Università Commerciale "Luigi Bocconi"

PhD School

PhD program in Economics and Finance

Cycle: XXXI

Disciplinary Field (code): SECS-P01

Overreacting Beliefs in Finance and Economics

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PhD Thesis by

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ID number: 3008101

Academic Year: 2019/2020

Abstract

This thesis investigates overreacting beliefs in Finance and Economics. Chapter 1 investigates overreaction to news in the term structure of interest rates. We find evidence of overreaction whose intensity is increasing with maturity, causing excess volatility of long term interest rates. We incorporate non rational beliefs into an otherwise standard asset pricing model and we show that it captures excess volatility of asset prices as well as forecast errors predictability. The second Chapter investigates the consequences of over-reacting beliefs when agents interact, via the observation of past actions of others. Even though individually overreaction entails a loss (in the MSE sense), at the aggregate level it injects more private information into the economy, thereby increasing stability and avoiding informational cascades. The third Chapter investigates the foundations of overreaction to information in a constrained Bayesian updating framework. We show that a bound on the surprise an agent can experience from the data implies an overweighting of current information and ultimately overreaction to information.

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Acknowledgements

I am truly grateful to my advisor Nicola Gennaioli for encouraging me in studying beautiful and challenging problems during my PhD. His passion and scientific rigor are priceless and truly inspiring. I benefited tremendously from his forward-looking advices and his patience. For those reasons, I feel extremely lucky to have such guidance.

I thank Anna Battauz, Simone Cerreira Vioglio, Carlo Favero, Massimo Guidolin, Fulvio Ortu, Barbara Rindi, Stefano Rossi and Claudio Tebaldi for all discussions, thoughts and teaching experiences we shared during these years. I thank Andrea, Blanka, Giacomo, Ilaria, Jean Gabriel, Matteo, Riccardo, Ziijan and all friends and collegues I met in Bocconi during my PhD. In particular, I thank Ekaterina and Francesco.

I thank Andrei Shleifer for hosting me at Harvard University and for continuous encouraging and support. I thank Andrei, Francesca, Giorgio, Nicola, Paul, Pedro, Peter, Pierr-Luc, Spencer and Toto for an amazing behavioral reading group. I also thanks all researchers and friends I met in Cambridge, in particular Elisa, Jacob, Jesper, Micheal and all the Alien group of visitors.

I thank Matteo Marsili for introducing me to the fascinating world of Economics and Finance. I thank my family for their support, and in particular Samuele. I thank my friends Benedicta, Pilu, Silvia, Maio, Martino and Alessia, Alice, Enrico, Giordano and Giorgio for unbounded friendship.

Introduction

This thesis consists of three chapters. Each of them is a self-contained paper.

- *Increasing Overreaction and Excess Volatility of Long Rates*;

  Abstract Giglio and Kelly (2018) find that the volatility of long-term rates is too large relative to that of short-term rates for a large class of rational expectations models. I assess the possibility that such excess volatility may come from investor beliefs. I use survey data on analyst expectations and data on market beliefs recovered from observed yields using the methodology of Ross (2015). I obtain three main findings. First, the two datasets reveal a remarkably similar pattern of *horizon dependent* departures from rationality: expectations about long rates over-react relative to expectations about short rates. Second, a model of diagnostic expectations rationalizes this horizon dependent belief distortions and generates excess volatility of long term rates. Third, when calibrated to the data, this model accounts from roughly 80% of the excess volatility puzzle for a reasonable value of the diagnosticity parameter.

- *Learning, Overreaction and the Wisdom of the Crowd* joint with M. Bizzarri;

  Abstract We study the classical sequential social learning problem in a setting where agents depart from the standard Bayesian updating rule. We consider the case of over-reacting - as well under-reacting - individual posterior beliefs, two well known biases in beliefs updating (Benjamin, 2019). Agent posterior beliefs over-react (or under-react) to the current information according to how much it is surprising relative to past information. We study the interplay of distorted posterior beliefs and social learning. We find that in a context with fine grained signals the biases do not impact on the eventual learning, while in a context with coarse signals, such as in the cascades setting of Banerjee (1992), over-reaction can make it easier for agents to learn, because past actions of others become more informative, hence a moderate level of over-reaction is socially optimal.

- *Bounded surprise and overreaction to news* joint with N. Gennaioli.

  Abstract We study the mechanism of over-reaction to news as an optimal Bayesian assessment, subject to a cognitive constraint. First, we reinterpret the Bayesian updating in information theoretic terms: we show that the Bayesian updating is an optimal trade-off between a cost of moving beliefs and an effort to accurately describe the world. Second, we show that a Bayesian agent with an upper bound on the surprise (e.g. a lower bound on accuracy) she can perceive from data, naturally exhibits over-reaction to news. Furthermore, the departures from the Bayesian updating rule is driven by the *representativeness* heuristics: the constrained Bayesian agent exaggerates the true likelihood of those models which better fit the data. Finally, we consider a noisy signal extension of the model and compare it with rational inattention and robust inference settings.

The following paper is not included as a chapter in the present manuscript.

- *Lost in diversification* joint with M.Bardoscia, M. Marsili and V.Volpati (published at *Comptes Rendus Physique*)

  Abstract. As financial instruments grow in complexity more and more information is neglected by risk optimization practices. This brings down a curtain of opacity on the origination of risk, that has been one of the main culprits in the 2007-2008 global financial crisis. We discuss how the loss of transparency may be quantified in bits, using information theoretic concepts. We find that i) financial transformations imply large information losses, ii) portfolios are more information sensitive than individual stocks only if fundamental analysis is sufficiently informative on the co-movement of assets, that iii) securitisation, in the relevant range of parameters, yields assets that are less information sensitive than the original stocks and that iv) when diversification (or securitisation) is at its best (i.e. when assets are uncorrelated) information losses are maximal. We also address the issue of whether pricing schemes can be introduced to deal with information losses. This is relevant for the transmission of incentives to gather information on the risk origination side. Within a simple mean variance scheme, we find that market incentives are not generally sufficient to make information harvesting sustainable.

Chapter 1

Increasing Overreaction and Excess Volatility of Long Rates

1.1 Introduction

Since Shiller (1981) celebrated analysis of the stock market, many studies have documented the so called *excess volatility of asset prices* relative to measures of fundamentals (De Bondt and Thaler (1985), LeRoy and Porter (1981) and Campbell and Shiller (1987) among others). There are two main explanations for this finding. One of them assumes rational expectations and emphasizes the role of discount rate variation (Campbell (2003), Cochrane (2011)). The other one relaxes rationality and views price volatility as reflecting excess volatility of beliefs. Consistent with the latter view, a growing body of work documents excess volatility of beliefs using survey data, e.g. Gennaioli and Shleifer (2018).

A recent paper by Giglio and Kelly (2018) assesses these hypotheses in the realm of the term structure of interest rates. This setting is appropriate because rational term structure models impose precise constraints on the amount of covariation of asset prices at different maturities, even after adjusting for discount rate variation. The key finding of this analysis is that long term interest rates are significantly more volatile with respect to short term interest rates relative to what rational models predict. This finding indicates that beliefs may be a key driver of excess volatility in an important setting such as the bond market where riskless rates are determined. This analysis raises three questions. First, can one go beyond assessing volatility in excess of discount rates and measure beliefs in the bond market? Second, if the answer is affirmative, can one assess which departure of rationality, if any, do these beliefs display? Third, what is the psychological foundation of this departure from rationality, and can it account for the excess volatility of long term rates, not only qualitatively but also quantitatively? This paper seeks to address these questions.

One immediate way to make progress is to proxy the beliefs of the bond market by using expectations data. Figure (1.1) reports a preliminary analysis using data from Blue Chip on professional forecasts of future rates at maturities of $1y$, $2y$, $5y$, $10y$, $20y$ and $30y$ (monthly frequency). For each maturity, I report the pooled correlation between the forecast revision made by the analyst[1] and his subsequent forecast error. Two features stand out in the data. First, at any maturity the average analyst over-reacts to news: when he revises his forecast of future rates up, reality systematically falls below his revised expectation. This is captured by the fact that in Figure (1.1) all correlations are negative. Second, there is more over-reaction for longer term interest rates, namely the coefficient becomes more negative for longer maturities. This latter aspect especially resonates with the possibility that long-term expectations may move too much with news, causing excess volatility in long-term rates relative to short-rates.

To analyze systematically this possibility, in Section 2 I introduce non-rational beliefs into an otherwise standard affine term structure model. In particular, I allow for distortions in belief formation that are maturity dependent. In this analysis, I show that stronger over-reaction of beliefs for longer maturities, as displayed in Figure (1.1), indeed gives rise to excess volatility of equilibrium long term rates relative to short term ones.

[1] The time t forecast revision of a variable X_{t+m} ($m > 0$) is defined as the difference between the time t forecast of X_{t+m} and the time $t-1$ forecast of X_{t+m}.

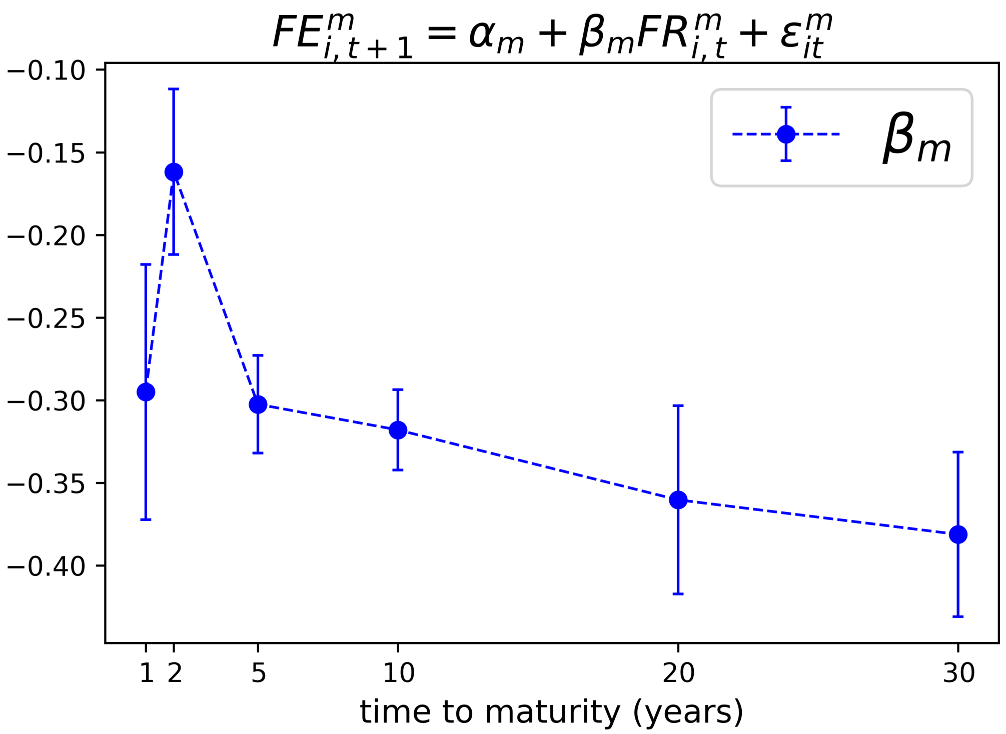


Figure 1.1: Sensitivity of forecast error to past forecast revision using monthly professional forecasts of (annualized) interest rates.

I also show that, within the class of affine models, there is a precise mapping between excess volatility of interest rates at a given maturity and the correlation between forecast revisions and forecast errors at the same maturity. That is, there is a sort of "duality" between the estimates in Figure (1.1) and the amount of interest rate volatility measured from equilibrium yields.

The remainder of the paper systematically studies this connection. One limitation in the analysis of Figure (1.1) concerns the use of survey data. On the one hand, such data are available only for a small subset of maturities, frequencies, and time periods. One would ideally want to have a richer term structure to fully assess the volatility of beliefs. On the other hand, the beliefs of professional forecasters may not be representative about the beliefs of bond traders. Ideally, one would want to measure the beliefs of the marginal investor, and this is clearly not available in conventional datasets.

To address these measurement issues, in Section 3 I use a method developed by Ross (2015) to extract market beliefs from equilibrium yields. This method rests on two assumptions: i) the dynamic of risk factors moving interest rates is stationary, and ii) the stochastic discount factor is path independent. Borovička et al. (2016) criticized assumption ii), showing that it does not hold for asset pricing models that contains long run risks (Bansal and Yaron (2004)) or that more generally include a martingale component in the SDF. Against this critique, I show that even if recovered beliefs are misspecified in the Borovička et al. (2016) sense, the use of Ross' method is still informative with respect to the maturity dependent over- (or under-) reaction that Figure (1.1) exemplifies. The intuition is that maturity dependent information processing distortions entail a violation of the law of iterated expectations, and such violation cannot be obtained under any rational asset pricing model, including those incorporating long run risks or more generally martingale components in the SDF.

I thus proceed to implement Ross' method for the yield curve, and describe the main patterns of this data in Section 4. The recovered beliefs exhibit, similarly to the professional forecasts, increasing over-reaction at long maturities (greater than $10y$). To validate the use of Ross recovered beliefs, I systematically compare the latter to professional forecasts. The two measured of beliefs display a remarkably strong and positive correlation, which suggests that recovered beliefs are not noise, and instead capture systematic features of market expectations. Interestingly, I find that, compared to the cross section of professional forecasts, Ross recovered beliefs weigh more heavily unbiased and accurate beliefs (relative to equally weighted averages). This property can be traced back to the working of arbitrage capital (Buraschi et al. (2018)), which partially (but not fully) offsets individual beliefs distortions.

Having validated Ross recovered beliefs, and having confirmed the broad pattern of maturity increasing over-reaction, In Section 5 I ask: what is the psychology of maturity increasing over-reaction? And can this form of belief distortions account quantitatively for excess volatility in long term rates? To answer the first question, I adapt to a term structure setting the diagnostic expectations model of Bordalo et al. (2017). This model features over-reaction and it is grounded on the Tversky and Kahneman (1974) representativeness heuristic. The basic principle of diagnostic beliefs is "kernel of truth": beliefs move in the right direction but exaggerate true features of the data generating process. I show that in a term structure model the kernel of truth logic naturally yields maturity increasing over-reaction. The intuition is that a diagnostic agent will over-react more strongly to a given signal when there is large fundamental uncertainty, which is indeed the case for longer term rates. The model provides therefore a foundation for increasingly over-reacting beliefs and for the violation of the law of iterated expectations, and in a way that is disciplined by the parameters of the data generating process. The latter property proves useful for quantification.

I then conclude the analysis by calibrating the diagnostic expectations model and by quantifying its ability to capture excess volatility in beliefs and in interest rates. I find that: i) the diagnosticity distortion parameter calibrated from recovered beliefs is consistent with previous literature (Bordalo et al. (2018a)), ii) such parameter matches fairly well the documented average over-reaction of analyst forecasts, and iii) it accounts for roughly 80% of the excess volatility in interest rates documented by Giglio and Kelly (2018).

My paper contributes to a growing body of research aimed at *testing* the rational expectation assumption with beliefs data. Bordalo et al. (2018a) finds evidence of over-reaction to information in several *individual* macroeconomic and financial time series. Piazzesi and Schneider (2011) finds that traditional factors (level and slope) are perceived as more persistent by financial analysts and Cieslak (2018) finds that systematic errors in short rate expectations drive bond returns predictability as opposed to time varying risk premia. Brooks et al. (2018) shows that beliefs of professional forecaster over-react to FOMC announcements, generating post-announcement drift and excess sensitivity of long term rates. Relative to these papers, I use Ross method to recover market beliefs, offer a parsimonious characterization of belief distortions based on diagnostic expectations, and quantify the model's ability to account for the volatility anomaly.

This paper also relates to the debate about the recovery theorem. Martin and Ross (2019) investigates the recovery theorem in fixed income markets, where the assumption of stationary and Markovian state variables may be more plausible, relative to equity markets. My setting is an empirical counterpart of Martin and Ross (2019), where I test for the rationality of recovered beliefs. Jensen et al. (2019) generalize the recovery theorem to non Markovian as well as non stationary settings. Qin et al. (2018) propose an empirical test for the degeneracy of the martingale component of the SDF in the context of US treasury bonds and reject it, but the test assumes rational expectations. Similarly to Qin et al. (2018), I implement the Ross recovery theorem using the pricing measure from the estimation of a standard $\mathbb{Q}$-affine term structure model. This class of models is flexible since it does not entail assumptions on the physical measure (Le et al. (2010)). Close to the intuition of the recovery theorem, Augenblick and Lazarus (2017) provide evidences of excess movements of stock market prices, particularly strong for long horizons.

My contribution to this literature is to consider a pattern, maturity increasing overreaction, that is robust to the Borovička et al. (2016) misspecification. Furthermore, while work on recovery is mostly methodological, I offer a systematic characterization of belief formation and account of the excess volatility pattern.

The paper unfolds as follows: in Section 2, I introduce increasingly over-reacting beliefs into an otherwise standard affine model for the term structure, and I show that in this economy the error predictability of Figure (1.1) and the excess volatility of Giglio and Kelly (2018) are closely related. In Section 3, I discuss the recovery theorem and its empirical implementation. In Section 4, I compare recovered beliefs with survey data. In Section 5, I introduce a model of beliefs formation and I quantitatively assess the increasingly over-reacting beliefs channel for excess volatility. In Section 6, I provide robustness checks. Section 7 concludes. All proofs are in Appendices.

1.2 Over-reaction to news and excess volatility

Figure (1.1) shows that analysts expectations about future interest rates tend to over-react to news, more strongly so for longer maturities. This section formally shows that, within the conventional class of $\mathbb{Q}$-affine models, there is a direct link between such horizon-dependent over reaction in beliefs and the excess volatility of long term interest rates documented by Giglio and Kelly (2018). This connection informs my empirical analysis, aimed at: i) measuring market beliefs, ii) characterizing their departure from rationality, and iii) quantifying the latter's impact on the excess volatility of interest rates. I illustrate the basic logic in a one factor economy, while, for the empirical analysis, I consider multiple factors.

Consider a frictionless market, where zero coupon bonds with different maturities are traded. $P_{t,m}$ denotes the price at time t of a zero coupon bond with time to maturity m. The yield to maturity, $y_{t,m}$, is defined as:

$$P_{t,m} = e^{-m \cdot y_{t,m}},$$

and the yield curve at time t equals the collection of yields $\{y_{t,m}\}_{m \geq 0}$.

Denote the one period interest rate prevailing at time s by r_s (short rate henceforth). Then, the average one period interest rate obtained on an investment at maturity m is equal to $r_{t,m} := \frac{1}{m} \sum_{i=0}^{m-1} r_{t+i}$. In a world with deterministic interest rates, the yield obtained from investing at maturity m is simply equal to the interest rate at that maturity: $y_{t,m} = r_{t,m}$. If instead the short rate is exposed to risk factors (e.g. GDP growth) the interest rate obtained from an investment at maturity m is not known with certainty. Specifically, assume that the short rate r_s is a function of the risk factor at time s, which is denoted by X_s. Then, while at time t the current short rate r_t is known, longer maturity rates $r_{t,m}$ are not known because the dynamics of the risk factor is stochastic. As a result, the price of the bond and hence the yield to maturity $y_{t,m}$ at t is influenced both by *expectations* about future rates and by *risk aversion*. We now characterize these expectations, and in particular their departure from rationality. Next we show how the same expectations affect – together with risk aversion – the yield curve in equilibrium.

Increasingly Over-Reacting Expectations

As in conventional affine models, the short rate is an affine function of the risk factor X_t, which for simplicity we assume to follow a stationary AR(1). Then, the dynamics of the short rate is fully described by:

$$\begin{cases} r_t = \delta_0 + \delta_1 X_t \\ X_t = \rho^{\mathbb{P}} X_{t-1} + \sigma^{\mathbb{P}} \varepsilon_t^{\mathbb{P}}, \end{cases} \tag{1.1}$$

where $\varepsilon_t^{\mathbb{P}}$ is an i.i.d. shock. In this notation, superscript $\mathbb{P}$ captures the so called "physical measure", or data generating process. As a consequence, the dynamics of interest rates at maturity m, $r_{t,m} := \frac{1}{m}\sum_{i=0}^{m-1} r_{t+i}$, reads:

$$r_{t,m} = \delta_0 + b_m^{\mathbb{P}} X_t + \varepsilon_{t,m}^{\mathbb{P}},$$

where $b_m^{\mathbb{P}} := \frac{\delta_1}{m}\sum_{i=0}^{m-1}(\rho^{\mathbb{P}})^i$ and $\varepsilon_{t,m}^{\mathbb{P}} := \frac{\delta_1}{m}\sum_{k=1}^{m-1}\sum_{i=1}^{k}(\rho^{\mathbb{P}})^{k-i}\sigma^{\mathbb{P}}\varepsilon_{t+i}^{\mathbb{P}}$. The coefficients $b_m^{\mathbb{P}}$ capture the sensitivity of interest rates at maturity m, $r_{t,m}$, to current information (X_t) under the physical measure. This sensitivity *geometrically* decays to zero as the maturity increases. $\varepsilon_{t,m}^{\mathbb{P}}$ captures fundamental risk about interest rates at maturity m.

We allow expectations of future interest rates, to depart from rationality. In particular, we allow them to over-react to the current state in a maturity dependent fashion, as suggested by Figure (1.1). Formally, we assume that at time t the market perceives interest rates at maturity m to be:

$$r^{\theta}_{t,m} = \delta_0 + b^{\mathbb{P}}_m(1 + \psi^{\theta}_m)X_t + \sigma^{\theta}_m \varepsilon^{\mathbb{P}}_{t,m}, \tag{1.2}$$

where θ denotes departures from rational expectations. There are two such departures. First, the variance of fundamental shocks is potentially distorted, with $\sigma^{\theta}_m \neq 1$. Second, and more important, beliefs of future rates may display excess sensitivity ($\psi^{\theta}_m \geq 0$) to the current state. The beliefs in Equation (1.2) arise when agents perceive interest rate shocks $\varepsilon^{\mathbb{P}^{\theta}}_{t,m} := \sigma^{\theta}_m \varepsilon^{\mathbb{P}}_{t,m} + \psi^{\theta}_m b^{\mathbb{P}}_m X_t$ so that perceived news are distorted toward the current state X_t. $\mathbb{P}^{\theta}$ denotes a distorted data generating process measure for the factor, and the derivation of Equation (1.2) from the distorted factor dynamics is performed in Appendix A.

Equation (1.2) implies that expectations formed at time t about interest rates at maturity m take the convenient intuitive form:

$$\mathbb{E}^{\mathbb{P}}_t\left[r^{\theta}_{t,m}\right] =: \underbrace{\mathbb{E}^{\mathbb{P}^{\theta}}_t\left[r_{t,m}\right]}_{\text{distorted expectations}} = \underbrace{\mathbb{E}^{\mathbb{P}}_t\left[r_{t,m}\right]}_{\text{rational expecations}} + \psi^{\theta}_m \underbrace{\left(\mathbb{E}^{\mathbb{P}}_t\left[r_{t,m}\right] - \mathbb{E}^{\mathbb{P}}\left[r_{t,m}\right]\right)}_{\text{news relative to the average}}. \tag{1.3}$$

The distorted expectation $\mathbb{E}^{\mathbb{P}^{\theta}}_t[\cdot]$ is equal to the rational expectation plus an adjustment in the direction of the news, where the latter is captured by $\mathbb{E}^{\mathbb{P}}_t\left[r_{t,m}\right] - \mathbb{E}^{\mathbb{P}}\left[r_{t,m}\right]$.

Departures from rationality at maturity m are parameterized by the coefficient $\psi^{\theta}_m \geq 0$. When $\psi^{\theta}_m = 0$, expectations are rational. When $\psi^{\theta}_m > 0$ agents overract to news. That is, they over-estimate future rates in states truly indicative of higher than average future rates: $\mathbb{E}^{\mathbb{P}}_t\left[r_{t,m}\right] > \mathbb{E}^{\mathbb{P}}\left[r_{t,m}\right]$. By contrast, agents under-estimate future rates in states truly indicative of lower than average future rates $\mathbb{E}^{\mathbb{P}}_t\left[r_{t,m}\right] < \mathbb{E}^{\mathbb{P}}\left[r_{t,m}\right]$.[2]

[2] The comparison with average information can be generalized to the comparison with a weighted sum of past predictions. This case includes the model of diagnostic expectations of Bordalo et al. (2018b) and it is discussed in Section 6.

I allow the distortion parameter ψ_m^θ to be maturity-dependent. For now, I leave such dependence unspecified. In Section 5, however, I show that under Gaussian noise, Equation (1.3) naturally follows from the diagnostic expectation model of Bordalo et al. (2018b). Such model also yields horizon dependent distortion parameters ψ_m^θ that are in turn functions of a more primitive distortion parameter θ. This is why I denote distorted expectations using θ.

1.2.1 Risk Aversion and the Yield Curve

Consider a market forming expectations according to Equation (1.2). What does the yield curve looks like? To answer this question, one must also consider investors' risk aversion, which affects – together with beliefs – required rates of return. To capture risk aversion, consider a stochastic discount factor (SDF) $M_{t,m}$ discounting more heavily future cash flows occurring in states in which the representative investor is poorer. Then the price of a maturity m zero coupon bond, and hence yields $y_{t,m}$, is pinned down by the discounted expected payoff[3]:

$$P_{t,m}^\theta = \mathbb{E}_t^{\mathbb{P}^\theta}\left[M_{t,m}\right] := \mathbb{E}_t^{\mathbb{Q}^\theta}\left[e^{-m\cdot r_{t,m}}\right], \tag{1.4}$$

where the so called *risk neutral* probability measure $\mathbb{Q}^\theta$ inflates the perceived probability of states in which the representative investor is poor. By using $\mathbb{Q}^\theta$ the economic analyst can account for risk aversion while using the convenient analytics prevailing under risk neutrality.

By the fundamental asset pricing equation, given a stochastic discount factor $M_{t,m}$, the

[3] Note that under the maturity dependent over-reaction model we have that the law of iterated expectations fails, in the sense that $\mathbb{E}_t^{\mathbb{P}^\theta}\left[M_{t,t+2}\right] \neq \mathbb{E}_t^{\mathbb{P}^\theta}\left[M_{t,t+1}\mathbb{E}_{t+1}^{\mathbb{P}^\theta}[M_{t+1,t+2}]\right]$. We need therefore to take a stance about valuation. In line with the logic of Figure (1.1), we assume that the market forecasts future rates and price future states with a "buy and hold" valuation approach, namely we set $P_{t,t+2} = \mathbb{E}_t^{\mathbb{P}^\theta}\left[M_{t,t+2}\right]$.

risk neutral measure density $f_{\mathbb{Q}^\theta}(X_{t+m}|X_t)$ associated to it is implicitly defined by the condition:

$$f_{\mathbb{Q}^\theta}(X_{t+m}|X_t)e^{-m\cdot r_{t,m}} = M_{t,m}f_{\mathbb{P}^\theta}(X_{t+m}|X_t),$$

which captures the optimality condition for a market with distorted beliefs captured by $\mathbb{P}^\theta$. Here I emphasize that under $\mathbb{P}^\theta$ the factor is still an AR(1) with with maturity dependent distorted persistence and volatility (the full analytics is performed in appendix A). To use the machinery of affine term structure models, we assume that the stochastic discount factor is such that the distorted and risk neutral dynamics $\mathbb{Q}^\theta$ for the factor is AR(1), with maturity dependent persistence and volatility. In this case, the risk adjustment to beliefs $\mathbb{P}^\theta$ yields the following distorted and risk neutral dynamics for interest rates at maturity m:

$$r^\theta_{t,m} = \delta_0 + b^{\mathbb{Q}}_m X_t + \psi^\theta_m b^{\mathbb{Q}}_m X_t + \varepsilon^{\mathbb{Q}}_{t,m},$$

where $b^{\mathbb{Q}}_m := \frac{\delta_1}{m}\sum_{i=0}^{m-1}(\rho^{\mathbb{Q}})^i$ and $\varepsilon^{\mathbb{Q}}_{t,m} := \frac{\delta_1}{m}\sum_{k=2}^{m}\sum_{i=0}^{k-2}(\rho^{\mathbb{Q}})^i\sigma^{\mathbb{P}}\varepsilon^{\mathbb{Q}}_{t+1+i}$. $\rho^{\mathbb{Q}} \neq \rho^{\mathbb{P}}$ is the risk neutral persistence of the factor under rational expectations, while $\varepsilon^{\mathbb{Q}}_{t+1}$ is a zero mean shock with risk neutral variance $\sigma^{\mathbb{Q}} \neq \sigma^{\mathbb{P}}$, again under rational expectations. $b^{\mathbb{Q}}_m$ is the risk neutral sensitivity of future rates to X_t and $\varepsilon^{\mathbb{Q}}_{t,m} := \frac{\delta_1}{m}\sum_{k=1}^{m-1}\sum_{i=1}^{k-1}(\rho^{\mathbb{Q}})^{k-1}\sigma^{\mathbb{Q}}\varepsilon^{\mathbb{Q}}_{t+i}$ is the risk neutral shock at maturity m, both under rational expectations. Importantly, under the belief distortions of Equation (1.2) the risk adjusted dynamics of interest rates still exhibit overreaction (ψ^θ_m) to the current state X_t.

Proposition 1. *Assume homoskedastic $\mathbb{Q}$-shocks. The yield curve under over-reacting beliefs is equal to:*

$$y^\theta_{t,m} = -\frac{1}{m}\log\mathbb{E}^{\mathbb{Q}^\theta}_t[e^{-m\cdot r_{t,m}}] = a^\theta_m + (b^{\mathbb{Q}}_m + \psi^\theta_m b^{\mathbb{P}}_m)X_t, \tag{1.5}$$

where:

$$\begin{aligned} b_m^{\mathbb{Q}} &= \frac{\delta_1}{m}\sum_{i=0}^{m-1}(\rho^{\ \mathbb{Q}})^i, \\ a_m^{\theta} &= \delta_0 - \frac{1}{m}\log \mathbb{E}^{\mathbb{Q}}[e^{-m\cdot\sigma_m^{\theta}\varepsilon_{t,m}^{\mathbb{Q}}}]. \end{aligned}$$

When expectations are rational, namely $\psi_m^{\theta} = 0$, Equation (1.5) is the signature of *affine term structure models* (see Duffee (2013) for a review). The coefficients a_m^{θ} and $b_m^{\mathbb{Q}}$ are maturity dependent. Interest rates volatility is shaped by the sensitivity $b_m^{\mathbb{Q}}$ of yields to changes in the factor X_t. The more persistent is the factor, namely the higher is $\rho^{\mathbb{Q}}$, the more sensitive is the interest rate at any given maturity to news, namely the higher is $b_m^{\mathbb{Q}}$. On the other hand, because the factor is stationary[4], $|\rho|^{\mathbb{Q}} < 1$, long term rates should be less sensitive than short term rates, namely $b_m^{\mathbb{Q}}$ declines with m. The Giglio and Kelly (2018) excess volatility puzzle is rooted in the maturity profile of the $b_m^{\mathbb{Q}}$ terms: they decay too slowly relative to what is implied by the persistence $\rho^{\mathbb{Q}}$ of the factors. The issue is then: can the horizon dependent over-reaction of Figure 1.1 rationalize this finding?[5]

With over-reacting beliefs, $\psi_m^{\theta} > 0$, equilibrium yields retain a tractable form. Over-reaction in beliefs enhances the sensitivity of yields to the risk factor relative to a rational world, namely $b_m^{\mathbb{Q}} \rightarrow b_m^{\mathbb{Q}} + \psi_m^{\theta} b_m^{\mathbb{P}}$. This formula connects over-reaction in beliefs and excess volatility in interest rates.

To see this connection more precisely, consider expectations about future rates, computed using the distorted physical measure $\mathbb{P}^{\theta}$ and consider the measured volatility of

[4] Stationarity of interest rates is an empirically established fact, see Giglio and Kelly (2018).

[5] A different yet important issue is the extent to which $\mathbb{Q}$-affine models correctly capture the yield curve dynamics. To this regard, it is worth noting that: i) empirically, at each maturity m, the same risk factors explain the variability of yields to that maturity, $y_{t,m}$, with R^2 close to one as shown in Section 3, ii) Giglio and Kelly (2018) show that quadratic $\mathbb{Q}$-specification, stationary long memory process nor regime switching models cannot account for the excess volatility and iii) $\mathbb{Q}$-affine models allows highly non linear $\mathbb{P}$ -dynamics as well as non standard SDFs, provided that their products yields $\mathbb{Q}$ affinity.

interest rates. I obtain the following result.

Theorem 1. *(Over-reaction and excess volatility).*

Under expectations (1.3) and the affine setting:

i) (Increasing Overreaction) the CG coefficients β_m obtained by regressing the forecast error made at maturity m using $\mathbb{P}^\theta$ with the forecast revision under $\mathbb{P}^\theta$ about the same maturity are equal to:

$$\beta_m = -c\frac{\psi_m^\theta}{1+\psi_m^\theta},$$

where c is a positive, maturity independent, constant;

ii) (Excess Volatility) the volatility of yields at maturity m relative to the volatility of the short rate is equal to:

$$\frac{\mathbb{V}^{\mathbb{P}}[y_{t,m}^\theta]}{\mathbb{V}^{\mathbb{P}}[y_{t,1}^\theta]} = \left(\frac{b_m^{\mathbb{Q}} + \psi_m^\theta b_m^{\mathbb{P}}}{b_1^{\mathbb{Q}} + \psi_1^\theta b_1^{\mathbb{P}}}\right)^2 > \frac{\mathbb{V}^{\mathbb{P}}[y_{t,m}]}{\mathbb{V}^{\mathbb{P}}[y_{t,1}^\theta]},$$

where $\mathbb{V}^{\mathbb{P}}[y_{t,m}^\theta]$ is the measured volatility of yields while $\mathbb{V}^{\mathbb{P}}[y_{t,m}]$ is the volatility arising under rational expectations. Over reaction to news yields both the pattern in Figure (1.1) and excess volatility of long term rates if and only if ψ_m^θ is positive and increases in maturity m.

This result conveys two important messages. First over-reaction to news increasing in the horizon m reconciles the observed patterns in forecast errors and excess volatility in long term rates. This is a general result, and holds beyond the more restrictive assumptions of this Section. As I show in the proof of Theorem (1), such connection holds under general $\mathbb{Q}$-affine models, where $\mathbb{P}$ is Markovian and $\mathbb{Q}$ is AR(1), provided expectations about future interest rates over-react according to Equation (1.3).

Second, and crucially, Theorem (1) says that in the conventional class of $\mathbb{Q}$-affine term structure models, when the data generating process $\mathbb{P}$ for the risk factors is itself

AR(1), the maturity dependent over-reacting beliefs in Equation (1.2) create a precise link between the over-reaction coefficient measured using beliefs data and the excess volatility detected from prices. They are both pinned down by the same maturity increasing distortion ψ_m^θ.

The remainder of the paper assesses empirically these predictions. In Sections 3 and 4 I start tackling this challenge by offering a measure of market beliefs. The survey evidence in Figure (1.1) has two pitfalls. First, it has only a limited number of maturities. Second, and most important, the beliefs of professional forecasters may not be representative of the beliefs of market participants. To overcome these issues, I use methods developed by Ross to extract information about beliefs from asset prices in conventional affine models[6].

In Section 5 I add the assumption that the physical measure $\mathbb{P}$ is a Gaussian AR(1). In this case, the over-reaction parameters ψ_m^θ can be conveniently founded using the diagnostic expectations model (Bordalo et al. (2018b)). This allows me to obtain an estimate of their distortion parameter θ that can be benchmarked to existing estimates. Furthermore, Theorem (1) highlights a duality between over-reaction in beliefs and excess volatility in interest rates, so that ψ_m^θ can be estimated both from beliefs data and directly from yields. I use this duality to assess which conclusions about ψ_m^θ are most robust.

1.3 Beyond survey data: Ross Recovery Theorem and the term structure of beliefs

The method of Ross (2015), rests on two assumptions: i) the underlying state of the economy follows a stationary Markovian process, both under $\mathbb{P}$ and under $\mathbb{Q}$, and ii) the SDF $M_{t,m}$ is *path independent*, namely it depends only on the final value X_{t+m} and on the initial value X_t of the state variable, and not on the *path* from t to $t+m$. Assumption i) is an approximation, but of significant empirical power: it is widely recognized that few state

[6] I will consider prices of US treasury bonds which are directly related to interest rates and yields.

variables drive the yield curve (see Duffee (2013)) and stationarity is not rejected in the data (see Giglio and Kelly (2018) and Martin and Ross (2019)). Assumption ii) is more controversial: Borovička et al. (2016) shows that path independence is not met in *long run risk* asset pricing models. In this case, the method of Ross (2015) does not recover market beliefs, but beliefs adjusted for a martingale component. This is an important criticism but, as I show below, my analysis is immune to it because the increasingly over-reacting beliefs in (1.3) can still be recovered from the data.

To grasp how Ross's method works, consider an Arrow-Debreu security that pays one dollar if next period's state is j (assume for simplicity that there is a finite number N of states, which correspond to factor values in my setting). Under rational expectations, if the current state is i, the price of such Arrow Debreu security is equal to:

$$\mathbb{A}_{ij} = M_{ij}\mathbb{P}_{ij},$$

where $\mathbb{P}_{ij} := \mathbb{P}(X_{t+1} = j | X_t = i)$ is the physical probability of transitioning from state i to state j and M_{ij} is the stochastic discount factor (SDF) capturing the marginal rate of substitution between current consumption in i and future consumption in j. Due to risk aversion, M_{ij} overweights bad states, attaching a higher price to a safe bond because it pays out in them.

In the presence of non-rational beliefs, the fundamental asset pricing equation implies that the price of the same Arrow Debreu security is equal to:

$$\mathbb{A}^{\theta}_{ij} = M_{ij}\mathbb{P}^{\theta}_{ij}.$$

An econometrician observing Arrow Debreu prices may recover market beliefs, be they rational $\mathbb{P}_{ij}$ or distorted $\mathbb{P}^{\theta}_{ij}$, by performing an "inverse risk adjustment" to the prices themselves. Ross has shown that such inverse risk adjustment is indeed possible, so that market beliefs can be recovered from prices, under assumptions i) and ii) above. Ross's

$$\begin{tikzcd}
\mathbb{P} \arrow[r, leftrightarrow, "\text{Risk}"] \arrow[d, leftrightarrow, dashed, "\text{Belief distortions}"'] & \mathbb{A} \arrow[d, leftrightarrow, dashed] \\
\mathbb{P}^{\theta} \arrow[r, leftrightarrow] & \mathbb{A}^{\theta}
\end{tikzcd}$$

Figure 1.2: The blue line adjusts the data generating process $\mathbb{P}$ for preferences, which determines Arrow-Debreu prices $\mathbb{A}$. The red line distorts the data generating process $\mathbb{P}$, thus defining beliefs $\mathbb{P}^{\theta}$, thus generating Arrow-Debreu prices $\mathbb{A}^{\theta}$.

method has been so far used under the assumption of rational expectations.

As displayed in Figure (1.2) above, when market beliefs may not be rational, the use of Ross' method entails an ambiguity: the econometrician does not know if Arrow Debreu prices are $\mathbb{A}$ (they reflect rational beliefs) or if they are $\mathbb{A}^{\theta}$ (they reflect non rational beliefs). Here I proceed as follows: I use Ross' method, and then performed on the recovered market beliefs some statistical tests of rationality, considering in particular the predictability of forecast errors. The outcome of this test tells us if we are in top or bottom row of Figure (1.2).

Before carrying out the analysis, consider again the critical path independence assumption i). When the SDF is *path independent*, it means that there exists a constant δ and function z of the state such that the one period SDF can be written as:

$$M_{ij} = \delta \frac{z_i}{z_j}.$$

This property is satisfied by conventional CRRA preferences (over stationary variables). However, this assumption has been criticized by Borovička et al. (2016), who show that it assumes away the kind of *long-run* risk adjustments from the SDF that are embedded in many conventional consumption based asset pricing models (CRRA when consumption exhibit stochastic trends, long run risk models as well as habit formation models)[7]. As they show, assuming path independence in these cases is akin to neglecting the fact that the SDF also contains a *martingale* component that changes over time. Does such misspecification invalidate the detection of the horizon dependent over-reacting beliefs that I consider here? To answer this question, suppose that the SDF has a martingale component. Then, by applying Ross method we would not recover the true market beliefs $\mathbb{P}^\theta$ but the beliefs $\tilde{\mathbb{P}}^\theta$ contaminated by misspecification. I then obtain the following result.

Theorem 2. *Consider the $\mathbb{Q}$-affine setting, non rational expectations as in (1.3) and suppose that the martingale component of the SDF is non degenerate, so that the econometrician detects $\tilde{\mathbb{P}}^\theta$, not $\mathbb{P}^\theta$. Then, the following are equivalent:*

i) the CG coefficients estimated by regressing forecast errors of Ross recovered beliefs on forecast revisions of Ross recovered beliefs vary with maturity m.

ii) $\mathbb{E}^\theta[\cdot]$ violates LIE.

Moreover, the CG coefficients obtained from Ross recovered beliefs are negative and decreasing:

$$\beta_m^\theta = -c' \frac{\psi_m^\theta}{1+\psi_m^\theta},$$

where $c' > 0$ is a maturity independent constant.

With misspecification à la Borovička et al. (2016), the econometrician applying Ross' method no longer recovers exact beliefs. Crucially, however, rationality tests still allow her to recover the horizon dependent over-reaction, critical for my analysis, and detected in Figure (1.1) for professional forecasters. The intuition is that horizon-dependent distortions entail a strong violation of rationality, namely a violation of the law of iterated

[7] However Walden (2017) explicitly showed that relabeling the state variables may lead to the assumption to be met, in the case of CRRA utility with non stationary consumption.

expectations.[8] As such, it cannot be accounted by any rational expectations models, including those featuring long run risk. In this respect, the Ross recovery theorem remains useful for spotting horizon dependent beliefs distortions.

1.3.1 Empirical implementation

To apply Ross' method, we need Arrow Debreu prices. These are not directly observed but they can be inferred from market prices. To do so, note that one period ahead Arrow-Debreu prices can also be written as state by state discounted risk neutral probabilities:

$$\mathbb{A}^{\theta}_{ij} = \mathbb{Q}^{\theta}_{ij} e^{-r(i)},$$

where $\mathbb{Q}^{\theta}_{ij}$ is the risk neutral probability of transitioning from state i to state j. The above formula includes both the case of rational expectations ($\theta = 0$) as well as non rational ones ($\theta \neq 0$).

To compute the risk neutral probabilities $\mathbb{Q}^{\theta}_{ij}$ I first estimate the risk neutral parameters of a three factor affine model. Then, I discretize the state space using the Rouwenhorst method (Cooley (1995)), compute the discretized transition probabilities $\mathbb{Q}^{\theta}_{ij}$ and finally I discount them by the known short rate. This procedure yields $\mathbb{A}^{\theta}_{ij}$.

I follow conventional methods and consider as factors the first three principal components of the yield curve (see Duffee (2013)). The construction of the factors is discussed in Appendix B. The three factor $\mathbb{Q}^{\theta}$-dynamics takes the form:

$$\begin{cases} r_t = \delta_0 + \delta_1^{\top} \mathbf{X}_t \\ \mathbf{X}_{t+1} = \rho^{\mathbb{Q}^{\theta}} \mathbf{X_t} + \Sigma^{C} \varepsilon^{\mathbb{Q}^{\theta}}_{t+1}, \end{cases}$$

where $\varepsilon^{\mathbb{Q}^{\theta}}_t$ are i.i.d. shocks. The matrix $\rho^{\mathbb{Q}^{\theta}}$ is assumed to be diagonal, $\rho^{\mathbb{Q}^{\theta}} =$

[8] Formally, the LIE is a mathematical theorem which holds true for every probability measure. The inconsistency of $\mathbb{P}^{\theta}$ at different maturities is mathematically due to the fact that $\mathbb{P}^{\theta}$ is a different probability measure for each maturity, as defined by condition (1.3).

$diag(\rho_1^{\mathbb{Q}^\theta}, \rho_2^{\mathbb{Q}^\theta}, \rho_3^{\mathbb{Q}^\theta})$.[9] Σ^C is the Cholesky decomposition of the variance-covariance matrix of the residuals, Σ. Following Cochrane and Piazzesi (2009), I assume that Σ coincides under $\mathbb{P}^\theta$ and $\mathbb{Q}^\theta$, so I can estimate it with the variance covariance matrix of residuals in a $VAR(1)$ for $\mathbf{X}_t$. Finally, the entries of the diagonal matrix $\rho^{\mathbb{Q}^\theta}$ are estimated by matching observed yields.

Due to my emphasis on maturity dependent over-reaction, I perform this estimation maturity by maturity *independently*, since I do not want to impose restrictions across maturities. The estimation sample is taken as the first half of the sample, while robustness to sub-samples are discussed in Section 6.

Summarizing the procedure:

1. $\hat{\delta}_0$, $\hat{\delta}_1$ are the OLS estimates of the regression of the short rate on the three factors.

2. $\hat{\Sigma}$ is estimated as the variance covariance matrix of the residuals of a VAR(1) for the factors. $\hat{\Sigma}^C$ is the corresponding Cholesky decomposition.

3. The risk neutral parameters of the factor dynamics are estimated, *maturity by maturity*, as:

$$\hat{\rho}_m^{\mathbb{Q}^\theta} := \arg\min_{\rho^{\mathbb{Q}^\theta}} \frac{1}{T} \sum_t \left(y_{t,m} - \bar{y}_{m,t} \right)^2 = \arg\min_{\rho^{\mathbb{Q}^\theta}} \frac{1}{T} \sum_t \left(\hat{a}_m^\theta + b_m^{\mathbb{Q}^\theta} X_t - \bar{y}_{m,t} \right)^2$$

   where $b_m^{\mathbb{Q}^\theta}$ is a function of the the parameters $\rho^{\mathbb{Q}^\theta}$, whose analytic expression in computed in Appendix A.

4. One period ahead Arrow-Debreu prices are finally estimated using yields to maturity m only as fitting the average yield curve:

$$\hat{\mathbb{A}}_{ij}^{(m)} = \hat{\mathbb{Q}}_{ij}^{(m)} e^{-\hat{r}(i)}.$$

[9] This restriction is necessary to achieve identification of the matrix $\rho^{\mathbb{Q}^\theta}$ assuming that the noise is Gaussian. In this case, a Gaussian transition probability density is in fact characterized by 9 independent parameters, which parametrize the mean and the covariance matrix (see Hamilton and Wu (2012)).

A final step before recovering beliefs in needed. The Recovery Theorem entails an eigenvalue problem for the Arrow-Debreu matrix, while the affine specification relies on continuous variables. As I discussed before, to tackle this issue I discretize the continuous state space of $\mathbf{X}_t$ by using the Rouwenhorst method (Cooley (1995)), which represents the state of art in approximating an $AR(1)$ process with a finite state space Markov chain. The method generates a Markov chain which matches mean, variance and autocorrelation of the original $AR(1)$ process. These are the moments I am interested in: the mean of the factor determines the behavior of the average yield curve, the autocorrelation and the variance determine the term structure of volatilities. Further details about the implementation are discussed in Appendix B.

1.4 Recovered Beliefs, Survey data and Rationality tests

I now study recovered beliefs and compare those with survey data from professional forecasters. The latter step helps me validate the analysis because the two sources of data are highly independent.

1.4.1 Data

US treasury yields

Gürkaynak et al. (2007) provides (and keep updated) nominal, annualized, zero coupon bond yields with yearly maturities from 1 year to 30 years. Gürkaynak et al. (2007) infer the yield curves time series from observed prices of fixed income instruments. The data are jointly available at all maturities from 11-25-1985 to 12-31-2016, at the daily frequency.

The Blue Chip Survey of Professional forecasters

The Blue chip survey of professional forecasters contains forecasts about yields to maturities $1, 2, 5, 10, 20$ and $30y$ from leading financial institutions, which are flagged in the dataset. Forecasts with maturity $1, 2, 5$ and 10 years are available from January 1984, forecasts with maturity $30y$ are available starting from January 2000, while forecasts with maturity $20y$ are available starting from January 2004. Forecasts about the next quarter yield curve are reported at the monthly frequency, so the prediction horizon oscillates between 2 and 6 months[10]. I remove from the sample those forecasters who reply to the survey for a short time period only (less then 5 years). At each time t, I remove those replies to the survey which contain answers for less than 3 prediction horizons: what I am mostly interested in are indeed forecasts at different horizons. Finally, for each prediction horizon, I remove outliers which are defined as observations in the first and last percentile of the distribution of forecasts.

1.4.2 Tests of rationality

Tests of rationality involve the predictability of the forecast error, defined, for each available maturity as:

$$FE_{t+1}[y_{t+1,m}] = \bar{y}_{t+1,m} - \hat{y}_{t+1,m|t},$$

where $\bar{y}_{t+1,m}$ is the observed yield and $\hat{y}_{t+1,m|t}$ is the prediction of yields with maturity m at time $t+1$ done at time t. I compute forecast errors using both survey data and recovered beliefs. In the former case, each $\hat{y}_{t+1,m|t}$ has multiple observations across different forecasters and the forecast error is forecaster specific. In the latter case, the forecast is

[10] The unpredictability of the forecast error which is implied by the rational expectation hypothesis is in principle unaltered by the moving forecast horizon. However, to ameliorate concerns regarding changes in expectations due to changes of the prediction horizon, I consider time fixed effects in the robustness section.

computed as:

$$\hat{y}_{t+1,m|t} := \sum_{\mathbf{X}_{t+1} \in \text{Grid}} \underbrace{\mathbb{P}^{\theta}(\mathbf{X}_{t+1}|\mathbf{X}_t)}_{\text{Recovered beliefs}} \times \underbrace{(\hat{a}_m^{\theta} + (\hat{b}_m^{\mathbb{Q}^{\theta}})^{\top}\mathbf{X}_{t+1})}_{\text{Emprical affine mapping}}.$$

Expectations about future yields are conveniently decomposed into distorted expectations about factors (identified using the recovery theorem) and the pricing function (empirical affine mappings). The affine mapping is known because $\hat{a}_m^{\theta}$ and $\hat{b}_m^{\mathbb{Q}^{\theta}}$ have been estimated as discussed in Section 3. To simplify notation, $\hat{y}_{t+1,m|t}$ does not carry superscript θ.

Under the null hypothesis of full information rational expectations, the forecast error should not be predictable on the basis of past information. But what is past information? Following Coibion and Gorodnichenko (2015), I define information at time t by the *forecast revision*:

$$FR_t[y_{t+1,m}] := \hat{y}_{t+1,m|t} - \hat{y}_{t+1,m|t-1}.$$

The logic underlying this definition is that *information moves beliefs*: if no information is observed at time t, then there is no revision in beliefs, i.e. $FR_t[y_{t+1,m}] = 0$.

For each maturity available, I then run the regression:

$$FE_{t+1}[y_{t+1,m}] = \alpha_m + \beta_m FR_t[y_{t+1,m}] + \varepsilon_{t+1,m}.$$

I call β_m "CG coefficient" from Coibion and Gorodnichenko (2015). A positive $\beta_m > 0$ is interpreted as *under-reaction* to information at maturity m. In this case, when beliefs about yields at horizon m are revised upward, they systematically under-estimate realized yields. That is, beliefs are not revised enough. On the contrary, $\beta_m < 0$ is interpreted as *over-reaction* to information at maturity m: when beliefs about yields at horizon m are revised upward, they systematically over-estimate realized yields. That

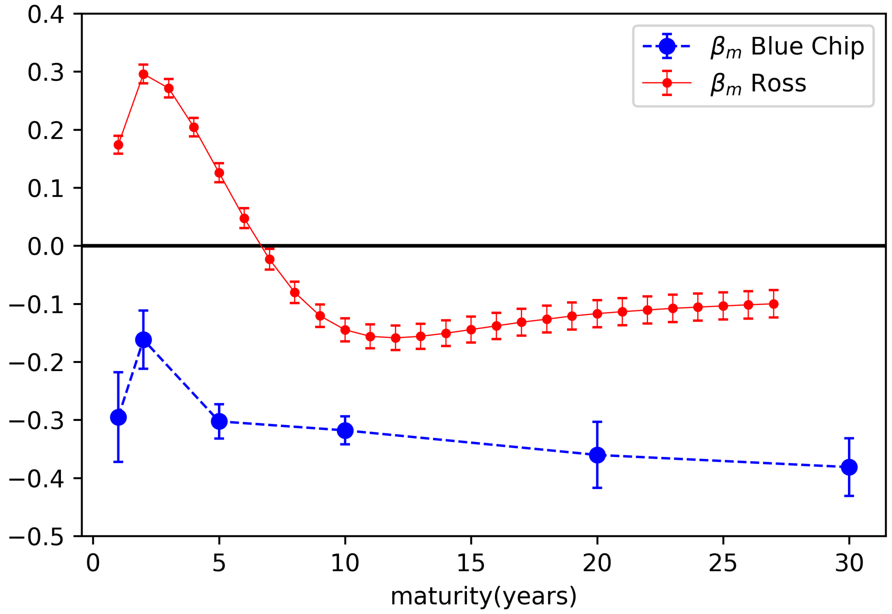


Figure 1.3: Slope of FE on FR using recovered beliefs (red) and using the Blue Chip dataset (pooled OLS). Confidence intervals are computed at 5%.

is, beliefs are revised too much. The CG coefficients obtained with the Ross recovered beliefs for maturities ranging of $2, 3, \ldots 30$ years[11] and by pooled estimation of professional forecasters data are shown in Figure 1.3.

Recovered beliefs also exhibit maturity dependent reaction to information. Just like the beliefs of professional forecasters, they over-react more for long term yields than for short term ones. Thus, the key message of excess reaction to information of long run beliefs relative to short run beliefs is consistent across survey data and Ross recovered beliefs. The main difference between the two datasets arises because Ross recovered beliefs exhibit a pattern of under-reaction to information at the short end of the yield curve and over-reaction to information at the long end. Professional forecaster data, on the contrary,

[11] The $1y$ yield is used as short rate and therefore predictions to this horizon cannot be computed.

exhibit over-reaction only [12] [13].

Ross recovered beliefs capture market beliefs, and the marginal investor is potentially different from the average forecaster. For this reason the qualitative similarity of predictability patterns across the two datasets is surprising: there is no mechanical reason for it, and this suggests that stronger over-reaction for longer horizons may be a robust feature of beliefs. We can even more directly compare Ross recovered beliefs and professional forecasts by correlating forecast revisions and the level of forecasts across the two datasets (considering the mean as well as the median forecast). Figure (1.4) shows that the two datasets are quite aligned along both criteria[14].

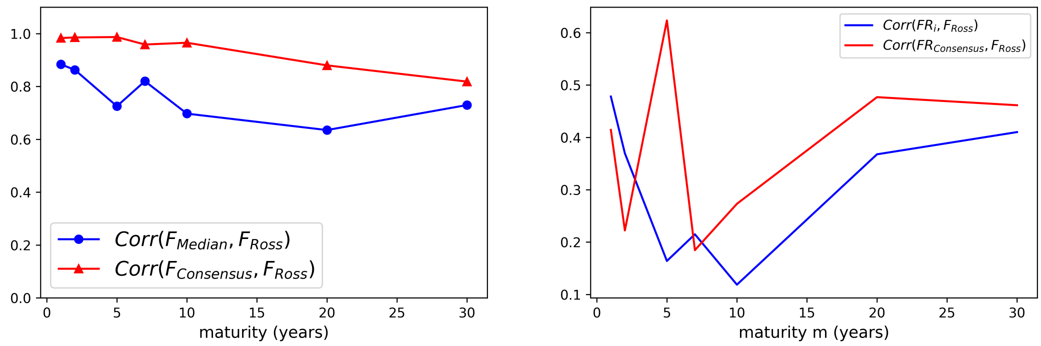


Figure 1.4: Left panel: correlation between mean forecasts (red triangle), median forecasts (blue circle) and Ross recovered forecasts as a function of the maturity. Right panel: correlation between mean forecast revision (red triangle), median forecast revision (blue circle) and Ross recovered forecast revisions as a function of the maturity.

The strong, positive correlation between Ross recovered beliefs and professional forecasts is important. It indicates that the two types of beliefs data are not noise, and thus

[12] In the robustness Section, I consider different specifications of the regression performed with professional forecasters data, including time fixed effects, forecasters fixed effects as well as single and double clustering of standard errors. The results are consistent.

[13] In Bordalo et al. (2018a), the authors find under-reaction at maturities shorter than one year with Blue Chip data, at the quarterly frequency. Here, I do not consider those maturities for the sake of comparison with recovered beliefs.

[14] For the comparison, I aggregated recovered beliefs from the daily to the monthly frequency.

they capture significant features of market beliefs. Of course, the benefit of Ross recovered beliefs is that, in addition to being more tightly linked to the marginal investor, they are available for all maturities and frequencies.

Are market beliefs, as measured with the Ross method, better or worse than the beliefs of analysts? There are two possible metrics to asses this: the predictability of the forecast error (which captures biases in forecasting) and the mean square error (which capture *accuracy* in forecasting, i.e. bias plus precision). In Figure (1.5), I plot the estimated distribution of distortions (biases) using a Gaussian kernel density estimator.

The distribution of forecaster distortions is not symmetric around rationality (i.e. $\beta = 0$): the majority of forecasters over-react to news. This is reflected in the fact that the average bias, captured by the pooled regression in the introduction and reported also in Figure (1.5) is negative. The above figure also reports as a benchmark the CG coefficient of the consensus forecast, defined as the predictability of errors for the median analyst forecast. Note that the consensus always under-reacts to news, a fact that has been previously documented in Bordalo et al. (2018a), where the authors also reconcile it with it with individual analyst over-reaction[15].

The distortion of the Ross recovered forecast lies in between the median forecaster and the average bias, and it is closer to rationality ($\beta = 0$) than both. Thus, regarding the bias dimension, this analysis suggests that the Ross recovered beliefs weigh more unbiased forecasts, relative to the average professional forecaster bias. This may be due to arbitrage capital moving partly, though not fully, towards less biased investors.

Second, I investigate the accuracy of forecasters, as measured by the mean square error in prediction. In Figure (1.5), I consider two groups of forecasters. The top 25% most accurate forecasters and the bottom 25% least accurate forecasters. In the data, the former forecasters are rational, namely, their CG coefficients are statistically indistinguishable

[15] The intuition is that when forecasters observe different noisy signals, stemming for instance from heterogeneous information sets, each analysts over-reacts to his own news, but does not react at all to the signal of other analysts. This second effect can be so strong that the consensus forecast under-reacts to the consensus revision even if each analyst over-reacts to his own information.

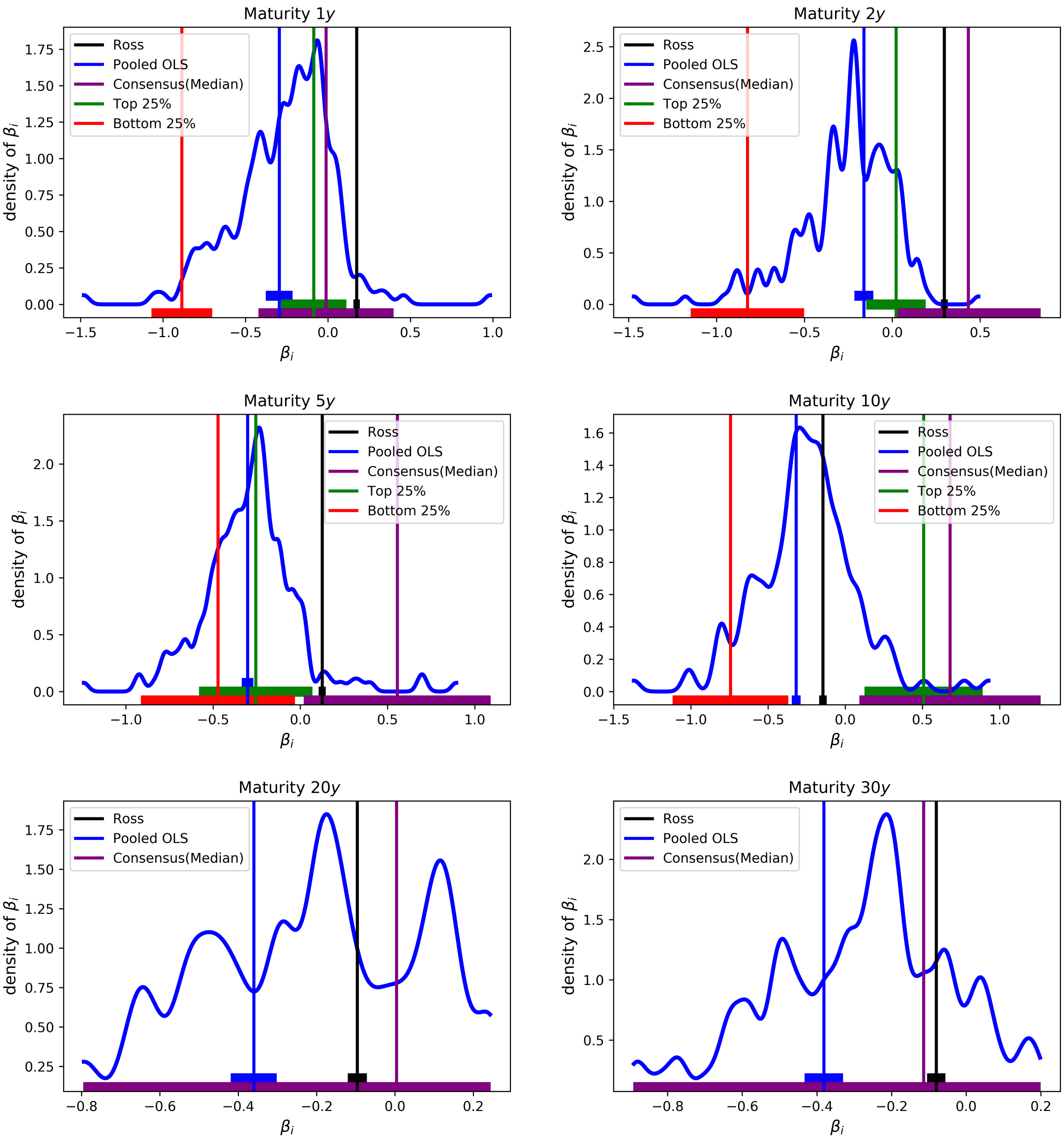


Figure 1.5: Density of forecasters distortions (CG coefficients) for different maturities. Distortions from recovered beliefs are shown in black, pooled distortions in blue, consensus (median) distortions in purple, best forecasters distortions (top quartile according to the mean square error criterion) in green, worse forecasters distortions (bottom quartile according to the mean square error criterion) in red. Confidence intervals are computed at the 5% level.

for zero, while the worst 25% are highly over-reacting. Thus, there is a positive relation between imprecision and bias in forecasting. A direct comparison between the correlations of Ross recovered forecasts and top/worse 25% (ranked according to the MSE criterion) is offered in Figure (1.6).

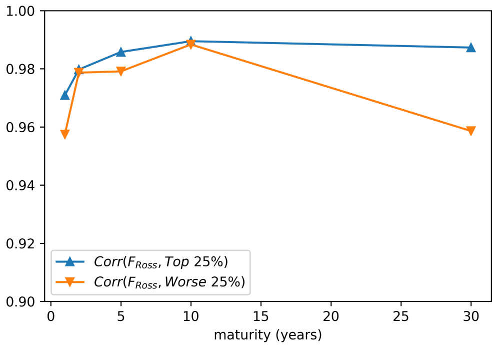


Figure 1.6: Correlation between mean forecast of top 25% forecasters (ranked according to the MSE criterion) and Ross recovered forecasts (blue triangle), Correlation between mean forecast of worse 25% forecasters (ranked according to the MSE criterion) and Ross recovered forecasts (orange triangle).

Figure (1.6) suggests that the Ross recovered forecast slightly over-weights more accurate views

Overall this analysis suggests that Ross forecasts weight more both more unbiased forecasters and more accurate forecasters.

Broadly speaking, this Section conveys the following messages. First, market beliefs recovered using Ross method display the same maturity increasing over-reaction displayed by survey data. Second, market beliefs and survey data are highly positively correlated. Third, market beliefs are less biased and more accurate than the average professional forecaster. This indicates that our recovered market beliefs capture systematic patterns in beliefs, and that arbitrage helps reduce the impact of highly biased forecasters on asset prices, consistent with basic asset pricing theory.

Having validated the recovered beliefs, several questions emerge. First, why do forecasts over-react in this maturity dependent way? What is the psychological foundation? Second, can the over-reaction detected from beliefs account for the excess volatility of interest rates along the lines of Theorem (1)? To make progress, I specify a realistic model of belief formation, based on the diagnostic expectation model of Bordalo et al. (2018b), and I show that it offers an answer to both questions: it yields maturity dependent over-reaction, it can *quantitatively* account for belief distortions *and* this in turn captures a big chunck of the excess volatility of interest rates documented by Giglio and Kelly (2018).

1.5 The Horizon Dependent Diagnostic Expectations Model

Using the diagnostic expectations model of Bordalo et al. (2018b), I now rationalize maturity-increasing over-reaction using a single parsimonious departure of beliefs from rationality, captured by the diagnosticity parameter θ that I estimate and use to quantify the explanatory power of the model for excess volatility of interest rates. As argued in Section 2, the duality between belief distortions and over-reaction in interest rates offers two separate methods for estimating θ, one based on beliefs data, another based on yields. I show that these two independent strategies yield similar estimates for θ that are consistent with previous estimates, and that can quantitatively account for excess volatility of interest rates and for the predictability of the forecast errors.

1.5.1 Diagnostic Expectations

Diagnostic expectations are based on Kahneman and Tversky's (Tversky and Kahneman (1974)) representativeness heuristics in probability judgments. Representativeness captures the idea that, when making a conditional probabilistic assessment, humans typically over-weight representative (or *diagnostic*) traits, defined as the traits that are objectively more frequent in such group relative to a comparison group. A conventional example is the exaggeration of the probability that an Irish person is red haired, because this hair color is relatively more frequent in Ireland than elsewhere (although even in Ireland it is unlikely in absolute terms). This heuristics has been widely documented in the psychology and cognitive science literature, since the seminal work of Tversky and Kahneman (1974) and it has recently been adapted to a dynamic setting by Bordalo et al. (2018b). When forecasting, economic agents exaggerate objectively positive news relative to a benchmark prediction, shaped by past information.

To capture this idea in my setting, consider the following *diagnostic* distribution at time t of interest rates with maturity m, $r_{t+m} = \frac{1}{m}\sum_{i=1}^{m-1} r_{t+i}$:

$$f_{\mathbb{P}^\theta}(r_{t,m}|\mathbf{X}_t) \propto f_{\mathbb{P}}(r_{t,m}|\mathbf{X}_t)\left(\frac{f_{\mathbb{P}}(r_{t,m}|\mathbf{X}_t)}{f_{\mathbb{P}}(r_{t,m})}\right)^\theta,$$

where $f_{\mathbb{P}}(\cdot)$ is the unconditional or long run distribution of $r_{t,m}$.

The diagnostic distribution of $r_{t,m}$ at time t re-weights the correct density $f_{\mathbb{P}}(\cdot|\mathbf{X}_t)$ via the likelihood ratio $\left(\frac{f_{\mathbb{P}}(r_{t+m}|\mathbf{X}_t)}{f_{\mathbb{P}}(r_{t,m})}\right)$ to the power θ. Investors over-weight future values of interest rates that are more likely under current information relative to the average information, where the latter is captured by the long run distribution. The parameter $\theta > 0$ in the diagnostic distribution quantifies the degree of over-reaction. The larger is θ the more outcomes whose likelihood has increased are overweighted in beliefs.

My model differs in two dimensions to the model of Bordalo et al. (2018b). First, in Bordalo et al. (2018b), the authors compare rational forecasts at time t with rational forecast at time $t-1$, while I use the average information as comparison. This is technically convenient because my model preserves Markovianity, which greatly simplifies the identification of recovered beliefs in Section 3. In Section 6, I show, theoretically, that results are qualitatively robust to the specification used in Bordalo et al. (2018b). Second, and more important, the benchmark distribution in Bordalo et al. (2018b) is assumed to have the same volatility as the rational forecast. This assumption is innocuous when considering a fixed horizon as in Bordalo et al. (2018b) and it greatly simplifies the math. However, this assumption misses an important intuition: namely that diagnostic distortions should depend on the underlying uncertainty about the economic environment, as discussed in Gennaioli and Shleifer (2018). I now show that allowing for this role of uncertainty is highly relevant here, because it implies the violation of the law of iterated expectations taking the form of maturity increasing over-reaction.

The diagnostic distribution can be conveniently computed for linear and Gaussian dynamics. Assume that the $\mathbb{P}$ dynamics of the factor is:

$$\begin{cases} r_t = \delta_0 + \delta_1^\top \mathbf{X} \\ \mathbf{X}_t = \rho^{\mathbb{P}} \mathbf{X}_{t-1} + \Sigma^C \varepsilon_t^{\mathbb{P}}, \end{cases}$$

where $\mathbf{X}_t \overset{\mathbb{P}}{\sim} VAR(1)(\rho^{\mathbb{P}}, \Sigma)$, Σ^C is a lower triangular matrix, $\varepsilon_t^{\mathbb{P}}$ are i.i.d. Gaussian shocks and $\Sigma := \Sigma^C \Sigma^{\top^\top}$ is the one period variance-covariance matrix. Here, in order to derive a parsimonious expression for increasing over-reaction, I impose additional assumptions relative to $\mathbb{Q}$-affine setting (the latter is the only assumption I needed so far). Specifically, I assume a VAR(1) Gaussian dynamics for the factor, under the physical measure $\mathbb{P}$. Then, the diagnostic distribution of interest rates at maturity m is characterized as follows.

Theorem 3. *(Diagnostic distribution) Given $\mathbf{X}_t \overset{\mathbb{P}}{\sim} VAR(1)(\rho^{\mathbb{P}}, \Sigma)$, under Gaussian noise, the diagnostic distribution of interest rates to maturity m, $r_{t,m}$, $f_{\mathbb{P}}^{\theta}(r_{t+m}|\mathbf{X}_t)$, is Gaussian, with mean:*

$$\mathbb{E}_t^{\mathbb{P}^\theta}[r_{t,m}] = \mathbb{E}_t^{\mathbb{P}}[r_{t,m}] + \psi_m^\theta \left(\mathbb{E}_t^{\mathbb{P}}[r_{t,m}] - \mathbb{E}^{\mathbb{P}}[r_{t,m}] \right) \tag{1.6}$$

and variance:

$$\mathbb{V}_t^{\mathbb{P}^\theta}[r_{t,m}] = \left(\frac{\theta+1}{\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]} - \frac{\theta}{\mathbb{V}^{\mathbb{P}}[r_{t,m}]} \right)^{-1}, \tag{1.7}$$

where

$$\psi_m^\theta := \frac{\theta \frac{\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]}{\mathbb{V}^{\mathbb{P}}[r_{t,m}]}}{1 + \theta - \theta \frac{\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]}{\mathbb{V}^{\mathbb{P}}[r_{t,m}]}}, \tag{1.8}$$

$$\mathbb{E}_t^{\mathbb{P}}[r_{t+m}] = \frac{1}{m} \left(\delta_0 + \delta_1^\top \left(\sum_{i=0}^{m-1} \rho^{\mathbb{P}^i} \right) \mathbf{X}_t \right), \tag{1.9}$$

$$\mathbb{E}^{\mathbb{P}}[r_{t,m}] = \delta_0, \tag{1.10}$$

$$\mathbb{V}_t^{\mathbb{P}}[r_{t,m}] = \frac{1}{m^2} (\delta_1)^\top \left(\sum_{i=0}^{2} \left(\sum_{i=0}^{m-2} \rho^{\mathbb{P}^{2i}} \Sigma \right) \right) \delta_1 \tag{1.11}$$

and

$$\mathbb{V}^{\mathbb{P}}[r_{t,m}] = \mathbb{E}[\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]] + \mathbb{V}[\mathbb{E}_t^{\mathbb{P}}[r_{t,m}]] \tag{1.12}$$

$$= \mathbb{V}_t^{\mathbb{P}}[r_{t,m}] + \frac{1}{m^2} (\delta_1)^\top \left(\sum_{i=0}^{m-1} \rho^{\mathbb{P}^i} \right) \Sigma \left(\sum_{i=0}^{m-1} \rho^{\mathbb{P}^i} \right) \delta_1 \tag{1.13}$$

As evident from Equation (1.6), the diagnostic model endogeneizes the maturity increasing over-reaction assumed in reduced from in Section 2. The coefficient ψ_m^θ in formula (1.8), that characterizes the departures from rationality in Section 2, it now pinned down by: i) the scalar parameter θ, ii) the maturity m, and iii) the true conditional and unconditional variances $\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]$ and $\mathbb{V}^{\mathbb{P}}[r_{t,m}]$. Belief distortions starts from zero at $m = 0$, then become positive at $m = 1$ and increase monotonically as $m \to \infty$, approaching a finite limiting value equal to θ.

The intuition for this result is simple: as the maturity m increases, fundamental uncertainty is higher. As a result, the tails of the distribution are prominent, so they more easily come to mind after information makes them more likely. That is, after good news, the right tail becomes very representative for long maturities and it is highly overweighted. As a result, expectations are too optimistic. After bad news, the left tail becomes very representative for long maturities and is highly overweighted. As a result, expectations become too pessimistic. In both cases, beliefs over-react and they do so more for longer maturities.

Theorem (2) also shows that in reduced form, for a given value of ψ_m^θ, the diagnostic model yields the same expectations for interest rates assumed in Section 2 and it therefore yields the same rule for equilibrium yields as in Theorem (1):

$$y_{t,m}^\theta = a_m^\theta + (b_m^{\mathbb{Q}})^\top \mathbf{X}_t + \psi_m^\theta (b_m^{\mathbb{P}})^\top \mathbf{X}_t,$$

where however we consider the realistic multi-factor setting and the coefficients $b_m^{\mathbb{P}}$ and $b_m^{\mathbb{Q}}$ are now vectors, with different entries for different factors. Of course, the distortion ψ_m^θ now depends on the data generating process, on maturity m, and on diagnosticity θ. This aspect places restrictions in the quantitative analysis.

1.5.2 Calibration

Theorem (1) suggests two independent routes to estimate the same primitive parameter θ. First, I retrieve θ using the profile of CG coefficients β_m, obtained with recovered beliefs. I match β_m with their theoretical counterparts, as derived in Theorem (1). I can then assess the fitting ability of such estimated θ on excess volatility. Second, I follow the mirror route: I estimate θ by matching excess volatility, and then assess its ability in fitting the profile of CG coefficients.

For the first exercise, I estimate θ that best matches the profile of CG coefficients. Note that the fit here cannot be perfect: in the short end of the curve Ross recovered beliefs under-react, a pattern which cannot be rationalized under diagnostic expectations, namely $\theta > 0$. The estimated value obtained when using this method fulfills:

$$\hat{\theta} = \arg\min_{\theta} \sum_{i=2}^{30} \left(\hat{\beta}_m - \beta_m^{\theta} \right)^2 \approx 0.7.$$

Second, I consider excess volatility. The diagnostic parameter θ disciplines excess volatility: the excess sensitivity of yields to factors is controlled by ψ_m^{θ}, which is a function of the rational conditional variance of interest rates with maturity m and the rational unconditional variance of interest rates with the same maturity. The variance of yields in a diagnostic world relates to the variance of yields in a rational world as:

$$\underbrace{\mathbb{V}^{\mathbb{P}}[y_{m,t}^{\theta}]}_{\text{data}} = \underbrace{\mathbb{V}^{\mathbb{P}}[y_{t,m}]}_{\text{RE model}} + \psi_m^{\theta} (b_m^{\mathbb{P}})^{\top} \Sigma b_m^{\mathbb{P}},$$

From this observation, I can calibrate the diagnostic parameter θ from the excess volatility of yields itself. Indeed, the variance of rational yields of the right hand side can be estimated by fitting the short end of the yield curve as in Giglio and Kelly (2018), while $b_m^{\mathbb{P}}$ is can be computed using the estimated persistence $\rho^{\mathbb{P}}$ obtained from a VAR(1) estimation of the $\mathbb{P}$-factor dynamics[16]. Therefore, I estimate θ as:

$$\hat{\theta} = \arg\min_{\theta} \sum_{i=1}^{30} \left(\mathbb{V}^{\mathbb{P}}[y_{m,t}^{\theta}] - \mathbb{V}^{\mathbb{P}}[y_{t,m}^{\theta=0}] - \psi_m^{\theta} (b_m^{\mathbb{P}})^{\top} \Sigma b_m^{\mathbb{P}})^2 \right)^2 \approx 0.47.$$

The two estimates differ, but remarkably they are close to previous estimates of parameter θ obtained using different data and different methodologies. For instance, Bordalo et al. (2018a) use survey forecasts of professional forecasters for many macro-financial variables and shows that typically there is over-reaction with magnitude $\theta \sim 0.6$, and equal to $\theta \sim 0.48$ for medium to long term interest rates. This is an additional confirmation that in my setting Ross recovery captures robust patterns of beliefs. To grasp the

[16] I used the methodology of Section 3 and imposed consistency across short maturities, namely: $\hat{\rho}^{\mathbb{Q}} := \arg\min_{\rho^{\mathbb{Q}}} \left(\frac{1}{T\bar{M}} \sum_{t,m} \left(y_{t,m} - \bar{y}_{t,m} \right)^2 \right)$. $\bar{M}$ is set as equal to 5: it quantifies how short the short end of the yield curve is. Robustnesses relative to the parameters to be included in the short end show consistency of the results.

quantitative meaning of the estimated values of θ, consider the benchmark $\theta = 1$. In this case, distorted forecasts of long maturity rates are equal to the rational forecast plus the revision. Assuming that the baseline rational forecast for the long run rate is around 2%, after the arrival of news indicative of a higher rational forecast of 3%, the diagnostic forecast will be then 4%. Distortions are thus sizable. This back-of-the-envelope calculation shows that the numbers at play are economically relevant.

Having estimated values for the distortion parameter θ, we can now evaluate the accuracy of the diagnostic model for excess volatility in interest rates. Figure (1.7) reports the volatilities of yields at different maturities obtained under a three factor affine rational model, namely a counter-factual model setting $\theta = 0$, together with the variance of yields obtained from the same model in which we plug the estimated $\theta \approx 0.47$ and $\theta \approx 0.7$.

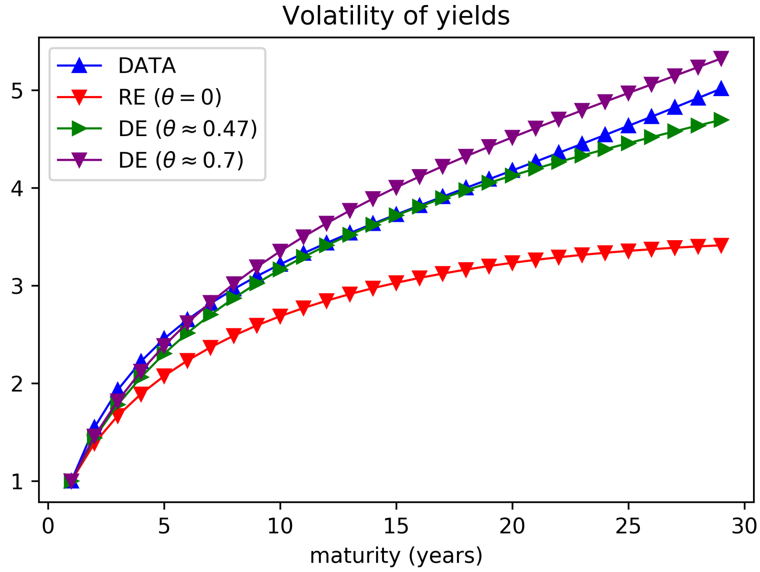


Figure 1.7: Excess volatility in the data (blue up triangle) versus the fit of a rational expectations affine model (red down triangle) and the diagnostic expectations affine model with $\theta \approx 0.47$ (green right triangle) and $\theta \approx 0.7$ (purple left triangle).

Figure (1.7) shows that diagnostic expectations capture much of the variation of the yield curve. At the longest maturity available, diagnostic expectations fit roughly

$\sqrt{\frac{\mathbb{V}^{\mathbb{P}}[y^{\hat{\theta}}_{t,30}]}{\mathbb{V}^{\mathbb{P}}[y^{\theta}_{t,30}]}} \approx 82\%$ of excess volatility. Considering the distortion parameter estimated from the profile of CG coefficients, $\theta \approx 0.7$, provide an explanation of ≈ 55 % of the excess volatility. This latter case shows that information implied from CG coefficients actually does help explaining excess volatility.

The symmetric question is, as suggested by Theorem (1), how much of the predictability of the forecast error can be explained from a distorting parameter θ which is inferred from excess volatility? The link between the forecast error predictability (CG coefficients) and the over-reaction is the one in Theorem (2), where the coefficients ψ^{θ}_{m} have now been derived from the diagnostic expectations model.

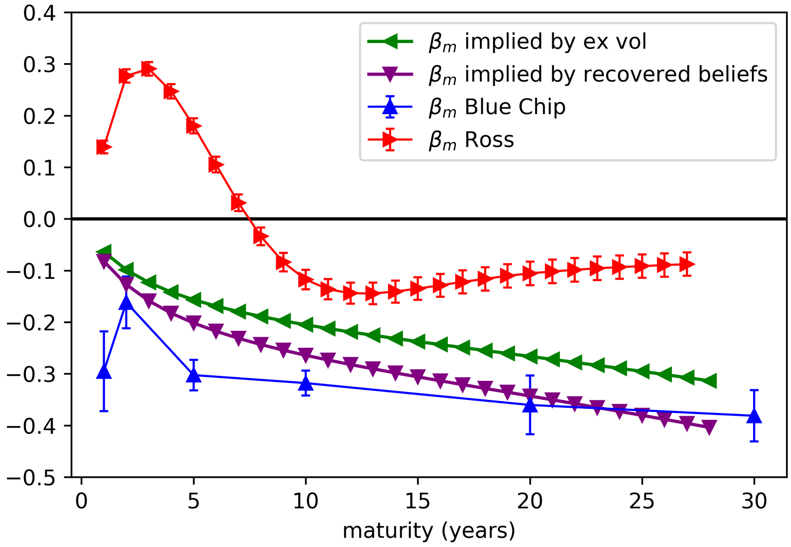


Figure 1.8: CG coefficients from Blue Chip data (blue up triangle), recovered beliefs (red right triangle) and implied by the calibrated diagnostic expectation model for $\hat{\theta} \approx 0.47$ (green left triangle) and $\hat{\theta} \approx 0.7$ (purple down triangle).

Figure (1.8) shows the term structure of CG coefficients, in the data (survey data in Blue, Ross recovered beliefs in red) as well as the β_m profiles implied by the calibrated parameters $\hat{\theta} \approx 0.47$ from excess volatility (green) and the calibrated distorted parameter $\hat{\theta} \approx 0.7$ from CG coefficients (purple). The fit is not expected to be perfect because under-reaction cannot be reconciled directly with the diagnostic expectation model. Indeed, within this model, the CG coefficients should be negative. The presence of under-reaction, particularly at short maturities, has been previously documented by different authors (Bouchaud et al. (2019), Bordalo et al. (2018a)). Other forces may be at play other than representativeness, such as disagreement and informational frictions, which needs to be further investigate. When considering the whole term structure, however, there is a clear pattern of increasing *reaction* to news, and, in particular, or over-reaction at long maturities.

1.6 Robustness

In this Section, I discuss theoretical and empirical robustness to the recovery theorem. Then, I consider different specifications of the forecast error predictability tests. Finally, I consider alternative specifications of the diagnostic expectations model.

1.6.1 Empirical implementation of the recovery theorem

Discretization and sampling frequency

Figure (1.9) shows that the number of states chosen for the discretization for each factor ($N = 25, 50, 75, 100$) and the sampling frequency (daily versus monthly) do not qualitatively affect the CG coefficients curve.

Error predictability in different sub-samples

The level of the nominal yield curve is close to unit root process, especially at high frequency (e.g. daily). Does possible non stationarity relates to under/over-reaction to news? Figure (**??**) shows that the decreasing pattern in CG coefficients survives in different subsamples.

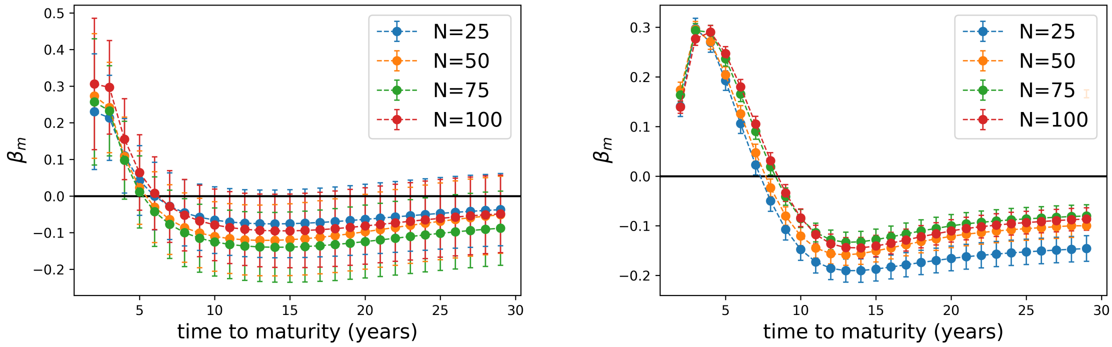
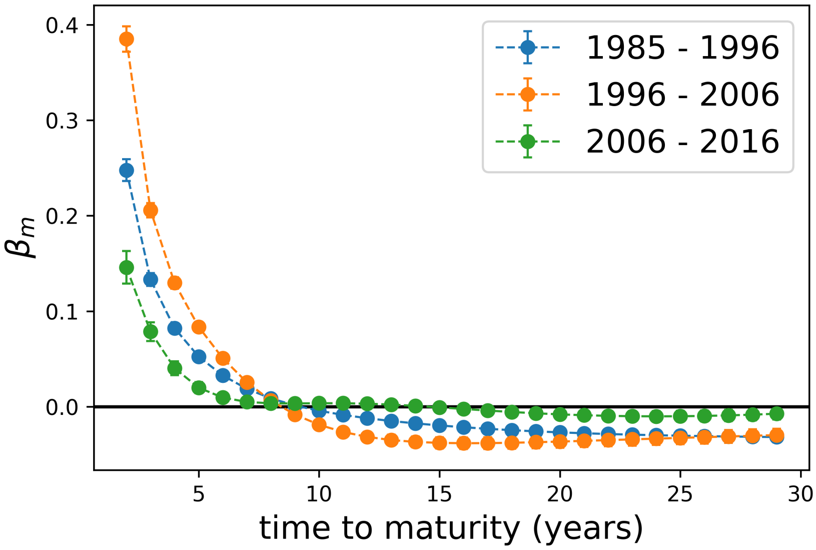


Figure 1.9: Left panel: CG coefficients at the monthly frequency. Right panel: CG coefficients at the daily frequency.



1.6.2 Predictability of the forecast error, different tests

News have been defined as the difference between the rational forecast and the average forecast, in the diagnostic expectation model used. Here I consider CG like tests, where the forecast revision is defined as the difference between the rational forecast and the long run forecast. The figure shows qualitative agreement, both using survey data and using recovered beliefs.

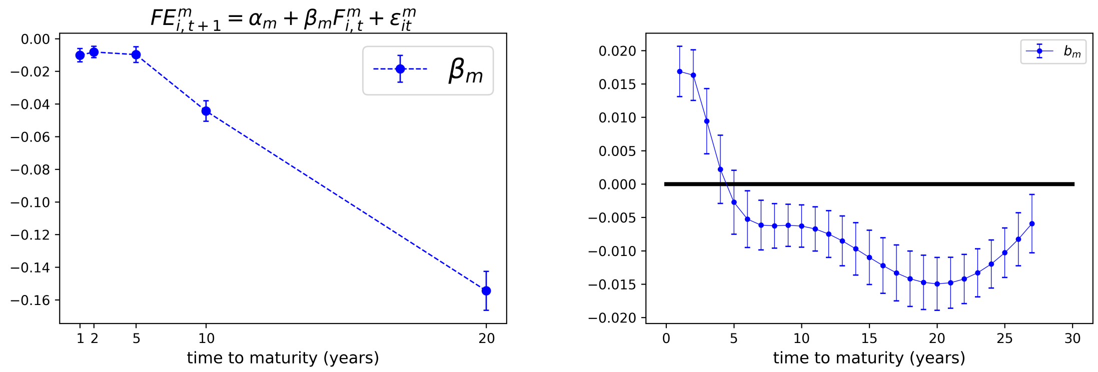


Figure 1.10: Left panel: Blue Chip Financial. Right panel: Ross recovered beliefs

1.6.3 Diagnostic Expectations: different benchmark distributions

One important degree of freedom in the specification of the diagnostic expectation model is the choice of the *comparison* distribution. Here, I have chosen the unconditional or long run distribution of future rates. This is a convenient choice because the diagnostic distribution remains Markovian, namely $r_{t,m}$ depends on $\mathbf{X}_t$ but not on $\mathbf{X}_{t-1}, \mathbf{X}_{t-2}, \dots$. It is however important to investigate what changes with different benchmark distributions.

Over-reaction relative to time $t-1$ prediction

Consider the following specification, inspired by Bordalo et al. (2018b)[17]:

$$f_{\mathbb{P}^\theta}(r_{t,m}|\mathbf{X}_t) \propto f_{\mathbb{P}}(r_{t,m}|\mathbf{X}_t) \left(\frac{f_{\mathbb{P}}(r_{t,m}|\mathbf{X}_t)}{f_{\mathbb{P}}(r_{t,m}|\mathbf{X}_{t-1})} \right)^\theta .$$

The diagnostic distribution of $r_{t,m}$ in this case is still Gaussian, with mean:

[17] In Bordalo et al. (2018b), the authors consider the convenient benchmark distribution as $f_{\mathbb{P}}(r_{t,m}|\mathbf{X}_t := \mathbf{X}_{t-1})$. This simplifies the algebra of diagnostic expectations: only the first moment is distorted and the distortion is independent of the maturity. This assumption is innocuous from a single horizon perspective, which is the setting of Bordalo et al. (2018b), yet it forces the law of iterated expectations to hold, differently from slight perturbations of the benchmark distribution. Therefore, this simplifying assumption is not appropriate for my setting.

$$\mathbb{E}_t^{\mathbb{P}^\theta}[r_{t,m}] = \mathbb{E}_t^{\mathbb{P}}[r_{t,m}] + \psi_m^\theta \left(\mathbb{E}_t^{\mathbb{P}}[r_{t,m}] - \mathbb{E}_{t-1}^{\mathbb{P}}[r_{t,m}] \right)$$

and variance:

$$\mathbb{V}_t^{\mathbb{P}^\theta}[r_{t,m}] = \left(\frac{\theta+1}{\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]} - \frac{\theta}{\mathbb{V}_{t-1}^{\mathbb{P}}[r_{t,m}]} \right)^{-1},$$

where

$$\psi_m^\theta := \frac{\theta \frac{\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]}{\mathbb{V}_{t-1}^{\mathbb{P}}[r_{t,m}]}}{1+\theta-\theta \frac{\mathbb{V}_t{}^{\mathbb{P}}[r_{t,m}]}{\mathbb{V}_{t-1}^{\mathbb{P}}[r_{t,m}]}}.$$

Also in this case, the distortion coefficients ψ_m^θ starts at zero (namely $\lim_{m\to 0} \psi_m^\theta = 0$), asymptotically approaches θ (namely $\lim_{m\to\infty} \psi_m^\theta = \theta$) and they are increasing. Moving to risk neutral distorted expectations, the diagnostic yield curve reads:

$$y_{t,m}^\theta = a_m^\theta + (b_m^{\mathbb{Q}})^\top \mathbf{X}_t + \psi_m^\theta \left((b_m^{\mathbb{P}})^\top \mathbf{X}_t - (b_{m+1}^{\mathbb{P}})^\top \mathbf{X}_{t-1} \right)$$

In this case, the relation between the volatility of the yields in a diagnostic world, relative to the rational one is modified by forecast revision of interest rates. After good news yields are higher while after bad news are lower, relative to the RE case. Unconditionally, yields display higher variance in the diagnostic case, since the extra term $\psi_m^\theta (b_m^{\mathbb{P}})^\top \mathbf{X}_t - (b_{m+1}^{\mathbb{P}})^\top \mathbf{X}_{t-1}$ is positively correlated with $(b_m^{\mathbb{Q}})^\top \mathbf{X}_t$. This is so because $b_{m+1}^{\mathbb{P}} < b_m^{\mathbb{P}}$ and the $\mathbb{P}$ persistence is on average smaller than one (or in the on linear case it is assumed the be smaller than one on average). they display higher volatility.

Over-reaction relative to time $t-k$ prediction ($k>1$)

In this case the logic of the case $k=1$ still goes through with:

$$\psi_m^\theta := \frac{\theta\frac{\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]}{\mathbb{V}_{t-k}^{\mathbb{P}}[r_{t,m}]}}{1+\theta-\theta\frac{\mathbb{V}_t{}^{\mathbb{P}}[r_{t,m}]}{\mathbb{V}_{t-k}^{\mathbb{P}}[r_{t,m}]}}.$$

and:

$$y_{t,m}^\theta = a_m^\theta + \psi_m^\theta\left((b_m^{\mathbb{Q}})^\top \mathbf{X}_{t+1} - (b^{\mathbb{P}})_{m+k}^\top \mathbf{X}_{t-k}\right).$$

Over-reaction relative to a weighted average of past predictions

Yesterday information and average information are two useful benchmark, yet one may consider a more "colorful" memory. I consider the following specification, inspired by Bordalo et al. (2018b), Internet Appendix):

$$f_{\mathbb{P}^\theta}(r_{t,m}|\mathbf{X}_t) \propto f_{\mathbb{P}}(r_{t,m}|\mathbf{X}_t)\prod_{k=1}^{M}\left(\frac{f_{\mathbb{P}}(r_{t,m}|\mathbf{X}_t)}{f_{\mathbb{P}}(r_{t,m}|\mathbf{X}_{t-k})}\right)^{\theta a_k}.$$

where $0 \le a_k \le 1$ are positive weights on past information such that $\sum_k a_k = 1$ and $1 \le M \le \infty$.[18] The diagnostic distribution of $r_{t,m}$ in this case is still Gaussian, with mean:

[18] When $M=\infty$ suitable regularity conditions need to be assumed for convergence.

$$\mathbb{E}_t^{\mathbb{P}^\theta}[r_{t,m}] = \mathbb{E}_t^{\mathbb{P}}[r_{t,m}] + \psi_m^\theta \left(\mathbb{E}_t^{\mathbb{P}}[r_{t,m}] - \sum_{k=1}^{M} a_k \mathbb{E}_{t-k}^{\mathbb{P}}[r_{t,m}] \right)$$

and variance:

$$\mathbb{V}_t^{\mathbb{P}^\theta}[r_{t,m}] = \left(\frac{\theta+1}{\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]} - \theta \sum_{k=1}^{M} \frac{a_k}{\mathbb{V}_{t-k}^{\mathbb{P}}[r_{t,m}]} \right)^{-1},$$

where

$$\psi_m^\theta := \frac{\theta \mathbb{V}^{\mathbb{P}}[r_{t,m}] \sum_{k=1}^{M} \frac{a_k}{\mathbb{V}_{t-k}^{\mathbb{P}}[r_{t,m}]}}{1 + \theta - \theta \mathbb{V}^{\mathbb{P}}[r_{t,m}] \sum_{k=1}^{M} \frac{a_k}{\mathbb{V}_{t-k}^{\mathbb{P}}[r_{t,m}]}}.$$

Also in this case, the distortion coefficients ψ_m^θ starts at zero (namely $\lim_{m\to 0} \psi_m^\theta = 0$), asymptotically approaches θ (namely $\lim_{m\to\infty} \psi_m^\theta = \theta$) and they are increasing. Moving to risk neutral distorted expectations,the diagnostic yield curve reads:

$$y_{t,m}^\theta = a_m^\theta + \psi_m^\theta \left((b_m^{\mathbb{Q}})^\top \mathbf{X}_{t+1} - \sum_{k=1}^{M} a_k (b^{\mathbb{P}})_{m+k}^\top \mathbf{X}_{t-k} \right).$$

The diagnostic yield curve is highly non Markovian in this case, yet it still feature the excess volatility pattern.

1.7 Conclusions

This paper shows empirically that beliefs distortions increases with maturity (decreasing CG coefficients) and that this is tightly linked to the excess volatility in the term structure of asset prices documented by Giglio and Kelly (2018) under the class of affine term structure models. The crucial property that beliefs fail in the data and that accounts for excess volatility is the law of iterated expectations. I show that a diagnostic affine model, can quantitatively capture the variation in excess volatility and the distortions in the individual beliefs. In the model agents over-react differently for different levels of the fundamental uncertainty, which naturally varies in the context of term structure. This approach is a first step toward two research direction: the investigation of the effects of higher moments in beliefs distortions and the incorporation of non rational beliefs into quantitative finance models. At the aggregate level, I show the beliefs dynamics mixes under-reaction at short maturities and over-reaction at long-maturities. A foundation of such cross-over needs further investigations. Methodologically, this paper implements empirically the Ross Recovery theorem in the context of the term structure of interest rates, and show that, despite the identification problem raised by Borovička et al. (2016), recovered beliefs can spot inter-temporal belief inconsistencies. This is important to augment survey data with asset prices information and it is portable to different domains, such as the study of the equity term structure.

A Affine term structure models with overreacting beliefs

Consider first the standard $\mathbb{Q}$-affine term structure models, which are defined by the two following ingredients. First, few factors (or state variables) $\mathbf{X}_t = (X_{1,t}, X_{2,t}, \cdots)$ drive the short rate in an affine fashion and, second, the $\mathbb{Q}$-dynamics (assuming no arbitrage) of the factors is a VAR(1) with homoskedastic shocks.

$$\begin{cases} r_t = \delta_0 + \delta_1^\top \mathbf{X}_t \\ \mathbf{X}_t = \rho^{\mathbb{Q}} \mathbf{X}_{t-1} + \Sigma^{\mathbb{Q}} \varepsilon_t^{\mathbb{Q}}. \end{cases}$$

This defines the class of $\mathbb{Q}$-affine models I consider, together with sufficient regularity conditions for the quantities computed to be well defined. The convenient affine specification and linear dynamics implies that prices are exponentially affine in the factors:

$$\begin{aligned} y_{t,m} &= -\frac{1}{m} \log P_{t,m} = -\frac{1}{m} \mathbb{E}_t^{\mathbb{Q}} \left[e^{m \cdot r_{t,m}} \right] \\ &= \delta_0 - \log \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\varepsilon_{t,m}^{\mathbb{Q}}} \right] + \frac{\delta_1^\top}{m} \sum_{i=0}^{m-1} (\rho^{\mathbb{Q}})^i \mathbf{X}_t, \end{aligned}$$

where $b_m^{\mathbb{Q}} = \frac{\delta_1^\top}{m} \sum_{i=0}^{m-1} (\rho^{\mathbb{Q}})^i$ and $\varepsilon_{t,m}^{\mathbb{Q}} = \sum_{k=1}^{m-1} \sum_{i=1}^{l} (\rho^{\mathbb{Q}})^{l-1} \Sigma^{\mathbb{Q}} \varepsilon_{t+i}$. The previous expression is affine since $\mathbb{Q}$ shocks are independent of time t information and therefore the cumulant generating function in the previous expression is maturity dependent by not state dependent. This setting is quite general since I do not have assumptions about the physical measure nor about the SDF other than technical regularity conditions, which are worked out in Le et al. (2010).

First, I discuss how this setting relates to the empirical analysis of Section 3 and Section 4 and then how this setting relates to the theoretical model of beliefs formation developed in Section 2. I do so by incrementally adding structure and assumptions needed, relative to the $\mathbb{Q}$-affine benchmark so far discussed.

In section 3, I apply the recovery theorem independently estimating the $\mathbb{Q}$ measure at different maturities. This amounts to detect distortions in the sensitivity coefficients $b_m^{\mathbb{Q}} = \frac{\delta_1^\top}{m}\sum_{i=0}^{m-1}(\rho^{\mathbb{Q}})^i$. Theorem (1) shows that overreacting beliefs can both generate such distortions, which, in turn, account for the Giglio and Kelly (2018) excess volatility puzzle and explain the predictability of forecast error documented with survey data.

How do beliefs generate such distortions? Assume that, under the physical measure $\mathbb{P}$, factors evolves in a Markovian fashion:

$$\mathbf{X}_{t+m} = f^{(m)}(\mathbf{X}_t)\mathbf{X}_t + \Sigma^{\mathbb{P}}\varepsilon^{\mathbb{P}}_{t+m,C},$$

where $f^{(m)}(\mathbf{X}_{t-1}) = \underbrace{f(f(\cdots}_{\text{m times}}(\mathbf{X}_{t-1})$ denotes a Markovian, yet possibly non linear dynamics for the factors and $\varepsilon^{\mathbb{P}}_{t+m,C} = \sum_{i=0}^{m-2} f^{(i+1)}(\mathbf{X}_t)\Sigma^{\mathbb{P}}\varepsilon^{\mathbb{P}}_{t+1+i}$ denotes the cumulated shock to maturity m, which is heteroskedastic if the $\mathbb{P}$-dynamics is non linear. Then, assume that, when considering horizon m, the market distorts the dynamics of the factors in a maturity m dependent fashion. Formally, $\forall\ l \leq m$:

$$\mathbf{X}_{t+l} = (1+\psi^\theta_m)f^{(l)}(\mathbf{X}_t)\mathbf{X}_t + \sigma^\theta_m\varepsilon^{\mathbb{P}}_{t+l,C}.$$

The local persistences of all fundamentals up to time $t+m$ are inflated if $\psi^\theta_m > 0$. Equivalently, cumulated shocks on the factors at time $t+l$ when forming expectations at maturity $m \geq l$ are distorted as:

$$\varepsilon^{\mathbb{P}^\theta}_{t+l,C} = \sigma^\theta_m\varepsilon^{\mathbb{P}}_{t+l,C} + \psi^\theta_m f^{(l)}(\mathbf{X}_t)\mathbf{X}_t.$$

Then, the distorted interest rates to maturity m at time t reads:

$$r_{t,m} = \frac{1}{m}\sum_{i=0}^{m-1} r_{t+i} = \delta_0 + (1+\psi^\theta_m)b^{\mathbb{P}}_m(\mathbf{X}_t)\mathbf{X}_t + \sigma^\theta_m\varepsilon^{\mathbb{P}}_{t+m,C},$$

where $b_m^{\mathbb{P}}$ as well as cumulated shocks to interest rates are defined as in Section 2. Then, I assume that the SDF is such that:

$$\mathbf{X}_{t+l} = (\rho^{\mathbb{Q}})^l \mathbf{X}_t + \psi_m^\theta f^{(l)}(\mathbf{X}_t)\mathbf{X}_t + \sigma_m^\theta \varepsilon_{t+l,C}^{\mathbb{Q}}.$$

Equivalently, shocks on the factor at time $t+l$ when forming risk adjusted expectations at maturity $m \geq l$ are distorted as:

$$\varepsilon_{t+l,C}^{\mathbb{Q}^\theta} = \sigma_m^\theta \varepsilon_{t+l,C}^{\mathbb{Q}} + \psi_m^\theta b_m^{\mathbb{P}}(\mathbf{X}_t)\mathbf{X}_t.$$

which implies that:

$$r_{t,m} = \frac{1}{m}\sum_{i=0}^{m} r_{t+i} = \delta_0 + b_m^{\mathbb{Q}}\mathbf{X}_t + \psi_m^\theta b_m^{\mathbb{P}}(\mathbf{X}_t)\mathbf{X}_t + \sigma_m^\theta \varepsilon_{t,m}^{\mathbb{Q}}.$$

This expression settles the ground to test both excess volatility of yields ($b_m^{\mathbb{P}}(\mathbf{X}_t)$ positively and increasingly in m contributing to the volatility of yields) and increasingly over-reacting beliefs ($\psi_m^\theta > 0$ and increasing in m).

B Estimation

Factor construction

I use the data available from Gürkaynak et al. (2007), who provide a daily estimate of the yield curve, from ??, with maturities $1y$, $2y$, $\ldots$ 30y

Factors are constructed as follows. First estimate of the variance covariance matrix as:

$$\frac{1}{T}\sum_{t=1}^{T}\underbrace{y_t y_t^\top}_{30\times 30}.$$

Then, I consider the first three principal components of the variance covariance matrix, $\underbrace{PC_1}_{30\times 1}$. PC_1 and PC_3. Principal components are unconditionally orthogonal. For $i = 1, 2, 3$, I build factor X_i as the projection of the yield curve into PC_i:

$$X_{t,i} := \langle PC_i, y_t \rangle.$$

Principal components and factors are plotted in Figure **??**.

Parameter Estimation

We want to estimate the parameters of the risk neutral dynamics:

$$y_{t+1} = \delta_0 + \delta_1^\top X_t$$
$$\mathbf{X}_t = \rho \mathbf{X}_{t-1} + \Sigma^C \varepsilon_t^{\mathbb{Q}}$$

where is the upper triangular Cholesky decomposition of the one period variance-covariance Σ. This is assumed not to be stochastic and to be the same under $\mathbb{Q}$ and under $\mathbb{P}$. Therefore is can be estimated via OLS.

In order to estimate the remaining parameters ρ_1, ρ_2 and ρ_3, let me compute the affine relation for yields. The price at time t of a bond with time to maturity $m \geq 1$ is:

$$
\begin{aligned}
P_{t,m} = e^{-y_{t,m}m} &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-(r_{t+1}+\dots+r_{t+m})}\right] = \\
&= \exp -\delta_0 m - \left(\sum_{i=1}^{3} \delta_{1,i} \sum_{k=0}^{m-1} \rho_i^k\right) X_{t,i} \mathbb{E}_t^{\mathbb{Q}}\left[\exp - \sum_{i=0}^{m-2} \delta_1^\top \Sigma^C \rho^i\right].
\end{aligned}
$$

Moving from prices to yields:

$$
y_{t,m} = -\frac{\log P_{t,m}}{m} = \delta_0 + \frac{\delta_1^\top \sum_{k=0}^{m-1} \rho^k}{m} \mathbf{X}_t - \frac{\log \mathbb{E}_t^{\mathbb{Q}}\left[\exp - \sum_{i=0}^{m-2} \delta_1^\top \Sigma^C \rho^i\right]}{m}
$$

Note that the maturity m need to be expressed in the same units of the sampling frequencies. For example, for monthly data and yearly maturities for $1y$ to $30y$, the maturities should be expressed in months, $m = 12 \times 1, \dots 12 \times 30$.

Let us know discuss hot to estimate model's parameters via OLS. First, we get $\hat{\delta}_0, \hat{\delta}_1^\top$ via OLS estimation of:

$$
y_t^1 := r_t = \delta_0 + \delta_1^\top X_t.
$$

Thus we can estimate $\rho_{i,\tau}$ as:

$$
\hat{\rho} := \arg\min_{\rho_{i,\tau}} \left(\frac{1}{T} \sum_{t=1}^{T} \sum_{m=1}^{30} y_{t,m}(\rho) - \bar{y}_{m,t}\right)^2,
$$

where $y_{t,m}(\rho)$ is computed via the affine relation, while $\bar{y}_{m,t}$ denote the data. Similarly, for computing, maturity by maturity parameters, I compute:

$$\hat{\rho}_m := \arg\min_{\rho_{i,\tau}} \left(\frac{1}{T} \sum_{t=1}^{T} y_{t,m}(\rho) - \bar{y}_{m,t} \right)^2 .$$

Discretization

I need to discretize the state space in order to compute the Arrow-Debreu price matrix. There are well known methods to efficiently discretize an $AR(1)$. In the case of multivariate processes with not trivial correlations, the following transformation provide a set of uncorrelated equation:

$$\mathbf{X}_t \longrightarrow \mathbf{Z}_t := \Sigma^{C^{-1}} \mathbf{X}_t.$$

Now, consider the VAR(1) $\mathbb{Q}$-dynamics:

$$\mathbf{X}_t = \rho \mathbf{X}_{t-1} + \Sigma^C \varepsilon_t^{\mathbb{Q}}.$$

Multiplying both sides by $\Sigma^{C^{-1}}$, I get:

$$\mathbf{Z}_t = \rho \mathbf{Z}_{t-1} + \varepsilon_t^{\mathbb{Q}}.$$

This is a system of three independent $AR(1)$ that can be independently discretized. After the discretization I come back to the real factor with the inverse transformation to the discretized version of $\mathbf{Z}_t$, say $\mathbf{Z}_t^D$:

$$\mathbf{X}_t^D \longleftarrow \Sigma^C \mathbf{Z}_t^D .$$

I rely on the Rouwenhorst discretization method, which matches conditional and unconditional first and second moments, and is the state of the art among those technique. I use 10 levels for each factor, thus having overall 1000 states. This choice does not seems to affect the results.

C Tables

	Y1	Y2	Y3	Y4	Y5	Y6	Y7	Y8	Y9	Y10
const	3.71***	3.97***	4.21***	4.43***	4.63***	4.81***	4.97***	5.11***	5.23***	5.80***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$X_{1,t}$	0.21***	0.22***	0.22***	0.21***	0.21***	0.20***	0.20***	0.19***	0.19***	0.16***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$X_{2,t}$	0.52***	0.42***	0.33***	0.25***	0.19***	0.13***	0.08***	0.04***	0.01***	-0.16***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$X_{3,t}$	0.25***	0.16***	0.08***	0.01***	-0.05***	-0.11***	-0.15***	-0.18***	-0.21***	0.36***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R-squared	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
No. observations	7903	7903	7903	7903	7903	7903	7903	7903	7903	7903

Table 1: Regressions of yields on factors

	Y11	Y12	Y13	Y14	Y15	Y16	Y17	Y18	Y19	Y20
const	5.43***	5.51***	5.58***	5.64***	5.69***	5.73***	5.76***	5.79***	5.81***	5.83***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$X_{1,t}$	0.18***	0.18***	0.18***	0.18***	0.18***	0.17***	0.17***	0.17***	0.17***	0.17***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$X_{2,t}$	-0.04***	-0.06***	-0.07***	-0.09***	-0.10***	-0.11***	-0.12***	-0.12***	-0.13***	-0.13***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$X_{3,t}$	-0.22***	-0.22***	-0.21***	-0.19***	-0.17***	-0.14***	-0.11***	-0.08***	-0.05***	-0.01***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R-squared	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
No. observations	7903	7903	7903	7903	7903	7903	7903	7903	7903	7903

	Y21	Y22	Y23	Y24	Y25	Y26	Y27	Y28	Y29	Y30
const	5.84***	5.84***	5.85***	5.85***	5.84***	5.84***	5.83***	5.82***	5.81***	5.80***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$X_{1,t}$	0.17***	0.17***	0.17***	0.17***	0.17***	0.17***	0.16***	0.16***	0.16***	0.16***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$X_{2,t}$	-0.14***	-0.14***	-0.14***	-0.15***	-0.15***	-0.15***	-0.15***	-0.15***	-0.16***	-0.16***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
$X_{3,t}$	0.03***	0.06***	0.10***	0.14***	0.18***	0.22***	0.25***	0.29***	0.33***	0.36***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R-squared	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
No. observations	7903	7903	7903	7903	7903	7903	7903	7903	7903	7903

The estimated covariance matrix of the factors is:

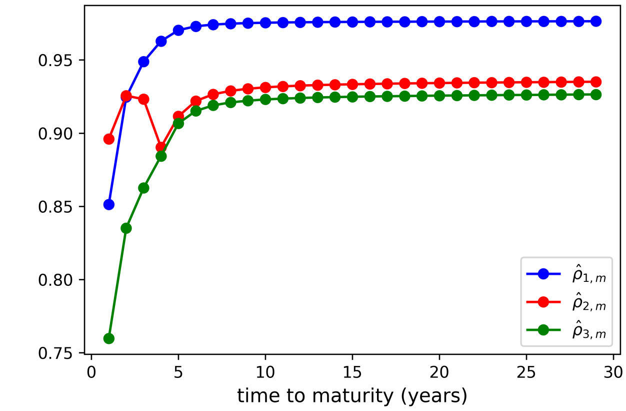
$$\Sigma = \begin{pmatrix} 1. & -0.333506 & -0.221877 \\ -0.333506 & 1 & 0.033376 \\ -0.221877 & -0.333506 & 1 \\ & & \end{pmatrix}.$$

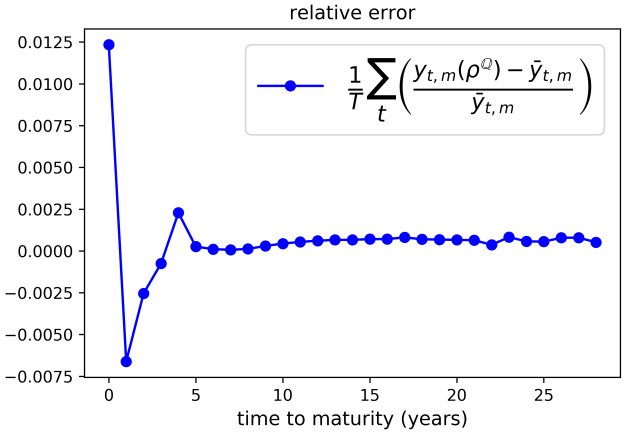
The Cholesky decomposition of the estimated covariance matrix of the factors is:

$$\Sigma^C = \begin{pmatrix} 1. & 0. & 0. \\ -0.333506 & 0.94274798 & 0 \\ -0.221877 & -0.0430882 & 0.974122171 \\ & & \end{pmatrix}.$$

Average estimated ρ :

$$\rho^{\mathbb{Q}} = \begin{pmatrix} 0.968190 & 0 & 0 \\ 0 & 0.92868686 & 0 \\ 0 & 0 & 0.911450 \\ & & \end{pmatrix}$$





D The Recovery Theorem

Let me now show how the *path independence* assumption helps with the identification of beliefs. Under path independence the fundamental asset pricing equation can be written as:

$$\mathbb{A}^{\theta}_{ij}z_j = \delta z_i \mathbb{P}^{\theta}_{ij}.$$

Noticing that probabilities add up to one, one gets:

$$\sum_j \mathbb{A}^{\theta}_{ij} z_j = \delta z_i.$$

This is an eigenvalue problem from the Arrow-Debreu matrix. By Perron-Frobenious theorem[19] (see Meyer (2000)), the Arrow-Debreu matrix has unique positive eigenvector (which can be identified with z) corresponding to the largest eigenvalue (which can be identified with δ). Therefore, under *path independent SDF* it is possible to identify beliefs as:

$$\mathbb{P}^{\theta}_{ij} = \mathbb{A}^{\theta}_{ij} \frac{1}{\delta} \frac{z_j}{z_i}.$$

So far, I exploited only *one period* Arrow-Debreu prices. Do prices of *multi-period* Arrow-Debreu secutiries provide additional information? If the law of iterated expectation holds, then long term beliefs are *iterations* of short term belies:

$$\mathbb{P}_m = \underbrace{\mathbb{P} \times \cdots \times \mathbb{P}}_{\text{m times}} = \mathbb{P}^m,$$

where $\mathbb{P}^m$ is the m-th power of the one period transition probability matrix $\mathbb{P}$. In this case one period Arrow-Debreu prices are sufficient to identify the term structure of beliefs. However, if the law holds true or not is ultimately an empirical question: moreover, a testable implication of the law of iterated expectation is the fact that the term structure of CG coefficients needs to be flat (2). Given a panel of bond prices $\{P_{t,m}\}_{t,m}$, Arrow-Debreu prices to maturity m can be therefore estimated by relying of the time series $\{P_{m,t}\}_t$ only. Using the recovery theorem, the econometrician can access the term structure of beliefs

[19] The Perron-Frobenious theorem assumes a positive and irreducible matrix. The Arrow-Debreu matrix is positive and irreducible under no arbitrage.

$\mathbb{P}^\theta_m$, where $[\mathbb{P}^\theta_m]_{ij}$ is the *believed* probability of transitioning from state i to state j in m steps and test the *horizon dependence* of beliefs as discussed in (2).

E Proofs

Proposition 1

Proof. From the last equation in Appendix A, it is straightforward to compute:

$$y_{t,m} = -\frac{1}{m}\mathbb{E}_t^{\mathbb{Q}^\theta}\left[e^{-m \cdot r_{t,m}}\right] = \delta_0 - \frac{1}{m}\log \mathbb{E}^{\mathbb{Q}}\left[e^{-\sigma_m^\theta \varepsilon_{t,m}^{\mathbb{Q}}}\right] + b_m^{\mathbb{Q}}\mathbf{X}_t + \psi_m^\theta b_m^{\mathbb{P}}(\mathbf{X}_t)\mathbf{X}_t.$$

The expectations is the cumulant generating function is independent of time t information because $\mathbb{Q}$ shocks are homoskedastic. □

Theorem 1 (Increasing over-reaction and excess volatility)

Proof. First, in order to compute the forecast of future yields (which are an endogenous variable) at maturity m, note that the agent first compute distorted forecasts of factors and then she plugs those in into the pricing equation which relates yields to factors. Therefore, there is a compounding effect of distortion coefficients, which will appear in the following calculations. The following approximation will be helpful. Consider the term:

$$\frac{b_m^\top Cov(F_t[\mathbf{X}_{t+1}], F_t[\mathbf{X}_{t+1}] - F_{t-k}[\mathbf{X}_{t+1}])b_m}{b_m^\top \mathbb{V}[F_t[\mathbf{X}_{t+1}] - F_{t-k}[\mathbf{X}_{t+1}]]b_m},$$

where $F_t[\mathbf{X}_{t+1}] - F_{t-k}[\mathbf{X}_{t+1}]$ is the forecast revision, when the reference distribution in the expectation model is the $k-$lagged one. The case $k = \infty$ corresponds to taking the unconditional (or historical) average as benchmark. Convex combination of the past forecasts can be easily handled as well. In this proof, $b_m := b_m^{\mathbb{P}}(\mathbf{X}_t)$ for convenience of notation. If the DGP is a $VAR(1)$ with diagonal persistence matrix ρ, then:

$$\frac{b_m^\top Cov(F_t[\mathbf{X}_{t+1}], F_t[\mathbf{X}_{t+1}] - F_{t-k}[\mathbf{X}_{t+1}])b_m}{b_m^\top \mathbb{V}[F_t[\mathbf{X}_{t+1}] - F_{t-k}[\mathbf{X}_{t+1}]]b_m} = 1 - \frac{b_m^\top (\rho^2 - \rho^{1+k})\mathbb{V}[\mathbf{X}_t]b_m}{b_m^\top \mathbb{V}[F_t[\mathbf{X}_{t+1}] - F_{t-k}[\mathbf{X}_{t+1}]]b_m}. \quad (14)$$

Therefore, the term is approximately equal to one, for highly persistent processes. The argument goes through also for non linear processes, provided that the local persistence $\rho(\mathbf{X}_t)$ is, on average, close to one. Moreover, for a fixed persistence matrix ρ, the deviation from one, asymptotically vanishes for $m \to \infty$. This is so because each entry of b_m convergences geometrically for large maturities. The approximation is exact also in the special case of exactly equally persistent factors or in the special case of a single factor model.

$i)$ Under rational expectations. $\beta_m = 0$ because the forecast error is unpredictable on the basis of past information.

Under maturity independent distortion ψ^θ:

$$\beta_m = \frac{Cov\left(FE_{t+1}^\theta[y_{t+1,m}^\theta], FR_t^\theta[y_{t+1,m}^\theta]\right)}{\mathbb{V}\left[FR_t^\theta[y_{t+1,m}^\theta]\right]} = \frac{-\psi^\theta(1+\psi^\theta)}{(1+\psi^\theta)^2} \frac{(b_m^\theta)^\top Cov\left(F_t[\mathbf{X}_{t+1}], FR_t[\mathbf{X}_{t+1}]\right) b_m^\theta}{(b_m^\theta)^\top \mathbb{V}[FR_t[\mathbf{X}_{t+1}]] b_m^\theta}.$$

Under average reference forecast or under one factor model or under approximation (14) the second fraction reduces to a positive constant. Therefore β_m is negative $\iff$ $\psi^\theta > 0$.

Under maturity dependent distortion ψ_m^θ:

$$\begin{aligned}\beta_m &= \frac{Cov\left(FE_{t+1}^\theta[y_{t+1,m}^\theta], FR_t^\theta[y_{t+1,m}^\theta]\right)}{\mathbb{V}\left[FR^\theta[y_{t+1,m}^\theta]\right]} \\ &= \frac{-\psi_{m+1}^\theta(1+\psi_m^\theta)}{(1+\psi_m^\theta)^2} \frac{(b_m^\theta)^\top Cov\left(F_t[\mathbf{X}_{t+1}], F_t[\mathbf{X}_{t+1}] - \frac{\psi_{m+2}^\theta}{\psi_{m+1}^\theta} F_{t-1}[\mathbf{X}_{t+1}]\right) b_m^\theta}{(b_m^\theta)^\top \mathbb{V}[F_t[\mathbf{X}_{t+1}] - \frac{\psi_{m+2}^\theta}{\psi_{m+1}^\theta} F_{t-1}[\mathbf{X}_{t+1}]] b_m^\theta}.\end{aligned}$$

Under average reference forecast or under one factor model the second fraction reduces to a positive constant. Under approximation (14) and assuming $\frac{\psi_{m+2}}{\psi_{m+1}}\rho_i < 1$ for each entry of the persistence matrix ρ, the second fraction reduces to a positive constant. β^{θ}_m is thus negative and increasing $\iff$ $\psi^{\theta}_m > 0$ and ψ^{θ}_m is increasing in m.

ii) It follows from the expression in Proposition (1).

□

Theorem 2

Proof. *i*) Calculations

ii)

1.) Under rational expectations, the misspecified CG coefficients, computed with forecasts identified by the recovery theorem and distorted by the martingale component of the SDF, are:

$$\widetilde{\beta}_m = \frac{Cov\left(\widetilde{FE}_{t+1}[y_{t+1,m}], \widetilde{FR}_t[y_{t+1,m}]\right)}{\mathbb{V}\left[\widetilde{FR}_t[y_{t+1,m}]\right]}.$$

First, consider the one period ahead misspecified forecast. I drop the additive constant $a^{\mathbb{Q}}_m$ in the following calculations, because it is inessential for the computation of covariances.

$$\begin{aligned}\widetilde{F}_t[\mathbf{X}_{t+1}] &= b^{\top}_m\mathbb{E}_t\left[h_{t+1}\mathbf{X}_{t+1}\right] = b^{\top}_m\mathbb{E}_t\left[\mathbf{X}_{t+1}\right]\mathbb{E}_t\left[h_{t+1}\right] + b^{\top}_m\mathrm{Cov}_t\left(h_{t+1}, \mathbf{X}_{t+1}\right)\\ &= b^{\top}_m\mathbb{E}_t\left[\mathbf{X}_{t+1}\right] + b^{\top}_m\mathrm{Cov}_t\left(h_{t+1}, \mathbf{X}_{t+1}\right).\end{aligned}$$

Therefore, the misspecified forecast error reads:

$$\widetilde{FE}_{t+1}[y_{t+1,m}] = y_{t+1,m} - \widetilde{\mathbb{E}}_t\left[y_{t+1,m}\right] = FE_{t+1}[y_{t+1,m}] - b^{\top}_m Cov_t\left(h_{t+1}, \mathbf{X}_{t+1}\right).$$

Similarly, the two period ahead misspecified forecast reads:

$$\widetilde{F}_{t-1}[y_{t+1,1}] = b_m^\top \mathbb{E}_{t-1}\left[h_{t+1}\mathbf{X}_{t+1}\right] = b_m^\top \mathbb{E}_{t-1}\left[\mathbf{X}_{t+1}\right]\mathbb{E}_{t-1}\left[h_{t+1}\right] + b_m^\top \text{Cov}_{t-1}\left(h_{t+1}, \mathbf{X}_{t+1}\right)$$
$$= b_m^\top \mathbb{E}_{t-1}\left[\mathbf{X}_{t+1}\right] + b_m^\top \text{Cov}_{t-1}\left(h_{t+1}, \mathbf{X}_{t+1}\right).$$

The forecast revision of yields with maturity m reads:

$$\widetilde{FR}_t[y_{t+1,m}] = FR_t[y_{t+1,m}] + b_m^\top\left(Cov_t\left(h_{t+1}, \mathbf{X}_{t+1}\right) - Cov_{t-1}\left(h_{t+1}, \mathbf{X}_{t+1}\right)\right).$$

The covariance between forecast error and forecast revision reads:

$$Cov\left(\widetilde{FE}_{t+1}[y_{t+1}], \widetilde{FR}_t[y_{t+1,m}]\right) = b_m^\top B_1 b_m,$$

where:

$$B_1 := -\text{Cov}\left(\text{Cov}_t\left(h_{t+1}, \mathbf{X}_{t+1}\right), \widetilde{FR}_t[\mathbf{X}_{t+1}]\right).$$

I used the fact that under rational expectations the forecast error and the forecast revision are orthogonal. Note that the term B_1 does not depend on the maturity m. The variance of the forecast revision reads:

$$\mathbb{V}\left[\widetilde{FR}_t[y_{t+1,m}]\right] = b_m^\top\left(FR_t[\mathbf{X}_{t+1}] + B_2\right)b_m,$$

where:

$$B_2 = \mathbb{V}\left[\text{Cov}_t\left(h_{t+1}, \mathbf{X}_{t+1}\right) - Cov_{t-1}\left(h_{t+1}, \mathbf{X}_{t+1}\right)\right]$$
$$+ 2Cov\left(FR_t[\mathbf{X}_{t+1}], Cov_t\left(h_{t+1}, \mathbf{X}_{t+1}\right) - Cov_{t-1}\left(h_{t+1}, \mathbf{X}_{t+1}\right)\right).$$

The misspecified CG coefficients therefore read:

$$\widetilde{\beta}_m = \frac{b_m^\top \left(Cov\left(FE_{t+1}[\mathbf{X}_{t+1}], FR_t[\mathbf{X}_t]\right) + B_1\right) b_m}{b_m^\top \left(\mathbb{V}[FR_t[\mathbf{X}_{t+1}]] + B_2\right) b_m} = \frac{b_m^\top B_1 b_m}{b_m^\top \left(\mathbb{V}[FR_t[\mathbf{X}_{t+1}]] + B_2\right) b_m},$$

which, under approximation (14), does not depend on m. The denominator is positive, since it is a variance. The numerator is positive in the case of standard consumption based asset pricing models[20]. In this case, the econometrician detects a flat term structure of CG coefficients, even though there are not departures from RE. Conversely, a non trivial term structure of CG regression coefficients reveal to the econometrician that beliefs are not rational.

2. Non rational expectations. First, consider the one period ahead misspecified forecast. I drop the additive constant $a_m^{\mathbb{Q}^\theta}$ in the following calculations, because it is inessential for the computation of covariances.

$$\widetilde{FE}^{\theta}_{t+1,m} := \bar{y}^{\theta}_{t+1,m} - \mathbb{E}_t^{\widetilde{\mathbb{P}}^\theta}\left[y^\theta_{t+1,m}\right] = \bar{y}^{\theta}_{t+1,m} - (1+\psi^\theta_{m+1})\mathbb{E}_t^{\widetilde{\mathbb{P}}}\left[y^\theta_{t+1,m}\right]$$

$$\bar{y}^{\theta}_{t+1,m} - \mathbb{E}_t^{\widetilde{\mathbb{P}}^\theta}\left[y^\theta_{t+1,m}\right] = \bar{y}^{\theta}_{t+1,m} - (1+\psi^\theta_{m+1})(\mathbb{E}_t^{\widetilde{\mathbb{P}}}\left[y^\theta_{t+1,m}\right] + \mathbb{E}_t^{\mathbb{P}}\left[y^\theta_{t+1,m}\right] - \mathbb{E}_t^{\mathbb{P}}\left[y^\theta_{t+1,m}\right])$$
$$= FE^\theta_{t+1,m} + \left(\mathbb{E}_t^{\mathbb{P}}\left[y^\theta_{t+1,m}\right] - \mathbb{E}_t^{\widetilde{\mathbb{P}}}\left[y^\theta_{t+1,m}\right]\right) - \psi^\theta_{m+1}\mathbb{E}_t^{\widetilde{\mathbb{P}}}\left[y^\theta_{t+1,m}\right].$$

Distorted and misspecified CB coefficients reads:

$$\widetilde{\beta}^\theta_m = \beta^\theta_m + \frac{{b^\theta_m}^\top \left(Cov\left(F_{t+1}[\mathbf{X}_{t+1}] - \widetilde{F}_t[\mathbf{X}_{t+1}], \widetilde{F}_t[\mathbf{X}_{t+1}] - \frac{\psi_{m+2}}{\psi_{m+1}}\widetilde{F}_{t-1}[\mathbf{X}_{t+1}]\right) + B_1\right) b^\theta_m}{{b^\theta_m}^\top \left(\mathbb{V}[\widetilde{FR}_t[\mathbf{X}_{t+1}]] + B_2\right) b^\theta_m}$$

the second term, under one factor model or under approximation (14) does not depend

[20] In standard consumption based asset pricing models, the SDF is negatively correlated with consumption growth, which corresponds to a linear combination of the factors. Therefore, when positive news arrives, the first term in the unconditional covariance in B_1 decreases, while the forecast revision increases. The unconditional covariance is therefore negative and thus B_1 is positive.

on m. The coefficients $\widetilde{\beta}_m^\theta$ are negative and decreasing iff ψ_m^θ is positive and increasing and provided that $\frac{(b_m^\theta)^\top \mathbb{V}[\widetilde{F}_t[\mathbf{X}_{t+1}]]b_m^\theta}{(b_m^\theta)^\top Cov(\widetilde{F}_t[\mathbf{X}_{t+1}],\widetilde{F}_{t-1}[\mathbf{X}_{t+1}]b_m^\theta} > \frac{\psi_2^\theta}{\psi_1^\theta}$. Differences of distorting coefficients remain identified.

□

Theorem (3 ($\mathbb{P}$-diagnostic expectations)

Proof.

$$r_{t,m} \overset{\mathbb{P}}{\sim} \mathcal{N}\left(\delta_0, \frac{1}{m^2}(\delta_1)^\top \Sigma(1-(\rho^{\mathbb{P}})^2))^{-1}\delta_1\right)$$

and

$$r_{t+m}|\mathbf{X}_t \overset{\mathbb{P}}{\sim} \mathcal{N}\left(\delta_0 + (\delta_1)^\top \left(\sum_{i=0}^{m-1}(\rho^{\mathbb{P}})^i\right)\mathbf{X}_t, \frac{1}{m^2}(\delta_1)^\top \Sigma \left(\sum_{i=0}^{m-2}(\rho^{\mathbb{P}})^{2i}\right)\delta_1\right).$$

We want to compute the distribution:

$$f_{\mathbb{P}^\theta}(r_{t+m}|\mathbf{X}_t) \propto f_{\mathbb{P}}(r_{t+m}|\mathbf{X}_t)\left(\frac{f_{\mathbb{P}}(r_{t+m}|\mathbf{X}_t)}{f_{\mathbb{P}}(r_{t+m})}\right)^\theta. \tag{15}$$

First observe that, given $G_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $G_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ with $\sigma_2^2 > \sigma_1^2$:

$$\frac{1}{\int_{\mathbb{R}} f_{G_2}(x)\left(\frac{f_{G_1}(x)}{f_{G_2(x)}}\right)^\theta dx} f_{G_2}(x)\left(\frac{f_{G_1}(x)}{f_{G_2(x)}}\right)^\theta$$

is a Gaussian pdf with mean:

$$\mu_1 + \frac{\theta\frac{\sigma_1^2}{\sigma_2^2}}{1+\theta-\theta\frac{\sigma_1^2}{\sigma_2^2}}(\mu_1-\mu_2),$$

and variance:

$$\left(\frac{1+\theta}{\sigma_1^2}-\frac{\theta}{\sigma_2^2}\right)^{-1}.$$

Then, apply the previous computation to $r_{t,m}$. □

Rational pricing of zero coupon bonds

Consider the arbitrage free price of zero coupon bonds (ZCB) at time t with time to maturity $t+m$:

$$P_{t,m} = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r_{t,m}}\right],$$

where, as usual, $r_{t,m} = \frac{\sum_{i=0}^{m-1} r_{t+i}}{m}$ and r_t denotes the short rate at time t. Assume that the short rate is an affine function of a set of factors, $r_t = \delta_0 + \delta_1^\top \mathbf{X}_t$, which evolve as a VAR(1) under the risk neutral measure:

$$\mathbf{X}_{t+1} = \rho^{\mathbb{Q}}\mathbf{X}_t + \Sigma^{\mathbb{Q}^C}\varepsilon_{t+1}^{\mathbb{Q}}.$$

$\Sigma^{\mathbb{Q}^C}$ is upper triangular, $\rho^{\mathbb{Q}}$ is diagonal and $\Sigma^{\mathbb{Q}} := (\Sigma^{\mathbb{Q}^C})^\top\Sigma^{\mathbb{Q}^C}$ is the variance covariance matrix of the residuals. Then, $r_{t,m}$ has conditional risk neutral mean and variances given by:

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}[r_{t,m}] &= \delta_0 + \frac{\delta_1^\top \sum_{i=0}^{m-1}(\rho^{\mathbb{Q}})^i \mathbf{X}_t}{m}\\ \mathbb{V}_t^{\mathbb{Q}}[r_{t,m}] &= \frac{\delta_1^\top \sum_{i=0}^{m-2}\sum_{j=0}^{i}(\rho^{\mathbb{Q}})^{2j}\Sigma^{\mathbb{Q}}\delta_1}{m^2}\end{aligned}$$

Consider the following class of additive $\mathbb{P}$-dynamics for the factors.

$$\mathbf{X}_{t+1} = f^{\mathbb{P}}(\mathbf{X}_t) + \Sigma^{\mathbb{P}^C}\varepsilon_{t+1}^{\mathbb{P}},$$

where $\Sigma^{\mathbb{P}^C}$ is upper triangular, $f^{\mathbb{P}}(\mathbf{X_t})$ is diagonal and $\Sigma^{\mathbb{P}} := (\Sigma^{\mathbb{P}^C})^\top\Sigma^{\mathbb{P}^C}$ is the variance covariance matrix of the residuals. The $\mathbb{P}$ dynamics is assumed to be Markovian as the risk neutral one and with additive noise. It not assumed to be linear. The linear case is simply recovered imposing $f^{\mathbb{P}}(\mathbf{X}_t) = \rho^{\mathbb{P}}\mathbf{X}_t$. Then, $r_{t,m}$ has conditional physical mean and variances given by:

$$\begin{aligned}\mathbb{E}_t^{\mathbb{P}}[r_{t,m}] &= \delta_0 + \frac{\delta_1^\top \sum_{i=0}^{m-1}(f_i^{\mathbb{P}}(\mathbf{X}_t)}{m}\\ \mathbb{V}_t^{\mathbb{P}}[r_{t,m}] &= \frac{\delta_1^\top \sum_{i=0}^{m-2}\sum_{j=0}^{i}(f_j^{\mathbb{P}})^{2}\Sigma^{\mathbb{P}}\delta_1}{m^2},\end{aligned}$$

where $f_0^{\mathbb{P}}(\mathbf{X}_t) = \mathbf{X}_t$, $f_1^{\mathbb{P}}(\mathbf{X}_t) = f^{\mathbb{P}}(\mathbf{X}_t)$ and, for $i > 1$, $f_i^{\mathbb{P}}(\mathbf{X}_t) = \underbrace{f^{\mathbb{P}}(\dots f^{\mathbb{P}}}_{i \text{ times}}(\mathbf{X}_t))$.

Thus, under this class of asset pricing models the following change of measure applies:

$$\begin{aligned}\Sigma^{\mathbb{Q}^C}\varepsilon_{t+1}^{\mathbb{Q}} &= \Sigma^{\mathbb{P}^C}\varepsilon_{t+1}^{\mathbb{P}} + (f^{\mathbb{P}}(\mathbf{X}_t) - \rho^{\mathbb{Q}}\mathbf{X}_t)\\ &= \Sigma^{\mathbb{P}^C}\varepsilon_{t+1}^{\mathbb{P}} + \mathbb{E}_t^{\mathbb{P}}\left[\mathbf{X}_{t+1}\right] - \mathbb{E}_t^{\mathbb{Q}}\left[\mathbf{X}_{t+1}\right].\end{aligned}$$

Effective shocks to interest rates with maturity m, $r_{t,m} - \mathbb{E}_t^k[r_{t,m}]$, for $k = \mathbb{P}, \mathbb{Q}$ are given by:

$$\delta_1^\top \sum_{i=0}^{m-2}\sum_{j=0}^{i}(\rho^{\mathbb{Q}})^j\Sigma^{\mathbb{Q}}\varepsilon^{\mathbb{P}}_{t+i} = \delta_1^\top \sum_{i=0}^{m-2}\sum_{j=0}^{i} f_j^{\mathbb{P}}(\mathbf{X}_t)\Sigma^{\mathbb{P}}\varepsilon^{\mathbb{P}}_{t+i} + m\left(\delta_1^\top\sum_{i=0}^{m-1} f_i^{\mathbb{P}}(\mathbf{X}_t) - \delta_1^\top \sum_{i=0}^{m-1}(\rho^{\mathbb{Q}})^i\mathbf{X}_t\right),$$

or:

$$\sqrt{\mathbb{V}_t^{\mathbb{Q}}[r_{t,m}]}\varepsilon^{\mathbb{Q}}_{t,m} = \sqrt{\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]}\varepsilon^{\mathbb{P}}_{t,m} + \mathbb{E}_t^{\mathbb{P}}[r_{t,m}] - \mathbb{E}_t^{\mathbb{Q}}[r_{t,m}], \tag{16}$$

where: $\varepsilon^k_{t,m} := \frac{\sum_{i=1}^{m-1}\sum_{j=0}^{i}\varepsilon^k_{t+j}}{m}$, for $k = \mathbb{P}, \mathbb{Q}$.

Pricing of zero coupon bonds with over-reacting beliefs

First consider the case in which beliefs distortions only affects linearly the first moment: $\mathbb{E}_t^{\mathbb{P}^\theta}[r_{t,m}] := a_{t,m}\mathbb{E}^{\mathbb{P}}[r_{t,m}] + b_{t,m}$, where $a_{t,m}, b_{t,m}$ and known at time t. Then, an agent which believes $\mathbb{E}_t^{\mathbb{P}^\theta}[r_{t,m}]$ is the correct mean, while having the same preferences as in the rational case, will adjust beliefs as:

$$\sqrt{\mathbb{V}_t^{\mathbb{Q}}[r_{t,m}]}\varepsilon^{\mathbb{Q}}_{t,m} = \sqrt{\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]}\varepsilon^{\mathbb{P}}_{t,m} + a_{t,m}\mathbb{E}_t^{\mathbb{P}}[r_{t,m}] + b_{t,m} - \mathbb{E}_t^{\mathbb{Q}^\theta}[r_{t,m}].$$

The risk neutral distorted expectation $\mathbb{E}_t^{\mathbb{Q}^\theta}[r_{t,m}]$ is determined from the condition that the change of measure (preference adjustment) is unchanged:

$$\mathbb{E}_t^{\mathbb{Q}^\theta}[r_{t,m}] = \mathbb{E}_t^{\mathbb{Q}}[r_{t,m}] + (a_{t,m} - 1)\mathbb{E}_t^{\mathbb{P}}[r_{t,m}] + b_{t,m}.$$

Consider now the case in which the variance is also distorted, in particular it is scaled as: $\mathbb{V}_t^{\mathbb{P}^\theta}[r_{t,m}] = c_{t,m}^2\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]$, where $c_{t,m}^2$ is known at time t. Then:

$$\sqrt{\mathbb{V}_t^{\mathbb{Q}^\theta}[r_{t,m}]}\varepsilon^{\mathbb{Q}}_{t,m} = c_{t,m}\sqrt{\mathbb{V}_t^{\mathbb{P}}[r_{t,m}]}\varepsilon^{\mathbb{P}}_{t,m} + a_{t,m}\mathbb{E}_t^{\mathbb{P}}[r_{t,m}] + b_{t,m} - \mathbb{E}_t^{\mathbb{Q}^\theta}[r_{t,m}].$$

Then:

$$\mathbb{V}_t^{\mathbb{Q}^\theta}[r_{t,m}] = \mathbb{V}_t^{\mathbb{Q}}[r_{t,m}]c_{t,m},$$

and:

$$\mathbb{E}_t^{\mathbb{Q}^\theta}[r_{t,m}] = \mathbb{E}_t^{\mathbb{Q}}[r_{t,m}] + (a_{t,m} - 1)\mathbb{E}_t^{\mathbb{P}}[r_{t,m}] + b_{t,m}.$$

Note that if $\mathbb{P}^\theta$ satisfies the law of iterated expectations, then:

$$\mathbb{E}_{t-k}^{\mathbb{P}}[\mathbb{E}_t^{\mathbb{P}^\theta}[r_{t,m}] - \mathbb{E}_t^{\mathbb{P}}[r_{t,m}]] = (a_{t-k,m} - 1)\mathbb{E}_{t-k}^{\mathbb{P}}[r_{t-k,m}] + b_{t-k,m} = \mathbb{E}_{t-k}^{\mathbb{P}^\theta}[r_{t-k,m}] - \mathbb{E}_{t-k}^{\mathbb{P}}[r_{t-k,m}].$$

This means that the conditional bias $\mathbb{E}_t^{\mathbb{P}^\theta}[r_{t,m}] - \mathbb{E}_t^{\mathbb{P}}[r_{t,m}]$ is a $\mathbb{P}-$martingale and it is therefore unpredictable. Similarly, temporary misspricing is not predictable. In the case the law is violated instead there is, in principle, predictable misspricing (arbitrage opportunities).

F Additional Results

Figure (1.5) shows that distortions from recovered beliefs are a combination of the "average" distortions (i.e. the distortion from the pooled regression) and of the distortion of the "average" (i.e. the consensus forecast), at least statistically. To quantitatively predict the aggregate outcome, one would need a specific model of aggregation, which is left for future work. The aforementioned mechanism proposed by Bordalo et al. (2018a) is however consistent with some evidence from the cross-sectional dispersion of forecasters: Figure (11) shows that the average cross-sectional dispersions decrease with maturity. Views are more aligned for predictions at longer horizons than for predictions at shorter horizons[21]. However, at longer horizon, data are much less rich, which may weaken the claim. In fact, as shown in Figure (1.5), the maturities $20y$ and $30y$, it is not possible to categorize forecasters along the mean square error dimension; also, for the consensus forecast, confidence intervals are huge.

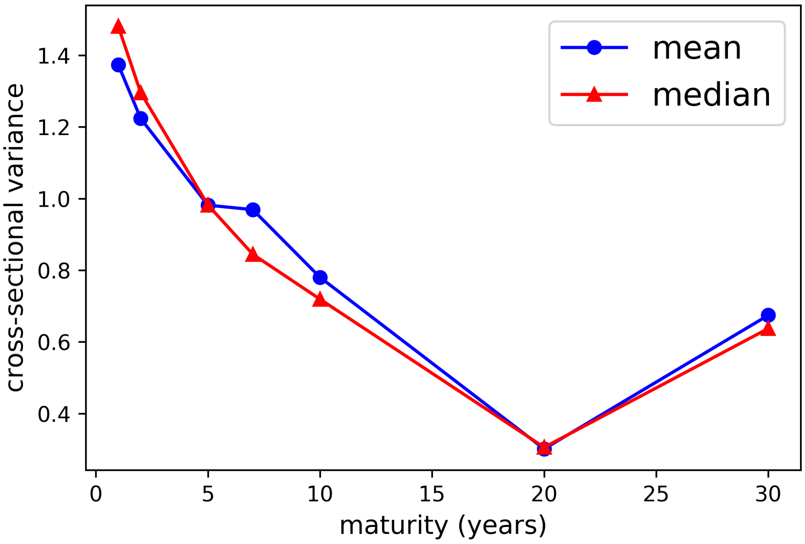


Figure 11: Time mean (blue circle) and median (red triangle) of analysts dispersion (cross sectional standard deviation) as a function of the maturity.

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Chapter 2

Learning, Overreaction and the Wisdom of the Crowd

2.1 Introduction

Departures from the standard Bayesian updating rule are well documented using both experiments and survey data (e.g. Benjamin (2019)). For example, in financial markets, the typical departure is that of over-optimistic beliefs, when good news are observed (e.g. Bordalo et al. (2018a)). However, much of the literature focuses on decisions taken by agents in isolation, abstracting away from another fundamental economic fact: interaction. On the other hand, the study of interaction in the formation of expectations and learning has been widely studied (Golub and Sadler (2017)), both under Bayesian behavior (Banerjee (1992), Acemoglu et al. (2011), Bala and Goyal (1998)) and under simple mechanical updating rules, such as averaging neighbors beliefs (e.g. Golub and Jackson (2010)). The literature on social networks has focused on how the network structure of interactions facilitates or forbids reaching consensus and learning. Recently, departures from the Bayesian paradigm has been investigated in the context of social learning. For example, Molavi et al. (2018) consider the case of beliefs updating with imperfect recall.

Here we bridge the gap by considering a simple model of sequential learning where agents - due to information processing limitations - depart from full Bayesian rationality. Agents exhibit under/over-reaction to signals. Our main contribution is to characterize the information externalities caused by departures from Bayesian rationality. Specifically, we find that over-reaction to news entails a positive externality, which partially heal the informational cascade phenomenon, thereby increasing social informational efficiency. This is surprising because one may expect individual biases to be socially inefficient.

We first introduce a model of non Bayesian updating, which features two specific observed biases: *over-reaction* or *under-reaction* to information. While the origin of the two mechanisms is thought to be different in nature [1] our model includes both cases. In the case of over-reaction, our model is a learning analog of the diagnostic expectation model of Bordalo et al. (2018b), which is a model of extrapolative predictions. We then apply the model to study the learning problem of an isolated agent. We find that in the long-run limit, the agent learns the true state of the world as in the Bayesian case. We show that however the mean square loss of biased agents - a measure of individual inefficiency - is not symmetric: under-reaction to information lead to greater losses than over-reaction to information. In both cases, however, it is sub-optimal to have biased expectations.

What happens instead when agents interact? We focus on the stylized case of a sequential decision task, as in the cascade literature started by Banerjee (1992) and Bikhchandani et al. (1992). At each time step, an agent is born and she has to take a binary action (e.g. buy or sell a financial asset) which corresponds to guess the true state of nature, which is binary as well (e.g. the fundamental value of the asset). Her information set consists of a private signal and past actions of other agents (e.g. past buy/sell orders). In

[1] Under-reaction to information can be rationalized by costly information acquisition. On the contrary, over-reaction to information may be grounded in the Tversky and Kahneman (1974) representativeness heuristic.

the Bayesian setting, this framework leads to the phenomenon of *informational cascades*: when the actions of previous agents are aligned enough, then future private signals become irrelevant and each future agent is stuck in a specific action. This may result in a cascade of wrong guesses. This is so because the mapping between private signals and the history of actions (which is what is observed by future agents) is highly non injective. Much of the information in the economy remain unexploited. We consider our model of non Bayesian updating and we find that over-reaction helps injecting more information into the history of actions, which is then exploited by future agents. We find that there exists a unique socially optimal level of over-reaction, which maximizes the probability of learning the true state of the world.

This insight also clarify that departures from Bayesian rationality - overreaction in particular - may be see from an evolutionary perspective as optimal with respect to the objective of social informational efficiency.

2.2 A non Bayesian learning model

Consider an agent that has to *learn* the state of the world ω, on which she has a prior belief, with density $p_0(\omega)$. The agent observes a signal X, whose likelihood is known to be $l(X|\omega)$. The Bayesian updating operator takes as inputs the prior density, the likelihood function and the observed signal and it prescribes to move beliefs about ω from $p_0(\omega)$ to the Bayesian posterior:

$$\mathcal{BU}(l, p_0)(\omega) := \frac{l(X|\omega)p_0(\omega)}{\int l(X|\omega')p_0(\omega')d\omega'}. \tag{2.1}$$

We propose the following distorted updating rule:

$$\mathcal{BU}^\theta(l, p_0)(\omega) = \frac{l(X|\omega)^{1+\theta}p_0(\omega)}{\int l(X|\omega')^{1+\theta}p_0(\omega')d\omega'}. \tag{2.2}$$

The scalar parameter $\theta > -1$ controls the departure from the Bayesian case. When $\theta > 0$ the model delivers *over-reaction* to information and it is a learning analog of the diagnostic expectation model of Bordalo et al. (2018b). To see this point, consider the case $\theta > 0$ and rewrite expression (2.2) as:

$$\mathcal{BU}^{\theta}(l, p_0)(\omega) = \frac{1}{Z}\mathcal{BU}(l, p_0)(\omega)\left(\frac{\mathcal{BU}(l, p_0)(\omega)}{p_0(\omega)}\right)^{\theta}, \tag{2.3}$$

where Z is a normalization constant. The previous formula says that states ω which are more likely under $\mathcal{BU}(l, p_0)(\omega)$ than under $p_0(\omega)$, i.e. *representative* states, are over-weighted. On the contrary, states ω which are less likely under $\mathcal{BU}(l, p_0)(\omega)$ than under $p_0(\omega)$, are under-weighted. Thus, we say that for $\theta > 0$ posterior beliefs over-react to information. On the contrary, for $-1 < \theta < 0$, posterior beliefs under-react to information. When facing multiple data, $X_1, \ldots, X_t$, agents can update beliefs sequentially: in at each step, the prior belief is the precious step distorted posterior belief. Alternatively agents could update their beliefs only once, after observing the string $X_1, \ldots, X_t$. Define the distorted updating given t observations as:

$$\mathcal{BU}^{\theta}_t(l, p_0)(\omega) = \frac{l(X_1, \ldots X_t|\omega)^{1+\theta} p_0(\omega)}{\int l(X_1, \ldots X_t|\omega')^{1+\theta} p_0(\omega') d\omega'}.$$

Then, the following consistency result shows that it is irrelevant which of the two strategy is implemented.

Theorem 4. *For $k \in \{1, \ldots, t-1\}$:*

$$\mathcal{BU}^{\theta}_t(l, p_0)(\omega) = \mathcal{BU}^{\theta}_{t-k}(l, \mathcal{BU}^{\theta}_k(l, p_0))(\omega)$$

Does learning take place? Expression (2.3) suggests the the Bayesian and the diagnostic distribution are connecting by a continuous transformation, which preserve convergence.

Theorem 5. *Call ω^* the true value of ω.*
Learning occurs under Bayesian updating, i.e. $\mathcal{BU}_t(l, p_0) \xrightarrow{d} \delta_{\omega^*}$ as $t \to \infty$ if and only if it occurs under distorted beliefs, i.e. $\mathcal{BU}^\theta_t(l, p_0) \xrightarrow{d} \delta_{\omega^*}$ as $t \to \infty$.

We now characterize the loss from using distorted posterior beliefs. Assume that the goal of the agent is to minimize the sum of future discounted losses:

$$\sum_{t^0}^{\infty} \beta^t \mathbb{E}_t(\omega - \mathbb{E}^\theta_t \omega)^2.$$

Then, the following results characterizes the losses occurring with distorted beliefs.

Theorem 6. *Under the updating model (2.2), the (cumulative) mean square error reads:*

$$-\log(1-\beta) + \sum_{t=1}^{\infty} \beta^t (\mathbb{E}_t \omega - \mathbb{E}^\theta_t \omega)^2.$$

As expected, distorted beliefs are, in general, sub-optimal since $(\mathbb{E}_t \omega - \mathbb{E}^\theta_t \omega)^2 > 0$. Thus, in a world with distorted beliefs, an isolated agents eventually learn (or does not) if the only if the Bayesian agent does. We now move to a concrete example to gain more intuition.

2.2.1 Learning the mean from Gaussian i.d.d. draws

Suppose that an agent observes *iid* realizations of $X \sim \mathcal{N}(\mu^{true}, \sigma^2)$. She knows the variance σ^2 and she has to learn the mean. Given $X \sim \mathcal{N}(\mu^{true}, \sigma^2)$ and prior $p_0(\mu) \sim$

$\mathcal{N}(\mu_0, \sigma_0^2)$, and t observations $X_1, \ldots, X_t$, we have:

$$p_t(\mu) := \mathcal{BU}_t(l, p_0) \sim \mathcal{N}\left(\left(\frac{1}{\sigma_0^2} + \frac{t}{\sigma^2}\right)^{-1}\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^t X_i}{\sigma^2}\right), \left(\frac{1}{\sigma_0^2} + \frac{t}{\sigma^2}\right)^{-1}\right).$$

The distorted posterior distribution is:

$$p_t^\theta(\mu) = \mathcal{BU}_t^\theta(l, p_0) = \mathcal{N}\left(\left(\frac{1}{\sigma_0^2} + \frac{t(1+\theta)}{\sigma^2}\right)^{-1}\left(\frac{\mu_0}{\sigma_0^2} + \frac{(1+\theta)\sum_{i=1}^t X_i}{\sigma^2}\right), \left(\frac{1}{\sigma_0^2} + \frac{t(1+\theta)}{\sigma^2}\right)^{-1}\right),$$

since the only effect of the distortion is to modify the variance of the likelihood function from σ^2 to $\frac{\sigma^2}{1+\theta}$. Thus, the variance of the posterior diagnostic distribution is:

$$\mathbb{V}[\mu_t^\theta] = \left(\frac{1}{\sigma_0^2} + \frac{(\theta+1)t}{\sigma^2}\right)^{-1} \sim \frac{\sigma^2}{(\theta+1)t},$$

which is smaller then the variance of the Bayesian posterior. Also, for large t, convergence to the truth is guaranteed by theorem (5). What about the mean square error? As shown in Appendix, in the Gaussian case the loss reads:

$$(\mathbb{E}_t^\theta \omega - \mathbb{E}_t \omega)^2 \sim \frac{1}{t^2}\frac{\theta^2}{(\theta+1)^2}\left((\mu_0 - \omega)^2 + \sigma_t^2\right).$$

The bias term depends on: the initial prior, the variance of the posterior under the bayesian updating, and the term $\frac{\theta^2}{(\theta+1)^2}$, which is asymmetric: under-reaction makes it increase very fast above 1, while even for strong over-reaction it is always smaller than 1. Hence it seems that underreaction is much worse in terms of learning than overreaction. Inspection of the calculations reveal that the numerator comes from the error in prediction, while the denominator from the precision. So the error tends to increase with bias, while the precision is increasing in overreaction. With one agent, though, the numerator always prevails, so that the optimal θ is 0. We now introduce social interactions.

2.3 Sequential learning and efficiency

In this section, we apply our behavioral model of learning to a simple social learning environment, to show that overreaction can be more socially efficient than bayesian updating.

Let us consider the simplest model of informational cascades, analogous to Banerjee (1992). There is a binary state of the world $\omega \in \{0,1\}$ which agents have to guess. Formally, agents are infinite and indexed with natural numbers $i = 1, 2, \ldots$. They act sequentially, first observing a private signal s_i and then choosing a public action a_i. Both actions and signals are binary $a_i, s_i \in \{0,1\}$ and we assume that $Pr(s_i = 1|\omega = 1) = q > \frac{1}{2}$ (i.e. signals are informative). Symmetrically (for convenience) let $Pr(s_i = 0|\omega = 0) = q > \frac{1}{2}$. Agents have a common prior $Pr(\omega = 1) = p$.

Agent i information set includes all actions of past agents $(a_1, \ldots, a_{i-1})$ and his own private signal s_i. Each agent has a utility v_1 from choosing action $a_i = 1$ if the correct state is $\omega = 1$ and v_0 from choosing the correct action when the state is $\omega = 0$, and they want to maximize their expected payoff. This means that, e.g., Mr 1, when observing signal s_1 will form the following posterior:

$$Pr^\theta(\omega = 1|s_1 = 1) = Pr(\omega = 1|s_1 = 1)\left(\frac{Pr(\omega = 1|s_1 = 1)}{Pr(\omega = 1)}\right)^\theta \frac{1}{Z(\theta, s_1 = 1)} = \frac{p}{p + (1-p)\left(\frac{1-q}{q}\right)^{1+\theta}}$$

and he will choose action $a_1 = 1$ if and only if:

$$v_1 Pr^\theta(\omega = 1|s_1 = 1) > v_0 Pr^\theta(\omega = 0|s_1 = 1),$$

which is equivalent to:

$$Pr^\theta(\omega = 1|s_1) > \frac{v_0}{v_0 + v_1}.$$

In the following, we will be interested in regions in the parameter space, so we do not need to specify tie-breaking rules. We define $\tau = \frac{v_0}{v_0+v_1}$.

We defined over and under-reaction relative to the estimation of the parameter done with past information. When considering interacting agents, we have two sources of information: private signals and actions of others. In the following, we are treating actions of others and the private signal in a symmetric way, as past information. This means that Mr 2 will compute his posterior as:

$$Pr^{\theta}(\omega = 1|s_1, a_1, s_2) = Pr(\omega = 1|s_1, a_1, s_2)\left(\frac{Pr(\omega = 1|s_1, a_1, s_2)}{Pr(\omega = 1, s_1, a_1)}\right)^{\theta}\frac{1}{Z(\theta, s_1, a_1, s_2)}$$

To understand what are the implications of the distortion for cascades and learning, let us start with the following definition.

Definition 2.3.1. *The* Informational efficient region *(IE) is the set of parameters given by the union of:*

$$\theta + 1 \geq \frac{\ln\left(\frac{p}{1-p}\left(\frac{1}{\tau} - 1\right)\right)}{\ln\frac{1-q}{q}} \qquad p \geq \tau$$

and

$$\theta + 1 \leq \frac{\ln\left(\frac{p}{1-p}\left(\frac{1}{\tau} - 1\right)\right)}{\ln\frac{q}{1-q}} \qquad p < \tau$$

The Informational efficient region is the region of parameters such that Mr 1 plays $a_1 = s_1$ and therefore "communicate" his private signal to future agents. Outside the efficient region, agent 1 instead chooses the action consistent with his prior regardless the signal. This is crucial in characterizing the behavior of the model. The following proposition describes such behavior.

Proposition 2.3.1. *If the parameters are in the Informational efficient region, then:*

If $p \geq \tau$ *If the first signal is 1, there is a cascade on 1. If the first two signals are* $(0, 0)$ *there is a cascade on 0. If the first two signals are* $(0, 1)$*, then the third agent faces the same problem of agent 1. The probability of learning is* $Pr(a_{\infty} = \omega) = \frac{pq+(1-p)q^2}{1-q(1-q)}$.

If $p < \tau$ *There is a cascade on 1 if the first signals are* $(1, 1)$*, there is a cascade on 0 if the first signal is 0. If the first two signals are* $(1, 0)$*, then the third agent faces the same problem of agent 1. The probability of learning is* $Pr(a_\infty = \omega) = \frac{pq^2+(1-p)q}{1-q(1-q)}$.

If the parameters are outside of the Informational efficient region, then:

1. *If* $\tau > p$*, then all agents play 0 with probability 1 and the probability of learning is* $1 - p$.

2. *If* $\tau \leq p$*, then then all agents play 1 with probability 1, and the probability of learning is* p.

By the form of the results, we can already see that a larger θ creates more room for learning, by enlarging the Informational efficient region. In the following, we make this argument formal. For simplicity in this section we fix $\tau = \frac{1}{2}$.

A way to quantify the size of the parameter space is to think of the parameters p and q as drawn before the process starts. from a distribution μ, with full support on $(0, 1) \times (\frac{1}{2}, 1)$. Denote $a_\infty = \lim_{t \to \infty} a_t$. Consider regions as $R_1 = IE \cup \{p > \tau\}$, $R_2 = IE \cup \{p \leq \tau\}$, $N_1 = \overline{IE} \cup \{p > \tau\}$, and $N_2 = \overline{IE} \cup \{p \leq \tau\}$. Then the ex-ante probability of learning the correct state of the world is:

$$
\begin{aligned}
&Pr(a_\infty = \omega) = \\
&= \int \left(pI_{N_1} + (1-p)I_{N_2} + \frac{pq + (1-p)q^2}{1 - q(1-q)} I_{R_1} + \frac{pq^2 + (1-p)q}{1 - q(1-q)} I_{R_2} \right) \mathrm{d}\mu,
\end{aligned}
$$

where $I(\cdot)$ represents the indicator function.

Let us define a level of θ *ex-ante efficient* if it achieves the maximum of this probability. The following figures illustrate the situation. In figure 2.1 we draw the region where Mr1

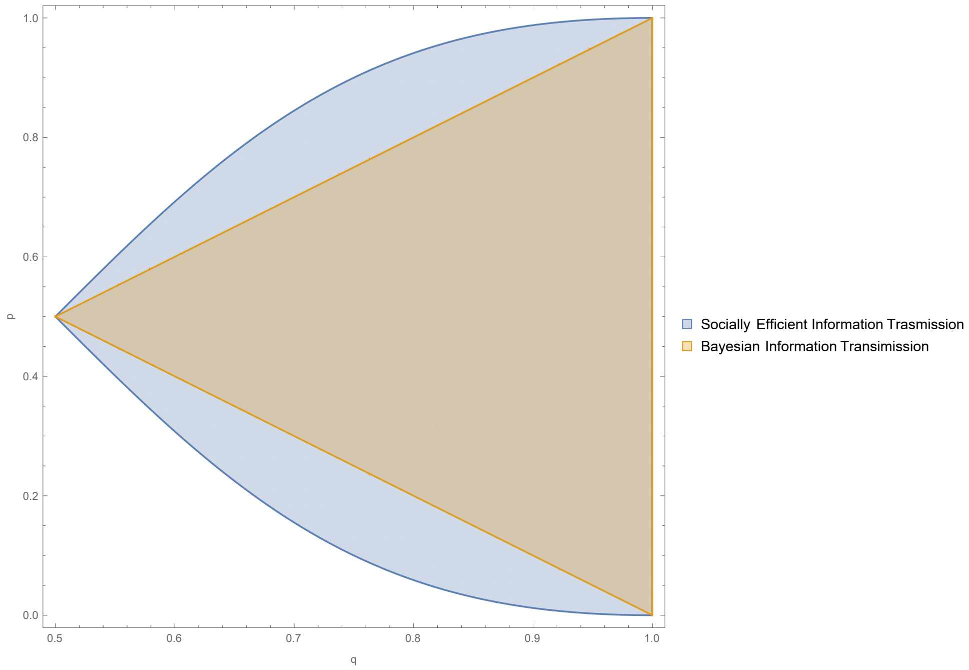

Figure 2.1: The areas of the parameter space where revealing is efficient, and the area where a Bayesian agent reveals. The figure shows that the Bayesian updating fails to be socially efficient: there is a region where agent 1 does not reveal but it would be socially optimal to do so.

playing $a_1 = s_1$ is *socially* efficient, and the region where (Bayesian) Mr1 playing $a_1 = s_1$ is *individually* efficient (i.e. optimal). As is clear from the figure, a Bayesian updating rule does not maximize the learning probability: there is a region where it would be socially efficient that Mr1 plays $a_1 = s_1$, but a Bayesian agent, since he does not internalize the information externality on other agents, does not. In figure 2.2, we plot instead the informational efficient regions for different values of the parameter θ: we can see that there are moderate values of overreaction that increases the probability of learning. In the following proposition, we show that there is a value of θ that actually achieve ex-ante efficiency.

Theorem 7. *If the parameters p, q are drawn from a distribution μ with full support on $(0,1) \times (\frac{1}{2}, 1)$, the distorted updating with $\theta = 1$ is ex-ante efficient.*

Given the importance of the result, we report the proof here in the main text.

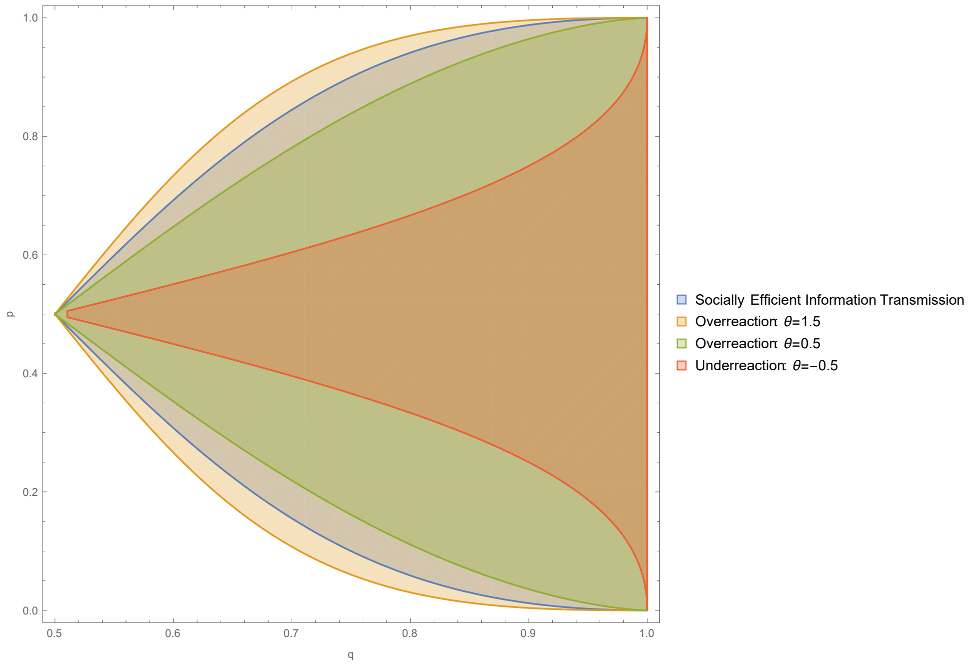

Figure 2.2: The areas of the parameter space where revealing is efficient, and the area where agent with different degree of distortion reveal. It is clear that underreaction is always worse than Bayesian, while a moderate overreaction can be socially better than Bayesian.

Proof. Let us focus on the subset of the parameter space where $\{p > \tau\}$, the reasoning for $p < \tau$ is analogous. The the ex-ante probability of learning is:

$$\int \left(pI\left(\theta + 1 \leq \frac{\ln\left(\frac{p}{1-p}\right)}{\ln \frac{q}{1-q}}\right) + \frac{pq + (1-p)q^2}{1 - q(1-q)} I\left(\theta + 1 > \frac{\ln\left(\frac{p}{1-p}\right)}{\ln \frac{q}{1-q}}\right)\right) \mathrm{d}\mu.$$

In the Informational efficient region IE, the probability of learning is $\frac{pq+(1-p)q^2}{1-q(1-q)}$, while outside is just p. The probability of learning is higher inside the Revelation region if and only if:

$$p < \frac{pq + (1-p)q^2}{1 - q(1-q)},$$

which is equivalent to:

$$p < \frac{q^2}{(1-q)^2 + q^2}.$$

Now we can rewrite the condition defining the IE region:

$$p < \frac{q^{\theta+1}}{(1-q)^{\theta+1} + q^{\theta+1}}.$$

Depending on θ, $\frac{q^2}{(1-q)^2+q^2}$ can be larger or smaller than $\frac{q^{\theta+1}}{(1-q)^{\theta+1}+q^{\theta+1}}$, with equality for $q = \frac{1}{2}$ $q = 1$, and for all q if $\theta = 1$.

If $\theta < 1$, it means that there is a region outside the IE region (with positive mass, because of the full support assumption on μ) with probability of learning p, strictly smaller than the corresponding probability if it belonged to the IE, hence by increasing θ the probability of learning would increase. If $\theta > 1$, on the contrary, there is a region that belongs to IE where the probability of learning is smaller than p. Hence, the maximum is achieved for $\theta = 1$.

□

A Proofs

Theorem (4)

Proof.

$$\begin{aligned}
&\mathcal{BU}_t(l,p_0)(\omega) = \frac{l(X_1,\ldots,X_t|\omega)p_0(\omega)}{\int d\omega' l(X_1,\ldots,X_t|\omega')p_0(\omega')}\\
&= \frac{l(X_\tau,\ldots,X_t|\omega,X_1,\ldots,X_{\tau-1})l(X_1,\ldots,X_{\tau-1}|\omega')p_0(\omega)}{\int l(X_\tau,\ldots,X_t|\omega',X_1,\ldots,X_{\tau-1})l(X_1,\ldots,X_{\tau-1}|\omega')p_0(\omega')d\omega'} \times \frac{\int l(X_1,\ldots,X_{\tau-1}|\omega')p_0(\omega')d\omega'}{\int l(X_1,\ldots,X_{\tau-1}|\omega')p_0(\omega')d\omega'}\\
&= \frac{l(X_\tau,\ldots,X_t|\omega,X_1,\ldots,X_\tau)\mathcal{BU}_\tau(l,p_0)(\omega)}{\int l(X_\tau,\ldots,X_t|\omega',X_1,\ldots,X_\tau)\mathcal{BU}_\tau(l,p_0)(\omega')d\omega'}\\
&= \mathcal{BU}_{t-\tau}(l,\mathcal{BU}_\tau(l,p_0)(\omega))(\omega)
\end{aligned}$$

Remaining close to the diagnostic expectations literature, we can assume that the reference group for representativeness be the information collected until period t. The diagnostic Bayesian operator is defined for $t = 1$ as:

$$\mathcal{BU}_1^\theta(l,p_0)(\omega) = \frac{1}{\int \mathcal{BU}_1(l,p_0)\left(\frac{\mathcal{BU}_1(l,p_0)(\omega')}{p_0(\omega')}\right)^\theta d\omega'}\mathcal{BU}_1(l,p_0)\left(\frac{\mathcal{BU}_1(l,p_0)(\omega)}{p_0(\omega)}\right)^\theta.$$

For $t > 1$ it is defined as:

$$\begin{aligned}
\mathcal{BU}_t^\theta(l,p_0)(\omega) = &\frac{1}{\int \mathcal{BU}_t(l,\mathcal{BU}_{t-1}^\theta(l,p_0))(\omega')\left(\frac{\mathcal{BU}_t(l,\mathcal{BU}_{t-1}^\theta(l,p_0))(\omega')}{\mathcal{BU}_{t-1}^\theta(l,p_0)(\omega')}\right)^\theta d\omega'}\times\\
&\times \mathcal{BU}_t(l,\mathcal{BU}_{t-1}^\theta(l,p_0))(\omega)\left(\frac{\mathcal{BU}_t(l,\mathcal{BU}_{t-1}^\theta(l,p_0))(\omega)}{\mathcal{BU}_{t-1}^\theta(l,p_0)(\omega)}\right)^\theta.
\end{aligned}$$

Note that this can be rewritten as:

$$\begin{aligned}
\mathcal{BU}^{\theta}_t(l,p_0)(\omega) &\propto \mathcal{BU}_t(l,\mathcal{BU}^{\theta}_{t-1}(l,p_0))(\omega)\left(\frac{\mathcal{BU}_t(l,\mathcal{BU}^{\theta}_{t-1}(l,p_0))(\omega)}{\mathcal{BU}^{\theta}_{t-1}(l,p_0)(\omega)}\right)^{\theta}\\
&\propto \mathcal{BU}_t(l,\mathcal{BU}^{\theta}_{t-1}(l,p_0))(\omega)\left(\frac{l(X_t|X_1,\dots,X_{t-1},\omega)^{\theta}\mathcal{BU}^{\theta}_{t-1}(l,p_0)(\omega)}{\mathcal{BU}^{\theta}_{t-1}(l,p_0)(\omega)}\right)^{\theta}\\
&\propto l(X_t|X_1,\dots,X_{t-1},\omega)^{1+\theta}\mathcal{BU}^{\theta}_{t-1}(l,p_0)(\omega)\\
&\propto \mathcal{BU}_t(l^{1+\theta},p_0)(\omega).
\end{aligned}$$

It turns out that this is a model of over-inference. The prior is corrected processed by the diagnostic operator, while the likelihood is over-weighted.

Note that for any $1 < \tau < t$:

$$\mathcal{BU}^{\theta}_{t-\tau}(l,\mathcal{BU}^{\theta}_{\tau}(l,p_0)(\omega))(\omega) = \mathcal{BU}_{t-\tau}(l^{1+\theta},\mathcal{BU}_{\tau}(l^{1+\theta},p_0)(\omega))(\omega) = \mathcal{BU}(l^{1+\theta},p_0)(\omega) = \mathcal{BU}^{\theta}_t(l,p_0)(\omega).$$

This says that sequential updating or "one shot" updating are equivalent. □

Theorem (6)

Proof. One way to state the optimality of the Bayesian updating is to say that the Bayesian posterior mean μ_t is the best predictor according to the quadratic loss function, $(\omega-\mu_t)^2$. This means that if an agent uses a different predictor, say $\mu^{\theta}_t = \mathbb{E}^{\theta}_t\omega$, then this is not the correct posterior mean, hence the expected utility of such an agent is:

$$-\mathbb{E}_t(\omega - \mathbb{E}^{\theta}_t\omega)^2 \leq -\mathbb{E}_t(\omega - \mathbb{E}_t\omega)^2.$$

This is a reduced form reasoning, in that it uses just the conditional expectations. If, as it is used in standard learning exercises, we assume that agents myopically optimize their quadratic utility at each time period (or any utility which has the conditional expectation of X_{t+1} as the optimal point), then we get that their intertemporal utility is in expectation smaller at every period:

$$-\mathbb{E}_0 \sum_t \beta^t (\omega - \mathbb{E}_t^\theta \omega)^2 \leq -\mathbb{E}_0 \sum_t \beta^t (\omega - \mathbb{E}_t \omega)^2.$$

We can understand better this discrepancy. First of all, applying the law of iterated expectations we can show that:

$$-\mathbb{E}_0 \sum_t \beta^t (\omega - \mathbb{E}_t^\theta \omega)^2 = -\mathbb{E}_0 \sum_t \beta^t \mathbb{E}_t (\omega - \mathbb{E}_t^\theta \omega)^2.$$

Consider the time t term:

$$(\omega - \mathbb{E}_t^\theta \omega)^2 = (\omega - \mathbb{E}_t \omega)^2 + (\mathbb{E}_t^\theta \omega - \mathbb{E}_t \omega)^2 + 2(\omega - \mathbb{E}_t \omega)(\mathbb{E}_t^\theta \omega - \mathbb{E}_t \omega)$$

and in expectation:

$$\mathbb{E}_t (\omega - \mathbb{E}_t^\theta \omega)^2 = \mathbb{V}_t \omega + \mathbb{E}_t (\mathbb{E}_t^\theta \omega - \mathbb{E}_t^P \omega)^2,$$

which shows that the disutility of an error has two components: the (im)precision of the rational posterior plus the discrepancy of the diagnostic from the bayesian posterior. □

MSE for Example 1.

$$\mathbb{E}_t^\theta \omega - \mathbb{E}_t \omega = \frac{\theta (t\mu_0 - \sum^t X_s)}{(t+1)(t(\theta+1)+1)}$$

and so, if t large:

$$(\mathbb{E}_t^\theta \omega - \mathbb{E}_t \omega)^2 = \frac{\theta^2}{(t(\theta+1)+1)^2} \mathbb{E}_t \left(\frac{t\mu_0 - \sum^t X_s}{t+1} \right)^2 \sim \frac{\theta^2}{(t^2(\theta+1))^2} \mathbb{E}_t \left(\mu_0 - \frac{\sum^t X_s}{t} \right)^2.$$

Moreover:

$$\mathbb{E}_t \left(\mu_0 - \frac{\sum^t X_s}{t} \right)^2 = \mathbb{E}_t \left(\mu_0 - \omega \right)^2 + \mathbb{E}_t \left(\omega - \frac{\sum^t X_s}{t} \right)^2.$$

Proof of Proposition 2.3.1

Mr 1 will play action $a_1 = 1$ after observing signal $s_1 = 1$ if and only if his posterior is higher than τ, that is:

$$\frac{pq^{1+\theta}}{pq^{1+\theta} + (1-p)(1-q)^{1+\theta}} \geq \tau$$

$$\iff 1 + \frac{1-p}{p}\left(\frac{1-q}{q}\right)^{1+\theta} \leq \frac{1}{\tau}$$

$$\iff (1+\theta)\log\left(\frac{1-q}{q}\right) \leq \log\left(\frac{1}{\tau}-1\right)\frac{p}{1-p}$$

$$\iff \theta + 1 \geq \frac{\ln\left(\frac{p}{1-p}\left(\frac{1}{\tau}-1\right)\right)}{\ln\frac{1-q}{q}} = \frac{-\ln\left(\frac{p}{1-p}\right)+\ln\left(\frac{\tau}{1-\tau}\right)}{\ln\frac{q}{1-q}}.$$

In the last line we used $q > \frac{1}{2}$ to change the inequality sign. This condition is always true if $p \geq \tau$, given that $\theta > -1$. Its interpretation is that if Mr 1 is ex ante indifferent or in favor of alternative 1, then if he observes signal $s_1 = 1$, he plays $a_1 = 1$ for any value of θ. On the contrary if $p < \tau$, meaning that the agent is ex ante in favor of alternative 0, when he observes signal $s_1 = 1$ he might or might not play action $a_1 = 1$, depending on the parameter values. The condition above says that the bigger θ is, the bigger the set of parameters under which Mr1 revises the prior and plays action $a_1 = 1$.

Similarly, after seeing $s_1 = 0$, Mr 1 will play action $a_1 = 1$ if and only if:

$$\frac{p(1-q)^{1+\theta}}{p(1-q)^{1+\theta} + (1-p)q^{1+\theta}} \geq \tau$$

$$\iff \theta + 1 \leq \frac{\ln\left(\frac{p}{1-p}\left(\frac{1}{\tau}-1\right)\right)}{\ln\frac{q}{1-q}} = \frac{\ln\left(\frac{p}{1-p}\right)-\ln\left(\frac{\tau}{1-\tau}\right)}{\ln\frac{q}{1-q}}.$$

This is never the case never if $p < \tau$, which means that if the agent is in favor of alternative 0 and then he sees the signal $s_1 = 0$, he never revises his opinion. On the contrary, depending on θ, the opposite case may be true. Call the above condition 2. Note that

the space of parameter such that condition two is violated increases with θ.
Summing up: if the agent sees $s_1 = 1$, then he plays $a_1 = 1$ the if either $\tau \leq p$ or $\tau > p$ and condition 1 is satisfied. If the agent sees $s_1 = 0$, then he plays $a_1 = 0$ if $\tau > p$ or $\tau \leq p$ and condition 2 is violated.

The behavior of Mr 2 will depend on which conditions are satisfied. Consider the informationally efficient region IE (for agent 1) defined as:

$$IE = \{(p, \tau) \in [0, 1] \times \mathbb{R}_+ | \, (\tau \leq p \text{ and condition 2 is true}) \, \text{or} \, (\tau > p \text{ and condition 1 is true }) \}.$$

If the parameters lie inside IE, then Mr 2 can perfectly infer Mr1 signal by observing his action, since $a_1 = s_1$. Thus Mr 2 effectively observes two signals.
Consider first the case $p > \tau$, namely the prior is in favor of 1. In this case if $s_1 = 1$ than Mr2 will do his Bayesian updating, leading him to play action $a_2 = 1$ regardless of his signal. Similarly for subsequent agents: a cascade therefore starts on state 1 in this case. If instead $a_1 = s_1 = 0$, then if $s_2 = 0$ Mr2 will do his Bayesian updating, leading him to play action $a_2 = 0$ regardless of his signal. Similarly for subsequent agents: a cascade therefore starts on state 0 in this case. Finally if $a_1 = s_1 = 0$ and $s_2 = 1$, then Mr2 will do his Bayesian updating, leading him to play action $a_2 = 1$. However in this case a cascade does not start immediately, as Mr 3 faces the same problem of Mr1. Here the intuition is straightforward: opposite signals s_1 and s_2 cancel out, and therefore $Mr2$ only relies on his prior belief. The case $p < \tau$ is symmetric.
Resuming the dynamics is characterized as follows:

- if $p > \tau$ then the probability of learning is:

$$Pr(a_\infty = \omega) = pPr(a_\infty = 1) + (1-p)Pr(a_\infty = 0) = \left(pq + (1-p)q^2\right)\left(\sum_{i=0}^{\infty}(q(1-q))^i\right);$$

- if $p < \tau$ then the probability of learning is:

$$Pr(a_\infty = \omega) = pPr(a_\infty = 1)+(1-p)Pr(a_\infty = 0) = \left(pq^2 + (1-p)q\right)\left(\sum_{i=0}^{\infty}(q(1-q))^i\right);$$

Resuming, if $s_1 = s_2$, then $a_2 = s_2$; if instead $s_1 \neq s_2$ then M2 2 will stick to his prior belief. In the former case Mr 3 will also play $a_3 = s_2$ regardless of his signal (if $s_3 = s_2 = s_1$ this is true since $IE_1 \subseteq IE_2 \subseteq IE_3$; if $s_3 \neq s_2 = s_1$ then Mr3 problem is the same problem of Mr1, therefore $a_3 = a_3$); in the latter case Mr3 problem is the same problem of Mr 1, therefore $a_3 = s_3$.
Therefore, if the parameters lie IE_1, then i

plays 1 if $p \geq \tau$, and 0 viceversa. This means that if $p \geq \tau$ and the first agent revealed his signal to be 1, then the second agent will always play 1, and this will not be informative for Mr 3, which will act as if he observed only the signal of the first agent. On the contrary, if the first agent revealed his signal to be 0 and still $p \geq \tau$, the second agent reveals his signal, and Mr 3 updates consequently. Hence, if the conditions on θ for Mr 1 are satisfied, the first agent reveals and the second follows if observes the same, and if observes a different signal it depends on the prior, as should be. If the conditions are satisfied for the first but not the second agent, it means that the first agent actually does not reveal information, hence the second agent actually behaves as the first, and it means that he will not reveal anything either, and we have the applicable cascade (because all subsequent agents will follow).

f Mr 1 plays 1 regardless of the signal s_1 observed, then Mr 2 has no updating to do, and will act as if she were the first of the line. This happens with probability $1 - q$.

Hence, if agents are all homogeneous, there is a trivial cascade on 1, and the probability of learning is p.

Mr 3 if he observes 2 identical zeros will ignore his private signals, and we have a cascade on 0 (the conditions on θ are trivially satisfied). If he observes 3 different signals, will follow the most frequent. Anyway, the first 2 signals are sufficient to determine which cascade we have.

B Additional Material

Large deviations

In the calculations above only the variance and the error matter. If we consider general concave utility functions $u(-(a-\omega)^2)$ (or general "risk aversion"), instead we have that the term t of the sum is:

$$\mathbb{E}_t^P u\left(-(\mathbb{E}_t^{P'}\omega-\omega)^2\right) = \mathbb{E}_t^P u\left(-\left((\omega-\mathbb{E}_t^P\omega)^2 + (\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega)^2 + 2(\omega-\mathbb{E}_t^P\omega)(\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega)\right)\right)$$

$$\leq u\left(-\mathbb{E}_t^P\left((\omega-\mathbb{E}_t^P\omega)^2 + (\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega)^2 + 2(\omega-\mathbb{E}_t^P\omega)(\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega)\right)\right)$$
$$= u\left(-Var_t^P\omega - \mathbb{E}_t(\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega)^2\right)$$

so now the expression obtained above in this case are just useful as upper bounds on the utility. The correct utility involves the term $\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega$. We know that as n becomes large, by the large deviations principle $P(|\sum Y_n-\mu_0|>a)\sim e^{-\frac{a^2}{2}}$ if the Y are standard normal i.i.d. Hence:

$$P(|\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega|) = P(\frac{\theta(t\mu_0-\sum^t X_s)}{(t+1)(t(\theta+1)+1)}>a) =$$

$$P(|\mu_0-\frac{\sum^t X_s}{t}|>a\frac{(t+1)(t(\theta+1)+1)}{t\theta})\sim e^{-\frac{1}{2\sigma^2}\left(a\frac{(\theta+1)}{\theta}\right)^2 t^2}$$

so the variance of the distribution of large deviations is proportional to $\frac{\theta^2}{(\theta+1)^2}$, the same term as before, with same intuitions: underreaction leads to much worse large deviations.

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Chapter 3

Bounded Surprise and Overreaction to Information

3.1 Introduction

The representativeness heuristics is a biased probability judgment proposed by Tversky and Kahneman (1974). It accounts for departures from Bayesian rationality observed in experiments. The heuristics stipulates that, when facing conditional probabilistic assessments of feature X in group G, $p(X|G)$, agents over-weight *representative* or *diagnostic* features X, which are defined as those which occur more frequently in group G relative to a comparison group G_0. The textbook example is that of assessing the frequency of red-haired individuals among the Irish population. The feature of being "red-haired" is representative of the Irish population G (relative to the rest of the world polulation G_0) and it is therefore over-estimated. While it is instructive to look at extreme cases in which the heuristics provides errors in probabilistic judgments, the rule is understood as an approximately accurate and effortless assessments in typical situations Shah and Oppenheimer (2008), and as an effective adaptive rule, efficiently working in typical situations (Tversky and Kahneman (1974), Gigerenzer and Gaissmaier (2011)). A different view tries to rationalize the heuristics at a deeper level. Tenenbaum et al. (2001) quantifies the representativeness in a Bayesian framework as the evidence in favor of a model, given a dataset, relative to the average evidence. Abbott et al. (2011) links representativeness to categorization algorithms developed in the machine learning literature. Gennaioli and Shleifer (2010) and Bordalo et al. (2017) provide a foundation of the representativeness heuristics based on selective retrieval from memory. More broadly, however: can one think about representativeness in a constrained optimal sense? What are the specific costs it allows to save upon, and what does it sacrifice, so that errors emerge as a byproduct? To answer those questions we reinterpret the representativeness heuristics as formalized by Tenenbaum et al. (2001) and Bordalo et al. (2016) through the lenses of information theory. In particular, we use the notion of *description length*, which quantifies how many *bits* of information one needs to describe a dataset X under hypothesis G, using a probability distribution $p(X|G)$, and reinterpret it as a *cost*. We show that representative events X are in fact those for which the description length under G is shorter than the description length under G_0. This suggests that, in fact, representativeness saves upon *bits* of information needed to describe the data, at the cost of making errors in specific and occasional situations. Under the representativeness heuristics, an agent will compress the dataset more relative to a Bayesian agent.

To assess the representativeness heuristics as a constrained optimal rule, however, one should clarify first in which sense Bayesian rationality is optimal and investigate cognitive constraints on top. This logic underlie the structure of the current paper. First, we consider a standard Bayesian inference scheme and we re-frame it as the optimal solution of a trade-off between a *cost* of moving beliefs and the *accuracy* of describing the data. When an agent is constrained to a fixed number of *bits* to accurately describe the world (i.e. there is an upper bound on the surprise an agent can experience looking at the data) then, the optimal constrained Bayesian inference follows the representativeness heuristics rule.

Distorted predictive distributions are shown to be a Bayesian prediction analog of the diagnostic expectations model of Bordalo et al. (2018b). Finally a comparison with the related frameworks of rational inattention and robust inference is highlighted.

3.2 Bayesian updating and departures from rationality

An agent considers an exogenously specified class of models, denoted by $\mathcal{M}$, in order to rationalize the behavior of some time series $\mathbf{X}_{1,t} = (X_1, \dots, X_t) \in \mathcal{X}^t$. For instance, $\mathbf{X}_{1,t}$ is the time series of log-returns on a market index, and the agent is considering as conceivable models, autoregressive processes of order 1, with persistence ρ and volatility σ. In this case $\mathcal{M} = \{(\rho, \sigma) \in \mathbb{R} \times \mathbb{R}_+\}$. At time $t = 0$, the agent has some prior belief (density) p_0 over possible conceived models. Absence of prior information coming from the data, can be easily modeled resorting to uninformative priors, also known in the literature as Jeffreys priors[1]. $l_0(X_1|m)$ denotes the time $t = 0$ probability of observing X_1, given a model $m \in \mathcal{M}$; it can be seen as a function of m, for a given X_1 and within this perspective it is also called the time $t = 0$ likelihood of model m. After observing X_1, the agent revises her beliefs. The Bayesian updating rule prescribes the following revision of beliefs:

$$p_1(m) \propto l_0(X_1|m)p_0(m). \tag{3.1}$$

Models which are more consistent with the observed datum X_1, gain weight in the posterior belief, relative to the weight they have in the prior belief. Mathematically, one can see the Bayesian updating rule as a simple application of the Bayes theorem only assuming that the agent is equipped with a *joint* probability distribution over the product space

[1] In the case of a finite set $\mathcal{X}$, this would simply lead to choose a uniform distribution.

of data and models. Indeed, in this case, it easy to see that $p_1(m)$ is the conditional density of model m given datum X_1. The normalization constant of the posterior belief is denoted by $l_0(X_1)$ and it is therefore the marginal density of datum X_1, also known as marginal likelihood. $l_0(X_1|m)$ has a dual interpretation in terms of *inference* (for given X_1, it is the likelihood of model m) and *prediction* (for given model m it is the density of X_1). There are many situations in which, however, belief assessments depart this scheme. For instance, predictions of financial forecasters are found to depart from the Bayesian setting in Bordalo et al. (2018a). After good news, forecasters become too optimistic and symmetrically after bad news, leading to predictable errors. Bordalo et al. (2018b) introduced a simple model of distorted predictions, the diagnostic expectation model, which is based on the Tversky and Kahneman (1974) representativeness heuristics. The model is remarkably in agreement with the observed behavior of forecasters data. The diagnostic expectation model exaggerates correct predictions relative to a baseline prediction. Bordalo et al. (2018b) introduce distorted forecasting distribution at time t, given model m, of the form:

$$l_t^\theta(X_{t+1}|m) \propto l_t(X_{t+1}|m) \left(\frac{l_t(X_{t+1}|m)}{B_t(X_{t+1}|m)} \right)^\theta , \tag{3.2}$$

where $\theta > 0$ quantifies the departure between the Bayesian forecast and diagnostic one, while $B_t(X_{t+1}|m)$ is a baseline prediction of X_{t+1}. For instance it may be specified as being constant or as the prediction done with past information, i.e: $B_t(X_{t+1}|m) = l_{t-1}(X_{t+1}|m)$. In the latter case, $l_t^\theta(X_{t+1}|m)$ overweights values of X_{t+1} which are more likely under model m and current information relatively to assessments done with model m and past information. From the point of view of Bayesian inference, one can also interpret the right hand side of expression (3.2) as a *pseudo-likelihood*, which, for a given X_{t+1} over-wights in the posterior beliefs of those models that explain better the data, relatively to the past.

This suggest the following non-Bayesian updating rule:

$$p_1(m) \propto l_0^\theta(X_1|m)p_0(m). \tag{3.3}$$

where $l_0^\theta(X_1|m)$ induces a change of measure from prior beliefs to posterior beliefs which departs from the Bayesian one. Here, we *are not* assuming that the agent is equipped with a *joint* probability distribution over the product space of data and models and, as such, expression (3.3) is a legitimate change of measure, as any other change of measure is. We ask: i) in which sense is the Bayesian change of beliefs (3.1) *optimal* and ii) can we interpret (3.3) as an optimal constrained Bayesian updating? The answers, formally developed in the next sections, are: i) the Bayesian updating optimally trades-off a *cost* of moving beliefs and the ability to *fit* the data. With a cognitive constraint on the description length used from time t to time $t+1$, departures from the Bayesian updating of the form of (3.3) emerge as optimal constrained rules.

Relax now the assumption that the agent is equipped with a *joint* probability distribution over the product space of models and data. A time $t = 0$ (and similarly for subsequent times), the agent has a prior belief over models, $p_0(m)$, and she observes some datum X_1. How should the agent revise her beliefs? Next section introduce a variational *optimization* framework to shed light on beliefs updating in terms of optimal information processing. We will recover the Bayesian updating as well as model (3.3) as special cases.

3.3 Bayesian updating as optimal information processing

3.3.1 Notation

Information theory started with the seminal work of Claude Shannon, Shannon (1948), aimed at quantifying the information content a random variable. It has been imported into economics with the seminal work of Sims (2003).

We employ the same language, first to reinterpret Bayesian updating from the perspective of optimal information processing, while later to discuss non Bayesian updating as solutions of constrained problems. Finally, we will discuss the relation with the rational inattention setting as well as with the one of robust inference. We introduce some notation first.

Definition 1. *A divergence between between two equivalent (i.e. with same support) probability distributions*[2] f *and* g *is a non negative valued function,* $D(f,g) \geq 0$*, such that* $D(f,g) = 0$ *if and only if* $f = g$ *(almost everywhere).*

A divergence between two equivalent probability distributions f and g, is a measure of discrepancy between the two. It is not, in general, a symmetric function of f and g, and it does satisfy, in general, the triangular inequality. As such, a divergence is a weaker concept than that of a *distance*. We now introduce an important and well known instance of divergences, together with its main properties (for additional details, see Cover and Thomas (2012)).

Proposition 2. *Consider the the Kullback-Leibler (KL) divergence (or relative entropy)*

[2] We stick to probability densities for the sake of having a simpler notation.

between f and g, defined as:

$$D_{KL}[f,g] = \int_{supp(g)} f(x) \log \frac{f(x)}{g(x)} dx = \int_{supp(g)} \frac{f(x)}{g(x)} \log \frac{f(x)}{g(x)} g(x) dx.$$

The Kullback-Leibler satisfies the following properties.

1. $D_{KL}[f,g] \geq 0$.

2. $D_{KL}[f,g] = 0$ *if and only if* $f = g$.

3. $D_{KL}[f,g]$ *is convex in* (f,g).

4. D_{KL} *does not satisfy the triangular inequality.*

It can be interpreted as quantifying the average log-likelihood experienced by an agent that thinks that the data are drawn from g, while in fact they are draw from f. The Kullback-Leibler divergence is closely related to the concept of *description length* or *surprise.*

Definition 2. *The* surprise *of* $\mathbf{X}_t$ *associated to model* m *is:*

$$-\log l_0(\mathbf{X}_t|m).$$

This is quantity in used in Bayesian statistics for model comparison as as well as in information theory for data compression Cover and Thomas (2012).
When the probability of the data given model m is close to one, the surprise is low: typical events are expected and therefore not surprising. On the contrary, when the probability of the data given model m is close to zero, the surprise is high: rare events are surprising. Note that here the measure of surprise also depends on the model m considered.

Finally, the representativeness of the data, given a model relative to benchmark B, for a given data set $\mathbf{X}_t$ is defined in line with (Tenenbaum et al. (2001)) as an increasing function of the likelihood ratio:

$$\frac{f(\mathbf{X}_t|m)}{B(\mathbf{X}_t)}.$$

Taking the log, representativeness can be in fact interpreted as the difference between the surprise of the data given model m and a benchmark assessment for data surprise, such as the average surprise $B(\mathbf{X}_t) = f(\mathbf{X}_t)$ or the surprise associated with a benchmark model m^*: $B(\mathbf{X}_t) = l(\mathbf{X}_t|m^*)$. Describing data more surprising than the norm is costly: the representativeness precisely quantifies this cost[3].

In the context of one period Bayesian updating at time t, it is convenient to consider the KL divergence between a candidate posterior distribution f and the prior distribution p_t:

$$D_{KL}[f, p_t] = \int_{\mathcal{M}} f(m) \log \frac{f(m)}{p_t(m)} \mathrm{d}m.$$

In the case of Bayesian updating (3.1), it reads:

$$\begin{aligned} 0 \leq D_{KL}[p_{t+1}, p_t] &= \int_{\mathcal{M}} p_t(m) \frac{l_t(X_{t+1}|m)}{l_t(X_{t+1})} \log \frac{l_t(X_{t+1})|m)}{l_t(X_{t+1})} \mathrm{d}m \\ &= -\log(l_t(X_{t+1})) - \Big(\mathbb{E}_{t+1}\left[-\log l_t(X_{t+1}|m)\right]\Big) \\ &\leq (\mathbb{E}_t - \mathbb{E}_{t+1})\left[-\log l_t(X_{t+1}|m)\right]. \end{aligned}$$

The previous expression shows that a Bayesian agent moves her beliefs to avoid surprise. Indeed, if X_{t+1} is equally likely under any model $m \in \mathcal{M}$, then it is not informative: the posterior belief will equal the prior belief and $D_{KL}[p_{t+1}, p_t] = 0$. This shows that

[3] In information theory, the surprise it is also connected to the notion of description lenght, which is the amount of physical resources needed to correctly describe on average a stochastic process which follows model m. Representativeness thus can only be interpreted by a gain/loss of resources when moving from the benchmark presentation fo the data to the one dictated by model m

Bayesian updating reduce the average surprise of the data. Why this is optimal?
Next section builds on this intuition in order to provide a variational approach to belief updating.

3.3.2 Bayesian updating: an information theoretic perspective

Consider an economic agent as a machine, whose goal is to came up with models which predict well. Suppose that at time t, the agent has a reasonable representation of the world, modeled as beliefs $p_t(m)$, which rationalize possible mechanisms explaining the data. At time $t+1$, the agent observes some new datum X_{t+1}. We show that the Bayesian updating of beliefs in light of new information is the solution of the following variational optimization problem.[4]

Theorem 8 (Bayesian updating as optimal information processing)**.** *The Bayesian posterior (3.1) is the solution to the following variational optimization problem:*

$$p_{t+1} = \arg\min_f \Big(D_{KL}(f||p_t) + \mathbb{E}_f[-\log l_t(X_{t+1}|\cdot)] \Big) , \tag{3.4}$$

subject to the normalization constraint: $\int_{\mathcal{M}} f(m)dm = 1$.

The Bayesian posterior is therefore an optimal trade-off between two forces. One force is the relative entropy $D_{KL}(f||p_t)$, which discipline the cost of moving beliefs. It quantifies how costly moving beliefs from p_t to f is. The second force is the average surprise (or description length) of X_{t+1} given model m. The average is taken with respect to candidate posterior distributions f. It quantifies how much surprising the data are to the agent on average, given agents class of models $\mathcal{M}$ and possible posterior beliefs. To quantitatively assess the interpretation of the two forces, consider the two terms in isolation. Consider first an agent that only experiences the first cost. She has a well calibrated prior and she

[4] For the following, we will assume the probability densities to be regular enough such that all integrals considered are well defined.

sticks to that. Therefore, she solves:

$$p_{t+1} = \arg\min_f D_{KL}(f||p_t),$$

subject to the normalization constraint: $\int_{\mathcal{M}} f(m)dm = 1$. It is clear that the obvious solution to this problem is $p_{t+1} = p_t$. This approach completely disregard the surprise coming from the data. It is reliable if the accuracy of representation p_t is high or if there is some reason to distrust the data.

Consider, on the contrary, an agent that only wants to best fit the data she sees. She solves:

$$p_{t+1} = \arg\min_f \mathbb{E}_f[-\log l_t(X_{t+1}|\cdot)]$$

subject to the normalization constraint: $\int_{\mathcal{M}} f_{t+1}(m)dm = 1$. After a technical assumption[5] and a bit of algebra (see Appendix B), one can show that the solution to this problem is:

$$p_{t+1} \propto \delta(m - m^{ML}(\mathbf{X}_{t+1})),$$

where $m^{ML}(\mathbf{X}_{t+1})$ is the maximum likelihood estimator:

$$m^{ML}(X_{t+1}) = \arg\max_{m\in\mathcal{M}} l(\mathbf{X}_{t+1}|m),$$

and $\delta(m{-}m^{ML}(X_{t+1}))$ is a Dirac delta distribution centered in the maximum likelihood

[5] This variational optimization problem is linear and as such one cannot rely on first order conditions to find the solution. However, the original problem is convex instead: one can therefore regularize (i.e. convexify) the problem by adding the additional term $\epsilon D_{KL}(f||p_t)$ in the objective function, find the optimal solution and finally take the limit $\epsilon \to 0$.

estimator. Appendix A provides its definition; intuitively it can be thought as a Gaussian distribution peaked at $m^{ML}(\mathbf{X}_{t+1})$ and with vanishingly small variance.

In this second case, the agent completely ignore the cost of moving beliefs, in fact she tries to best fit the data, within the class $\mathcal{M}$ of models. She therefore chooses posterior beliefs such that actual data are maximally explained. This approach is reliable if the sample is large enough and if the data generating process is well approximated by one of the models the agent conceives.

Thus, Bayesian updating optimally trades-off a good description of the data (low surprise) and and stable beliefs (low divergence). Next, consider the optimal updating solution of an agent which attaches different weights to those two forces as captured by an exogenously specified parameter θ:

$$p_{t+1} = \arg\min_f \left(D_{KL}(f||p_t) + (1+\theta)\mathbb{E}_f[-\log l_t(X_{t+1}|\cdot)] \right), \tag{3.5}$$

subject to the normalization constraint: $\int_{\mathcal{M}} f(m)dm = 1$. The next theorem characterizes the solution to problem (3.5).

Theorem 9 (Non Bayesian updating as optimal constrained rule)**.** *For $\theta \geq -1$, the posterior beliefs which solve the following variational optimization problem:*

$$p_{t+1} = \arg\min_f \left(D_{KL}(f||p_t) + (1+\theta)\mathbb{E}_f[-\log l_t(X_{t+1}|\cdot)] \right),$$

subject to the normalization constraint: $\int_{\mathcal{M}} f(m)dm = 1$, is:

$$p_{t+1}(m) \propto p_t l_t(X_{t+1}|m) \left(l_t(X_{t+1}|m)\right)^{\theta}.$$

Moreover:

$$\lim_{\theta \to -1} p_{t+1}(m) = p_t(m)$$

$$\lim_{\theta \to 0} p_{t+1}(m) \propto p_t(m) l_t(X_t + 1|m)$$

$$\lim_{\theta \to \infty} p_{t+1}(m) \propto \delta(m - m^{ML}(\mathbf{X}_{t+1}))$$

and:

$$D_{KL}[\lim_{\theta \to -1} p_{t+1}, p_t] = 0$$

$$D_{KL}[\lim_{\theta \to 0} p_{t+1}, p_t] > 0$$

$$D_{KL}[\lim_{\theta \to \infty} p_{t+1}, p_t] = +\infty.$$

The previous theorem characterizes a class of information theoretic variational optimization problems which are built on the trade-off between minimizing costly beliefs movements and avoiding surprise (i.e. fitting well the data).
Can we understand the representativeness distorted Bayesian updating (3.3) in terms of an optimal updating rule?

Theorem 10 (Constrained Bayesian updating and representativeness heuristics)**.** *Consider the Lagrangian which gives rise to the Bayesian updating:*

$$p_{t+1} = \arg\min_f \Big(D_{KL}(f||p_t) + \mathbb{E}_f[-\log l_t(X_{t+1}|\cdot)] \Big) ,$$

subject to the normalization constraint: $\int_{\mathcal{M}} f(m)dm = 1$*, and to following constraint:*

$$\mathbb{E}_f[-\log l_t(X_{t+1}|\cdot)] = \mathbb{E}_f[-\log B_t(X_{t+1}|\cdot)]$$

where $\mathbb{E}_f[-\log B_t(X_{t+1}|\cdot)]$ *is a constraint on the average surprise allowed. Then:*

$$p_{t+1}(m) \propto p_t l_t(X_{t+1}|m) \left(\frac{l_t(X_{t+1}|m)}{B_t(X_{t+1}|m)} \right)^{\theta_{t+1}}$$

where θ_{t+1} *is the Lagrange multiplier associated to the limited surprise constraint, which, in general, depend on* $\mathbf{X}_{t+1}$*. Moreover,* θ_{t+1} *is the solution of the equation:*

$$\left(\frac{d \log Z_t(X_{t+1}, \lambda)}{d\lambda} \right) |_{\lambda=\theta} = 0$$

where:

$$Z_t(X_{t+1}, \lambda) = \int_{\mathcal{M}} p_t(m) l_t(X_{t+1}|m) \left(\frac{l_t(X_{t+1}|m)}{B_t(X_{t+1}|m)} \right)^{\lambda} dm.$$

When the average description length allowed is smaller then the Bayesian one, i.e. $\mathbb{E}_{p_{t+1}}[-\log l_t(X_{t+1}|\cdot)] > \mathbb{E}_f[-\log B_t(X_{t+1}|\cdot)]$, the agent optimally move beliefs further away from the prior relative to the Bayesian case, in order to better fit the data and therefore she experiences less surprise. As such, her posterior will over-weight those models m that, on average, fit better the data than $B_t(X_{t+1}|m)$, as prescribed by the representativeness heuristics. In this case, $\theta_{t+1} > 0$. On the contrary, when the average description length is forced to be larger than the Bayesian one, i.e. $\mathbb{E}_{p_{t+1}}[-\log l_t(X_{t+1}|\cdot)] < \mathbb{E}_f[-\log B_t(X_{t+1}|\cdot)]$, then $-1 < \theta_{t+1} < 0$ and the agent updates less, relative to the Bayesian case. Let us consider some concrete examples.

Example 1

Consider the case of a constant surprise bound $\log B_t(X_{t+1}|m) = C$. Then, the solution to the optimization problem, in the case that the surprise bound is saturated is:

$$p_{t+1} \propto l_t(X_{t+1}|m)\left(l_t(X_{t+1}|m)\right)^{\theta}.$$

In this case the agent exaggerates the posterior weight, relative to the Bayesian case, of those models m which better fit the data. Suppose the data is a string of binary symbols (e.g. toin cosses), namely $\mathcal{X} = \{0,1\}$, and suppose that the agent conceives only i.i.d. models: $\mathcal{M} = \{p = Pr(X_t = 1), p \in [0,1]\}$. Assume the her prior is a uniform (or symmetric) distribution on $\mathcal{M}$ and that C is small enough so that $\theta > 0$. Then, after observing a string of data $X_{1,t}$, with more ones than zeros, namely such that $\sum_{i=1}^{t} X_i > \frac{t}{2}$, she over-weights, relative to a Bayesian agent, models with high p.

Example 2

Consider the case $B_t(X_{t+1}|m) = l_{t-1}(X_{t+1}|m)$. This case compares the actually experienced average surprise at time t, with a benchmark one that the agent would experience if no fundamental news about the process X realizes from time t to time $t+1$. This seems closely related to the *diagnostic expectation* model of Bordalo et al. (2018b). Consider indeed the class of autoregressive processes of order one, with zero long run mean (without loss of generality) and persistence $\rho \in [0,1]$. Assume moreover, for simplicity, that the agent has dogmatic correct beliefs about the volatility σ, while she forms beliefs about the persistence $\rho \in [0,1]$ of the process. Thus $\mathcal{M} = \{\rho \in [0,1]\}$. In this case, the agent over-weights models m under which the realized value X_{t+1} is more likely under time t information, relative to time $t-1$ information. Therefore, after positive news, i.e. $X_{t+1} > X_t$, the agent will over-weight the persistence of the process, relative to the Bayesian case.

3.3.3 From updating to predictions

We now compare Bayesian forecasts versus distorted forecasts. Under a standard quadratic loss function, the Bayesian agent exploits the marginal likelihood for predictions at time t about X_{t+1}[6]:

$$F_t[X_{t+1}] = \int_{\mathcal{X}} X_{t+1} l_t(X_{t+1}) dX_{t+1} = \int_{\mathcal{X}} X_{t+1} \int_{\mathcal{M}} l_t(X_{t+1}|m) p_t(m) dm \, dX_{t+1}.$$

The θ-distorted agent uses the θ-distorted to compute predictions:

$$F_t^\theta[X_{t+1}] := \int_{\mathcal{X}} X_{t+1} l_t^\theta(X_{t+1}) dX_{t+1} := \int_{\mathcal{X}} X_{t+1} \frac{\int_{\mathcal{M}} l_t(X_{t+1}|m') \left(\frac{l_t(X_{t+1}|m')}{B_t(X_{t+1}|m')}\right)^\theta p_t(m') dm'}{\int_{\mathcal{X}} l_t(X'_{t+1}|m) \frac{l_t(X'_{t+1}|m)}{B_t(X'_{t+1}|m)} dX'_{t+1}} dX_{t+1}.$$

The details of forecasting departure from the Bayesian agent depend on the class of models used by the agent, $\mathcal{M}$ and on the specification of the bound on the surprise $B_t(X_{t+1}|m)$. Let us consider some examples.

Exmaple 2 (continued)

Consider again example 2, with $l(X_{t+1}|m) \sim \mathcal{N}(\rho X_t, \sigma^2)$ and $B_t(X_{t+1}|m) = \mathcal{N}(\rho^2 X_{t-1}, \sigma^2)$. In this case, we get:

[6] We use the quadratic function here for the purpose of simplicity, because it is by far the most popular loss function. Different loss functions implies forecasting rules different from the conditional expectations, which can however be computed given beliefs and likelihood function.

$$
\begin{aligned}
F_t^\theta[X_{t+1}] &= \int_{\mathcal{X}} X_{t+1} \frac{\int_{\mathcal{M}} l_t(X_{t+1}|m) \left(\frac{l_t(X_{t+1}|m)}{B_t(X_{t+1}|m)}\right)^\theta p_t(m)dm}{\int_{\mathcal{X}} l_t(X'_{t+1}|m)\frac{l_t(X'_{t+1}|m)}{B_t(X'_{t+1}|m)}dX'_{t+1}} dX_{t+1} \\
&= \int_{\mathcal{X}} X_{t+1} \int_{\mathcal{M}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{(X_{t+1}-\rho(1+\theta)X_t+\theta\rho^2 X_{t-1})} p_t(\rho)d\rho dX_{t+1} \\
&= \int_{\mathcal{M}} \left(\int X_{t+1} \frac{1}{\sqrt{2\pi\sigma^2}} e^{(X_{t+1}-\rho X_t-\theta\sigma\rho\epsilon_t)} dX_{t+1} \right) p_t(\rho)d\rho \\
&= \int_{\mathcal{M}} (\rho X_t + \theta\rho\sigma\epsilon_t)\, p_t(\rho)d\rho = \underbrace{\mathbb{E}_t[\rho]X_t}_{\text{Bayesian forecast}} + \theta\mathbb{E}_t[\rho]\sigma\epsilon_t
\end{aligned}
$$

This is an exact analog of the diagnostic expectation model, where however she attaches beliefs to the persistence ρ of the process. The Bayesian as well as the distorted forecasts depend on the first subjective moment only of the persistence parameter. Moreover, the expression reduces exactly to the one of the diagnostic expectation model in the limiting case in which there a exists a true persistence ρ^* and the agent knows such value, i.e. $p_t(\rho) = \delta(\rho - \rho^*)$. This provides a foundation for distorted forecasting rules even in the absence of learning.

3.4 Noisy signals and bounded surprise

Consider the effect of exogenously specified noise, or *partial observation*. Consider an agent which observes a noisy version of X_{t+1}, denoted by $X^\eta_{t+1} \sim \eta_{t+1}$, such that $\mathbb{E}_t[X^\eta_{t+1}] = X_{t+1}$: The Bayesian updating reads:

$$p_{t+1}(m) \propto l_t(X_{t+1}|X^\eta_{t+1}, m)p_t(m).$$

The agent has to filter the time series of interest $\mathbf{X}_{1,t}$ out of the observed noisy signals, other than learn the data generating process.

For the sake of simplicity we consider the filtering problem as known to the agent, namely the agent knows the map $X_t^\eta = X_t^\eta(X_t)$, while in full generality this task may entail an additional learning problem. How beliefs movement compare with the Bayesian agent? First, we show that, the average description length (or average surprise) in a noisy world is larger than the one in a noiseless world:

$$\int_{\mathcal{M}} f(m) \left(\int_{\mathcal{X}} \left[-\log(l_t(X_{t+1}|\eta_{t+1}, m)) \right] f_t(\eta_{t+1}|m) d\eta_{t+1} \right) dm \geq$$
$$\int_{\mathcal{M}} f(m) \left(-\log \left(\int_{\mathcal{X}} l_t(X_{t+1}|\eta_{t+1}, m) f_t(\eta_{t+1}|m) d\eta_{t+1} \right) \right) dm = \mathbb{E}_f[-\log l_t(X_{t+1}|, m)],$$

where f is a candidate posterior distribution. This shows that, under noise, the expected description length is larger than in the noiseless case. For a fixed resource constraint of the expected description length, our setting predicts that the more the environment is noisy the more the agent relies of the representativeness heuristics, compared to the Bayesian case. This result can also be interpreted in terms of a sharp change of the noisiness of signals, which makes more likely judge according to representativeness.

3.5 Relation to model misspecification and ambiguity aversion

We sketch an intriguing connection with the setting of Hansen and Sargent (2008) of robust prediction. In the simplest version of such setting, the agent recognizes that the class of models $\mathcal{M}$ is limited and it is likely that it does not include the true model generating the data. A agent may therefore look for *robust* posterior beliefs. In out framework we did not take a stand on the true model being in $\mathcal{M}$ or not, yet we implicitly assume that the agent is not concerned about robustnesses with respect to the class of models considered. Robust posterior beliefs are constructed by: first performing a standard Bayesian updating and then looking into an exogenously fixed neighbor of the Bayesian posterior, for alternative posterior beliefs. This recipe will heal the problem of choosing posterior beliefs under which the the objective of the agent is not stable. When estimating a model, the agent minimizes the Bayesian expectation of a loss function:

$$\min_f \mathbb{E}_f[U(X_{t+1})],$$

where $U(X_{t+1}, m)$ is a generic loss function, with $U_t(X_{t+1}) = -\log l(X_{t+1}|m)$ recovering our setting. The minimization problem is subject to the normalization constraint $\int_{\mathcal{M}} f(m)dm = 1$ and the robustness constraint:

$$D_{KL}(f||p^B_{t+1}) \leq C.$$

The constraint explores a neighbor of the Bayesian p^B_{t+1}, with the purpose of being robust in the case of model misspecification. The solution to this optimization problem when $U_t(X_{t+1}) = -\log l(X_{t+1}|m)$ is:

$$p^{ROB}_{t+1} \propto p^{\lambda}_{t+1},$$

where however $-1 < \lambda < 0$, thus leading to a choice of cautious, or tempered, posterior. There are two crucial differences with our setting. First, a negative λ implies that the agents is conservative as opposed to the behavior dictated by the representativeness heuristics. Second the framework does not specify a benchmark distribution and as such departures from Bayesian rationality do not depend on past information, or, more generally, on the relevant context.

3.6 Conclusion

We showed that the representativeness heuristic follows from a Bayesian updating framework, where however the agent has an upper bound she can experience from the data. To do so, we first reinterpreted Bayesian updating as an optimal information processing problem. We showed that it is the trade-off between: i) the cost of moving beliefs and ii) and the fitting ability (surprise). Introducing a constraint on the surprise, due to cognitive finite resources, the agent naturally distorts Bayesian assessments of *representative* features in order to the met the resource capacity. Predictions in this setting feature systematic biases, which in general depend on the form of the constraint. We recovered over-reaction to news - as featured in the diagnostic expectation model of Bordalo et al. (2018b) - when the coding cost bound is adaptive and the depends on the past history. Finally we commented on the relation with rational inattention and model misspecification literatures.

A The Dirac delta distribution

Suppose $x_{t+1} \sim \mathcal{N}(\mu, \sigma^2)$. We want to capture the idea that we perfectly observe x_{t+1}. Intuitively, this is done by considering the limit $\sigma \to 0$. The result need to be carefully defined mathematically. However, it turns out that the the limit distribution is a degenerate distribution with no mass everywhere except on μ. Such distribution is denoted with the symbol $\delta(y-\mu)$ and it is called Dirac delta distribution. Let us recall two main and intuitive properties of the delta distribution:

1. Normalization:

$$\int_{\mathbb{R}} \delta(y-\mu)\mathrm{d}y = 1.$$

2. Expectation of a ("well behaved") function f:

$$\int_{\mathbb{R}} f(y)\delta(y-\mu)\mathrm{d}y = f(\mu).$$

Property 1. recall the usual normalization for probability densities. Property 2. can be easily interpreted thinking that the support of the Dirac delta is $\{\mu\}$, yet it is normalized.

B Proofs

Proof. Consider the Lagrangian for the variational problem:

$$\mathcal{L}_t[f,\lambda,\theta] := \int_{\mathcal{M}} f(m)\log\frac{f(m)}{p_t(m)}\mathrm{d}m - \int_{\mathcal{M}} f(m)\log l_t(X_{t+1}|m)dm - \lambda\left(\int_{\mathcal{M}} f(m)\mathrm{d}m - 1\right)$$
$$-\,\theta\int_{\mathcal{M}} f(m)\log\frac{l_t(X_{t+1}|m)}{B_t(X_{t+1}|m)}dm.$$

The third term is the normalization constraint, while the fourth one is the surprise constraint. Consider the variation $\delta\mathcal{L}$ relative to f:

$$\delta\mathcal{L} = \int_{\mathcal{M}}\left(\log\frac{f(m)}{p_t(m)} + 1 - \lambda - l_t(X_{t+1}|m) - \theta\log\frac{l_t(X_{t+1}|m)}{B_t(X_{t+1}|m)}\right)\mathrm{d}m.$$

Setting $\delta\mathcal{L}=0$, we get:

$$p_{t+1}(m) \propto p_t(m)l_t(X_{t+1}|m)e^{\theta\log\left(\frac{l_t(X_{t+1}|m)}{B_t(X_{t+1}|m)}\right)}.$$

The normalization constants, λ and θ are both dependent on X_{t+1} and on the past history, since the constraints are. We can get rid of λ, since the posterior distribution can be normalized ex-post. Let us discuss θ. Using the description length constraint:

$$\begin{aligned}
&\int_{\mathcal{M}} p_{t+1}(m)\log\frac{l_t(X_{t+1}|m)}{B_t(X_{t+1}|m)}\mathrm{d}m = 0\\
\iff\ &\int_{\mathcal{M}}\frac{l_t(X_{t+1}|m)}{Z_t(\theta,X_{t+1})}\left(\frac{l_t(X_{t+1}|m)}{B_t(X_{t+1}|m)}\right)^{\theta} p_t(m)\log\frac{l_t(X_{t+1}|m)}{B_t(X_{t+1}|m)}\mathrm{d}m = 0\\
\iff\ &\frac{d\log Z_t(\theta,X_{t+1})}{d\theta} = 0.
\end{aligned}$$

This provides us with an equation for θ. We can investigate the limiting cases, $\theta \to 0$, $\theta \to -1$ and $\theta \to \infty$.

$$\begin{aligned}
&\lim_{\theta\to-1} p_{t+1} = p_t \\
&\lim_{\theta\to 0} p_{t+1} = p_t \\
&\lim_{\theta\to-1} p_{t+1}(m) \propto p_t(m) l_t(X_{t+1}|m)\delta(m - m_t^*(X_{t+1}))
\end{aligned}$$

where:

$$m_t^*(X_{t+1}) = \arg\max_{m\in\mathcal{M}} \frac{l_t(X_{t+1}|m)}{B_t(X_{t+1}|m)}$$

□

C Relation to Maximum Entropy Method

The maximum entropy principle (or minimum relative entropy principle) provides a behavioral foundation on how to move beliefs for a non ideological rational agent. This literature has been initiated by Jeynes (see Jaynes (2003)). The principle says: given the prior belief, the agent should revise it only in light of new information. In particular, if no information arrives, the agent should not move her beliefs at all. If, on the contrary, some information is observed, the agent should move as minimally as possible her beliefs, constrained by the observed evidence. Let us formalize those thoughts. Later, we will prove the equivalence with the Bayesian updating.

Additional notation: joint prior and joint posterior

1. Joint prior distribution: $p_t(\theta, \bar{\mathbf{x}}_t, x_{t+1})$.

2. Joint candidate posterior distribution: $p_{t+1}(\theta, \bar{\mathbf{x}}_t, x_{t+1})$.

Note that joint prior and candidates posterior agree on what happened up to time t (i.e. the marginal distribution integrating on θ and x_{t+1} is degenerate).[7]

We will see that the Basyesian posterior will also agree on what happens at time $t+1$ (consistency with observed data).

Note also that both the joint prior and the joint posteriors are defined on the space $\Theta \times \mathcal{X}$ [8]. $\mathcal{X}$ denotes the space on which X_t takes values. For example, if X_t follows an $AR(1)$ process, then $\mathcal{X} = \mathbb{R}$.

[7] Actually, for the time $t+1$ updating, the prior is completely free to be specified. It does not need to be consistent with information up to time t. Here, we consider for convenience a sequencial bayesian updating setting.

[8] Or we can define them on the space $\Theta \times \mathcal{X}^{t+1}$, where the marginal distributions integrating on θ and x_{t+1} are degenerate. This is an esthetic choice.

Let us now discuss what we mean with *the posterior being constrained by the evidence.* We mean that after observing $x_{t+1} = \bar{x}_{t+1}$, its distribution is degenerate, namely $x_{t+1} \sim \delta(x_{t+1} - \bar{x}_{t+1})$. This information *should* be taken into account into the posterior, while it is not incorporated into the prior (since $x_{t+1} = \bar{x}_{t+1}$ is not observed yet and thus x_{t+1} is a random variable from the perspective of time t.)

Formally, the joint posterior needs to satisfy the constraint:

$$\int_{\Theta} p_{t+1}(\theta', \bar{\mathbf{x}}_t, x_{t+1}) \mathrm{d}\theta' = \delta(x_{t+1} - \bar{x}_{t+1}).$$

Note that we assumed both prior and posterior need to be consistent with information observed up to time t, namely:

$$\int_{\Theta \times \mathcal{X}} p_{t+1}(\theta', \mathbf{x}_t, x'_{t+1}) \mathrm{d}\theta' \mathrm{d}x'_{t+1} = \delta(\mathbf{x}_t - \bar{\mathbf{x}}_t).$$

and for the prior:

$$\int_{\Theta \times \mathcal{X}} p_t(\theta', \mathbf{x}_t, x'_{t+1}) \mathrm{d}\theta' \mathrm{d}x'_{t+1} = \delta(\mathbf{x}_t - \bar{\mathbf{x}}_t).$$

Theorem 11 (Bayesian updating minimizes relative entropy subject to new information)**.**

Consider the optimization problem[9]*:*

$$\min_{p_{t+1}} D_{KL}[p_{t+1}, p_t]$$

[9] This is a variational problem, the posterior pdf has to satisfy regularity conditions for the problem to be well posed.

subject to the constraints:

1. *Normalization:*

$$\int_{\boldsymbol{\Theta}\times\mathcal{X}} p_{t+1}(\theta', \bar{\boldsymbol{x}}_t, x'_{t+1})\, d\theta'\, dx'_{t+1} = 1.$$

2. *Consistency with observed data:*

$$\int_{\boldsymbol{\Theta}} p_{t+1}(\theta', \bar{\boldsymbol{x}}_t, x_{t+1})\, d\theta' = \delta(x_{t+1} - \bar{x}_{t+1}).$$

Denote as p^*_{t+1} the solution to the minimization problem. Then $p^*_{t+1}(\theta|\bar{\boldsymbol{x}}_t, \bar{x}_{t+1})$ is the Bayesian posterior.

D Processsing consistency

We comment on different ways of processing evidences from the data. The agent in fact could update her beliefs after each datum she observes, or she could update beliefs less frequently, i.e. after a number of observations. He and Xiao (2017) dub the property of Bayesian posterior beliefs of being invariant with respect to the way a string of data is processed, as *processing consistency.* A pseudo-likelihood[10] satisfies processing consistency if and only if, for any subset $\mathcal{P}_k$ of $\{1, \dots, T+1\}$ with k elements:

$$\log l_1^{\theta}(X_{T+1}|m) = \sum_{i \in \mathcal{P}_k} \log l_{t_i}^{\theta}(X_{t_{i+1}}|m).$$

The Bayesian updating is not the only updating which satisfies it (for instance, model (3.3) with constant B does), yet the processing consistency impose the following strong restriction on updating:

$$\log B_1(X_{T+1}|m) = \sum_{i \in \mathcal{P}_k} \log B_{t_i}(X_{t_{i+1}}|m).$$

The diagnostic expectation model of Bordalo et al. (2018b) does not feature such consistency, in general: updating does not depend on the data only but also on the *updating path*, which may set the benchmark distribution $B_t(X_{t+1}|m)$ at time time t. As such, one should not expect this property to hold in the diagnostic expectation model. We will not consider the choice of *when* updating, while we focus on the features of the constrained updating we will introduce.

[10] note that, given the arbitrariness of $B_t(X_{t+1}|m)$ there is no loss of generality in considering as general the updating dictated by (3.3).

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