Stanley's character polynomials and coloured factorisations in the symmetric group

Journal Article

Citation:

Rattan, A. (2008) Stanley's character polynomials and coloured factorisations in the symmetric group - *Journal of Combinatorial Theory, Series A*, 115(4), pp. 535-546

Publisher version available at: http://dx.doi.org/10.1016/j.jcta.2007.06.008

Stanley's character polynomials and coloured factorizations in the symmetric group

A. Rattan
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA, 02139
email: arattan@math.mit.edu

February 10, 2007

Abstract

In Stanley [8], the author introduces polynomials which help evaluate symmetric group characters and conjectures that the coefficients of the polynomials are positive. In [9], the same author gives a conjectured combinatorial interpretation for the coefficients of the polynomials. Here, we prove the conjecture for the terms of highest degree.

1 Introduction

A *partition* is a weakly ordered list of positive integers $\lambda = \lambda_1\lambda_2\ldots\lambda_k$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. The integers $\lambda_1, \ldots, \lambda_k$ are called the *parts* of the partition λ, and we denote the number of parts by $\ell(\lambda) = k$. If $\lambda_1 + \ldots + \lambda_k = d$, then λ is a partition of d, and we write $\lambda \vdash d$. We denote by $\mathcal{P}$ the set of all partitions, including the single partition of 0 (which has no parts). For partitions $\omega, \lambda \vdash n$, let $\chi_\omega(\lambda)$ be the character of the irreducible representation of the symmetric group $\mathfrak{S}_n$ indexed by ω, and evaluated on the conjugacy class $\mathcal{C}_\lambda$ of $\mathfrak{S}_n$, where C_λ is the class of all permutations whose disjoint cycle lengths are specified by the parts of λ. For a permutation α, we use the notation $\kappa(\alpha)$ to denote the number of cycles of α. We use the convention that permutation are multiplied from right to left.

Various scalings of irreducible symmetric group characters have been considered in the recent literature, one of which is the central object of this paper. Suppose that $\mu \vdash k$ and $k \leq n$. For the conjugacy class $\mathcal{C}_{\mu 1^{n-k}}$, the *normalized character* is given by

$$\widehat{\chi}_\omega(\mu\, 1^{n-k}) = (n)_k \frac{\chi_\omega(\mu\, 1^{n-k})}{\chi_\omega(1^n)},$$

where $(n)_k$ is the falling factorial $n(n-1)\cdots(n-k+1)$. The normalized character has been the topic of much recent literature and has been shown to have connections with combinatorics and free probability, see for example [1, 2, 4, 7, 8].

The subject of this paper is a particular polynomial expression for the normalized character, introduced in Stanley [8]. Consider the partition of n with p_i parts of size q_i, for i from 1 to m, with q_1 the largest part. Thus, $p_1, p_2, \ldots, p_m$ are positive integers and $q_1 \geq q_2 \geq \cdots \geq q_m$ (see Figure 1). We denote this partition of n by $\mathbf{p} \times \mathbf{q}$. Define the expression F_k in indeterminates $p_1, \ldots, p_m, q_1, \ldots, q_m$ by

$$F_k(p_1, p_2, \ldots, p_m;\ q_1, q_2, \ldots, q_m) = \widehat{\chi}_{\mathbf{p}\times\mathbf{q}}(k\, 1^{n-k}). \qquad (1)$$

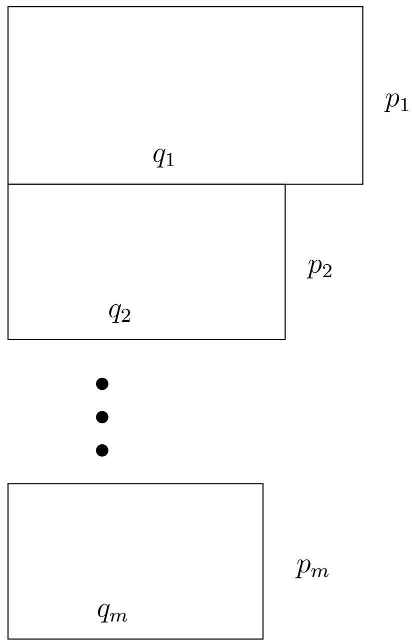

Figure 1: The shape $\mathbf{p} \times \mathbf{q}$.

We often use $\mathbf{p}$ for $(p_1, \ldots, p_m)$ and $\mathbf{q}$ for $(q_1, \ldots, q_m)$, giving us the notation $F_k(\mathbf{p};\ \mathbf{q})$ for $F_k(p_1, p_2, \ldots, p_m;\ q_1, q_2, \ldots, q_m)$. The following theorem appears in Stanley [8, Proposition 1].

Theorem 1.1 (Stanley). *$F_k(\mathbf{p};\ \mathbf{q})$ is a polynomial in the p's and q's such that $(-1)^k F_k(1, 1, \ldots, 1;\ -1, -1, \ldots, -1) = (k + m - 1)_k$.*

In light of this theorem, we call the polynomials in (1) *Stanley's character polynomials*. These polynomials are the main objects in this paper. For example, for the case $m = 2$, the first two polynomials are

$$F_1(a, p;\ b, q) = -ab - pq,$$
$$F_2(a, p;\ b, q) = -a^2 b + ab^2 - 2\,apq - p^2 q + pq^2$$

where we have set $p_1 = a, p_2 = p, q_1 = b$ and $q_2 = q$. Also in Stanley [8] the author states that if one defines $F_\mu(\mathbf{p};\ \mathbf{q})$ as

$$F_\mu(\mathbf{p};\ \mathbf{q}) = \widehat{\chi}_{\mathbf{p}\times\mathbf{q}}(\mu\ 1^{n-k}), \tag{2}$$

where $\mu \vdash k \leq n$ then $F_\mu(\mathbf{p};\ \mathbf{q})$ is also polynomial. We emphasize that $F_\mu(\mathbf{p};\ \mathbf{q})$ is independent of n as it is, formally, a polynomial in indeterminates in $p_1, \ldots, p_m$ and $q_1, \ldots, q_m$. Our understanding is that the polynomial $F_\mu(\mathbf{p};\ \mathbf{q})$ evaluates to the normalized character in (2) when evaluated at a shape $\mathbf{p} \times \mathbf{q}$ that is a partition of $n \geq k$.

In Stanley [8], the author conjectures that $(-1)^k F_\mu(\mathbf{p}; -\mathbf{q})$ has positive coefficients. This conjecture has only been proved in the case $m = 1$ for general μ (see [8, Theorem 1.1]); in particular, the conjecture is not known to be true for $m > 1$ even when μ has one part; that is, the conjecture is unknown even for $(-1)^k F_k(\mathbf{p}; -\mathbf{q})$. Some partial results showing positivity of the coefficients of $(-1)^k F_k(\mathbf{p}; -\mathbf{q})$ were given in Rattan [5], but otherwise little is known about these polynomials. Recently, Stanley [9] has a conjectured combinatorial interpretation for $F_\mu(\mathbf{p};\ \mathbf{q})$, which we now explain.

Let $[m]$ be the set $\{1, 2, \ldots, m\}$ and $\mathfrak{S}_k^{(m)}$ be the set of permutations of the set $[k]$ whose cycles are coloured by $[m]$. Formally, if $C_k(\alpha)$ is the set of cycles in α then members of $\mathfrak{S}_k^{(m)}$ are ordered pairs (α, ψ) where $\alpha \in \mathfrak{S}_k$ and $\psi : C_k(\alpha) \longrightarrow [m]$. Define a product $\circ : \mathfrak{S}_k^{(m)} \times \mathfrak{S}_k \longrightarrow \mathfrak{S}_k^{(m)}$ by the following: for $(\alpha, \psi) \in \mathfrak{S}_k^{(m)}$, $\beta \in \mathfrak{S}_k$ and $(\alpha, \psi) \circ \beta = (\gamma, \nu)$, where

1. $\gamma = \alpha\beta$, and
2. If $u = (u_1\ u_2\ \cdots u_t)$ is a cycle of γ and $C^\alpha(u_i)$ is the cycle of α containing the symbol u_i then
$$\nu(u) = \max_{1 \le i \le t} \{\psi(C^\alpha(u_i))\}.$$

In words, $\nu(u) = t$, where t is the largest value of $\psi(w)$, and where w ranges over all cycles in α with an element in common with u (see [9, Page 3] for an example). For $(\alpha, \psi) \in \mathfrak{S}_k^{(m)}$ let $\kappa^{(m)}(\alpha, \psi) = (\kappa_1^{(m)}(\alpha, \psi), \kappa_2^{(m)}(\alpha, \psi), \ldots)$ where $\kappa_i^{(m)}(\alpha, \psi)$ is the number of cycles of α coloured i and for $\mathbf{p} = (p_1, p_2, \ldots, p_m)$, $\mathbf{p}^{\kappa^{(m)}(\alpha,\psi)} = \prod_i p_i^{\kappa_i^{(m)}(\alpha,\psi)}$. We can now state the conjecture we address in this paper, found in Stanley [9].

Conjecture 1.2 (Stanley). *Suppose that $\mu \vdash k$, and let ω_μ be some fixed element in the conjugacy class C_μ in $\mathfrak{S}_k$. Then*
$$(-1)^k F_\mu(\mathbf{p};\ -\mathbf{q}) = \sum_{(\alpha,\psi) \in \mathfrak{S}_k^{(m)}} \mathbf{p}^{\kappa^{(m)}(\alpha,\psi)} \mathbf{q}^{\kappa^{(m)}((\alpha,\psi)\circ\omega_\mu)}.$$

As stated earlier, in Stanley [8, Theorem 1.1] Conjecture 1.2 has been proved for $m = 1$ (note that this corresponds to factorizations without any colours), but otherwise this conjecture remains open. However, for arbitrary m, it is shown in [8] that
$$F_k(\mathbf{p};\ \mathbf{q}) = -\frac{1}{k}[x^{-1}]_\infty (x)_k \prod_{j=1}^{m} \frac{(x - (q_j + p_j + p_{j+1} + \cdots + p_m))_k}{(x - (q_j + p_{j+1} + p_{j+2} + \cdots + p_m))_k}, \tag{3}$$
where for an expression $g(x)$ the notation $[x^{-1}]_\infty g(x)$ is the coefficient of $1/x$ when $g(x)$ is expanded in powers of $1/x$. From this it follows (see [8, Proposition 2]) that if $G_k(\mathbf{p};\ \mathbf{q})$ are the terms of highest degree in $F_k(\mathbf{p};\ \mathbf{q})$ and $G_{\mathbf{p};\,\mathbf{q}}(x) = 1 + \sum_{k=0}^{\infty} G_k(\mathbf{p};\ \mathbf{q})x^{k+1}$ then
$$G_{\mathbf{p};\,\mathbf{q}}(x) = 1 + \sum_{i \ge 1} G_{i-1}(\mathbf{p};\ \mathbf{q})x^i = \frac{x}{\left(x \prod_{j=1}^{m} \frac{(1-(q_j+p_{j+1}+\cdots+p_m)x)}{(1-(q_j+p_j+\cdots+p_m)x)}\right)^{\langle -1 \rangle}}. \tag{4}$$
where $\langle -1 \rangle$ denotes compositional inverse. It easily follows from (4) that $G_{\mathbf{p};\,\mathbf{q}}(x)$ satisfies
$$-G_{\mathbf{p};-\mathbf{q}}(-x) \prod_{j=1}^{m} \frac{(-G_{\mathbf{p};-\mathbf{q}}(-x) - (p_j + \cdots + p_m)x + q_j x)}{(-G_{\mathbf{p};-\mathbf{q}}(-x) - (p_{j+1} + \cdots + p_m)x + q_j x)} = -1 \tag{5}$$
It is (5) that we will eventually use to prove our main theorem. It is known for $m = 1$ (Stanley [8, Page 9]) that the series $G_{p;\,q}(x)$ is the generating series for *top* factorizations in the symmetric group; namely, we have
$$G_{p;\,q}(x) = 1 + px + \sum_{k \ge 1} x^{k+1} \sum_{\substack{u \in \mathfrak{S}_k \\ \kappa(u)+\kappa(u\omega_k)=k+1}} (-1)^k p^{\kappa(u)} (-q)^{\kappa(u\cdot\omega_k)}.$$

where ω_k is used for $\omega_{(1\ 2\ \cdots\ k)}$. Here, we have $\kappa(u) = \kappa^{(1)}(u)$ is the number of cycles of u. We call products of the type in the previous sum i.e. products of permutations $\alpha\beta = \gamma$ in $\mathfrak{S}_k$ such $\kappa(\alpha) + \kappa(\beta) = k + \kappa(\gamma)$ *top products*, *top factorizations* or *minimal factorizations*. Such factorizations are an extremal case; namely, if $\alpha, \beta, \gamma \in \mathfrak{S}_k$ and $\alpha\beta = \gamma$ then

$$\kappa(\alpha) + \kappa(\beta) \leq k + \kappa(\gamma) \tag{6}$$

(see Goulden and Jackson [3]).

Set,

$$\mathrm{TopFact}_{\mathbf{p};\,\mathbf{q}}(x) = \sum_{k\geq 1} x^{k+1} \sum_{\substack{(\alpha,\psi)\in\mathfrak{S}_k^{(m)} \\ \kappa(\alpha)+\kappa(\alpha\omega_k)=k+1}} \mathbf{p}^{\kappa^{(m)}(\alpha,\psi)} \mathbf{q}^{\kappa^{(m)}((\alpha,\psi)\circ\omega_k)}.$$

The following are the two main theorems of this paper. We will see that Corollary 1.4 follows from Theorem 1.3.

Theorem 1.3 (Main Theorem). *Conjecture 1.2 holds for the term of highest degree in* $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$*; that is,*

$$-G_{\mathbf{p};-\mathbf{q}}(-x) = -1 + (p_1 + p_2 + \cdots + p_m)x + \mathrm{TopFact}_{\mathbf{p};\,\mathbf{q}}(x). \tag{7}$$

Corollary 1.4 (Main Corollary). *For any partition* $\mu \vdash k$*, Conjecture 1.2 holds for the terms of highest degree in* $(-1)^k F_\mu(\mathbf{p};\ -\mathbf{q})$.

We prove the main theorems at the end of Section 4.

2 The Goulden-Jackson construction for top factorizations

In Goulden and Jackson [3], the authors give a construction for top factorizations in the symmetric group in terms of black and white plane edge rooted trees. Namely, they give a bijection between products of permutations $\alpha\beta = (1\ 2\ \cdots k)$ in $\mathfrak{S}_k$ such that $\kappa(\alpha) + \kappa(\beta) = k + 1$ and edge rooted plane trees on $k + 1$ vertices, with vertices coloured black and white such that adjacent vertices receive different colours. The correspondence is very simple to state; in the tree, label the edges beginning with the root edge (which obtains the label 1). The edges are labelled in numerical order by travelling around the tree, keeping the tree to the right and labelling an edge only when traversed from its white vertex to its black vertex. From each white vertex a cycle is obtained for the permutation α by considering the sequence of edges incident with the vertex in a clockwise direction. Likewise, the cycles of β are obtained from the black vertices. In Figure 2, the plane tree given corresponds to the pair $\alpha = (1\ 6\ 8\ 9)(2\ 5)(3)(4)(7)(10)(11)$ and $\beta = (1\ 5)(2\ 3\ 4)(6\ 7)(8)(9\ 10\ 11)$. One can easily check that $\alpha\beta = (1\ 2\ \cdots\ 11)$.

It is easy to see from the above construction, that top *coloured* factorizations are obtained in the following way. Let $(\alpha, \psi) \in \mathfrak{S}_k^{(m)}$ and β be such that $\alpha\beta = (1\ 2\ \cdots k)$. Now, using the construction of Goulden and Jackson, from α and β create a black and white plane edge rooted tree. As the white vertices of the tree correspond to cycles of α, give the white vertices an additional colour i for $1 \leq i \leq m$ using ψ. For a black vertex v, an additional colour i for $1 \leq i \leq m$ is given with the rule that v obtains colour j, where j is the maximum colour amongst all neighbours of v. Thus, to be clear, vertices have two types of colours; they are either black or white and they have a colour i, with $1 \leq i \leq m$. As the black vertices determine the cycles of β, the labels i determine a function $\phi : C_k(\beta) \longrightarrow [m]$. Note that ϕ clearly determines a function $\phi' : C_k(\beta^{-1}) \longrightarrow [m]$ by $\phi'(u^{-1}) = \phi(u)$ for

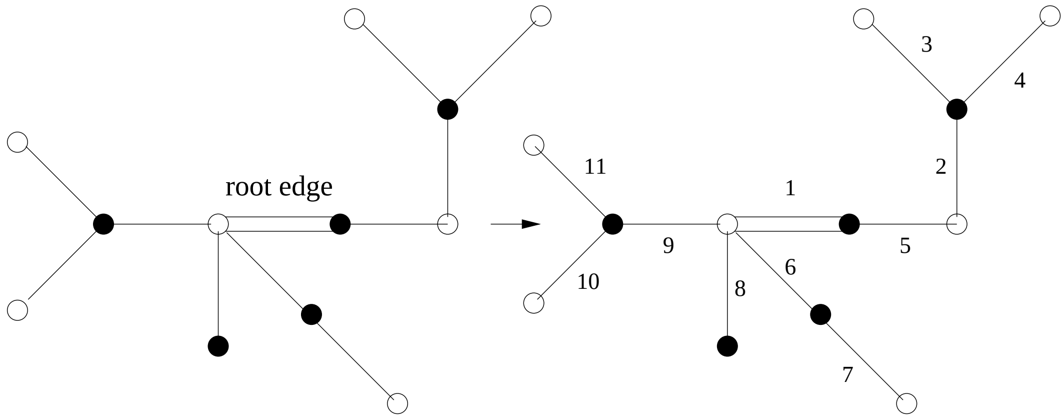

Figure 2: On the left is a black and white plane edge rooted tree. Using the description in the first paragraph of Section 2 to label the edges of the tree on the left, we obtain the tree on the right. A clockwise rotation around each white vertex gives a cycle of the permutation $\alpha = (1\ 6\ 8\ 9)(2\ 5)(3)(4)(7)(10)(11)$, and likewise for the black vertices and $\beta = (1\ 5)(2\ 3\ 4)(6\ 7)(8)(9\ 10\ 11)$.

any cycle u of β. One can easily check that $(\alpha, \psi) \circ (1\ 2\ \cdots k)^{-1} = (\beta^{-1}, \phi')$ (see Figure 3). We will, therefore, call the set of black and white plane edge rooted trees with this additional colour restriction *coloured black and white plane edge rooted trees* and denote this class by $\mathcal{T}$. Define for a tree $T \in \mathcal{T}$ the weights $\omega_w^{(m)}(T) = (\omega_{w_1}^{(m)}(T), \omega_{w_2}^{(m)}(T), \dots)$ and $\omega_b^{(m)}(T) = (\omega_{b_1}^{(m)}(T), \omega_{b_2}^{(m)}(T), \dots)$ where $\omega_{w_i}^{(m)}(T)$ and $\omega_{b_i}^{(m)}(T)$ are the number of white, respectively black, vertices in T coloured i. As usual, let $\mathbf{p}^{\omega_w^{(m)}(T)} = \prod_i p_i^{\omega_{w_i}^{(m)}(T)}$ and $\mathbf{q}^{\omega_b^{(m)}(T)} = \prod_i q_i^{\omega_{b_i}^{(m)}(T)}$, and define

$$T_{\mathbf{p};\,\mathbf{q}}(x) = \sum_{T \in \mathcal{T}} \mathbf{p}^{\omega_w^{(m)}(T)} \mathbf{q}^{\omega_b^{(m)}(T)} x^{\text{number of vertices of } T}.$$

Evidently, we have the following proposition from the above discussion.

Proposition 2.1.

$$T_{\mathbf{p};\,\mathbf{q}}(x) = \mathrm{TopFact}_{\mathbf{p};\,\mathbf{q}}(x).$$

In the following sections we show that $T_{\mathbf{p};\,\mathbf{q}}(x) + (p_1 + p_2 + \cdots + p_m)x - 1 = -G_{\mathbf{p};\,\mathbf{q}}(-x)$, which proves that (7) holds by Proposition 2.1.

3 Planted Trees

It is clear that the class of trees $\mathcal{T}$ is in bijective correspondence with the following class. Let $\mathcal{B}_i$ be the set of *coloured* plane *planted* trees whose planted vertex is coloured black (the planted vertex does not otherwise have a colour i), the vertex adjacent to the planted vertex, which we call the *root*, is white and coloured with the colour i and the colouring of the rest of the tree is consistent with the class of trees in $\mathcal{T}$. The planted vertex gives a linear order to the edges connecting the root to its children (see Figure 4). Define $\mathcal{W}_i$ analogously; that is, $\mathcal{W}_i$ is the class of plane planted trees with planted vertex coloured white (but with no colour i) and with the black root vertex coloured i. In both these classes of trees, a vertex

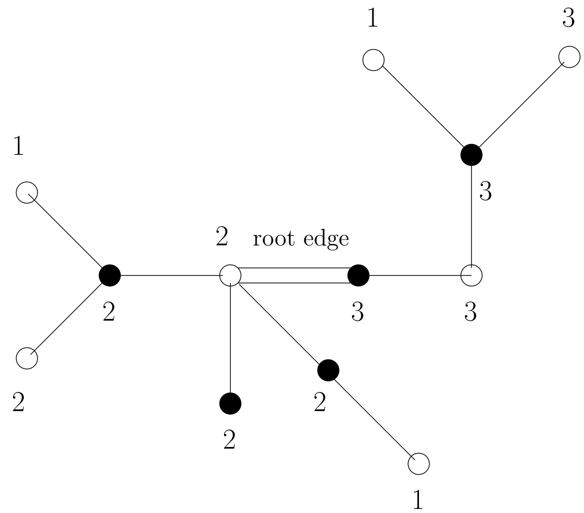

Figure 3: The coloured black and white plane edge rooted tree in this figure is the same as the one on the left in Figure 2, except with colours. Its edges would be labelled as the tree on the right in Figure 2. Thus, this tree corresponds to (α, ψ) and (β, ν), where $\alpha = (1\ 6\ 8\ 9)(2\ 5)(3)(4)(7)(10)(11)$ and $\beta = (1\ 5)(2\ 3\ 4)(6\ 7)(8)(9\ 10\ 11)$, and where the cycles $(1\ 6\ 8\ 9)$ and (4) of α are coloured 2 and 3 by ψ, respectively. Here, we assume $m \geq 3$.

v is the parent of a vertex w and w is, likewise, called a child of v if v and w are connected by an edge and v is on the unique path joining w to the planted vertex of the tree. A tree in the class $\mathcal{B}_2$ is given in Figure 4. Define the generating series

$$\begin{aligned} B_i(x) &= \sum_{T \in \mathcal{B}_i} \mathbf{p}^{\omega_w^{(m)}(T)} \mathbf{q}^{\omega_b^{(m)}(T)} x^{\text{number of non planted vertices of } T} \\ W_i(x) &= \sum_{T \in \mathcal{W}_i} \mathbf{p}^{\omega_w^{(m)}(T)} \mathbf{q}^{\omega_b^{(m)}(T)} x^{\text{number of non planted vertices of } T} \end{aligned} \tag{8}$$

If we let $\mathcal{P}_i$ be the class of white vertices with label i then it is easy to see that

$$\bigcup_{i=1}^{m} \mathcal{P}_i \bigcup \mathcal{T} = \bigcup_{i=1}^{m} \mathcal{B}_i,$$

from which it follows

$$T_{\mathbf{p};\,\mathbf{q}}(x) + (p_1 + p_2 + \cdots + p_m)x = \sum_{i=1}^{m} B_i(x)$$

Thus, in order to find an expression for the generating series $T_{\mathbf{p};\,\mathbf{q}}(x)$ we find one for $\sum_{i=1}^{m} B_i(x)$ (see Figure 4). For convenience, we set $I_{\mathbf{p};\,\mathbf{q}}(x)$ be the previous generating series; that is, we set

$$I_{\mathbf{p};\,\mathbf{q}}(x) = T_{\mathbf{p};\,\mathbf{q}}(x) + (p_1 + p_2 + \cdots + p_m)x = \sum_{i=1}^{m} B_i(x) \tag{9}$$

We now find relations between the classes $\mathcal{B}_i$ and $\mathcal{W}_i$ for $1 \leq i \leq m$. In order to do this we introduce a final class of trees. Define the class of *improperly* coloured trees, denoted $\hat{\mathcal{W}}_i$, planted at a white coloured vertex (but otherwise does not have a colour i) and a black

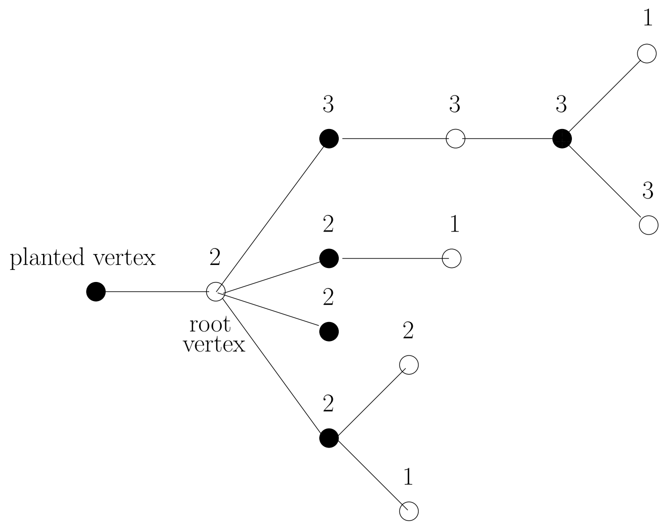

Figure 4: The coloured plane planted tree in $\mathcal{B}_2$ (here, we assume that $m \geq 3$) that corresponds to the coloured plane tree in Figure 3. The tree is obtained by attaching a planted vertex to the white vertex incident with the root edge of the tree in Figure 3. The root edge in Figure 3 becomes the first of the linearly ordered edges (from top to bottom) emanating from the root.

root coloured i. The white children of the black root vertex can only be coloured with the colours of $1, 2, \ldots, i-1$ (hence, the name *improperly* coloured). Also note, we insist that the black root has non-empty subtree below it (otherwise, such a tree would *not* be improperly labelled). Define the generating series $\hat{W}_i(x)$ of the class $\hat{\mathcal{W}}_i$ analogously to the series $W_i(x)$. For $i = 1$, the class $\hat{\mathcal{W}}_i$ is empty and its corresponding generating series is $\hat{W}_1(x) = 0$. We shall see their importance in the next section.

4 Decomposition of the classes $\mathcal{B}_i$ and $\mathcal{W}_i$ and the proof of the Main Theorems

We begin by discussing the decomposition of the class $\mathcal{B}_i$, for $1 \leq i \leq m$. Recall, a tree in this class has a planted black vertex adjacent to a white root vertex with the colour i. Every child of the root vertex is black and because of the colouring rule requiring the colour of a black vertex to be the largest colour amongst it white neighbours, the colours $i, i+1, \ldots, m$ are the possible colours for the children of the white root vertex. Each of these black children have subtrees to which they are attached, and can be made into a planted $\mathcal{W}_j$ tree for $i \leq j \leq m$ by attaching a planted white vertex to each of the black children (see, for example, Figure 5). Note, however, we may also obtain an improperly labelled tree $\hat{\mathcal{W}}_j$ for $i \leq j \leq m$ (as in, for example, the second subtree counting from the top in Figure 5). The final caveat here is that any black child of the white root vertex with colour strictly greater than i, must have a non-trivial subtree below it (for otherwise that black vertex would be not be properly coloured). Since the subtrees are linearly ordered, we see that $B_i(x)$ satisfies

$$B_i(x) = \frac{p_i x}{1 - \Big(W_i(x) + \hat{W}_i(x) + (W_{i+1}(x) - q_{i+1}x) + \cdots + (W_m(x) - q_m x)\Big)}. \tag{10}$$

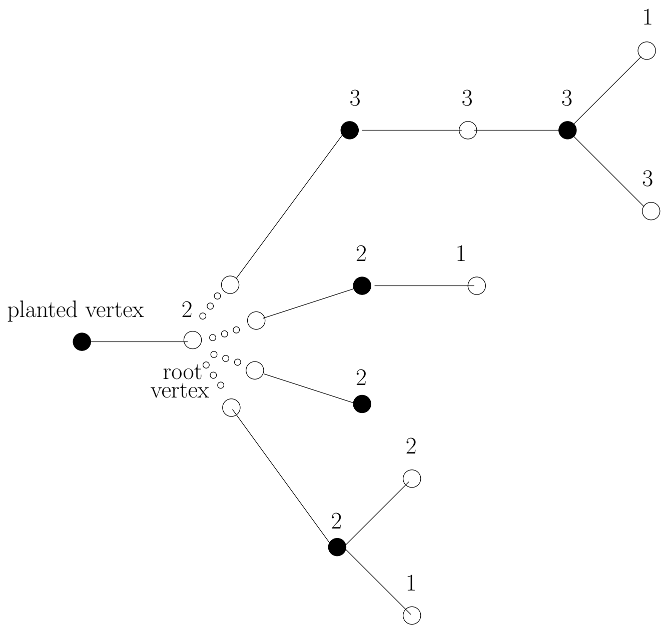

Figure 5: The coloured plane planted tree in $\mathcal{B}_2$ given in Figure 4 decomposed into two $\mathcal{W}_2$, one $\mathcal{W}_3$ and one $\hat{\mathcal{W}}_2$ trees.

We repeat that $\hat{W}_1(x) = 0$. We can similarly find expressions for the generating series $\hat{W}_i(x)$ and $W_i(x)$. Beginning with $\hat{W}_i(x)$, for $2 \le i \le m$, we have by definition a tree in the class $\hat{\mathcal{W}}_i$ has a white planted vertex and black root vertex with colour i. The black root vertex has white children coloured with $1, 2, \dots i-1$. Also by definition, the black root vertex must have a non-trivial subtree beneath it. Since the subtrees are linearly ordered, we have

$$\hat{W}_i(x) = \frac{(q_i x)(B_1(x) + B_2(x) + \cdots + B_{i-1}(x))}{1 - (B_1(x) + B_2(x) + \cdots + B_{i-1}(x))}. \tag{11}$$

For the trees $\mathcal{W}_i$, by definition they are trees with a white planted vertex and a black root vertex with colour i and are properly coloured; that is, the black root vertex has no white children with colour greater than i. The black root vertex may have a trivial subtree beneath it, but if it does not it must have at least one white child coloured i (in order be properly coloured). As these are the only restrictions, we see

$$\begin{aligned} W_i(x) &= \frac{q_i x}{1 - (B_1(x) + B_2(x) + \cdots + B_i(x))} - \frac{(q_i x)(B_1(x) + B_2(x) + \cdots + B_{i-1}(x))}{1 - (B_1(x) + B_2(x) + \cdots + B_{i-1}(x))} \\ &= \frac{q_i x}{1 - (B_1(x) + B_2(x) + \cdots + B_i(x))} - \hat{W}_i(x). \end{aligned} \tag{12}$$

We now show that $I_{\mathbf{p};\,\mathbf{q}}(x)$ given in (9) satisfies the same equation as (5).

Lemma 4.1.

$$B_i(x) = \frac{p_i x(B_1(x) + \cdots + B_i(x) - 1)}{I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{i+1} + \cdots + p_m)x + q_i x}.$$

Proof. Our proof is by induction on i beginning at $i = m$. For $i = m$ we have by (10)

$$\begin{aligned} B_m(x) &= \frac{p_m x}{1 - (W_m(x) + \hat{W}_m(x))} \\ &= \frac{p_m x}{1 - \frac{q_m x}{1-(B_1(x)+\cdots+B_m(x))}} \\ &= \frac{p_m x(B_1(x) + \cdots + B_m(x) - 1)}{I_{\mathbf{p};\,\mathbf{q}}(x) - 1 + q_m x}, \end{aligned}$$

completing the base case.

Now suppose that our conclusion is true for $i = t$. We wish to show our conclusion holds for $i = t - 1$. By the induction hypothesis, we have

$$B_t(x) = \frac{p_t x(B_1(x) + \cdots + B_t(x) - 1)}{I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{t+1} + \cdots + p_m)x + q_t x}.$$

and from (10) we have

$$B_{t-1}(x) = \frac{p_{t-1}x}{1 - \Big(W_{t-1}(x) + \hat{W}_{t-1}(x) + (W_t(x) - q_t x) + \cdots + (W_m(x) - q_m x)\Big)}.$$

from which we obtain

$$\begin{aligned} B_{t-1}(x) &= \frac{-p_{t-1}x}{\begin{array}{l}\Big(W_{t-1}(x) + \hat{W}_{t-1}(x) - \hat{W}_t(x) - q_t x \\ \qquad + \Big(W_t(x) + \hat{W}_t(x) + W_{t+1}(x) - q_{t+1}x + \cdots + W_m(x) - q_m x\Big)\Big) - 1\end{array}} \\ &= \frac{-p_{t-1}x}{\frac{q_{t-1}x}{1-(B_1(x)+\cdots+B_{t-1}(x))} - \frac{q_t x}{1-(B_1(x)+\cdots+B_{t-1}(x))} + \frac{I_{\mathbf{p};\,\mathbf{q}}(x)-1-(p_{t+1}+\cdots+p_m)x+q_t x}{1-(B_1(x)+\cdots+B_t(x))}}, \end{aligned} \tag{13}$$

where the last summand in the denominator of (13) follows from (10) with $i = t$ and the induction hypothesis. Continuing to simplify, (13) becomes

$$\begin{aligned} &\frac{-p_{t-1}x(1 - (B_1(x) + \cdots + B_t(x)))(1 - (B_1(x) + \cdots + B_{t-1}(x)))}{\begin{array}{l}(1 - (B_1(x) + \cdots + B_t(x)))\,((q_{t-1}x - q_t x) + (I_{\mathbf{p};\,\mathbf{q}}(x) - 1 \\ \qquad -(p_{t+1} + \cdots + p_m)x + q_t x)) + B_t(x)(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{t+1} + \cdots + p_m)x + q_t x)\end{array}} \\ &= \frac{-p_{t-1}x(1 - (B_1(x) + \cdots + B_t(x)))(1 - (B_1(x) + \cdots + B_{t-1}(x)))}{\begin{array}{l}(1 - (B_1(x) + \cdots + B_t(x)))(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{t+1} + \cdots + p_m)x + q_{t-1}x) \\ \qquad\qquad + p_t x(B_1(x) + \cdots + B_t(x) - 1)\end{array}} \\ &= \frac{p_{t-1}x(B_1(x) + \cdots + B_{t-1}(x) - 1)}{I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_t + p_{t+1} + \cdots + p_m)x + q_{t-1}x}, \end{aligned}$$

completing the proof. □

Lemma 4.2. *For $0 \leq i \leq m$ the following equation holds:*

$$B_1(x) + \cdots + B_i(x) - 1 = (I_{\mathbf{p};\,\mathbf{q}}(x) - 1) \prod_{j=i+1}^{m} \frac{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_j + \cdots + p_m)x + q_j x)}{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{j+1} + \cdots + p_m)x + q_j x)}$$

Proof. The proof is by induction on i, beginning at $i = m$. The case $i = m$ is trivial.

Now suppose for $i = t$ the statement of this lemma is true; that is,

$$B_1(x) + \cdots + B_t(x) - 1 = (I_{\mathbf{p};\,\mathbf{q}}(x) - 1) \prod_{j=t+1}^{m} \frac{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_j + \cdots + p_m)x + q_j x)}{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{j+1} + \cdots + p_m)x + q_j x)}.$$

By Lemma 4.1, we have

$$B_t(x) = \frac{p_t x\,(B_1(x) + \cdots + B_t(x) - 1)}{I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{t+1} + \cdots + p_m)x + q_t x)}.$$

But,

$$\begin{aligned}
&B_1(x) + \cdots + B_{t-1}(x) - 1 \\
&\qquad = B_1(x) + \cdots + B_t(x) - 1 - \frac{p_t x\,(B_1(x) + \cdots + B_t(x) - 1)}{I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{t+1} + \cdots + p_m)x + q_t x} \\
&\qquad = (B_1(x) + \cdots + B_t(x) - 1)\left(1 - \frac{p_t x}{I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{t+1} + \cdots + p_m)x + q_t x}\right) \\
&\qquad = \left((I_{\mathbf{p};\,\mathbf{q}}(x) - 1) \prod_{j=t+1}^{m} \frac{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_j + \cdots + p_m)x + q_j x)}{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{j+1} + \cdots + p_m)x + q_j x)}\right) \\
&\qquad\qquad\qquad \cdot \frac{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_t + \cdots + p_m)x + q_t x)}{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{t+1} + \cdots + p_m)x + q_t x)} \\
&\qquad = (I_{\mathbf{p};\,\mathbf{q}}(x) - 1) \prod_{j=t}^{m} \frac{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_j + \cdots + p_m)x + q_j x)}{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{j+1} + \cdots + p_m)x + q_j x)},
\end{aligned}$$

completing the proof. □

We now give a proof of the main theorems.

Proof of Theorem 1.3. From Lemma 4.2, we have by setting $i = 0$

$$(I_{\mathbf{p};\,\mathbf{q}}(x) - 1) \prod_{j=1}^{m} \frac{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_j + \cdots + p_m)x + q_j x)}{(I_{\mathbf{p};\,\mathbf{q}}(x) - 1 - (p_{j+1} + \cdots + p_m)x + q_j x)} = -1.$$

Thus, $I_{\mathbf{p};\,\mathbf{q}}(x) - 1$ and $-G_{\mathbf{p};-\mathbf{q}}(-x)$ both satisfy (5). Combining this with (9) and Proposition 2.1 gives the result. □

Proof of Corollary 1.4 If $\mu = \mu_1\mu_2\ldots\mu_\ell \vdash k$, then the terms of highest degree in $(-1)^k F_\mu(\mathbf{p}; -\mathbf{q})$, which have degree $k + \ell(\mu)$, are given by $\prod_{i=1}^{\ell}(-1)^{\mu_i} G_{\mu_i}(\mathbf{p}; -\mathbf{q})$ (see Śniady [7, Theorem 4.9] and [6, Theorem 9] and references therein. To see how Kerov polynomials are used to obtain characters of the symmetric group and Stanley's polynomials, see Rattan [5]). Assuming that $\alpha\beta = \gamma$ in $\mathfrak{S}_k$ where γ has cycle type μ and $\kappa(\alpha) + \kappa(\beta) = k + \kappa(\gamma)$ then the product $\alpha\beta = \gamma$ necessarily decomposes into $\kappa(\gamma)$ products of the form $\alpha_i\beta_i = \gamma_i$, where

1. γ_i is a cycle of γ;
2. $\alpha_i = \alpha_i^1\alpha_i^2 \cdots \alpha_i^s$ and $\beta_i = \beta_i^1\beta_i^2 \cdots \beta_i^t$, where α_i^j and β_i^j are cycles in α and β, respectively; the cycles α_i^j and β_i^j are precisely the cycles in α and β that contain elements in the support of γ_i; i.e. the elements h of $\{1, 2, \ldots, k\}$ such that $\gamma_i(h) \neq h$;
3. if the length of γ_i is k_i then $\kappa(\alpha_i) + \kappa(\beta_i) = k_i + 1$

(see Goulden and Jackson [3, Section 4]). From this it follows that

$$\sum_{\substack{(\alpha,\psi)\in\mathfrak{S}_k^{(m)}\\ \kappa(\alpha)+\kappa(\alpha\omega_\mu)=k+\kappa(\mu)}} \mathbf{p}^{\kappa^{(m)}(\alpha,\psi)}\mathbf{q}^{\kappa^{(m)}((\alpha,\psi)\circ\omega_\mu)} = \prod_{i=1}^{\ell(\mu)}(-1)^{\mu_i}G_{\mu_i}(\mathbf{p};-\mathbf{q}).$$

completing the proof. □

Acknowledgements

This work was supported by a *Natural Sciences and Engineering Research Council of Canada* Postdoctoral Fellowship. I would like to thank Richard Stanley for communicating his conjecture to me and for some helpful discussions. I would also like to thank Karola Meszaros and the anonymous referee for their useful comments on the previous version of this manuscript.

References

[1] P. Biane. Characters of symmetric groups and free cumulants. *Asymptotic Combinatorics with Applications to Mathematical Physics, A. Vershik (Ed.), Springer Lecture Notes in Mathematics*, 1815:185–200, 2003.

[2] P. Biane. Representations of the symmetric groups and free probability. *Advances in Mathematics*, 138:126–181, 1998.

[3] I. P. Goulden and D. M. Jackson. The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group. *European J. Combin.*, 13 (5):357–365, 1992. ISSN 0195-6698.

[4] I.P. Goulden and A. Rattan. An explicit form for Kerov's character polynomials. *Trans. Amer. Math. Soc.*, 359:3669–3685, 2007.

[5] A. Rattan. Positivity results for Stanley's character polynomials. *Journal of Algebra*, 308:26–43, February 2007.

[6] P. Śniady. Free probability and representations of large symmetric groups, arXiv:math.CO/0304275.

[7] P. Śniady. Asymptotics of characters of symmetric groups, genus expansion and free probability. *Discrete Math.*, 306:624–665, 2006.

[8] R. P. Stanley. Irreducible symmetric group characters of rectangular shape. *Séminaire Lothar. Combin.*, 50:B50d, 11pp, 2003.

[9] R. P. Stanley. A conjectured combinatorial interpretation of the normalized irreducible character values of the symmetric group. math.CO/0606467, 2006.