned} &\int_\Omega \eta^4 (1+|Du|^2)^{\frac{\gamma}{2}} \sum_{i,j,\ell,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du) u^\alpha_{x_\ell x_i} u^\beta_{x_\ell x_j} \, dx \\ &\quad \le 4\sigma \int_\Omega \eta^4 a(x)(1+|Du|^2)^{\frac{p-2+\gamma}{2}} |D^2u|^2 \, dx + C \int_\Omega \eta^3 |D\eta| k(x)(1+|Du|^2)^{\frac{q+\gamma}{2}} \, dx \\ &\qquad + C_\sigma(1+\gamma^2) \int_\Omega \eta^4 \frac{k^2(x)}{a(x)} (1+|Du|^2)^{\frac{2q-p+\gamma}{2}} \, dx + C \int_\Omega \eta^2 |D\eta|^2 (1+|Du|^2)^{\frac{q+\gamma}{2}} \, dx. \end{aligned} \tag{3.15}$$

Now, using the ellipticity condition in (1.3) to estimate the left-hand side of (3.15), we get

$$
\begin{aligned}
c_2 \int_\Omega \eta^4 a(x)(1+|Du|^2)^{\frac{p-2+\gamma}{2}}|D^2u|^2\,dx
&\le 4\sigma \int_\Omega \eta^4 a(x)(1+|Du|^2)^{\frac{p-2+\gamma}{2}}|D^2u|^2\,dx + C\int_\Omega \eta^3|D\eta|k(x)(1+|Du|^2)^{\frac{q+\gamma}{2}}\,dx\\
&\quad + C_\sigma(1+\gamma^2)\int_\Omega \eta^4\frac{k^2(x)}{a(x)}(1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\,dx + C\int_\Omega \eta^2|D\eta|^2(1+|Du|^2)^{\frac{q+\gamma}{2}}\,dx.
\end{aligned}
$$

By $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and Proposition 3.2, the first integral in the right-hand side of previous estimate is finite.

By choosing $\sigma = \frac{c_2}{8}$, we can reabsorb the first integral in the right-hand side by the left-hand side, thus getting

$$
\begin{aligned}
\int_\Omega \eta^4 a(x)(1+|Du|^2)^{\frac{p-2+\gamma}{2}}|D^2u|^2\,dx
&\le C\int_\Omega \eta^3|D\eta|k(x)(1+|Du|^2)^{\frac{q+\gamma}{2}}\,dx + C(1+\gamma^2)\int_\Omega \eta^4\frac{k^2(x)}{a(x)}(1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\,dx\\
&\quad + C\int_\Omega \eta^2|D\eta|^2(1+|Du|^2)^{\frac{q+\gamma}{2}}\,dx\\
&\le C\int_\Omega (\eta^2|D\eta|^2+|D\eta|^4)a(x)(1+|Du|^2)^{\frac{p+\gamma}{2}}\,dx\\
&\quad + C(\gamma+1)^2\int_\Omega \eta^4\frac{k^2(x)}{a(x)}(1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\,dx,
\end{aligned}
\tag{3.16}
$$

where we used Young's inequality again. Now, we note that

$$
\eta^4 a(x)\big|D\big((1+|Du|^2)^{\frac{p+\gamma}{4}}\big)\big|^2 \le c(p+\gamma)^2 a(x)\eta^4(1+|Du|^2)^{\frac{p-2+\gamma}{2}}|D^2u|^2,
$$

and so, fixing $\frac{R_0}{2} \le \rho < t' < t < R < R_0$ with R_0 such that $B_{R_0} \Subset \Omega$ and choosing $\eta \in C_0^\infty(B_t)$ a cut-off function between $B_{t'}$ and B_t, by the assumption $a^{-1} \in L^s_{\mathrm{loc}}(\Omega)$, we can use the Sobolev type inequality of Lemma 3.1 with $w = \eta^2(1+|Du|^2)^{\frac{p+\gamma}{4}}$, $\lambda = a$ and $p = 2$, thus obtaining

$$
\begin{aligned}
\Big(\int_{B_t}\big(\eta^2(1+|Du|^2)^{\frac{p+\gamma}{4}}\big)^{(\frac{2s}{s+1})^*}\,dx\Big)^{\frac{2}{(\frac{2s}{s+1})^*}} &\le \frac{c}{(t-t')^2}\int_{B_t} a(x)(1+|Du|^2)^{\frac{p+\gamma}{2}}\,dx\\
&\quad + c(p+\gamma)^2\int_{B_t}\eta^4 a(x)(1+|Du|^2)^{\frac{p-2+\gamma}{2}}|D^2u|^2\,dx,
\end{aligned}
$$

with a constant c depending on n and $\|a^{-1}\|_{L^s(B_t)}$.

Using (3.16) to estimate the last integral in the previous inequality, we obtain

$$
\begin{aligned}
\Big(\int_{B_t}\eta^{\frac{4ns}{n(s+1)-2s}}(1+|Du|^2)^{\frac{(p+\gamma)ns}{2(n(s+1)-2s)}}\,dx\Big)^{\frac{n(s+1)-2s}{ns}}
&\le c\Big(\frac{(p+\gamma)^2}{(t-t')^2}+\frac{(p+\gamma)^2}{(t-t')^4}\Big)\int_{B_t} a(x)(1+|Du|^2)^{\frac{p+\gamma}{2}}\,dx\\
&\quad + c(p+\gamma)^4\int_{B_t}\frac{k^2(x)}{a(x)}(1+|Du|^2)^{\frac{2q-p+\gamma}{2}}\,dx\\
&\le c\Big(\frac{(p+\gamma)^2}{(t-t')^2}+\frac{(p+\gamma)^2}{(t-t')^4}\Big)\int_{B_t} a(x)(1+|Du|^2)^{\frac{p+\gamma}{2}}\,dx\\
&\quad + c(p+\gamma)^4\Big(\int_{B_t}\frac{1}{a^s}\,dx\Big)^{\frac{1}{s}}\Big(\int_{B_t}k^r\,dx\Big)^{\frac{2}{r}}\Big(\int_{B_t}(1+|Du|^2)^{\frac{(2q-p+\gamma)rs}{2(rs-2s-r)}}\,dx\Big)^{\frac{rs-2s-r}{rs}},
\end{aligned}
\tag{3.17}
$$

where we used assumption (1.5) and Hölder's inequality with exponents s, $\frac{r}{2}$ and $\frac{rs}{rs-2s-r}$.

Using the properties of η, we obtain

$$\begin{aligned}\Big(\int_{B_{t'}}(1+|Du|^2)^{\frac{(p+\gamma)ns}{2(n(s+1)-2s)}}\,dx\Big)^{\frac{n(s+1)-2s}{ns}} &\le c(p+\gamma)^4\|a^{-1}\|_{L^s(B_{R_0})}\|k\|^2_{L^r(B_{R_0})}\Big(\int_{B_t}(1+|Du|^2)^{\frac{(2q-p+\gamma)rs}{2(rs-2s-r)}}\,dx\Big)^{\frac{rs-2s-r}{rs}}\\&\quad+c\Big(\frac{(p+\gamma)^2}{(t-t')^2}+\frac{(p+\gamma)^2}{(t-t')^4}\Big)\int_{B_t}a(x)(1+|Du|^2)^{\frac{p+\gamma}{2}}\,dx\\&\le c(p+\gamma)^4\|a^{-1}\|_{L^s(B_{R_0})}\|k\|^2_{L^r(B_{R_0})}\Big(\int_{B_t}(1+|Du|^2)^{\frac{(2q-p+\gamma)rs}{2(rs-2s-r)}}\,dx\Big)^{\frac{rs-2s-r}{rs}}\\&\quad+c\Big(\frac{(p+\gamma)^2}{(t-t')^2}+\frac{(p+\gamma)^2}{(t-t')^4}\Big)\|a\|_{L^{\frac{rs}{2s+r}}(B_{R_0})}\Big(\int_{B_t}(1+|Du|^2)^{\frac{(p+\gamma)rs}{2(rs-2s-r)}}\,dx\Big)^{\frac{rs-2s-r}{rs}},\end{aligned}$$

where we used that, by assumption (1.3), $a\in L^\infty_{\mathrm{loc}}(\Omega)$. Setting

$$\mathcal{K}_{R_0}=1+\|a^{-1}\|_{L^s(B_{R_0})}\|k\|^2_{L^r(B_{R_0})}+\|a\|_{L^{\frac{rs}{2s+r}}(B_{R_0})}\tag{3.18}$$

and assuming without loss of generality that $t-t'<1$, we can write the previous estimate as follows:

$$\begin{aligned}\Big(\int_{B_{t'}}(1+|Du|^2)^{\frac{(p+\gamma)ns}{2(n(s+1)-2s)}}\,dx\Big)^{\frac{n(s+1)-2s}{ns}} &\le c(p+\gamma)^4\mathcal{K}_{R_0}\Big(\int_{B_t}(1+|Du|^2)^{\frac{(2q-p+\gamma)rs}{2(rs-2s-r)}}\,dx\Big)^{\frac{rs-2s-r}{rs}}\\&\quad+c(p+\gamma)^2\frac{\mathcal{K}_{R_0}}{(t-t')^4}\Big(\int_{B_t}(1+|Du|^2)^{\frac{(p+\gamma)rs}{2(rs-2s-r)}}\,dx\Big)^{\frac{rs-2s-r}{rs}},\end{aligned}$$

and using the a priori assumption $u\in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, we get

$$\begin{aligned}\Big(\int_{B_{t'}}(1+|Du|^2)^{\frac{(p+\gamma)ns}{2(n(s+1)-2s)}}\,dx\Big)^{\frac{n(s+1)-2s}{ns}}\\&\le c(p+\gamma)^4\mathcal{K}_{R_0}\Big(\|Du\|^{2(q-p)}_{L^\infty(B_R)}+\frac{1}{(t-t')^4}\Big)\Big(\int_{B_t}(1+|Du|^2)^{\frac{(p+\gamma)rs}{2(rs-2s-r)}}\,dx\Big)^{\frac{rs-2s-r}{rs}}.\end{aligned}\tag{3.19}$$

Setting now

$$m:=\frac{rs}{rs-2s-r}\tag{3.20}$$

and noting that

$$\frac{ns}{n(s+1)-2s}=\frac{1}{2}\Big(\frac{2s}{s+1}\Big)^*=:\frac{2^*_s}{2},\tag{3.21}$$

we can write (3.19) as follows:

$$\Big(\int_{B_{t'}}\big((1+|Du|^2)^{\frac{(p+\gamma)m}{2}}\big)^{\frac{2^*_s}{2m}}\,dx\Big)^{\frac{2m}{2^*_s}}\le c(p+\gamma)^{4m}\mathcal{K}^m_{R_0}\frac{\|Du\|^{2(q-p)m}_{L^\infty(B_R)}}{(t-t')^{4m}}\int_{B_t}(1+|Du|^2)^{\frac{(p+\gamma)m}{2}}\,dx,\tag{3.22}$$

where, without loss of generality, we supposed $\|Du\|^{2(q-p)m}_{L^\infty(B_R)}\ge 1$.

By (2.10) and definitions (3.20) and (3.21), we have

$$\frac{2^*_s}{2m}>1.$$

Define the increasing sequence of exponents

$$p_0=pm,\quad p_i=p_{i-1}\Big(\frac{2^*_s}{2m}\Big)=p_0\Big(\frac{2^*_s}{2m}\Big)^i$$

and the decreasing sequence of radii by setting

$$\rho_i = \rho + \frac{R-\rho}{2^i}.$$

As we will prove (see (3.30) below) the right-hand side of (3.22) is finite for $\gamma = 0$. Then, for every $\rho < \rho_{i+1} < \rho_i < R$, we may iterate it on the concentric balls B_{ρ_i} with exponents p_i, thus obtaining

$$\begin{aligned}\Big(\int_{B_{\rho_{i+1}}} (1+|Du|^2)^{\frac{p_{i+1}}{2}}\,dx\Big)^{\frac{1}{p_{i+1}}} &\le \prod_{j=0}^{i}\Big(C^m\mathcal{K}_{R_0}^m\frac{p_j^{4m}\|Du\|_{L^\infty(B_R)}^{2m(q-p)}}{(\rho_j-\rho_{j+1})^{4m}}\Big)^{\frac{1}{p_j}}\Big(\int_{B_R}(1+|Du|^2)^{\frac{p_0}{2}}\,dx\Big)^{\frac{1}{p_0}}\\ &= \prod_{j=0}^{i}\Big(C^m\mathcal{K}_{R_0}^m\frac{4^{jm}p_j^{4m}\|Du\|_{L^\infty(B_R)}^{2(q-p)m}}{(R-\rho)^{4m}}\Big)^{\frac{1}{p_j}}\Big(\int_{B_R}(1+|Du|^2)^{\frac{p_0}{2}}\,dx\Big)^{\frac{1}{p_0}}\\ &= \prod_{j=0}^{i}(4^{jm}p_j^{4m})^{\frac{1}{p_j}}\prod_{j=0}^{i}\Big(\frac{C^m\mathcal{K}_{R_0}^m\|Du\|_{L^\infty(B_R)}^{2(q-p)m}}{(R-\rho)^{4m}}\Big)^{\frac{1}{p_j}}\Big(\int_{B_R}(1+|Du|^2)^{\frac{p_0}{2}}\,dx\Big)^{\frac{1}{p_0}}.\end{aligned} \tag{3.23}$$

We have that

$$\prod_{j=0}^{i}(4^{jm}p_j^{4m})^{\frac{1}{p_j}} = \exp\Big(\sum_{j=0}^{i}\frac{1}{p_j}\log(4^{jm}p_j^{4m})\Big) \le \exp\Big(\sum_{j=0}^{\infty}\frac{1}{p_j}\log(4^{jm}p_j^{4m})\Big) \le c(n,r,s)$$

and

$$\begin{aligned}\lim_{i\to+\infty}\prod_{j=0}^{i}\Big(\frac{C^m\mathcal{K}_{R_0}^m\|Du\|_{L^\infty(B_R)}^{2(q-p)m}}{(R-\rho)^{4m}}\Big)^{\frac{1}{p_j}} &= \lim_{i\to+\infty}\Big(\frac{C\mathcal{K}_{R_0}\|Du\|_{L^\infty(B_R)}^{2(q-p)}}{(R-\rho)^4}\Big)^{\sum_{j=0}^{i}\frac{m}{p_j}}\\ &= \Big(\frac{C\mathcal{K}_{R_0}\|Du\|_{L^\infty(B_R)}^{2(q-p)}}{(R-\rho)^4}\Big)^{\sum_{j=0}^{\infty}\frac{m}{p_j}} = \Big(\frac{C\mathcal{K}_{R_0}\|Du\|_{L^\infty(B_R)}^{2(q-p)}}{(R-\rho)^4}\Big)^{\frac{2_s^*}{p(2_s^*-2m)}},\end{aligned}$$

where, recalling that $p_0 = pm$, we used in the last equality that

$$\sum_{j=0}^{\infty}\frac{m}{p_j} = \frac{m}{p_0}\frac{1}{1-\frac{2m}{2_s^*}} = \frac{2_s^*}{p(2_s^*-2m)}.$$

Therefore, we can let $i\to\infty$ in (3.23), thus getting

$$\|Du\|_{L^\infty(B_\rho)} \le C(n,r,p)\Big(\frac{\mathcal{K}_{R_0}}{(R-\rho)^4}\Big)^{\frac{2_s^*}{p(2_s^*-2m)}}\|Du\|_{L^\infty(B_R)}^{\frac{2(q-p)2_s^*}{p(2_s^*-2m)}}\Big(\int_{B_R}(1+|Du|^2)^{\frac{pm}{2}}\,dx\Big)^{\frac{1}{pm}}.$$

Since assumption (1.6) implies that

$$\frac{2(q-p)2_s^*}{p(2_s^*-2m)} < 1,$$

we can use Young's inequality with exponents

$$\frac{p(2_s^*-2m)}{2(q-p)2_s^*} > 1 \quad\text{and}\quad \frac{p(2_s^*-2m)}{p(2_s^*-2m)-2(q-p)2_s^*}$$

to deduce that

$$\|Du\|_{L^\infty(B_\rho)} \le \frac{1}{2}\|Du\|_{L^\infty(B_R)} + C(n,r,p,s)\Big(\frac{\mathcal{K}_{R_0}}{(R-\rho)^4}\Big)^{\vartheta}\Big(\int_{B_R}(1+|Du|^2)^{\frac{pm}{2}}\,dx\Big)^{\varsigma}, \tag{3.24}$$

with $\vartheta = \vartheta(p,q,n,r,s)$ and $\varsigma = \varsigma(p,q,n,r,s)$.

We now estimate the last integral. By definition of m and by the assumption on s, that implies $s > \frac{nr}{r-n}$, we get

$$m < \frac{ns}{n(s+1)-2s}.$$

Thus, by Hölder's inequality,

$$\int_{B_R} (1+|Du|^2)^{\frac{pm}{2}}\,dx \le c(R_0,n,r,s)\Big(\int_{B_R}(1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\Big)^{\frac{r(n(s+1)-2s)}{n(rs-2s-r)}}. \tag{3.25}$$

This last integral can be estimated by using (3.17) with $\gamma = 0$. Indeed, let us re-define t', t and η as follows: consider $R \le t' < t \le 2R - \rho \le R_0$ and η a cut-off function, $\eta \equiv 1$ on $B_{t'}$ and $\operatorname{supp}\eta \subset B_t$. By (3.19) with $\gamma = 0$,

$$\begin{aligned}\Big(\int_{B_{t'}}(1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\Big)^{\frac{n(s+1)-2s}{ns}} &\le c\Big(\frac{p^2}{(t-t')^2}+\frac{p^2}{(t-t')^4}\Big)\int_{B_{R_0}} a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx\\ &\quad + cp^4\mathcal{K}_{B_{R_0}}\Big(\int_{B_t}(1+|Du|^2)^{\frac{(2q-p)rs}{2(rs-2s-r)}}\,dx\Big)^{\frac{rs-2s-r}{rs}}.\end{aligned} \tag{3.26}$$

If we denote

$$\tau := \frac{(2q-p)rs}{rs-2s-r}, \quad \tau_1 := \frac{nps}{n(s+1)-2s}, \quad \tau_2 := \frac{ps}{s+1},$$

by (1.6) and $s > \frac{rn}{r-n}$, we get

$$\frac{\tau}{\tau_1} < 1 < \frac{\tau}{\tau_2}.$$

Therefore, there exists $\theta \in (0,1)$ such that

$$1 = \theta\frac{\tau}{\tau_1} + (1-\theta)\frac{\tau}{\tau_2}.$$

The precise value of θ is

$$\theta = \frac{ns(qr-pr+p)+qrn}{rs(2q-p)}. \tag{3.27}$$

By Hölder's inequality with exponents $\frac{\tau_1}{\theta\tau}$ and $\frac{\tau_2}{(1-\theta)\tau}$, we get

$$\begin{aligned}\Big(\int_{B_t}(1+|Du|^2)^{\frac{(2q-p)rs}{2(rs-2s-r)}}\,dx\Big)^{\frac{rs-2s-r}{rs}} &= \Big(\int_{B_t}(1+|Du|^2)^{\theta\frac{\tau}{2}+(1-\theta)\frac{\tau}{2}}\,dx\Big)^{\frac{2q-p}{\tau}}\\ &\le \Big(\int_{B_t}(1+|Du|^2)^{\frac{\tau_1}{2}}\,dx\Big)^{\frac{(2q-p)\theta}{\tau_1}}\Big(\int_{B_t}(1+|Du|^2)^{\frac{\tau_2}{2}}\,dx\Big)^{\frac{(2q-p)(1-\theta)}{\tau_2}}.\end{aligned}$$

Hence, we can use the inequality above to estimate the last integral of (3.26) to deduce that

$$\begin{aligned}\Big(\int_{B_{t'}}(1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\Big)^{\frac{n(s+1)-2s}{ns}} &\le c\Big(\frac{p^2}{(t-t')^2}+\frac{p^2}{(t-t')^4}\Big)\int_{B_{R_0}} a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx\\ &\quad + C\mathcal{K}_{B_{R_0}}\Big(\int_{B_t}(1+|Du|^2)^{\frac{ps}{2(s+1)}}\,dx\Big)^{\frac{(1-\theta)(2q-p)(s+1)}{ps}}\\ &\quad \times\Big(\int_{B_t}(1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\Big)^{\frac{\theta(ns+n-2s)(2q-p)}{nps}}.\end{aligned} \tag{3.28}$$

Note that, again by (1.6) and (3.27), we have

$$\frac{\theta(2q-p)}{p} < 1.$$

We can use Young's inequality in the last term of (3.28) with exponents $\frac{p}{p-\theta(2q-p)}$ and $\frac{p}{\theta(2q-p)}$ to obtain that, for every $\sigma < 1$,

$$\begin{aligned}\Big(\int_{B_{t'}}(1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\Big)^{\frac{n(s+1)-2s}{ns}} &\le C\Big(\frac{p^2}{(t-t')^2}+\frac{p^2}{(t-t')^4}\Big)\int_{B_{R_0}} a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx\\ &\quad + C_\sigma\mathcal{K}_{B_{R_0}}^{\frac{p}{p-\theta(2q-p)}}\Big(\int_{B_t}(1+|Du|^2)^{\frac{ps}{2(s+1)}}\,dx\Big)^{\frac{(1-\theta)(2q-p)}{p-\theta(2q-p)}\frac{s+1}{s}}\\ &\quad + \sigma\Big(\int_{B_t}(1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\Big)^{\frac{n(s+1)-2s}{ns}}.\end{aligned}$$

By applying Lemma 2.5, and noting that $2R-\rho-R=R-\rho$, we conclude that

$$\Big(\int_{B_R}(1+|Du|^2)^{\frac{pns}{2(n(s+1)-2s)}}\,dx\Big)^{\frac{n(s+1)-2s}{ns}} \le C\Big(\frac{p^2}{(R-\rho)^2}+\frac{p^2}{(R-\rho)^4}\Big)\int_{B_{R_0}} a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx + C\mathcal{K}_{B_{R_0}}^{\frac{p}{p-\theta(2q-p)}}\Big(\int_{B_{R_0}}(1+|Du|^2)^{\frac{ps}{2(s+1)}}\,dx\Big)^{\frac{(1-\theta)(2q-p)}{p-\theta(2q-p)}\frac{s+1}{s}}. \tag{3.29}$$

Collecting (3.25) and (3.29), we obtain

$$\int_{B_R}(1+|Du|^2)^{\frac{pm}{2}}\,dx \le C\Big(\frac{p^2}{(R-\rho)^2}+\frac{p^2}{(R-\rho)^4}\Big)\int_{B_{R_0}} a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx + C\mathcal{K}_{B_{R_0}}^{\frac{p}{p-\theta(2q-p)}}\Big(\int_{B_{R_0}}(1+|Du|^2)^{\frac{ps}{2(s+1)}}\,dx\Big)^{\frac{(1-\theta)(2q-p)}{p-\theta(2q-p)}\frac{s+1}{s}}. \tag{3.30}$$

Notice that the right-hand side is finite because u is a local minimizer and (1.4) and (1.10) hold. This inequality, together with (3.24), implies

$$\|Du\|_{L^\infty(B_\rho)} \le \frac{1}{2}\|Du\|_{L^\infty(B_R)} + C\Big(\frac{\mathcal{K}_{B_{R_0}}}{(R-\rho)^8}\Big)^{\theta}\Big(\int_{B_{R_0}} a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx\Big)^{\tilde{\theta}} + C\Big(\frac{\mathcal{K}_{B_{R_0}}}{(R-\rho)^8}\Big)^{\theta}\Big(\int_{B_{R_0}}(1+|Du|^2)^{\frac{ps}{2(s+1)}}\,dx\Big)^{\tilde{\varsigma}}$$

with the constant C depending on the data. Applying Lemma 2.5, we conclude the proof of estimate (3.5). Now, we write estimate (3.16) for $\gamma=0$ and for a cut-off function $\eta\in C_0^\infty(B_{\frac{R}{2}})$, $\eta=1$ on B_ρ for some $\rho<\frac{R}{2}$. This yields

$$\begin{aligned}\int_{B_\rho} a(x)(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx &\le C(R)\int_{B_{\frac{R}{2}}} a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx + C\int_{B_{\frac{R}{2}}}\frac{k^2(x)}{a(x)}(1+|Du|^2)^{\frac{2q-p}{2}}\,dx\\ &\le C(R)\int_{B_{\frac{R}{2}}} f(x,Du)\,dx + C\|1+|Du|\|_{L^\infty(B_{\frac{R}{2}})}^{2q-p}\int_{B_{\frac{R}{2}}}\frac{k^2(x)}{a(x)}\,dx\\ &\le C(R)\int_{B_{\frac{R}{2}}} f(x,Du)\,dx + C(R)\|1+|Du|\|_{L^\infty(B_{\frac{R}{2}})}^{2q-p}\Big(\int_{B_{\frac{R}{2}}}k^r(x)\,dx\Big)^{\frac{2}{r}}\Big(\int_{B_{\frac{R}{2}}}\frac{1}{a^s(x)}\,dx\Big)^{\frac{1}{s}},\end{aligned}$$

where we used Hölder's inequality since $\frac{1}{s}+\frac{2}{r}<1$ by assumptions. Using (3.5) to estimate the L^∞ norm of $|Du|$ and recalling the definition of $\mathcal{K}_{R_0}$ at (3.18), we get

$$\int_{B_\rho} a(x)(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx \le c\Big(\int_{B_R}1+f(x,Du)\,dx\Big)^{\tilde{\varrho}},$$

i.e. (3.6), with c depending on $p, r, s, n, \rho, R, \mathcal{K}_{R_0}$. □

4 Proof of Theorem 1.1

Using the previous results and an approximation procedure, we can prove of our main result.

Proof of Theorem 1.1. For $f(x,\xi)$ satisfying assumptions (1.3)–(1.6), let us introduce the sequence

$$f_h(x,\xi)=f(x,\xi)+\frac{1}{h}(1+|\xi|^2)^{\frac{ps}{2(s+1)}}.$$

Note that $f_h(x,\xi)$ satisfies the following set of conditions:

$$\frac{1}{h}(1+|\xi|^2)^{\frac{ps}{2(s+1)}} \le f_h(x,\xi) \le (1+L)(1+|\xi|^2)^{\frac{q}{2}}, \tag{4.1}$$

$$\frac{c_1}{h}(1+|\xi|^2)^{\frac{ps}{2(s+1)}-2}|\lambda|^2 \le \langle D_{\xi\xi}f_h(x,\xi)\lambda,\lambda\rangle, \tag{4.2}$$

$$|D_{\xi\xi}f_h(x,\xi)| \le c_2(1+L)(1+|\xi|^2)^{\frac{q-2}{2}}, \tag{4.3}$$

$$|D_{\xi x}f_h(x,\xi)| \le k(x)(1+|\xi|^2)^{\frac{q-1}{2}} \tag{4.4}$$

for some constants $c_1, c_2 > 0$, for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{N\times n}$.

Now, fix a ball $B_R \Subset \Omega$, and let $v_h \in W^{1,\frac{ps}{s+1}}(B_R,\mathbb{R}^N)$ be the unique solution to the problem

$$\min\Big\{\int_{B_R} f_h(x,Dv)\,dx : v_h \in u + W_0^{1,\frac{ps}{s+1}}(B_R,\mathbb{R}^N)\Big\}.$$

Since $f_h(x,\xi)$ satisfies (4.1), (4.2), (4.3) (4.4) with $k \in L^r$, $r > n$, and (1.6) holds, then by the result in [23], we have $v_h \in W^{1,\infty}_{\mathrm{loc}}(B_R)$. Therefore, it is legitimate to apply Proposition 3.2 to obtain that $V_p(Dv_h) \in W^{1,\frac{2s}{s+1}}_{\mathrm{loc}}(B_R)$ and $a(x)|D(V_p(Dv_h))|^2 \in L^1_{\mathrm{loc}}(B_R)$.

Since $f_h(x,\xi)$ satisfies (4.1), by the minimality of v_h, we get

$$\begin{aligned}
\int_{B_R}|Dv_h|^{\frac{ps}{s+1}}\,dx &\le c_s\int_{B_R} a(x)|Dv_h|^p + c_s\int_{B_R}\frac{1}{a^s(x)}\,dx \le c_s\int_{B_R} f_h(x,Dv_h)\,dx + c_s\int_{B_R}\frac{1}{a^s(x)}\,dx\\
&\le c_s\int_{B_R} f_h(x,Du)\,dx + c_s\int_{B_R}\frac{1}{a^s(x)}\,dx\\
&\le c_s\int_{B_R} f(x,Du)\,dx + \frac{c_s}{h}\int_{B_R}(1+|Du|)^{\frac{ps}{s+1}}\,dx + c_s\int_{B_R}\frac{1}{a^s(x)}\,dx\\
&\le c_s\int_{B_R} f(x,Du)\,dx + c_s\int_{B_R}(1+|Du|)^{\frac{ps}{s+1}}\,dx + c_s\int_{B_R}\frac{1}{a^s(x)}\,dx.
\end{aligned}$$

Therefore, the sequence v_h is bounded in $W^{1,\frac{ps}{s+1}}(B_R)$, so there exists $v \in u + W_0^{1,\frac{ps}{s+1}}(B_R)$ such that, up to subsequences,

$$v_h \rightharpoonup v \quad \text{weakly in } W^{1,\frac{ps}{s+1}}(B_R).$$

On the other hand, we can apply Theorem 3.3 to $f_h(x,\xi)$ since the assumptions are satisfied, with L replaced by $1+L$. Thus, it is legitimate to apply estimates (3.5) and (3.6) to the solutions v_h to obtain

$$\begin{aligned}
\|Dv_h\|_{L^\infty(B_\rho)} &\le C\mathcal{K}_R^{\tilde\vartheta}\Big(\int_{B_R}(1+f_h(x,Dv_h))\,dx\Big)^{\tilde\varsigma} \le C\mathcal{K}_R^{\tilde\vartheta}\Big(\int_{B_R}(1+f_h(x,Du))\,dx\Big)^{\tilde\varsigma}\\
&= C\mathcal{K}_R^{\tilde\vartheta}\Big(\int_{B_R}(1+f(x,Du)+\frac{1}{h}(1+|Du|^2)^{\frac{ps}{2(s+1)}})\,dx\Big)^{\tilde\varsigma}\\
&\le C\mathcal{K}_R^{\tilde\vartheta}\Big(\int_{B_R}(1+f(x,Du)+(1+|Du|^2)^{\frac{ps}{2(s+1)}})\,dx\Big)^{\tilde\varsigma},
\end{aligned} \tag{4.5}$$

with $C, \tilde\vartheta, \tilde\varsigma$ independent of h and $0<\rho<R$. Therefore, up to subsequences,

$$v_h \rightharpoonup v \quad \text{weakly* in } W^{1,\infty}(B_\rho). \tag{4.6}$$

Our next aim is to show that $v = u$. The lower semicontinuity of $u \mapsto \int_{B_R} f(x,Du)$ and the minimality of v_h imply

$$\begin{aligned}
\int_{B_R} f(x,Dv)\,dx &\le \liminf_h \int_{B_R} f(x,Dv_h)\,dx \le \liminf_h \int_{B_R} f_h(x,Dv_h)\,dx \le \liminf_h \int_{B_R} f_h(x,Du)\,dx\\
&= \liminf_h \int_{B_R}(f(x,Du)+\frac{1}{h}(1+|Du|^2)^{\frac{ps}{2(s+1)}})\,dx = \int_{B_R} f(x,Du)\,dx.
\end{aligned}$$

The strict convexity of f yields that $u = v$. Therefore, passing to the limit as $h \to \infty$ in (4.5), we get

$$\|Du\|_{L^\infty(B_\rho)} \le C\mathcal{K}_R^{\tilde{\vartheta}}\Big(\int_{B_R} (1 + f(x, Du) + (1 + |Du|^2)^{\frac{ps}{2(s+1)}})\,dx\Big)^{\tilde{\varsigma}},$$

i.e. (1.7). Moreover, it is legitimate to apply estimate (3.6) to each v_h, thus getting

$$\begin{aligned}\int_{B_\rho} a(x)(1 + |Dv_h|^2)^{\frac{p-2}{2}}|D^2 v_h|^2\,dx &\le c\Big(\int_{B_R} (1 + f_h(x, Dv_h))\,dx\Big)^{\tilde{\varrho}} \le c\Big(\int_{B_R} (1 + f_h(x, Du))\,dx\Big)^{\tilde{\varrho}} \\ &= c\Big(\int_{B_R}\Big(1 + f(x, Du) + \frac{1}{h}(1 + |Du|^2)^{\frac{ps}{2(s+1)}}\Big)\,dx\Big)^{\tilde{\varrho}},\end{aligned}$$

where we used the minimality of v_h and the definition of $f_h(x, \xi)$. By Remark 2.2, the above estimate implies that the sequence $x \mapsto \sqrt{a(x)}D(V_p(Dv_h))$ is bounded in the $L^2(B_\rho)$-norm. Using also (4.6) and that $v = u$, we get that, up to subsequences, this sequence converges in the weak topology of $L^2(B_\rho)$-norm to $\sqrt{a(x)}D(V_p(Du))$. By the lower semicontinuity of the L^2-norm with respect to the weak convergence, we conclude that

$$\begin{aligned}\int_{B_\rho} a(x)|D(V_p(Du))|^2\,dx &\le \liminf_h \int_{B_\rho} a(x)(1 + |Dv_h|^2)^{\frac{p-2}{2}}|D^2 v_h|^2\,dx \\ &\le c\liminf_h\Big(\int_{B_R}\Big(1 + f(x, Du) + \frac{1}{h}(1 + |Du|^2)^{\frac{ps}{2(s+1)}}\Big)\,dx\Big)^{\tilde{\varrho}} \\ &= \Big(\int_{B_R} (1 + f(x, Du))\,dx\Big)^{\tilde{\varrho}},\end{aligned}$$

i.e. (1.8). □

Acknowledgment: We thank the referee for carefully reading the manuscript, for the valuable remarks and suggestions. The authors are members of GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica), whose support is warmly acknowledged.

Funding: A. Passarelli di Napoli has been supported by Università di Napoli "Federico II" through the project FRA 000022--ALTRI_CDA_75_2021_FRA_PASSARELLI.

References

[1] A. K. Balci, L. Diening, R. Giova and A. Passarelli di Napoli, Elliptic equations with degenerate weights, preprint (2020), https://arxiv.org/abs/2003.10380v1.

[2] P. Baroni, M. Colombo and G. Mingione, Regularity for general functionals with double phase, *Calc. Var. Partial Differential Equations* **57** (2018), no. 2, Paper No. 62.

[3] P. Bella and M. Schäffner, Local boundedness and Harnack inequality for solutions of linear nonuniformly elliptic equations, *Comm. Pure Appl. Math.* **74** (2021), no. 3, 453–477.

[4] S. Biagi, G. Cupini and E. Mascolo, Regularity of quasi-minimizers for non-uniformly elliptic integrals, *J. Math. Anal. Appl.* **485** (2020), no. 2, Article ID 123838.

[5] G. Buttazzo and M. Belloni, A survey on old and recent results about the gap phenomenon in the calculus of variations, in: *Recent Developments in Well-Posed Variational Problems*, Math. Appl. 331, Kluwer Academic, Dordrecht (1995), 1–27.

[6] M. Carozza, G. Moscariello and A. Passarelli di Napoli, Higher integrability for minimizers of anisotropic functionals, *Discrete Contin. Dyn. Syst. Ser. B* **11** (2009), no. 1, 43–55.

[7] A. Cianchi and V. G. Maz'ya, Global Lipschitz regularity for a class of quasilinear elliptic equations, *Comm. Partial Differential Equations* **36** (2011), no. 1, 100–133.

[8] M. Colombo and G. Mingione, Bounded minimisers of double phase variational integrals, *Arch. Ration. Mech. Anal.* **218** (2015), no. 1, 219–273.

[9] M. Colombo and G. Mingione, Regularity for double phase variational problems, *Arch. Ration. Mech. Anal.* **215** (2015), no. 2, 443–496.

[10] D. Cruz-Uribe, P. Di Gironimo and C. Sbordone, On the continuity of solutions to degenerate elliptic equations, *J. Differential Equations* **250** (2011), no. 6, 2671–2686.

[11] G. Cupini, F. Giannetti, R. Giova and A. Passarelli di Napoli, Regularity results for vectorial minimizers of a class of degenerate convex integrals, *J. Differential Equations* **265** (2018), no. 9, 4375–4416.

[12] G. Cupini, M. Guidorzi and E. Mascolo, Regularity of minimizers of vectorial integrals with p-q growth, *Nonlinear Anal.* **54** (2003), no. 4, 591–616.

[13] G. Cupini, P. Marcellini and E. Mascolo, Local boundedness of solutions to quasilinear elliptic systems, *Manuscripta Math.* **137** (2012), no. 3–4, 287–315.

[14] G. Cupini, P. Marcellini and E. Mascolo, Existence and regularity for elliptic equations under p, q-growth, *Adv. Differential Equations* **19** (2014), 693–724.

[15] G. Cupini, P. Marcellini and E. Mascolo, Regularity of minimizers under limit growth conditions, *Nonlinear Anal.* **153** (2017), 294–310.

[16] G. Cupini, P. Marcellini and E. Mascolo, Nonuniformly elliptic energy integrals with p, q-growth, *Nonlinear Anal.* **177** (2018), 312–324.

[17] C. De Filippis and G. Mingione, On the regularity of minima of non-autonomous functionals, *J. Geom. Anal.* **30** (2020), no. 2, 1584–1626.

[18] C. De Filippis and J. Oh, Regularity for multi-phase variational problems, *J. Differential Equations* **267** (2019), no. 3, 1631–1670.

[19] E. De Giorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, *Boll. Unione Mat. Ital. (4)* **1** (1968), 135–137.

[20] T. Di Marco and P. Marcellini, A-priori gradient bound for elliptic systems under either slow or fast growth conditions, *Calc. Var. Partial Differential Equations* **59** (2020), no. 4, Paper No. 120.

[21] F. Duzaar and G. Mingione, Local Lipschitz regularity for degenerate elliptic systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), no. 6, 1361–1396.

[22] M. Eleuteri, P. Marcellini and E. Mascolo, Lipschitz continuity for energy integrals with variable exponents, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **27** (2016), no. 1, 61–87.

[23] M. Eleuteri, P. Marcellini and E. Mascolo, Lipschitz estimates for systems with ellipticity conditions at infinity, *Ann. Mat. Pura Appl. (4)* **195** (2016), no. 5, 1575–1603.

[24] M. Eleuteri, P. Marcellini and E. Mascolo, Regularity for scalar integrals without structure conditions, *Adv. Calc. Var.* **13** (2020), no. 3, 279–300.

[25] M. Eleuteri, P. Marcellini, E. Mascolo and S. Perrotta, Local Lipschitz continuity for energy integrals with slow growth, *Ann. Mat. Pura Appl. (4)* (2021), DOI 10.1007/s10231-021-01147-w.

[26] A. Esposito, F. Leonetti and P. V. Petricca, Absence of Lavrentiev gap for non-autonomous functionals with (p, q)-growth, *Adv. Nonlinear Anal.* **8** (2019), no. 1, 73–78.

[27] L. Esposito, F. Leonetti and G. Mingione, Regularity results for minimizers of irregular integrals with (p, q) growth, *Forum Math.* **14** (2002), no. 2, 245–272.

[28] L. Esposito, F. Leonetti and G. Mingione, Sharp regularity for functionals with (p, q) growth, *J. Differential Equations* **204** (2004), no. 1, 5–55.

[29] E. B. Fabes, C. E. Kenig and R. P. Serapioni, The local regularity of solutions of degenerate elliptic equations, *Comm. Partial Differential Equations* **7** (1982), no. 1, 77–116.

[30] R. Giova and A. Passarelli di Napoli, Regularity results for a priori bounded minimizers of non-autonomous functionals with discontinuous coefficients, *Adv. Calc. Var.* **12** (2019), no. 1, 85–110.

[31] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, River Edge, 2003.

[32] T. Iwaniec, L. Migliaccio, G. Moscariello and A. Passarelli di Napoli, A priori estimates for nonlinear elliptic complexes, *Adv. Differential Equations* **8** (2003), no. 5, 513–546.

[33] T. Iwaniec and C. Sbordone, Quasiharmonic fields, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **18** (2001), no. 5, 519–572.

[34] P. Marcellini, Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals, *Manuscripta Math.* **51** (1985), no. 1–3, 1–28.

[35] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions, *Arch. Ration. Mech. Anal.* **105** (1989), no. 3, 267–284.

[36] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Differential Equations* **90** (1991), no. 1, 1–30.

[37] P. Marcellini, Everywhere regularity for a class of elliptic systems without growth conditions, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **23** (1996), no. 1, 1–25.

[38] P. Marcellini, A variational approach to parabolic equations under general and p, q-growth conditions, *Nonlinear Anal.* **194** (2020), Article ID 111456.

[39] P. Marcellini, Regularity under general and p, q-growth conditions, *Discrete Contin. Dyn. Syst. Ser. S* **13** (2020), no. 7, 2009–2031.

[40] P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, *J. Math. Anal. Appl.* **501** (2021), no. 1, Article ID 124408.
[41] P. Marcellini and G. Papi, Nonlinear elliptic systems with general growth, *J. Differential Equations* **221** (2006), no. 2, 412–443.
[42] G. Mingione, Regularity of minima: An invitation to the dark side of the calculus of variations, *Appl. Math.* **51** (2006), no. 4, 355–426.
[43] C. Mooney and O. Savin, Some singular minimizers in low dimensions in the calculus of variations, *Arch. Ration. Mech. Anal.* **221** (2016), no. 1, 1–22.
[44] J. Moser, A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations, *Comm. Pure Appl. Math.* **13** (1960), 457–468.
[45] A. Passarelli di Napoli, Higher differentiability of minimizers of variational integrals with Sobolev coefficients, *Adv. Calc. Var.* **7** (2014), no. 1, 59–89.
[46] A. Passarelli di Napoli, Higher differentiability of solutions of elliptic systems with Sobolev coefficients: The case $p = n = 2$, *Potential Anal.* **41** (2014), no. 3, 715–735.
[47] A. Passarelli di Napoli, Regularity results for non-autonomous variational integrals with discontinuous coefficients, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **26** (2015), no. 4, 475–496.
[48] M. Pingen, Regularity results for degenerate elliptic systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25** (2008), no. 2, 369–380.
[49] V. Šverák and X. Yan, A singular minimizer of a smooth strongly convex functional in three dimensions, *Calc. Var. Partial Differential Equations* **10** (2000), no. 3, 213–221.
[50] N. S. Trudinger, On the regularity of generalized solutions of linear, non-uniformly elliptic equations, *Arch. Rational Mech. Anal.* **42** (1971), 50–62.
[51] N. S. Trudinger, Linear elliptic operators with measurable coefficients, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **27** (1973), 265–308.
[52] V. V. Zhikov, On Lavrentiev's phenomenon, *Russian J. Math. Phys.* **3** (1995), no. 2, 249–269.