BIRATIONAL GEOMETRY OF MODULI OF CURVES WITH AN S_3-COVER

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ABSTRACT. We consider the space $\mathcal{R}_{g,S_3}^{S_3}$ of curves with a connected S_3-cover, proving that for any odd genus $g \geq 13$ this moduli is of general type. Furthermore we develop a set of tools that are essential in approaching the case of G-covers for any finite group G.

Keywords: curves; moduli; covers; spin curves; principal bundles; admissible covers; twisted covers.

1. INTRODUCTION

The goal of this paper, alongside its companion [20], is to analyze the birational geometry of the moduli space of curves equipped with a G-cover, where G is any finite group. More specifically, here we prove that the moduli space $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ of genus g connected twisted S_3-covers is of general type for any odd genus $g \geq 13$.

In a series of seminal works, Eisenbud, Harris and Mumford proved that the moduli $\mathcal{M}_g$ of genus g smooth curves is a variety of general type for any genus $g > 23$ (see [22, 21, 13]). The behavior is different for low genus, and in fact there is a large literature showing that $\mathcal{M}_g$ is unirational for $g \leq 14$ and rationally connected for $g = 15$ (see for instance [3, 30, 8]). Recently, Farkas, Jensen and Payne showed that also $\mathcal{M}_{22}$ and $\mathcal{M}_{23}$ are of generaly type [15].

Many finite covers of $\mathcal{M}_g$ with a modular interpretation, have the same behavior: there is a transition genus such that for any higher value the variety is of general type. The interesting property is that in many cases the transition genus is strictly lower. Farkas and Ludwig proved in [16] that the space $\mathcal{R}_{g,\boldsymbol{\mu}_2}^{\boldsymbol{\mu}_2}$ of curves with a non-trivial 2-torsion bundle is of general type for any genus $g \geq 14$ and $g \neq 15$; Chiodo, Eisenbud, Farkas and Schreyer (see [11]) proved the same in the case of 3-torsion bundles for $g \geq 12$; Farkas and Verra approached in [18] the case of odd spin curves, which is of general type for $g \geq 12$.

In this paper we focus on the moduli $\mathcal{R}_{g,G}$ of curves with a principal G-bundle for G finite group. We build its compactification $\overline{\mathcal{R}}_{g,G}$ as the moduli of twisted G-covers, that is balanced representable maps $\phi \colon \mathsf{C} \to BG$ where C is a Deligne-Mumford stack such that its coarse space is a stable curve C, with non-trivial stabilizer only at some nodes. We properly introduce twisted G-covers in §2.1, and in §2.2 we recall the equivalence with the notion of admissible G-cover. In order to evaluate the Kodaira dimension of $\overline{\mathcal{R}}_{g,G}$ it is necessary to consider a desingularization $\widehat{\mathcal{R}}_{g,G} \to \overline{\mathcal{R}}_{g,G}$, but it is difficult to evaluate the dimension of the space of n-canonical sections $H^0(\widehat{\mathcal{R}}_{g,G}, nK)$. Following the approach of [22], we intend to focus in those cases where it is the same to evaluate the dimension of the space of n-canonical sections over the regular locus of $\overline{\mathcal{R}}_{g,G}$, that is those cases such that

$$H^0\left(\widehat{\mathcal{R}}_{g,G}, nK_{\widehat{\mathcal{R}}_{g,G}}\right) = H^0\left(\overline{\mathcal{R}}_{g,G}^{\mathrm{reg}}, nK_{\overline{\mathcal{R}}_{g,G}^{\mathrm{reg}}}\right).$$

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A pluricanonical form defined locally over $\overline{\mathcal{R}}_{g,G}^{\mathrm{reg}}$, extends uniquely to $\widehat{\mathcal{R}}_{g,G}$ in smooth points and in the so called canonical singularities (for a treatment of canonical and non-canonical singularities see [27]). Therefore, we need an analysis of the locus of non-canonical singularities, as done by Chiodo and Farkas in [12] for level ℓ curves (that is curves with an ℓ-torsion bundle), and by the author in [19] for the moduli space $\mathcal{R}_{g,\ell}^k$ with a line bundle L such that $L^{\otimes \ell} \cong \omega^{\otimes k}$.

We search for a result allowing a global unique extension of pluricanonical forms. Consider the natural project $\pi \colon \overline{\mathcal{R}}_{g,G} \to \overline{\mathcal{M}}_g$, we know from the author work [20] that the locus of non-canonical singularities is bipartitioned as $\mathrm{Sing}^{\mathrm{nc}}\overline{\mathcal{R}}_{g,G} = T_{g,G} \cup J_{g,G}$, where the T-locus is defined as $T_{g,G} := \mathrm{Sing}^{\mathrm{nc}}\overline{\mathcal{R}}_{g,G} \cap \pi^{-1} \mathrm{Sing}^{\mathrm{nc}}\overline{\mathcal{M}}_g$, while $J_{g,G}$ is the locus of "new" singularities that are not in the preimage of $\overline{\mathcal{M}}_g$ singularities. We recall that the J-locus is empty for $\overline{\mathcal{R}}_{g,S_3}$, a fact proven in [20, Theorem 5.2.1].

In Theorems 4.1.2 and 4.2.5 we prove the following.

Theorem. *Consider a desingularization $\widehat{\mathcal{R}}_{g,G} \to \overline{\mathcal{R}}_{g,G}$. If G is abelian and $J_{g,G} = \varnothing$, then*

$$H^0\left(\widehat{\mathcal{R}}_{g,G}, nK_{\widehat{\mathcal{R}}_{g,G}}\right) = H^0\left(\overline{\mathcal{R}}_{g,G}^{\mathrm{reg}}, nK_{\overline{\mathcal{R}}_{g,G}^{\mathrm{reg}}}\right)$$

for n sufficiently big and divisible. The same is true when $G = S_3$ is the symmetric group of order 3.

The extension result is obtained via a non-trivial generalization of Harris-Mumford techniques used in [22], which is particularly tricky in the case $G = S_3$ because the covering $T_{g,S_3} \to \mathrm{Sing}^{\mathrm{nc}}\overline{\mathcal{M}}_g$ is non-étale.

Finally, the main result of this paper concerns the component of $\overline{\mathcal{R}}_{g,S_3}$ parametrizing connected twisted S_3-covers, that we denote by $\overline{\mathcal{R}}_{g,S_3}^{S_3}$. The other components are finite covers of the well known moduli of level curves of order 2 and 3. In order to prove the bigness of the canonical divisor of $\overline{\mathcal{R}}_{g,S_3}^{S_3}$, for every odd genus $g = 2i+1$ we write it down as a linear combination

$$K_{\overline{\mathcal{R}}_{g,S_3}^{S_3}} = \alpha \cdot \overline{\mathcal{U}}_g + \beta \cdot (\pi^* \mathcal{M}_{g,i+1}^1) + E + \gamma \cdot \lambda \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{R}}_{g,S_3}^{S_3}),$$

where α, β, γ are real coefficients, E is a boundary effective divisor and λ is the Hodge class, which is known to be big. Furthermore, the divisor $\mathcal{U}_g$ is the jumping locus for the Koszul cohomology $K_{i,1}$ of a particular rank 2 vector bundle (see Definition 5.2.4), its effectiveness is proved in Proposition 5.2.6, while $\pi^* \mathcal{M}_{g,i+1}^1$ is the lifting of an effective $\mathcal{M}_g$ divisor introduced by Harris and Mumford (see Definition 5.2.12). Therefore $\overline{\mathcal{U}}_g$ and $\pi^* \mathcal{M}_{g,i+1}^1$ are both effective. In Proposition 5.2.13 we prove that the coefficients α, β, γ can be chosen all positive for $i > 5$, and this implies the bigness of λ and the space $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ being of general type.

Theorem. *The moduli space $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ is of general type for every odd genus $g \geq 13$.*

In §2 we introduce the compactification of $\mathcal{R}_{g,G}$ via twisted G-covers and admissible G-covers. In §3 we describe the boundary of $\overline{\mathcal{R}}_{g,S_3}$ and evaluate its canonical divisor. In §4 the extension of pluricanonical forms is shown and in §5 we build the Koszul divisor and conclude the proof.

2. Twisted and admissible covers

In this section we recall the notions of twisted G-cover and admissible G-cover, and their equivalence in Theorem 2.2.15. In the following of the manuscript, we will adopt preferably the twisted cover notation in Section 4 and 5, and the admissible cover notation in Section 3.

In §2.2 we also introduce the notion of admissible $\mathcal{H}$-cover, where $\mathcal{H}$ is any subgroup conjugacy class of G. This concept is central in classifying the components and boundary divisors of any moduli space $\overline{\mathcal{R}}_{g,G}$, as we do in Section 3, in particular for the case $G = S_3$. In order to better understand these subjects, we also recall a series of group action tools.

Finally, in §2.3 we introduce a description of admissible G-covers via a monodromy-type approach. This description allows to treat admissible G-covers as particular classes of group morphisms.

2.1. **Introduction to twisted G-covers.**

Definition 2.1.1 (Twisted curve). A twisted n-marked S-curve is a diagram

$$\Sigma_1, \Sigma_2, \ldots, \Sigma_n \quad \subset \quad \begin{array}{c} \mathsf{C} \\ \downarrow \\ C \\ \downarrow \\ S. \end{array}$$

(1) C is a Deligne-Mumford stack, proper over S, and étale locally it is a nodal curve over S;
(2) the $\Sigma_i \subset \mathsf{C}$ are disjoint closed substacks in the smooth locus of $\mathsf{C} \to S$ for all i;
(3) $\Sigma_i \to S$ is an étale gerbe for all i;
(4) $\mathsf{C} \to C$ exhibits C as the coarse space of C, and it is an isomorphism over C_{gen}.

We recall that, given a scheme U and a finite abelian group $\boldsymbol{\mu}$ acting on U, the stack $[U/\boldsymbol{\mu}]$ is the category of principal $\boldsymbol{\mu}$-bundles $E \to T$, for any scheme T, equipped with a $\boldsymbol{\mu}$-equivariant morphism $f \colon E \to U$. The stack $[U/\boldsymbol{\mu}]$ is a proper Deligne-Mumford stack and has a natural morphism to its coarse scheme $U/\boldsymbol{\mu}$.

By the definition of twisted curve we get the local pictures:
- *At a marking*, morphism $\mathsf{C} \to C \to S$ is locally isomorphic to

$$\left[\operatorname{Spec} A[x']/\boldsymbol{\mu}_r\right] \to \operatorname{Spec} A[x] \to \operatorname{Spec} A$$

  for some normal ring A and some integer $r > 0$. Here $x = (x')^r$, and $\boldsymbol{\mu}_r$ is the cyclic group of order r acting on $\operatorname{Spec} A[x']$ by the action $\xi \colon x' \mapsto \xi x'$ for any $\xi \in \boldsymbol{\mu}_r$.

- *At a node*, morphism $\mathsf{C} \to C \to S$ is locally isomorphic to

$$\left[\mathrm{Spec}\left(\frac{A[x', y']}{(x'y' - a)} \right) / \boldsymbol{\mu}_r \right] \to \mathrm{Spec}\left(\frac{A[x, y]}{(xy - a^\ell)} \right) \to \mathrm{Spec}\, A$$

  for some integer $r > 0$ and $a \in A$. Here $x = (x')^\ell$, $y = (y')^\ell$. The group $\boldsymbol{\mu}_r$ acts by the action

$$\xi \colon (x', y') \mapsto (\xi x', \xi^m y')$$

  where m is an element of $\mathbb{Z}/r$ and ξ is a primitive rth root of the unit. The action is called *balanced* if $m \equiv -1 \mod r$. A curve with balanced action at every node is called a balanced curve.

Definition 2.1.2 (Twisted G-cover)**.** Given an n-marked twisted balanced curve $(\Sigma_1, \dots, \Sigma_n; \mathsf{C} \to C \to S)$, a twisted G-cover is a representable stack morphism $\phi \colon \mathsf{C} \to BG$, *i.e.* an object of the category $\mathrm{Fun}(\mathsf{C}, BG)$ which moreover is representable.

We observe that by adding the balancing hypothesis, we are using a slightly different notion of twisted G-cover with respect to [1].

Definition 2.1.3. We consider the category $\mathcal{B}^{\mathrm{bal}}_{g,n}(G)$. The objects of $\mathcal{B}^{\mathrm{bal}}_{g,n}(G)$ are twisted n-marked S-curves of genus g with a twisted G-cover, for any scheme S.

Consider two twisted G-covers $\phi' \colon \mathsf{C}' \to BG$ and $\phi \colon \mathsf{C} \to BG$ over the twisted n-marked curves C' and C respectively. A morphism $(\mathsf{C}', \phi') \to (\mathsf{C}, \phi)$ is a pair (f, α) such that $\mathsf{f} \colon \mathsf{C}' \to \mathsf{C}$ is a morphism of n-marked twisted curves, and $\alpha \colon \phi' \to \phi \circ \mathsf{f}$ is an isomorphism in $\mathrm{Fun}(\mathsf{C}', BG)$.

2.2. Introduction to admissible G-covers.

Definition 2.2.1 (Admissible cover)**.** Given a nodal S-curve $X \to S$ with marked points, an admissible cover $u \colon F \to X$ is a morphism such that:

(1) the composition $F \to S$ is a nodal S-curve;
(2) given a geometric point $\bar{s} \in S$, every node of $F_{\bar{s}}$ maps via u to a node of $X_{\bar{s}}$;
(3) the restriction $F|_{X_{\mathrm{gen}}} \to X_{\mathrm{gen}}$ is an étale cover of degree d;
(4) given a geometric point $\bar{s} \in S$, the local picture of $F_{\bar{s}} \xrightarrow{u} X_{\bar{s}}$ at a point of $F_{\bar{s}}$ mapping to a marked point of X is isomorphic to

$$\mathrm{Spec}\, A[x'] \to \mathrm{Spec}\, A[x] \to \mathrm{Spec}\, A,$$

  for some normal ring A, an integer $r > 0$ and $u^* x = (x')^r$;
(5) the local picture of $F_{\bar{s}} \xrightarrow{u} X_{\bar{s}}$ at a node of $F_{\bar{s}}$ is isomorphic to

$$\mathrm{Spec}\left(\frac{A[x', y']}{(x'y' - a)} \right) \to \mathrm{Spec}\left(\frac{A[x, y]}{(xy - a^r)} \right) \to \mathrm{Spec}\, A,$$

  for some integer $r > 0$ and an element $a \in A$, $u^* x = (x')^r$ and $u^* y = (y')^r$.

The category $\mathrm{Adm}_{g,n,d}$ of n-pointed stable curves of genus g with an admissible cover of degree d, is a proper Deligne-Mumford stack.

Consider $F \to C$ an admissible cover of a nodal curve C, a G-action on F such that the restriction $F|_{C_{\mathrm{gen}}} \to C_{\mathrm{gen}}$ is a principal G-bundle, a smooth point p of C and a preimage $\tilde{p} \in F$ of p. We denote by $H_{\tilde{p}} \subset G$ the stabilizer of $\tilde{p}$. The G-action induces a primitive character $\chi_{\tilde{p}} \colon H_{\tilde{p}} \to \mathrm{GL}(T_{\tilde{p}} F) = \mathbb{C}^*$. In the set of pairs (H, χ), with H a G subgroup and $\chi \colon H \to \mathbb{C}^*$ a character, we introduce the equivalence relation $(H, \chi) \sim (H', \chi')$ if and only if there exists $s \in G$ such that $H' = sHs^{-1}$ and $\chi' = \chi^s$, where $\chi^s \colon h \mapsto \chi(s^{-1}hs)$ is the conjugated

character to χ. Consider a point $\tilde{p}$ on F with stabilizer $H_{\tilde{p}}$ and associated character $\chi_{\tilde{p}}$. We observe that for any point $s \cdot \tilde{p}$ of the same fiber, $H_{s \cdot \tilde{p}} = s H_{\tilde{p}} s^{-1}$ and $\chi_{s \cdot \tilde{p}} = \chi_{\tilde{p}}^s$. Therefore the equivalence class of the pair $(H_{\tilde{p}}, \chi_{\tilde{p}})$ only depends on the point p.

Definition 2.2.2. For any smooth point $\tilde{p}$ on F, we call *local index* the associated pair $(H_{\tilde{p}}, \chi_{\tilde{p}})$. For any smooth point $p \in C$, the conjugacy class of the local index of any $\tilde{p}$ in F_p is called the *G-type* at p, following the notation in [7]. We denote the G-type by $[\![H_p, \chi_p]\!]$, where H_p is the stabilizer of one of the points in F_p, and χ_p the associated character.

Lemma 2.2.3 (see [20, Lemma 2.3.3]). *Consider $u \colon F \to C$ an admissible cover of a nodal curve C such that the restriction $F|_{C_{\mathrm{gen}}} \to C_{\mathrm{gen}}$ is a principal G-bundle. If $\tilde{p} \in F$ is one of the preimages of a node or a marked point, then the stabilizer $H_{\tilde{p}}$ is a cyclic group.*

Observe that the set of characters $\chi \colon \boldsymbol{\mu}_r \to \mathbb{C}^*$ of a cyclic group, is the group $\mathbb{Z}/r\mathbb{Z}$. In particular, the character associated to $k \in \mathbb{Z}/r\mathbb{Z}$ maps $\xi \mapsto \xi^k$ for any ξ rth root of the unit. In the case of a node $\tilde{p} \in F$, we observe that $H_{\tilde{p}}$ acts independently on the two branches U_1 and U_2. We denote by $\chi_{\tilde{p}}^{(1)}$ and $\chi_{\tilde{p}}^{(2)}$ the characters of these actions.

Definition 2.2.4. The G-action at node $\tilde{p}$ is balanced when $\chi_{\tilde{p}}^{(1)} = -\chi_{\tilde{p}}^{(2)}$, that is they are opposite as elements of $\mathbb{Z}/r\mathbb{Z}$.

Definition 2.2.5 (Admissible G-cover). Take G finite group, an admissible cover $u \colon F \to C$ of a nodal curve C is an admissible G-cover if

(1) the restriction $u|_{C_{\mathrm{gen}}} \colon F|_{C_{\mathrm{gen}}} \to C_{\mathrm{gen}}$ is a principal G-bundle. This implies, by Lemma 2.2.3, that for every node or marked point $\tilde{p} \in F$, the stabilizer $H_{\tilde{p}}$ is a cyclic group;
(2) the action of $H_{\tilde{p}}$ is balanced for every node $\tilde{p} \in F$.

This notion was developed by Abramovich, Corti and Vistoli in [1], and also by Jarvis, Kaufmann and Kimura in [23].

Definition 2.2.6. We call $\mathrm{Adm}_{g,n}^G$ the stack of stable curves of genus g with n marked points and equipped with an admissible G-cover.

Remark 2.2.7. For any cyclic subgroup $H \subset G$, we choose the privileged root $\exp(2\pi i/|H|)$. After this choice, The datum of (H, χ), is equivalent to the datum of the H generator $h = \chi^{-1}(e^{2\pi i/|H|})$. As a consequence, the conjugacy class $[\![H, \chi]\!]$ is identified with the conjugacy class $[\![h]\!]$ of h in G.

Definition 2.2.8. Given an admissible G-cover $F \to C$ over an n-marked stable curve, the series $[\![h_1]\!]$, $[\![h_2]\!]$, ..., $[\![h_n]\!]$, of the G-types of the singular fibers over the marked points, is called *Hurwitz datum* of the cover. The stack of admissible G-covers of genus g with a given Hurwitz datum is denoted by $\mathrm{Adm}_{g,[\![h_1]\!],\dots,[\![h_n]\!]}^G$.

Remark 2.2.9. Given an admissible G-cover $F \to C$, if p is a node of C and $\tilde{p}$ one of its preimages on F, then the local index of $\tilde{p}$ and the G-type of p are well defined once we fix a privileged branch of p. Switching the branches sends the local index and the G-type in their inverses.

We denote by $\mathcal{T}(F)$ the set of connected components of an admissible G-cover $F \to C$, and naturally it inherits the G-cover action ψ. The conjugacy class of the stabilizer $H_E \subset G$ of any connected component E, is independent of the choice of E.

We call $\mathcal{T}(G)$ the set of equivalence classes in $\mathrm{Sub}(G)$ with respect to conjugation. Then, for every admissible G-cover there exists a canonical class $\mathcal{H}$ in $\mathcal{T}(G)$ and a canonical surjective map $\mathcal{T}(F) \twoheadrightarrow \mathcal{H}$ sending any component in its stabilizer. About the theory of group actions that we use in this work, we also refer to [20, §2.1.2].

Definition 2.2.10. Consider two subgroup conjugacy classes $\mathcal{H}_1, \mathcal{H}_2$ in $\mathcal{T}(G)$, we say that $\mathcal{H}_2$ is a subclass of $\mathcal{H}_1$, denoted by $\mathcal{H}_2 \leq \mathcal{H}_1$, if for one element $H_2 \in \mathcal{H}_2$ (and hence for all), there exists $H_1 \in \mathcal{H}_1$ such that H_2 is a subgroup of H_1. If the inclusion is strict, then $\mathcal{H}_2$ is a strict subclass of $\mathcal{H}_1$ and the notation is $\mathcal{H}_2 < \mathcal{H}_1$.

Definition 2.2.11. We denote by $\mathrm{Hom}^G(\mathcal{T}(F), G)$ the set of maps $v \colon \mathcal{T}(F) \to G$ such that $v(\psi(g, E)) = g \cdot v(E) \cdot g^{-1}$.

Definition 2.2.12. Consider a subgroup conjugacy class $\mathcal{H}$ of G. An admissible $\mathcal{H}$-cover is an admissible G-cover such that every connected component has stabilizer in $\mathcal{H}$.

Definition 2.2.13. We denote by $\mathrm{Adm}_g^{G,\mathcal{H}}$ the stack of admissible $\mathcal{H}$-covers over stable curves of genus g, and we denote by $\mathrm{Adm}_{g,[\![h_1]\!],\ldots,[\![h_n]\!]}^{G,\mathcal{H}}$ the stack of admissible $\mathcal{H}$-cover with Hurwitz datum $[\![h_1]\!], \ldots, [\![h_n]\!]$ over the n marked points.

Proposition 2.2.14 (see in [20, Proposition 2.3.14]). *Consider* $(C; p_1, \ldots, p_n)$ *a nodal n-marked curve, and $F \to C$ an admissible G-cover, then*

$$\mathrm{Aut}_{\mathrm{Adm}}(C, F) = \mathrm{Hom}^G(\mathcal{T}(F), G).$$

We introduced the two categories $\mathcal{B}_g^{\mathrm{bal}}(G)$ (see Definition 2.1.3) and Adm_g^G with the purpose of "well" defining the notion of principal G-bundle over stable non-smooth curves. These two categories are proven isomorphic in [1].

Theorem 2.2.15 (see [1, Theorem 4.3.2]). *There exists a base preserving equivalence between $\mathcal{B}_g^{\mathrm{bal}}(G)$ and Adm_g^G, therefore in particular they are isomorphic Deligne-Mumford stacks.*

Remark 2.2.16. From now on we will use the notation $\overline{\mathbf{R}}_{g,G} = \mathcal{B}_g^{\mathrm{bal}}(G) = \mathrm{Adm}_g^G$ for the moduli stack of curves of genus g equipped with an admissible G-cover, and $\overline{\mathcal{R}}_{g,G}$ for its coarse space. Analogously, we will use the notation $\overline{\mathbf{R}}_{g,G}^{\mathcal{H}} = \mathrm{Adm}_g^{G,\mathcal{H}}$, and $\overline{\mathcal{R}}_{g,G}^{\mathcal{H}}$, for the moduli stack of admissible $\mathcal{H}$-covers over stable curves of genus g, and its coarse space.

Remark 2.2.17. In the following, we will say that a twisted G-cover (C, ϕ) "is" an admissible G-cover $F \to C$ (or the other way around), meaning that $F \to C$ is the naturally associated admissible G-cover to (C, ϕ).

Remark 2.2.18. We recall the local description of $\overline{\mathcal{R}}_{g,G}$, following [20, Remark 2.2.8]. For any twisted G-cover (C, ϕ), the local picture of $\overline{\mathcal{R}}_{g,G}$ at $[\mathsf{C}, \phi]$ is $\mathrm{Def}(\mathsf{C}, \phi)/\mathrm{Aut}(\mathsf{C}, \phi)$, where the action of the automorphism group is induced by the universal property. If we consider the universal deformation $\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\,\mathsf{C})$ of C alongside with its nodes, this is naturally identified with $\mathrm{Def}(C; \mathrm{Sing}\,C)$ where C is the coarse space of C. If $C_1, \ldots, C_V$ are the irreducible components of C, $\overline{C}_i$ their normalizations, $D_i \subset \overline{C}_i$ the divisors of the the preimages of the nodes, then

$$\mathrm{Def}(\mathsf{C}; \mathrm{Sing}\,\mathsf{C}) = \mathrm{Def}(C; \mathrm{Sing}\,C) = \bigoplus_{i=1}^{V} \mathrm{Def}(\overline{C}_i; D_i) = \bigoplus_{i=1}^{V} H^1(\overline{C}_i, T_{\overline{C}_i}(-D_i)).$$

We denote by $q_1, \ldots, q_\delta$ the nodes of C, and we observe that we have a canonical splitting

$$\operatorname{Def}(\mathsf{C}) / \operatorname{Def}(\mathsf{C}; \operatorname{Sing} \mathsf{C}) = \bigoplus_{j=1}^{\delta} R_j.$$

For every j, $R_j \cong \mathbb{A}^1$. These are the universal deformations (or smoothings) of nodes q_j with the associated stabilizers at C. We recall that for the coarse curve C, we have the splitting $\operatorname{Def}(C) / \operatorname{Def}(C; \operatorname{Sing} C) = \bigoplus_{j=1}^{\delta} M_j$, where again $M_j \cong \mathbb{A}^1$ and there exists a canonical morphisms $R_j \to M_j$ of degree r_j, the cardinality of the q_j stabilizer, and branched at the origin. In particular we call t_j the coordinate of M_j, and $\tilde{t}_j$ the coordinate of R_j such that $t_j = (\tilde{t}_j)^{r_j}$.

2.3. Monodromy description of admissible G-covers.

Consider a smooth curve C of genus g and n marked points $p_1, \ldots, p_n$, the fundamental group of $C_{\mathrm{gen}} = C \backslash \{p_1, \ldots, p_n\}$ has $2g + n$ generators $\alpha_1, \alpha_2, \ldots \alpha_g, \beta_1 \ldots, \beta_g, \gamma_1, \ldots, \gamma_n$. These are represented in the figure below, where the arrows with the same label are identified respecting the orientation. As we can also see in the figure, these generators respect the following relation,

$$(2.1) \qquad \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} \cdot \gamma_1 \cdots \gamma_n = 1.$$

This is called the canonical representation of the fundamental group of a genus g smooth curve.
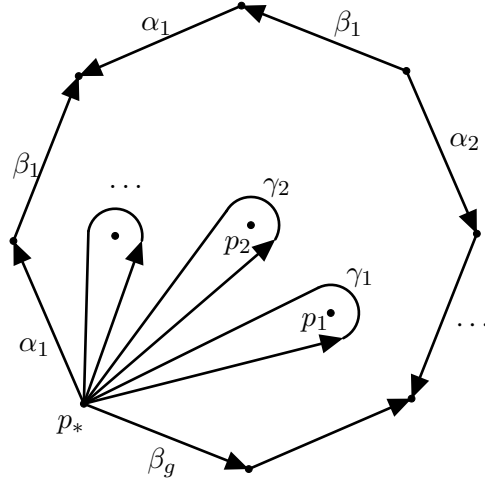
It is possible to describe admissible G-covers over smooth curves by the monodromy action, as done for example in [7, §2.3] and [28, §3.5]. Consider a smooth curve C, a generic point p_* on it and an admissible G-cover $F \to C$. We denote the points of the fiber F_{p_*} by $\tilde{p}_*^{(g)}$ for any $g \in G$, in such a way that $g \cdot \tilde{p}_*^{(1)} = \tilde{p}_*^{(g)}$. This induces a group morphism $\pi_1(C_{\mathrm{gen}}, p_*) \to G$. This monodromy morphism is well defined up to relabelling the points $\tilde{p}_*^{(g)}$, i.e. up to G conjugation. The following proposition is a rephrasing of [7, Lemma 2.6].

Proposition 2.3.1. *Given a smooth n-marked curve $(C; p_1, \ldots, p_n)$ and a point p_* on its generic locus C_{gen}, the set of isomorphism classes of admissible G-covers on C is naturally in bijection with the set of conjugacy classes of maps*

$$\varpi \colon \pi_1(C_{\mathrm{gen}}, p_*) \to G.$$

Remark 2.3.2. We also point out that, as represented in the figure, the monodromy of γ_i at any point $p_*^{(g)}$, with $g \in G$, is given by a small circular lacet around the deleted point p_i. Therefore by definition of G-type, if $[\![h_i]\!]$ is the G-type of p_i, then $[\![\varpi(\gamma_i)]\!] = [\![h_i]\!]$.

We consider now the case of an irreducible curve C with one type 0 node, *i.e.* an autointersection node. If $\overline{C}$ is the C normalization, we denote by p_1 and p_2 the preimages of the node on $\overline{C}$. An admissible G-cover $F \to C$ induces an admissible G-cover $\overline{F} \to \overline{C}$ on the 2-marked genus $g-1$ curve $(\overline{C}; p_1, p_2)$ and by what we said above, there exists an associated morphism $\varpi \colon \pi_1(\overline{C}_{\mathrm{gen}}) \to G$ such that $[\![\varpi(\gamma_1)]\!] = [\![\varpi(\gamma_2)^{-1}]\!]$. As we obtain C from $\overline{C}$ by gluing p_1 to p_2, F is obtained by gluing the fibers $\overline{F}_{p_1}$ and $\overline{F}_{p_2}$.

Remark 2.3.3. In the last case, consider a path $\gamma_{1,2}$ on $\overline{C}$ from p_1 to p_2. Gluing $\overline{F}_{p_1}$ to $\overline{F}_{p_2}$ is equivalent to lift $\gamma_{1,2}$ and therefore to give a monodromy factor h_γ well defined up to conjugation. Taking the twisted G-cover point of view, a point of $\overline{F}_{p_1}$ has to be sent by h_γ to a point with the same local index (see [20, §2.2.2]). For example if we take a point with local index $w \in [\![\varpi(\gamma_1)]\!]$, then the local index of its image is $h_\gamma w h_\gamma^{-1}$. This means that w and h_γ must commute. This condition allows to define h_γ up to conjugation and is also sufficient to make the gluing.

Remark 2.3.4. If C is a nodal curve with a type 0 node as above, the fundamental group of the normalization $\pi_1(\overline{C}_{\mathrm{gen}})$ is naturally a subgroup of $\pi_1(C)$, therefore the generators $\alpha_1, \ldots, \beta_{g-1}$ are $\pi_1(C)$ elements. Consider a slight deformation $F' \to C'$ of $F \to C$, smoothing the node, such that C' is a smooth curve. This induces a natural morphism $\pi_1(C) \to \pi_1(C')$, in particular the α_i, β_i from $\pi_1(\overline{C}_{\mathrm{gen}})$ are generators in the canonical representation of $\pi_1(C')$, and we can complete this representation in such a way that the class of lacet γ_1 becomes the element α_g in $\pi_1(C')$. The generator β_g comes from the gluing data of the nodal fiber.

Following this monodromy approach, and in order to describe the components of $\overline{\mathbf{R}}_{g,S_3}$ in §3.1.1, it is useful to introduce a notion of group morphism defined up to conjugation.

Definition 2.3.5. Consider two groups A, G, and the equivalence relation $\sim$ such that for two morphisms $\varphi, \varphi' \colon A \to G$, $\varphi \sim \varphi'$ if there exists $g \in G$, $\varphi' = g \cdot \varphi \cdot g^{-1}$.

For any conjugacy class $\mathcal{H} \in \mathcal{T}(G)$, we denote by $\mathrm{Hom}_{\mathrm{S}}(A, \mathcal{H})$ the subset of $\mathrm{Hom}(A, G)/\sim$ induced by the morphisms whose image is a subgroup in $\mathcal{H}$.

We will use the notation $\varphi \colon A \overset{G}{\rightsquigarrow} \mathcal{H}$ for an element of $\mathrm{Hom}_{\mathrm{S}}(A, \mathcal{H})$, that is a group morphism $A \to G$ defined up to conjugation whose image is in $\mathcal{H}$.

3. Structure of $\overline{\mathcal{R}}_{g,S_3}$

In the first section we study the boundary of $\overline{\mathcal{R}}_{g,S_3}$, *i.e.* the locus $\overline{\mathcal{R}}_{g,G} \backslash \mathcal{R}_{g,G}$. In the other two sections we describe the canonical divisor of this space as a combination of the Hodge divisor and boundary divisors.

3.1. **Moduli boundary of $\overline{\mathcal{R}}_{g,S_3}$.** For every subgroup conjugacy class $\mathcal{H} \in \mathcal{T}(G)$, we consider the moduli stack $\overline{\mathbf{R}}_{g,G}^{\mathcal{H}}$ of admissible $\mathcal{H}$-covers over stable curves of genus g, *i.e.* admissible G-covers such that the stabilizer of every connected component of a cover is in the class $\mathcal{H}$ (see Definition 2.2.12 and Remark 2.2.16). These are all pairwise disconnected substacks of $\overline{\mathbf{R}}_{g,G}$.

3.1.1. *Components of $\overline{\mathcal{R}}_{g,S_3}$.* We observe that S_3 has 6 subgroups and 4 subgroup classes.

- The trivial subgroup $(1) \subset S_3$. In this case $\overline{\mathcal{R}}^1_{g,S_3}$ is isomorphic to $\overline{\mathcal{M}}_g$.
- The three subgroups T_1, T_2, T_3 of order 2, generated respectively by transpositions $(23), (13), (12)$. These subgroups all stay in the same conjugacy class $T = \{T_1, T_2, T_3\}$. We are going to show that the stack $\mathbf{R}^T_{g,S_3}$ is isomorphic to $\mathbf{R}^{\boldsymbol{\mu}_2}_{g,\boldsymbol{\mu}_2} \subset \mathbf{R}^0_{g,2}$, the moduli stack of twisted curves equipped with a non-trivial square root of the trivial bundle.

  Indeed, any admissible connected $\boldsymbol{\mu}_2$-cover $E \to C$ is equivalent to the data (C, ρ) of a curve C and $\rho \in \mathrm{Hom}_S(\pi_1(C), \boldsymbol{\mu}_2)$, as stated by Proposition 2.3.1 and using the notation of Definition 2.3.5. There exists an isomorphism $\sigma_2 \colon \boldsymbol{\mu}_2 \overset{S_3}{\rightsquigarrow} T$ (see again Definition 2.3.5), and this induces a set bijection

  $$\mathrm{Hom}_S(\pi_1(C), \boldsymbol{\mu}_2) \cong \mathrm{Hom}_S(\pi_1(C), T).$$

  Therefore the map $\mathbf{R}^{\boldsymbol{\mu}_2}_{g,\boldsymbol{\mu}_2} \to \mathbf{R}^T_{g,S_3}$ defined by $(C, \rho) \mapsto (C, \sigma_2 \circ \rho)$ is an isomorphism. We additionally observe that given an admissible T-cover $F \to C$ and any connected component $E \subset F$, the map $E \to C$ is an admissible connected $\boldsymbol{\mu}_2$-cover.

  We observe that the isomorphism of moduli stacks extends to $\overline{\mathbf{R}}^T_{g,S_3} \cong \overline{\mathbf{R}}^{\boldsymbol{\mu}_2}_{g,\boldsymbol{\mu}_2}$, but the construction at the boundary introduces an additional complexity which is useless for the purpose of this work.
- The normal subgroup $N \subset S_3$, a cyclic group generated by the 3-cycle (123). With a little abuse of notation we call N the class $\{N\} \in \mathcal{T}(S_3)$. Consider the moduli stack $\mathbf{R}^{\boldsymbol{\mu}_3}_{g,\boldsymbol{\mu}_3} \subset \mathbf{R}^0_{g,\boldsymbol{\mu}_3}$ of twisted curves equipped with a non-trivial third root of the trivial bundle. We are going to prove that there exists a $2:1$ map $\mathbf{R}^{\boldsymbol{\mu}_3}_{g,\boldsymbol{\mu}_3} \to \mathbf{R}^N_{g,S_3}$.

  Indeed, any admissible connected $\boldsymbol{\mu}_3$-cover $E \to C$ is equivalent to the data (C, ρ) of a curve C and an element $\rho \in \mathrm{Hom}_S(\pi_1(C), \boldsymbol{\mu}_3)$. There exists an isomorphism $\sigma_3 \colon \boldsymbol{\mu}_3 \overset{S_3}{\rightsquigarrow} N$, and this induces a set surjection

  $$\mathrm{Hom}_S(\pi_1(C), \boldsymbol{\mu}_3) \twoheadrightarrow \mathrm{Hom}_S(\pi_1(C), N).$$

  If $\wedge 2 \colon \boldsymbol{\mu}_3 \to \boldsymbol{\mu}_3$ is the second power map, then $\wedge 2$ also acts as an involution on $\mathrm{Hom}_S(\pi_1(C), \boldsymbol{\mu}_3)$. We observe that if $\rho \colon \pi_1(C) \overset{\boldsymbol{\mu}_3}{\rightsquigarrow} \boldsymbol{\mu}_3$, then $\sigma_3 \circ \rho = \sigma_3 \circ (\wedge 2) \circ \rho$. In fact, the following is true

  $$\mathrm{Hom}_S(\pi_1(C), \boldsymbol{\mu}_3)/(\wedge 2) \cong \mathrm{Hom}_S(\pi_1(C), N).$$

  Therefore the map $(C, \rho) \mapsto (C, \sigma_3 \circ \rho)$ is the $2:1$ map we were searching. As before the map extends to $\overline{\mathbf{R}}^{\boldsymbol{\mu}_3}_{g,\boldsymbol{\mu}_3} \to \overline{\mathbf{R}}^N_{g,S_3}$.
- The group S_3 itself. In this case the stack $\overline{\mathcal{R}}^{S_3}_{g,S_3}$ is the moduli of curves equipped with a connected admissible S_3-cover. This is the "really new" component of moduli space $\overline{\mathcal{R}}_{g,S_3}$, and our analysis will focus on it.

Furthermore, we observe that there exists a canonical map $\overline{\mathbf{R}}_{g,S_3} \to \overline{\mathbf{R}}_{g,\boldsymbol{\mu}_2}$. Any admissible G-cover $F \to C$ is equivalent to the data of a curve C plus an element of $\mathrm{Hom}_S(\pi_1(C), S_3)$. The quotient $S_3 \twoheadrightarrow S_3/N = \boldsymbol{\mu}_2$ induces a surjection $\mathrm{Hom}_S(\pi_1(C), S_3) \twoheadrightarrow \mathrm{Hom}_S(\pi_1(C), \boldsymbol{\mu}_2)$ and therefore the map above.

3.1.2. *The boundary divisors.* To classify the boundary divisors of $\overline{\mathcal{R}}_{g,G}$, that is the divisors filling the locus $\overline{\mathcal{R}}_{g,G} \backslash \mathcal{R}_{g,G}$, we start by recalling the boundary divisors of $\overline{\mathcal{M}}_g$.

Definition 3.1.1. A node q of a nodal curve C is a disconnecting node if the partial normalization of C at q is a disconnected scheme. This disconnected scheme has two connected components noted C_1 and C_2. Consider C of genus g and q a disconnecting node, q is of type i, with $1 \leq i \leq \lfloor g/2 \rfloor$, if C_1 and C_2 have genus i and $g - i$. If q is a non-disconnecting node, then it is called a node of type 0.

Definition 3.1.2. For every i with $1 \leq i \leq \lfloor g/2 \rfloor$, the divisor $\Delta_i \subset \overline{\mathcal{M}}_g$ is the locus of curves with a disconnecting node of type i. For $i = 0$, the divisor $\Delta_0 \subset \overline{\mathcal{M}}_g$ is the locus of curves with a node of type 0, or equivalently the closure of the locus of nodal irreducible curves.

For every i we denote by δ_i the class of Δ_i in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$. We consider the natural morphism $\pi \colon \overline{\mathcal{R}}_{g,G} \to \overline{\mathcal{M}}_g$ and look at the preimages $\pi^{-1}(\Delta_i)$ for every i. The intersections of this preimages with the connected components of $\overline{\mathcal{R}}_{g,G}$ are the boundary components that we are going to consider.

We start by focusing on the loci of curves with a disconnecting node, *i.e.* the preimages of Δ_i with $i \neq 0$. In what follows for any curve C in Δ_i we denote by q the node who separates the components C_1, of genus i, and C_2, of genus $g - i$. As a consequence $C_1 \sqcup C_2 \to C$ is the partial normalization of C at q, and we denote by q_1 and q_2 the preimages of q respectively on C_1 and C_2.

There exists, at the level of the moduli space $\overline{\mathcal{M}}_g$, a natural gluing map

$$\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g,$$

defined in such a way that a pair of points $([C_1, q_1], [C_2, q_2])$, where C_1 and C_2 have genus i and $g - i$ respectively, is sent on the point associated to the nodal curve

$$C := (C_1 \sqcup C_2)/(q_1 \sim q_2).$$

We recall that any admissible G-cover on C induces two admissible G-covers F_1 on $(C_1; q_1)$ and F_2 on $(C_2; q_2)$ such that the G-types $[\![h_1]\!]$ and $[\![h_2]\!]$ on q_1 and q_2 are one the inverse of the other, $[\![h_1]\!] = [\![h_2^{-1}]\!]$.

Definition 3.1.3. Consider two subgroup conjugacy classes $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{T}(G)$, a conjugacy class $[\![h]\!]$ in $[\![G]\!]$ and i such that $1 \leq i \leq \lfloor g/2 \rfloor$. We denote by $\Delta_{i,[\![h]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$ the locus in $\overline{\mathcal{R}}_{g,G}$ of curves C with a node q of type i and with an admissible G-cover $F \to C$ such that if we call $F_1 \to (C_1; q_1)$ and $F_2 \to (C_2; q_2)$ the restrictions of F, then these are an admissible $\mathcal{H}_1$-cover of $(C_1; q_1)$ and an admissible $\mathcal{H}_2$-cover of $(C_2; q_2)$ respectively (see Definition 2.2.12), and the G-type at q, with respect to the branch of C_1, is $[\![h]\!]$. In order to simplify the notation we will omit the G-type when it is trivial.

The loci $\Delta_{i,[\![h]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$ are (not necessarily connected) divisors of the moduli space $\overline{\mathcal{R}}_{g,G}$.

For example in the case of $\overline{\mathcal{R}}_{g,S_3}$ we have the following classes:

- Δ_i^{1,S_3}, $\Delta_i^{S_3,1}$, $\Delta_i^{1,T}$, $\Delta_i^{T,1}$, $\Delta_i^{1,N}$, $\Delta_i^{N,1}$ and $\Delta_i^{1,1} \cong \Delta_i$ are the cases of admissible G-covers which are trivial over C_1 or C_2;
- $\Delta_i^{T,T}, \Delta_i^{T,N}, \Delta_i^{N,T}, \Delta_i^{T,S_3}, \Delta_i^{N,N}, \Delta_i^{N,S_3}, \Delta_i^{S_3,N}$ and $\Delta_i^{S_3,S_3}$ are the other cases with trivial G-type at the node q;
- $\Delta_{i,c_3}^{S_3,S_3}$ is the only case with non-trivial stabilizer at q. Here we denoted by c_3 the conjugacy class $[\![123]\!] = \{(123), (132)\}$.

Remark 3.1.4. We observe that by Equation (2.1) and Remark 2.3.2, it is possible to have non-trivial stabilizer at q only if for both covers $F_1 \to C_1$ and $F_2 \to C_2$, the associated morphisms $\pi_1(C_i) \to S_3$ are surjective, and therefore both are connected admissible S_3-covers.

Finally, we consider the preimage of Δ_0, *i.e.* the locus of curves with a node of type 0. We start by working on a covering stack of $\pi^{-1}(\Delta_0)$.

Definition 3.1.5. Category D_0 has for objects the data of a curve C with a node q of type 0 with an admissible G-cover $F \to C$ and a privileged branch at q. Morphisms are admissible G-cover morphisms preserving the privileged branch.

Remark 3.1.6. Category D_0 is a Deligne-Mumford stack. Its coarse space $\mathcal{D}_0$ comes with a natural $2:1$ morphism $\gamma \colon \mathcal{D}_0 \to \pi^{-1}(\Delta_0)$.

In what follows we consider a curve C with a node q of type 0, we call $\mathsf{nor} \colon \overline{C} \to C$ the partial normalization of C at q. Given any admissible G-cover $F \to C$, the pullback nor^*F over $\overline{C}$ is still an admissible G-cover.

Definition 3.1.7. Consider two conjugacy classes $\mathcal{H}_1, \mathcal{H}_2$ in $\mathcal{T}(G)$ such that $\mathcal{H}_2 \leq \mathcal{H}_1$ (see Definition 2.2.10). The category $\mathsf{D}_{0,[\![h]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$ is a full subcategory of D_0. Its objects are stable curves C with a node q of type 0, a privileged branch at q and an admissible $\mathcal{H}_1$-cover $F \to C$ such that the pullback $\mathsf{nor}^*F \to \overline{C}$ is an admissible $\mathcal{H}_2$-cover on $\overline{C}$, and the G-type of F at q with respect to the privileged branch is $[\![h]\!]$. In case $h = 1$ we will omit its notation. The category $\mathsf{D}_{0,[\![h]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$ is again a Deligne-Mumford stack and we denote its coarse space by $\mathcal{D}_{0,[\![h]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$.

Remark 3.1.8. By Equation (2.1) and Remark 2.3.2, there exists a compatibility condition for the stack $\mathsf{D}_{0,[\![h]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$ to be non-empty. There must exist an element h in the class $[\![h]\!]$ and a subgroup H_2 in the class $\mathcal{H}_2$ such that h lies in the subgroup of commutators of H_2.

Remark 3.1.9. There exists a natural automorphism $\mathsf{inv} \colon \mathsf{D}_0 \to \mathsf{D}_0$, which sends any curve with an admissibile G-cover to the same curve and G-cover but changing the privileged branch at the node. This sends isomorphically $\mathsf{D}_{0,[\![h]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$ in $\mathsf{D}_{0,[\![h^{-1}]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$.

Definition 3.1.10. For any group G we consider the inverse relation in the set of conjugacy classes $[\![G]\!]$, that is $[\![h]\!] \sim [\![h']\!]$ if and only if $[\![h]\!] = [\![h']\!]$ or $[\![h]\!] = [\![h'^{-1}]\!]$. Then we define the set $\underline{[\![G]\!]}$ as $\underline{[\![G]\!]} := [\![G]\!]/\sim$. we denote by $\underline{[\![h]\!]}$ the class in $\underline{[\![G]\!]}$ of any element $h \in G$.

We observe that any point $[C, F]$ of $\mathcal{D}_0$ and its image $\mathsf{inv}([C, F])$ are sent by $\gamma \colon \mathcal{D}_0 \to \pi^{-1}(\Delta_0)$ to the same point. This means that the image of $\mathcal{D}_{0,[\![h]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$ via γ depends only on $\mathcal{H}_1, \mathcal{H}_2$ and on the class $\underline{[\![h]\!]} \in \underline{[\![G]\!]}$. Equivalently, for all $[\![h]\!] \in [\![G]\!]$, $\mathcal{D}_{0,[\![h]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$ and $\mathcal{D}_{0,[\![h^{-1}]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$ have the same image.

Definition 3.1.11. For every class $\mathcal{H}_1, \mathcal{H}_2$ in $\mathcal{T}(G)$ and $\underline{[\![h]\!]}$ in $\underline{[\![G]\!]}$, the locus $\Delta_{0,[\![h]\!]}^{\mathcal{H}_1, \mathcal{H}_2}$ is the image of $\mathcal{D}_{0,\underline{[\![h]\!]}}^{\mathcal{H}_1, \mathcal{H}_2}$ via γ for any G-type $[\![h]\!] \in \underline{[\![h]\!]}$. As before we will omit to note $\underline{[\![h]\!]}$ in the case of the trivial class $\underline{[\![1]\!]}$.

As before, we list the divisors in $\pi^{-1}(\Delta_0)$:

- $\Delta_0^{N,1}$, $\Delta_0^{T,1}$ and $\Delta_0^{1,1} \cong \Delta_0$ are the divisor of admissible G-covers $F \to C$ such that $\mathsf{nor}^*F \to \overline{C}$ is trivial;

- $\Delta_0^{T,T}$, $\Delta_0^{S_3,T}$, $\Delta_0^{N,N}$, $\Delta_0^{S_3,N}$ and $\Delta_0^{S_3,S_3}$ are the other cases with trivial associated G-type at node q;
- with another small abuse of notation, we define $c_2 := [\![12]\!] = \{(12),(13),(23)\}$ and $c_3 := [\![123]\!] = \{(123),(132)\}$. Then, $\Delta_{0,c_2}^{T,T}$, $\Delta_{0,c_2}^{S_3,S_3}$, $\Delta_{0,c_3}^{N,N}$ and $\Delta_{0,c_3}^{S_3,S_3}$ are the cases with non-trivial stabilizer at q.

Observe that the divisors $\Delta_{0,c_2}^{S_3,T}$ and $\Delta_{0,c_3}^{S_3,N}$ are both empty as a direct consequence of Remark 2.3.4.

3.2. **The canonical divisor.** To evaluate the canonical divisor we start by evaluating the pullbacks of the $\overline{\mathcal{M}}_g$ boundary divisors. We denote by $\delta_{i,[\![h]\!]}^{\mathcal{H}_1,\mathcal{H}_2}$ the class of the divisor $\Delta_{i,[\![h]\!]}^{\mathcal{H}_1,\mathcal{H}_2}$ in the ring $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{R}}_{g,S_3})$. Furthermore, we introduce some other notation to simplify the formulas that will follow.

- For any $i \geq 1$ we define
$$\delta_i' := \sum \delta_i^{\mathcal{H}_1,\mathcal{H}_2},$$
where the sum is over al the divisors with trivial S_3-type at the node.
- For $i = 0$ we define
$$\delta_0' := \sum_{\mathcal{H}_2 \leq \mathcal{H}_1} \delta_0^{\mathcal{H}_1,\mathcal{H}_2},$$
$$\delta_{0,c_2} := \delta_{0,c_2}^{T,T} + \delta_{0,c_2}^{S_3,S_3},$$
$$\delta_{0,c_3} := \delta_{0,c_3}^{N,N} + \delta_{0,c_3}^{S_3,S_3}.$$

Given the classes δ_i in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$, and the natural morphism $\pi\colon \overline{\mathcal{R}}_{g,S_3} \to \overline{\mathcal{M}}_g$, we consider the pullbacks $\pi^*\delta_i$ to the ring $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{R}}_{g,S_3})$.

Lemma 3.2.1. *If $i > 1$, then*
$$\pi^*\delta_i = \delta_i' + 3\delta_{i,c_3}^{S_3,S_3}.$$

Proof. This relation is true also in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathbf{R}}_{g,S_3})$. We consider a general S_3-cover (C,ϕ) in the divisor $\Delta_{i,c_3}^{S_3,S_3}$, and we denote by C the coarse space of the curve and by q the type i node. By construction we have
$$\frac{\mathrm{Aut}(\mathsf{C},\phi)}{\mathrm{Aut}(C)} = \mathrm{Aut}(\mathsf{C}_q) = \boldsymbol{\mu}_3.$$
This implies that the morphism $\overline{\mathbf{R}}_{g,S_3} \to \overline{\mathbf{M}}_g$ is 3-ramified over $\Delta_{i,c_3}^{S_3,S_3}$. Over Δ_i but outside $\Delta_{i,c_3}^{S_3,S_3}$ the morphism is étale, and the thesis follows. $\square$

Lemma 3.2.2. *If $i = 1$, then*
$$\pi^*\delta_1 = \delta_1' + 3\delta_{1,c_3}^{S_3,S_3}.$$

Proof. The case of a curve C with an elliptic tail, is different from the case $i > 1$ because there exists an elliptic tail involution on C. This involution always lifts to an admissible S_3-cover. Indeed, an admissible S_3-cover over an elliptic curve (E, p_*) is the datum of a morphism $\pi_1(E, p_*) \to S_3$ defined up to conjugation, and this is always preserved by the involution. Thus, $\pi^*\Delta_1 = \Delta_1' + \Delta_{1,c_3}^{S_3,S_3}$. To obtain the result we observe that, as in the case of Lemma 3.2.1, the morphism $\overline{\mathbf{R}}_{g,S_3} \to \overline{\mathbf{M}}_g$ is 3-ramified over $\Delta_{1,c_3}^{S_3,S_3}$. $\square$

Lemma 3.2.3. *If $i = 0$, then*

$$\pi^*\delta_0 = \delta'_0 + 2\delta_{0,c_2} + 3\delta_{0,c_3}.$$

Proof. Similarly to what observed in the case of Lemma 3.2.1, the morphism $\overline{\mathbf{R}}_{g,S_3} \to \overline{\mathbf{M}}_g$ is 2-ramified over Δ_{0,c_2}, 3-ramified over Δ_{0,c_3} and étale elsewhere over Δ_0. □

From the Harris and Mumford work [22], we know the evaluation of the canonical divisor $K_{\overline{\mathcal{M}}_g}$ of the moduli space $\overline{\mathcal{M}}_g$. Knowing that the morphism $\pi\colon \overline{\mathcal{R}}_{g,S_3} \to \overline{\mathcal{M}}_g$ is étale outside the divisors Δ_i and a sublocus of codimension greater than 1, we can now evaluate the canonical divisor $K_{\overline{\mathcal{R}}_{g,S_3}}$.

Lemma 3.2.4. *On the smooth variety $\overline{\mathcal{R}}_{g,S_3}^{\mathrm{reg}}$, the sublocus of regular points, we have the following evaluation of the canonical divisor,*

$$K_{\overline{\mathcal{R}}_{g,S_3}} = 13\lambda - (2\delta'_0 + 3\delta_{0,c_2} + 4\delta_{0,c_3}) - (3\delta'_1 + 7\delta^{S_3,S_3}_{1,c_3}) - \sum_{i=2}^{\lfloor g/2 \rfloor} (2\delta'_i + 4\delta^{S_3,S_3}_{i,c_3}).$$

Proof. As proved in [22, Theorem 2 bis, p.52], on $\overline{\mathcal{M}}_g^{\mathrm{reg}}$ we have

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor g/2 \rfloor}.$$

We know, from the description above, that the ramification divisor of $\pi\colon \overline{\mathcal{R}}_{g,S_3} \to \overline{\mathcal{M}}_g$ is

$$R = \delta_{0,c_2} + 2\delta_{0,c_3} + 2\sum_{i=1}^{\lfloor g/2 \rfloor} \delta^{S_3,S_3}_{i,c_3}.$$

By Hurwitz formula we have $K_{\overline{\mathcal{R}}_{g,S_3}} = \pi^* K_{\overline{\mathcal{M}}_g} + R$, this is true on $\pi^{-1}(\overline{\mathcal{M}}_g^{\mathrm{reg}})$, but this locus and $\overline{\mathcal{R}}_{g,S_3}^{\mathrm{reg}}$ differ by a codimension 2 locus, therefore the evaluation is unchanged on $\overline{\mathcal{R}}_{g,S_3}^{\mathrm{reg}}$. Precisely, if π is non-étale in a point $[C, F] \in \overline{\mathcal{R}}_{g,S_3}$ outside the $\pi^{-1}(\Delta_i)$, then necessarily C is an irreducible curve with a non-trivial automorphism group. The thesis follows by the evaluations of the divisors $\pi^*\delta_i$ of Lemmata 3.2.1, 3.2.2 and 3.2.3. □

3.3. The subspace of covers on irreducible curves.

We consider the full substack $\widetilde{\mathbf{R}}_{g,S_3}$ of $\overline{\mathbf{R}}_{g,S_3}$ of irreducible stable curves of genus g with an admissible S_3-cover. We denote by $\widetilde{\mathcal{R}}_{g,S_3}$ its coarse space. We will prove that the bigness of the canonical divisor $K_{\overline{\mathcal{R}}_{g,S_3}}$ over $\widetilde{\mathcal{R}}_{g,S_3}$ implies the bigness over $\overline{\mathbf{R}}_{g,S_3}$. As a consequence, it suffices to prove that $K_{\overline{\mathcal{R}}_{g,S_3}}$ is big over $\widetilde{\mathcal{R}}_{g,S_3}$, to prove $\overline{\mathcal{R}}_{g,S_3}$ being of general type.

We build some pencils filling up specific divisors of $\overline{\mathcal{R}}_{g,S_3}^{S_3}$. To do this we follow [16] and [17]. Given a general K3 surface X of degree $2i - 2$ in $\mathbb{P}^i$, the map $\mathrm{Bl}_{i^2}(X) \to \mathbb{P}^1$ is a family of genus i stable curves lying on X, where $\mathrm{Bl}_{i^2}(X)$ is the blowup of X in i^2 points. This induces a pencil $B \subset \overline{\mathcal{M}}_i$. Moreover, there exists at least one section σ on B, therefore for every genus $g > i$ we can glue along σ a fixed 1-marked curve (C_2, p_*) of genus $g - i$, thus inducing a pencil $B_i \subset \overline{\mathcal{M}}_g$. The pencils B_i fill up the divisors Δ_i except for $i = 10$, if $i = 10$ then the B_{10} fill up a divisor $\mathcal{Z}$ in Δ_{10} which is the locus of smooth curves of genus 10 lying on a K3 surface and attached to a curve of genus $g - 10$. We denote by $\mathcal{Z}'$ the preimage of $\mathcal{Z}$ in $\overline{\mathcal{R}}_{g,S_3}^{S_3}$.

Lemma 3.3.1. *Consider an effective divisor E in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{R}_{g,S_3}^{S_3})$ such that*

$$[\overline{E}] = a \cdot \lambda - b_0' \cdot \delta_0' - b_{0,c_2} \cdot \delta_{0,c_2} - b_{0,c_3} \cdot \delta_{0,c_3} - b_1' \cdot \delta_1' - b_{1,c_3}^{S_3,S_3} \cdot \delta_{1,c_3}^{S_3,S_3} - \sum_{i=2}^{\lfloor g/2 \rfloor} (b_i' \cdot \delta_i' + b_{i,c_3}^{S_3,S_3} \cdot \delta_{i,c_3}^{S_3,S_3})$$

in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{R}}_{g,S_3}^{S_3})$, where the a and b are rational coefficients.

If $a \leq 13$, $b_0' \geq 2$, $b_{0,c_2} \geq 3$ and $b_{0,c_3} \geq 4$, then $b_i' \geq 3$ and $b_{i,c_3}^{S_3,S_3} > 7$ for all $i \geq 1$ and $i \neq 10$. The same is true for $i = 10$ if $\overline{E}$ does not contain $\mathcal{Z}'$.

Proof. By [17, Lemma 2.4] we have

$$B_i \cdot \lambda = i + 1, \quad B_i \cdot \delta_0 = 6i + 18, \quad B_i \cdot \delta_i = -1, \quad B_i \cdot \delta_j = 0 \; \forall j \neq 0, i.$$

We introduce some pencils lying in the preimage of B_i with respect to the natural projection $\pi \colon \overline{\mathcal{R}}_{g,S_3}^{S_3} \to \overline{\mathcal{M}}_g$.

The pencil $A_i^{T,N}$ is the preimage of B_i in $\Delta_i^{T,N}$. It is obtained by taking any 1-marked stable curve C_1 of genus i with an admissible T-cover, and gluing it to a fixed general 1-marked curve C_2 of genus $g - i$ with an admissible N-cover. We remark that the gluing is uniquely defined.

The pencil $A_{i,c_3}^{S_3,S_3}$ is the preimage of B_i in $\Delta_{i,c_3}^{S_3,S_3}$. It is obtained by taking any 1-marked stable curve (C_1, p_*') of genus i with an admissible connected S_3-cover and S_3-type equal to c_3 at p_*', and gluing it to a fixed general 1-marked curve (C_2, p_*) of genus $g-i$ with an admissible connected S_3 cover and the same S_3-type at p_*. Again, the gluing is unique.

We can write down some intersection numbers for the $A_i^{T,N}$:

- $A_i^{T,N} \cdot \lambda = (2^{2i} - 1)(i + 1)$, which is true because $\pi^* \lambda_{\overline{\mathcal{M}}_g} = \lambda_{\overline{\mathcal{R}}_{g,S_3}}$ and moreover $\pi_* A_i^{T,N} = B_i \cdot \deg(A_i^{T,N}/B_i)$;
- $A_i^{T,N} \cdot \delta_0' = (2^{2i-1} + 1)(6i + 18)$, $A_i^{T,N} \cdot \delta_{0,c_2} = (2^{2i-1} - 2)(6i + 18)$, $A_i^{T,N} \cdot \delta_{0,c_3} = 0$. The third equality is clear. The second one is obtained by counting the admissible T-covers over a curve of genus $i-1$ and by multiplying for the 2 possible gluing factors at the node. The first equality is obtained by difference;
- $A_i^{T,N} \cdot \delta_{j,c_3}^{S_3,S_3} = 0$ for all j and $A_i^{T,N} \cdot \delta_j' = 0$ for all $j \neq i$ and $A_i^{T,N} \cdot \delta_i' = -(2^{2i} - 1)$ because it is the same of $A_i^{T,N} \cdot \pi^* \delta_i$.

Consider an effective divisor E of $\mathcal{R}_{g,S_3}^{S_3}$, as the pencils $A_i^{T,N}$ fill up the boundary divisor $\Delta_i^{T,N}$ for $i \neq 10$, we have $A_i^{T,N} \cdot \overline{E} \geq 0$ that is, by the relations above,

$$(i + 1) \cdot a - \frac{2^{2i-1} + 1}{2^{2i} - 1} \cdot (6i + 18) \cdot b_0' - \frac{2^{2i-1} - 2}{2^{2i} - 1} \cdot (6i + 18) \cdot b_{0,c_2} + b_i' \geq 0,$$

which implies $b_i' \geq 3$ for all $i \geq 1$. The same is true for $i = 10$ if $\overline{E}$ does not contain the locus $\mathcal{Z}'$.

We define $d := \deg(A_{i,c_3}^{S_3,S_3})$. As in the previous case, also for $A_{i,c_3}^{S_3,S_3}$ we have the equalities

- $A_{i,c_3}^{S_3,S_3} \cdot \lambda = d \cdot (i + 1)$;
- $A_{i,c_3}^{S_3,S_3} \cdot \delta_{i,c_3}^{S_3,S_3} = -d$;
- $A_{i,c_3}^{S_3,S_3} \cdot \delta_j' = 0$ for all j and $A_{i,c_3}^{S_3,S_3} \cdot \delta_{j,c_3}^{S_3,S_3} = 0$ for all $j \neq i$.

Similarly we have $A_{i,c_3}^{S_3,S_3} \cdot (\delta_0' + \delta_{0,c_2} + \delta_{0,c_3}) = A_{i,c_3}^{S_3,S_3} \cdot \pi^* \delta_0 = d \cdot (6i + 18)$.

The following inequality is also true

(3.1)
$$A_{i,c_3}^{S_3,S_3} \cdot (\delta_{0,c_2} + \delta_{0,c_3}) \geq A_{i,c_3}^{S_3,S_3} \cdot \delta_0'.$$

For any effective divisor E of $\mathcal{R}_{g,S_3}^{S_3}$, as the pencils $A_{i,c_3}^{S_3,S_3}$ fill up a connected component of the boundary divisor $\Delta_{i,c_3}^{S_3,S_3}$ for $i \neq 10$, we have $A_{i,c_3}^{S_3,S_3} \cdot \overline{E} \geq 0$. This implies, by the inequalities above,

$$d \cdot b_{i,c_3}^{S_3,S_3} \geq A_{i,c_3}^{S_3,S_3} \cdot (2\delta_0' + 3\delta_{0,c_2} + 4\delta_{0,c_3}) - d \cdot (i+1) \cdot a \geq$$

$$\frac{5}{2} \cdot A_{i,c_3}^{S_3,S_3} \cdot (\delta_0' + \delta_{0,c_2} + \delta_{0,c_3}) - d \cdot (i+1) \cdot a > d \cdot 7.$$

The same is true if $i = 10$ and $\overline{E}$ does not contain the locus $\mathcal{Z}'$.

It remains to prove the inequality (3.1). Consider a 1-marked curve (C_1, p'_*) of genus i with an autointersection node q, we call $\overline{C}_1$ the partial normalization at q. By Remarks 2.3.2 and 2.3.3, an admissible S_3-cover on (C_1, p'_*) is equivalent to an admissible S_3-cover on $(\overline{C}_1, p'_*)$ plus the data of the S_3-type $[\![h_1]\!]$ at q and a gluing factor h_γ at the same node (defined up to conjugation), such that h_γ is in the centralizer of h_1.

Consider the pencil $A_{i,c_3}^{S_3,S_3}$ obtained by joining any such curve (C_1, p'_*) to a fixed general 1-marked curve (C_2, p_*) of genus $g - i$ with a connected admissible S_3-cover on it. Any curve in the (finite) intersection $A_{i,c_3}^{S_3,S_3} \cap \Delta_{0,c_2}$ induces a curve in the intersection $A_{i,c_3}^{S_3,S_3} \cap \Delta_0'$ by putting a trivial S_3-type at q instead of $[\![h_1]\!]$. The same is true for $A_{i,c_3}^{S_3,S_3} \cap \Delta_{0,c_3}$. All the points of $A_{i,c_3}^{S_3,S_3} \cap \Delta_0'$ are obtained at least once via these operations of S_3-type trivialization. This proves the inequality. $\square$

Proposition 3.3.2. *If*
$$K_{\overline{\mathcal{R}}_{g,S_3}} = a' \cdot \lambda + E'$$

on $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$, where a' is a positive coefficient and E' an effective divisor not containing $\mathcal{Z}'$, then the canonical divisor is big over the space $\overline{\mathcal{R}}_{g,S_3}^{S_3,\mathrm{reg}}$.

Proof. The equation in the hypothesis implies that the canonical divisor is big on $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$, because λ is a big divisor. If we consider the cloture $\overline{E}'$ of E' on the space $\overline{\mathcal{R}}_{g,S_3}^{S_3,\mathrm{reg}}$, we observe that after Lemma 3.2.4, it respects the hypothesis of Lemma 3.3.1, and therefore there exists another effective boundary divisor $E'' \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{R}}_{g,S_3}^{S_3,\mathrm{reg}})$ such that $K_{\overline{\mathcal{R}}_{g,S_3}} = a' \cdot \lambda + \overline{E}' + E''$ on $\overline{\mathcal{R}}_{g,S_3}^{S_3,\mathrm{reg}}$, and the proof is completed. $\square$

4. Extension of pluricanonical forms

We recall that to every twisted G-cover (C, ϕ) is uniquely associated an admissible G-cover $F \to C$ and *vice versa*, see Remark 2.2.17. In this and the following sections, we will mainly use the notation of twisted G-covers.

In order to evaluate the Kodaira dimension of any moduli space $\overline{\mathcal{R}}_{g,G}$, we want to prove an extension result of pluricanonical forms, as done for example by Harris and Mumford for $\overline{\mathcal{M}}_g$ (see [22]) and by Chiodo and Farkas for $\overline{\mathcal{R}}_{g,\ell}^0$ with $\ell < 5$ and $\ell = 6$ (see [12]). In particular

given a desingularization $\widehat{\mathcal{R}}_{g,G} \to \overline{\mathcal{R}}_{g,G}$, and denoting by $\overline{\mathcal{R}}_{g,G}^{\text{reg}}$ the sublocus of regular points, we know that $H^0\left(\widehat{\mathcal{R}}_{g,G}, nK_{\widehat{\mathcal{R}}_{g,G}}\right) \subset H^0\left(\overline{\mathcal{R}}_{g,G}^{\text{reg}}, nK_{\overline{\mathcal{R}}_{g,G}}\right)$ and we would like to prove

$$H^0\left(\widehat{\mathcal{R}}_{g,G}, nK_{\widehat{\mathcal{R}}_{g,G}}\right) = H^0\left(\overline{\mathcal{R}}_{g,G}^{\text{reg}}, nK_{\overline{\mathcal{R}}_{g,G}}\right)$$

for n sufficiently big and divisible. This condition is verified locally for smooth points and canonical singularities, it remains to treat the non-canonical locus $\text{Sing}^{\text{nc}}\,\overline{\mathcal{R}}_{g,G}$.

First, we recall the structure of $\text{Sing}^{\text{nc}}\,\overline{\mathcal{M}}_g$. Consider a curve $[C] \in \overline{\mathcal{M}}_g$, then $[C]$ is a non-canonical singularity if and only if C admits an elliptic tail automorphism of order 6. This means that C has an irreducible component which is an elliptic curve E such that $E \cap \overline{C \backslash E}$ is a single point and E admits an automorphism of order 6. In the following, we denote by (E, p_*) the elliptic tail of a curve C, where p_* is the preimage of the node. Moreover we define $C_1 := \overline{C \backslash E}$ and we mark the preimage p'_* of the node on C_1. With this notation $C = C_1 \cup E$.

Following again [20], we introduce the T-curves over $\mathcal{R}_{g,G}$.

Definition 4.0.1 (T-curve). A twisted G-cover (C, ϕ) is a T-curve if there exists an automorphism $\mathsf{a} \in \underline{\text{Aut}}(\mathsf{C}, \phi)$ such that its coarsening a is an elliptic tail automorphism of order 6. The locus of T-curves in $\overline{\mathcal{R}}_{g,G}$ is denoted by $T_{g,G}$.

In order to introduce J-curves, we recall the definition of the age invariant associated to a linear automorphism. Given a finite order automorphism $\mathsf{h} \in \text{GL}(m)$, we can diagonalize it as $\mathsf{h} = \text{Diag}\left(\xi_{r_1}^{a_1}, \ldots, \xi_{r_m}^{a_m}\right)$, where ξ_{r_i} is an r_ith root of the unit. Then, its age is defined as $\text{age}(\mathsf{h}) := \sum(a_i/r_i)$. A finite subgroup of $\text{GL}(m)$ with no quasireflections is junior if it contains a non-trivial element h with $\text{age}(\mathsf{h}) < 1$. While the notion of age depends on the choice of the roots ξ_{r_i}, the notion of junior group is independent from this choice. For a wider introduction to the age invariant see [20, §5.1.1].

Definition 4.0.2 (J-curve). A twisted G-cover (C, ϕ) is a J-curve if the group

$$\underline{\text{Aut}}_C(\mathsf{C}, \phi)/\text{QR}_C(\mathsf{C}, \phi),$$

which is the group of ghosts quotiented by its subgroup of quasireflections, is junior (see [20, Definition 4.1.2] for the notion of ghost automorphism). The locus of J-curves in $\overline{\mathcal{R}}_{g,G}$ is denoted by $J_{g,G}$.

As proved in [20, Theorem 5.1.8], $\text{Sing}^{\text{nc}}\,\overline{\mathcal{R}}_{g,G}$) is the union of $T_{g,G}$ and $J_{g,G}$. We consider the cases where the second one is empty, and therefore we treat the T-locus by generalizing the Harris-Mumford technique for $\overline{\mathcal{M}}_g$. In particular, we focus in the case of G abelian group and in the case of $G = S_3$ the symmetric group of order 3.

4.1. The case G abelian group.

Lemma 4.1.1. *Consider a twisted G-cover (C, ϕ) with G finite abelian group. If (C, ϕ) is a T-curve, then the restriction of the cover to the elliptic tail is trivial.*

Proof. By Proposition 2.3.1, the set of admissible G-covers over the elliptic curve (E, p_*) (that is the set of twisted G-covers over the same curve) is in bijection with the set of maps $\varpi \colon \pi_1(E_{\text{gen}}, p_*) \to G$. If (E, p_*) admits an order 6 automorphism a_6, then $E \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \cdot \Omega)$, where Ω is a primitive 6th root of the unit and p_* is the origin. Therefore a_6 acts on E as

multiplication by Ω. The fundamental group $\pi_1(E, p_*) \subset \pi_1(E_{\text{gen}}, p_*)$ is generated by a and b which are the classes of the two laces γ_a and γ_b such that

$$\gamma_a \colon [0,1] \to \mathbb{C} \colon \ \gamma_a(t) = t$$
$$\gamma_b \colon [0,1] \to \mathbb{C} \colon \ \gamma_b(t) = t \cdot \Omega.$$

We have as a consequence $\mathsf{a}_6(a) = b$, and $\mathsf{a}_6(b) = ba^{-1}$. Therefore if we call ϖ' the map $\mathsf{a}_6^*\varpi$, by Proposition 2.3.1, a_6 lifts to the cover if and only if $\varpi' = \varpi$. This is true if and only if $\varpi \equiv 1$, *i.e.* the restriction of (C, ϕ) to the elliptic tail must be trivial. $\qquad\square$

Theorem 4.1.2. *In the case of a moduli space $\overline{\mathcal{R}}_{g,G}$ of twisted G-covers with G finite abelian group, we consider a desingularization $\widehat{\mathcal{R}}_{g,G} \to \overline{\mathcal{R}}_{g,G}$. If the locus $J_{g,G} \subset \overline{\mathcal{R}}_{g,G}$ is empty, then*

$$H^0(\overline{\mathcal{R}}_{g,G}^{\text{reg}}, nK_{\overline{\mathcal{R}}_{g,G}^{\text{reg}}}) = H^0(\widehat{\mathcal{R}}_{g,G}, nK_{\widehat{\mathcal{R}}_{g,G}}),$$

for n sufficiently big and divisibile.

In [22] the same is proved for the moduli space $\overline{\mathcal{M}}_g$. The idea is the following. Consider a general non-canonical singularity of $\overline{\mathcal{M}}_g$, that is a point $[C]$ where $C = C_1 \cup E$, C_1 is smooth of genus $g-1$ and without automorphisms, and E is an elliptic tail admitting an order 6 automorphism. Consider the operation of gluing any elliptic tail E' at C_1 along the same node. This gives an immersion

$$(4.1) \qquad\qquad\qquad\qquad \Psi \colon \overline{\mathcal{M}}_{1,1} \hookrightarrow \overline{\mathcal{M}}_g,$$

and the image of Ψ passes through the point $[C]$. Furthermore, there exists a neighborhood $S = S([C])$ of $\text{Im}\,\Psi$ in $\overline{\mathcal{M}}_g$ with the following properties:

(1) it exists a smooth $(3g-3)$-dimensional variety B and a birational morphism $g \colon S \to B$;
(2) it exists a subvariety $Z \subset B$ of codimension 2 such that $g^{-1}(B \backslash Z) \cong B \backslash Z$;
(3) as $B \backslash Z \subset S \subset \overline{\mathcal{M}}_g$, we have $B \backslash Z \subset \overline{\mathcal{M}}_g^0 \subset \overline{\mathcal{M}}_g^{\text{reg}}$, where $\overline{\mathcal{M}}_g^0$ is the subspace of stable curves with trivial automorphism group.

$$\begin{array}{ccccc}
\text{Im}\,\Psi & \lhook\joinrel\longrightarrow & S([C]) & \lhook\joinrel\longrightarrow & \overline{\mathcal{M}}_g \\
& g\downarrow & \nwarrow & & \big\uparrow \\
& B & \longleftarrow \ B/Z & \lhook\joinrel\longrightarrow & \overline{\mathcal{M}}_g^0
\end{array}$$

This allows to conclude. Indeed, for every pluricanonical form ω on $S([C])^{\text{reg}}$, we consider its restriction to $B \backslash Z$, this extends to the smooth variety B and pullbacks to a desingularization $\widehat{S}([C]) \to S([C]) \to B$. We use the Ludwig approach for $\overline{\mathcal{R}}_{g,\boldsymbol{\mu}_2}$ developed in [24]. In order to complete this, we need a generalization of the age tools and the age criterion [20, Proposition 5.1.4].

Proposition 4.1.3 (see Appendix 1 to §1 of [22])**.** *Consider a complex vector space $V \cong \mathbb{C}^n$, $\mathfrak{G} \subset \text{GL}(V)$ finite subgroup, a desingularization $\widehat{V/\mathfrak{G}} \to V/\mathfrak{G}$ and a $\mathfrak{G}$-invariant pluricanonical form ω on V. Consider an element h in $\mathfrak{G}$, $V^0 \subset V$ the subset where $\mathfrak{G}$ acts freely and $\text{Fix}(\mathsf{h}) \subset V$ the fixed point set of h. By an abuse of notation we denote by $\text{Fix}(\mathsf{h})$ also the image of the fixed point set in $V/\mathfrak{G}$. Let $U \subset V/\mathfrak{G}$ be an open subset such that $V^0/\mathfrak{G} \subset U$ and such that for every h with $\text{age}\,\mathsf{h} < 1$ (with respect to some primitive root of the unit), the intersection $U \cap \text{Fix}(\mathsf{h})$ is non-empty. We denote by $\widehat{U} \subset \widehat{V/\mathfrak{G}}$ the preimage of U under the desingularization.*

If ω, as a meromorphic form on $\widehat{V/\mathfrak{G}}$, is holomorphic on $\widehat{U}$, then it is holomorphic on $\widehat{V/\mathfrak{G}}$.

Proof of Theorem 4.1.2. Consider a pluricanonical form ω on $\overline{\mathcal{R}}_{g,G}^{\mathrm{reg}}$. We show that ω lifts to a desingularization of an open neighborhood of every point $[\mathsf{C}, \phi]$ of $\overline{\mathcal{R}}_{g,G}$.

If $[\mathsf{C}, \phi]$ is a canonical singularity this is obvious by definition.

If $[\mathsf{C}, \phi]$ is a non-canonical singularity, at first we consider the case of a general non-canonical singularity. As $\mathrm{Sing}^{\mathrm{nc}} \overline{\mathcal{R}}_{g,G} = T_{g,G}$, then by Lemma 4.1.1 a general point $[\mathsf{C}, \phi] \in T_{g,G}$ is a T-curve C whose coarse space C has two irreducible components (E, p_*), an elliptic tail, and (C_1, p'_*) of genus $g - 1$. Moreover, if (C_1, ϕ_1) and $(\mathsf{E}, \phi_{\mathsf{E}})$ are the restrictions, then $(\mathsf{E}, \phi_{\mathsf{E}})$ is the trivial cover of E, $\mathsf{C}_1 = C_1$ and $\underline{\mathrm{Aut}}(\mathsf{C}_1, \phi_1)$ is trivial.

Once we fix the twisted G-cover (C_1, ϕ_1), we consider the morphism $\Psi_1 \colon \overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{R}}_{g,G}$ sending any point $[E']$ of $\overline{\mathcal{M}}_{1,1}$ to the point $[C, \phi]$ obtained by joining C_1 and E' along their marked points, and by considering the G-cover ϕ such that $\phi|_{C_1} = \phi_1$ and $\phi|_{E'}$ is trivial.

Following [24], we see that the projection $\pi \colon \overline{\mathcal{R}}_{g,G} \to \overline{\mathcal{M}}_g$ sends $\mathrm{Im}\,\Psi_1$ isomorphically on $\mathrm{Im}\,\Psi$, and $\pi|_{\mathrm{Im}\,\Psi}$ is a local isomorphism. Indeed, $\mathrm{Def}(C; \phi) = \mathrm{Def}(C)$ and for every point of $\mathrm{Im}\,\Psi_1$ the automorphism group $\underline{\mathrm{Aut}}(C, \phi)$ is isomorphic to $\mathrm{Aut}(C)$. Therefore if we consider the neighborhood $S([C])$ of $[C] \in \mathrm{Im}\,\Psi$ introduced by Harris and Mumford, then up to shrinking $\pi^{-1}S([C]) \cong S([C])$, we have a neighborhood of $[C, \phi]$ with the same properties.

It remains to consider the case of any non-canonical singularity $[\mathsf{C}, \phi]$. Here C is a twisted curve such that $\mathsf{C} = \mathsf{C}_1 \cup \bigcup \mathsf{E}^{(i)}$, where the $\mathsf{E}^{(i)}$ are all the elliptic tails of C admitting an elliptic tail automorphism of order 6. Again we follow the last part of the Ludwig's demonstration of [24, Theorem 4.1]. We consider for each i a small deformation $(\mathsf{C}^{(i)}, \phi^{(i)})$ of (C, ϕ) which fixes the ith elliptic tail. That is, $\mathsf{C}^{(i)} = \mathsf{C}_1^{(i)} \cup \mathsf{E}^{(i)}$ where $\mathsf{C}_1^{(i)}$ is irreducible. Moreover, the twisted G-cover admits no non-trivial automorphism over $\mathsf{C}_1^{(i)}$ and it is unchanged over $\mathsf{E}^{(i)}$. By the previous point we consider $S^{(i)} := S([\mathsf{C}^{(i)}, \phi^{(i)}])$. Up to shrinking the open subsets $S^{(i)}$, they are all disjoint. Given the point $[\mathsf{C}, \phi]$ of $\overline{\mathcal{R}}_{g,G}$, we consider the local picture of its universal deformation $V := \mathrm{Def}(\mathsf{C}, \phi) \cong \mathbb{C}^{3g-3}$, and recall that the local picture at $[\mathsf{C}, \phi]$ is the same of $V/\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ at the origin. We define

$$S([\mathsf{C}, \phi]) := (V/\underline{\mathrm{Aut}}(\mathsf{C}, \phi)) \cup \left(\bigcup S^{(i)} \right).$$

If V^0 is the V subset where $\underline{\mathrm{Aut}}(\mathsf{C}, \phi)$ acts freely, let $U \subset S([\mathsf{C}, \phi])$ be the set

$$U := (V^0/\underline{\mathrm{Aut}}(\mathsf{C}, \phi)) \cup \left((V/\underline{\mathrm{Aut}}(\mathsf{C}, \phi)) \cap \bigcup S^{(i)} \right).$$

If ω is a pluricanonical holomorphic form on S^{reg}, then it extends to $\widehat{S}^{(i)}$ by definition of the neighborhoods $S^{(i)}$. Moreover, by applying Proposition 4.1.3 to the subset U, ω extends to $\widehat{\dfrac{V}{\underline{\mathrm{Aut}}(\mathsf{C}, \phi)}}$, and therefore to the whole $\widehat{S}$. $\square$

Remark 4.1.4. If we consider $G = \boldsymbol{\mu}_\ell$, we observe that this is the case (treated in [12]) of the moduli space $\overline{\mathcal{R}}_{g,\ell}^0$ of curves equipped with a line bundle which is an ℓth root of the trivial bundle. The proof above applies with minor adjustments also to the case of the moduli space $\overline{\mathcal{R}}_{g,\ell}^k$ for any k (see [19]), that is the moduli space of curves with an ℓth root of ω^k. Therefore the extension result is true also in this case if the J-locus $J_{g,\ell}^k$ of $\overline{\mathcal{R}}_{g,\ell}^k$ is empty.

4.2. **The case $G = S_3$.** We know from [20] that in $\overline{\mathcal{R}}_{g,S_3}$ the non-canonical singular locus coincides with the T-locus.

Lemma 4.2.1. *Consider a twisted S_3-cover (C, ϕ) which is a T-curve. If E is an elliptic tail admitting an elliptic tail automorphism a_6 of order 6, then the restriction $(\mathsf{E}, \phi_\mathsf{E})$ is a trivial cover or an admissible N-cover, i.e. in this last case it has two connected components and trivial S_3-type on the marked point p_*.*

Proof. We follow the same approach of Lemma 4.1.1. By Proposition 2.3.1, the set of admissible S_3-covers on E is in bijection with the conjugacy classes of maps $\varpi \colon \pi_1(E_{\mathrm{gen}}, p_*) \to S_3$. As before we note that if (E, p_*) is an elliptic curve admitting and order 6 automorphism, then $E \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \cdot \Omega)$, where Ω is a primitive 6th root of the unit and p_* is the origin. Here a_6 acts as multiplication by Ω. If a and b are defined as before, then again $\mathsf{a}_6(a) = b$ and $\mathsf{a}_6(b) = ba^{-1}$. If $\varpi' = \mathsf{a}_6^* \varpi$, then by Proposition 2.3.1, a_6 lifts to the cover if and only if $\varpi' = h \cdot \varpi \cdot h^{-1}$ for some element $h \in S_3$. This is true in the case $\varpi \equiv 1$ and for every group morphism $\varpi \colon \pi_1(E) \twoheadrightarrow N \subset S_3$, where N is the normal non-trivial subgroup of S_3, that is the group generated by any 3-cycle. By definition of admissible N-cover, the thesis follows. $\square$

As observed in §3.1.1, there exists an isomorphism, $\overline{\mathcal{R}}_{1,S_3}^1 \cong \overline{\mathcal{M}}_{1,1}$. We remarked in the same section that there exists a natural $2 : 1$ morphism $\overline{\mathcal{R}}_{1,\boldsymbol{\mu}_3}^{\boldsymbol{\mu}_3} \to \overline{\mathcal{R}}_{1,S_3}^N$. As the natural morphism $\overline{\mathcal{R}}_{1,\boldsymbol{\mu}_3}^{\boldsymbol{\mu}_3} \to \overline{\mathcal{M}}_{1,1}$ is $8 : 1$, we have that $\Psi_N \colon \overline{\mathcal{R}}_{1,S_3}^N \to \overline{\mathcal{M}}_{1,1}$ is a $4 : 1$ morphism.

Remark 4.2.2. Consider (E, p_*) a general elliptic curve, we list the four preimages of $[E, p_*] \in \overline{\mathcal{M}}_{1,1}$ with respect to Ψ_N. We characterize every class with a representative $\varpi \colon \pi_1(E, p_*) \twoheadrightarrow N$ of the correspondent conjugacy class of morphisms. We recall that $\pi_1(E, p_*)$ is a free abelian group generated by a and b.

	$\varpi \colon \pi_1(E, p_*) \twoheadrightarrow N$
(i)	$a \mapsto 1; \quad b \mapsto (123)$
(ii)	$a \mapsto (123); \quad b \mapsto (123)$
(iii)	$a \mapsto (123); \quad b \mapsto (132)$
(iv)	$a \mapsto (123); \quad b \mapsto 1$

We recall that $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}^1$, $\overline{\mathcal{R}}_{1,S_3}^N$ is a connected curve and we are interested in finding the branch points for Ψ_N. We observe that the only automorphism of a general elliptic curve is the natural involution i, and this always lifts to any admissible S_3-cover. Following Remark 2.2.18, we detect a branch point in three cases.

- The elliptic curve E_4 admitting an automorphism a_4 of order 4. Automorphism a_4 acts on $\pi_1(E_4, p_*)$ by sending $a \mapsto b$ and $b \mapsto a^{-1}$, therefore it does not lift to any admissible N-cover: in particular it exchanges classes (i) and (iv), and also classes (ii) and (iii). Therefore over $[E_4] \in \overline{\mathcal{M}}_{1,1}$ we have two branch points of order 2 that we denote by $[E_4']$ and $[E_4'']$.
- The elliptic curve E_6 admitting an automorphism a_6 of order 6. Automorphism a_6 acts on $\pi_1(E_6, p_*)$ by sending $a \mapsto b$ and $b \mapsto ba^{-1}$, therefore it lifts to the cover (iii). Moreover, it sends (i) to (ii), (ii) to (iv) and (iv) to (i). Therefore over $[E_6] \in \overline{\mathcal{M}}_{1,1}$ we have one branch point $[E_6']$ of order 3, and moreover we have another preimage point $[E_6'']$.

- The curve E_0 with an autointersection node q_1, whose normalization $\overline{E}_0$ is a rational curve. By Remark 2.2.18, $[E_0] \in \overline{\mathcal{M}}_{1,1}$ has two preimages via Ψ_N, one is obtained by putting an order 3 stabilizer at q_1, we call this preimage $[E_0']$ and it is an order 3 branch point. The other preimage, called $[E_0'']$, is associated to an admissible N-cover with trivial stabilizer at q_1.

Lemma 4.2.3. *The moduli space $\overline{\mathcal{R}}_{1,S_3}^N$ is isomorphic to $\mathbb{P}^1$.*

Proof. This is a consequence of Hurwitz formula plus the observations we just listed. $\square$

We consider a general genus $g-1$ curve with one marked point (C_1, p_*') and a twisted S_3-cover (C_1, ϕ_1) with trivial S_3-type at p_*'. For any twisted G-cover $(\mathsf{E}, \phi_\mathsf{E})$ which is an admissible N-cover over an elliptic curve (E, p_*), we glue (C_1, ϕ_1) and $(\mathsf{E}, \phi_\mathsf{E})$ along their marked points, to get a node q. We obtain a map $\Upsilon \colon \overline{\mathcal{R}}_{1,S_3}^N \to \overline{\mathcal{R}}_{g,S_3}$ which is an isomorphism into the image and fits into the diagram

$$
\begin{array}{ccc}
\overline{\mathcal{R}}_{1,S_3}^N & \overset{\Upsilon}{\hookrightarrow} & \overline{\mathcal{R}}_{g,S_3} \\
\Psi_N \downarrow & & \downarrow \pi \\
\overline{\mathcal{M}}_{1,1} & \underset{\Psi}{\hookrightarrow} & \overline{\mathcal{M}}_g.
\end{array}
$$

Therefore the projection $\pi|_{\mathrm{Im}\,\Upsilon} \colon \mathrm{Im}\,\Upsilon \to \mathrm{Im}\,\Psi$ is the $4:1$ morphism we described above.

Remark 4.2.4. Before stating the main theorem of this section, we describe the local picture of π at the points of $\mathrm{Im}\,\Upsilon$. We call (C, ϕ) the admissibile S_3-cover obtained via the gluing. By Remark 2.2.18, the local picture of $\overline{\mathcal{R}}_{g,S_3}$ at $[\mathsf{C}, \phi]$ is

$$
\left(\mathrm{Def}(C_1, p_*') \oplus \mathbb{A}^1_{\tilde{t}_q} \oplus \mathbb{A}^1_{\tilde{t}_E} \right) / \underline{\mathrm{Aut}}(\mathsf{C}, \phi).
$$

Here $\tilde{t}_q$ and $\tilde{t}_E$ are the (non-canonical) coordinates associated to the smoothing of node q and to the deformation of $(\mathsf{E}, \phi_\mathsf{E})$. At the same time the local picture of $\overline{\mathcal{M}}_g$ at $[C]$ is

$$
\left(\mathrm{Def}(C_1, p_*') \oplus \mathbb{A}^1_{t_q} \oplus \mathbb{A}^1_{t_E} \right) / \mathrm{Aut}(C).
$$

Here t_q and t_E are associated to the smoothing of node q and the deformation of the elliptic curve E. As the S_3-type at q is trivial, $\tilde{t}_q = t_q$, while for the other coordinate there are different cases. As the automorphism group acts non-trivially only on t_q, t_E and $\tilde{t}_E$, we focus on these coordinates. Moreover we recall that the canonical elliptic tail involution of E, acts trivially.

We consider again the case treated by Harris and Mumford in [22], the birational morphism $g \colon S([C]) \to B$ at a general non-canonical singularity $[C]$. This means that the associated curve C is the junction of (C_1, p_*'), a genus $g-1$ smooth automorphism free curve, and (E, p_*), an elliptic tail. By construction $S([C]) \cong S_1 \times \mathrm{Def}(C_1, p_*')$ and $B \cong B_1 \times \mathrm{Def}(C_1, p_*')$, where S_1 is a singular surface, B_1 is a smooth surface and g is the identity on the component $\mathrm{Def}(C_1, p_*')$. We denote by D the projection of $\mathrm{Im}\,\Psi$ on S_1. In particular the coordinates t_q, t_E span exactly the tangent space to S_1 at any point of D.

If $[\mathsf{C}, \phi]$ is a general non-canononical singularity or $\overline{\mathcal{R}}_{g,S_3}$, then the curve coarse space C is a union $C_1 \cup E$ as before. We denote by $S([\mathsf{C}, \phi])$ the component of $\pi^{-1}(S([C]))$ containing $[\mathsf{C}, \phi]$. By Lemma 4.2.1, the restriction ϕ_E is trivial or an admissible N-cover, we focus in the second case and we observe that as a consequence $\mathrm{Im}\,\Upsilon \subset S([\mathsf{C}, \phi])$. Following the construction

above, $S([\mathsf{C}, \phi]) = S_2 \times \mathrm{Def}(C_1, p'_*)$, where S_2 is a singular surface. We call D' the projection of $\mathrm{Im}\,\Upsilon$ on S_2. With an abuse of notation we call $\pi \colon S_2 \to S_1$ the natural projection.

In the following list we describe the local picture of $S_2 \to S_1$ at any point. For the sake of simplicity we identify D' with $\overline{\mathcal{R}}^N_{1,S_3}$ and D with $\overline{\mathcal{M}}_{1,1}$. We use the notation $\left(\frac{a_1}{r_1}, \cdots, \frac{a_m}{r_m}\right)$ to denote the diagonal matrix $\mathrm{Diag}(\xi_{r_1}^{a_1}, \ldots, \xi_{r_m}^{a_m})$, where $\xi_r = \exp(2\pi i / r)$ is the privileged rth root of the unit.

- At the two points $[E'_4], [E''_4] \in S_2$, the automorphism a_4 does not lift. It acts as $t_q \mapsto \xi_2 t_q$ and $t_E \mapsto \xi_2 t_E$ locally at $[E_4] \in S_1$, therefore the local picture of π at $[E'_4]$ and $[R''_4]$ is the canonical projection $\mathbb{C}^2 \to \mathbb{C}^2 / \left(\frac{1}{2}, \frac{1}{2}\right)$.
- At the two points $[E'_6], [E''_6] \in S_2$, the automorphism a_6 does not lift to $[E'_6]$ but it lifts to $[E''_6]$. It acts as $t_q \mapsto \xi_3 t_q$ and $t_E \mapsto \xi_3 t_E$ at $[E_6] \in S_1$, therefore the local picture of π at $[E'_6]$ is the projection $\mathbb{C}^2 \to \mathbb{C}^2 / \left(\frac{1}{3}, \frac{1}{3}\right)$, and the local picture of π at $[E''_6]$ is the identity $\mathbb{C}^2 / \left(\frac{1}{3}, \frac{1}{3}\right) \to \mathbb{C}^2 / \left(\frac{1}{3}, \frac{1}{3}\right)$.
- At the two points $[E'_0], [E''_0] \in S_2$, we have that $t_E = \tilde{t}_E^3$ in the case of $[E'_0]$ because of the definition of admissible S_3-cover. Instead in the case of $[E''_0]$, $t_E = \tilde{t}_E$. In both cases there are no additional automorphisms, therefore the local picture of π at $[E'_0]$ is the projection $\mathbb{C}^2 \to \mathbb{C}^2 / \left(1, \frac{1}{3}\right) \cong \mathbb{C}^2$, and the local picture at $[E''_0]$ is the identity $\mathbb{C}^2 \to \mathbb{C}^2$.
- Elsewhere on D' and on the whole surface S_2, the projection π is étale, therefore its local picture is the identity $\mathbb{C}^2 \to \mathbb{C}^2$.

Theorem 4.2.5. *Consider a desingularization $\widehat{\mathcal{R}}_{g,S_3} \to \overline{\mathcal{R}}_{g,S_3}$ of the moduli space of genus g curves equipped with an admissible S_3-cover. Then,*

$$H^0\left(\overline{\mathcal{R}}^{\mathrm{reg}}_{g,S_3}, nK_{\overline{\mathcal{R}}^{\mathrm{reg}}_{g,S_3}}\right) = H^0\left(\widehat{\mathcal{R}}_{g,S_3}, nK_{\widehat{\mathcal{R}}_{g,S_3}}\right)$$

for n sufficiently big and divisible.

Proof. We are going to prove the result for a general non-canonical singularity of $\overline{\mathcal{R}}_{g,S_3}$. As a consequence, the extension of pluricanonical forms is true also for any non-canonical singularity via a patchwork of "good" neighborhoods analogous to what we do in the proof of Theorem 4.1.2.

Consider a general non-canonical singularity $[\mathsf{C}, \phi] \in \overline{\mathcal{R}}_{g,S_3}$, therefore $\mathsf{C} = C_1 \cup \mathsf{E}$ as described before and by Lemma 4.2.1 the restriction ϕ_E is trivial or it is an admissible N-cover. In the first case, $[\mathsf{C}, \phi]$ is in a component of $\pi^{-1}(\mathrm{Im}\,\Psi)$ which is isomorphic to $\mathrm{Im}\,\Psi$ and the construction of $S([\mathsf{C}, \phi])$ is the same of Theorem 4.1.2. In the second case, $[\mathsf{C}, \phi]$ is in $\mathrm{Im}\,\Upsilon$ and we are going to show a neighborhood $S([\mathsf{C}, \phi]) = S_2 \times \mathrm{Def}(C_1, p'_*)$ and a birational morphism $S_2 \to B_2$ such that B_2 has at most canonical singularities.

Consider the blowup at the points $[E_4], [E_6] \in D \subset S_1$. We denote by A_4 and A_6 the two corresponding exceptional curves, by the description of singularities we gave above, the autointersection numbers of these divisors are $A_4^2 = -2$ and $A_6^2 = -3$. We denote by S_1^* the blown up surface and by $\overline{D}$ the strict transform of D after the blowups. We know by the Harris-Mumford result resumed above, that there exists a contraction of the curve $A_4 \cup A_6 \cup \overline{D}$, and the contracted surface is smooth. By [6, Theorem II.11] we must have a sequence of (-1)-curve contractions, and this happens if and only if $\overline{D}^2 = -1$, and all the three curves are rational.

Consider the blowup at the points $[E_4'], [E_4''], [E_6'], [E_6''] \in D' \subset S_2$. We denote by A_4', A_4'', A_6', A_6'' the corresponding exceptional divisors, by S_2^* the blown up surface and by $\overline{D}'$ the strict transform of D'. From the description of the singularities we gave above, we know $(A_4')^2 = (A_4'')^2 = (A_6')^2 = -1$ and $(A_6'')^2 = -3$. Moreover, after the blowups there exists a morphism $\tilde{\pi} \colon S^{*}2 \to S^{*}1$ of degree 4 and whose ramification locus is $2A_6'$ (plus some component of codimension 2). Therefore $\tilde{\pi}^*\overline{D} = 4\overline{D}'$, and so $(\overline{D}')^2 = -4$. By costruction all the A_i' and A_i'' are rational curves and $\overline{D}'$ is a rational curve too by Lemma 4.2.3. We obtain that there exists a birational morphism $S_2 \to B_2$ contracting the curve $A_4' \cup A_4'' \cup A_6' \cup A_6'' \cup D'$. Indeed, by [5, Theorem III.5.1] there exists such a contraction and B_2 has a singularity of type $\mathbb{C}^2/\left(\frac{1}{2}, \frac{1}{2}\right)$, that is a canonical singularity.

We obtained a contraction $g' \colon S([\mathsf{C}, \phi]) \to B' = B_2 \times \mathrm{Def}(C_1, p_*')$ such that B' has only canonical singularities, moreover there exists a locus $Z' \subset B'$ of codimension 2 such that $(g')^{-1}(B' \backslash Z') \cong B' \backslash Z'$. By construction $(g')^{-1}(B' \backslash Z')$ is naturally a subset of $\overline{\mathcal{R}}_{g,S_3}$ and this allows to conclude as in the previous case. □

5. Evaluating the Kodaira dimension

In order to calculate the Kodaira dimension, we need to develop some calculations in the tautological ring of the moduli space. In §5.1 we develop Grothendieck Riemann-Roch type calculations for vector bundles. In §5.2 we apply it to evaluate the canonical divisor $K_{\overline{\mathcal{R}}_{g,S_3}}$ and prove its bigness over $\overline{\mathcal{R}}_{g,S_3}^{S_3}$, the connected component of $\overline{\mathcal{R}}_{g,S_3}$ of connected S_3-covers.

5.1. Adapted Grothendieck Riemann-Roch.

5.1.1. *Tautological classes.* In this section we recall some well known *tautological* classes in the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ of the moduli space of curves, and their generalizations to $\overline{\mathcal{R}}_{g,G}$. For a wider survey of the tautological relations and the tautological rings structure see [4, §17], [26] and [14].

There exists two natural morphisms "coming from the geometry of curves" on the moduli spaces $\overline{\mathcal{M}}_{g,n}$. With this we mean that we can define them using the modular interpretation of the space.

- The *forgetful morphism* is a morphism
$$\mu \colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$$
sending any geometric point $[C; p_1, \ldots, p_n]$ to the same marked stable curve without the last point, $[C; p_1, \ldots, p_{n-1}]$.
- The *gluing morphisms* are of two types
$$\iota \colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2} \quad \text{and} \quad \iota \colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}.$$
In the first case $[C; p_1, \ldots, p_{n_1+1}] \times [C'; p_1', \ldots, p_{n_2+1}']$ is sent two the junction of C and C' along the marked points p_{n_1+1} and p_{n_2+1}'. The new curve maintains all the other marked points. In the second case $[C; p_1, \ldots, p_{n+2}]$ is sent to the quotient curve $C/(p_{n+1} \sim p_{n+2})$ with the same other marked points.

Definition 5.1.1. The system of tautological rings $R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$ with g, n varying on the non-negative integers, is the smallest system of $\mathbb{Q}$-algebras closed under the pushforwards of the forgetful morphisms and the gluing morphisms.

We define n tautological ψ-classes inside the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$. We consider the universal family $u\colon \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, where $\mathcal{C}_{g,n}$ is a Deligne-Mumford stack such that every geometric fiber of u is isomorphic to the associated n-marked stable curve, and there exist n sections $\sigma_1, \ldots, \sigma_n \colon \overline{\mathcal{M}}_{g,n} \to \mathcal{C}_{g,n}$. For every $i = 1, \ldots, n$ the line bundle $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g,n}$ is the ith cotangent line bundle, $\mathcal{L}_i := \sigma_i^*(T_u^\vee)$. Then we define

$$\psi_i := c_1(\mathcal{L}_i) \in A^1(\overline{\mathcal{M}}_{g,n}).$$

Remark 5.1.2. The ψ-classes are in the tautological ring $R^1(\overline{\mathcal{M}}_{g,n})$, as showed for example in [14].

There are two other type of classes that are very important for our analysis and belong to the tautological ring. To introduce κ-classes we consider the log-canonical line bundle on $\mathcal{C}_{g,n}$, $\omega_u^{\log} := \omega_u(\sigma_1 + \cdots + \sigma_n)$. Therefore we have

$$\kappa_d := u_*\left(c_1(\omega_u^{\log})^{d+1}\right).$$

As before the class κ_d is well defined in the Chow ring $A^d(\overline{\mathcal{M}}_{g,n})$. We state without proof the known fact that $\kappa_d = \mu_*(\psi_{n+1}^{d+1})$ and being ψ_{n+1} in the tautological ring, the κ-classes is contained in the tautological ring too.

The Hodge bundle over $\overline{\mathcal{M}}_{g,n}$ is the rank g vector bundle $\mathcal{E} := u_*\omega_u$, *i.e.* the vector bundle whose fiber at $[C; p_1, \ldots, p_n]$ is $H^0(C; \omega_C)$. The *Hodge class* is

$$\lambda := c_1(\mathcal{E}) \in A^1(\overline{\mathcal{M}}_{g,n}).$$

The Hodge class is proved to be in the tautological ring $R^1(\overline{\mathcal{M}}_{g,n})$ in [25].

We define the universal family also on $\overline{\mathcal{R}}_{g,G}$ and we use the same notation $u\colon \mathcal{C}_{g,G} \to \overline{\mathcal{R}}_{g,G}$. We will see in the next section that this family is equipped with a universal twisted G-cover. On $A^*\left(\overline{\mathcal{R}}_{g,G}\right)$ it is possible to define as before the κ classes and the Hodge class λ.

5.1.2. *Using Grothendieck Riemann-Roch in the Chow ring of $\mathcal{R}_{g,G}$.* The universal family of curves over $\mathcal{C}_{g,G} \to \overline{\mathcal{R}}_{g,G}$ is equipped with a universal twisted G-cover $\Phi\colon \mathcal{C}_{g,G} \to BG$. In particular for every geometric point $[\mathsf{C}, \phi]$ of $\overline{\mathcal{R}}_{g,G}$, the restriction of Φ to the associated geometric fiber is isomorphic to the twisted G-cover $\phi\colon \mathsf{C} \to BG$.

We consider the singular locus $\mathcal{N} \subset \mathcal{C}_{g,G}$ of the universal family, whose points are the nodes of $\mathcal{C}_{g,G}$ fibers. Furthermore, we consider the stack $\mathcal{N}'$ whose points are nodes equipped with the choice of a privileged branch. There exists a natural étale double cover $\mathcal{N}' \to \mathcal{N}$, and an involution $\varepsilon\colon \mathcal{N}' \to \mathcal{N}'$ associated to the branch switch. There exists also a natural decomposition of $\mathcal{N}'$: given a conjugacy class $[\![h]\!]$ in $[\![G]\!]$, we denote by $\mathcal{N}'_{i,[\![h]\!]} \subset \mathcal{N}'$ the substack of nodes such that the associated privileged branch is in a component of genus i, and it has $[\![h]\!]$ as G-type. In the case of a node of type 0, the component is $\mathcal{N}'_{0,[\![h]\!]}$. Therefore

$$\mathcal{N}' = \bigsqcup_{\substack{0 \le i \le g-1, \\ [\![h]\!] \in [\![G]\!]}} \mathcal{N}'_{i,[\![h]\!]}.$$

We denote the natural projection by $j\colon \mathcal{N}' \to \overline{\mathcal{R}}_{g,G}$. Furthermore, we denote by $j_{i,[\![h]\!]}$ the restriction of the map j to the component $\mathcal{N}'_{i,[\![h]\!]}$. We finally define the classes ψ and ψ' on $\mathcal{N}'$:

$$\psi := c_1(T_{\mathcal{N}'}^\vee); \quad \psi' := c_1(\varepsilon^* T_{\mathcal{N}'}^\vee).$$

In order to evaluate the Chern character of some line bundle pushforwards, we state a generalization of the Grothendieck Riemann-Roch formula, following the approach of Chiodo in [9]. In the case treated by Chiodo, this vector bundle is the universal root of the trivial (or canonical) line bundle. Here we generalize to the case of any vector bundle coming from a representation of the group G.

Consider W, a dimension w representation of group G, then W can be regarded as a vector bundle on BG. We consider the universal cover $\Phi\colon \mathcal{C}_{g,G} \to BG$, then the pullback

$$W_{\mathcal{C}_{g,G}} := \Phi^* W$$

is a vector bundle of rank w on the universal family $\mathcal{C}_{g,G}$. From now on we will use the more compact notation $W_{\mathcal{C}}$.

We observe that the projection $j_{i,[\![h]\!]}\colon \mathcal{N}'_{i,[\![h]\!]} \to \overline{\mathcal{R}}_{g,G}$ is locally isomorphic to $B\boldsymbol{\mu}_r \to \operatorname{Spec}\mathbb{C}$ at every point of $\overline{\mathcal{R}}_{g,G}$, where $r = r(h)$ is the order of class $[\![h]\!]$ in G. We follow the approach of [29, §2.2] to decompose the restriction of $W_{\mathcal{C}}$ to any locus $\mathcal{N}'_{i,[\![h]\!]}$. The local picture of $\mathcal{C}_{g,G}$ at any point of $\mathcal{N}'_{i,[\![h]\!]}$ is a Deligne-Mumford stack $[U/\boldsymbol{\mu}_r]$ where U is an affine scheme, therefore the vector bundle restriction $W_{\mathcal{C}}|_{\mathcal{N}'_{i,[\![h]\!]}}$ is a $\boldsymbol{\mu}_r$-equivariant vector bundle. Following [20, §2.3.2], $\boldsymbol{\mu}_r$ acts naturally on $W_{\mathcal{C}}|_{\mathcal{N}'_{i,[\![h]\!]}}$, so there exists a decomposition in subbundles,

$$W_{\mathcal{C}}|_{\mathcal{N}'_{i,[\![h]\!]}} = W_0 \oplus W_1 \oplus \cdots \oplus W_{r-1},$$

where W_k is the eigen-subbundle with eigenvalue ξ_r^k for all $k = 0, \ldots, r-1$. We denote by $w_{i,[\![h]\!]}(k)$ the rank of W_k, and clearly these integers satisfy the equation $\sum_k w_{i,[\![h]\!]}(k) = w$. We also recall the Bernoulli polynomials $B_d(x)$ defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{d=0}^{\infty} B_d(x)\frac{t^d}{d!}.$$

The Bernoulli numbers $B_d := B_d(0)$ are the evaluations of the Bernoulli polynomials at 0. With this setting we can state the following.

Proposition 5.1.3. *On $\overline{\mathcal{R}}_{g,G}$ we have the following evaluation for the degree d component of the Chern character of $Ru_* W_{\mathcal{C}}$.*

$$\operatorname{ch}_d(Ru_* W_{\mathcal{C}}) = \frac{w \cdot B_{d+1}}{(d+1)!}\kappa_d +$$

$$+\frac{1}{2} \cdot \sum_{\substack{0 \leq i \leq g-1, \\ [\![h]\!] \in [\![G]\!]}} \left(\sum_{0 \leq k < r(h)} \left(\frac{r(h)^2 \cdot w_{i,[\![h]\!]}(k) \cdot B_{d+1}\left(k/r(h)\right)}{(d+1)!} \right) \cdot (j_{i,[\![h]\!]})_* \left(\sum_{a+a'=d-1} \psi^a(-\psi')^{a'} \right) \right).$$

This formula follows directly from Tseng formula (7.3.6.1) in [29]. In Tseng notation the morphism ev_{n+1} is the u morphism, the representation W is denoted by F and moreover

$$(u_*(\operatorname{ch}(ev^*W)Td^\wedge(L_{n+1})))_d$$

is the first term in our formula, the one with κ classes. The ψ_i-classes terms are associated to marked points and therefore are absent in our formula. Finally, the terms A_m (see [29, Definition 4.1.2]) give the last term of our formula.

5.2. **Bigness of the canonical divisor.** We consider the moduli space $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ of genus g curves equipped with a connected admissible S_3-cover. This is a component of $\overline{\mathcal{R}}_{g,S_3}$.

The goal of this section is to prove that the canonical divisor of $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ is big over the subspace $\widetilde{\mathcal{R}}_{g,S_3}^{S_3} \subset \overline{\mathcal{R}}_{g,S_3}^{S_3}$ of covers on irreducible curves, for every odd genus $g > 11$. By Proposition 3.3.2, this implies that $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ is of general type for every odd genus $g \geq 13$.

The approach follows the strategy of [11] for $\overline{\mathcal{R}}_{g,\boldsymbol{\mu}_3}^{\boldsymbol{\mu}_3}$. We write down the canonical divisor as a sum

$$(5.1) \qquad K = \alpha \cdot \mathcal{U} + \beta \cdot \mathsf{M} + E + \gamma \cdot \lambda \in \mathrm{Pic}_{\mathbb{Q}}(\widetilde{\mathcal{R}}_{g,S_3}^{S_3}).$$

Here $\mathcal{U}, \mathsf{M}, E$ are effective divisors, λ is the hodge class, α, β positive coefficients and γ a strictly positive coefficient.

5.2.1. *Basic notions of syzygy theory.* The divisor $\mathcal{U}$ in equation (5.1) will be defined following the approach of Chiodo-Eisenbud-Farkas-Schreyer paper [11]. It is the locus of curves with "extra" syzygies with respect to a particular vector bundle, and it has a determinantal structure over an open subset of $\overline{\mathcal{R}}_{g,S_3}^{S_3}$. To properly define this, we recall some fundamental notions of syzygy theory over stable curves, following the notations of Aprodu-Farkas paper [2].

Consider a finitely generated graded module N over the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$. The module has a minimal free resolution $0 \leftarrow N \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$, where

$$F_i = \sum_j S(-i-j)^{b_{i,j}}.$$

The numbers $b_{i,j}$ are well defined and are called the Betti numbers of N, moreover we have

$$b_{i,j} = \dim(\mathrm{Tor}_i^S(N; \mathbb{C}))_{i+j}.$$

Remark 5.2.1. In an irreducible flat family of modules N_t, the Betti numbers $b_{i,j}(N_t)$ are semicontinuous, and simultaneously take minimal values on an open set. The jumping locus for their values is where one of the values $b_{i,j}(N_t)$ is bigger than this minimum.

For every stable curve we consider a line bundle $L \in \mathrm{Pic}(C)$, a sheaf $\mathcal{F}$ on C, the polynomial ring $S := \mathrm{Sym}\, H^0(C; L)$ and the graded S-module

$$\Gamma_C(\mathcal{F}; L) := \bigoplus_{n \in \mathbb{Z}} H^0(C; \mathcal{F} \otimes L^{\otimes n}).$$

From now on we will use the Green notation by calling

$$K_{i,j}(C; \mathcal{F}, L) := (\mathrm{Tor}_i^S(\Gamma_C(\mathcal{F}, L); \mathbb{C}))_{i+j}.$$

One fundamental idea of syzygy theory is that the vector space $K_{i,j}(C; \mathcal{F}, L)$ can be evaluated via a minimal S resolution of $\mathbb{C}$, where the latter is seen as a graded S-module. To do this we introduce the vector bundle M_L of rank $H^0(L) - 1$ over C, via the following short exact sequence

$$(5.2) \qquad 1 \to M_L \to H^0(C; L) \otimes \mathcal{O}_C \xrightarrow{\mathrm{ev}} L \to 1.$$

Using this vector bundle we can state the following lemma

Lemma 5.2.2 (see [2, Theorem 2.6]). *Consider C a stable curve, L a line bundle on it, $\mathcal{F}$ a coherent sheaf on it and m a positive integer. Then,*

$$K_{m,1}(C; \mathcal{F}, L) = H^0\left(\bigwedge^m M_L \otimes \mathcal{F} \otimes L\right).$$

5.2.2. *The Koszul divisor.* The symmetric group S_3 has 3 irreducible representations.
  (1) The trivial representation $\mathbb{I}\colon S_3 \to \mathrm{GL}(1;\mathbb{C}) = \mathbb{C}^*$;
  (2) the parity representation $\epsilon\colon S_3 \to \mathbb{C}^*$, which sends even elements to 1 and odd elements to -1;
  (3) given a vector space $R \cong \mathbb{C}^2$, the representation $\rho\colon S_3 \to \mathrm{GL}(R) \cong \mathrm{GL}(2;\mathbb{C})$ such that

$$\rho((12)) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \quad \rho((23)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \rho((13)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Remark 5.2.3. In particular, if we consider the tautological representation over the vector space $P := \langle v_1, v_2, v_3 \rangle_{\mathbb{C}}$ such that S_3 acts naturally by permutation, then P is the direct sum of the trivial representation and R, $P = \mathbb{C}_{\mathbb{I}} \oplus R$. At the same time, if we consider the regular representation $\mathbb{C}[S_3]$ of dimension $|S_3| = 6$, we have $\mathbb{C}[S_3] = \mathbb{C}_{\mathbb{I}} \oplus \mathbb{C}_{\epsilon} \oplus R^{\oplus 2}$.

As explained above, we consider R as a line bundle over the stack BS_3. Given a twisted G-cover $[\mathsf{C}, \phi]$, the pullback $R_{\mathsf{C}} := \phi^* R$ is a rank 2 vector bundle over C. In particular if $\mathsf{C} = C$ is a scheme theoretic curve, R_C is a usual scheme theoretic vector bundle.

We consider the case of odd genus $g = 2i + 1$, and we focus on the Koszul cohomology $K_{i,1}$. We introduce the locus $\mathcal{U}_g$ as the locus with non-zero cohomology $K_{i,1}$. Supposing that the minimal value assumed by $\dim K_{i,1}$ on $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ is 0, this is therefore the jumping locus for $K_{i,1}$. In particular we will show that $\mathcal{U}_g$ is a virtual divisor, *i.e.* $\mathcal{U}_g$ is an effective divisor if and only if $\dim K_{i,1}$ takes value 0 on a general curve, or equivalently on at least one curve.

Definition 5.2.4. Given an odd genus $g = 2i + 1$,

$$\mathcal{U}_g := \left\{ [C, \phi] \in \mathcal{R}_{g,S_3}^{S_3} \mid K_{i,1}(C; R_C, \omega_C) \neq 0 \right\} \subset \overline{\mathcal{R}}_{g,S_3}^{S_3}.$$

We want to show that $\mathcal{U}_g$ is a virtual divisor. For any stable curve C, by Lemma 5.2.2 applied in the case $L = \omega_C$ with the vector bundle R_C as the sheaf $\mathcal{F}$, we have

$$K_{i,1}(C; R_C, \omega_C) = H^0 \left(\bigwedge^i M_\omega \otimes \omega_C \otimes R_C \right).$$

We can reformulate the definition of $\mathcal{U}_g$ with another scheme theoretic condition. Consider the Equation (5.2) in the case of the canonical line bundle $L = \omega_C$. As ω_C is a line bundle, by a well known property we have the short exact sequence

$$0 \to \bigwedge^i M_\omega \otimes \omega_C \otimes R_C \to \bigwedge^i H^0(\omega_C) \otimes \omega_C \otimes R_C \to \bigwedge^{i-1} M_\omega \otimes \omega_C^{\otimes 2} \otimes R_C \to 0.$$

Passing to the long exact sequence we have

$$0 \to H^0 \left(\bigwedge^i M_\omega \otimes \omega_C \otimes R_C \right) \to \bigwedge^i H^0(\omega_C) \otimes H^0(\omega_C \otimes R_C) \xrightarrow{\Lambda} H^0 \left(\bigwedge^{i-1} M_\omega \otimes \omega_C^{\otimes 2} \otimes R_C \right)$$

Proposition 5.2.5. *The two vector spaces*

$$\bigwedge^i H^0(\omega_C) \otimes H^0(\omega_C \otimes R_C) \quad and \quad H^0 \left(\bigwedge^{i-1} M_\omega \otimes \omega_C^{\otimes 2} \otimes R_C \right)$$

have the same dimension. As a consequence any point $[C, \phi]$ of $\mathcal{R}^{S_3}_{g,S_3}$ is in $\mathcal{U}_g$ if and only if the associated Λ morphism is not an isomorphism.

Proof. We start by proving that

(5.3)
$$\bigwedge^i H^0(\omega_C) \otimes H^1(\omega_C \otimes R_C) = H^1\left(\bigwedge^{i-1} M_\omega \otimes \omega_C^{\otimes 2} \otimes R_C\right) = 0.$$

We observe that this two terms fit in the long exact sequence

$$\cdots \to \bigwedge^i H^0(\omega_C) \otimes H^1(\omega_C \otimes R_C) \to H^1\left(\bigwedge^{i-1} M_\omega \otimes \omega_C^{\otimes 2} \otimes R_C\right) \to H^2\left(\bigwedge^i M_\omega \otimes \omega_C \otimes R_C\right) \to \cdots$$

and because the last term is 0, it suffices to prove that $H^1(\omega_C \otimes R_C) = H^0(R^\vee) = 0$.

Consider the group quotient $S_3/N = \boldsymbol{\mu}_2$, it induces a twisted $\boldsymbol{\mu}_2$-cover $C \xrightarrow{\phi} BS_3 \to B\boldsymbol{\mu}_2$. This is equivalent to a principal $\boldsymbol{\mu}_2$-bundle $\varphi \colon F \to C$. As R_C is defined via the representation $\rho \colon S_3 \to \mathrm{GL}(R) \cong \mathrm{GL}(2; \mathbb{C})$, its pullback on F is given by the restriction $\rho|_N$ which is a representation of $N \cong \boldsymbol{\mu}_3$, which decomposes as direct sum of two irreducible $\boldsymbol{\mu}_3$ representations of rank 1. We observe in particular that $\varphi^* R_C^\vee = \eta \oplus \eta^{\otimes 2}$ where η is a non-trivial line bundle such that $\eta^{\otimes 3} \cong \mathcal{O}_F$. Therefore $H^0(\eta) = H^0(\eta^{\otimes 2}) = 0$ and *a fortiori* $H^0(R_C^\vee) = 0$.

Since the first cohomology group is trivial, we have that the Euler characteristic of both vector bundles coincides with the dimension of their spaces of global sections. In particular, if $\mathcal{E}$ is one of these vector bundles, $H^0(\mathcal{E}) = \chi(\mathcal{E}) = \deg(\mathcal{E}) + \mathrm{rk}(\mathcal{E})(1 - g)$.

- If $\mathcal{E} = \bigwedge^i H^0(\omega_C) \otimes H^0(\omega_C \otimes R_C)$, then

$$\dim(\mathcal{E}) = \dim\left(\bigwedge^i H^0(\omega_C)\right) \cdot \dim\left(H^0(\omega_C \otimes R_C)\right).$$

  The first term is simply $\binom{g}{i}$ because $h^0(\omega_C) = g$. For the second term we have $h^0(\omega_C \otimes R_C) = \chi(\omega_C \otimes R_C) = \deg(\omega_C \otimes \mathcal{R}_C) + 2 - 2g = 2g - 2$. Therefore

$$\dim\left(\bigwedge^i H^0(\omega_C) \otimes H^0(\omega_C \otimes R_C)\right) = 4i\binom{2i + 1}{i}.$$

- If $\mathcal{E} = H^0(\bigwedge^{i-1} M_\omega \otimes \omega_C^{\otimes 2} \otimes R_C)$, then

$$\deg(\mathcal{E}) = 2 \cdot \deg\left(\bigwedge^{i-1} M_\omega\right) + \binom{g-1}{i-1} \cdot \deg(\omega_C^{\otimes 2} \otimes R_C) = 4(3i + 1)\binom{2i}{i-1}.$$

  Knowing that $\mathrm{rk}(\mathcal{E}) = 2 \cdot \binom{g-1}{i-1}$, we have

$$\dim(\mathcal{E}) = 4\binom{2i}{i-1}(2i + 1) = 4i\binom{2i+1}{i},$$

  and this completes the proof.

$\square$

Proposition 5.2.6. *Consider a general hyperelliptic curve (C, p) of genus $g \geq 2$ with a Weierstrass point. Then there exists a twisted S_3-cover $\phi \colon C \to BG$ such that, if R_C is the vector bundle of rank 2 associated to the R representation via ϕ, then $H^0(C, R_C((g-1)p)) = 0$.*

This proves the effectiveness of $\mathcal{U}_g$ by following the approach of [11, Theorem 2.3]: via the proposition we prove that $[C, \phi]$ is a point outside $\mathcal{U}_g$ in the case of genus $g = 2i + 1$, and therefore that $\mathcal{U}_g$ is effective. Indeed, $M_{\omega_C} = \mathcal{O}_C(-2p)^{\oplus g-1}$ and therefore

$$\bigwedge^i M_\omega \otimes \omega_C \otimes R_C = (R_C((g-1)p))^{\oplus \binom{g-1}{i}}.$$

This means that $[C, \phi] \notin \mathcal{U}_g$ if and only if $H^0(R_C((g-1)p)) = 0$.

Proof. We recall that R_C is obtained by pulling back via ϕ the irreducible representation $\rho \colon S_3 \to \mathrm{GL}(R) \cong \mathrm{GL}(2; \mathbb{C})$. Consider the normal subgroup $N \subset S_3$, and the group quotient $S_3/N = \boldsymbol{\mu}_2$. This induces a twisted $\boldsymbol{\mu}_2$-cover $C \xrightarrow{\phi} BS_3 \to B\boldsymbol{\mu}_2$. As C is a scheme theoretic curve, this is simply a principal $\boldsymbol{\mu}_2$-bundle $\pi \colon F \to C$. We use the notation $W := R_C((g-1)p)$ and we want to show that π^*W decomposes as direct sum of two line bundles. By construction the image of $F \to C \to BS_3$ is BN, that is π^*W is induced by a representation of $N \cong \boldsymbol{\mu}_3$. The group $\boldsymbol{\mu}_3$ has 3 irreducible representations: the trivial one, the identity $\eta \colon \boldsymbol{\mu}_3 \hookrightarrow \mathbb{C}^*$ and $\eta^{\otimes 2}$. The restriction $\rho|_N$ decomposes as $\eta \oplus \eta^{\otimes 2}$. With a little abuse of notation we denote by η and $\eta^{\otimes 2}$ the order 3 line bundles induced by these representations. Moreover, we call $A_1 := \eta((g-1)p)$ and $A_2 := \eta^{\otimes 2}((g-1)p)$.

By construction we have the decomposition $\pi^*W = \pi^*A_1 \oplus \pi^*A_2$. We observe that $H^0(W) = H^0(\pi^*W)_+$, that is the space of sections which are invariant by the natural involution of F. If $H^0(\pi^*W)_+ \neq 0$, then one between $H^0(\pi^*A_1)_+$ and $H^0(\pi^*A_2)_+$ is non-empty, but $H^0(\pi^*A_j)_+ = H^0(A_j)$ and by [11, Theorem 2.3] there exists (C, ϕ) such that both $H^0(A_j)$ are empty.

To prove this last point, observe that by [11, Theorem 2.3] there exists a point a point $[C, \nu]$ outside the divisor $\mathcal{U}_{g,3} \subset \overline{\mathcal{R}}_{g,\boldsymbol{\mu}_3}^{\boldsymbol{\mu}_3}$, where ν is a non-trivial third root of $\mathcal{O}_C$, or equivalently a non-trivial principal $\boldsymbol{\mu}_3$-bundle on C. As the locus of hyperelliptic curves with a Weiestrass point and a third root, is not included in $\mathcal{U}_{g,3}$, then by dimensional considerations there exists an hyperelliptic curve C such that $[C, \nu] \notin \mathcal{U}_{g,3}$ for any ν. And this completes the proof. $\square$

We use the result of Proposition 5.2.5 to evaluate the class of $\mathcal{U}_g$ in the Chow ring, and also the class of its closure on the space $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$. Consider the universal family $u \colon \mathcal{C}_{g,S_3} \to \overline{\mathcal{R}}_{g,S_3}^{S_3}$ and the universal rank 2 vector bundle $R_{\mathcal{C}}$ associated to the representation R. Introduce the vector bundle $\mathfrak{M}_u$ on $\overline{\mathcal{R}}_{g,S_3}^{S_3}$ defined by the short exact sequence

$$0 \to \mathfrak{M}_u \to u^*(u_*\omega_u) \to \omega_u \to 0.$$

The geometric fiber of $\mathfrak{M}_u$ over any point $[\mathsf{C}, \phi]$ of $\mathcal{R}_{g,S_3}^{S_3}$ is the previously defined vector bundle M_ω.

Definition 5.2.7. We introduce a series of sheaves on $\overline{\mathcal{R}}_{g,S_3}^{S_3}$,

$$\mathbb{E}_{j,b} := u_* \left(\bigwedge^j \mathfrak{M}_u \otimes \omega_u^{\otimes b} \otimes R_{\mathcal{C}} \right),$$

with $j \geq 0$ and $b \geq 1$ integers.

Remark 5.2.8. To prove that these sheaves are locally free on $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$, by Grauert's Theorem it suffices to prove that $h^1(M_\omega \otimes \omega_{\mathsf{C}}^{\otimes b} \otimes R_{\mathsf{C}}) = 0$ for every twisted S_3-cover (C, ϕ) in $\widetilde{\mathcal{R}}_{g,S_3}^{S_3}$. As showed in the proof of Proposition 5.2.5, this reduces to prove $h^0((\omega_{\mathsf{C}}^\vee)^{\otimes(b-1)} \otimes R_{\mathsf{C}}^\vee) = 0$ for all $b \geq 1$. In the proof of Proposition 5.2.5 we showed this equality for $b = 1$ and a scheme

theoretic curve $\mathsf{C} = C$, but the same proof works for every $b \geq 1$ and for every twisted curve C such that its coarse space is irreducible.

By Proposition 5.2.5, on $\mathcal{R}^{S_3}_{g,S_3}$ the locus $\mathcal{U}_g$ is the degeneration locus of a morphism between the vector bundles $\mathbb{E}_{i-1,2}$ and $\bigwedge^i \mathbb{E} \otimes \mathbb{E}_{0,1}$, where $\mathbb{E}$ is the Hodge bundle $u_* \omega_u$. In the following we will evaluate this degeneration locus on the space $\widetilde{\mathcal{R}}^{S_3}_{g,S_3}$. We denote by $\overline{\mathcal{U}}_g$ the closure of the locus $\mathcal{U}_g$ on the space $\widetilde{\mathcal{R}}^{S_3}_{g,S_3}$.

Lemma 5.2.9. *Given an odd genus $g = 2i + 1$, the class of $\overline{\mathcal{U}}_g$ in $\mathrm{Pic}_{\mathbb{Q}}(\widetilde{\mathcal{R}}^{S_3}_{g,S_3})$ is*

$$[\overline{\mathcal{U}}_g] = c_1\left(\mathrm{Hom}\left(\mathbb{E}_{i-1,2}, \bigwedge^i \mathbb{E} \otimes \mathbb{E}_{0,1}\right)\right) = \mathrm{rk}(\mathbb{E}_{i-1,2}) \cdot \left(\sum_{b=0}^{i} (-1)^{b+1} c_1\left(\bigwedge^{i-b} \mathbb{E} \otimes \mathbb{E}_{0,b+1}\right)\right).$$

Proof. Given two vector bundles $\mathcal{A}$ and $\mathcal{B}$ over $\widetilde{\mathcal{R}}^{S_3}_{g,S_3}$, the morphism vector bundle $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ is isomorphic to $\mathcal{B} \otimes \mathcal{A}^\vee$ and therefore $c_1(\mathrm{Hom}(\mathcal{A}, \mathcal{B})) = \mathrm{rk}(\mathcal{A}) c_1(\mathcal{B}) - \mathrm{rk}(\mathcal{B}) c_1(\mathcal{A})$. To conclude we observe that by the definition of the vector bundles $\mathbb{E}_{j,b}$, they fit in the short exact sequences

$$0 \to \mathbb{E}_{j,b+1-j} \to \bigwedge^j \mathbb{E} \otimes \mathbb{E}_{0,b+1-j} \to \mathbb{E}_{j-1,b+2-j} \to 0,$$

for all $j \geq 0$ and $b \geq j$. $\square$

Lemma 5.2.10. *The first Chern class of $\mathbb{E}_{0,b}$ is*

$$c_1(\mathbb{E}_{0,b}) = 2\lambda + 2\binom{b}{2}\kappa_1 - \frac{1}{4}\delta_0^T - \frac{2}{3}\delta_0^N \in \mathrm{Pic}_{\mathbb{Q}}(\widetilde{\mathcal{R}}^{S_3}_{g,S_3}).$$

Proof. This is a direct application of Proposition 5.1.3 in the evaluation of $\mathrm{ch}_1(Ru_* W_{\mathcal{C}})$. By [10] we have

$$\lambda = \mathrm{ch}_1(u_* \omega_u) = \frac{B_2}{2}\kappa_1 + \frac{1}{2} \cdot \sum_{\substack{0 \leq i \leq g-1, \\ [\![h]\!] \in [\![G]\!]}} r(h) \cdot (j_{i,[\![h]\!]})_* \left(\frac{B_2}{2} \cdot \sum_{a+a'=d-1} \psi^a (-\psi')^{a'}\right).$$

To complete the proof we only need the eigenvalues decomposition of $R((12))$ and $R((123))$, where $R \colon S_3 \to \mathrm{GL}(\mathbb{C}, 2)$ is the irreducible S_3 representation of dimension 2. $\square$

With these lemmata we can develop the calculations to evaluate $[\overline{\mathcal{U}}_g]$ in terms of the Hodge class and the boundary classes.

Proposition 5.2.11. *In the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\widetilde{\mathcal{R}}^{S_3}_{g,S_3})$ we have,*

$$[\overline{\mathcal{U}}_g] = \mathrm{rk}(\mathbb{E}_{i-1,2}) \cdot 2 \cdot \binom{2i-2}{i-1}\left(\frac{2(3i+1)}{i}\lambda - \delta_0' - \left(\frac{6i+1}{4i}\right)\delta_0^T - \left(\frac{5i+2}{3i}\right)\delta_0^N\right).$$

Proof. From the result of Lemma 5.2.10, we have that

$$c_1\left(\bigwedge^{i-b} \mathbb{E} \otimes \mathbb{E}_{0,b+1}\right) = \mathrm{rk}(\bigwedge^{i-b} \mathbb{E}) \cdot c_1(\mathbb{E}_{0,b+1}) + \mathrm{rk}(\mathbb{E}_{0,b+1}) \cdot c_1(\bigwedge^{i-b} \mathbb{E}) =$$

$$= \binom{g}{i-b} \cdot \left(2\lambda + 2\binom{b+1}{2}\kappa_1 - \frac{1}{4}\delta_0^T - \frac{2}{3}\delta_0^N\right) + 2(2b+1)(g-1) \cdot \binom{g-1}{i-b-1}\lambda.$$

This, thanks to Lemma 5.2.9, allows to conclude the evaluation.   □

We are ready to introduce the divisor **M** of Equation (5.1) over $\overline{\mathcal{R}}_{g,S_3}^{S_3}$, with $g = 2i+1$ odd genus. Harris and Mumford introduced in [22] the following divisor.

Definition 5.2.12. If $W_d^r(C)$ is the set of complete linear series over C of degree d and dimension at least r, it defines the locus

$$\mathcal{M}_{g,i+1}^i := \{[C] \in \mathcal{M}_g \mid W_{i+1}^1(C) \neq \varnothing\} \subset \mathcal{M}_g.$$

In the same paper they proved

$$\mathcal{M}_{g,i+1}^i = c' \cdot \left(\frac{6(i+2)}{i+1}\lambda - \delta_0\right) \in \text{Pic}_{\mathbb{Q}}(\widetilde{\mathcal{M}}_g),$$

where $\widetilde{\mathcal{M}}_g \subset \overline{\mathcal{M}}_g$ is the locus of irreducible stable curves, and c' is a positive coefficient. Then,

$$[\pi^*\mathcal{M}_{g,i+1}^1] = c' \cdot \left(\frac{6(i+2)}{i+1}\lambda - \delta_0' - 2\delta_0^T - 3\delta_0^N\right) \in \text{Pic}_Q(\widetilde{\mathcal{R}}_{g,S_3}^{S_3}),$$

where π is the natural projection $\widetilde{\mathcal{R}}_{g,S_3}^{S_3} \to \widetilde{\mathcal{M}}_g$.

Summarizing,

(5.4) $[K_{\widetilde{\mathcal{R}}_{g,S_3}}] = \alpha \cdot [\overline{\mathcal{U}}_g] + \beta \cdot [\pi^*\mathcal{M}_{g,i+1}^1] + E + \gamma \cdot \lambda,$

for every odd genus $g = 2i+1$, with α and β positive coefficients and E an effective sum of boundary divisors.

Proposition 5.2.13. *In Equation (5.4) the γ coefficient can be chosen strictly positive for any $i > 5$.*

Proof. By scaling appropriately every coefficient, the equation is equivalent to choosing a real number $s \in [0,1]$ such that

$$s \cdot \left(\frac{2(3i+1)}{i}\lambda - \delta_0' - \left(\frac{6i+1}{4i}\right)\delta_0^T - \left(\frac{5i+2}{3i}\right)\delta_0^N\right) +$$

$$+(1-s) \cdot \left(\frac{6(i+2)}{i+1}\lambda - \delta_0' - 2\delta_0^T - 3\delta_0^N\right) + E + \gamma \cdot \lambda =$$

$$= \frac{13}{2}\lambda - \delta_0' - \frac{3}{2}\delta_0^T - 2\delta_0^N.$$

For E to be an effective divisor we must have $s \cdot \left(\frac{6i+1}{4i}\right) + (1-s) \cdot 2 \geq \frac{3}{2}$ and $s \cdot \left(\frac{5i+2}{3i}\right) + (1-s) \cdot 3 \geq 2$, and therefore $s \leq \frac{3i}{4i-2}$ is a necessary and sufficient condition for E to be an effective boundary divisor.

To complete the proof we evaluate the γ coefficient,

$$s \cdot \left(\frac{6i+2}{i}\right) + (1-s) \cdot \left(\frac{6i+12}{i+1}\right) + \gamma = \frac{13}{2}.$$

After calculations this gives $\gamma = \frac{i-11}{2(i+1)} + s \cdot \frac{4i-2}{i(i+1)}$, which means a maximal possible value of

$$\gamma = \frac{i-5}{2(i+1)},$$

which is positive if and only if $i > 5$.   □

After this proposition, the canonical divisor over $\widetilde{\mathcal{R}}^{S_3}_{g,S_3}$ is big for every odd genus $g \geq 13$. Then, thanks to Proposition 3.3.2, the moduli space $\overline{\mathcal{R}}^{S_3}_{g,S_3}$ is of general type for every odd genus $g \geq 13$.

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