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Essays on American Options

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Abstract

This thesis deals with the pricing of American equity options exposed to correlated interest rate and equity risks.
The first article, *American options on high dividend securities: a numerical investigation* by F. Rotondi, investigates the Monte Carlo-based algorithm proposed by Longstaff and Schwartz (2001) to price American options. I show how this algorithm might deliver biased results when valuing American options that start out of the money, especially if the dividend yield of the underlying is high. I propose two workarounds to correct for this bias and I numerically show their strength.
The second article, *American options and stochastic interest rates* by A. Battauz and F. Rotondi introduces a novel lattice-based approach to evaluate American option within the Vasicek model, namely a market model with mean-reverting stochastic interest rates. Interestingly, interest rates are not assumed to be necessarily positive and non standard optimal exercise policy of American call and put options arise when interest rates are just mildly negative. The third article, *Barrier options under correlated equity and interest rate risks* by F. Rotondi deals with derivatives with barrier features within a market model with both equity and interest rate risk. Exploiting lattice-based algorithm, I price European and American knock-in and knock-out contracts with both a discrete and a continuous monitoring. Then, I calibrate the model to current European data and I document how models that assume either a constant interest rate, or strictly positive stochastic interest rates or uncorrelated interest rates deliver sizeable pricing errors.

Contents

Introduction

This thesis deals with the pricing of American equity options within several market models. Whereas European options can be exercised only at maturity, American ones can be exercise at any time up to their maturity. As a consequence, the holder of an American option faces each time an optimization problem as she has to compare what she gets right now exercising the option to what she would get exercising it later on in the future. This recursive optimization problem makes the evaluation of American options quite challenging even in the simplest market models like the pure diffusive one factor model of Black-Scholes. When introducing more complicated, but yet more realistic, market models and when considering more exotic derivatives the current literature offers no solution. This thesis aims at solving some open problems when dealing with American options within market models with more than one risky factor.

This thesis consists of three chapters. Each of them is a self-contained paper, namely

- *American options on high dividend securities: a numerical investigation* by Francesco Rotondi (published);
- *American options and stochastic interest rates* by Anna Battauz and Francesco Rotondi (submitted);
- *Barrier options under correlated equity and interest rate risks* by Francesco Rotondi.

The abstracts of the three papers are collected below.

American options on high dividend securities: a numerical investigation: I document a sizeable bias that might arise when valuing out of the money American options via the Least Square Method proposed by Longstaff and Schwartz (2001). The key point of this algorithm is the regression-based estimate of the continuation value of an American option. If this regression is ill-posed, the procedure might deliver biased results. The price of the American option might even fall below the price of its European counterpart. For call options, this is likely to occur when the dividend yield of the underlying is high. This distortion is documented within the standard Black–Scholes–Merton model as well as within its most common extensions (the jump-diffusion, the stochastic volatility and the stochastic interest rates models). Finally, I propose two easy and effective workarounds that fix this distortion.

American options and stochastic interest rates: We study finite-maturity American equity options in a stochastic mean-reverting diffusive interest rate framework. We allow for a non-zero correlation between the innovations driving the equity price and the interest rate. Importantly, we also allow for the interest rate to assume negative values, which is the case for some investment grade government bonds in Europe in recent years. In this setting we focus on American equity call and put options and characterize analytically their two-dimensional free boundary, i.e. the underlying equity and the interest rate values that trigger the optimal exercise of the option before maturity. We show that non-standard double continuation regions may appear, extending the findings documented in the literature in a constant interest rate framework. Moreover, we contribute by developing a bivariate discretization of the equity price and interest rate processes that converges in distribution as the time step shrinks. The discretization, described by a recombining quadrinomial tree, allows us to compute American equity options' prices and their related free boundaries. In particular, we document the existence of non-standard optimal exercise policies for American call options on a non-dividend-paying equity. We also verify the existence of a non-standard double continuation region for American equity options, and provide a detailed analysis of the associated free boundaries with respect to the time and the current interest rate variables.

Barrier options under correlated equity and interest rate risks: I study European and American equity derivatives with barrier features exposed to correlated equity and interest rate risks. The interest rate is modelled as a diffusive mean-reverting stochastic process and its zero lower bound is removed in order to resemble current European market conditions. Using novel lattice-based pricing techniques, I carry out a throughout analysis of discretely and continuously monitored European and American knock-in and knock-out options solving the related numerical issues. I show how models assuming 1) constant interest rates, 2) independent stochastic interest rates, 3) strictly positive stochastic interest rates deliver sizeable relative pricing errors with respect to my benchmark. The relative pricing errors, that in some cases reach values around 15%, are larger when dealing with path-dependent options, whose dependence on the comovements of the equity and the interest rate is stronger.

1

American Options on High Dividend Securities: a numerical investigation

1.1 Introduction

Whereas[1] most of the exchange-traded options on global financial indexes can be exercised only at maturity, thus being European-style options, the vast majority of equity options are American-style, as they can be exercised at any time up to their maturity. Evaluation of American-style options is, therefore, of crucial relevance in the financial industry.[2]

The fair pricing of this kind of claims is tricky even within simple market models due to their embedded optimization problem. In fact, since the holder of an American option has the right to exercise it at any time up to maturity, she will do so at the moment in which the expected payoff of the option is maximum. Therefore, virtually at each instant in time, she has to compare what she would get by the immediate exercise of the option to what she would get in the future if she waits and exercises the option later on. In turns, what she would get in the future depends on her future decisions: this recursive structure of the decision problem makes the evaluation of American claims quite complicated. See, e.g., Detemple (2014) for an extensive review of the main pricing methods of American-style derivatives.

Longstaff and Schwartz (2001) proposed the Least Square Methods (LSM henceforth), an interesting, fast and flexible Monte Carlo-based algorithm to price American options. The key point of the LMS is a regression-based approximation of the continuation value for the American option, which overcomes the well known issue of recursively estimating the conditional expectation of future optimal exercises. At maturity, the option is exercised whenever in the money. Then, the optimal policy is retrieved going backward by comparing the immediate exercise payoff with the continuation value, which is approximated by the fitted values of a pathwise regression of all future payoffs on the immediate payoff. Since this regression is run on the paths along which the option is in the money, if there are too few of them, the regression is ill-posed and produces biased estimates of the continuation value. This propagates recursively and the final estimate of the price of the American option might be biased. Such bias might be so large that the American option price falls below the price of its European counterpart, delivering a price that violates the no arbitrage assumption as American options are always worth more than their European counterparts due to the early exercise premium.

If the option starts even mildly out of the money, the probability that the underlying reverts

[1]This article is published in *Risks*, 2019, **7**(2), 59.

[2]See, e.g., the *CBOE Market Statistics* annual report released by the Chicago Board Options Exchange, the largest trading market for derivatives. In 2016, the overall dollar value of all the equity options traded at the CBOE was roughly equal to $66 billion with an average of 1.35 million equity options traded daily corresponding to 205 million call options and 135 million put through the year.

back to the in the money region depends on the drift of the risky asset. This drift depends in turns on the risk-free interest rate, on the dividend yield and on the volatility of the equity. Under realistic combinations of these parameters, the probability that initially out of the money option ends up in the in the money region is quite low. This damages the regression-based estimation of the continuation value of the American option thus altering (usually lowering) the final price. As an example, a high dividend yield relative to a low risk-free interest rate depresses the drift of a lognormal risky asset and, therefore, the stock is expected to decrease. In this case, the evaluation of an American option on this stock through the LSM, would most likely deliver a biased result.

My analysis builds on the large literature about the weaknesses and the related improvements of the Monte Carlo-regression based methods for pricing American options. Among others, García (2003) and Kan and Reesor (2012) analysed and corrected the biases of the algorithm due to suboptimal exercise decisions whereas, as an example, Belomestny et al. (2015) and Fabozzi et al. (2017) proposed further improvements of the original LSM.

Out of the money options play a key role in hedging strategies to protect investors against sudden drop/peak of the equity. Furthermore, Carr and Madan (2001) showed how to replicate any derivative whose payoff is a smooth function of the underlying at maturity with fixed positions in the bond, the stock itself and out of the money European call and put options. As European options on equity are quite illiquid, American ones are used in practice (delivering a small deviation from the perfect replication). Therefore, the correct evaluation of American out of the money options is relevant as well.

The remaining of the paper is organized as follows. Section 1.2 analyses the aforementioned flaw of the LSM in the standard diffusive framework of Black–Scholes–Merton. The following three sections address this issue within the three most common extensions of the standard diffusive framework. Section 1.3 deals with the jump-diffusion model, Section 1.4 with the stochastic interest rate framework and, finally, Section 1.5 with the stochastic volatility one. Section 1.6 concludes.

1.2 American Equity Options, Constant Interest Rates

I first analyse the LSM in a simple diffusive framework, as the one of Black and Scholes (1973) and Merton (1973). The risk-free interest rate is assumed to be deterministic. In Section 1.2.1, I first review the LSM and I highlight the possible flaws that might arise when valuing out of the money American option. Then, I propose a possible workaround to overcome them. In Section 1.2.2, I propose some numerical example to quantify the size of the flaws and to show how the workaround delivers correct results.

1.2.1 Theoretical Framework

1.2.1.1 The Primary Assets

Assume that the market is arbitrage-free and let $r \in (-1, +\infty)$ be the constant prevailing risk-free interest rate[3]. The risk-free interest rate is capitalized through a traded bond with price $B(t) = e^{rt}$. Consider a traded lognormal risky security S whose price dynamics under the[4] risk-neutral probability measure $\mathbb{Q}$ solve the following stochastic differential equation (SDE henceforth):

$$\mathrm{d}S(t) = (r-q)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t), \quad S(0) = S_0 \tag{1.1}$$

with $t \in \mathbb{R}^+$, where W is a $\mathbb{Q}$-Brownian motion, $\sigma \in \mathbb{R}^+$ is the constant volatility of the security, q its continuous dividend yield and $S_0 \in \mathbb{R}^+$ is its current price at $t = 0$. It is well known that the solution to Equation (1.1) delivers the following explicit expression for the price of the risky security

$$S(t) = S_0 \exp\left[\left(r - q - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right], \quad t \geq 0. \tag{1.2}$$

Notice that the continuously compounded rate of return over $[0, t]$ on S,

$$\ln\frac{S(t)}{S_0} = \left(r - q - \frac{\sigma^2}{2}\right)t + \sigma W(t)$$

has two contributions: the first one is deterministic and depends on the drift $\mu^{\mathbb{Q}} := r - q - \sigma^2/2$ of the security; the second one is normally distributed with zero mean and variance equal to $\sigma^2 t$. Globally, the expected rate of return over $[0, t]$ is therefore normally distributed with mean $\mu^{\mathbb{Q}}t$ and variance $\sigma^2 t$. Furthermore, as time goes by, the deterministic component $\mu^{\mathbb{Q}}t$ prevails over the random one $\sigma W(t)$[5]. Therefore, the investor expects the security to appreciate as time goes by proportionally to the constant drift $\mu^{\mathbb{Q}}$.

1.2.1.2 The Derivatives

Let $f(S)$, $S \in \mathbb{R}^+$, be the payoff of a derivative written on $S(t)$. For a thorough analysis of many derivatives in this diffusion framework, see, e.g., Björk (2009). I restrict my investigation only to

[3] I advisedly allow for $r \in [-1, 0]$ in order to possibly replicate also the current situation of the Eurozone where "risk-free" government bonds, such as German ones, display negative yield up to few years maturities.

[4] Under these assumptions, the market is actually also complete; therefore, the risk-neutral measure $\mathbb{Q}$ is unique.

[5] It holds true (see, e.g., Revuz and Yor (2001)) that $\limsup_{t\to+\infty} \frac{W(t)}{\sqrt{2t\log_2 t}} = 1$ and $\liminf_{t\to+\infty} \frac{W(t)}{\sqrt{2t\log_2 t}} = -1$ almost surely; as $\pm\sqrt{2t\log_2 t} = o(t)$, $\limsup_{t\to+\infty} \frac{\mu^{\mathbb{Q}}t + \sigma W(t)}{\mu^{\mathbb{Q}}t} = 1$.

plain vanilla options; these are derivatives whose payoff can be cashed in by investors if (and only if) it is positive and that depends only on the current value of the underlying. The two instances of these options I will investigate are the call options, with $f(S) = (S - K)^+$, and the put options, with $f(S) = (K - S)^+$, where in both cases K, the strike price of the option, is the constant quantity specified on the contract at which the holder of the option has the right to buy or to sell, respectively, the underlying[6].

European-style options can be exercised only at maturity $T \in \mathbb{R}^+$. As their payoff at maturity is $f(S(T))$, the fundamental no-arbitrage pricing equation gives the value of these options at any time t, from inception, $t = 0$, up to maturity T

$$\pi_f^E(t) = \mathbb{E}^{\mathbb{Q}}\left[f(S(T))\frac{B(t)}{B(T)} \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}}\left[f(S(T))e^{-r(T-t)} \middle| \mathcal{F}_t \right]. \tag{1.3}$$

For call and put options, $\pi_f(t)$ admits a closed form solution, the celebrated Black–Scholes–Merton formula first derived by Black and Scholes (1973) and Merton (1973).

American-style options can be exercised at any time up to their maturity $T \in \mathbb{R}^+$. If exercised at $t \in [0, T]$, their payoff is $f(S(t))$. Clearly, a rational investor would exercise an American option when the payoff it delivers is the greatest possible. Therefore, the value of an American option at any time t is

$$\pi_f^A(t) = \operatorname{ess}\sup_{\tau \in [t,T]} \mathbb{E}^{\mathbb{Q}}\left[f(S(\tau))e^{-r(\tau-t)} \middle| \mathcal{F}_t \right], \tag{1.4}$$

where the essential supremum accounts for the fact that the sup is taken on an (uncountable) family of random variables defined up to zero-probability sets. In other words, the value of the American option is determined by the optimal stopping time τ that maximizes the discounted payoff. It is well known that $\pi_f^A(t)$ admits closed form expressions for neither call nor put options.

The evaluation of American options has, therefore, to rely on numerical techniques. As greatly summed up by Detemple (2014), there are broadly three valuation approaches to tackle this issue:

- the variational inequality approach, which generalizes the Black-Scholes PDE and translates into a free boundary problem;

- the lattice approach, inspired by the seminal work of Cox et al. (1979), who discretized the evolution of the underlying asset S and evaluated the American option backward along this discretization; and

[6] $(x)^* := \max\{0, x\}$ denotes the positive part. The holder of an option will exercise it if and only if it delivers a positive payoff; if this is not the case, the option will not be exercised and its payoff is floored at zero.

- the least square method, first introduced by Longstaff and Schwartz (2001), who exploited a Monte Carlo simulation to recursively estimate the expected future payoff of the American option.

The present work focuses on the last approach, the LSM, as it is widely used thanks to the simplicity and the flexibility of its algorithm. Nevertheless, I show that, when implemented without few shrewdnesses, the LSM might deliver biased results under some realistic combinations of assets' parameters.

1.2.1.3 The LSM

The general LSM is effectively illustrated in Section 2 of Longstaff and Schwartz (2001). For ease of reading, I briefly recall here its working flow.

First, the LSM considers a uniform discretization of the investment window $[0, T]$ and evaluates a Bermudan-style option that can be exercised at any discrete monitoring date of the time partition. Then, a large number of sample paths of the security S is simulated; each of them is monitored at every possible exercise date.

The LSM runs backward in time. At maturity T, the option is exercised only along the paths in which it ends up in the money. Therefore, the optimal exercise policy at T is known along all the paths and simply prescribes to exercise the option when it is in the money. At any intermediate monitoring date t_i, the holder of the option considers the immediate payoff she would get by the early exercise of the option, $f(S(t_i))$. Along all the paths in which the immediate exercise is positive[7], the holder of the option has to decide whether she is better off by exercising it right at t_i or by waiting and exercising it later on. In other words, she has to compare the immediate exercise value of the option to its continuation value. This continuation value is the discounted expected value of the option as if it were optimally exercised from t_{i+1} on and can be expressed by a conditional expected value as follows

$$\mathbb{E}^{\mathbb{Q}}\left[\pi_f^A(t_{i+1})e^{-r(t_{i+1}-t_i)}\middle|\mathcal{F}_{t_i}\right]. \tag{1.5}$$

In this discrete time backward recursion, the optimal exercise policy has been found along all the paths from T to t_{i+1}; consequently, the value of the American option along all the paths is known as well at t_{i+1}. As the key point of the algorithm, the LSM regresses pathwise the discounted values of the American option at t_{i+1} on some polynomials in the immediate exercises

[7] If the payoff from the immediate exercise at t_i of the option is zero, a rational investor would not exercise it and she would surely hold it on waiting for a positive payoff later on.

values $f(S(t_i)) \in \mathcal{F}_{t_i}$[8]. In other words, it is assumed that

$$\mathbb{E}^{\mathbb{Q}}\left[\pi_f^A(t_{i+1})e^{-r(t_{i+1}-t_i)}\middle|\mathcal{F}_{t_i}\right] = \sum_{m\in\mathbb{N}} \beta_m L_m\left(f(S(t_i))\right), \tag{1.6}$$

where $\{L_m(\cdot)\}_{m\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{L}^2$.

The intuition here is that the variation across the paths in which the option is in the money at t_i conveys some information about the value of the option after a "short" time has passed. The continuation values along all the paths are then obtained as the fitted values of the regression in Equation (1.6). Once the continuation value of the option is known along all the paths, it can be compared to the immediate exercise and the optimal exercise policy is updated for all the paths. As the optimal exercise policy is now known from t_i to T, the algorithm moves backward considering the choice the option holder faces at t_{i-1}.

Once the optimal policy has been derived also at $t = 0$, the value of the Bermudan–American option is obtained by the average of the discounted cashflows (which may occur at different instants in time depending on the particular path of S) across all the paths.

The LSM method is as powerful as flexible. Its implementation is indeed quite straightforward, requires few lines of code and is reasonably fast in delivering the results. The scope of the LSM is almost unbounded: as it works pathwise, the investor just needs to be able to simulate path by path the relevant processes in order to exploit it. This is why it is important to avoid all the possible flaws that come with it.

1.2.1.4 Possible Flaws of the LSM

As already pointed out, the key point of the algorithm is the regression-based approximation of the continuation value that overcomes the well known issue of the recursive estimation of the conditional expectation of future optimal exercises.

If the regression in Equation (1.6) is ill-posed[9], there might be severe consequences in the estimation of the continuation value of the option and, consequently, on the updating of the optimal exercise policy and, ultimately, on the value of the option at $t = 0$.

I investigate two issues that might affect negatively the regression in Equation (1.6) at some t_i:

[8] Notice that, if the conditional expectation of two random variables, $\mathbb{E}[Y|X]$, is an element of the $\mathcal{L}^2$ space, since $\mathcal{L}^2$ is an Hilbert space, $\mathbb{E}[Y|X]$ can be represented as a linear combination of the elements of an orthonormal basis of the space.

[9] See, e.g., Wooldridge (2013), Section 2.2, for a careful explanation of the linear regression model and of the related necessary assumptions for its unbiased and efficient estimation.

1. there are fewer in the money paths than the number M of the polynomial taken from the orthonormal basis of $\mathcal{L}^2$; and

2. the paths along which the option is in the money deliver very low immediate exercise values that translate into a rank-deficient matrix of regressors, especially when high order polynomials are considered.

Consider the linear model $Y = X\beta + \varepsilon$ with $Y, \varepsilon \in \mathbb{R}^{N\times 1}$, $X \in \mathbb{R}^{N\times M}$, $\beta \in \mathbb{R}^{M\times 1}$ where M is the number of regressors, namely the explanatory variables, N is the number of observations and β is the vector of parameters one is interested in. One crucial assumption for the least square estimator of β, $\hat{\beta} = (X'X)^{-1}X'Y$, to be efficient is that $rank(X) = M$, namely that all the regressors are linearly independent. As, by definition, $rank(X) \leq \min\{M, N\}$, if there are not enough observation, namely if $N < M$, the aforementioned hypothesis cannot hold true by construction and the estimate $\hat{\beta}$ one gets might display quite large variances and be far from the true value.

In the LSM, at each time step t_i, N represents the number of the in the money paths and M is the number of polynomials included. If the number of the in the money paths is less than the number of polynomials included, the estimates of the continuation values are not reliable. This might happen when the option starts out of the money. As concretely shown in the following subsection, when the maturity of the option is short and the time step small, there are few in the money paths at the first monitoring dates. The option is clearly not optimally exercised at these monitoring dates. Nevertheless, the imprecise estimate of β one gets at these dates might distort the continuation value and make it even negative. If this were the case, the algorithm would prescribe to exercise the option immediately as a seemingly null payoff is still better than a negative one. Clearly, this would drastically affect the final value of the Bermudan–American option.

The very same dramatic outcome can be reached if, at any t_i, there are enough in the money paths but the immediate exercise values along them is too close to zero. If this was the case, the matrix of the regressors can still have no full rank as the high grade polynomials[10] get closer and closer to zero. This would make one or more regressor equal to the null vector which gives no contribution to the rank of the matrix of regressors, which, in turns, would become rank deficient delivering the problems outlined above.

These issues are not explicitly debated in Longstaff and Schwartz (2001) probably because of the few possible exercise dates, 50 per year, they allow and the few basis functions, the first three

[10]For all the possible choices of basis functions $\{L_m(X)\}_{m\in\mathbb{N}}$, it holds that $\lim_{m\to+\infty, X\to 0} L_m(X) = 0$; see Chapter 22 of Abramowitz and Stegun (1970) for a comprehensive review of the basis functions of $\mathcal{L}^2$.

Laguerre polynomials, they exploit in the regressions.

As Clément et al. (2002) proved, the value of the Bermudan option obtained through the LSM converges to the no-arbitrage price of the related American option as the following three quantities jointly tend to infinity: the number of time steps, n, the number of simulated paths, $NSim$, and the number of basis, m, exploited in the regression in Equation (1.6). When implementing the LSM, one has to choose finite values of the three aforementioned parameters for sake of feasibility.

As can be seen in Figure 1.1 Left, for the evaluation of American option in the standard diffusive framework, the LSM needs at least $NSim \geq 10^4$ and $n/T \geq 125$ in order to obtain relative errors smaller than a percentage point. Interestingly, as can be seen in Figure 1.1 Right, it also turn out that adding more basis function does not improve the estimate of the continuation value but it rather slows the algorithm and increases the probability that one of the two pitfalls described above manifests.[11]

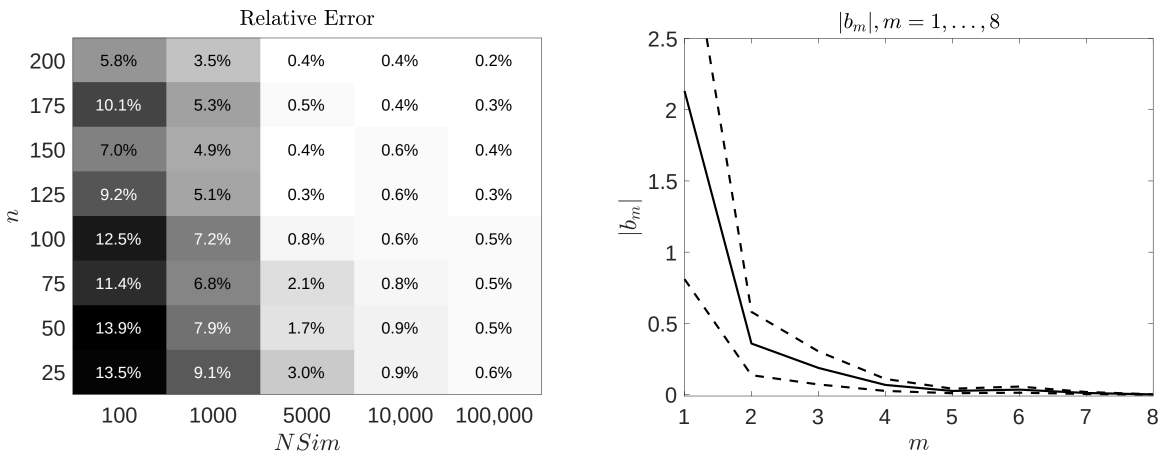

Figure 1.1: (**Left panel**): Relative pricing errors with respect to the binomial tree of an American call option in the standard Black–Scholes–Merton model with $S_0 = 100$, $K/S_0 = 105$, $r = 3\%$, $q = 4\%$, $T = 1$, $m = 6$; the darker the cell, the higher the relative error; (**Right panel**): absolute value of the average of the first 8 betas of the regression in Equation (1.6) for the evaluation of the previous American call option. Twenty independent Monte Carlo (MC henceforth) simulations were run.

[11] The same analysis can be carried out within any of the three extensions to the standard diffusive model: No relevant differences arise though.

1.2.1.5 Fixing of the Possible Flaws of the LSM

To fix the issues described above, I propose two workarounds: the first is finance-based whereas the second is econometric-based[12].

The first one prescribes to estimate the continuation values along all the paths as the original LSM algorithm says, namely as the fitted values of the (possibly ill-posed) regression, and then to floor these estimates pathwise with the current value of the European option obtained by the Black–Scholes–Merton formula. This can be seen as a financial sanity check: the expected payoff that the holder of the American option considers in her exercise decision cannot be lower than the one she would get by exercising the option at maturity.

The second one prescribes to run a constrained regression where the continuation value is forced to be non-negative. Since the flaws of the LSM are likely to arise when the early exercise of the option is never optimal, preventing the continuation value from being negative is enough to correctly postpone the exercise of the American option.

Both e workarounds fully solve the issue pointed out above. The following subsection shows it by means of multiple numerical examples.

1.2.2 Numerical Investigation

As pointed out in the previous subsection, the issues with the LSM are more likely to arise when the option is out of the money.

I focus my numerical investigation on the two most traded options: the American call and the American put option. Besides the level of the initial moneyness at which the option is written, also the particular choice of the other parameters plays a role in the determining whether the underlying is expected to move towards the in the money/out of the money region.

The call option, namely when $f(S) = (S - K)^+$, is out of the money at t if $S(t) < K$. Conversely, the put option, with $f(S) = (K - S)^+$, is out of the money at t if $S(t) > K$. Notice that, if the options share the same parameters, these two events are clearly complementary: the call option is out of the money if and only if the related put option is in the money. As it can be directly derived by Equation (1.2), the unconditional risk-neutral probability evaluated at $t = 0$ that the call option is out of the money at a given $t \in (0, T]$ is

$$\mathbb{Q}\left(S(t) < K\right) = N\left(\frac{\ln\frac{K}{S(0)} - \left(r - q - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right) =: N(-d_2), \tag{1.7}$$

[12] I'm grateful to an anonymous referee for suggesting this extremely simple and effective econometric-based workaround.

where $N(\cdot) : \mathbb{R} \to (0, 1)$ is the cumulative distribution function of a 0–1 normal random variable and $d_2 := \frac{\ln \frac{S(0)}{K} + \left(r - q - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = \frac{\ln \frac{S(0)}{K} + \mu^{\mathbb{Q}} t}{\sigma\sqrt{t}}$ varies with t. Analogously, the unconditional risk-neutral probability evaluated at $t = 0$ that the put option is out of the money at a given is $N(d_2)$. Table 1.1 shows the sensitivities of these two probabilities to the parameters of the model.

Table 1.1: Sensitivities of $N(-d_2)/N(d_2)$, the risk-neutral initial probability that the call/put option ends out of the money at t, to the parameters of the model. + (resp. −) indicates a positive (resp. negative) sensitivity of the probability to the parameter under investigation. ? indicates that the sign of sensitivity of the probability to the parameter is not unique and might change. S_0 is always kept constant.

	$\boldsymbol{K}$	$\boldsymbol{r}$	$\boldsymbol{q}$	$\boldsymbol{\sigma}$
$N(-d_2)$	$+$	$-$	$+$	?
$N(d_2)$	$-$	$+$	$-$	?

The probability in Equation (1.7) and its complementary one are of great interest in investigating whether the pitfalls described in the previous subsection are likely to arise or not.

Consider the call option and fix a monitoring date t_i. If at the first step of the LSM one simulates $NSim$ paths of the underlying, then the call option is expected to be in the money at t_i along $NSim \cdot N(d_2)$ paths. If $NSim \cdot N(d_2) < M$, where M is the number of the basis polynomials included in the regression, or $NSim \cdot N(d_2) > M$ but along these very few paths the option is mildly in the money, the issues described above may arise. If the call option starts even a little bit out of the money, the number of in the money paths expected at the first monitoring dates is extremely low; this worsens for security with high dividend yield q and if the prevailing risk-free interest rate r is low.

Figure 1.2 shows the impact of the moneyness S_0/K on the probability that a call option is in the money at the first monitoring dates. As can be seen, few paths are expected to be in the money when the option has a moneyness roughly larger than 1.04.

Figure 1.3 shows the same probability that a call option on S that starts mildly out of the money ($K/S_0 = 105\%$) reaches the in the money region at the first monitoring dates for different (and realistic) values of r and q. As an example, if one million paths are generated ($NSim = 10^6$), none of them is expected to be in the money at the first three monitoring dates when $r = 3\%$ and $q = 4\%$; only three paths are expected to be in the money at the fourth monitoring date and, since usually $M \approx 6$ basis functions are exploited in the regression, one would introduce four times a bias in the estimate of the continuation value.

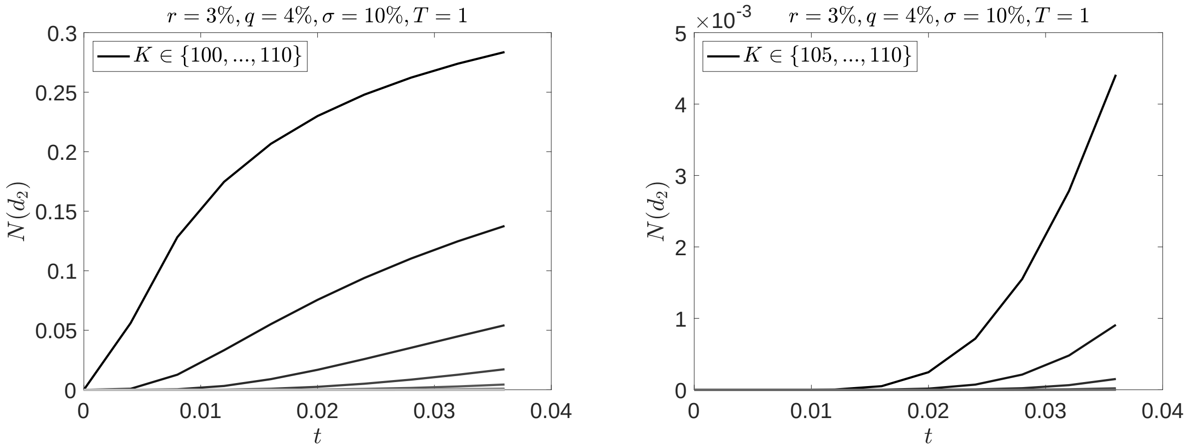

Figure 1.2: Probability that a call option on S is in the money at the first eight monitoring dates. Daily monitoring (250 dates for a $T = 1$ year maturity). The darker is the line, the higher is the strike price K. The panel on the right zooms in the one on the left focusing on more out of the money call options.

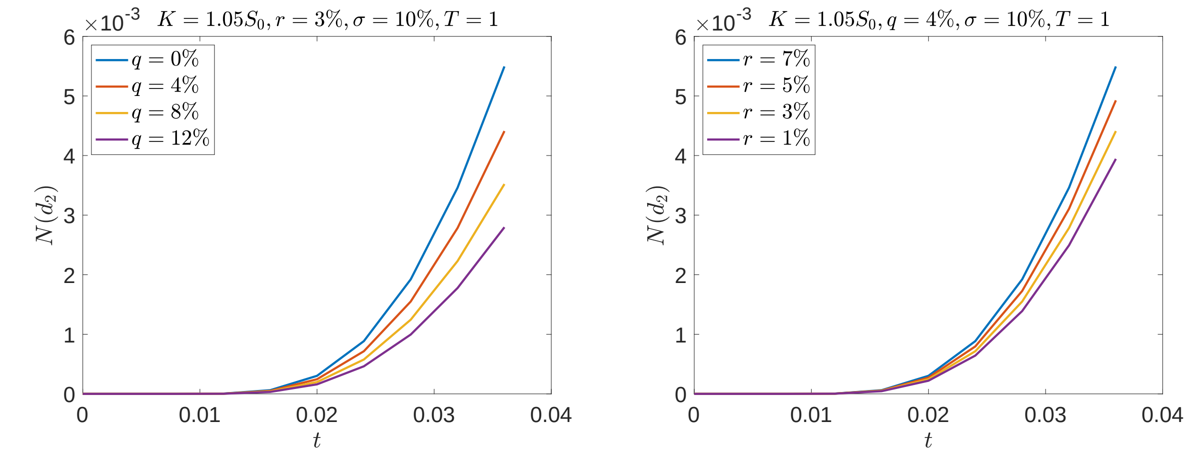

Figure 1.3: Probability that a call option on S is in the money at the first eight monitoring dates. Daily monitoring (250 dates for a $T = 1$ year maturity).

Table 1.2 shows some numerical examples. Three levels of moneyness are considered. In the first case, $K/S_0 = 1.02$, the LSM provides good results even without any correction, but in the case large dividend yield: nevertheless, the distortion here is quite small and the price of the American option is much larger than its European counterpart and quite close to the benchmark. However, the correction of the LSM fixes this issue and delivers coherent results. In the other two levels of moneyness, when the option starts a little bit more out of the money, the LSM without correction heavily underprices the American call delivering also large standard errors.

When any of the corrections to the LSM is implemented instead, the results basically coincide with the ones derived in the binomial model of Cox et al. (1979), which, for the large number of the steps considered, can be assumed as a benchmark. Numerical imprecisions appear also when early exercise is never optimal ($r \approx q$) and, therefore, the price of the European option and of the American are roughly the same. In this case, the tiny difference between the two is much smaller than the confidence interval of the LSM's estimates and such approach delivers unreliable results.

Completely analogous (and symmetric) results are obtained for out of the money put options, especially when the dividend yield q is low and the risk-free interest rate r is high. See, e.g., Carr and Chesney (1997); Detemple (2001) for a throughout discussion on the put-call symmetry for American-style options.

1.3 American Equity Options, Jump-Diffusion Model

In this Section, I propose a first variation of the standard Black–Scholes market. More specifically, I allow for stock price process to jump at random dates and with an idiosyncratic intensity. For the sake of simplicity, the risk-free interest rate is kept constant. The result is the well known jump-diffusion model first introduced by Merton (1976), which can be seen as the first generalization of the standard Black–Scholes–Merton model.

As in the previous section, Section 1.3.1 describes the theoretical aspects of the analysis, whereas Section 1.3.2 contains the related numerical examples.

1.3.1 Theoretical Framework: The Primary Assets and the Derivatives

Assume that the market is arbitrage free. Two assets are traded: the usual riskless bond $B(t) = e^{rt}$ and a risky asset S. To match some features of real market data like high peaks and heavy tails, it is convenient to relax the continuity hypothesis of the risky stock's price and allow it to jump at random times. This introduces a new source risk in the market: The jump risk. Following the seminal work of Merton (1976) and as greatly explained by Glasserman (2003), I postulate that, since jumps are assumed to be independent of each other and of the stock's level, the jump risk can be diversified away by investing in many difference stocks. Hence, the investors require no jump risk premium. This hypothesis being made, the risk-neutral measure $\mathbb{Q}$ becomes unique and derivatives on S can be priced uniquely.

Under $\mathbb{Q}$, the price process of S solves

$$\mathrm{d}S(t) = S(t^-)\Big((r-q)\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{Q}}(t) + \mathrm{d}J(t)\Big), \quad S(0) = S_0, \tag{1.8}$$

where $J(t) = \sum_{j=1}^{N(t)} (Y_j - 1)$ is a jump process, Y_js are i.i.d. positive random variables and $N(t)$ is a Poisson process with intensity $\lambda > 0$ that counts how many jumps occurred before t included (with the convention $N(0) = 0$). As outlined before, $W^{\mathbb{Q}}$, the Y_js and N are assumed to be independent of each other. Jumps arrive at random instants and the waiting time before to consecutive jumps is exponentially distributed with parameter λ. Furthermore, it is convenient to assume that the jumps Y_js are $\mathbb{Q}$-lognormally distributed with mean a and volatility b^2.

Under all of these assumptions and setting $m := \mathbb{E}^{\mathbb{Q}}\left[Y_j - 1\right] = e^{a+b^2/2} - 1$, the explicit solution of $S(t)$ in Equation (1.8) is

$$S(t) = S_0 \exp\left[\left(r - q - \lambda m - \frac{\sigma^2}{2}\right) t + \sigma W^{\mathbb{Q}}(t)\right] \prod_{j=1}^{N(t)} Y_j$$

and, conditioning on n jumps having occurred before t, namely, conditioning on $N(t) = n$,

$$S(t)|_{N(t)=n} = S_0 \exp\left[\left(r - q - \lambda m - \frac{\sigma^2}{2}\right) t + \sigma W^{\mathbb{Q}}(t) + an + b\sqrt{n} Z\right],$$

where Z is standard normal random variable independent of $W^{\mathbb{Q}}(t)$. It also holds true in distribution

$$S(t)|_{N(t)=n} = S_0 \exp\left[\left(r_n(t) - q - \frac{\sigma_n^2(t)}{2}\right) t + \sigma_n(t) W^{\mathbb{Q}}(t)\right] \tag{1.9}$$

with $r_n(t) := r - m\lambda + \frac{n}{t}\left(a + \frac{b^2}{2}\right)$ and $\sigma_n^2(t) := \sigma^2 + \frac{n}{t} b^2$. The deterministic drift of $S(t)|_{N(t)=n}$ here is $\mu^{\mathbb{Q}} = r - m\lambda + na/t - q - \sigma^2/2$, strictly lower than in the standard model when $a \geq 0$.

The price at t of European and American derivatives within the present jump-diffusion model are still given by the risk-neutral expected values in Equations (1.3) and (1.4). For the European call, with $f(S(T)) = (S(T) - K)^+$ and suppressing the argument of $r_n(t)$ and $\sigma_n^2(t)$, it holds

$$\pi_{JD}^E(t) = \sum_{n=0}^{\infty} e^{-\lambda'(T-t)} \frac{(\lambda'(T-t))^n}{n!} \pi_{BS}^E(t; r_n, \sigma_n) \tag{1.10}$$

with $\lambda' := \lambda(1 + m)$, $\pi_{BS}^E(t; r_n, \sigma_n) = S(t) e^{-q(T-t)} N(d_n) - K e^{-r(T-t)} N(d_n - \sigma_n \sqrt{T-t})$ and

$$d_n := \frac{1}{\sigma_n \sqrt{T-t}} \left[\ln \frac{S(t)}{K} + \left(r_n - q + \frac{\sigma_n^2}{2}\right)(T - t)\right]. \tag{1.11}$$

As usual, the pricing formula for a European put option can be retrieved by put-call parity.

The pricing of American options within the jump-diffusion model has to rely on numerical techniques instead, based on extensions of the celebrated Black–Scholes partial differential equation (see, e.g., Kinderlehrer and Stampacchia (2000) for a complete analysis on how to price American options through variational inequalities and Friedman (2003) for general solving schemes for PDEs and free boundary problems). Zhang (1997) provided an extremely useful characterization of a finite difference scheme to price American options in the jump-diffusion model.

1.3.2 Numerical Investigation

With the very same technique exploited to derive $\pi^E_{JD}(t)$ and starting from Equation (1.9), it can be shown that the risk-neutral probability that a call option on S is in the money at $t \in (0, T]$ is

$$\mathbb{Q}\left(S(t) > K\right) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} N(d_n - \sigma_n \sqrt{t}). \tag{1.12}$$

This probability is again increasing in r and decreasing in q, as Figure 1.4 shows. Nevertheless, as the drift of the underlying is now smaller due to the non negligible probability of a downward jump, this probability is slightly larger than the same one in the standard diffusive model.

This implies a lower expected growth of S that translates into smaller call option prices, as it can be seen comparing Tables 1.2 and 1.3 that share the same parameters.

As can be seen from the numerical examples in Table 1.3, the LSM works almost fine also at an intermediate level of out of moneyness, but when the dividend rate is too large.

This is coherent with the numerical figures of the previous section. Again, when the early exercise is almost never optimal, the price of the European option falls inside the confidence interval of the Monte Carlo estimate for the American price, making it not really reliable.

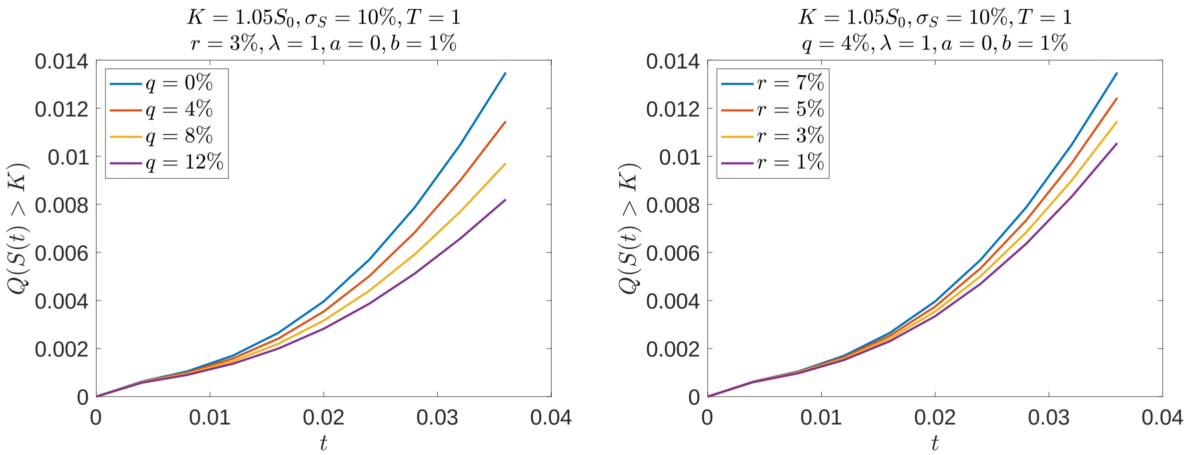

Figure 1.4: Probability that a call option on S in jump-diffusion framework is in the money at the first eight monitoring dates. Daily monitoring (250 dates for a $T = 1$ year maturity).

1.4 American Equity Options, Stochastic Interest Rates

In this section, I propose a second generalized market where the short-term risk free interest rate is stochastic. More specifically, I assume that the interest rate follows a mean-reverting stochastic process, as described first by the seminal work of Vasicek (1977). For the sake of simplicity, the volatilities of both the stock price process and of the locally risk-free interest rate are assumed to be constant. As in Sections 1.2 and 1.3, Section 1.4.1 describes the theoretical aspects of the analysis, whereas Section 1.4.2 contains the related numerical examples.

1.4.1 Theoretical Framework: The Primary Assets and the Derivatives

Assume that the market is arbitrage-free. The locally risk-free interest rate r follows an Ornstein–Uhlenbeck process. The locally risk-free rate is capitalized through a bond whose price at t is $B(t) = e^{\int_0^t r(s)\mathrm{d}s}$. A zero-coupon bond is traded in market as well. It pays out 1 at maturity T and its price at t is labeled by $p(t,T)$. As in the previous Section, this markets involves two sources of uncertainty: The standard diffusive market risk and the interest rate one. Nevertheless, since the investor can hedge from both through S and the T-bond, the market is complete and all the derivatives can be uniquely priced. The explicit formula for the price of the zero-coupon bond $p(t,T)$ can be found, for example, in Brigo and Mercurio (2007). Finally, a lognormal risky security S is traded. I allow for a non-zero correlation between the two processes. Under the risk-neutral measure $\mathbb{Q}$, the two solve the following SDEs:

$$\begin{aligned} \mathrm{d}S(t) &= S(t)\left((r(t)-q)\mathrm{d}t + \sigma_S \mathrm{d}W_S^{\mathbb{Q}}(t)\right), \quad S(0) = S_0 \\ \mathrm{d}r(t) &= \kappa\left(\theta - r(t)\right)\mathrm{d}t + \sigma_r \mathrm{d}W_r^{\mathbb{Q}}(t), \quad r(0) = r_0 \end{aligned} \tag{1.13}$$

with $\langle \mathrm{d}W_S^{\mathbb{Q}}(t), \mathrm{d}W_r^{\mathbb{Q}}(t)\rangle = \rho \mathrm{d}t$. According to standard notation, the new parameters in Equation (1.13) represent: $\sigma_S > 0$ the volatility of the risky asset, κ the speed of mean-reversion of the short-term interest rate, θ its long-run mean, $\sigma_r > 0$ the volatility of the short-term interest rate and $\rho \in [-1,1]$ the correlation between the Brownian shocks on S and r. The explicit solution to the SDEs in Equation (1.13) is

$$\begin{aligned} S(t) &= S_0 \exp\left[\int_0^t r(s)\mathrm{d}s - \left(q + \frac{\sigma_S^2}{2}\right)t + \sigma_S W_S(t)\right], \\ r(t) &= r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma_r \int_0^t e^{-\kappa(t-s)}\mathrm{d}W_r(s). \end{aligned} \tag{1.14}$$

As before, the contribution of the drift of S, $\int_0^t r(s)\mathrm{d}s - \left(q + \frac{\sigma_S^2}{2}\right)t$, prevails over its volatility part $\sigma_S W_S(t)$. Therefore, the expected behaviour of the paths of S depends mostly on the drift.

The pricing formulas for European and American derivatives with maturity $T \in \mathbb{R}^+$ and payoff

$f(S(\cdot))$ closely recall Equations (1.3) and (1.4):

$$\pi_f^E(t) = \mathbb{E}^{\mathbb{Q}}\left[f(S(T))\frac{B(t)}{B(T)} \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}}\left[f(S(T))e^{-\int_t^T r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right]$$

$$\pi_f^A(t) = \operatorname{ess}\sup_{\tau\in[t,T]} \mathbb{E}^{\mathbb{Q}}\left[f(S(\tau))e^{-\int_t^\tau r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right].$$

For European call and put options, the pricing formulas depart slightly from the standard Black–Scholes–Merton ones as now the variability of the locally risk-free interest rate has to be accounted for. The full derivation of the modified formulas can be found in the Appendix of Battauz and Rotondi (2019). For the European call option, $f(S(T)) = (S(T) - K)^+$ and it holds

$$\pi_f^E(t) = S(t)e^{-q(T-t)}N(\tilde{d}_1) - Kp(t,T)N(\tilde{d}_2) \tag{1.15}$$

with

$$\begin{aligned}
\tilde{d}_1 &= \frac{1}{\sqrt{\Sigma_{t,T}^2}}\left(\ln\frac{S(t)}{Kp(t,T)} + \frac{1}{2}\Sigma_{t,T}^2 - q(T-t)\right),\\
\tilde{d}_2 &= \tilde{d}_1 - \sqrt{\Sigma_{t,T}^2},\\
\Sigma_{t,T}^2 &= \sigma_S^2(T-t) + 2\sigma_S\sigma_r\rho\left(\frac{-1+e^{-\kappa(T-t)}+\kappa(T-t)}{k^2}\right) + \\
&\qquad -\sigma_r^2\left(\frac{3+e^{-2\kappa(T-t)}-4e^{-\kappa(T-t)}-2\kappa(T-t)}{2k^3}\right),
\end{aligned} \tag{1.16}$$

whereas the related formula for the European put option can be retrieved by put-call parity.

The extension of the pricing to American options is less trivial. The variational inequality approach can be generalized including the new state variable r but it becomes quite tricky. On the contrary, the generalization of the binomial tree of Cox et al. (1979) is less involved: Battauz and Rotondi (2019) proposed a quadrinomial tree that models the joint evolution of S and r within a lattice structure. This allows for a relatively simple and fast evaluation of American claims.

Longstaff and Schwartz (2001) already allowed in their original work for a stochastic interest rate, which actually changes a little the LSM described in the previous Section. Nevertheless, the valuation algorithm suffers from the same drawbacks of the constant interest rate framework as the following subsection shows.

1.4.2 Numerical Investigation

Since the core of the LSM is unchanged, the drawback spotted out in the previous section might affect the pricing exercise also in this framework. The risk-neutral probability that the option is in the money is still pivotal. Going through the proof of Equation (1.15), it turns out that, in the present framework, the risk-neutral probability evaluated at $t = 0$ that a call option is in the money at a given $t \in (0, T]$ is

$$\mathbb{Q}(S(t) > K) = N\left(\frac{1}{\sqrt{\Sigma_{0,t}^2}}\left(\ln\frac{S_0}{Kp(0,t)} + \frac{1}{2}\Sigma_{0,t}^2 - qt\right)\right) = N(\tilde{d}_2), \tag{1.17}$$

where $\Sigma_{0,t}^2$ is defined in Equation (1.16). At the first steps of the LSM, this probability is again extremely low if the option starts even a little bit out of the money.

Figure 1.5 shows that also when the short-term interest rate is stochastic, very few paths of a simulation are expected to be in the money at the first monitoring dates. Without the numerical correction proposed at the end of Section 1.2.1, the LSM is likely to provide again wrong estimates.

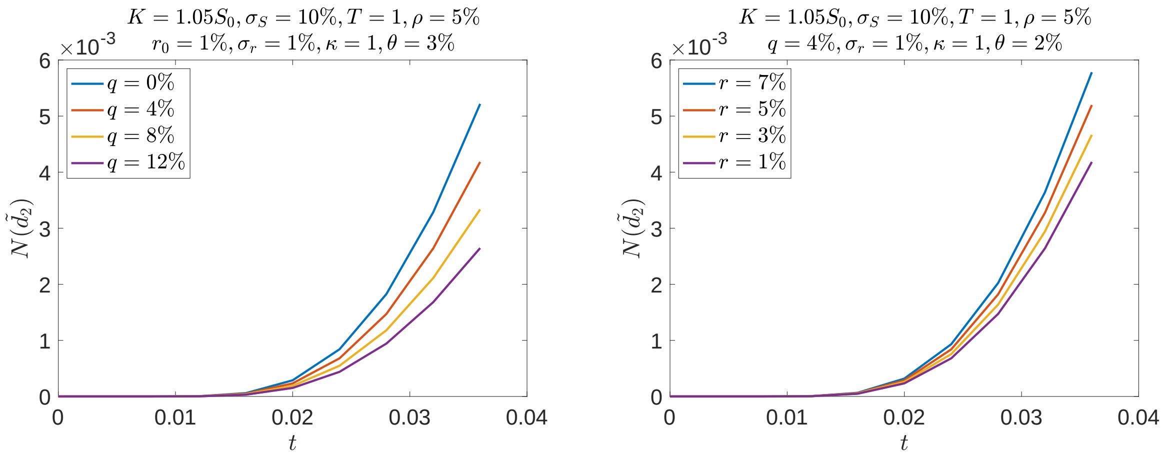

Figure 1.5: Probability that a call option on S in a stochastic interest rate framework is in the money at the first eight monitoring dates. Daily monitoring (250 dates for a $T = 1$ year maturity).

Table 1.4 shows some numerical examples that match the ones of Table 1.2.

First, it is interesting to notice that prices do not vary much when $r_0 = 3\%$ with respect to the ones obtained with a deterministic interest rate $r = 3\%$. This is due to the fact that, in the stochastic interest rate case, the long-run value is exactly $\theta = r_0$; therefore, r is simply expected to oscillate around $r_0 = \theta$ in a symmetric (and thus not very relevant) way. Prices are a little bit higher in the stochastic rate framework to account for the variability in the interest rate itself. This variability is relevant when r_0 is significantly different from θ. When $r_0 > \theta$, r is expected to decrease towards θ: This implies a lower expected drift of S and a smaller call option's price with respect to the standard case with a deterministic $r = r_0$. The converse holds true when $r_0 < \theta$, thus delivering a larger call option's price with respect to the standard model. If the pitfalls of the

LSM are not corrected as proposed, the prices of the American call options fall constantly below their European counterpart. If the correction is made, instead, the LSM produces results that are comparable to the benchmark.

1.5 American Equity Options, Stochastic Volatility

Finally, in this section, I propose a third generalization of the standard Black–Scholes market. More specifically, I allow for the volatility of the stock price process to be stochastic while setting, for the sake of simplicity, the risk-free interest rate to be constant. The result is the celebrated model of stochastic volatility first introduced by the seminal work of Heston (1993).

As in the previous sections, Section 1.5.1 describes the theoretical aspects of the analysis, whereas Section 1.5.2 contains the related numerical examples.

1.5.1 Theoretical Framework: The Primary Assets and the Derivatives

Assume that the market is arbitrage-free. The instantaneous variance ν of the stock process S follows a Cox-Ingersoll-Ross (CIR henceforth) process, namely, a mean-reverting, non-negative[13] stochastic process first introduced by Cox et al. (1985). In this setting, the market involves two state variables: the risky asset's price and the volatility level. Consequently, there are two types of risks: the standard diffusive risk associated to S and the new volatility risk associated to ν. Since only the risky stock S and the riskless bond $B(t) = e^{rt}$ are traded, the market is not complete as the investor cannot hedge from the volatility risk. Therefore, the risk neutral measure $\mathbb{Q}$ is not unique. Nevertheless, uniqueness of contingent claims' prices in this incomplete market is still attainable by making an assumption on the price of volatility risk. As proposed by Heston (1993), I assume that the price of volatility risk is proportional to the instantaneous volatility itself and I denote the constant of proportionality by λ_ν, which becomes another exogenous parameter of the model. Thanks to this assumption, the risk-neutral measure $\mathbb{Q}$ is unique and the processes that drive the markets solve the following SDEs

$$\begin{aligned} \mathrm{d}S(t) &= (r-q)S(t)\mathrm{d}t + \sqrt{\nu(t)}S(t)\mathrm{d}W_S^{\mathbb{Q}}(t), & S(0) = S_0 \\ \mathrm{d}\nu(t) &= \Big[\kappa\left(\nu_\infty - \nu(t)\right) - \lambda_\nu \nu(t)\Big]\mathrm{d}t + \xi\sqrt{\nu(t)}\mathrm{d}W_\nu^{\mathbb{Q}}(t), & \nu(0) = \nu_0 \end{aligned} \tag{1.18}$$

[13]With respect to the specification of $\nu(t)$ in Equation (1.18), if the so-called Feller condition holds true, namely if $2\kappa\theta > \xi^2$, $\nu(t)$ is also strictly positive almost surely.

with $\langle \mathrm{d}W_S^{\mathbb{Q}}, W_\nu^{\mathbb{Q}} \rangle = \rho \mathrm{d}t$. According to standard notation, the new parameters in Equation (1.18) represent: κ the speed of mean-reversion on the volatility, ν_∞ its long-run mean, ξ the volatility of the volatility process (the so called "vol of vol"), and $\rho \in [-1, 1]$ the correlation between the Brownian shocks on S and ν. Although $\nu(t)$ admits no explicit solution[14], the one for $S(t)$ in Equation (1.18) is

$$S(t) = S_0 \exp\left[\left(r - q - \frac{\nu(t)}{2}\right) t + \sqrt{\nu(t)} W_S^{\mathbb{Q}}(t)\right], \quad t \geq 0. \tag{1.19}$$

As before, the contribution of the drift of S, $\left(r - q - \frac{\nu(t)}{2}\right) t$ prevails over its diffusive part $\sqrt{\nu(t)} W_S^{\mathbb{Q}}(t)$ and it is still the main driver of its expected future behaviour.

The price at t of European and American options are still given by the risk-neutral expected values in Equation (1.3) and (1.4), where now the dynamics of S is shown by Equation (1.19).

For European options, the closed-form pricing formula are derived in the first section of Heston (1993). For the European call option, namely when the payoff function is $f(S(T)) = (S(T) - K)^+$, it holds at any $t \in [0, T]$

$$\pi_H^E(t) = S(t) e^{-q(T-t)} P_1 - K e^{-r(T-t)} P_2 \tag{1.20}$$

with

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \mathrm{Re}\left[\frac{1}{ix} \exp\Big(ix\left(S(t) - \ln(K)\right) + C_j(T-t, x) + D_j(T-t, x)\nu(t)\Big)\right] \mathrm{d}x, \quad j = 1, 2, \tag{1.21}$$

where i is the imaginary unit, Re$[\cdot]$ the real part operator and, for $j = 1, 2$,

$$\begin{aligned}
C_j(\tau, x) &= (r-q)xi\tau + \frac{\kappa \nu_\infty}{\xi^2}\left[(b_j - \rho\xi xi + d_j(x))\tau - 2ln\left(\frac{1 - g_j(x) e^{d_j(x)\tau}}{1 - g_j(x)}\right)\right], \\
D_j(\tau, x) &= \frac{b_j - \rho\xi xi + d_j(x)}{\xi^2}, \\
g_j(x) &= \frac{b_j - \rho\xi xi + d_j(x)}{b_j - \rho\xi xi - d_j(x)}, \\
d_j(x) &= \sqrt{(\rho\xi xi - b_j)^2 - \xi^2(2u_j xi - x^2)},
\end{aligned}$$

$$u_1 = 0.5, \quad u_2 = -0.5, \quad b_1 = \kappa + \lambda_\nu \nu - \rho\xi, \quad b_2 = \kappa + \lambda.$$

The analogous formula for the European put option can be retrieved by put-call parity.

The pricing of American option within this stochastic volatility framework is quite challenging. This is precisely one of the cases in which the LSM is of great help as, on the contrary, one can

[14] As effectively explained in Subsection 3.4 of Glasserman (2003), a CIR process is not explicitly solvable; nevertheless, it can be shown that it is distributed, up to a scale factor, as a non-central chi-squared random variable.

easily simulate the paths of Equation (1.18). The benchmark I will compare the performance of the modified LSM to is a finite difference approach to the free boundary problem for the American call option (see, e.g., Pascucci (2008) on this approach). The following numerical investigation focuses on the American call option in the Heston model; analogous results can be retrieved by the put–call symmetry for American options within the Heston model described by Battauz et al. (2014).

1.5.2 Numerical Investigation

By analogy with the standard Black–Scholes–Merton formula and as carefully described by Heston (1993), P_2 in Equation (1.20) represents the risk-neutral probability that an European call option on S closes in the money. As in the previous cases, this probability depends heavily on the drift of S, $\left(r - q - \frac{\nu(t)}{2}\right) t$, and it is increasing with respect to the risk-free interest rate and decreasing with respect to the dividend yield.

Figure 1.6 provides a graphical illustration of this intuition. Notice that, as in the previous cases, at the first monitoring dates, very few paths are expected to be in the money if the option is even mildly out of the money at inception. As before, this worsens when the drift of the underlying becomes negative.

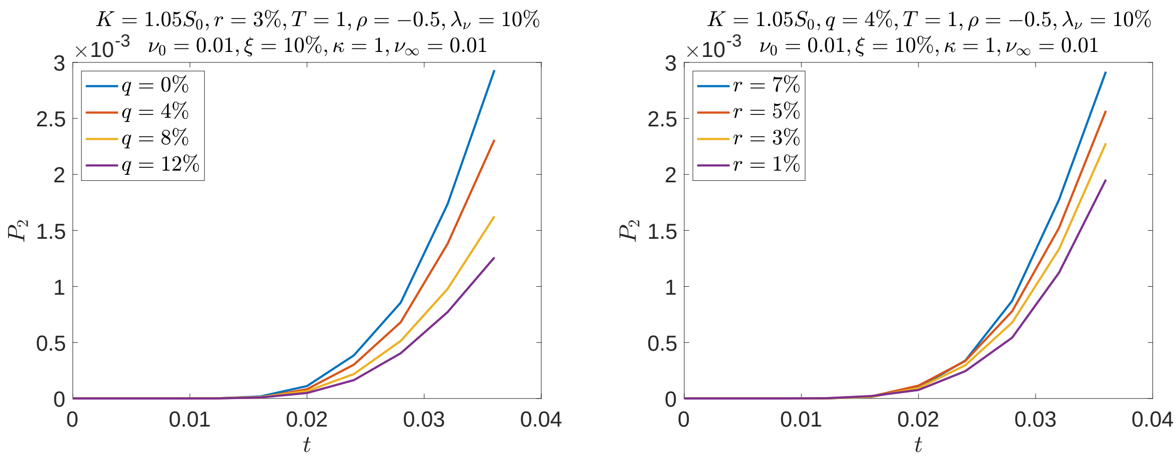

Figure 1.6: Probability that a call option on S in within the Heston model is in the money at the first eight monitoring dates. Daily monitoring (250 dates for a $T = 1$ year maturity).

Table 1.5 shows some numerical examples in the spirit of the previous sections. First, prices of the options are a little bit larger here with respect to the standard model due to the positive volatility risk premium. Then, the LSM fails in providing correct estimates in the very same cases of the standard model. Nevertheless, the correction is still effective.

1.6 Conclusions

The Least Square Methods proposed by Longstaff and Schwartz (2001) is one of the most widely used algorithms to price American equity options. I quantified and corrected a sizeable bias that could arise if the regression exploited for the approximation of the continuation value of the option is ill-posed. I showed that this might happen when the option is even mildly out of the money at inception and when the underlying is not likely to go back into the in the money region. For American call options, this is likely to happen when the risk-free interest rate is low and the dividend yield is higher within the all the most commons financial market models.

Table 1.2: Numerical results, Black–Scholes–Merton model. $S_0 = 100$, $T = 1$, $\sigma = 10\%$, 150 possible exercise dates per year. $\pi^E_{BS}(0)$: initial price of the European call option computed with the Black–Scholes–Merton formula. $\pi^A_{CRR}(0)$: initial price of the American call option computed with the Cox–Ross–Rubinstein binomial tree (average of the price obtained with 250 and 251 steps). $\pi^A_{\text{LSM}}(0)^*$ ($\pi^A_{\text{LSM}}(0)$): initial price of the American call option computed with the LSM not corrected (corrected) for the pitfalls previously described; the first six Laguerre polynomials are used for the regression: $L_m(x) = \frac{e^{\frac{x}{2}}}{m!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^m e^{-x})$, with $m = 0, \cdots, 6$; $NSim = 10^5$, standard errors obtained by 20 independent MC simulations. MC estimates that do not include the benchmark value within the confidence interval are denoted by *.

K/S_0	r	q	$\pi^E_{BS}(0)$	$\pi^A_{CRR}(0)$	$\pi^A_{\tilde{\text{LSM}}}(0)^*$	s.e.	$\pi^A_{\tilde{\text{LSM}}}(0)$	s.e.
102%	5%	4%	3.3955	3.3965	3.3953	(0.0031)	3.3942	(0.0026)
		6%	2.5558	2.6513	2.6504	(0.0023)	2.6489	(0.0035)
		8%	1.8765	2.1129	2.1133	(0.0034)	2.0134	(0.0021)
	3%	4%	2.6074	2.6881	2.6879	(0.0035)	2.6900	(0.0043)
		6%	1.9144	2.1376	2.1369	(0.0027)	2.1384	(0.0018)
		8%	1.3694	1.7149	1.7096 *	(0.0049)	1.7118	(0.0032)
	1%	4%	1.9531	2.1628	2.1602	(0.0032)	2.1609	(0.0040)
		6%	1.3971	1.7324	1.7343	(0.0025)	1.7248	(0.0041)
		8%	0.9724	1.3973	1.3896 *	(0.0052)	1.3976	(0.0027)
105%	5%	4%	2.2971	2.2983	1.7515 *	(0.3618)	2.2988	(0.0036)
		6%	1.6672	1.7209	1.4135 *	(0.4398)	1.7193	(0.0019)
		8%	1.1782	1.3048	0.9535 *	(0.3175)	1.3045	(0.0024)
	3%	4%	1.7009	1.7468	1.0236 *	(0.3757)	1.7494	(0.0057)
		6%	1.2020	1.3222	0.9634 *	(0.2135)	1.3204	(0.0025)
		8%	0.8262	1.0018	0.7122 *	(0.2588)	1.0093	(0.0029)
	1%	4%	1.2263	1.3399	0.8669 *	(0.2495)	1.3375	(0.0038)
		6%	0.8429	1.0140	0.7095 *	(0.2236)	1.0124	(0.0028)
		8%	0.5629	0.7667	0.3622 *	(0.2486)	0.7415	(0.0032)
108%	5%	4%	1.4930	1.4930	1.1524 *	(0.2163)	1.4921	(0.0037)
		6%	1.0433	1.0719	0.8451 *	(0.2461)	1.0695	(0.0039)
		8%	0.7088	0.7741	0.5428 *	(0.1856)	0.7745	(0.0026)
	3%	4%	1.0644	1.0890	0.6342 *	(0.3154)	1.0894	(0.0032)
		6%	0.7231	0.7853	0.5526 *	(0.2866)	0.7841	(0.0019)
		8%	0.4771	0.5635	0.3623 *	(0.1849)	0.5626	(0.0032)
	1%	4%	0.7377	0.7968	0.4255 *	(0.2121)	0.7965	(0.0025)
		6%	0.4868	0.5711	0.3574 *	(0.1937)	0.5723	(0.0021)
		8%	0.3117	0.4066	0.2114 *	(0.0984)	0.4073	(0.0024)

Table 1.3: Numerical results, jump-diffusion. $S_0 = 100$, $T = 1$, $\sigma = 10\%$, $a = 0$, $b = 1\%$, $\lambda = 1$; 150 possible exercise dates per year. $\pi^E_{JD}(0)$: initial price of the European call option computed with Equation (1.10). $\pi^A_{FD}(0)$: initial price of the American call option computed solving the free boundary problem by finite differences. $\pi^A_{\mathrm{L\tilde{S}M}}(0)^*$ ($\pi^A_{\mathrm{L\tilde{S}M}}(0)$): initial price of the American call option computed with the LSM not corrected (corrected) for the pitfalls previously described; the regression involves the first six Laguerre polynomials for the values of the immediate exercise payoff; $NSim = 10^5$, standard errors obtained by 20 independent MC simulations. MC estimates that do not include the benchmark value within the confidence interval are denoted by $\star$.

K/S_0	r	q	$\pi^E_{JD}(0)$	$\pi^A_{FD}(0)$	$\pi^A_{\mathrm{L\tilde{S}M}}(0)^*$	s.e.	$\pi^A_{\mathrm{L\tilde{S}M}}(0)$	s.e.
102%	5%	4%	2.7297	2.7321	2.7301	(0.0025)	2.7312	(0.0031)
		6%	2.1466	2.2184	2.2158	(0.0034)	2.2175	(0.0062)
		8%	1.6861	1.9121	1.9134	(0.0028)	1.9103	(0.0043)
	3%	4%	2.1900	2.2448	2.2442	(0.0033)	2.2456	(0.0037)
		6%	1.7201	1.9289	1.9293	(0.0024)	1.9301	(0.0024)
		8%	1.3339	1.7143	1.6887 $\star$	(0.0206)	1.7142	(0.0026)
	1%	4%	1.7549	1.9461	1.9436	(0.0025)	1.9458	(0.0019)
		6%	1.3608	1.7283	1.7274	(0.0031)	1.7296	(0.0032)
		8%	1.0425	1.5446	1.5101 $\star$	(0.0329)	1.5439	(0.0027)
105%	5%	4%	1.9863	1.9878	1.9857	(0.0034)	1.9885	(0.0024)
		6%	1.5511	1.5982	1.5998	(0.0026)	1.5997	(0.0036)
		8%	1.1962	1.3516	1.3519	(0.0012)	1.3489	(0.0042)
	3%	4%	1.5824	1.6182	1.6189	(0.0032)	1.6168	(0.0037)
		6%	1.2204	1.3640	1.3625	(0.0048)	1.3638	(0.0031)
		8%	0.9304	1.1968	1.1789 $\star$	(0.0107)	1.1999	(0.0048)
	1%	4%	1.2450	1.3766	1.3739	(0.0027)	1.3748	(0.0028)
		6%	0.9492	1.2068	1.2041	(0.0029)	1.2045	(0.0026)
		8%	0.7162	1.0671	0.6845 $\star$	(0.0714)	1.0701	(0.0052)
108%	5%	4%	1.4367	1.4376	1.1214 $\star$	(0.4147)	1.4369	(0.0028)
		6%	1.1027	1.1332	0.8427 $\star$	(0.2845)	1.1327	(0.0014)
		8%	0.8370	0.9414	0.6854 $\star$	(0.2656)	0.9402	(0.0023)
	3%	4%	1.1249	1.1480	0.9697 $\star$	(0.3274)	1.1459	(0.0043)
		6%	0.8539	0.9503	0.6369 $\star$	(0.1667)	0.9489	(0.0034)
		8%	0.6419	0.8234	0.4214 $\star$	(0.2546)	0.8251	(0.0037)
	1%	4%	0.8711	0.9595	0.6214 $\star$	(0.2652)	0.9602	(0.0026)
		6%	0.6549	0.8306	0.4886 $\star$	(0.2145)	0.8295	(0.0028)
		8%	0.4882	0.7265	0.3215 $\star$	(0.1341)	0.7255	(0.0017)

Table 1.4: Numerical results, Vasicek model. $S_0 = 100$, $T = 1$, $\sigma = 10\%$, $\sigma_r = 1\%$, $\kappa = 1$, $\theta = 3\%$; 150 possible exercise dates per year. $\pi^E_{\tilde{BS}}(0)$: initial price of the European call option computed with Equation (1.15). $\pi^A_{BR}(0)$: initial price of the American call option computed with the quadrinomial tree of Battauz and Rotondi (2019) (average of the price obtained with 150 and 151 steps). $\pi^A_{\tilde{\text{LSM}}}(0)^*$ ($\pi^A_{\tilde{\text{LSM}}}(0)$): initial price of the American call option computed with the LSM not corrected (corrected) for the pitfalls previously described; the regression involves the first four Laguerre polynomials for the values of the immediate exercise payoff and the first four Laguerre polynomials for the current value of r; $NSim = 10^5$, standard errors obtained by 20 independent MC simulations. MC estimates that do not include the benchmark value within the confidence interval are denoted by $\star$.

K/S_0	r_0	q	$\pi^E_{\tilde{BS}}(0)$	$\pi^A_{BR}(0)$	$\pi^A_{\tilde{\text{LSM}}}(0)^*$	s.e.	$\pi^A_{\tilde{\text{LSM}}}(0)$	s.e.
102%	5%	4%	3.1074	3.1424	3.1433	(0.0021)	3.1412	(0.0036)
		6%	2.3141	2.4910	2.4901	(0.0025)	2.4896	(0.0032)
		8%	1.6847	1.9970	1.9958	(0.0036)	1.9982	(0.0023)
	3%	4%	2.6170	2.7026	2.7011	(0.0029)	2.7028	(0.0017)
		6%	1.9232	2.1473	2.1452	(0.0039)	2.1464	(0.0026)
		8%	1.3772	1.7213	1.7195	(0.0036)	1.7215	(0.0019)
	1%	4%	2.1930	2.3223	2.3205	(0.0027)	2.3231	(0.0030)
		6%	1.5858	1.8467	1.8452	(0.0031)	1.8450	(0.0022)
		8%	1.1160	1.4803	1.4764 $\star$	(0.0032)	1.4816	(0.0020)
105%	5%	4%	2.0746	2.0950	1.4113 $\star$	(0.1999)	2.0941	(0.0030)
		6%	1.4935	1.5882	1.0263 $\star$	(0.2665)	1.5896	(0.0035)
		8%	1.0461	1.2088	0.7596 $\star$	(0.2133)	1.2092	(0.0031)
	3%	4%	1.7116	1.7553	1.2524 $\star$	(0.2932)	1.7569	(0.0051)
		6%	1.2111	1.3275	0.8142 $\star$	(0.3120)	1.3288	(0.0036)
		8%	0.8331	1.0053	0.6368 $\star$	(0.2743)	1.0033	(0.0042)
	1%	4%	1.3966	1.4653	0.9235 $\star$	(0.2293)	1.4658	(0.0036)
		6%	0.9709	1.1037	0.8097 $\star$	(0.1773)	1.1015	(0.0027)
		8%	0.6557	0.8312	0.4521 $\star$	(0.1884)	0.8323	(0.0017)
108%	5%	4%	1.3303	1.3430	0.9593 $\star$	(0.2273)	1.3411	(0.0027)
		6%	0.9213	0.9747	0.6695 $\star$	(0.2874)	0.9749	(0.0020)
		8%	0.6211	0.7055	0.4241 $\star$	(0.2132)	0.7063	(0.0031)
	3%	4%	1.0723	1.0966	0.7946 $\star$	(0.1537)	1.0967	(0.0037)
		6%	0.7297	0.7910	0.4215 $\star$	(0.1632)	0.7923	(0.0021)
		8%	0.4821	0.5675	0.2559 $\star$	(0.1241)	0.5670	(0.0023)
	1%	4%	0.8538	0.8903	0.4178 $\star$	(0.1896)	0.8914	(0.0025)
		6%	0.5703	0.6371	0.3558 $\star$	(0.1181)	0.6352	(0.0028)
		8%	0.3695	0.4528	0.2036 $\star$	(0.1087)	0.4531	(0.0012)

Table 1.5: Numerical results, Heston model. $S_0 = 100$, $T = 1$, $\nu_0 = 0.01$, $\xi = 10\%$, $\kappa = 1$, $\nu_\infty = 0.01$, $\lambda_\nu = 10\%$; 150 possible exercise dates per year. $\pi^E_H(0)$: initial price of the European call option computed with Equation (1.20). $\pi^A_{BR}(0)$: initial price of the American call option computed solving the free boundary problem by finite differences. $\pi^A_{\tilde{\mathrm{LSM}}}(0)^*$ ($\pi^A_{\tilde{\mathrm{LSM}}}(0)$): initial price of the American call option computed with the LSM not corrected (corrected) for the pitfalls previously described; the regression involves the first four Laguerre polynomials for the values of the immediate exercise payoff and the first four Laguerre polynomials for the current value of ν; $NSim = 10^5$, standard errors obtained by 20 independent MC simulations. MC estimates that do not include the benchmark value within the confidence interval are denoted by *.

K/S_0	r	q	$\pi^E_H(0)$	$\pi^A_{FD}(0)$	$\pi^A_{\tilde{\mathrm{LSM}}}(0)^*$	s.e.	$\pi^A_{\tilde{\mathrm{LSM}}}(0)$	s.e.
102%	5%	4%	3.2587	3.2588	3.2530	(0.0065)	3.2580	(0.0036)
		6%	2.3580	2.4389	2.4401	(0.0042)	2.4405	(0.0039)
		8%	1.6416	1.8626	1.8601	(0.0028)	1.8647	(0.0038)
	3%	4%	2.4057	2.4741	2.4809	(0.0070)	2.4762	(0.0024)
		6%	1.6748	1.8841	1.8802	(0.0045)	1.8890	(0.0040)
		8%	1.1198	1.4607	$1.4572^\star$	(0.0021)	1.4589	(0.0047)
	1%	4%	1.7087	1.9062	1.9063	(0.0023)	1.9058	(0.0036)
		6%	1.1424	1.4746	1.4736	(0.0013)	1.4759	(0.0028)
		8%	0.7326	1.1628	$1.1604^\star$	(0.0017)	1.1597	(0.0053)
105%	5%	4%	2.0737	2.0738	$1.4938^\star$	(0.3181)	2.0744	(0.0029)
		6%	1.4186	1.4820	$0.6268^\star$	(0.2667)	1.4835	(0.0032)
		8%	0.9316	1.0361	$0.5911^\star$	(0.1911)	1.0395	(0.0039)
	3%	4%	1.4473	1.4820	$0.6037^\star$	(0.2722)	1.4796	(0.0035)
		6%	0.9504	1.0500	$0.5094^\star$	(0.1900)	1.0524	(0.0027)
		8%	0.5976	0.7486	$0.4932^\star$	(0.1214)	0.7471	(0.0024)
	1%	4%	0.9697	1.0643	$0.5353^\star$	(0.1909)	1.0644	(0.0023)
		6%	0.6107	0.7574	$0.5181^\star$	(0.1281)	0.7568	(0.0017)
		8%	0.3686	0.5440	$0.2180^\star$	(0.1039)	0.5436	(0.0021)
108%	5%	4%	1.2346	1.2347	$0.3744^\star$	(0.1680)	1.2336	(0.0029)
		6%	0.7975	0.8168	$0.2408^\star$	(0.1277)	0.8176	(0.0035)
		8%	0.4940	0.5408	$0.2372^\star$	(0.0880)	0.5400	(0.0024)
	3%	4%	0.8136	0.8304	$0.2964^\star$	(0.1243)	0.8329	(0,0027)
		6%	0.5040	0.5488	$0.2497^\star$	(0.0823)	0.5508	(0.0028)
		8%	0.2992	0.3616	$0.1409^\star$	(0.0534)	0.3638	(0.0020)
	1%	4%	0.5142	0.5570	$0.4261^\star$	(0.1136)	0.5544	(0.0023)
		6%	0.3053	0.3664	$0.2439^\star$	(0.0935)	0.3691	(0.0018)
		8%	0.1736	0.2410	$0.1273^\star$	(0.0636)	0.2399	(0.0015)

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2

American Options and Stochastic Interest Rates

2.1 Introduction

In[1] an arbitrage-free financial market the role of the short-term interest rate is twofold: on one hand it represents the rate at which the equity price appreciates; on the other hand it drives the locally risk-free asset and the related discount rate. Therefore, neglecting the variability of short-term interest rates may induce significant mispricing on both interest rates and equity derivatives. This issue is particularly relevant when derivatives are path-dependent and therefore sensitive to the entire path of the short-term interest rate, and not just to its expected value at maturity. American equity call and put options, due to the optionality of their exercise policy, fall within this category. In fact, the holder of an American option has to timely chose when to cash in by exercising the option, balancing the effects from the discount rate and from the expected rate of return of the underlying asset. When both of these effects depend on a stochastic process, the valuation of the option becomes tricky. Our paper develops an extensive analysis of American call and put options written on equity with constant volatility in a stochastic interest rate framework of Vasicek type (see Vasicek (1977)). We employ the Vasicek mean-reverting model for the interest rate, because it allows the interest rate to assume mildly negative values, as the ones documented in recent years in the Eurozone[2]. The feasibility of negative interest rates within the Vasicek model, once a source of major criticism, has very recently become the reason of renewed interest in the model itself because of the aforementioned market circumstances. We also allow for a non-zero constant correlation between the Brownian innovations of the interest rate and the equity price processes. A positive (resp. negative) correlation between the interest rate and the equity price corresponds to a negative (resp. positive) correlation between the bond price and the equity price. After 2000 the market observed persistent negative stock-bond correlation as shown by Connolly et al. (2005). Perego and Vermeulen (2016) find that the correlation between equities and bonds is now consistently negative also in the Eurozone but for Southern Europe. Thus, in line with the recent empirical evidence, in our numerical examples we consider a positive correlation between the interest rate and the equity price.

The literature on American equity options has so far focused on alternative stochastic interest rates models, such as the CIR one, based on the seminal work of Cox et al. (1885) (See Medvedev and Scaillet (2010) and Boyarchenko and Levendorskiǐ (2013)). Our paper is, to our knowledge, the first that addresses the evaluation of American equity options in a stochastic interest rate framework of Vasicek type, allowing for the possibility of negative interest rates (see Detemple

[1]This is a joint work with Anna Battauz.

[2]It is widely accepted to proxy the risk-free rate in Europe by the recently negative rates of German bonds.

(2014) for an exhaustive review of the state of the art.[3])

We contribute to the literature by offering an intuitive and effective lattice method to compute the price,the optimal exercise policies and the related free boundaries of American equity options in the presence of market risk and interest rate risk. In the spirit of Cox et al. (1979), building on Nelson and Ramaswamy (1990), who provide a tree approximation for an univariate process, we construct a discrete joint approximation for the both the equity price and the interest rate processes. Hahn and Dyer (2008) develop a similar discretization for a correlated two-dimensional mean reverting process representing the price of two correlated commodities and they use it to evaluate the value of an oil and gas switching option. Our paper is different, as the mean reverting stochastic interest rate process enters the risk-neutral drift of our equity price, that has constant volatility and correlates with the interest rate. In this framework, we provide a throughout investigation of American equity call and put options and their free boundaries. Our findings contribute to the literature on American options with stochastic interest rates, that usually restricts on non-negative interest rates. In particular, we unveil two novel significant features of the free boundary that appear when the stochastic interest rate may take mildly negative values.

First, we show that for American put (resp. call) options the early exercise region is not always downward (resp. upward) connected. The early exercise region is downward (resp. upward) connected if optimal exercise at t of the put (resp. call) option for some underlying equity price implies optimal exercise at t for all lower (resp. greater) values of the underlying equity price. In a stochastic interest rate framework Detemple (2014) retrieves the free boundary by a discretization of an integral equation for the early exercise premium decomposition. For American call options he argues that the exercise region is connected in the upward direction. Our results show that this property holds true if interest rates are always non-negative, but may fail if the interest rates' positivity assumption is not satisfied. In this case, we document the existence of a non standard double continuation region first described by Battauz et al. (2015) in a constant interest rate framework. In particular, a non-standard additional continuation region appears where the

[3]Fabozzi et al. (2016) provide a new recent quasi-analytic method to price and hedge American options on a lognormal asset with constant interest rate. Jin et al. (2013) introduce a computationally effective pricing algorithm for American options in a multifactor setting. First attempts to evaluate the impact of stochastic interest rates on American derivatives date back to Amin and Bodurtha Jr. (1995) and Ho et al. (1997). Nevertheless, both of them proxy American with Bermudan options with few exercise dates. Although this allows them to obtain closed form solutions for both the price of the options and their optimal exercise policy, the approximation of a continuum of exercise dates by just a couple of possible exercise dates leads to a heavy mispricing of the options and provides no accurate insight on the free boundary.

option is most deeply in the money and the underlying pays a negative dividend.[4] Under these circumstances a mildly negative interest rate may lead to optimal postponment of the deeply in the money option as the holder is confident the option will still be in the money later and prefers to delay the cash-in.
Second, we show that early exercise may be optimal for an American call option even if the underlying equity does not pay any dividend. This happens when a mildly negative initial interest rate causes the underlying equity's drift to be negative as well, pushing the underlying equity towards the out of the money region. In this case, immediate exercise turns out to be optimal as soon as the option is sufficiently in the money. The critical equity price that triggers optimal early exercise is increasing with respect to the interest rate value, as the higher the interest rate, the higher the underlying equity drift, the lower the risk of ending up in the out of the money region for the call option, and thus the higher has to be the immediate payoff to be optimally exercised before maturity.

The remaining of the paper is organized as follows: in Section 2 we introduce the financial market and develop its lattice-based discretization, that we call *quadrinomial tree*. In Section 3 we deal with American put and call equity options in our stochastic interest rate enviroment, charachterizing their optimal exercise policies and the main analytical features of their free boundaries. We also provide numerical pricing results for the discretized market via our quadrinomial tree, showing the pricing differences from the standard constant interest rate case, and providing a graphical characterization of the free boundaries that confirm their analytical features in the continuous-time setting. Section 4 concludes. All proofs are in the Appendix.

2.2 The market and the Quadrinomial Tree

2.2.1 The Assets in the Market

Consider a stylized financial market in a continuous time framework with investment horizon $T > 0$. A risky security $S(t)$ is traded. Following the seminal work of Vasicek (1977), we assume a mean-reverting stochastic process for the prevailing short term interest rate on the market $r(t)$. We allow for a non zero correlation between the innovations of S and r. A market player can invest in the short-term interest rate, which is locally risk-free, through the money market account $B(t)$. The dynamics of the risky equity price, of the short-term interest rate and of the money market

[4]A negative dividend can be interpreted as a storage cost for commodities (e.g. gold or silver) or as the result of the interplay of domestic and foreign interest rates when evaluating options on foreign equities (see Battauz et al. (2018), (2015)).

account under the risk-neutral measure $\mathbb{Q}$ are:

$$
\begin{cases}
\dfrac{\mathrm{d}S(t)}{S(t)} = (r(t)-q)\mathrm{d}t + \sigma_S \mathrm{d}W_S^{\mathbb{Q}}(t) \\
\mathrm{d}r(t) = \kappa\left(\theta - r(t)\right)\mathrm{d}t + \sigma_r \mathrm{d}W_r^{\mathbb{Q}}(t) \\
\mathrm{d}B(t) = r(t)B(t)\mathrm{d}t
\end{cases}
\tag{2.1}
$$

with $\langle \mathrm{d}W_S^{\mathbb{Q}}(t), \mathrm{d}W_r^{\mathbb{Q}}(t)\rangle = \rho \mathrm{d}t$ and given some initial conditions $S(0) = S_0$, $r(0) = r_0$ and $B(0) = 1$. The parameter q is the constant annual dividend rate of the equity, $\sigma_S > 0$ the volatility of the equity price, κ the speed of mean-reversion of the short-term interest rate, θ its long-run mean, $\sigma_r > 0$ the volatility of the short-term interest rate and $\rho \in [-1, 1]$ the correlation between the Brownian shocks on S and r.

The stochastic differential equations (SDEs) of System (2.1) can be rewritten equivalently in the following vectorial specification:

$$
\begin{cases}
\dfrac{\mathrm{d}S(t)}{S(t)} = \mu_S \mathrm{d}t + \nu_S \cdot \mathrm{d}W^{\mathbb{Q}}(t) \\
\mathrm{d}r(t) = \mu_r \mathrm{d}t + \nu_r \cdot \mathrm{d}W^{\mathbb{Q}}(t)
\end{cases}
\tag{2.2}
$$

where $\mu_S = (r(t) - q)$, $\mu_r = \kappa(\theta - r(t))$, $\nu_S = [\sigma_S \quad 0]$, $\nu_r = [\sigma_r \rho \quad \sigma_r\sqrt{1-\rho^2}]$, $W^{\mathbb{Q}}(t) = \left[W_1^{\mathbb{Q}}(t) \quad W_2^{\mathbb{Q}}(t)\right]'$ is a standard two-dimensional Brownian motion and $\cdot$ is the matrix product.

The explicit solution to the System (2.1) is

$$
\begin{cases}
S(t) = S_0 \exp\left[\displaystyle\int_0^t r(s)\mathrm{d}s - \left(q + \frac{\sigma_S^2}{2}\right)t + \sigma_S W_S(t)\right] \\
r(t) = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma_r \displaystyle\int_0^t e^{-\kappa(t-s)}\mathrm{d}W_r(s) \\
B(t) = \exp\left[\displaystyle\int_0^t r(s)\mathrm{d}s\right]
\end{cases}
\tag{2.3}
$$

The zero-coupon bond with maturity T pays 1 at its holder at T and its price at $t \in (0, T)$ is labelled with $p(t, T)$. By no arbitrage valuation, we have

$$
p(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T)}\middle| \mathcal{F}_t\right] = e^{A(t,T) - B(t,T)r(t)},
\tag{2.4}
$$

where the deterministic functions $A(t, T)$ and $B(t, T)$ are defined in Section 3.2.1 of Brigo and Mercurio (2007).

In this fairly general pricing framework, the price of European put and call options on S can be derived in closed formulae. Indeed, exploiting a change of numèraire as described in Geman et al. (1995), it is possible to obtain the following:

Proposition 2.1 ***(Value of the European put/call equity option)*** *In the financial market specified in* (2.1)*, the price at* $t \in [0, T]$ *of an European put option on* S *with strike* K *is equal to*

$$\pi_E^{put}(t, S(t), r(t)) = Kp(t,T)N(-\tilde{d}_2) - S(t)e^{-q(T-t)}N(-\tilde{d}_1) \tag{2.5}$$

with[5]*:*

$$\begin{aligned}
\tilde{d}_1 &= \frac{1}{\sqrt{\Sigma_{t,T}^2}}\left(\ln\frac{S(t)}{Kp(t,T)} + \frac{1}{2}\Sigma_{t,T}^2 - q(T-t)\right), \\
\tilde{d}_2 &= \tilde{d}_1 - \sqrt{\Sigma_{t,T}^2}, \\
\Sigma_{t,T}^2 &= \sigma_S^2(T-t) + 2\sigma_S\sigma_r\rho\left(\frac{-1 + e^{-\kappa(T-t)} + \kappa(T-t)}{k^2}\right) + \\
&\qquad -\sigma_r^2\left(\frac{3 + e^{-2\kappa(T-t)} - 4e^{-\kappa(T-t)} - 2\kappa(T-t)}{2k^3}\right).
\end{aligned}$$

The price at $t \in [0, T]$ *of an European call option on* S *with strike* K *is equal to*

$$\pi_E^{call}(t, S(t), r(t)) = S(t)e^{-q(T-t)}N(\tilde{d}_1) - Kp(t,T)N(\tilde{d}_2). \tag{2.6}$$

Proof See Appendix 2.B. ■

Within this market model, Bernard et al. (2008) price European barrier options by properly approximating the hitting time of the equity price of the exogenously given barrier. Unfortunately the same approach does not work when the derivative is of American style, as the barrier has to be endogenously determined. For this reason we develop in the next session the lattice discretization of our market that will allow us to compute the American free boundaries.

2.2.2 The Quadrinomial Tree

In their seminal work, Cox et al. (1979) show how to discretize a lognormal risky security and how to easily exploit such a binomial discretization in order to evaluate derivatives written on the primary asset. Embedding this geometric Brownian motion case into a more general class of diffusion processes, Nelson and Ramaswamy (1990) propose a one-dimensional scheme to properly define a binomial process that approximates a one-dimensional diffusion process. They do so by matching the diffunsion's instantaneous drift and its variance and imposing a recombining structure to their discretized process.

[5]Notice that the current value of the interest rate $r(t)$ enters the current price of the zero-coupon bond $p(t, T)$ in $\tilde{d}_1$ and $\tilde{d}_2$.

The discretization via a tree/lattice structure of correlated processes, possibly of different kind, is more challenging. Gamba and Trigeorgis (2007) model two or more correlated geometric Brownian motion representing the price processes of different risky assets exploiting a log-transformation of the processes first and then an orthogonal decomposition of the shocks. In this way they are able to efficiently price derivatives on five correlated assets. Moving away from lognormality, Hahn and Dyer (2008) construct a quadrinomial lattice to approximate two mean-reverting processes in order to model two correlated one-factor commodity prices and evaluate derivatives on them.

We propose here a quadrinomial tree to jointly model a mean-reverting process for the short term interest rate as suggested first by Vasicek (1977) and the process for the risky equity's price with constant volatility and the drift that embeds the stochastic interest rate as in Equation (2.1). Non constant short-term interest rates are surely more suitable from an option pricing perspective and an Ornstein–Uhlenbeck process enables us to investigate some interesting features of options when the discount rate becomes slightly negative, as documented in the Euro zone in recent years.

We first show how to build the discretization of the processes $(S(t), r(t))$ described by (2.1) and then we address the convergence issue of the discretization itself.
We apply Itô's Lemma to $Y(t) := \ln(S(t))$ and we get

$$\begin{cases} \mathrm{d}Y(t) = \mu_Y \mathrm{d}t + \sigma_S \mathrm{d}W_S(t) \\ \mathrm{d}r(t) = \mu_r \mathrm{d}t + \sigma_r \mathrm{d}W_r(t) \end{cases} \tag{2.7}$$

where $\mu_Y := \left(r(t) - q - \frac{\sigma_S^2}{2} \right)$ and $\mu_r := \kappa(\theta - r(t))$. Again, the vectorial version of (2.2) is:

$$\mathrm{d}\begin{bmatrix} Y(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \left(r(t) - q - \frac{\sigma_S^2}{2} \right) \\ \kappa(\theta - r(t)) \end{bmatrix} \mathrm{d}t + \begin{bmatrix} \sigma_S & 0 \\ \sigma_r \rho & \sigma_r \sqrt{1-\rho^2} \end{bmatrix} \cdot \mathrm{d}W(t). \tag{2.8}$$

We refer here to the general technique of Section 11.3 of Stroock and Varadhan (1997) exploiting the very convenient notation introduced in Section 3.2.1 of Prigent (2003). For the ease of the reader we recall here their template. Consider the following bivariate SDE:

$$\mathrm{d}X(t) = \mu(x,t)\mathrm{d}t + \sigma(x,t) \cdot \mathrm{d}W(t) \tag{2.9}$$

where $X(t)_{t \geq 0} = (Y(t), r(t))_{t \geq 0}$, $W(t)$ is a standard two-dimensional Brownian motion, $\mu(x,t) : (\mathbb{R} \times \mathbb{R}^+) \times \mathbb{R}^+ \to \mathbb{R}^2$, $\sigma(x,t) : (\mathbb{R} \times \mathbb{R}^+) \times \mathbb{R}^+ \to \mathbb{R}^{2\times 2}$ and an initial condition $X(0) = (x_0, r_0)$ is given.
Consider a discrete uniform partition of the time interval $[0,T]$ like $\left\{ i\frac{T}{n}, i = 1, \ldots, n \right\}$ and define $\Delta t := \frac{T}{n}$. For each n consider a bivariate stochastic process $\{X_n\}$ on $[0,T]$ which is constant

between the nodes of the partition. At each node, both of the components of X_n jump up (or down) a certain distance with a certain probability. The sizes of the jumps and the probabilities are allowed to be time-dependent and state-contingent. Since after any jump each component of X_n can assume two new different values, there will be globally four possible outcomes after each jump. For sake of clarity, fix n and consider the generic i-th step of the bivariate discrete process $X_i = (Y_i, r_i)$. In the following step, the process can assume only one of the following four values:

$$(Y_{i+1}, r_{i+1}) = \begin{cases} (Y_i + \Delta Y^+, r_i + \Delta r^+) & \text{with probability } q_{uu} \\ (Y_i + \Delta Y^+, r_i + \Delta r^-) & \text{with probability } q_{ud} \\ (Y_i + \Delta Y^-, r_i + \Delta r^+) & \text{with probability } q_{du} \\ (Y_i + \Delta Y^-, r_i + \Delta r^-) & \text{with probability } q_{dd} \end{cases} \tag{2.10}$$

where $\Delta Y^\pm, \Delta r^\pm$ are the jumping increments and the four transition probabilities are both time-dependent and state-contingent. Figure (2.1) provides a graphical intuition for the bivariate binomial discretization over one step.

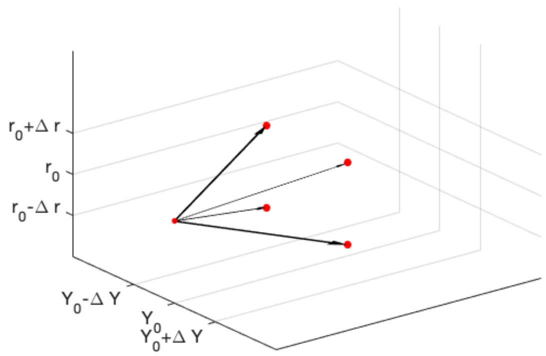

Figure 2.1: One step of the bivariate binomial discretization.

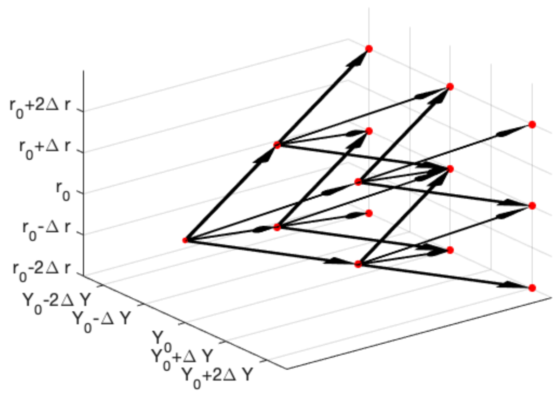

Figure 2.2: Two steps of the bivariate binomial discretization.

Globally, there are 8 parameters to pin down at each step: $\Delta Y^\pm$, $\Delta r^\pm$, q_{uu}, q_{ud}, q_{du} and q_{dd}. In order to obtain a discretization that converges in distribution to the solution of (2.9), we need to match the first two moments of $Y(t)$ and $r(t)$ as well as their cross variation. In this way we impose five conditions on the eight parameters we need to determine. One more constraint has to be imposed on the four transition probabilities that need to sum up to one. Finally, we may want to impose a recombining structure to our quadrinomial tree in order to preserve tractability. Setting $\Delta Y^- = -\Delta Y^+ := \Delta Y$ and $\Delta r^- = -\Delta r^+ := \Delta r$ makes the number of different outcomes of our discretization grow quadratically (and non exponentially) in the number of steps. Figure (2.2) provides a graphical intuition of this trick: starting from (Y_0, r_0), after two steps the bivariate binomial process may assume nine possible values, namely all the possible ordered couples of $\{Y_0 - 2\Delta Y, Y_0, Y_0 + 2\Delta Y\}$ and of $\{r_0 - 2\Delta r, r_0, r_0 + 2\Delta r\}$. Thus, for a generic number of time steps n, the final possible outcomes of the discretization are $(n+1)^2$ rather than 2^{n+1}, the number of possible outcomes along a non recombining tree.

We now derive the explicit expressions of the increments and of the transition probabilities of our discretization for the bivariate SDEs (2.2).

Matching the first two moments of $Y(t)$ and $S(t)$ as well as their cross-variation, neglecting the Δt-second order terms, imposing the proper constraint on the probabilities and imposing a recombining tree as explained above lead to the following system of eight equations in eight unknowns:

$$\begin{cases} \mathbb{E}_t[\Delta Y] = & (q_{uu} + q_{ud})\Delta Y^+ + (q_{du} + q_{dd})\Delta Y^- \stackrel{!}{=} \mu_Y \Delta t \\ \mathbb{E}_t[\Delta r] = & (q_{uu} + q_{du})\Delta r^+ + (q_{ud} + q_{dd})\Delta r^- \stackrel{!}{=} \mu_r \Delta t \\ \mathbb{E}_t[\Delta Y^2] = & (q_{uu} + q_{ud})(\Delta Y^+)^2 + (q_{du} + q_{dd})(\Delta Y^-)^2 \stackrel{!}{=} \sigma_S^2 \Delta t \\ \mathbb{E}_t[\Delta r^2] = & (q_{uu} + q_{du})(\Delta r^+)^2 + (q_{ud} + q_{dd})(\Delta r^-)^2 \stackrel{!}{=} \sigma_r^2 \Delta t \\ \mathbb{E}_t[\Delta Y \Delta r] = & q_{uu}\Delta Y^+\Delta r^+ + q_{ud}\Delta Y^+\Delta r^- + \\ & +q_{du}\Delta Y^-\Delta r^+ + q_{dd}\Delta Y^-\Delta r^- \stackrel{!}{=} \rho\sigma_S\sigma_r\Delta t \\ & q_{uu} + q_{ud} + q_{du} + q_{dd} \stackrel{!}{=} 1 \\ & \Delta Y^+ \stackrel{!}{=} -\Delta Y^- \\ & \Delta r^+ \stackrel{!}{=} -\Delta r^- \end{cases}$$

Imposing $\Delta Y^+ > \Delta Y^-$ and $\Delta r^+ > \Delta r^-$ we get:

$$\begin{aligned} \Delta Y^+ &= \sigma_S\sqrt{\Delta t} = -\Delta Y^- := \Delta Y \\ \Delta r^+ &= \sigma_r\sqrt{\Delta t} = -\Delta r^- := \Delta r \end{aligned} \tag{2.11}$$

$$\begin{aligned} q_{uu} &= \frac{\mu_Y\mu_r\Delta t + \mu_Y\Delta r + \mu_r\Delta Y + (1+\rho)\sigma_r\sigma_S}{4\sigma_r\sigma_S} \\ q_{ud} &= \frac{-\mu_Y\mu_r\Delta t + \mu_Y\Delta r - \mu_r\Delta Y + (1-\rho)\sigma_r\sigma_S}{4\sigma_r\sigma_S} \\ q_{du} &= \frac{-\mu_Y\mu_r\Delta t - \mu_Y\Delta r + \mu_r\Delta Y + (1-\rho)\sigma_r\sigma_S}{4\sigma_r\sigma_S} \\ q_{dd} &= \frac{\mu_Y\mu_r\Delta t - \mu_Y\Delta r - \mu_r\Delta Y + (1+\rho)\sigma_r\sigma_S}{4\sigma_r\sigma_S}. \end{aligned} \tag{2.12}$$

As noted in Nelson and Ramaswamy (1990), the four transition probabilities are not necessarily positive. In the limit, namely as $\Delta t \to 0$, we have $\Delta Y, \Delta r \to 0$ and, therefore, $q_{uu}, q_{dd} \to \frac{(1+\rho)}{4} > 0$ and $q_{ud}, q_{du} \to \frac{(1-\rho)}{4} > 0$. For $\Delta t > 0$, some of the four probabilities in (2.12) may become non-positive. In Appendix 2.A we derive conditions leading to four non-negative probabilities in (2.12). Summing up, negative probabilities arise for very extreme values of r, namely for values very far from the interest rate's long term mean. If r is way below zero and far from θ, further downward movements might are unlikely. If such an already unlikely movement is against the sign of the correlation between r and S, a negative probability might arise. Nevertheless, the model remains still arbitrage-free as this is a minor flaw of the numerical procedure.

Once we obtained a discretization of (2.2), we can map back the rate of return $Y(t)$ to the level $S(t)$ of the equity. In this way we have a lattice discretization of the solution of (2.2) and we name this discretization *quadrinomial tree*.
The following theorem shows that our quadrinomial tree converges in distribution to the solution of (2.2).

Theorem 2.2 ***(Convergence of the quadrinomial tree)*** *The bivariate discrete process* $(X_i)_i$ *defined in (2.10) with the parameters in (2.11) and (2.12) converges in distribution to* $X(t) = (Y(t), r(t))$.

Proof See Appendix 2.B. ∎

2.3 American Options

In this section we focus on American equity put (resp. call) options, whose final payoff is $\varphi(S) := (K-S)^+$ (resp. $\varphi(S) := (S-K)^+$). The value at $t \leq T$ of the American equity option with maturity T is:

$$\begin{aligned} V(t) &= ess \sup_{t\leq\tau\leq T} \mathbb{E}^{\mathbb{Q}}\left[\frac{B(t)}{B(\tau)}\varphi(S(\tau))\middle| \mathcal{F}_t\right] \\ &= ess \sup_{t\leq\tau\leq T} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^\tau r(s)\mathrm{d}s}\varphi(S(\tau))\middle| \mathcal{F}_t\right] \end{aligned} \tag{2.13}$$

where τ ranges among all possible stopping times (see for instance Chapter 21 in Björk (2009)). It is well known that Equation (2.13) admits no explicit formulation even in the standard Black and Scholes market. In our case, since r is stochastic and not independent of S, we cannot split the conditional expected value in (2.13) into two simpler separate ones. Nevertheless, the following proposition shows that the value of an American option $V(t)$ can be expressed as a deterministic function of time t (or, equivalently, of time to maturity $T-t$) and of the current value of both the underlying asset $S = S(t)$ and the short term interest rate $r = r(t)$. Moreover, this deterministic function inherits the same monotonicity properties with respect to t and S variables as in the constant interest rate enviroment. We also prove that the American equity put option is decreasing with respect to the current value of the interest rate r, whereas the American equity call option is increasing with respect to r. An increase in r has a direct effect on American equity options via the discounting of future cashflows, that becomes more severe. It has also an indirect effect, channelled through the equity drift, that increases if r increases. For an American equity put option this implies that the likelihood of lower payoffs increases. Thus an increase in r diminishes the value of an American equity put option. On the contrary, for an American equity call option, the drift increase determined by the increase of r pushes the underlying equity towards higher payoffs' regions, thus potentially increasing the call option value. We show that this positive effect prevails over the negative effect of the increased discounting, and the American call option is actually increasing with respect to r. This result is achieved by a change of the numeraire for the call option, that allows to disentangle the opposite effects of the interest rate's increase on the discounting and the call payoffs' likelihood.

Proposition 2.3 ***(Value of the American option as a deterministic function)*** *In the market described by* (2.1)*, the value of an American put option on S* (2.13) *is of the form:*

$$V(t) = F(t, S(t), r(t))$$

with $F : [0,T] \times \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+$ given by:

$$F(t,S,r) = \sup_{0\le\eta\le T-t} \mathbb{E}^{\mathbb{Q}} \left[\exp\left(-\int_0^\eta r(s)\mathrm{d}s \right) \cdot \right.$$

$$\left. \cdot \varphi\left(S\exp\left(\int_0^\eta r(s)\mathrm{d}s - \left(q + \frac{1}{2}\sigma_S^2 \right)\eta + \sigma_S W_S(\eta) \right) \right) \right]. \tag{2.14}$$

The function F is decreasing with respect to time t, convex with respect to S and increasing (resp. decreasing) in the call (resp. put) case. Moreover, F is increasing (resp. decreasing) in the call (resp. put) case with respect to r.

Proof See Appendix 2.B. ■

As the American equity option value is a deterministic function of (t, S, r), at each $t \in [0,T]$, the plane $(S, r) \in \mathbb{R}^+ \times \mathbb{R}$ can be divided into two complementary regions:

- the continuation region $CR(t) = \{(S,r) \in \mathbb{R}^+ \times \mathbb{R} : F(t,S,r) > \varphi(S)\}$, the set of couples (S, r) where it is optimal to continue the option at t; the r-section of the continuation region at t is $CR_r(t) = \{S \in \mathbb{R}^+ : F(t,S,r) > \varphi(S)\}$;

- the early exercise region $EER(t) = \{(S,r) \in \mathbb{R}^+ \times \mathbb{R} : F(t,S,r) = \varphi(S)\}$, the set of couples (S,r) where it is optimal to exercise the option at t; the r-section of the early exercise region at t is $EER_r(t) = \{S \in \mathbb{R}^+ : F(t,S,r) = \varphi(S)\}$.

The boundary separating the continuation region and the early exercise region as t varies in $[0,T]$ is a surface called *free boundary* in the three-dimensional space (t,S,r). In Theorem 2.5 we describe the main features of the free boundary surface, that can be single (the standard case) or double, generalizing the results of Battauz et al. (2015). In the constant interest rate enviroment, the American equity option value depends only on time t and the equity value S. Thus, the early exercise region and the continuation region are separated by a one-dimensional free boundary, called *critical price*. When the interest rate is positive, there is always a single (if any) free boundary. Consider the put option, in the limiting case where $S = 0$. Its value is $\sup_{0\le\eta\le T-t} \mathbb{E}^{\mathbb{Q}}\left[e^{-r\eta}(K-0)^+\right] = K$. Thus the value of the put option and its immediate payoff coincide at $S = 0$ for any t. Convexity and value dominance of the option value imply that, if early exercise is optimal at t for some underlying value S, then it is optimal to exercise as well for all values of the underlying that are smaller than S. The early exercise region is the area bounded from above by the single critical price $t \mapsto \overline{S}^*(t)$ and is therefore downward connected (see Figure 1 in Battauz et al. (2015)). On the contrary, if $r < 0$, then the American put value when $S = 0$ is $\sup_{0\le\eta\le T-t} \mathbb{E}^{\mathbb{Q}}\left[e^{-r\eta}(K-0)^+\right] = Ke^{-r(T-t)} > K$. Thus if early exercise is optimal at t, the exercise region stops strictly above $S = 0$. In a neighborhood of $S = 0$ a deeply in the money continuation region appears, bounded from above by the new lower critical price $t \mapsto \underline{S}^*(t)$ (see Figure 2 in Battauz et al. (2015) and the related comments for a discussion on the economic intuition behind this result). The exercise region for the American put option in this case is no more downward connected.

A similar reasoning applies also in our stochastic interest rate setting. If $S = 0$ the value of the American put option defined in Equation (2.14) becomes $F(t,0,r) = \sup_{0\le\eta\le T-t} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^\eta r(s)\mathrm{d}s\right)\cdot K\right]$. Suppose that there exist some deterministic η such that $p(0,\eta) > 1$. Then $F(t,0,r) \ge K \cdot p(0,\eta) > K$. In this case, if early exercise is optimal at t, r for some value of S, then the early exercise region will be bounded from below by a strictly positive equity value. A non-standard continuation region then appears when the put is most deeply in the money.

The following proposition formalizes this intuition for both American put and call options, and provides necessary conditions for the existence of optimal early exercise opportunities when the current interest rate value determines the existence of a zero-coupon-bond price greater than 1. This is very likely to occur when the current interest rate value is non-positive.

Proposition 2.4 ***(Asymptotic necessary conditions for the existence of optimal early***

exercise opportunities*) In the market described by* (2.1)*, at any point in time t and given the current value of the interest rate* $r(t) = r$*, suppose that*

[NC0] $r\alpha - \theta(\alpha + (T-t)) > 0$ *with* $\alpha = \frac{e^{-\kappa(T-t)}-1}{\kappa} \leq 0$

Then the following are jointly necessary conditions for the existence of optimal exercise opportuinites at t, for $T - t \approx 0$*:*

[NC1] the dividend yield is non positive, $q \leq 0$*;*

[NC2] for some S*,* $\pi_E(t, S, r) = \varphi(S)$*, where* $\pi_E(t, S, r)$ *is the value of the European put (resp. call) option defined in Proposition 2.1.*

Proof See Appendix 2.B. ■

The condition *[NC0]* is very likely satisfied when $r < 0$, as the long-run mean of the interest rate θ is commonly assumed to be positive. *[NC0]* is equivalent to have a bump in the yield-curve, namely to have an instant in time in which the price or a zero-coupon bond is larger than one. If that is the case, it might be optimal to exercise the American option at that moment to gain from the temporary increase in the value of money. The formal argument is described in the proof of the Proposition, in Appendix 2.B. *[NC1]* ensures that the discounted price of the risky security is not a supermartingale. If this was the case, we show in the proof that, under condition *[NC0]*, this would lead automatically to optimal exercise of the American put option at maturity only. For the American put option, if early exercise is optimal under condition *[NC0]*, then EER_r, the early exercise region section at r, is bounded by below by a strictly positive (non standard) lower boundary. A similar reasoning works for American equity call options. We remark that our results cannot be obtained from standard symmetry results for American options (see Battauz et al. (2015) and the references therein) due to the stochasticity of our interest rates. In the standard Black-Scholes case, the American put-call symmetry swaps the constant interest rate with the constant dividend yield. Being our interest rate stochastic and our dividend yield constant, such symmetry result is not viable.

Under *[NC0]*, *[NC2]* ensures that the price of the European option $\pi_E(t, S, r)$ does not dominate the immediate payoff value. If this was the case, the American option would dominate the immediate payoff value as well, thus preventing the existence of optimal early exercise opportunities. Although the formal proof of the necessary conditions in Proposition 2.4 requires the time to maturity to be small enough, we show in the following section that actually those conditions correctly identify nodes on the tree in which a double continuation region appears along the whole lifetime of the option (see Figure 2.5).

In the following theorem we describe the main properties of the free boundary surface, under the assumption that the early exercise region is non-empty. We distinguish between the standard case of a non-negative interest rate and the case of a negative interest rate, where unusual optimal continuation policies may appear.

Theorem 2.5 ***(The free-boundary surface)***

1. *Suppose $r \geq 0$ and assume that $EER_r(\bar{t})$ is non-empty for some $\bar{t} \in (0,T)$.*
   For the American put option
   $$\overline{S}^*(t,r) = \sup\left\{S \geq 0 : F(t,S,r) = \varphi(S)\right\} \tag{2.15}$$
   defines the (standard upper) free boundary and early exercise is optimal at any $t \geq \bar{t}$ for $S(t)$ and $r(t) = r$ if $S(t) \leq \overline{S}^*(t,r)$. The free boundary $\overline{S}^*(t,r)$ is increasing with respect to $t \geq \bar{t}$ and $r \geq 0$.

   For the American call option
   $$\underline{S}^*(t,r) = \inf\left\{S \geq 0 : F(t,S,r) = \varphi(S)\right\} \tag{2.16}$$
   defines the (standard lower) free boundary and early exercise is optimal at any $t \geq \bar{t}$ for $S(t)$ and $r(t) = r$ if $S(t) \geq \underline{S}^*(t,r)$. The free boundary $\underline{S}^*(t,r)$ is decreasing with respect to $t \geq \bar{t}$ and increasing with respect to $r \geq 0$.

2. *Suppose $r < 0$, $T - \bar{t} \approx 0$ and that the necessary conditions of Propositions 2.4 are satisfied with $q < 0$ and assume that $EER_r(\bar{t})$ is non-empty. Then the segment with extremes $[\underline{S}^*(t,r), \overline{S}^*(t,r)]$ (see Equations (2.15), (2.16)) is non-empty for any $t \in \left[\bar{t}, T\right]$. The option is optimally exercised at any $t \geq \bar{t}$ for $S(t)$ and $r(t) = r$ whenever $S(t) \in \left[\underline{S}^*(t,r), \overline{S}^*(t,r)\right]$. The lower free boundary, $\underline{S}^*(t,r)$, is decreasing with respect to t and the upper free boundary $\overline{S}^*(t,r)$ is increasing with respect to t for any $t \geq \bar{t}$.*
   For the American put
   $$\frac{rK}{q} \leq \underline{S}^*(t,r) < \overline{S}^*(t,r) \leq K.$$
   Their limits at maturity are $\lim_{t\to T} \overline{S}^*(t,r) = K = \overline{S}^*(T,r)$ and $\underline{S}^*(T^-,r) = \lim_{t\to T} \underline{S}^*(t,r) = \frac{rK}{q} > \underline{S}^*(T,r) = 0$. The lower free boundary, $\underline{S}^*(t,r)$, is decreasing with respect to r and the upper free boundary $\overline{S}^*(t,r)$ is increasing with respect to r.
   For the American call
   $$K \leq \underline{S}^*(t,r) < \overline{S}^*(t,r) \leq \frac{rK}{q}.$$
   Their limits at maturity are $\lim_{t\to T} \underline{S}^*(t,r) = K = \underline{S}^*(T,r)$ and $\overline{S}^*(T^-,r) = \lim_{t\to T} \overline{S}^*(t,r) = \frac{rK}{q} < \overline{S}^*(T,r) = +\infty$. The lower free boundary, $\underline{S}^*(t,r)$, is increasing with respect to r and the upper free boundary $\overline{S}^*(t,r)$ is decreasing with respect to r.

3. *Suppose $r < 0$ and $q = 0$. Then the early exercise region for the American put option at t is empty.*
   For the American call, suppose $EER_r(\bar{t})$ is non-empty for some $\bar{t} \in (0, T)$. Then early exercise is optimal at any $t \geq \bar{t}$ for $S(t)$ and $r(t) = r$ if $S(t) \geq \underline{S}^*(t, r)$ (see Equation (2.16)). The free boundary $\underline{S}^*(t, r)$ is decreasing with respect to $t \geq \bar{t}$ and increasing with respect to $r \geq 0$

Proof See Appendix 2.B. ■

An interesting extension

The quadrinomial tree proposed in the present Chapter can be also used for the pricing of American securities written on foreign securities. These kind of options are called *quanto options*.
Assume that there are two integrated, efficient, arbitrage-free and complete markets: a domestic and a foreign market. Two assets are traded in each market: a locally risk free bond B, and a risky security S. Assume that the prevailing risk-free rate of the foreign market, r_f, is constant whereas the one in the domestic market, $r_d(t)$, follows a mean-reverting stochastic process as in (2.1). Consequently, the SDEs (and the ODEs) that characterized this domestic-foreign market are

$$\begin{cases} \dfrac{\mathrm{d}S_d(t)}{S_d(t)} = (r_d(t) - q_d)\mathrm{d}t + \sigma_d \mathrm{d}W_d^{\mathbb{Q}} \\ \dfrac{\mathrm{d}S_f(t)}{S_f(t)} = (r_f - q_f)\mathrm{d}t + \sigma_f \mathrm{d}W_f^{\mathbb{Q}} \\ \dfrac{\mathrm{d}B_d(t)}{B_d(t)} = r_d(t)\mathrm{d}t \\ \dfrac{\mathrm{d}B_f(t)}{B_f(t)} = r_f \mathrm{d}t \\ \mathrm{d}r_d(t) = \kappa\left(\theta - r_d(t)\right)\mathrm{d}t + \sigma_r \mathrm{d}W_r^{\mathbb{Q}} \end{cases} \tag{2.17}$$

with $\langle W_f^{\mathbb{Q}}, \mathrm{d}W_r^{\mathbb{Q}} \rangle = \rho \mathrm{d}t$. Furthermore, assume that the exchange rate between the two currency is X, where 1 unit of the domestic currency is equivalent to X units of the foreign security. Given the dynamics in (2.17), it is natural to assume that X solves the following SDEs

$$\frac{\mathrm{d}X(t)}{X(t)} = \mu_X^{\mathbb{Q}}(t)\mathrm{d}t + \sigma_X \mathrm{d}W^{\mathbb{Q}}(t)$$

where $\sigma_X \in \mathbb{R}$ and $\mu_X^{\mathbb{Q}} = r_d(t) - r_f$ due to the interest rate parity.
Let $\varphi(S_f)$ be the payoff of an American option with maturity T written of S_f and traded in the domestic market. Its price at inception for an investor in the domestic market is

$$V(0) = ess \sup_{\tau \in [0,T]} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^\tau r_d(s)\mathrm{d}s} \varphi(S_f(\tau)) \right].$$

Nevertheless, S_f must be express in the domestic currency. Computing the stochastic differential of $S_f(t)X(t)$ allows us to determine the dynamics of the foreign risky security with respect to the domestic investor as

$$\frac{\mathrm{d}S_f(t)}{S_f(t)} = (r_d(t) - \tilde{q}_f)\mathrm{d}t + \left(\sigma_f + \sigma_X\right)\mathrm{d}W_f^{\mathbb{Q}}$$

where $\tilde{q}_f = r_f - q_f - \sigma_X\sigma_f$ is an artificial dividend yield that might well take negative values. Therefore, all of the results of the previous Section can be exploited also in this framework and a double continuation region can arise when valuing American quanto options.

2.3.1 Numerical examples

We now present and describe three illustrative numerical examples that show the optimal exercise strategies and the possible characterizations of the continuation region for the American put and call options in the market described by (2.1), highlighting the free boundary's features derived in Theorem (2.5).
We exploit our quadrinomial tree to evaluate American options by backward induction. Once the whole quadrinomial tree, namely all the couples (S, r) and the related transition probabilities, have been generated, we start from the values of the state variables S and r at maturity T. At maturity, the American option is exercised in all the nodes in which it is in the money; the resulting payoff is the value of the American option at T. At any other generic instant $t \in \{0, \Delta t, 2\Delta t, \ldots, T - \Delta t\}$, and for any couple $(S(t), r(t))$, we compute the immediate payoff $\varphi(S)$ and we compare it to the continuation value of the option. The continuation value is obtained as the discounted (by the current realization of $r(t)$) expected value (according the transition probabilities computed at $(S(t), r(t))$) of the four values of the American option at $t + \Delta t$ connected on the tree to the current node. From the comparison between the immediate exercise and the continuation value, we get the value of the American option in the node $(S(t), r(t))$. Going backward, we finally get the price of the American option at $t = 0$.
Theorem 2.2 showed that the quadrinomial tree we proposed converges in distribution to the bivariate process that solves (2.1), as the time step shrinks. Mulinacci and Pratelli (1998) prove that the convergence in distribution of the lattice-based approximation of the underlying state variables implies that the price of the American option evaluated according to the backward procedure described above converges to its theoretical value given by (2.13).

In all of the three following examples the parameters are: $T = 1$, $n = 125$, $S_0 = K = 1$, $\sigma_S = 0.15$, $r_0 = 0$, $\theta = 0.02$, $\kappa = 1$, $\sigma_r = 0.01$ and $\rho = 0.05$. The dividend yield q of the equity is the only parameter that varies across the examples: in the first one we set $q = 0$, in the second $q = 0.02$

and $q = -0.02$ in the last one.
For each example we:

- compute the value at inception of the European counterpart π_E obtained both with the formula of Proposition 2.1 and along the quadrinomial tree (the values obtained in the two ways are indistinguishable);
- compute the value at inception of the American option π_A along the quadrinomial tree;
- compute the price of the American option, $\pi_A^{r_0}$, evaluated along the standard binomial tree of Cox et al. (1979) with a deterministic interest rate $r = r_0 = 0$[6]. Our aim is to quantify the error that an "unsophisticated" investor would make by evaluating American options within a flat term structure framework rather than within a fluctuating one;
- graphically show how the single, or double (if any), free boundaries look like in the space (t, S, r). These graphs characterize the optimal exercise policy: at any t, the investor should look at the current values of $(S(t), r(t))$;
- graphically highlight the nodes of the quadrinomial tree where the necessary conditions of Proposition 2.4 are satisfied.

We first show the numerical results for the American put option that are summed up in Table 2.1.

Figure	q	π_E	π_A	$\pi_A^{r_0}$	$\lvert\pi_A - \pi_A^{r_0}\rvert/\pi_A$
2.3	0%	5.620%	5.712%	5.979%	4.67%
2.4	2%	6.565%	6.570%	6.962%	5.96%
2.5	-2%	4.763%	5.030%	5.230%	3.97%

Table 2.1: Results from the three numerical examples for the American put option.

First example: $q = 0\%$. If the underlying pays no dividend and its volatility is reasonably small, the expected drift of S basically coincides with $r(t) = r$. This splits the domain of r in two complementary regions according to the sign of r, as can be seen in the right panel of Figure 2.3

[6] we also evaluate the American option with a deterministic interest rate equal to the expected value of r over the investment period; namely, we also set $r = \mathbb{E}^{\mathbb{Q}}\left[r(T)\right] = 1.26\%$. This exercise delivers qualitatively similar results. With respect to the last column of Table 2.1, the relative errors in this case are, respectively, 4.58%, 4.64%, 4.52%.

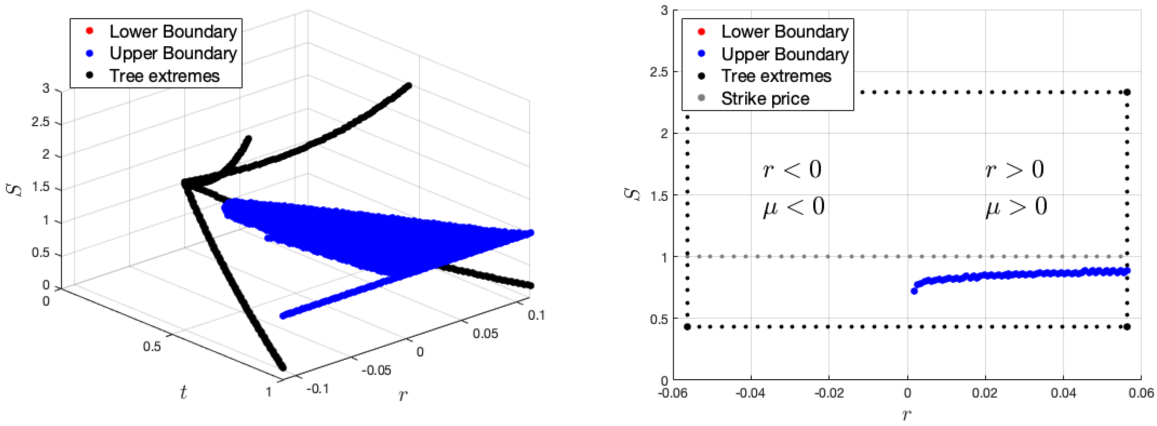

Figure 2.3: First example, American put: $q = 0\%$.

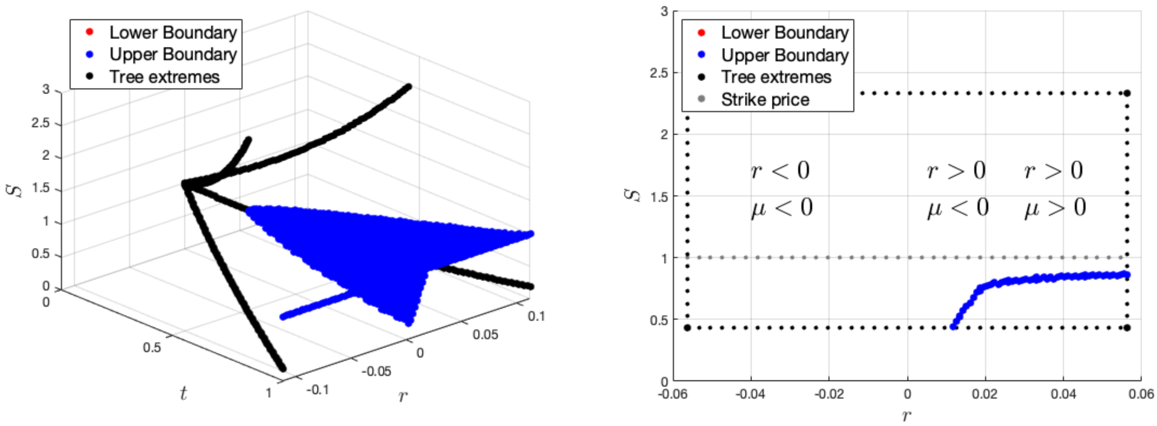

Figure 2.4: Second example, American put: $q = 2\%$.

(that displays the free boundary section at $t = \frac{T}{2}$). In the left region where r and $\mu = r - q - \frac{\sigma_S^2}{2}$ are both negative, the investor is willing to wait and postpone the exercise as much as possible in order to gain from both the negative discount rate and the implied expected depreciation of S. In the right region, on the contrary, where r and μ are both positive, we have the standard tradeoff between a positive discount rate (that makes the investor willing to exercise the option as soon as possible) and a negative expected drift of S (that makes the investor willing to wait for a larger payoff). This generates the standard upper boundary shown in the left panel of Figure 2.3. We notice that the standard upper boundary is increasing with respect to r. Indeed, early exercise is more profitable when r increases and S is likely to appreciate.

The investor who believes that the term structure is flat and evaluates the American put option with a constant discount rate equal to our r_0 makes a relative error almost equal to 5%. This

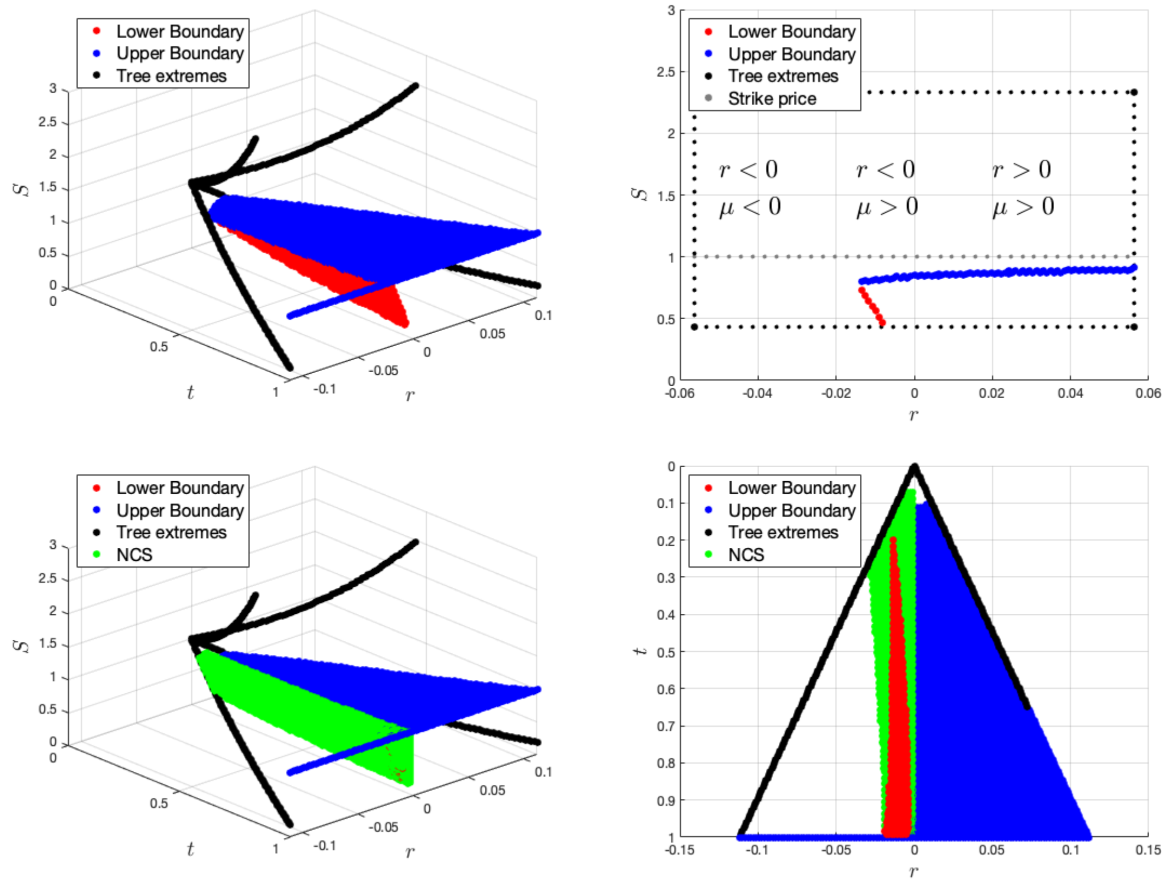

Figure 2.5: Third example, American put: $q = -2\%$. Green points in the bottom panels show the nodes of the quadrinomial tree in which necessary conditions *[NC0]*, *[NC1]* and *[NC2]* of Proposition 2.4 for a double continuation region hold simultaneously.

figure is economically significant as it is greater than the maximal error due to suboptimal exercise delay of the option as estimated[7] in Chockalingam and Feng (2015).

Second example: $q = 2\%$. If the underlying pays (positive) dividends, the drift of S is equal to r plus a negative quantity $(-q - \frac{\sigma_S^2}{2} < 0)$. This splits the domain of r into three complementary regions. The first one in which r and μ are both negative, the one in which r is positive but small so that μ is still negative, the last one in which r and μ are both positive. In the first one, the

[7] Our relative error of 4.67% in the first line of Table 2.1 corresponds to an absolute pricing error of 27.8 bps. This figure is indeed significant compared to the maximal error obtained in Figure 3 by Chockalingam and Feng (2015). In particular, Figure 3, second row, right column, in Chockalingam and Feng (2015), displays a pricing error of 4 bps, after a rescaling to unit moneyness and with volatility equal to 20%.

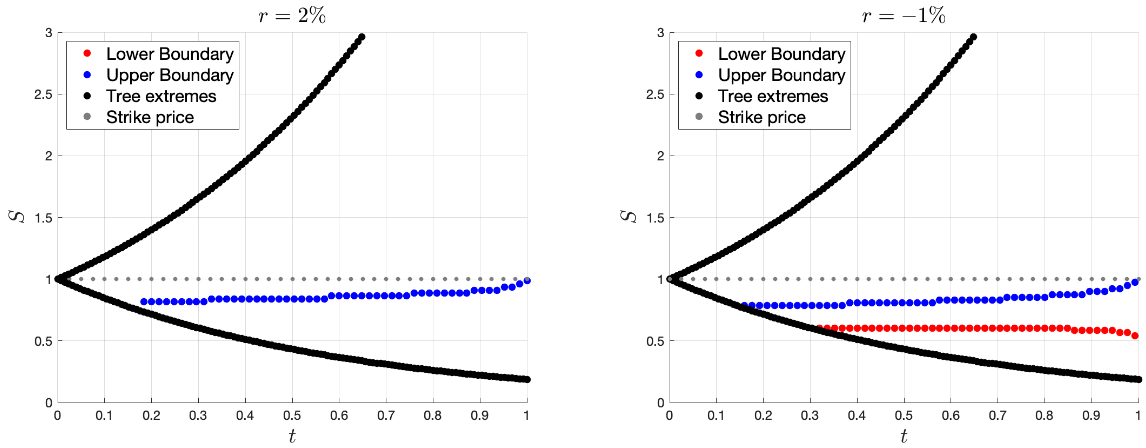

Figure 2.6: $r-$sections of free boundaries for the American put option. Left panel $r = 2\%$ and $q = 0\%$. Right panel $r = -1\%$ and $q = -2\%$.

option is optimally exercised at maturity, as before. In the middle region there is a new tradeoff: the investor would like to cash in as soon as possible due to $r > 0$ but the value of S is expected to decrease as $\mu < 0$. This allows for a standard upper boundary. The critical price below which the investor will exercise, though, becomes smaller as r approaches 0: as r decreases the threat of the positive discount rate weakens and, therefore, the investor would postpone the exercise unless the underlying reaches a very low level. In other words, if the discount is not that strong, the investor prefers to gain the relative high dividend yield keeping the asset as long as possible. In the last region, we find the standard behaviour already outlined in the first example.
The investor who believes that the term structure is flat and evaluates the American option with a constant interest rate makes here an even higher relative error than before (5.96%).

Third example: $q = -2\%$. In the case of negative dividends[8], the drift of S is equal to r plus a quantity which is now positive ($-q - \frac{\sigma_S^2}{2} > 0$). As a result, μ may be positive also when r is mildly negative. This splits again the domain of r into three complementary regions, as shown in the top-right panel of Figure 2.4: the one in which r and μ are both negative, the one in which r is negative but μ is positive and the last one in which r and μ are both positive. In the first region, the option is again optimally exercised at maturity as in the previous examples. In the middle section a double continuation region appears: this is the case in which the necessary conditions in Proposition 2.4 are satisfied as documented in the bottom panels of Figure 2.4. To the best of our knowledge, this is the first paper that documents the existence of a non standard double free boundary in a stochastic interest rates framework, generalizing the result obtained in the constant

[8]As previously discussed, negative dividends might model storage and insurance cost for commodities such as gold or domestic risk-neutral drifts of foreign equities in quanto options.

interest rates setting by Battauz et al. (2015). In the last region where both r and μ are positive, we find the standard behaviour already outlined in the first two examples.
We conclude our analysis of the American put option's free boundaries, by displaying in Figure 2.6 their time-dependence structure. In particular, we show that, for fixed values of r, the upper critical price of the American put is increasing with respect to time t whereas the lower critical price (if any) is decreasing, as already proved in Theorem 2.5 and documented in the constant interest rate framework by (Battauz et al., 2015).
In Appendix 2.C we also document the impact of the correlation on the American equity options' prices.

We now turn to the American call options. Numerical pricing results for the American call option in the same scenarios analysed above for the American put option are summed up in Table 2.2. We notice that in all cases the investor who believes that the term structure is flat and evaluates the American call option with a constant discount rate equal to our r_0 makes a relative error between 4% and 5%.
It is well known that American call options on non-paying dividend assets do not display any early exercise premium. This is true under usual market circumstances, i.e. when interest rates are non negative. In fact, in this case, the zero-coupon bonds of any maturity have initial prices that are smaller than one, i.e. $p(0,\tau) < 1$ for any $\tau \in [0,T]$. This implies that the option is optimally exercised at maturity only, as Jensen's inequality implies that

$$\mathbb{E}^{\mathbb{Q}}\left[\left(S(\tau)-K\right)^{+} e^{-\int_0^\tau r(s)\mathrm{d}s}\right] \geq \left(S(0)-Kp(0,\tau)\right)^{+} > \left(S(0)-K\right)^{+}.$$

The same holds true if S pays a negative dividend yield as $\mathbb{E}^{\mathbb{Q}}\left[S(\tau)e^{-\int_0^\tau r(s)\mathrm{d}s}\right] = S(0)e^{-q\tau} > S(0)$.
Within our framework, interest rates are not always positive and zero-coupon bonds may have initial prices larger than one. Thus, early exercise may be optimal under some circumstances as one can indeed see in the following first example.

Figure	q	π_E	π_A	$\pi_A^{r_0}$	$\lvert\pi_A - \pi_A^{r_0}\rvert/\pi_A$
2.7	0%	6.339%	6.339%	5.979%	5.69%
2.8	2%	5.314%	5.396%	5.163%	4.32%
2.9	-2%	7.511%	7.511%	7.102%	5.44%

Table 2.2: Results from the three numerical examples for the American call option.

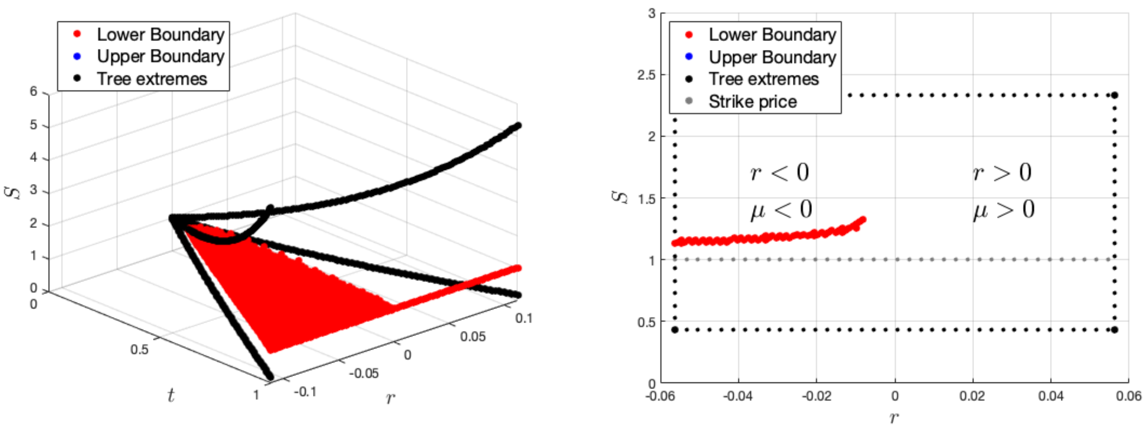

Figure 2.7: First example, American call: $q = 0\%$.

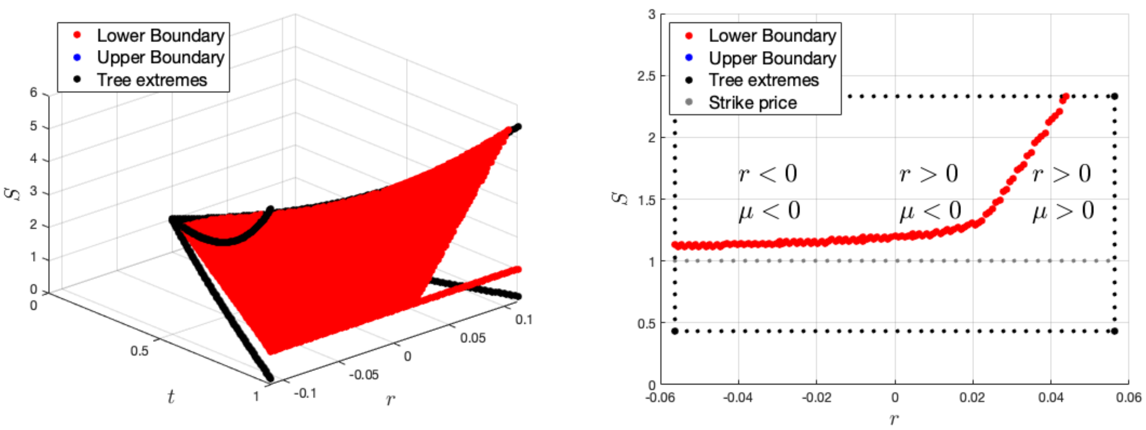

Figure 2.8: Second example, American call: $q = 2\%$.

First example: $q = 0\%$. As explained above, early exercise may be optimal in this case only if zero-coupon bonds display initial prices larger than one for some maturity. This is the case portraited in Figure 2.7, where a (standard lower) free boundary for the American call option is documented for initial interest rates values smaller than -1%. To our knowledge, this is the first paper that shows the existence of optimal early exercise opportunities for an American call option when the dividend yield is zero. We notice that the critical price, and thus the continuation region, is increasing in r, as the increasing drift μ of S pushes the option towards the in the money region. The impact of these optimal early exercise opportunities on the price of the option, however, is negligible because the risk-neutral probability of the equity price entering the early exercise region is quite small, as one can see from the first row of Table 2.2.

Second example: $q = 2\%$. When the dividend yield is positive, early exercises of the American

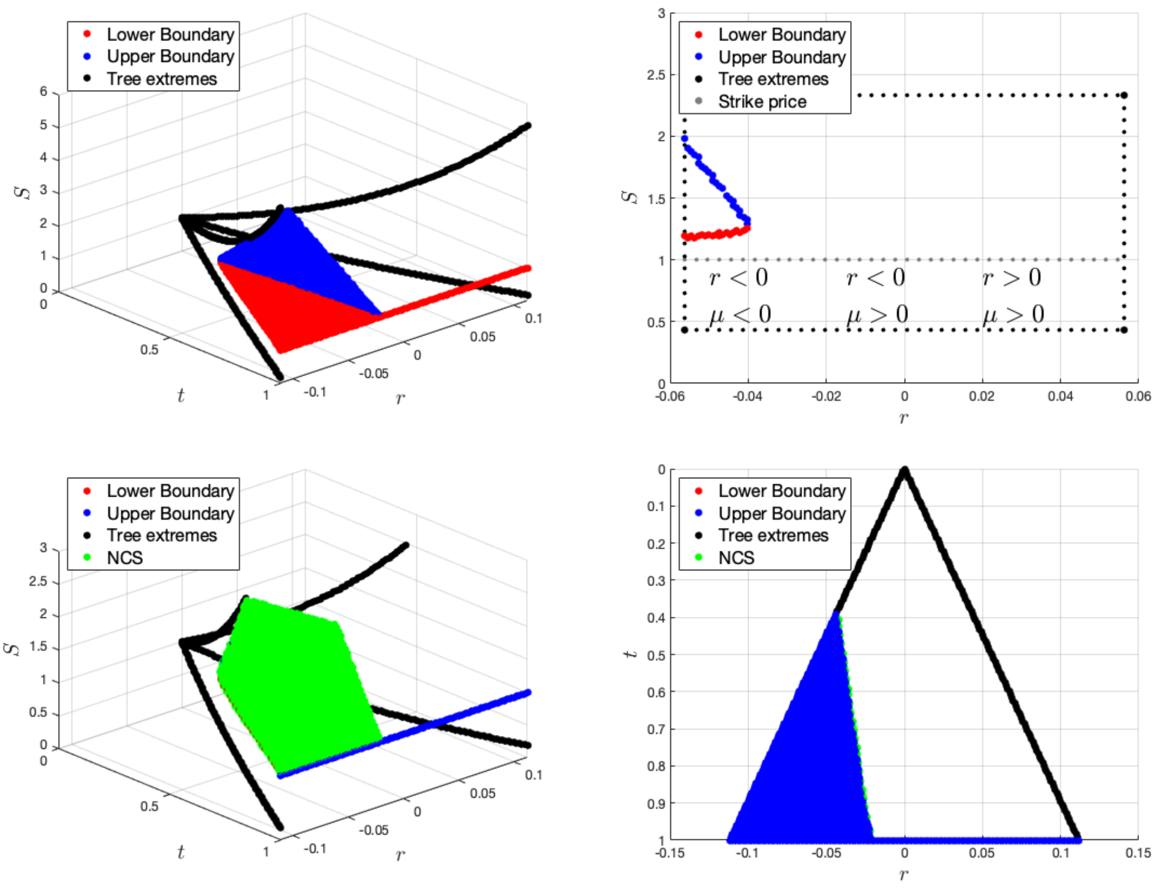

Figure 2.9: Third example, American call: $q = -2\%$. Green points in the bottom panels show the nodes of the quadrinomial tree in which necessary conditions *[NC0]*, *[NC1]* and *[NC2]* of Proposition 2.4 for a double continuation region hold simultaneously.

call option become profitable. In Figure 2.8 we document the existence of a (lower standard) free boundary that is again increasing in r. Interestingly, the slope of the free boundary becomes steeper when μ, the drift of S, turns positive, and the continuation region increases substantially as S is expected to appreciate. Consequently, early exercise in this case is optimal only if S is very deeply in the money.

Third example: $q = -2\%$. As already discussed for the American put option example, when the dividend yield is negative, the instantaneous drift of S, μ, is always positive but for very negative values of r. As a result, early exercise for the American call option is never optimal unless r is very negative. In this case, for negative values of r, a non standard early exercise region appears surrounded by two continuation regions (see the top panels of Figure 2.9). However, as in the first example with $q = 0\%$, the early exercise premium does not significantly contribute to the price of the American call option because the equity price enters the early exercise region with

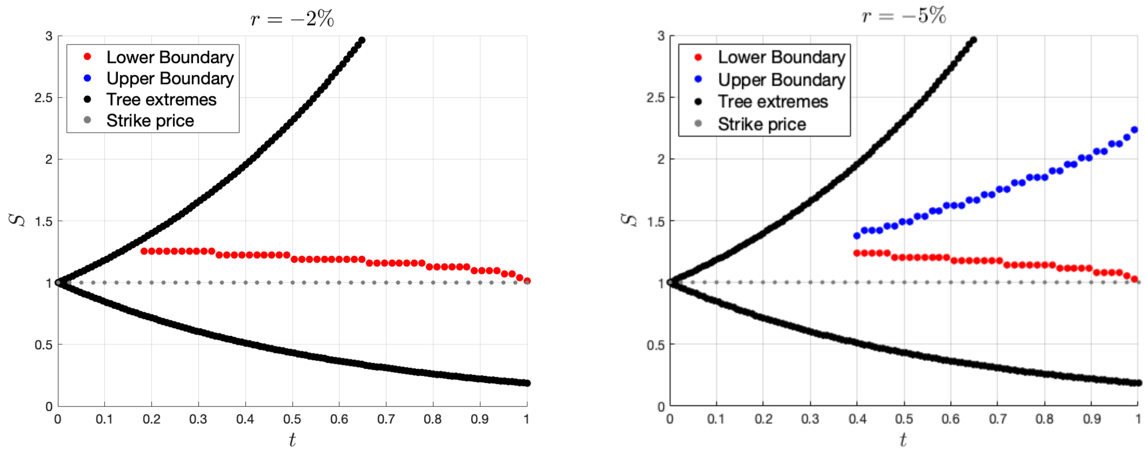

Figure 2.10: $r-$sections of free boundaries for the American call option. Left panel $r = -2\%$ and $q = 0\%$. Right panel $r = -5\%$ and $q = -2\%$.

a very small risk-neutral probability, as one can see from the third row of Table 2.2. The green dots in the bottom panels of Figure 2.9 mark the region where our necessary conditions for non standard early exercise of Proposition 2.4 are satisfied. We notice that this region overlaps very accurately with the area where early exercise is optimal as portrait in the top-left panel of Figure 2.9. We conclude our analysis of the American call option's free boundaries, by portraying in Figure 2.10 their time-dependence structure. In particular, we see that for American call options the upper critical price (if any) is increasing with respect to time t whereas the lower critical price is decreasing (see Figure 2.10), thus confirming the results of Theorem 2.5 and of (Battauz et al., 2015) in a constant interest rate framework.

2.4 Conclusions

In this paper we have studied American equity options in a correlated stochastic interest rate framework of Vasicek (1977) type. We have introduced a tractable lattice-based discretization of the equity price and interest rate processes by means of a quadrinomial tree. Our quadrinomial tree matches the joint discretized moments of the equity price and the stochastic interest rate and converges in distribution to the continuous time original processes. This allowed us to employ our quadrinomial tree to characterize the two-dimensional free boundary for American equity put and call options, that consists of the underlying asset and the interest rate values that trigger the optimal exercise of the option. Our results are in line with the existing literature when interest rates lie in the positive realm. In particular, for the American put options, the higher the dividend yield, the higher the benefits from deferring the option exercise. Moreover, in this case, the exercise region is downward connected with respect to the underlying asset value. On the contrary, when interest rate are likely to assume even mildly negative values, optimal exercise policies change, depending on the trade-off between the interest rate and the expected rate of return on the equity price. If such expected rate of return is negative, optimal exercise occurs at maturity only as the option goes (on average) deeper in the money as time goes by and the negative interest rates make the investor willing to cash in as late as possible. If the expected rate of return on the equity asset is positive, the option is expected to move towards the out of the money region. This effect is compensated by the preference to postponement due to negative interest rates. The trade-off results in a non-standard double continuation region that violates the aforementioned downward connectedness of the exercise region for American put option. We quantified the pricing error that an investor would make assuming a constant interest rate and therefore neglecting the variability (and the related risk) of the term structure. Finally, we documented similar non standard optimal exercise policies also for American call options. In particular, we find that early exercise of the American call option might be optimal even when the equity does not pay any dividend. This results confirm the analytical features of the free boundaries retrieved in Theorem 2.5 for the continuous-time framework described in System (2.1).

Appendix

2.A Bounds of the probabilities in the Quadrinomial Tree

Recall that at each t the four probabilities of an upward/downward movement of r/Y on the tree are:

$$\begin{aligned} q_{uu} &= \frac{\mu_Y \mu_r \Delta t + \mu_Y \Delta r^+ + \mu_r \Delta Y^+ + (1+\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\ q_{ud} &= \frac{-\mu_Y \mu_r \Delta t + \mu_Y \Delta r^+ - \mu_r \Delta Y^+ + (1-\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\ q_{du} &= \frac{-\mu_Y \mu_r \Delta t - \mu_Y \Delta r^+ + \mu_r \Delta Y^+ + (1-\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\ q_{dd} &= \frac{\mu_Y \mu_r \Delta t - \mu_Y \Delta r^+ - \mu_r \Delta Y^+ + (1+\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \end{aligned} \tag{2.A1}$$

with $\Delta r^+ = \sigma_r \sqrt{\Delta t}$, $\Delta Y^+ = \sigma_S \sqrt{\Delta t}$, $\mu_Y = r(t) - q - \frac{\sigma_S^2}{2}$ and $\mu_r = \kappa(\theta - r(t))$. From now on we light the notation writing r instead of $r(t)$. Nevertheless, it is crucial to remember that these probabilities are different for each node of the quadrinomial tree.

As already pointed out, the four probabilities sum up to one by construction. Unfortunately, they do not necessarily lie in (0,1). As a first control, we investigate what happens as the length of the time step goes to zero, namely, as $\Delta t \to 0$. We have

$$\lim_{\Delta t \to 0} q_{uu} = \lim_{\Delta t \to 0} q_{dd} = \frac{1+\rho}{4},$$

$$\lim_{\Delta t \to 0} q_{ud} = \lim_{\Delta t \to 0} q_{du} = \frac{1-\rho}{4}$$

which are all positive quantities (at least as $\rho \in (-1, 1)$). Therefore, the problem of having possibly negative probabilities is only due to the discretization procedure.

For instance, with $n = 250$ steps and $T = 1$ (that corresponds to $\Delta t = 0.004$), we need to impose the positivity constraint on all the four numerators in (2.A1).

Imposing $q_{uu} \geq 0$ and solving with respect to r leads to:

$$A_{uu} r^2 + B_{uu} r + C_{uu} \leq 0 \tag{2.A2}$$

where:

$$\begin{aligned}
A_{uu} = &\quad \kappa \\
B_{uu} = &\quad -\kappa\left(\theta + q + \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}}\right) - \frac{\sigma_r}{\sqrt{\Delta t}} \\
C_{uu} = &\quad -\kappa\theta\left(-q - \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}}\right) - \frac{\sigma_r}{\sqrt{\Delta t}}\left(-q - \frac{\sigma_S^2}{2}\right) - \frac{(1+\rho)\sigma_r\sigma_S}{\Delta t}.
\end{aligned}$$

Provided that the discriminant of Equation (2.A2) is positive, which surely holds true as $\Delta t \to 0$, the solution is $\underline{r}_{uu} \le r \le \overline{r}_{uu}$, where, of course,

$$\underline{r}_{uu} = \frac{-B_{uu} - \sqrt{B_{uu}^2 - 4A_{uu}C_{uu}}}{2A_{uu}} \quad \text{and} \quad \overline{r}_{uu} = \frac{-B_{uu} + \sqrt{B_{uu}^2 - 4A_{uu}C_{uu}}}{2A_{uu}}.$$

Similarly, we can work out all of the other probabilities.
Imposing $q_{ud} \ge 0$ leads to:

$$A_{ud}r^2 + B_{ud}r + C_{ud} \ge 0$$

where:

$$\begin{aligned}
A_{ud} = &\quad \kappa \\
B_{ud} = &\quad -\kappa\left(\theta + q + \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}}\right) + \frac{\sigma_r}{\sqrt{\Delta t}} \\
C_{ud} = &\quad -\kappa\theta\left(-q - \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}}\right) - \frac{\sigma_r}{\sqrt{\Delta t}}\left(q + \frac{\sigma_S^2}{2}\right) + \frac{(1-\rho)\sigma_r\sigma_S}{\Delta t},
\end{aligned}$$

that is solved by $r \le \underline{r}_{ud} \cup r \ge \overline{r}_{ud}$.
Imposing $q_{du} \ge 0$ leads to:

$$A_{du}r^2 + B_{du}r + C_{du} \ge 0$$

where:

$$\begin{aligned}
A_{du} = &\quad k \\
B_{du} = &\quad -\kappa\left(\theta + q + \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}}\right) - \frac{\sigma_r}{\sqrt{\Delta t}} \\
C_{du} = &\quad -\kappa\theta\left(-q - \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}}\right) + \frac{\sigma_r}{\sqrt{\Delta t}}\left(q + \frac{\sigma_S^2}{2}\right) + \frac{(1-\rho)\sigma_r\sigma_S}{\Delta t},
\end{aligned}$$

that is solved by $r \le \underline{r}_{du} \cup r \ge \overline{r}_{du}$.
Finally, imposing $q_{dd} \ge 0$ leads to:

$$A_{dd}r^2 + B_{dd}r + C_{dd} \le 0$$

where:

$$
\begin{aligned}
A_{dd} = &\quad \kappa \\
B_{dd} = &\; -\kappa\left(\theta + q + \frac{\sigma_S^2}{2} + \frac{\sigma_S}{\sqrt{\Delta t}}\right) + \frac{\sigma_r}{\sqrt{\Delta t}} \\
C_{dd} = &\; -\kappa\theta\left(-q - \frac{\sigma_S^2}{2} - \frac{\sigma_S}{\sqrt{\Delta t}}\right) + \frac{\sigma_r}{\sqrt{\Delta t}}\left(-q - \frac{\sigma_S^2}{2}\right) - \frac{(1+\rho)\sigma_r\sigma_S}{\Delta t}.
\end{aligned}
$$

that is solved by $\underline{r}_{dd} \leq r \leq \overline{r}_{dd}$.
Summing up, probabilities in (2.A1) stay positive as long as r satisfies:

$$
\begin{cases}
\underline{r}_{uu} \leq r \leq \overline{r}_{uu} \\
r \leq \underline{r}_{ud} \cup r \geq \overline{r}_{ud} \\
r \leq \underline{r}_{du} \cup r \geq \overline{r}_{du} \\
\underline{r}_{dd} \leq r \leq \overline{r}_{dd}
\end{cases}
$$

The solution to the previous system of inequalities depends on the sign of the correlation ρ. Given the sign of ρ, the eight extremes values $\underline{r}_{uu}, \underline{r}_{ud}, \ldots, \overline{r}_{du}, \overline{r}_{dd}$ always satisfy the same chain of inequalities. Furthermore, notice that this eight values depend only on the parameters of the model and not on t.
When $\rho \in (0,1]$, the only interval on which all of the inequalities hold true is $\overline{r}_{ud} \leq r \leq \underline{r}_{du}$ as it can be conveniently seen in Figure 2.A.1.
The intuition here is that when r and S move together and the discretization of r reaches values

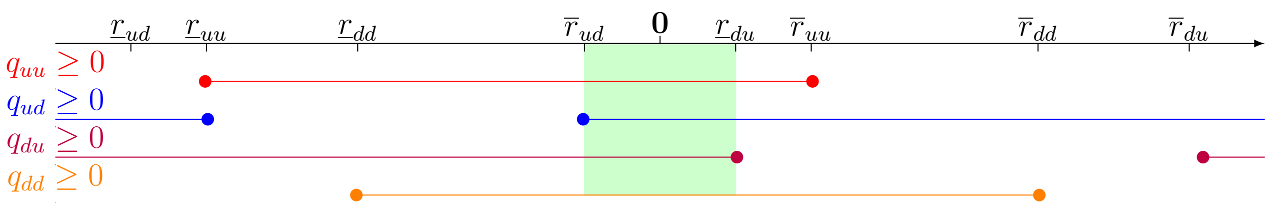

Figure 2.A.1: Graphical solution to the system of inequalities when $\rho \in (0,1]$.

far away from its long run mean θ, a further movement of r away from θ and in the opposite direction of S is extremely unlikely and, eventually, happens "with a negative probability".
If, for example, $r(0) = 0$, $\theta = 0.02$, $\sigma_r = 0.01$, $\kappa = 0.7$, $S(0) = 1$, $\sigma_S = 0.15$, $q = 0$, $\rho = 0.5$, $T = 1$ and $n = 125$, after $m = 100$ steps, namely at $t = m \cdot \Delta t = m \cdot \frac{T}{n} = 0.8$, $r(t)$ spans the interval [-0.0885, 0.0885] and $S(t)$ the interval [-1.3282, 1.3282], both of them assuming $m = 101$ different values. Hence, at $t = 0.8$ there are $101^2 = 10201$ possible nodes on tree. As an instance,

at the node $(S(t), r(t))_{t=0.8} = (0.5847, -0.0751)$ the four probabilities are:

$$q_{uu} = 0.4885$$
$$q_{ud} = -0.0143$$
$$q_{du} = 0.2780$$
$$q_{dd} = 0.2478.$$

Indeed, with the given parameters, probabilities are all positive as long as $\overline{r}_{ud} = -0.0660 \leq r(t) \leq 0.0861 = \underline{r}_{du}$, which is not our case. As $r(t)$ is extremely far away from its long-run mean and since $\rho > 0$ implies that r and S are likely to move together in the same direction, q_{ud}, namely the probability that r deviates even further from its long-run mean and also against S, becomes negative. Notice that $q_{ud} > q_{dd}$, meaning that the force that drives r towards its long-run mean prevails on the positive correlation between the two processes.
When such a scenario happens, we adjust the probabilities by setting the negative one to 0 and normalizing to 1 the others. From the example above we would then get:

$$q_{uu} = 0.4816$$
$$q_{ud} = 0$$
$$q_{du} = 0.2741$$
$$q_{dd} = 0.2443.$$

A very similar situation happens when $\rho \in [-1, 0)$ and the four probabilities stay positive as long as $\underline{r}_{dd} \leq r \leq \overline{r}_{uu}$. Figure 2.A.2 shows the solution to the system of inequalities in this case. Now q_{uu} or q_{dd} might become negative. This is due to the negative correlation: as r and S are likely to move in the opposite direction, when r is far away from its long-run mean, moving even further in the same direction of S may result in a negative probability. Again, we correct for such a phenomenon with the normalization described above.

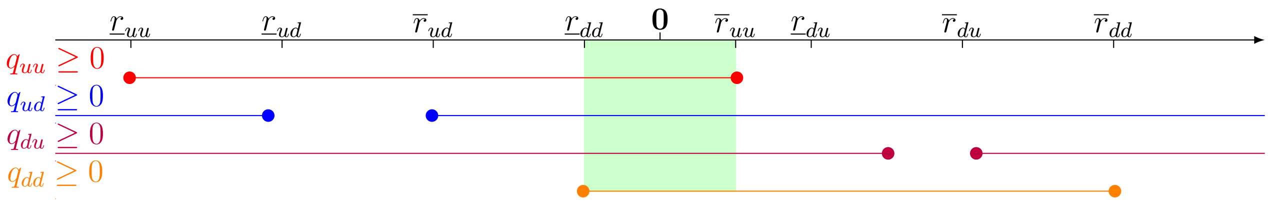

Figure 2.A.2: Graphical solution to the system of inequalities when $\rho \in [-1, 0)$.

For sake of completeness, we briefly discuss also the limit of zero correlation between r and S. When $\rho = 0$, $\underline{r}_{uu} = \underline{r}_{ud}$, $\overline{r}_{ud} = \underline{r}_{dd}$, $\overline{r}_{uu} = \underline{r}_{du}$ and $\overline{r}_{du} = \overline{r}_{dd}$. Hence, the two intervals we found for the two previous cases, $\overline{r}_{ud} \leq r \leq \underline{r}_{du}$ when $\rho \in (0,-1]$ and $\underline{r}_{dd} \leq r \leq \overline{r}_{uu}$ when $\rho \in [-1,0)$, coincide. When $\rho = 0$, probabilities stay positive as long as r belong to that interval.

Since the support of the discretization of $r(t)$ is known at each t, we can retrieve the maximum t before which no normalization of the probabilities is needed.
Given the two thresholds $\underline{r}$ and $\overline{r}$ (where $\underline{r} = \overline{r}_{ud}$, $\overline{r} = \underline{r}_{du}$ if $\rho > 0$ and $\underline{r} = \underline{r}_{dd}$, $\overline{r} = \overline{r}_{uu}$ if $\rho < 0$) we can set $\underline{t}$ and $\overline{t}$ as:

$$\underline{t} := \min_{s\in\{0,\Delta t,2\Delta t,\ldots,T\}} \left\{r(s) \geq \underline{r}\right\} \quad \text{and} \quad \overline{t} := \max_{s\in\{0,\Delta t,2\Delta t,\ldots,T\}} \left\{r(s) \leq \overline{r}\right\}.$$

Given the binomial structure of the discretization, after m steps we have we have:

$$r(0) - m\Delta r^- = r(0) - m\sigma_r\Delta t \leq r(t) \leq r(0) + m\sigma_r\Delta t = r(0) + m\Delta r^+$$

and, therefore, from

$$\begin{aligned} r(0) - \underline{m}\sigma_r\Delta t &\geq \underline{r} \\ r(0) + \overline{m}\sigma_r\Delta t &\leq \overline{r} \end{aligned} \tag{2.A3}$$

we can explicitly compute:

$$\underline{t} = \underline{m}\Delta t = \frac{r(0) - \underline{r}}{\sigma_r\sqrt{\Delta t}}\Delta t = \frac{r(0) - \underline{r}}{\sigma_r}\sqrt{\Delta t}$$

$$\overline{t} = \overline{m}\Delta t = \frac{\overline{r} - r(0)}{\sigma_r\sqrt{\Delta t}}\Delta t = \frac{\overline{r} - r(0)}{\sigma_r}\sqrt{\Delta t}.$$

Of course, neither $\underline{r}$, $\overline{r}$ nor $\underline{t}$, $\overline{t}$ are likely to correspond to any node of the discretized process $r(t)$ or to the discretized time line $\{0, \Delta t, 2\Delta t, \ldots, T\}$. In this case, we set $\underline{r}$, $\overline{r}$ and $\underline{t}$, $\overline{t}$ equal to the smallest values on the grid of $r(t)$ and t that satisfy the constraints in (2.A3). Going back to the numerical example above, we have that $\underline{t} = 0.5840$ and $\overline{t} = 0.7680$. A section of the quadrinomial tree in this case is displayed in Figure 2.A.3.

2.B Proofs of the Claims

Proof of Proposition 2.1: Value of the European put/call equity option. We first derive the price of the European put option at $t = 0$. As the payoff of the derivative depends only on the final value of S, the price at any t is obtained straightforwardly thanks to the Markovianity of (S, r) by replacing S_0 by $S(t)$, r_0 by $r(t)$ and T by $T - t$.
In the market described by (2.1), the risk-neutral price at $t = 0$ of the European put option on S

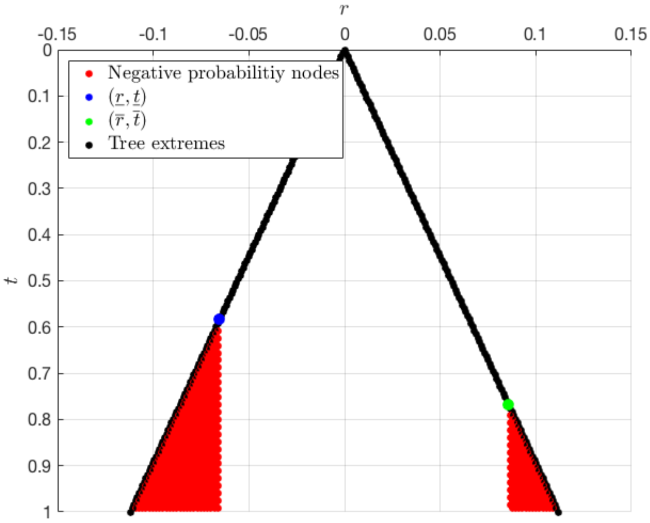

Figure 2.A.3: Section of the quadrinomial tree for $S = 0$. Red points indicate nodes at which one transition probability becomes negative. Parameters: $r(0) = 0$, $\theta = 0.02$, $\sigma_r = 0.01$, $\kappa = 0.7$, $S(0) = 1$, $\sigma_S = 0.15$, $q = 0$, $\rho = 0.5$, $T = 1$, $n = 125$.

with strike price K and maturity T is given by:

$$\begin{aligned}
\pi_E^{put}(0) &= \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(s)\mathrm{d}s}(K - S(T))^+] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{(K - S(T))}{B(T)}\mathbf{1}_{\{K-S(T)>0\}}\right] \\
&= K\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right] - \mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right].
\end{aligned} \tag{2.B1}$$

Since $B(T)$ depends on r which is correlated with S, in order to compute the two expected values we would need to know their joint distribution under $\mathbb{Q}$ and then evaluate a double integral. This turns out to be rather complicated. Nevertheless, we can greatly simplify the computation of the two expected values applying a change of numèraire.

We start from the first expectation in (2.B1). Consider the T-forward measure $\mathbb{Q}^T$, namely the martingale measure for the numèraire process $p(t, T)$. The Radon-Nikodym derivative of $\mathbb{Q}^T$ with respect to $\mathbb{Q}$ (whose numèraire process is the money market $B(t) = e^{\int_0^t r(s)\mathrm{d}s}$) is:

$$\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = L^T(t) = \frac{p(t, T)}{B(t)p(0, T)} \text{ on } \mathcal{F}_t, 0 \leq t \leq T.$$

As $p(T,T)=1$ and since $p(0,T)$ is a scalar, we get:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right] &= p(0,T)\mathbb{E}^{\mathbb{Q}}\left[\frac{p(T,T)}{B(T)p(0,T)}\mathbf{1}_{\{K-S(T)>0\}}\right]\\ &= p(0,T)\mathbb{E}^{\mathbb{Q}}[L^T(T)\mathbf{1}_{\{K-S(T)>0\}}]\\ &= p(0,T)\mathbb{E}^{\mathbb{Q}^T}[\mathbf{1}_{\{K-S(T)>0\}}]\\ &= p(0,T)\mathbb{Q}^T\left(S(T)<K\right)\\ &= p(0,T)\mathbb{Q}^T\left(\frac{S(T)}{p(T,T)}<K\right).\end{aligned}$$

By definition, under the T-forward measure $\mathbb{Q}^T$ the discounted process of the risky asset, when accounting for the dividend, $Z_{0,T}(t) := \dfrac{S(t)e^{qt}}{p(t,T)}$ is a martingale. Applying the multidimensional Itō's Lemma to $Z_{0,T}(t)$ under $\mathbb{Q}$ we get:

$$\mathrm{d}Z_{0,T}(t) = (\dots)\,\mathrm{d}t + (\nu_S + B(t,T)\nu_r)\cdot \mathrm{d}W^{\mathbb{Q}}(t)$$

where, $\nu_S = [\sigma_S \quad 0]$, $\nu_r = [\sigma_r\rho \quad \sigma_r\sqrt{1-\rho^2}]$ and $W^{\mathbb{Q}}(t) = [W_1^{\mathbb{Q}}(t) \quad W_2^{\mathbb{Q}}(t)]'$ is standard two-dimensional Brownian motion under $\mathbb{Q}$. Since the volatility process $\sigma_{0,T}(t) := \nu_S + B(t,T)\nu_r$ is constant, we can apply a suitable change of measure from $\mathbb{Q}$ to $\mathbb{Q}^T$ to get rid of the deterministic drift of $Z_{0,T}(t)$. Under the T-forward measure we get:

$$\mathrm{d}Z_{0,T}(t) = +\sigma_{0,T}(t)\cdot \mathrm{d}W^{\mathbb{Q}^T}(t)$$

as we expected. The process $Z_{0,T}(t)$ is now a geometric Brownian motion driven by a bi-dimensional Wiener process. Hence, its solution is:

$$\begin{aligned}Z_{0,T}(t) &= Z_{0,T}(0)\exp\left\{-\frac{1}{2}\int_0^t ||\sigma_{0,T}^2(s)||\mathrm{d}s + \int_0^t \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^T}(s)\right\}\\ &= \frac{S(0)}{p(0,T)}\exp\left\{-\frac{1}{2}\int_0^t ||\sigma_{0,T}^2(s)||\mathrm{d}s + \int_0^t \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^T}(s)\right\}.\end{aligned}$$

Notice that, due to Itō's Isometry,

$$-\frac{1}{2}\int_0^t ||\sigma_{0,T}^2(s)||\mathrm{d}s + \int_0^t \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^T}(s) \sim \mathcal{N}\left(-\frac{1}{2}\Sigma_{0,T}^2(t), \Sigma_{0,T}^2(t)\right)$$

where:

$$\begin{aligned}\Sigma_{0,T}^2(t) &:= \int_0^t ||\sigma_{0,T}^2(s)||\mathrm{d}s\\ &= \int_0^t \sigma_S^2 + 2\sigma_S\sigma_r\rho B(s,T) + B(s,T)^2\sigma_r^2\mathrm{d}s\\ &= \sigma_S^2 t + 2\sigma_S\sigma_r\rho\left(\frac{\kappa t - e^{-kT}\left(-1+e^{kt}\right)}{k^2}\right) + \sigma_r^2\left(\frac{e^{-2\kappa T}\left(-1+e^{2\kappa t}+4e^{\kappa T}-4^{\kappa(t+T)}+2e^{2\kappa T}\kappa t\right)}{2k^3}\right).\end{aligned}$$

When $t = T$, the expression above simplifies to:

$$\Sigma_{0,T}^2(T) = \sigma_S^2 T + 2\sigma_S\sigma_r\rho\left(\frac{-1+e^{-\kappa T}+\kappa T}{k^2}\right) + \sigma_r^2\left(-\frac{3+e^{-2\kappa T}-4e^{-\kappa T}-2\kappa T}{2k^3}\right).$$

Finally, we can compute the T-forward probability that the put option closes in the money as:

$$\begin{aligned}
\mathbb{Q}^T\left(\frac{S(T)}{p(T,T)} < K\right) &= \mathbb{Q}^T\left(\frac{S(T)e^{qT}}{p(T,T)} < Ke^{qT}\right) \\
&= \mathbb{Q}^T\left(Z_{0,T}(T) < Ke^{qT}\right) \\
&= \mathbb{Q}^T\left(\frac{S(0)}{p(0,T)}\exp\left\{-\frac{1}{2}\int_0^T ||\sigma_{0,T}||^2(s)\mathrm{d}s + \int_0^T \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^T}(s)\right\} < Ke^{qT}\right) \\
&= \mathbb{Q}^T\left(\mathcal{N}\left(-\frac{1}{2}\Sigma_{0,T}^2(T), \Sigma_{0,T}^2(T)\right) < \ln\frac{p(0,T)Ke^{qT}}{S(0)}\right) \\
&= N(-\tilde{d}_2)
\end{aligned}$$

with $\tilde{d}_2 = \frac{1}{\sqrt{\Sigma_{0,T}^2(T)}}\left(\ln\frac{S(0)}{p(0,T)K} - \frac{1}{2}\Sigma_{0,T}^2(T) - qT\right)$. Hence,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right] = p(0,T)N(-\tilde{d}_2).$$

We now turn to the second expected value in (2.B1). Consider the martingale measure $\mathbb{Q}^S$ with numèraire process $S(t)e^{qt}$. The Radon-Nikodym derivative of $\mathbb{Q}^S$ with respect to $\mathbb{Q}$ is:

$$\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}} = L^S(t) = \frac{S(t)e^{qt}}{S(0)B(t)} \text{ on } \mathcal{F}_t, 0 \le t \le T. \tag{2.B2}$$

As both $S(0)$ and e^{qT} are scalars, we have:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right] &= S(0)e^{-qT}\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)e^{qT}}{S(0)B(T)}\mathbf{1}_{\{K-S(T)>0\}}\right] \\
&= S(0)e^{-qT}\mathbb{E}^{\mathbb{Q}}\left[L^S(T)\mathbf{1}_{\{K-S(T)>0\}}\right] \\
&= S(0)e^{-qT}\mathbb{E}^{\mathbb{Q}^S}[\mathbf{1}_{\{K-S(T)>0\}}] \\
&= S(0)e^{-qT}\mathbb{Q}^S(S(T) < K).
\end{aligned}$$

Under $\mathbb{Q}^S$, the process $Y_{0,T}(t) := \frac{p(0,t)}{S(t)e^{qt}}$ is a martingale. Notice that $Y_{0,T}(t) = Z_{0,T}(t)^{-1}$. Then, Itō's Lemma tells us immediately that:

$$\mathrm{d}Y_{0,T}(t) = (\dots)\,\mathrm{d}t - (\nu_S + B(t,T)\nu_r)\cdot \mathrm{d}W^{\mathbb{Q}}(t)$$

and after a suitable change of measure we get that under $\mathbb{Q}^S$:

$$\mathrm{d}Y_{0,T}(t) = -\sigma_{0,T}(t)\cdot \mathrm{d}W^{\mathbb{Q}^S}(t).$$

As before, we get:

$$\begin{aligned}
Y_{0,T}(t) &= Y_{0,T}(0)\exp\left\{-\frac{1}{2}\int_0^t ||\sigma^2_{0,T}(s)||\mathrm{d}s - \int_0^t \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^2}(s)\right\} \\
&= \frac{p(0,T)}{S(0)}\exp\left\{-\frac{1}{2}\int_0^t ||\sigma^2_{0,T}(s)||\mathrm{d}s - \int_0^t \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^S}(s)\right\}.
\end{aligned}$$

and, since again $p(T,T)=1$, the $\mathbb{Q}^S$ probability that the put option closes in the money is:

$$\begin{aligned}
\mathbb{Q}^S\left(S(T)<K\right) &= \mathbb{Q}^S\left(\frac{1}{S(T)} > \frac{1}{K}\right) \\
&= \mathbb{Q}^S\left(\frac{p(T,T)}{S(T)e^{qT}} > \frac{1}{Ke^{qT}}\right) \\
&= \mathbb{Q}^S\left(Y_{0,T}(T) > \frac{1}{Ke^{qT}}\right) \\
&= \mathbb{Q}^T\left(\frac{p(0,T)}{S(0)}\exp\left\{-\frac{1}{2}\int_0^T ||\sigma^2_{0,T}(s)||\mathrm{d}s - \int_0^T \sigma_{0,T}(s)\cdot \mathrm{d}W^{\mathbb{Q}^T}(s)\right\} > \frac{1}{Ke^{qT}}\right) \\
&= \mathbb{Q}^T\left(\mathcal{N}\left(-\frac{1}{2}\Sigma^2_{0,T}(T), \Sigma^2_{0,T}(T)\right) > \ln\frac{S(0)}{p(0,T)Ke^{qT}}\right) \\
&= \mathbb{Q}^T\left(\mathcal{N}(0,1) > \frac{1}{\sqrt{\Sigma^2_{0,T}(T)}}\left(\ln\frac{S(0)}{p(0,T)K} + \frac{1}{2}\Sigma^2_{0,T}(T) - qT\right)\right) \\
&= N(-\tilde{d}_1)
\end{aligned}$$

where $\tilde{d}_1 = \dfrac{1}{\sqrt{\Sigma^2_{0,T}(T)}}\left(\ln\dfrac{S(0)}{p(0,T)K} + \dfrac{1}{2}\Sigma^2_{0,T}(T) - qT\right) = \tilde{d}_2 + \sqrt{\Sigma^2_{0,T}(T)}$. Hence,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)\mathbf{1}_{\{K-S(T)>0\}}}{B(T)}\right] = S(0)e^{-qT}N(-\tilde{d}_1).$$

Finally, putting everything together we find the initial price of the put option:

$$\begin{aligned}
\pi^{put}_E(0) &= K\mathbb{E}^{\mathbb{Q}}[p(0,T)\mathbf{1}_{\{K-S(T)>0\}}] - \mathbb{E}^{\mathbb{Q}}[p(0,T)S(T)\mathbf{1}_{\{K-S(T)>0\}}] \\
&= Kp(0,T)N(-\tilde{d}_2) - S(0)e^{-qT}N(-\tilde{d}_1).
\end{aligned}$$

The price of the related call option can be derived by the put-call parity that at $t\in[0,T]$ reads

$$\pi^{call}_E(t) = S(t)e^{-q(T-t)} - Kp(t,T) + \pi^{put}_E(t)$$

as $e^{-q(T-t)}$ units of $S(t)$ at time t lead to one unit of S at time T by continuously investing the dividends in S itself.

Proof of Theorem 2.2: Convergence of the quadrinomial tree. We now need to show that the bivariate discrete process $(X_i)_i$ defined in (2.10) with the parameters in (2.11) and (2.12) converges in distribution to $X(t) = (Y(t), r(t))$ that solves (2.7). With the notation of the general case in (2.9) and exploiting the result of Section 11.3 of Stroock and Varadhan (1997), the desired result holds true if the following four conditions are met:

(A1) the functions $\mu(x,t)$ and $\sigma(x,t)$ are continuous and $\sigma(x,t)$ is non negative;

(A2) with probability 1 a solution $(X_t)_t$ to the SDE:

$$X_t = X_0 + \int_0^t \mu(X_s, s)\mathrm{d}s + \int_0^t \sigma(X_s, s) \cdot \mathrm{d}W(s)$$

exists for $0 < t < +\infty$ and it is unique in law;

(A3) for all $\delta, T > 0$

$$\lim_{n\to+\infty} \sup_{||x||\le\delta, 0\le t\le T} |\Delta Y^{\pm}| = 0$$

$$\lim_{n\to+\infty} \sup_{||x||\le\delta, 0\le t\le T} |\Delta r^{\pm}| = 0;$$

(A4) let $X_{i,j}$ indicate the j-th entry of X_i and let $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$ be the filtration generated by the discrete bivariate process (X_i). Define:

$$\mu_i(x,t) := \begin{bmatrix} \mu_{i,1}(x,t) \\ \mu_{i,2}(x,t) \end{bmatrix} \text{ and } \sigma_i^2(x,t) := \begin{bmatrix} \sigma_{i,1}^2(x,t) \\ \sigma_{i,2}^2(x,t) \end{bmatrix}$$

where $\mu_{i,j}(x,t) = \dfrac{\mathbb{E}^{\mathbb{Q}}[X_{i+1,j} - X_{i,j}|\mathcal{F}_i]}{\frac{T}{n}}$ and $\sigma_{i,j}^2(x,t) = \dfrac{\mathbb{E}^{\mathbb{Q}}[(X_{i+1,j} - X_{i,j})^2|\mathcal{F}_i]}{\frac{T}{n}}$ for $j = 1, 2$. Let $\rho_i(x,t) = \dfrac{\mathbb{E}^{\mathbb{Q}}[(X_{i+1,1} - X_{i,1})(X_{i+1,2} - X_{i,2})|\mathcal{F}_i]}{\frac{T}{n}}$ and $\rho(x,t) = \sigma_1(x,t) \cdot \sigma_2(x,t)'$ where $\sigma_j(x,t)$ is the j-th row of $\sigma(x,t)$. Then, for all $\delta, T > 0$,

$$\lim_{n\to+\infty} \sup_{||x||\le\delta, 0\le t\le T} ||\mu_i(x,t) - \mu(x,t)|| = 0$$

$$\lim_{n\to+\infty} \sup_{||x||\le\delta, 0\le t\le T} ||\sigma_i^2(x,t) - \sigma^2(x,t) \cdot \mathbf{I}_2|| = 0$$

$$\lim_{n\to+\infty} \sup_{||x||\le\delta, 0\le t\le T} |\rho_i(x,t) - \rho(x,t)| = 0$$

where $\mathbf{I}_n$ is the column vector with all of the n entries equal to one.

For our quadrinomial tree we have $X_t = [Y(t), \quad r(t)]'$,

$$\mu(X_t, t) = \begin{bmatrix} \left(r(t) - q - \frac{\sigma_S^2}{2}\right) \\ \kappa(\theta - r(t)) \end{bmatrix} \quad \text{and} \quad \sigma(X_t, t) = \begin{bmatrix} \sigma_S & 0 \\ \sigma_r\rho & \sigma_r\sqrt{1-\rho^2} \end{bmatrix}.$$

Assumption (A1) trivially holds true.

Assumption (A2) holds true if the standard conditions for the existence and the uniqueness of the solution to an SDE are met. According, e.g., to Proposition 5.1 in Björk (2009), it is sufficient to show that there exists a constant K such that the following are satisfied for all $x_i = [y_i, \quad r_i]'$, $i = 1, 2$ and t:

$$||\mu(x_1,t) - \mu(x_2,t)|| \leq K||x_1 - x_2||,$$
$$||\sigma(x_1,t) - \sigma(x_2,t)|| \leq K||x_1 - x_2||,$$
$$||\mu(x_1,t)|| + ||\sigma(x_1,t)|| \leq K\left(1 + ||x_1||\right).$$

Notice that the second and the third conditions involve the operator norm of a matrix $A \in \mathbb{R}^n$ defined as $||A|| := \sup_{||x||=1}\{||A \cdot x|| : x \in \mathbb{R}^n\}$.

As $||\mu(x_1,t) - \mu(x_2,t)|| = \sqrt{1+\kappa^2}|r_1 - r_2|$ and $(r_1 - r_2)^2 \leq ||x_1 - x_2||^2$, the first condition is surely satisfied for any $K \geq \sqrt{1+\kappa^2}$. As $\sigma(x_i,t)$ is actually constant and independent of x_i and t, $||\sigma(x_1,t) - \sigma(x_2,t)|| = 0$ and the second condition is surely satisfied for any $K \geq 0$. Finally, as

$$||\sigma(x_1),t)|| = \sigma_S^2 + \rho^2\frac{\sigma_r^2}{2} + |\rho|\frac{\sigma_r}{2}\sqrt{4\sigma_s^2 + \sigma_r^2}$$

is constant and as

$$||\mu(x_1,t)|| = \sqrt{\left(r_1 - q - \frac{\sigma_S^2}{2}\right)^2 + \kappa^2(\theta - r_1)^2}$$

can be bounded from above by $\sqrt{2(1+\kappa^2)r_1^2}$, we have

$$||\mu(x_1,t)|| + ||\sigma(x_1,t)|| \leq \sqrt{2(1+\kappa^2)}||x_1|| + ||\sigma(x_1,t)|| \leq K(1 + ||x_1||)$$

for any $K \geq \max\{\sqrt{2(1+\kappa^2)}, ||\sigma(x_1,t)||\}$. As the three conditions hold true simultaneously for any $K \geq \max\{\sqrt{2(1+\kappa^2)}, ||\sigma(x_1,t)||\}$, assumption (A2) is satisfied.

As the increments of the bivariate discrete process $\Delta Y^{\pm} = \pm\sigma_S\sqrt{\Delta t} = \pm\sigma_S\sqrt{\frac{T}{n}}$, $\Delta r^{\pm} = \pm\sigma_r\sqrt{\Delta t} = \pm\sigma_r\sqrt{\frac{T}{n}}$ are constant and do not depend neither on x_i, $i = 1, 2$, nor on t,

$$\sup_{||x||\leq\delta, 0\leq t\leq T} |\Delta Y^{\pm}| = |\Delta Y^{\pm}| = \sigma_S\sqrt{\frac{T}{n}},$$
$$\sup_{||x||\leq\delta, 0\leq t\leq T} |\Delta r^{\pm}| = |\Delta r^{\pm}| = \sigma_r\sqrt{\frac{T}{n}}.$$

As both of the sups are infinitesimal with respect to n, (A3) holds true as well.

As the parameters in (2.11) and (2.12) of the bivariate discretization $X_i = (Y_i, r_i)$ are chosen in order to match the first two moments and the cross-variation of $X(t) = (Y(t), r(t))$, we have $\mu_i(x,t) = \mu(x,t)$, $\sigma_i^2(x,t) = \sigma^2(x,t) \cdot \mathbf{I}_2$ and $\rho_i(x,t) = \rho(x,t)$. Hence, assumption (A4) is satisfied by construction.

Theorem 11.3.3 of Stroock and Varadhan (1997) allows us to conclude.

Proof of Proposition 2.3: Value of the American option as a deterministic function. Let $\eta := \tau - t$. Then we can rewrite the value of the American option (2.13) as:

$$V(t) = ess \sup_{0\leq\eta\leq T-t} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+\eta} r(s)\mathrm{d}s}\, \varphi(S(t+\eta))\middle|\, \mathcal{F}_t\right]. \tag{2.B3}$$

Recalling the dynamics of the equity price conditional on $S(t) = S$ we can further rewrite $\tilde{V}(t)$ as:

$$\begin{aligned} V(t) = ess \sup_{0\leq\eta\leq T-t} \mathbb{E}^{\mathbb{Q}} &\left[\exp\left(-\int_t^{t+\eta} r(s)\mathrm{d}s\right)\cdot\right. \\ &\left.\cdot\varphi\left(S\exp\left(\int_t^{t+\eta} r(s)\mathrm{d}s - \left(q+\frac{1}{2}\sigma_S^2\right)\eta + \sigma_S(W_S(t+\eta) - W_S(t))\right)\right)\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

Therefore, V depends on the expectation of two random variables: the Brownian increment $W_S(t+\eta) - W_S(t)$ and the integral $\int_t^{t+\eta} r(s)\mathrm{d}s$, which appears both in the drift part of the underlying and in the discount factor. The first of the two random variables is $\mathcal{F}_t$-independent by definition:

$$W_S(t+\eta) - W_S(t) \perp\!\!\!\perp \mathcal{F}_t,$$

and, moreover,

$$W_S(t+\eta) - W_S(t) \overset{\mathbb{Q}}{\sim} W_S(\eta).$$

We now show that also $\int_t^{t+\eta} r(s)\mathrm{d}s$ is independent of $\mathcal{F}_t$ and that $\int_t^{t+\eta} r(s)\mathrm{d}s \overset{\mathbb{Q}}{\sim} \int_0^{\eta} r(s)\mathrm{d}s$ as well. Recalling the solution to the SDE driving the short term interest rate conditional on $r(t) = r$ we have:

$$\begin{aligned} \int_t^{t+\eta} r(s)\mathrm{d}s &= \int_t^{t+\eta}\left[re^{-\kappa(s-t)} + \theta(1-e^{-\kappa(s-t)}) + \sigma_r\int_t^s e^{-\kappa(s-y)}\mathrm{d}W_r(y)\right]\mathrm{d}s \\ &= -\frac{e^{-\kappa\eta}}{\kappa}(r-\theta) + \frac{r-\theta}{\kappa} + \theta\eta + \sigma_r\int_t^{t+\eta}\int_t^s e^{-\kappa(s-y)}\mathrm{d}W_r(y)\mathrm{d}s. \end{aligned}$$

The constant $\alpha := -\frac{e^{-\kappa\eta}}{\kappa}(r-\theta) + \frac{r-\theta}{\kappa} + \theta\eta$ does not depend on t. Exploiting the definition of stochastic integral with $\{t_i\}_{i=1,\ldots,N}$ such that $t_0 = t$, $t_N = t+\eta$ and $||\{t_i\}|| \to 0$, we get:

$$\int_t^{t+\eta} r(s)\mathrm{d}s = \alpha + \sigma_r\int_t^{t+\eta}\sum_{i=0}^{N-1} e^{-\kappa(s-t_i)}(W_r(t_{i+1}) - W_r(t_i))\mathrm{d}s.$$

Since $t_{i+1} > t_i > t$ for any $i = 1,\ldots,N-1$, $W_r(t_{i+1}) - W_r(t_i) \perp\!\!\!\perp \mathcal{F}_t$ by definition and for any value of s. Hence,

$$\sum_{i=0}^{N-1} e^{-\kappa(s-t_i)}(W_r(t_{i+1}) - W_r(t_i)) \perp\!\!\!\perp \mathcal{F}_t \quad \forall s.$$

Since the sum is independent of $\mathcal{F}_t$ for any s, the outer integral in ds preserves such independence. As a result,

$$\int_t^{t+\eta} r(s)\mathrm{d}s \perp\!\!\!\perp \mathcal{F}_t.$$

Furthermore, we need to show that the distribution of

$$\int_t^{t+\eta} r(s)\mathrm{d}s$$

does not depend on t. Recalling that:

$$\int_t^{t+\eta} r(s)\mathrm{d}s = \alpha + \sigma_r \int_t^{t+\eta} \int_t^s e^{-\kappa(s-y)}\mathrm{d}W_r(y)\mathrm{d}s$$

and setting $a := s - t$ in the outer integral in ds, we get:

$$\int_t^{t+\eta} r(s)\mathrm{d}s = \alpha + \sigma_r \int_0^{\eta} \int_t^{a+t} e^{-\kappa(a+t-y)}\mathrm{d}W_r(y)\mathrm{d}a.$$

The argument of the inner stochastic integral is deterministic in y and, therefore:

$$\begin{aligned}\int_t^{a+t} e^{-\kappa(s-y)}\mathrm{d}W_r(y) &\overset{\mathbb{Q}}{\sim} \mathcal{N}\left(0, \int_t^{a+t} e^{-2\kappa(s-y)}\mathrm{d}y\right) \\ &\overset{\mathbb{Q}}{\sim} \mathcal{N}\left(0, \frac{1}{2\kappa}\left(1 - e^{-2\kappa a}\right)\right),\end{aligned}$$

which does not depend on t. Thanks to a little abuse of notation we see that:

$$\int_t^{t+\eta} r(s)\mathrm{d}s = \alpha + \sigma_r \int_0^{\eta} \mathcal{N}\left(0, \frac{1}{2\kappa}\left(1 - e^{-2\kappa a}\right)\right)\mathrm{d}a,$$

where the right-hand side of the equation does not depend on t. Hence:

$$\int_t^{t+\eta} r(s)\mathrm{d}s \overset{\mathbb{Q}}{\sim} \int_0^{\eta} r(s)\mathrm{d}s.$$

We now go back to the original expression (2.B3). Thanks to the independence of $\mathcal{F}_t$, the conditional expected value turns into the unconditional one:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+\eta} r(s)\mathrm{d}s}\varphi(S(t+\eta))\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+\eta} r(s)\mathrm{d}s}\varphi(S(t+\eta))\right],$$

and setting $S(0) = S(t) = S$, $r(0) = r(t) = r$,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+\eta} r(s)\mathrm{d}s}\varphi(S(t+\eta))\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{\eta} r(s)\mathrm{d}s}\varphi(S(\eta))\right].$$

Therefore, the value on an American option on S defined in (2.13) reduces to

$$V(t) = F(t, S(t), r(t))$$

with

$$F(t, S, r) = \sup_{0\le\eta\le T-t} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^{\eta} r(s)\mathrm{d}s\right)\cdot\right.$$

$$\cdot\varphi\left(S\exp\left(\int_0^\eta r(s)\mathrm{d}s-\left(q+\frac{1}{2}\sigma_S^2\right)\eta+\sigma_S W_S(\eta)\right)\right)\Bigg].$$

where t enters only the upper bound of η, namely the time to maturity of the option. From this last expression it is immediate to see that F enjoys the same monotonicity properties of φ w.r.t.S, and that it is decreasing w.r.t. t, and convex w.r.t. S. For the put option we show now that F is decreasing in r. To this aim we rewrite

$$F(t,S,r)=\sup_{0\le\eta\le T-t}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^\eta r(s)\mathrm{d}s)}\left(K-Se^{\int_0^\eta r(s)\mathrm{d}s-\left(q+\frac{1}{2}\sigma_S^2\right)\eta+\sigma_S W_S(\eta)}\right)^+\right] \tag{2.B4}$$

where $r=r(0)$. If $r'>r$ then $F(t,S,r')\le F(t,S,r)$. In fact, $\int_0^\eta r(s)\mathrm{d}s$ started at $r(0)=r'>r$ is larger than $\int_0^\eta r(s)\mathrm{d}s$ started at $r(0)=r$. As the object of the expectation in (2.B4) is a decreasing function of $\int_0^\eta r(s)\mathrm{d}s$, we conclude that $F(t,S,r)$ is decreasing in r.

To show that the American call option is increasing with respect to r, we apply a change of numeraire to isolate the effect of the interest rate in the underlying drift only (as under the original risk neutral measure an increase in r has opposite effects in the discount factor and in the call's payoff).

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[(S(\tau)-K)^+e^{-\int_0^\tau r(s)\mathrm{d}s}\right]&=\mathbb{E}^{\mathbb{Q}}\left[\frac{S(\tau)e^{q\tau}}{S(0)B(\tau)}\left(\frac{1}{K}-\frac{1}{S(\tau)}\right)^+Ke^{-q\tau}S(0)\right]\\&=\mathbb{E}^{\mathbb{Q}}\left[L^S(\tau)\left(\frac{1}{K}-\frac{1}{S(\tau)}\right)^+Ke^{-q\tau}S(0)\right]\end{aligned}$$

where $L^S(\tau)$ is the Radon-Nikodym derivative of $\mathbb{Q}^S$ with respect to $\mathbb{Q}$ defined in (2.B2). Thus the call option is a put option under the new measure on K/S with strike $S(0)$ and interest rate q

$$\mathbb{E}^{\mathbb{Q}}\left[(S(\tau)-K)^+e^{-\int_0^\tau r(s)\mathrm{d}s}\right]=\mathbb{E}^{\mathbb{Q}^S}\left[\left(S(0)-\frac{K}{S(\tau)}\right)^+e^{-q\tau}\right]$$

Recalling the dynamics of the equity price and of the interest rate under $\mathbb{Q}$,

$$\begin{cases}\dfrac{\mathrm{d}S(t)}{S(t)}=(r(t)-q)\mathrm{d}t+[\sigma_S\quad 0]\cdot\mathrm{d}W^{\mathbb{Q}}(t)\\ \mathrm{d}r(t)=\kappa(\theta-r(t))\mathrm{d}t+[\sigma_r\rho\quad\sigma_r\sqrt{1-\rho^2}]\cdot\mathrm{d}W^{\mathbb{Q}}(t)\end{cases} \tag{2.B5}$$

Girsanov's theorem implies that $\mathrm{d}W^{\mathbb{Q}}(t)=\mathrm{d}W^{\mathbb{Q}^S}(t)+[\sigma_S\quad 0]'\mathrm{d}t$ and, therefore, (2.B5) becomes

$$\begin{cases}\dfrac{\mathrm{d}S(t)}{S(t)}=(r(t)-q+\sigma_S^2)\mathrm{d}t+[\sigma_S\quad 0]\cdot\mathrm{d}W^{\mathbb{Q}^S}(t)\\ \mathrm{d}r(t)=\kappa(\theta-r(t)+\frac{\rho\sigma_S\sigma_r}{\kappa})\mathrm{d}t+[\sigma_r\rho\quad\sigma_r\sqrt{1-\rho^2}]\cdot\mathrm{d}W^{\mathbb{Q}^S}(t)\end{cases} \tag{2.B6}$$

Ito's formula implies that

$$\mathrm{d}\left(\frac{1}{S(t)}\right) = \frac{1}{S(t)}\Big((q-r(t))\mathrm{d}t - [\sigma_S \quad 0]\cdot \mathrm{d}W^{\mathbb{Q}^S}(t)\Big)$$

and therfore the new underlying

$$\mathrm{d}\left(\frac{K}{S(t)}\right) = \frac{K}{S(t)}\Big((q-r(t))\mathrm{d}t - [\sigma_S \quad 0]\cdot \mathrm{d}W^{\mathbb{Q}^S}(t)\Big)$$

has drift $q-r(t)$. Thus the call option is a put option whose underlying under the new measure is

$$\frac{K}{S(t)} = \frac{K}{S(0)} e^{\int_0^\eta (q-r(s))\mathrm{d}s - \frac{1}{2}\sigma_S^2\eta - \sigma_S W_1^{\mathbb{Q}^S}(\eta)}$$

Thus, if the process r starts at $r(0) = r' > r$ the factor $\int_0^\eta (q-r(s))\mathrm{d}s$ is smaller than the one started at $r(0) = r$, and thus the put's payoff is larger, and the value of the American option larger as well. This shows that for the call option $r' > r$ implies $F(t,S,r') > F(t,S,r)$.

Proof of Proposition 2.4: Asymptotic necessary conditions for the existence of a double continuation region. When the interest rate is constant, the value at t of the American put option defined in Equation (2.14) becomes a deterministic function $G(t,S)$: $\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ of the time and the current value of the underlying:

$$G(t,S) = \sup_{0\le\eta\le T-t} \mathbb{E}^{\mathbb{Q}}\left[e^{-r\eta}\left(K - S\exp\left(\left(r - q - \frac{\sigma_S^2}{2}\right)\eta + \sigma_S W_S(\eta)\right)\right)^+\right]. \tag{2.B7}$$

As Battauz et al. (2015) show in Section 2, necessary conditions for the double continuation region to appear for a put option at a generic t are that the drift of S is positive and $G(t,0) > K$. This last condition is verified if there exists some η such that $p(0,\eta) > 1$, as discussed in the comments before Proposition 2.4. We now deduce *[NC0]* by imposing $p(0,\eta) > 1$ for some $\eta \in [0, T-t]$. Exploiting Jensen's inequality and the uniform integrability of $r(s)$, we get:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^\eta r(s)\mathrm{d}s\right)\right] \ge \exp\left(-\mathbb{E}^{\mathbb{Q}}\left[\int_0^\eta r(s)\mathrm{d}s\right]\right) = \exp\left(-\int_0^\eta \mathbb{E}^{\mathbb{Q}}\left[r(s)\right]\mathrm{d}s\right).$$

As before, thanks to (2.3), we have:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^\eta r(s)\mathrm{d}s\right)\right] \ge \exp\left(-\int_0^\eta re^{-\kappa s} + \theta(1-e^{-\kappa s})\mathrm{d}s\right) = \exp\left(r\alpha - \theta(\alpha+\eta)\right)$$

where we set $\alpha := \dfrac{e^{-\kappa\eta}-1}{\kappa}$. Notice that $\alpha \le 0$ for any κ and $\eta \in [0, T-t]$.
If $r\alpha - \theta(\alpha+\eta) > 0$, then $F(t,0,r) > K$.

For the American put option, under *[NC0]*, if *[NC1]* is not satisfied, i.e. $q > 0$, than the discounted risky security $\tilde{S}$ is driven by

$$\mathrm{d}\tilde{S}(t) = -q\mathrm{d}t + \sigma_S \mathrm{d}W_S^{\mathbb{Q}}(t),$$

and $\tilde{S}$ is a supermartingale. Thus, for any $t < \tau < T$,

$$\mathbb{E}^{\mathbb{Q}}\left[S(\tau)e^{-\int_t^\tau r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right] \leq S(t)$$

and, by Jensen's inequality,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[(K - S(\tau))^+ e^{-\int_t^\tau r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right] &\geq \left(K\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^\tau r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right] - S(t)e^{-q(\tau-t)} \right)^+ \\ &\geq (K - S(t))^+,\end{aligned}$$

where the last inequalities holds under *[NC0]*. This shows that, for the American put option, under *[NC0]*, if *[NC1]* is violated, early exercise is never optimal at t.
We deal now with the American call option. For $0 < \tau < T$, we have by Jensen's inequality,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[(S(\tau) - K)^+ e^{-\int_t^\tau r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right] &\geq \left(S(t)e^{-q(\tau-t)} - K\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^\tau r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right] \right)^+ \\ &= \left(S(t)e^{-q(\tau-t)} - Kp(t,\tau) \right)^+ \\ &\geq (S(t) - K)^+,\end{aligned}$$

if $q \leq 0$ and $p(t,\tau) \leq 1$. Therefore, to ensure the existence of optimal early exercise opportunities for the American call option, we must assume that $q > 0$, or $q \leq 0$ and $p(t,\tau) > 1$ for some τ.

Under *[NC0]*, if *[NC2]* is not satisfied, then $\pi_A(t,S,r) \geq \pi_E(t,S,r) > (K - S)^+$, that means that early exercise is never optimal at t.

Proof of Theorem 2.5. The case $r \geq 0$ is standard (see Detemple, 2014), and therefore we focus on $r < 0$. The continuity, the monotonicity of the r-sections of the put option's free boundaries with respect to t and S and their limits as $t \to T^-$ follow by adapting the proof of Theorem 2.3 in (Battauz et al., 2015) where now the operator $\mathcal{L}$ becomes $\mathcal{L}F = \frac{\partial F}{\partial S}S(r-q) + \frac{\partial F}{\partial r}\kappa(\theta - r) + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma_S^2 S^2 + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma_r^2 + \frac{\partial^2 F}{\partial r \partial S}\rho\sigma_r\sigma_S$. The monotonicity properties of the free boundaries with respect to r follow from the monotonicity properties of F. In fact, let $r' > r$, and assume $S \in EER_r$. Then $(K - S)^+ \leq F(t,S,r') \leq F(t,S,r) = (K - S)^+$, where the first inequality follows from value dominance and the second one from the fact that F is decreasing in r. Thus if $S \in EER_r$, then $S \in EER_r'$, and $EER_r' \supseteq EER_r$. By passing to the infimum (resp. supremum) we conclude that the lower (resp. upper) free boundary is decreasing (resp. increasing) with respect to r.

For the call option, we start from the monotonicity properties of the free boundaries with respect to r. As the call option is increasing in r, we have that if $r' > r$ and $S \in EER'_r$ then $(S-K)^+ \leq F(t,S,r) \leq F(t,S,r') = (K-S)^+$, where the first inequality follows from value dominance and the second one from the fact that F is increasing in r. This means that $EER'_r \subseteq EER_r$. By passing to the infimum (resp. supremum) we conclude that the lower (resp. upper) free boundary is increasing (resp. decreasing) with respect to r.

For the other call option's properties, we cannot simply adapt the proof of Theorem 3.3 in (Battauz et al., 2015), as it relies on a simmetry result in a constant interest rate enviroment that fails to be applicable to our setting. The monotonicity properties of $\underline{S}^*$ and $\overline{S}^*$ with respect to t follow from the fact that F is decreasing with respect to t, similarly to the put case. We then prove the inequalities satisfied by the free boundaries. In the EER the function F satisfies

$$\frac{\partial F}{\partial t} + \mathcal{L}F \leq rF \tag{2.B8}$$

On the EER in the call case $F(t,S,r) = S-K$ and therefore Equation (2.B8). simplifies to $1 \cdot S(r-q) \leq r(S-K)$. Thus $-Sq \leq -rK$ for all $S \in EER_r$, i.e. $S \leq \frac{r}{q}K$ for all $S \in EER_r$, as $q < 0$. By passing to the supremum we get $K \leq \underline{S}^*(t,r) < \overline{S}^*(t,r) \leq \frac{rK}{q}$.

At maturity $\underline{S}^*(T,r) = K$ and $\overline{S}^*(T,r) = +\infty$, as the option is exercised at T whenever $S(T) \geq K$.

We now show that $\underline{S}^*(T^-,r) = K$ and $\overline{S}^*(T^-,r) = \frac{rK}{q}$. By construction $\underline{S}^*(t,r) \geq K$ for all $t \in (\bar{t};T)$, and hence $\underline{S}^*(T^-,r) \geq K$. Suppose by contradiction that $\underline{S}^*(T^-,r) > K$. The set $(\bar{t};T) \times \left(K;\underline{S}^*(T^-,r)\right) \subset CR_r$ and therefore $(\mathcal{L}-r)F = -\frac{\partial}{\partial t}F \geq 0$, as F is decreasing w.r.t. t. As $t \uparrow T$ we have $(\mathcal{L}-r)F \to (\mathcal{L}-r)(S-K) = -qS + rK$ for $S \in \left(K;\underline{S}^*(T^-,r)\right)$. This implies $-qS + rK \geq 0$ for $S \in \left(K;\underline{S}^*(T^-,r)\right)$ and passing to the supremum over $S \in \left(K;\underline{S}^*(T^-,r)\right)$ this delivers $\underline{S}^*(T^-,r) \geq \frac{rK}{q}$ which is a contradiction. We deal now with the upper free boundary limit. Suppose (by contradiction) that $\overline{S}^*(T^-,r) < \frac{rK}{q}$. But then the set $(\bar{t};T) \times \left(\overline{S}^*(T^-,r);\frac{rK}{q}\right) \subset CR_r$ and $(\mathcal{L}-r)F = -\frac{\partial}{\partial t}F \geq 0$ for $S \in \left(\overline{S}^*(T^-,r);\frac{rK}{q}\right)$. As $t \uparrow T$ we have $(\mathcal{L}-r)F \to (\mathcal{L}-r)(S-K) = -qS + rK$ for $S \in \left(\overline{S}^*(T^-,r);\frac{rK}{q}\right)$ (here the limits are in distribution). Then $-qS + rK \geq 0$ for all $S \in \left(\overline{S}^*(T^-,r);\frac{rK}{q}\right)$ and therefore also for the infimum $-q\overline{S}^*(T^-,r) + rK \geq 0$ that implies the contradiction $\overline{S}^*(T^-,r) \geq \frac{rK}{q}$.

2.C Additional numerical analysis of the free boundary

In this section we provide additional numerical analysis of American equity options in our Vasicek interest rate framework.[9]

The impact of the correlation between S and r is addressed in Tables 2.C.1 and 2.C.2.

Tables 2.C.3 and 2.C.4 address the impact of the speed of mean reversion κ on the price of American call and put options. Tables 2.C.3 and 2.C.4 assume $\kappa = 0.25$, namely one forth of the baseline case whose pricing results are reported in Tables 2.1 and 2.2. We notice that, as the interest rates are now expected to move much slower, the relative pricing errors with respect to a model with constant interest rates are smaller. Interestingly, this is more pronounced for American put options. Furthermore, it is interesting to notice also how the free boundaries (if any) change for this new value of κ. Figure 2.C.1 shows two r-sections of the free boundaries for the call and the put option respectively when $q = -2\%$. We notice that, everything else being equal, with respect to the baseline graphs in Figures 2.6 and 2.10, the early exercise region widens for the call option but reduces for the put one.

Tables 2.C.5 and 2.C.6 address the impact of the volatility coefficient of the interest rate σ_r on the price of American and put options. In particular, Tables 2.C.5 and 2.C.6 assume $\sigma_r = 5\%$ while in the baseline model its value was equal to 1%. A larger volatility impact positively on the price of the American call option as its payoff is increasing in the value of S which, in turns, is increasing in the value of r. Therefore, relative errors with respect to a constant interest rates are much larger. For the American put option, the errors are comparable to the baseline case as the payoff of the option now does not benefit from possibly higher values of r. Furthermore, Figure 2.C.2 shows how the free boundaries look like when $q = -2\%$. As before, we notice that the early exercise regions widen.

[9] Medvedev and Scaillet (2010) introduce an analytical approach to price American options using a short-maturity asymptotic expansion. They perform a throughout numerical investigation for American call and put options with both stochastic CIR interest rates and stochastic underlying's volatility. Analogously, Boyarchenko and Levendorskiǐ (2013) consider a stochastic volatility equity and stochastic interest rates depending on two CIR factors, allowing for non-zero correlations between all the underlying processes. They provide a sophisticated iterative algorithm to price American derivatives that exploits a sequence of embedded perpetual options and their pricing results are in line with those of the Longstaff and Schwartz method and the asymptotics of Medvedev and Scaillet (2010). Pressacco et al. (2008) perform a throughout comparison between lattice and analytical approximation of the early exercise premium in the Black-Scholes framework.

ρ	q	π_E	π_A	$\pi_A^{r_0}$	$\lvert\pi_A - \pi_A^{r_0}\rvert/\pi_A$
American put option					
	0%	5.540%	5.674%	5.979%	5.37%
-50%	2%	6.486%	6.499%	6.962%	7.12%
	-2%	4.683%	5.002%	5.230%	4.55%
	0%	5.606%	5.705%	5.979%	4.79%
-5%	2%	6.551%	6.557%	6.962%	6.17%
	-2%	4.748%	5.025%	5.230%	4.07%
	0%	5.613%	5.709%	5.979%	4.72%
0%	2%	6.558%	6.563%	6.962%	6.08%
	-2%	4.755%	5.028%	5.230%	4.01%
	0%	5.620%	5.712%	5.979%	4.67%
5%	2%	6.565%	6.570%	6.962%	5.96%
	-2%	4.763%	5.030%	5.230%	3.97%
	0%	5.672%	5.745%	5.979%	4.06%
50%	2%	6.629%	6.630%	6.962%	5.00%
	-2%	4.827%	5.053%	5.230%	3.50%

Table 2.C.1: Results from the three numerical examples for the American put option.

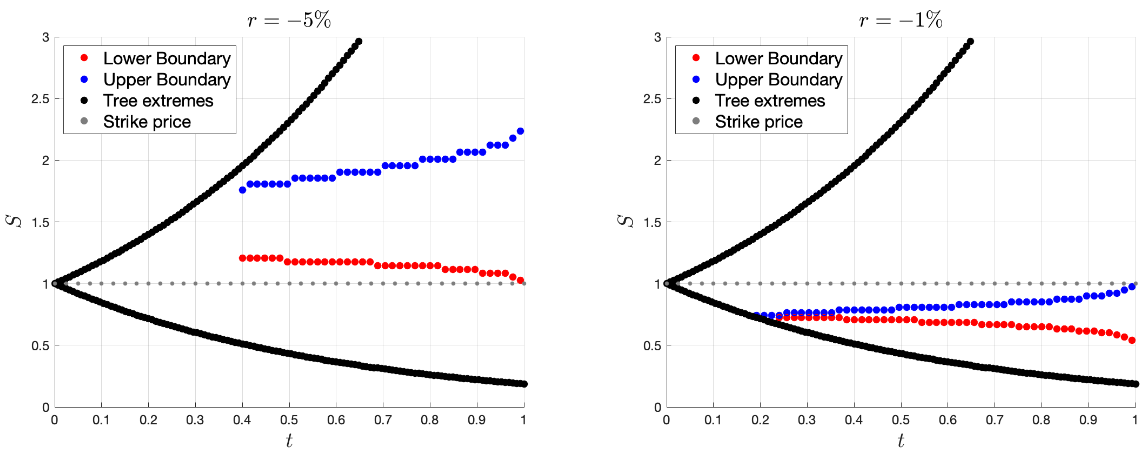

Figure 2.C.1: $r-$sections of free boundaries for the American call (left panel) and put (right panel) option with $\kappa = 0.25$ and $q = -2\%$.

ρ	q	π_E	π_A	$\pi_A^{r_0}$	$\lvert\pi_A - \pi_A^{r_0}\rvert/\pi_A$
American call option					
	0%	6.269%	6.271%	5.979%	4.66%
-50%	2%	5.235%	5.356%	5.163%	3.61%
	-2%	7.432%	7.432%	7.102%	4.44%
	0%	6.335%	6.336%	5.979%	5.64%
-5%	2%	5.230%	5.390%	5.163%	4.22%
	-2%	7.497%	7.497%	7.102%	5.26%
	0%	6.342%	6.343%	5.979%	5.75%
0%	2%	5.307%	5.393%	5.163%	4.27%
	-2%	7.505%	7.505%	7.102%	5.37%
	0%	6.339%	6.339%	5.979%	5.68%
5%	2%	5.314%	5.396%	5.163%	4.32%
	-2%	7.511%	7.511%	7.102%	5.45%
	0%	6.415%	6.415%	5.979%	6.80%
50%	2%	5.378%	5.431%	5.163%	4.94%
	-2%	7.576%	7.576%	7.102%	6.25%

Table 2.C.2: Results from the three numerical examples for the American call option.

Figure	q	π_E	π_A	$\pi_A^{r_0}$	$\lvert\pi_A - \pi_A^{r_0}\rvert/\pi_A$
-	0%	6.181%	6.193%	5.979%	3.46%
-	2%	5.170%	5.281%	5.163%	2.23%
2.C.1	-2%	7.317%	7.327%	7.102%	3.01%

Table 2.C.3: Results from the three numerical examples for the American call option with $\kappa = 0.25$.

Figure	q	π_E	π_A	$\pi_A^{r_0}$	$\lvert\pi_A - \pi_A^{r_0}\rvert/\pi_A$
-	0%	5.871%	5.883%	5.979%	1.63%
-	2%	6.841%	6.856%	6.962%	1.55%
2.C.1	-2%	4.986%	5.183%	5.230%	1.00%

Table 2.C.4: Results from the three numerical examples for the American put option with $\kappa = 0.25$.

Figure	q	π_E	π_A	$\pi_A^{r_0}$	$\lvert\pi_A - \pi_A^{r_0}\rvert/\pi_A$
-	0%	6.726%	6.746%	5.979%	11.37%
-	2%	5.686%	5.746%	5.163%	10.15%
2.C.2	-2%	7.886%	7.895%	7.102%	10.04%

Table 2.C.5: Results from the three numerical examples for the American call option with $\sigma_r = 5\%$.

Figure	q	π_E	π_A	$\pi_A^{r_0}$	$\lvert\pi_A - \pi_A^{r_0}\rvert/\pi_A$
-	0%	6.011%	6.192%	5.979%	3.44%
-	2%	6.982%	7.089%	6.962%	1.80%
2.C.2	-2%	5.125%	5.410%	5.230%	3.33%

Table 2.C.6: Results from the three numerical examples for the American put option with $\sigma_r = 5\%$.

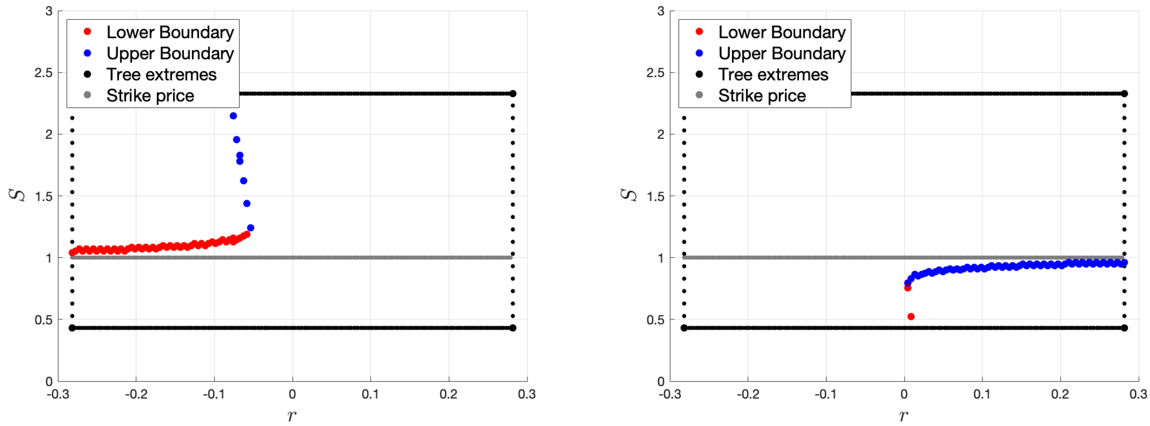

Figure 2.C.2: $r-$sections of free boundaries for the American call (left panel) and put (right panel) option with $\kappa = 0.25$ and $q = -2\%$.

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3

Barrier Options under Correlated Equity and Interest Rate Risks

3.1 Introduction

"You can't get a risk-free rate above zero, and everything else is relative"
M. Bradshow, head of global aggregate fixed income at Amundi,
Financial Times, August 13 2019

The risk-free interest rate plays a crucial role in derivative pricing; indeed, it is well known that the interest rate drives both the discount factor and the expected drift of risky equity processes under the risk neutral measure. Furthermore, current European market conditions challenge the traditional assumption of non negative interest rates. This assumption, which is pervasive in the economic and financial literature, underestimates the relevant impact negative interest rates may have on the discount rate and on the expected growth of equity price processes. Moreover, also the correlation between the interest rate and the equity risk factors, which is usually neglected, is of great relevance when dealing with derivatives pricing.
In the present paper I investigate the impact of stochastic, possibly negative and correlated interest rates on the price of derivatives with barrier features. Besides being actively traded by investors, this particular kind of path-dependent derivatives has indeed many applications also in corporate valuation theory.
Exploiting a Vasicek (1977) model, rather than a pure diffusive Black and Scholes (1973) or a Cox et al. (1885) model, I document critical mispricings that arise if constant, or exogenously positive, or uncorrelated interest rates are used to price this kind of path-dependent derivatives. This pricing errors are the result of non-trivial interplays of the two channels through which the interest rate process affects the pricing of barrier derivatives: 1) the discount rate, 2) the expected drift of the equity process under the risk neutral measure, which is the main driver of the probability that a given threshold is crossed or not.
Depending on the specific parameters of the option considered, relative pricing errors may be as large as 15%. For American options, this mispricing is easily observable from the different optimal exercise policies implied by the alternative models.

It is well known that financial derivatives are priced as if agents were risk-neutral. In such a world, the risk-free interest rate plays two roles: on one hand it represents the rate at which the risky equity appreciates and on the other hand it drives the dollar-value of time, namely the discount rate. As a consequence, a careful modelling of interest rates is crucial when pricing derivatives written on risky equity and a constant interest rate is clearly not a realistic assumption. Furthermore, current market conditions challenge also the so called *zero lower bound* assumption, namely the hypothesis that interest rates are always non negative. As of late 2019, yields on European AAA-rated sovereign bonds are negative up to very long maturities[1] and from 2010 to 2016 also the US 3 Month Treasury Bill's yield has been critically close to zero and, on some occasions, also below this lower bound, at least intradaily[2]. It is economically compelling, though, to assume that interest rates will revert back to the positive domain sooner or later. Therefore, capturing this expected path is of great relevance when pricing equity derivatives.

On a parallel note, traditional macroeconomic and fiscal policy literature (see, e.g. Woodford (2003)) postulates a significant correlation between risk-free interest rates and returns on the stock market: according to the traditional view, if the market is performing bad, interest rates should be kept low to boost investments; on the contrary, when the stock market is expanding, interest rates should be kept high in order to curb the market and avoid bubbles or sudden harmful drops. Evidence of this correlation is mixed: although the empirical literature almost unanimously agrees on its statistical significance, its sign appears to be a more debatable issue and to surely change over time. Nevertheless, and regardless of the sign, the correlation between interest rates and the risky equity plays a relevant role when pricing derivatives whose payoff depends on both the equity price and the interest rate. In this perspective, path dependent derivatives are surely the ones more exposed to the comovements of the risky equity and of in interest rate as their payoff depends on the whole trajectories of both of them.

Considering the wide class of traded path-dependent derivatives, barrier options are surely among the most relevant ones. Generally speaking, barrier options are financial derivatives, written on an underlying, whose payoff depends on whether the underlying's price or value crosses or not an exogenous threshold (the so-called *barrier*) during the life of the option. Derivatives with barrier features have been studied since the very begging of mathematical finance and, indeed, the first barrier option is discussed and priced in the seminal work of Merton (1973)[3]. Since the payoff of these derivatives is paid out if and only if the barrier event occurs, they trade at a lower price than their plain Vanilla counterparts. This discount makes derivatives with barrier features

[1]More precisely, as of October 1 2019, yields of AAA-rated Euro area central government bonds are negative up to 28 years; source: European Central Bank, www.ecb.europa.eu.

[2]Source: www.cnbc.com.

[3]Section 26.9 of Hull (2018) and Chapter 18 of Björk (2009) are entirely devoted to barrier options and may serve as an effective introduction to this kind of derivatives. Standard literature references for barrier options include Rich (1994), Ritchken (1995) and Cheuk and Vorst (1996) that carry on their analysis within the standard Black-Scholes diffusive framework and its traditional lattice approximations, namely the binomial and the trinomial tree. More recent works, like Fusai and Recchioni (2007), Bernard et al. (2008) and Carr and Crosby (2010), expand the class of financial market models considered accounting for local or stochastic volatility, jumps and stochastic interest rates.

appealing. Although pure barrier options are often traded over the counter, many other derivatives that embed barrier features are traded on exchanges: exchange-traded notes in US market and certificates in the European one are prominent examples of this kind of derivatives.

Besides being directly traded or embedded in more complex securities, barrier options have many applications to several valuation problems outside derivatives pricing. For example, barrier options pricing is relevant when valuing executive stock options (ESOs henceforth), which are options granted to a company's executives as part of their compensation and whose value depend on the performance of the company itself [4]. These options are usually at-the-money American call options on the company's stock granted for free to executives. As explained in Carr (1995), these options are long-lived (usually 10 years) and they might display also an initial vesting period (usually 3 years) during which their early exercise it is not allowed. Hull and White (2004a) provide sound evidence that ESOs are exercised when the stock price reaches a certain multiple of the strike price. Therefore, this kind of ESOs can be treated and priced as American up-and-out call option with an exogenous barrier that, if crossed, makes the option worthless: as a consequence, the optimal exercise policy for this kind of option is to exercise it when the stock price approaches the barrier and the probability of been knocked-out is high.
Derivatives with barrier features are also widely considered in capital structure theory, especially when valuing corporate securities. Since the influential work of Black and Cox (1976), barrier options have been recognised as a useful tool when modelling features of debt and equity. As an example, bondholders might have the legal right to claim the property of the assets if the firm is performing poorly, namely, if the firm's asset value falls below a given threshold. Therefore, this feature can be modelled and priced within debt securities as a down-and-out barrier option. Furthermore, Brockman and Turtle (2003) and Wong and Choi (2009) argue and prove empirically that barrier option models perform well when used to estimate the default barrier of a firm, namely the firm's asset level below which the firm is forced to fill for bankruptcy.
Finally, derivatives with barrier features have many applications also in the evaluation of certain insurance financial products. As an example, Grosen and Jørgensen (2002) show how having a regulatory authority that monitor solvency requirements of insurance companies can be efficiently embedded in a barrier options framework.

Guided by all of these applications, I study barrier options exposed to correlated market and interest rate risks. Building on Battauz and Rotondi (2019), I propose lattice-based algorithms to price European and American discretely monitored knock-in/knock-out options. Furthermore, I

[4]ESOs have been extensively discussed in the literature. Classical references include the seminal works of Yermack (1995) and Core and Guay (2002), whereas Devers et al. (2007) provide an effective review of the state of the art.

show how to efficiently extend these algorithms to price also continuously monitored options, reducing the well known biases[5] Then, I document the sizeable impact of both assuming a stochastic term structure and a non-zero correlation between the equity price and the interest rate by means of several numerical examples.

The rest of the paper is organized as follows. Section 3.2 contains the description of the financial market model considered, of the primary assets traded therein and of the barrier options I introduce in the market. Pricing techniques for these options are then formalized. Section 3.3 contains numerical examples of the results in the previous Section. More precisely, the market model is first calibrated to real European market data. Then illustrative numerical examples deliver the relative pricing errors of several alternative models with respect to mine. Finally, two concrete examples of my framework are illustrated: a European equity-linked note and an American executive stock option. Section 3.4 concludes.

3.2 Barrier Options

The current Section contains the theoretical contribution of my paper. The financial market model I choose and the assets and derivatives traded therein are described in Subsection 3.2.1. Subsection 3.2.2 provides the pricing algorithm for discretely monitored barrier options whereas Subsection 3.2.3 deals with their continuously monitored counterparts and tackles all the related numerical issues. The formal proofs of all the propositions stated are deferred to Appendix 3.A.

3.2.1 The market: primary assets and derivatives

Short-term interest rates have been modelled in many ways in the literature[6]. Restricting to one-factor models, the most famous ones include the generalized Brownian motion of Merton (1973), the mean-reverting stochastic process of Vasicek (1977), the so-called square-root diffusion of Cox et al. (1885) and their extensions proposed by Ho and Lee (1986) and Hull and White (1990). Since it is economic compelling to assume some stationarity of the interest rates (at least up to a trend) and since I have to remove the zero lower bound, I choose the mean-reverting model in Vasicek (1977).

I consider a simplified financial market with just two (correlated) risk factors/sources of randomness: a “equity” risk factor driving an underlying’s risky price process $S(t)$ and a interest rate

[5]See, e.g., Chapter 26, Section 7 of Hull (2018) that affect lattice-based pricing of barrier derivatives.
[6]As a standard reference, Chapter 23 of Björk (2009) is entirely devoted to short rate models.

risk factor driving a short-term interest rate process $r(t)$. I assume that the financial market is both arbitrage-free and complete. As a consequence[7], there exists a unique risk-neutral measure $\mathbb{Q}$. Following Vasicek (1977), I assume that, under $\mathbb{Q}$, the short-term interest rate r follows a mean reverting stochastic process and that the continuously compounded log return of the underlying's risky price process S follows a generalized Brownian motion. Therefore, the stochastic differential equations (SDEs henceforth) characterizing the financial markets are

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{S(t)} &= (r(t) - q)\mathrm{d}t + \sigma_S \mathrm{d}W_S^{\mathbb{Q}}(t) \\ \mathrm{d}r(t) &= \kappa\left(\theta - r(t)\right)\mathrm{d}t + \sigma_r \mathrm{d}W_r^{\mathbb{Q}}(t) \\ \mathrm{d}B(t) &= r(t)B(t)\mathrm{d}t \end{cases} \tag{3.1}$$

where q is the constant dividend yield of the underlying, σ_S is its volatility, κ is the speed of mean-reversion of the interest rate process, θ its long-term mean and σ_r its volatility. $B(t)$ is the money market account[8] the allows the investor to capitalize the interest rate r. The solution of the SDEs in (3.1) is

$$\begin{cases} S(t) &= S(0)\exp\left[\displaystyle\int_0^t r(s)\mathrm{d}s - \left(q + \frac{\sigma_S^2}{2}\right)t + \sigma_S W_S(t)\right] \\ r(t) &= r(0)e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma_r \displaystyle\int_0^t e^{-\kappa(t-s)}\mathrm{d}W_r(s) \\ B(t) &= \exp\left[\displaystyle\int_0^t r(s)\mathrm{d}s\right] \end{cases} \tag{3.2}$$

To actually complete the market, I assume that a continuum of τ-zero coupon bonds is traded in the market with $\tau \in (0, T]$. Consider a generic T-zero coupon bond. This asset pays 1 to its holder at maturity T and its price at $t \in (0, T)$ is labelled with $p(t, T)$. By no arbitrage valuation, we have

$$p(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T)}\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left[-\int_0^T r(s)\mathrm{d}s\right]\middle|\mathcal{F}_t\right],$$

that admits a closed formula solution as derived in Section 3.2.1 of Brigo and Mercurio (2007):

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)} \tag{3.3}$$

[7]See Delbaen and Schachermayer (1994) for the general version of the First Fundamental Theorem of Asset Pricing in a continuous time framework.

[8]$B(t)$ serves as numéraire of the unique risk-neutral measure $\mathbb{Q}$. Therefore, any traded security discounted by B has to be a $\mathbb{Q}$-martingale.

where:

$$
\begin{aligned}
B(t,T) &= \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\\
A(t,T) &= \left(\theta - \frac{\sigma_r^2}{2\kappa^2}\right)\left(B(t,T) - (T-t)\right) - \frac{\sigma_r^2 B^2(t,T)}{4\kappa}.
\end{aligned}
$$

Within this market model, at $t < T$ the investor can trade (without frictions) S, B, and $p(t,\tau)$, $\tau \leq T$: these are the *primary assets* of the market.

Derivatives are traded in the market as well and, since $\mathbb{Q}$ is unique, they can be priced uniquely. If $\varphi(S(T))$ is the terminal payoff of a European derivative written on S, its unique price at $t \leq T$ is given by

$$
\pi_\varphi^E(t) = \mathbb{E}\left[e^{-\int_t^T r(s)\mathrm{d}s}\varphi(S(T)) \middle| \mathcal{F}_t \right]. \tag{3.4}
$$

If, on the contrary, the derivative is of American- style, its holder can cash in its payoff $\varphi(S(\tau))$ at any time τ before the maturity T. The rational holder of the American derivative will timely exercise it when its payoff is maximum. Therefore, the unique price at $t \leq T$ of the American derivative is given by

$$
\pi_\varphi^A(t) = \operatorname{ess}\sup_{\tau\in[t,T]} \mathbb{E}\left[e^{-\int_t^\tau r(s)\mathrm{d}s}\varphi(S(\tau)) \middle| \mathcal{F}_t \right]. \tag{3.5}
$$

Whichever the payoff function φ, the expected value in (3.4) is rarely known explicitly and the one in (3.5) is actually never known explicitly.

As they will be useful later one, I acknowledge that the price of plain Vanilla European put and call options admits an explicit expression. In particular[9], within the financial market specified in (3.1), the price at $t \in [0,T]$ of an European put option on S with strike K is equal to

$$
\pi_{put}^E(t, S(t), r(t)) = Kp(t,T)N(-\tilde{d}_2) - S(t)e^{-q(T-t)}N(-\tilde{d}_1) \tag{3.6}
$$

with:

$$
\begin{aligned}
\tilde{d}_1 &= \frac{1}{\sqrt{\Sigma_{t,T}^2}}\left(\ln\frac{S(t)}{Kp(t,T)} + \frac{1}{2}\Sigma_{t,T}^2 - q(T-t)\right),\\
\tilde{d}_2 &= \tilde{d}_1 - \sqrt{\Sigma_{t,T}^2},\\
\Sigma_{t,T}^2 &= \sigma_S^2(T-t) + 2\sigma_S\sigma_r\rho\left(\frac{-1 + e^{-\kappa(T-t)} + \kappa(T-t)}{k^2}\right) + \\
&\qquad -\sigma_r^2\left(\frac{3 + e^{-2\kappa(T-t)} - 4e^{-\kappa(T-t)} - 2\kappa(T-t)}{2k^3}\right).
\end{aligned}
$$

[9]See Battauz and Rotondi (2019) for the full derivation.

The price at $t \in [0, T]$ of an European call option on S with strike K is equal to

$$\pi^E_{call}(t, S(t), r(t)) = S(t)e^{-q(T-t)}N(\tilde{d_1}) - Kp(t, T)N(\tilde{d_2}). \tag{3.7}$$

I now turn my analysis to derivatives with barrier features.

A barrier option is an exotic derivative whose payoff depends on whether the underlying reaches (or not) a pre-specified threshold during the life of the option.
As the payoff of barrier options is triggered by the highest/lowest value reached by the underlying during the life of the option, let

$$m(T) := \min_{t\in[0,T]} S(t) \tag{3.8}$$

denote the *running minimum* of S, namely the lowest value reached by S in the time window $[0, T]$ and let

$$M(T) := \max_{t\in[0,T]} S(t) \tag{3.9}$$

denote the *running maximum* of S, namely the highest value reached by S in the time window $[0, T]$. If S is a geometric Brownian motion (namely if $r(t)$ in (3.1) is constant), the marginal densities of $m(t)/M(t)$ and the joint ones $(m(t), S(t))/(M(t), S(t))$ are known and can be found in Section 18.1 of Björk (2009). If $r(t)$ in (3.1) is stochastic, none of the aforementioned distributions is known explicitly.
Let $\varphi(S(T))$ a generic function of the underlying at maturity. There exist two basic kinds of barrier options:

- the *knock-out barrier option* that grants the payoff to its holder if and only if the underlying never crosses a given threshold during the life of the option; assuming that $S_0 > B$, the payoff of a knock-out barrier option at maturity T can be written as

  $$X(T) = \varphi(S(T))\mathbb{1}\left(\min_{t\in[0,T]} S(t) > B\right) = \varphi(S(T))\mathbb{1}\left(m(T) > B\right), \tag{3.10}$$

  where $\mathbb{1}(A)$ denotes the indicator function of the event A.

- the *knock-in barrier option* that grants the payoff to its holder if and only if the underlying reaches a given threshold during the life of the option; assuming that $S_0 < B$, the payoff of a knock-in barrier option at maturity T can be written as

  $$X(T) = \varphi(S(T))\mathbb{1}\left(\max_{t\in[0,T]} S(t) > B\right) = \varphi(S(T))\mathbb{1}\left(M(T) > B\right). \tag{3.11}$$

Similar payoffs can be defined for knock-out barrier options if $S_0 < B$ and for knock-in ones if $S_0 > B$. Finally, a third kind of barrier option that involves both knock-out and knock-in features is often traded:

- the *double barrier option* that grants the payoff to its holder if and only if the underlying strictly lies always between two barriers during the life of the option; assuming $B_L < S_0 < B_U$, the payoff of a double barrier option at maturity T can be written as

$$\begin{aligned} X(T) &= \varphi(S(T))\mathbb{1}\left(\min_{t\in[0,T]} S(t) > B_L \cap \max_{t\in[0,T]} S(t) < B_U\right) \\ &= \varphi(S(T))\mathbb{1}\left(m(T) > B_L \cap M(T) < B_U\right). \end{aligned} \tag{3.12}$$

Before moving on to the main pricing results of the paper, it is worth to notice that barrier options are surely not the only path-dependent derivatives that can be priced with the methodologies hereafter. Indeed, these methodologies work for any kind of European/American derivative that meets the following conditions:

- the payoff of the derivative must depend only on a function f of the current value of the state variables and of their past realizations ;
- it must be possible to evaluate f at $t + \Delta t$ starting solely from f at t and the value of the state variables at $t + \Delta t$.

As an instance, lookback options or Asian options both meet the two requirements above and could be priced exploiting similar methodologies.

3.2.2 Discretely monitored barrier options

Assume that the derivative's contract requires a discrete monitoring of the underlying. Namely, assume that the contract specifies a finite set of monitoring dates $\{t_i\}_{i=0,\ldots,N}$ at which the level of the underlying is checked. Usually, short-term derivatives require a daily monitoring of the underlying at the closing price, whereas long-term derivative, such as equity-linked notes, might prescribe monthly or even semi-annual monitoring.
Even when S is lognormally distributed, the distribution of the discrete versions of the running maximum (resp. minimum) of S, namely $\max_{t\in\{t_0,\ldots,t_N\}} S(t)$ (resp. $\min_{t\in\{t_0,\ldots,t_N\}} S(t)$), is not known explicitly. Neither do they when $r(t)$ is stochastic as in my market model (3.1). Therefore, the pricing of discretely monitored barrier options has to rely on numerical techniques.
I first focus my attention on European and American knock-out contracts that can both be obtained by backward recursion along the monitoring dates. Then, I move to European and American knock-in contracts. I first state the general in-out parity that readily delivers the price of the European knock-in contract and then I describe a numerical algorithm to price American knock-in derivatives.

3.2.2.1 Knock-out contracts

Without loss of generality, I focus my analysis on so-called *down-and-out options* which are knock-out barrier options with $S_0 > B$; the name derives from the fact that if at any of the monitoring dates S falls *down*, below B, the barrier option goes *out* of the money and its payoff at maturity is null.
Consider a discretely monitored European down-and-out option. Similarly to (3.10), its payoff at maturity T is

$$X_{DO}(T) = \varphi(S(T))\mathbb{1}\left(\min_{t\in\{t_0,\dots,t_N\}} S(t) > B\right) \tag{3.13}$$

where $\{t_0 = 0, \dots, t_N = T\}$ is a set of monitoring dates specified in the contract and $B < S_0$ is the knock-out barrier. The value of this derivative at inception can be computed by backward induction as the following Proposition shows.

Proposition 3.1 ***(Price of an European down-and-out option).*** *The price at inception of the European derivative in (3.13) is given by $v^E_{DO}(0)$ where*

$$v^E_{DO}(t_N) = \varphi(S(T))\mathbb{1}\left(S(T) > B\right) \tag{3.14}$$

and

$$v^E_{DO}(t_i) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_{t_i}^{t_{i+1}} r(s)\mathrm{d}s\right) v^E_{DO}(t_{i+1})\mathbb{1}\left(S(t_i) > B\right)\middle| \mathcal{F}_{t_i}\right] \tag{3.15}$$

for $i = N-1, \dots, 0$.

Proof See Appendix 3.A. ■

It is interesting to notice that Proposition 3.1 just prescribes to check whether the underlying is above the barrier date by date: it is not necessary to keep track of the whole path of S. On the contrary, only a local check is needed. This is due to the fact that

$$\mathbb{1}\left(\min_{t\in\{t_0,\dots,t_N\}} S(t) > B\right) = \prod_{i=0,\dots,N} \mathbb{1}\left(S(t_i) > B\right), \tag{3.16}$$

and therefore, if S is always above the barrier, necessarily also its running minimum is.
Proposition 3.1 provides the value of the down-and-out option at maturity only. Nevertheless, the price of the same option at any intermediate monitoring date t_i, $i = 0, \dots, N$, conditional on not having been knocked out before, is given by

$$\pi^E_{\varphi,DO}(t_i) = \begin{cases} v^E_{DO}(t_i) & \text{if } \min_{t\in\{t_i,\dots,t_N\}} S(t) > B \\ 0 & \text{else.} \end{cases}$$

Namely, the intermediate values of $v_{DO}^E(t_i)$ in Proposition 3.1 represent also the prices of the option conditional on not having crossed the barrier at previous monitoring dates.
Consider now the American version of the discretely monitored down-and-out option. Its holder has now the additional opportunity to exercise the option and cash in the payoff $\varphi(S(t_i))$ at any monitoring date $\{t_i\}_{i=0,\ldots,T}$, provided that the underlying never crossed the barrier before t_i. The following Proposition shows how to price this American-style down-and-out barrier option.

Proposition 3.2 ***(Price of an American down-and-out option).*** *The price at inception of the American down-and-out derivative with payoff $\varphi(S(t_i))$ is given by $v_{DO}^A(0)$ where*

$$v_{DO}^A(t_N) = \varphi(S(T))\mathbb{1}\left(S(T) > B\right)$$

and

$$v_{DO}^A(t_i) = \max\left\{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_i}^{t_{i+1}} r(s)\mathrm{d}s} v_{DO}^A(t_{i+1})\middle| \left(S(t_i), r(t_i)\right)\right], \varphi(S(t_i))\right\}\mathbb{1}\left(S(t_i) > B\right)$$

for $i = N-1,\ldots,0$.

Proof See Appendix 3.A. ∎

As before, Proposition 3.2 explicitly provides the price of the American down-and-out option only at inception. Nevertheless, the price of the American option at any intermediate t_i, $i = 0,\ldots,N$, conditional on not having been knocked-out before, is given by

$$\pi_{\varphi,DO}^A(t_i) = \begin{cases} v_{DO}^A(t_i) \text{ if } \min\limits_{t\in\{t_i,\ldots,t_N\}} S(t) > B \\ 0 \qquad\quad \text{else.}\end{cases}$$

3.2.2.2 Knock-in contracts

As for the knock-out contracts, I focus my analysis on the so-called *down-and-in options* which are knock-in barrier options with $S_0 > B$; the name derives from the fact that if at any of the monitoring dates S falls *down*, below B, the holder of the option goes *in* and she can hence claim the payoff of the option whenever positive.
Consider a discretely monitored European down-and-in option. Similarly to (3.11), its payoff at maturity T is

$$X_{DI}(T) = \varphi(S(T))\mathbb{1}\left(\min_{t\in\{t_0,\ldots,t_N\}} S(t) < B\right) \tag{3.17}$$

where $\{t_0 = 0,\ldots,t_N = T\}$ is a set of monitoring dates specified in the contract and $B < S_0$ is the knocking-in barrier.

Unfortunately, for in contracts an equivalence between indicator functions similar to (3.16) does not hold as

$$\mathbb{1}\left(\min_{t\in\{t_0,\dots,t_N\}} S(t) < B\right) \neq \prod_{i=0,\dots,N} \mathbb{1}\left(S(t_i) < B\right).$$

Indeed, it is not necessary that S is below the barrier at *all* the monitoring dates to receive the final payoff. The sufficient condition is that S is below the barrier at least once.

Nevertheless, the following simple parity result allows to retrieve the price of European down-and-in options from the price of the related down-and-out one and of the plain Vanilla European option with payoff $\varphi(S(T))$.

Proposition 3.3 ***(European in-out parity).*** *The price of the European knock-out option with payoff $\varphi(S(T))$ and of the European knock-in option with the same payoff satisfy*

$$\pi^E_{\varphi,DO}(t) + \pi^E_{\varphi,DI}(t) = \pi^E_{\varphi}(t), \quad \forall t \in [0,T]$$

where $\pi^E_{\varphi}(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\mathrm{d}s}\varphi(S(T))\middle|\mathcal{F}_t\right]$ is the no-arbitrage price of the plain European derivative that pays $\varphi(S(T))$ at T.

Proof See Appendix 3.A. ■

Unfortunately, this Proposition does not apply to American-style barrier options as

$$\begin{aligned}\pi^A_{\varphi} &= \sup_{\tau\in\{t_0,\dots,t_n\}} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^\tau r(s)\mathrm{d}s}\left(X_{\varphi,DO}(\tau) + X_{\varphi,DI}(\tau)\right)\right] \\ &\neq \underbrace{\sup_{\tau\in\{t_0,\dots,t_n\}} \mathbb{E}^{\mathbb{Q}}\left[\frac{X_{\varphi,DO}(\tau)}{e^{\int_0^\tau r(s)\mathrm{d}s}}\right]}_{\pi^A_{\varphi,DO}} + \underbrace{\sup_{\tau\in\{t_0,\dots,t_n\}} \mathbb{E}^{\mathbb{Q}}\left[\frac{X_{\varphi,DI}(\tau)}{e^{\int_0^\tau r(s)\mathrm{d}s}}\right]}_{\pi^A_{\varphi,DI}}.\end{aligned}$$

Actually, as π^A_{φ}, $\pi^A_{\varphi,DO}$ and $\pi^A_{\varphi,DI}$ are all non negative numbers, it holds $\pi^A_{\varphi} \leq \pi^A_{\varphi,DO} + \pi^A_{\varphi,DI}$ and we can quantify the *deviation from parity* of American options, if any, as

$$DFP = \pi^A_{\varphi,DO} + \pi^A_{\varphi,DI} - \pi^A_{\varphi}. \tag{3.18}$$

DFP is clearly zero for European options and it is non negative for American ones.

First attempts to price these American in-contracts date back to Gao et al. (2000) that extended the traditional free boundary approach to American options accounting for the new boundary condition dictated by the barrier. Nevertheless, this technique is developed in a continuous-time (and continuous monitoring) standard diffusive market, under the assumption of constant interest rate. More recently, Jun and Ku (2015), inspired by Ingersoll (1998), tackle the same issue exploiting an approximation of the barrier feature of the American option by a convenient mix of digital (or binary) contracts.

I follow and extend the approach first suggested by Hull and White (1993) and Ritchken et al. (1993), the so-called *forward shooting grid* (FSG henceforth). The key tool is the "Markovianization" of the payoff that simplifies the evaluation of the American options' continuation value. Indeed, the following Proposition shows how the continuation value of the American option at t_i does not depend on whole history contained in $\mathcal{F}_{t_i}$ but only on the current value of the two state variables $(S(t_i), r(t_i))$ and of the auxiliary state variable $m(t_i)$, the running minimum.

Proposition 3.4 ***(Price of an American down-and-in option).*** *The price at inception of the American down-and-in derivative with payoff $\varphi(S(t_i))$ is given by $v^A_{DI}(0)$ where*

$$v^A_{DI}(t_N) = \varphi(S(T))\mathbb{1}\left(m(T) \leq B\right)$$

and

$$v^A_{DI}(t_i) = \max\left\{\varphi(S(t_i))\mathbb{1}\left(m(t_i) \leq B\right), \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_i}^{t_{i+1}} r(s)\mathrm{d}s} v^A_{DI}(t_{i+1})\middle| (S(t_i), r(t_i), m(t_i))\right]\right\}$$

for $i = N-1, \ldots, 0$.

Proof See Appendix 3.A. ■

In particular, the dependence of $S(t_{i+1})$ and $r(t_{i+1})$ on $S(t_i)$ and $r(t_i)$ is trivial; on the contrary, $m(t_{i+1})$ depends on $m(t_i)$ (and on $S(t_i)$) since

$$m(t_{i+1}) = \min\{m(t_i), S(t_{i+1})\}. \tag{3.19}$$

Assuming a generic discretization of the market model (3.1), this forward-shooting grid approach works as follows. Exploiting a *forward* induction, I first consider each node $(S(t_i), r(t_i)$ and I compute all the compatible possible values of $m(t_i)$, namely, the minimum values reached along all possible paths that connects $(S(0), r(0))$ to $(S(t_i), r(t_i))$.

As $\{S(t_i)\} = \left\{S(0)\exp\left(M\sigma_S\sqrt{\Delta t}\right)\right\}_{M\in\{-i,-i+2,\ldots,i-2,i\}}$, it holds

$$\{m(t_i)\} = \begin{cases} \left\{S(0)\exp\left(-J\sigma_S\sqrt{\Delta t}\right)\right\}_{J=0,\ldots,\frac{i-M}{2}} & M \geq 0 \\ \left\{S(t_i)\exp\left(-J\sigma_S\sqrt{\Delta t}\right)\right\}_{J=0,\ldots,\frac{i+M}{2}} & M < 0 \end{cases}$$

Therefore, each node of the lattice is enlarged to $(S(t_i), r(t_i), m(t_i)_1, \ldots, m(t_i)_J)$ with $J \leq i/2$. Then, I start the *backward* recursion part of the algorithm. At $t_N = T$ and for each $(S(t_N), r(t_N), m(t_N)_1, \ldots, m(t_N)_J)$ I compute the payoff of the option according to each possible value of the running minimum thus getting at most $i/2$ values of the option. Then, at t_{N-1}, I consider each triplet $(S(t_{N-1}), r(t_N - 1), m(t_N - 1)_j)_{j \leq N/2}$, I compute the immediate exercise value of the option and I look for the compatible following nodes along the tree exploiting (3.19). Then, I compute the continuation value discounting back the values of the options in the nodes I found. As the number of possible values of the running minimum compatible with a generic $S(t_i)$ is decreasing in i, $S(0)$ will have only one compatible running minimum, $m(0) = S(0)$. In this way, I obtain the unique price of the option.

3.2.3 Continuously monitored barrier options

The extension from discretely monitored barrier options to their continuously monitored counterparts relies on the convergence of the discrete approximation chosen for the market model (3.1).

Proposition 3.5 ***(Convergence of the continuously monitored barrier options)*** *Assume that $(S(t_i), r(t_i))_{i=0,\dots,N}$ is a discrete time stochastic process that converges in distribution to $(S(t), r(t))$ in (3.2) as $N \to +\infty$. Then the price of the down-and-out/down-and-in European/American barrier options discretely monitored at $\{S_i(t)\}_{i=0,\dots,N}$ evaluated by means of Propositions 3.1, 3.2, 3.3, 3.4 converges to the price of the related continuously monitored barrier options as $N \to +\infty$.*

Proof See Appendix 3.A. ■

The discrete approximation of the market model (3.1) I will work with is the *quadrinomial tree* proposed in Section 2.2 of Battauz and Rotondi (2019). Loosely speaking, the quadrinomial tree adds an extra dimension to the binomial tree of Cox et al. (1979) to account for the evolution of the stochastic interest rate. More precisely, the continuous time stochastic process $(S(t), r(t))$ over $[0, T]$ in (3.1) is approximated by a discrete time stochastic process $(\tilde{S}(t_i), \tilde{r}(t_i))$ over the uniform discrete partition $\{i\frac{T}{n}\}_{i=0,\dots,n}$, where $n \in \mathbb{N}$ is the number of time steps chosen. At each step the discrete process evolves according to

$$\left(\tilde{S}(t_{i+1}), \tilde{r}(t_{i+1})\right) = \begin{cases} \left(\tilde{S}(t_i)e^{\Delta Y}, \tilde{r}(t_i) + \Delta r\right) & \text{with probability } q_{uu} \\ \left(\tilde{S}(t_i)e^{\Delta Y}, \tilde{r}(t_i) - \Delta r\right) & \text{with probability } q_{ud} \\ \left(\tilde{S}(t_i)e^{-\Delta Y}, \tilde{r}(t_i) + \Delta r\right) & \text{with probability } q_{du} \\ \left(\tilde{S}(t_i)e^{-\Delta Y}, \tilde{r}(t_i) - \Delta r\right) & \text{with probability } q_{dd} \end{cases} \tag{3.20}$$

where, for sake of readability, the explit expressions of the parameters ΔY, Δr and of the four transition probabilities are deferred to Appendix 3.C. It is worth to notice that, differently from the standard binomial tree of Cox et al. (1979), the transition probabilities of the quadrinomial tree are state-dependent and time contingent. Furthermore, notice that the quadrinomial tree is recombining (namely, it has a lattice structure) and the number of possible states grows quadratically in the number n of steps. Finally, it holds that the quadrinomial tree converges in distribution to the continuous time processes that solve system (3.1) as the time step shrinks or, equivalently, as the number of steps n grows to infinity.

Of course, any finite n chosen to retain computational feasibility leads to a discretization error in the discrete time approximation of the state variables $(S(t), r(t))$ and thus in the related option pricing. I analyse the resulting numerical issues in the following subsection.

3.2.3.1 Numerical issues

When dealing with numerical methods for option pricing, the accuracy and the stability of algorithms proposed are of great relevance. As an example, it is well known that the price of at-the-money Vanilla European call and put options obtained along the binomial tree of Cox et al. (1979) overestimates or underestimates the related Black-Scholes price depending on the parity of the finite number of steps chosen for the tree. A straightforward correction of this inaccuracy is to average out the prices obtained along the tree using n steps first and then $n+1$ steps.
As effectively explained in Section 6.4 of Glasserman (2003), dealing with barrier features remarkably slows down the rate of convergence option prices due to the singularity of the distribution of the running maximum/minimum even in the standard Black-Scholes framework.
Without loss of generality, consider a down and out contract. When approximating a continuous monitoring with a discrete one, the main issue is that the possibility that the barrier is hit between two subsequent monitoring dates (both of them above the barrier) is neglected. Therefore, the discrete monitoring version of a down and out contract systematically overestimate its continuous monitoring version. The size of this error is directly proportional to the distance between the barrier and the nearest nodes of S on the tree as it can be seen graphically in Figure 3.1. To eliminate this source of error I propose to conveniently choose the number of discretization steps so that the barrier falls precisely on one node of the tree. Let n be the number of time steps. At maturity T the possible values of S on the tree are of the form

$$S(T) = S(0) \exp\left(m\sigma_S \sqrt{\frac{T}{n}} \right) \qquad \text{with } m \in \mathbb{Z}, |m| \leq n.$$

Therefore, the barrier level B is precisely on one node of the tree if

$$n(m) = m^2 \sigma_S^2 T \left(\ln \frac{B}{S(0)} \right)^{-2}$$

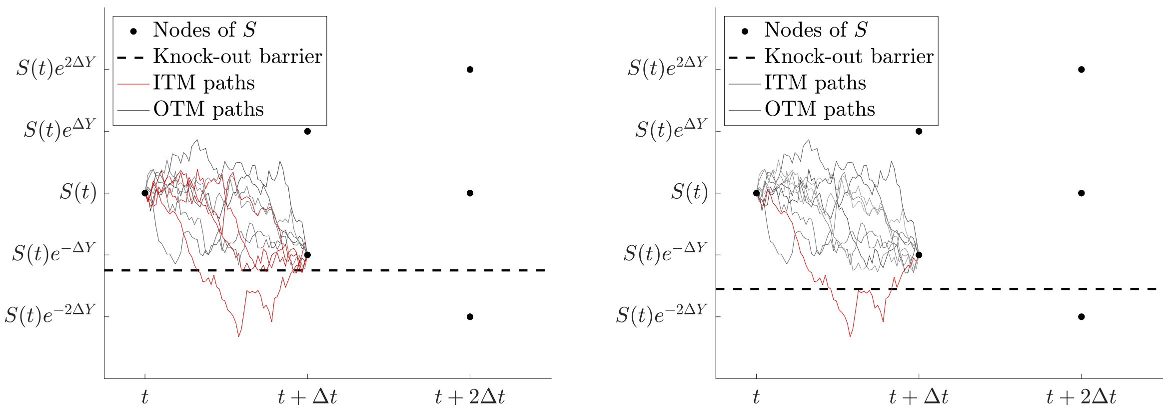

Figure 3.1: Possible paths of S between $S(t)$ and $S(t+\Delta t)$. If the barrier is close to a node of the tree (left panel) then it more likely that a continuous path of S gets knocked-out before going back to $S(t+\Delta t)$: in the illustrative example this happens for 4 out of 10 paths. If the barrier is farther from the nodes (right panel), the probability of the aforementioned event is lower: in the illustrative example it happens for just 1 out of 10 paths.

with $m \in \mathbb{Z}$ and $m \leq n$. Granting at least a monitoring each two days, the "optimal" number of time steps to choose is therefore $n(m^*)$ with

$$m^* := \min_{m \in \mathbb{Z}, m \leq n} m^2 \sigma_S^2 T \left(\ln \frac{B}{S(0)} \right)^{-2} > 125 \cdot T \tag{3.21}$$

This workaround allows to efficiently price barrier options along a lattice even with a relative small number of time steps.

As an illustrative example consider a European down-and-out put option on S with $S(0) = K = 100$, $B = 90$ and $T = 1$. The parameters of S and r are as in Table 3.1 (which displays the result of the model's calibration exercise performed in Section 3.3.1). Figure 3.2 shows $\pi^E_{put,DO}$, the price at inception of a continuously monitored European down-and-out put option computed by means of different techniques. The benchmark is a Monte Carlo estimate (solid black line) that exploits 10^7 paths and 250 monitoring dates, and whose confidence interval is highlighted by the two dashed lines. Notice that the Monte Carlo estimate is only available when pricing European options: the price of American ones can not be estimated directly by Monte Carlo. The performance of different lattice-based approaches, as a function of the number of time steps chosen, are compared. The lattice considered is always the quadrinomial tree of Battauz and Rotondi (2019).

With no adjustments, the algorithm suggested in Proposition 3.1 heavily overestimate the value of the continuously monitored option. Even with more than 250 time steps, the error one might

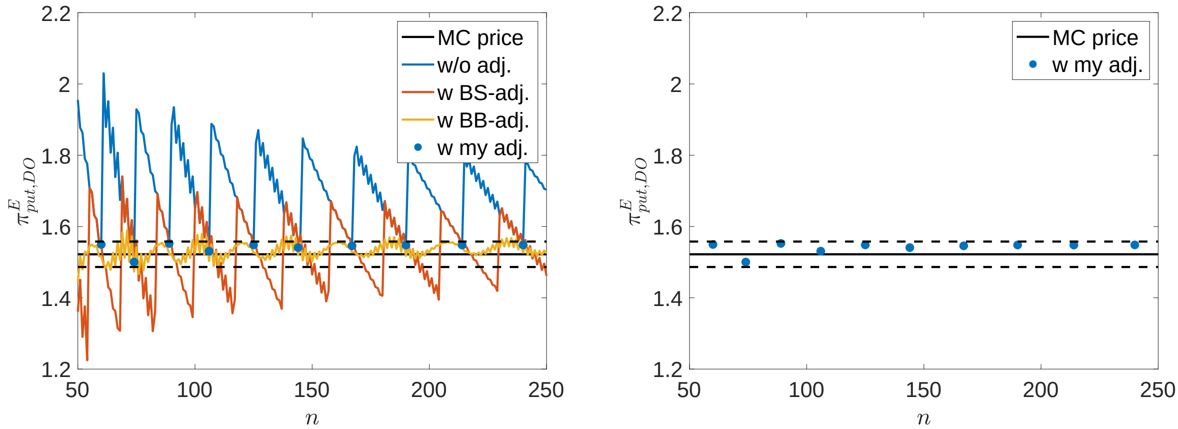

Figure 3.2: Lattice-based prices vat inception of a continuously monitored European down-and-out put option as the number of time steps increases.

get is still sizeable as the blue line in the left panel of Figure 3.2 suggests. Broadie et al. (1997) suggest a *barrier shifting adjustment* (BS-adj.) that improves the estimate of the price as the orange line in the left panel of Figure 3.2 shows. The intuition behind this correction is that shifting the barrier down a little bit reduces the probability that the barrier itself is hit between two subsequent nodes that lay above the barrier. A even more effective workaround is proposed by Glasserman (2003) in Section 6.4. This *Brownian bridge adjustment* (BB-adj.) prescribes to correct for the probability that the underlying price process falls below the barrier conditional of starting and ending at two subsequent nodes above the barrier. As the yellow line in left panel of Figure 3.2 shows, this BB-adjustment delivers prices that are very close to the benchmark even for a reasonably small number of time steps. Finally, the choice of the optimal number of time steps I proposed in Equation (3.21) precisely picks out the points of the plain algorithm without any adjustment that are as close as possible to the benchmark. As it can be seen in the right panel of Figure 3.2, the optimal numbers of steps I suggest to consider deliver stable results even from a small number of steps. Therefore, when pricing a continuously monitored barrier option I will always choose the number of time steps according to (3.21).

3.2.4 Alternative market models

Although fairly general, the market model in (3.1) is commonly modified along two directions:

- setting a zero lower bound for the interest rate process;
- relaxing the hypothesis of constant volatility (at least) for the underlying equity price process.

Macroeconomic literature has focused a lot on the zero lower bound of interest rates, mostly due to its implications on the effectiveness of central banks' monetary policy. Indeed, it has been argued that when (nominal) interest rates are close to zero, "traditional" monetary policy interventions, like lowering the nominal interest rates, fail. On the contrary, under these particular circumstances, only "unconventional" policies like large-scale asset purchases and forward guidance seem to be effective to boost the economy. See the groundbreaking theoretical work of McCallum (2000) and the further empiric considerations of Hamilton and Wu (2012) and Wu and Xia (2016), among others. Furthermore, earning a negative rate on the riskless asset (namely, saving \$100 in the bank account today and withdrawing \$99 in one year) is a contingency that also asset pricing literature usually avoided: indeed, in such a circumstance, the investor is rather assumed to "hide the cash under the mattress" and preserve the nominal value of her asset. Nevertheless, nowadays these assumptions are challenged by real market circumstances.
Due to these reasons, though, short-term rate models with strictly positive interest rates have been preferred over the ones that allowed for negative values. Among the latter ones, the most famous and the most widely used is the celebrated CIR model proposed by Cox et al. (1885). Therefore, I will include this model as an alternative benchmark in my numerical examples.
With respect to the other natural extension of the market model in (3.1), it is well known that a constant volatility for the underlying equity process is not consistent with many empirical stylized facts in option pricing such as the volatility smile. In order to overcome this limitation, two extensions have been proposed: local volatility models, like the constant elasticity of variance (CEV, henceforth) model[10], and stochastic volatility ones, like the celebrated model of Heston (1993). The main difference between local volatility and stochastic volatility models is that the latter introduces a new risk factor that needs to be modelled independently; on the contrary, local volatility models assume that the volatility of the underlying process depends on the level of the underlying itself and therefore introduces no new state variables. The CEV model captures the so-called *leverage effect*, which is a pattern observed in equity markets: the volatility of an equity is usually low when the equity is performing well and, on the contrary, its volatility increases when the equity is performing poorly. Therefore, for sake of simplicity, I will add the CEV model for the underlying as another alternative benchmark in my numerical examples.

Accounting for the zero lower bound of the interest rate process and for the local volatility of the underlying equity price process leads to a generalized financial where the risk factors are

[10]The CEV model was developed in an unpublished draft titled "Notes on option pricing I: constant elasticity of diffusions" by J. Cox in 1975. See Linetsky and Mendoza (2009) for a throughout review of the CEV model.

driven by the following SDEs

$$\begin{cases} \mathrm{d}S(t) = S(t)(r(t) - q)\mathrm{d}t + \sigma_S S(t)^\gamma \mathrm{d}W_S^{\mathbb{Q}}(t) \\ \mathrm{d}r(t) = \kappa \left(\theta - r(t)\right) \mathrm{d}t + \sigma_r r(t)^\beta \mathrm{d}W_r^{\mathbb{Q}}(t) \end{cases} \tag{3.22}$$

with $\gamma, \beta > 0$. Notice that $\beta = 0.5$ delivers the CIR model of Cox et al. (1885). Both of the SDEs in (3.22) admit no explicit solution[11] like (3.2). Nevertheless, they can still be fitted in the quadrinomial tree of Battauz and Rotondi (2019). See the formal derivation of the lattice in this case in Appendix 3.C.

3.3 Empirics

In this Section I carry out a throughout analysis of barrier options pricing within my market model as opposed to several other benchmarks. In order to assume parameters that are as more realistic as possible, in Subsection 3.3.1 I calibrate the market model in (3.1) to European data. Then, I use the resulting parameters as input for the wide numerical exercised proposed in Subsection 3.3.2 and I quantify the pricing errors generated by assuming constant/ non correlated/ strictly positive interest rates. Finally, Subsection 3.3.3 contains two applications of the techniques developed in this paper to more sophisticated option pricing problems.

3.3.1 Calibration

I calibrate the model described by (3.1) to European market data as of November 2018 the 30th. Full details of this calibration exercise are provided in Appendix 3.B.

The five parameters that have to be calibrated are κ, θ, σ_r, σ_S, ρ. I select the EuroStoxx 50 Index as equity price process and the three months yield of AAA-rated Euro-denominated bonds as the interest rate process.

I collect the prices of several frequently traded European call and put options on the EuroStoxx along with the prices of AAA-rated Euro-denominated bonds of many maturities. Then, I look for the parameters that minimize the distance between the current market prices and the ones implied by the model ((3.6) and (3.7) for the options and (3.3) for the bonds). See Appendix 3.B for further details on the options used in the calibration exercise.

The result of this calibration exercise is shown in Table 3.1. Figure (3.1) shows graphically the remarkably small distance between the prices implied by the calibrated model and the real market

[11] See Section 3.4 of Glasserman (2003) for the exact and approximated simulation schemes of this kind of processes. Those can be useful when exploiting Monte Carlo-based pricing techniques.

κ	θ	σ_r	σ_S	ρ
11.99%	3.16%	1.54%	14.37%	85.42%

Table 3.1: Results of the calibration of market model (3.1).

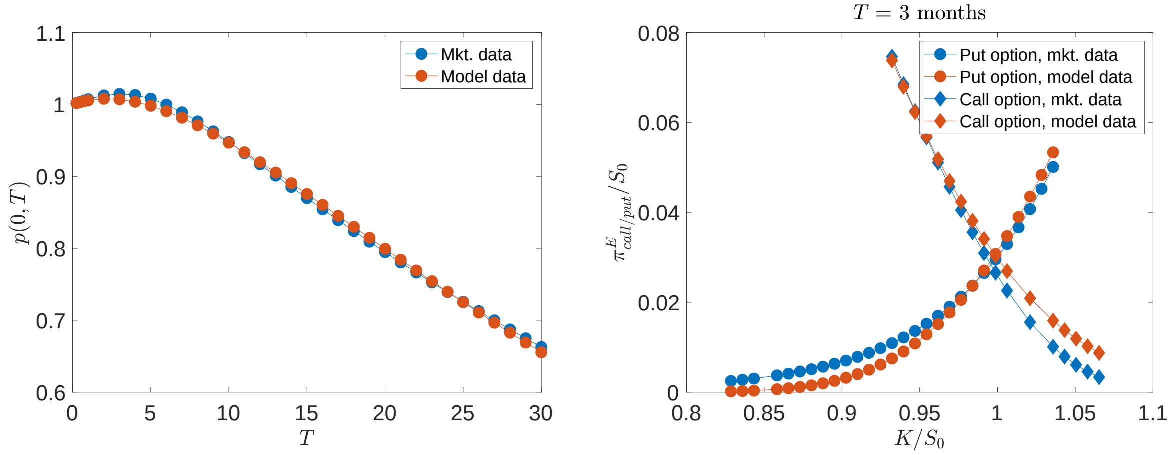

Figure 3.1: Market prices of bonds and options (blue markers) and prices obtained from the calibrated model (orange markers). Left panel: prices of AAA-rated Euro-denominated zero coupon bonds; right panel: prices of call and put options on EuroStoxx 50 with 3 months maturity.

ones.

Table 3.1 suggests that he mean reversion towards the long run interest rate value is slow and the process itself displays low volatility. Nevertheless, both of the coefficients are quite relevant when dealing with long maturity derivatives as it will be clear in the following examples. The correlation between the equity and the bonds is estimated to be largely positive, which reflects the current situation of a negative correlation between the interest rates and the equity prices (as recently documented by, e.g., Perego and Vermeulen (2016)). Finally, the value of the volatility of the equity is coherent with the levels of the Stoxx 50 Volatility Index that aims to measure the volatility of the EuroStoxx 50 as the CBOE VIX index does analogously for the S&P 500.

3.3.2 Numerical examples

Table 3.2 and Table 3.3 collect the results of the main numerical example.

The options priced in Table 3.2 are European-style whereas the ones in Table 3.3 are American-style. Three different payoff function $\varphi(S)$ are considered each time: the digital payoff, constantly equal to 1, the put's and the call's payoff. Furthermore, three different barrier events are considered: no barrier event at all (thus leading to a plain Vanilla option); the down-and-out event that makes the investor obtain the payoff if the barrier is not crossed throughout the life of the option; the down-and-in event the makes the investor obtain the payoff if and only if the barrier is touched at least once. Considering all the possible combinations, nine different options are priced. Finally, these nine options are priced within the following seven different market models. The baseline case is the Vasicek market model in (3.1) with the parameters in Table 3.1. Then, in order to quantify the error an unsophisticated investor who neglects interest rate risk would make, pure diffusive models, with constant interest rates, are considered: the interest rate is set equal to $r(0)$, r^*, the yield to maturity of the AAA-rated bond expiring at T, and θ^*, the estimated long-term mean. Furthermore, in order to quantify the error an investor who neglects the correlation between the two risk factors or wrongly flips its sign would make, I consider again the baseline model with ρ equal to either zero or the opposite of the estimated parameter. Finally, the other more prominent alternative to the baseline model, namely the CIR model introduced in Subsection 3.2.4 is considered.

The other parameters of the numerical exercise are: $S(0) = K = 100$, where K is the strike of the call/put options. $T = 2$, that, together with $L = 70$, delivers, according to (3.21), an optimal number of time steps $m^* = 259$.

For each option, and across all models, the tables display options' prices along with relative errors with respect to the baseline case. Furthermore, for American options, also the Early Exercise Premium (EEP henceforth) is computed. The EEP is the difference between the price of any American option and its European counterpart and it represents the dollar-value of the early exercise opportunity provided by American-style contracts.

The following considerations can be made:

- although less interesting from the applications point of view, European digital options offer useful insights. The Vanilla versions of the these digital options are basically zero coupon bonds. Their price depends solely on the interest rate and, consequently, it is not sensitive to the correlation between the risk factors. In this case r impacts only the discount rate and assuming a constant interest rate (as long as not too different from $r(0)$) does not change the price much. As a consequence, relative errors with respect to the benchmark case are small.

- On the other hand, the payoff of European down-and-out and down-and-in digital options depends also on the path of the underlying equity process through the barrier event. As an instance, the price of a digital down-and-out option is given by

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(s)\mathrm{d}s}\mathbb{1}\left(m(T) > L\right)\right],$$

European Options

payoff	model	Vanilla price	Vanilla rel. error	Down-and-out price	Down-and-out rel. error	Down-and-in price	Down-and-in rel. error
Digital	baseline	0.9950		0.5077		0.4873	
	$r(t) \equiv r(0)$	1.0131	1.82 %	0.5215	2.72 %	0.4916	0.88 %
	$r(t) \equiv r^*$	1.0121	1.72 %	0.5219	2.80 %	0.4902	0.60 %
	$r(t) \equiv \theta^*$	0.9388	5.65 %	0.5446	7.27 %	0.3942	19.11 %
	$\rho = 0$	0.9950	0.00 %	0.5233	3.07 %	0.4717	3.20 %
	$\rho = -\rho^*$	0.9950	0.00 %	0.5412	6.60 %	0.4538	6.87 %
	CIR	0.9997	0.47 %	0.5258	3.57 %	0.4739	2.75 %
Put	baseline	17.9580		1.6715		16.2865	
	$r(t) \equiv r(0)$	17.8640	0.52 %	1.8784	12.38 %	15.9856	1.85 %
	$r(t) \equiv r^*$	17.8021	0.87 %	1.8757	12.22 %	15.9264	2.21 %
	$r(t) \equiv \theta^*$	13.5771	24.40 %	1.6608	0.64 %	11.9163	26.83 %
	$\rho = 0$	16.8462	6.19 %	1.7955	7.42 %	15.0507	7.59 %
	$\rho = -\rho^*$	15.6479	12.86 %	1.9410	16.12 %	13.7069	15.84 %
	CIR	17.0602	5.00 %	1.8414	10.16 %	15.2188	6.56 %
Call	baseline	16.4712		15.8051		0.6661	
	$r(t) \equiv r(0)$	14.5723	11.53 %	13.9732	11.59 %	0.5991	10.06 %
	$r(t) \equiv r^*$	14.6117	11.29 %	14.0117	11.35 %	0.6000	9.92 %
	$r(t) \equiv \theta^*$	17.7203	7.58 %	17.0606	7.94 %	0.6597	0.96 %
	$\rho = 0$	15.3589	6.75 %	14.7136	6.91 %	0.6453	3.12 %
	$\rho = -\rho^*$	14.1599	14.03 %	13.5358	14.36 %	0.6241	6.31 %
	CIR	15.1125	8.25 %	14.5007	8.25 %	0.6118	8.15 %

Table 3.2: European options prices. $S(0) = 100$, $q = 1\%$ and $T = 2$. The payoff at maturity of digital options is $\varphi(S(T)) = 1$; of (at the money) put options is $\varphi(S(T)) = (S(0) - S(T))^+$; of (at the money) call options is $\varphi(S(T)) = (S(T) - S(0))^+$. The baseline model is the Vasicek market model in (3.1) with the parameters specified in Table 3.1. The models $r(t) \equiv r(0)/r^* = -0.60\%/\theta^*$ are pure diffusive models (Black-Scholes ones) where the interest rate is constant and equal respectively to its initial value/ the yield on T-bond riskless bond/ the long-run mean. The models $\rho = 0/-\rho^*$ differ from the baseline one only with respect to the correlation which is assumed to be 0/ the opposite of the result from the calibration exercise. The CIR model is the one in (3.22) with $\gamma = 0.5$. The barrier is $L = 70$.

American Options

payoff	model	Vanilla			Down-and-out			Down-and-in			DFP
		price	rel. error	EEP	price	rel. error	EEP_{DO}	price	rel. error	EEP_{DI}	
Digital	baseline	1.0138		0.02	1.0138		0.51	0.4890		0.00	0.49
	$r(t) \equiv r(0)$	1.0131	0.07 %	0.00	1.0131	0.07 %	0.49	0.4916	0.53 %	0.00	0.49
	$r(t) \equiv r^*$	1.0121	0.17 %	0.00	1.0121	0.17 %	0.49	0.4902	0.25 %	0.00	0.49
	$r(t) \equiv \theta^*$	1.0000	1.36 %	0.06	1.0000	1.36 %	0.46	0.4072	16.73 %	0.01	0.41
	$\rho = 0$	1.0138	0.00 %	0.02	1.0133	0.05 %	0.49	0.4778	2.29 %	0.01	0.48
	$\rho = -\rho^*$	1.0138	0.00 %	0.02	1.0137	0.01 %	0.47	0.4671	4.48 %	0.01	0.47
	CIR	1.0000	1.36 %	0.00	1.0000	1.36 %	0.47	0.4741	3.05 %	0.00	0.47
Put	baseline	17.9807		0.02	15.5266		13.86	17.9774		1.69	15.52
	$r(t) \equiv r(0)$	17.8640	0.65 %	0.00	16.0052	3.08 %	14.13	17.8578	0.67 %	1.87	16.00
	$r(t) \equiv r^*$	17.8021	0.99 %	0.00	15.9742	2.88 %	14.10	17.7961	1.01 %	1.87	15.97
	$r(t) \equiv \theta^*$	14.1292	21.42 %	0.55	13.7137	11.68 %	12.05	14.1292	21.41 %	2.21	13.71
	$\rho = 0$	17.1479	4.63 %	0.30	15.5886	0.40 %	13.79	17.1429	4.64 %	2.09	15.58
	$\rho = -\rho^*$	16.4252	8.65 %	0.78	15.6273	0.65 %	13.69	16.4232	8.65 %	2.72	15.63
	CIR	17.0602	5.12 %	0.00	17.0602	9.88 %	15.22	15.4178	14.24 %	0.20	15.42
Call	baseline	16.4753		0.00	15.8091		0.00	0.6661		0.00	0.00
	$r(t) \equiv r(0)$	14.8508	9.86 %	0.28	14.2449	9.89 %	0.27	0.7245	8.77 %	0.13	0.12
	$r(t) \equiv r^*$	14.8828	9.67 %	0.27	14.2763	9.70 %	0.26	0.7350	10.34 %	0.13	0.13
	$r(t) \equiv \theta^*$	17.7205	7.56 %	0.00	17.0607	7.92 %	0.00	0.6597	0.96 %	0.00	0.00
	$\rho = 0$	15.5759	5.46 %	0.22	14.9255	5.59 %	0.21	0.7053	5.89 %	0.06	0.05
	$\rho = -\rho^*$	14.8354	9.95 %	0.68	14.1976	10.19 %	0.66	0.7241	8.71 %	0.10	0.09
	CIR	15.2841	7.23 %	0.17	15.2841	3.32 %	0.78	0.6818	2.36 %	0.07	0.68

Table 3.3: American options prices. $S(0) = 100$, $q = 1\%$ and $T = 2$. The immediate payoff of digital options is $\varphi(S(t)) = 1$; of (at the money) put options is $\varphi(S(t)) = (S(0) - S(t))^+$; of (at the money) call options is $\varphi(S(t)) = (S(t) - S(0))^+$. The baseline model is the Vasicek market model in (3.1) with the parameters specified in Table 3.1. The models $r(t) \equiv r(0)/r^* = -0.60\%/\theta^*$ are pure diffusive models (Black-Scholes ones) where the interest rate is constant and equal respectively to its initial value/ the yield on T-bond riskless bond/ the long-run mean. The models $\rho = 0/-\rho^*$ differ from the baseline one only with respect to the correlation which is assumed to be 0/ the opposite of the result from the calibration exercise. The CIR model is the one in (3.22) with $\gamma = 0.5$. The barrier is $L = 70$. The EEP is the Early Exercise Premium, namely the difference between the price of the American option and of its European counterpart. The deviation from parity, DFP, is defined in Equation (3.18).

where, as usual, $m(T)$ is the running minimum of the equity price process S. In this case, r impacts both the discount rate and the expected drift of S that is the main driver of the probability that the barrier is reached. The interplay of these two effects delivers larger relative pricing error. If a constant interest rate is considered, the relative pricing errors can be as large as 2.80%. If, on the contrary, the correlation between the effects is ignored or if its sign is flipped, relative pricing errors rise up to 6.87%.

- When moving to European put and call options, relative pricing errors with respect to the benchmark models widen. This is due to a third effect of r as now the payoff of the option depends explicitly on the level of the underlying at T and not just only on the probability of the barrier event, if any. Overall, if a constant interest rates (either $r(0)$ or r^*) is chosen, the relative pricing error ranges from 0.52% (for the Vanilla put) to 12.38% (for the down-and-out put). If the interest rate is assumed to be positive and a CIR model is used, the relative pricing error is on average equal to 8%. If correlation between the risk factors is neglected or, even worse, its sign is flipped, the relative pricing error ranges from 3.12% (for the down-and-in call option) to 16.12% (for the down-and-out put option).

- American options carry always a non negative early exercise premium as shown in Table 3.3. In general, this premium is particularly sizeable for out contracts as, if the equity price process approaches the barrier and the threat of being knocked out becomes relevant, the holder of the American contract will surely exercise it whereas the holder of the European one has no such a possibility. In the present example, the related deviation from in-out parity is particularly large for put options as, if the underlying equity process goes down, the put is in the money and, if of American-style, it can be profitably exercised before the barrier is touched.

- Relative pricing errors with respect to the baseline case are slightly smaller for American options, due to the early exercise possibility. Nevertheless, optimal exercise policies (namely the instant in time at which the option is optimally exercised) may vary a lot, as the second example of the following Section shows. Overall, assuming a flat term structure with r equal to either $r(0)$ or r generates relative pricing errors that range between 0.65% (for the Vanilla put option) and 10.34% (for the down-and-in call option). On the other hand, a wrong choice of the correlation leads to relative pricing errors between 0.4% and 10.19%.

Appendix 3.C contains similar numerical exercises with longer-lived options ($T = 5$ years) and different barrier levels ($L = 80$, $L = 60$).

t (years)	CF_t
0	-N
1	$C_1 \mathbb{1}(\min_{\tau \in I_1} S(\tau) > S(0))$
$\dots$	$\dots$
T	$C_T \mathbb{1}(\min_{\tau \in I_M} S(\tau) > S(0))$
T	N

Table 3.4: Cashflow of the equity-linked note of the first example. $C = [30\ 37\ 44\ 51\ 58\ 65\ 72]$ and, assuming 250 trading days per year, $I_i = [i - \frac{7}{250}, i]$ for $i = 1, \dots, 7$.

3.3.3 Applications

First example: a equity-linked note

This first example I make is about an actually traded equity-linked note[12] written on the EuroStoxx 50, that I label by S. The note, whose full name is *"Standard long barrier protected digital certificate on EuroStoxx 50 index due 11.28.2025"* was announced on November 30 2018 when the index was standing at $S(0) = 3173.13$. Its price at inception was set equal to $N = 1000$[13]. Since this is a capital protected note, N is given back to the investor at T. Using the same notation of Section 3.2, the cashflow of the equity-liked note is shown in Table 3.4.

Looking at its cashflow, this note can be seen as the combination of seven correlated down-and-out digital options on the same underlying and therefore, it can be priced using the algorithm developed in the previous Section. Table 3.5 displays the pricing results from the first example.

As it can be seen from the left panel, the baseline model, namely the Vasicek one, delivers a small relative pricing error (1.96%) with respect to the market price.

As this note can be seen as a combination of digital down-and-out option, the main drivers of its price are the discount rate and the expected drift of the underlying. In particular,

- if $r(t) \equiv r(0)$, since r_0 is constantly mildly negative, the discount rate is always greater than one and the present value of the expected cashflows is too high. Even though the note is supposed to go out of the money due to a negative expected drift, the first effect prevails and the note is way overpriced;

[12]This note, "certificato" in Italian, is issued by Banca IMI S.p.a., an Italian financial company, and its unique identification code is `XS1898262578`. The official webpage (in English) of this product is https://www.bancaimi.prodottiequotazioni.com.

[13]All the priced are quoted in Euros. Therefore, I suppress everywhere the currency symbol.

constant volatility			local volatility		
model	price	relative error	model	price	relative error
mkt. price	1000	-	mkt. price	1000	-
baseline	1019.60	1.96%	baseline	1035.25	3.53%
$r(t) \equiv r(0)$	1129.40	12.94%	$r(t) \equiv r(0)$	1153.67	15.37%
$r(t) \equiv r^*$	1089.32	8.93%	$r(t) \equiv r^*$	1102.41	10.24%
$r(t) \equiv \theta^*$	930.86	6.91%	$r(t) \equiv \theta^*$	945.58	5.44%
$\rho = 0$	1025.62	2.56%	$\rho = 0$	1046.17	4.62%
$\rho = -\rho^*$	1030.23	3.02%	$\rho = -\rho^*$	1053.62	5.36%
CIR	964.21	3.58%	CIR	978.36	2.16%

Table 3.5: Prices of the equity-linked note. The cashflow of the derivative is described in Table 3.4. $S(0) = L = 3173.13$, $q = 0\%$. The baseline market model is the Vasicek model with the parameters specified in Table 3.1. The models $r(t) \equiv r(0)/r^* = -0.23\%/\theta^*$ are pure diffusive models (Black-Scholes ones) where the interest rate is constant and equal respectively to its initial value/ the yield on T-bond riskless bond/ the long-run mean. The models $\rho = 0/-\rho^*$ differ from the baseline one only with respect to the correlation which is assumed to be 0/ the opposite of the result from the calibration exercise. For the CIR model in (3.22), $\beta = 0.5$ and $r(0) = 10^{-4} > 0$. For the local volatility model, $\gamma = 0.8$.

- if $r(t) \equiv \theta^*$, since the long-term mean of interest rate is a good proxy only for long maturities, the discount rate at near maturities is too severe. Even if the expected drift of the underlying is positive, thus pulling the price process towards the in the money region, the discounting effect prevails and the note is sensibly underpriced;
- the case $r(t) \equiv r^* = -0.23\%$ consistently delivers an intermediate pricing error;
- if ρ is set equal to zero or if its sign is flipped, the relative pricing error is not large: this is due to the fact this is a digital option whose payoff does not explicitly depend on the level of the underlying but just on its running minimum. Therefore, the comovements of r and S are less relevant within this framework;
- the CIR model forces the interest rate to assume strictly positive values. This translates into a stronger discount rate and the price is again underestimated.

Numerical results from the local volatility model are similar to the ones described above. In particular, prices are a little bit higher due to the variability of the volatility of the underlying price process. Nevertheless, the relative errors are coherent with the ones of the constant volatility case. Interestingly, the combination of the CIR and the CEV model delivers a remarkably small relative error.

Figure 3.C.1 in Appendix 3.C shows the sensitivities of the price at inception of the equity-linked note to the parameters of the models.

Finally, it interesting to discuss a possible replicating strategy of this note. As it has been already pointed out, this note can be seen as the sum of seven down-and-out European digital options. Therefore, the natural replicating strategy of this derivative is to go long on seven down-and-out European digital options with matching maturities and coupons. Nevertheless, this hedging strategy completely neglects the inter-temporal correlation between the barrier events: if the first coupon is paid, namely if the underlying is always above the barrier during the first monitoring period, then it is more like that it will stay above the barrier also during the other monitoring periods. On the contrary, digital options are priced independently of each other and thus they are overall more expensive than the note itself. As an example, such an hedging strategy for the note would cost 1112.11, thus delivering a 11.12% relative error.

Second example: an executive stock option

According to the Financial Accounting Standard Board (FASB henceforth), a firm is required to estimate and report the "fair" value of share options granted to its employees or executives. Nevertheless, FASB Statement No. 123[14], *Accounting for Stock-based Compensation*, which is the legal statement that establishes the accounting standards for share- and option-based compensations, leaves quite a lot of freedom with respect to the methods that should be used to estimate the "fair" value of this kind of compensation. In particular, paragraphs A13. and A14. of this statement read

> *This Statement does not specify a preference for a particular valuation technique or model in estimating the fair values of employee share options and similar instruments [...]. A lattice model (for example, a binomial model) and a closed-form model (for example, the Black-Scholes-Merton formula) are among the valuation techniques that meet the criteria required by this Statement for estimating the fair values of employee share options and similar instruments.*

Despite this freedom, the particular valuation technique and the assumptions made play a crucial role in estimating the "fair" value of these instruments. In particular, a constant interest rate (but

[14]The latest version of the FASB Statement No. 123 can be found at www.fasb.org.

also a constant volatility for the underlying equity process) is not realistic when valuing this kind of derivatives as they usually display very long maturities[15]. As a result, the error with respect a stochastic interest rate model might be sizeable as shown in the following realistic example.

The numerical example I make is about a fictional option granted to one of an European large bank's executives. As of November 2018 the 30th, the annual dividend yield was 1.8% and its implied volatility was 29.7%. The executive stock option considered is an at the money American call option with maturity $T = 5$ years and vesting period equal to $T^* = 1$ years. Following Hull and White (2004a) and Hull and White (2004b), I assume that the American option is exercised (when the vesting period is over) if the underlying reaches the value $B = M \cdot S(0)$. Therefore, this option can be seen as a continuously monitored up-and-out American call option and it can be priced through Proposition 3.2.

The only peculiarity of these executive options is that, most of the times, they can not be sold: if the manager wants to cash in, she has to exercise the option and sell the share she gets. As a consequence, when being fired or when voluntarily leaving the job, the executive will surely exercise her option if in the money. To account for this "forced" (and, thus, possibly suboptimal) exercise decision, the continuation value at $t_i > T^*$ of the American option is set equal to

$$(1-\alpha)\pi^A_{call,UO}(t_{i+1}) + \alpha(S(t_i) - K)^+ \mathbb{1}(S(t_i) < B)$$

namely, a weighted average of the standard continuation value and of the payoff from the immediate "forced" exercise that happens with probability α, the exit rate of the employee.

Table 3.6 collects the numerical results from this pricing exercise. Its structure mimics the one of Table 3.4 and the models considered are precisely the same.

The relative pricing error with respect to the baseline case is around 10%. In this case the payoff of the option depends explicitly also on the level of the underlying. Therefore, the effect of r on the expected drift of the underlying dominates the one on the discount rate: if r is constantly equal to $r(0)$, which is mildly negative, the price of the option is lower than in the baseline case as the equity is not expected to appreciate much even though the negative discount rate inflates all the expected cashflows. On the contrary, if r is set equal to θ^*, which is sensibly positive, the equity is expected to appreciate a lot and this translates into an overestimation of the baseline price. As it can be seen from the relative error when the correlation is neglected or its sign is flipped, the comovements of r and S plays a crucial role this time. The CIR model delivers the smallest relative error but it neglects a large fraction of the state space of r. This has a sizeable impact on the optimal exercise policy.

[15] See, e.g., Aboody et al. (2006) for a throughout analysis on why and how firms are likely to misreport the price of ESOs.

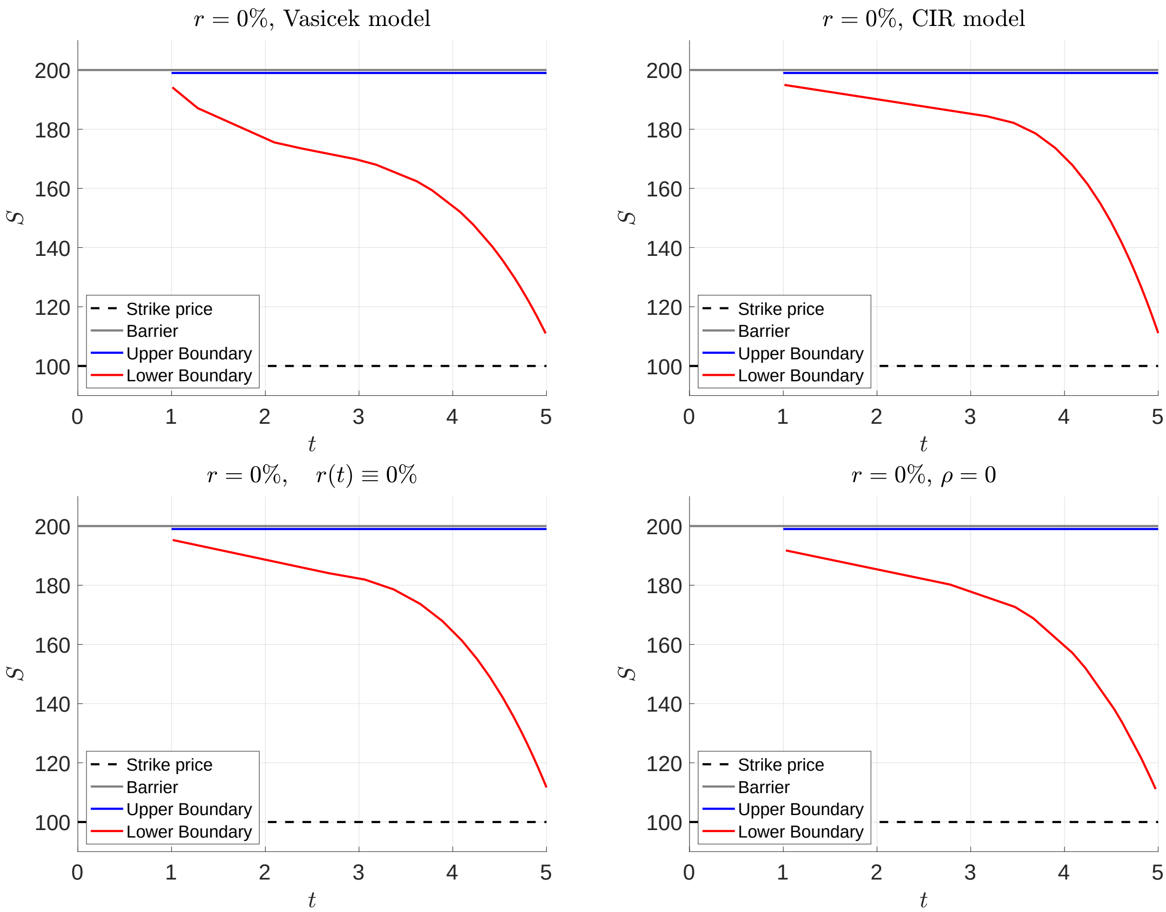

Figure 3.2: Sections at $r = 0.01\%$ of the free boundaries of the early exercise region of the American up-and-out call option of the second example. Top-left panel: baseline case, Vasicek model; top-right panel: CIR case. Bottom-left panel: $r(t) \equiv r_0$ case; bottom-right panel: Vasicek model with $\rho = 0$.

constant volatility			local volatility		
model	price	relative error	model	price	relative error
baseline	20.75	-	baseline	25.83	-
$r(t) \equiv r(0)$	18.50	10.84%	$r(t) \equiv r(0)$	22.12	14.36%
$r(t) \equiv r^*$	18.89	8.96%	$r(t) \equiv r^*$	23.47	9.14%
$r(t) \equiv \theta^*$	22.49	8.39%	$r(t) \equiv \theta^*$	24.27	6.04%
$\rho = 0$	19.46	6.22%	$\rho = 0$	22.85	11.54%
$\rho = -\rho^*$	18.07	12.92%	$\rho = -\rho^*$	21.42	17.07%
CIR	19.27	7.13%	CIR	21.50	16.76%

Table 3.6: Price of the up-and-out American call option with vesting period and exogenous exit rate described in the second example. Left panel: equity process with constant volatility. Right panel: equity process with local volatility, CEV model, $\gamma = 0.8$. Common parameters: $S(0) = K = 100$, $T = 5$, $q = 1.8\%$, $\sigma_S = 29.7\%$, $r^* = -0.23\%$, barrier level $B = 2S(0)$, vesting period $T^* = 1$, exit rate $\alpha = 5\%$; parameters of the interest rate process as in Table 3.1. For the CIR model in (3.22), $\beta = 0.5$ and $r(0) = 10^{-4} > 0$. For the CEV model, $\gamma = 0.8$.

Figure 3.2 shows the sections at $r(t) = 0\%$ of the *early exercise region* and of the *continuation region* within the alternative models. The interpretation of the plots is as follows: if $r(t) = 0\%$, the American call option is optimal exercised at t if $S(t)$ lies between the upper and the lower boundary (the blue and the red line, respectively). It is easy to see how the continuation region of the baseline case is wider than the one obtained with the CIR model: since the CIR model contemplates no negative values for r, r is therein expected to move towards its positive long-run level θ^*, that delivers a higher expected drift and, therefore, a more juicy payoff. On the contrary, in the baseline case r is likely to fall below zero: in this case, the executive may prefer to cash in even at lower value of S to gain from the negative discount and to protect herself from the depreciation of the underlying. As a small final remark, notice that the shape of the lower boundary of the early exercise region within the baseline case challenges traditional results on strict concavity of optimal stopping boundary for American options in constant interest rate frameworks. (see, e.g., Ekström (2004)).

Figure 3.C.2 in Appendix 3.C collects the sensitivities of the price at inception of the American call option with respect to the parameters of the model.

Third example: a Double Barrier Note on the S&P 500 Index

κ	θ	σ_r	σ_S	ρ
4.48%	5.65%	0.92%	17.08%	77.27%

Table 3.7: Results of the calibration of market model (3.1) in the US market.

While the first two examples deal with products traded in the European market, it is interesting to validate the model also in the other major market, namely the US market.

First of all I calibrate the market model (3.1) to US data as of November 2018 the 30th. Details of the calibration are provided in the second part of Appendix 3.B. The equity price process S I chose is the S&P 500 index while I proxy the short term interest rate process r with the United States Treasury Securities yield. The main difference between European market conditions is the actual level of the interest rate: in the US market I have $r_0 = 2.37\%$ while in the European market I had $r_0 = -0.6728\%$. The results of the calibration are displayed in Table 3.7. We notice that, with respect to the European counterparts in Table 3.1, the interest rate process reverts to its long-run mean more slowly and also with a smaller volatility coefficient. Nevertheless, both the volatility of the equity process and the correlation between the two risk factors are comparable to the European market.

The derivative I analyze is an at the money double barrier European option on the S&P 500 issued by a large American Bank[16]. The option was issued on February 21 2019 when the index closed at $S(0) = 2774.88$. Recalling the general definition of a double barrier option in (3.12), the payoff of this Note is

$$X(T) = \frac{1}{S(0)} \left| S(T) - S(0) \right| \mathbb{1} \left(m(T) > L \cap M(T) < U \right)$$

plus the initial capital. As it can be seen, if the underlying does not go out of the interval (L, U) throughout the life of the option, the payoff to its holder combines the one of a call and of a put option. In other words, the holder of the option always gets the relative distance between the initial and the final level, no matter the direction. The payoff to the holder of the option is therefore minimal if the underlying does not very much from the starting level and it is maximum if the underlying at maturity is far from the starting level (but still within (L, U)). The two barriers are continuously monitored.

At inception, the index stand at $S(0) = 2743.79$. The upper barrier is set equal to $U = 3329.86$ and the lower one is set equal to $L = 1803.67$. The maturity of the Note is in $T = 2$ years, February 22 2021. The price at inception of the Note was 100$, which coincides with the capital

[16]The full name of the product is “2 year Capital Protected Note with Double Barrier on the S&P 500 Index”. Its ISIN code is `XS0347496538`. The issuer is JP Morgan Chase Bank, one of the largest investment banks in the world.

constant volatility		
model	price	relative error
market price	100	
baseline	106.32	6.32%
$r(t) \equiv r(0)$	108.21	8.21%
$r(t) \equiv r^*$	109.18	9.18%
$r(t) \equiv \theta^*$	109.98	9.98%
$\rho = 0$	108.75	8.75%
$\rho = -\rho^*$	110.87	10.87%
CIR	105.23	5.23%

Table 3.8: Price of the American up-and-out call option on the S&P 500 described in the third example. Parameters: $S(0) = K = 2743.79$, $T = 2$.

that is surely given back at maturity to the investor.

Table 3.8 reports the pricing results for this third numerical example. Interestingly, the baseline model and all of the alternative ones overprice the Note. However, the relative pricing error of the baseline case is quite small. The CIR model here, is actually performing better whereas, as in the case of the equity-linked note of the first example any model assuming constant interest rates deliver worse results.

3.4 Conclusions

In the present paper I investigate barrier options, one of th widest class of path-dependent derivatives, exposed to both interest rate and equity risks. Interestingly, I allow for a non-zero correlation between the two and I remove the zero lower bound of the interest rates. More precisely, I focus on the Vasicek market model, where the interest rate is assumed to follow a mean-reverting stochastic process that can take also negative values. Using lattice-based pricing techniques, I develop pricing algorithms for European/American in and out contracts in the case of both discrete and continuous monitoring. Then, I quantify by means of several numerical examples the relative errors an unsophisticated investor who believes that interest rates are constant/ uncorrelated with the equity market/ bounded below from zero would make. It turns out that, depending on the specific option and on the parameters chosen, the relative pricing error can reach remarkably large values, exceeding also 15%.

Future research can be aimed at introducing jumps in the equity price process as their impact when the equity is close to the barrier is surely relevant.

Appendix

The Appendix is divided into 3 subsections:

- Subsection 3.A collects all the proofs of the propositions stated in the paper; for the ease of reading, I recall also the statements of the propositions when providing the proof;
- Subsection 3.B contains the full details of the calibration exercised briefly discussed in subsection 3.3.1;
- Subsection 3.C develops further details that are omitted in the main paper for sake of brevity and readability.

3.A Proofs of the propositions

Proposition 3.1. (Price of an European down-and-out option). The price at inception of the European derivative in (3.13) is given by $v^E_{DO}(0)$ where

$$v^E_{DO}(t_N) = \varphi(S(T))\mathbb{1}\left(S(T) > B\right)$$

and

$$v^E_{DO}(t_i) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_{t_i}^{t_{i+1}} r(s)\mathrm{d}s\right) v^E_{DO}(t_{i+1})\mathbb{1}\left(S(t_i) > B\right)\middle| \mathcal{F}_{t_i}\right]$$

for $i = N-1, \dots, 0$.

Proof Since

$$\mathbb{1}\left(\min_{t\in\{t_0,\dots,t_N\}} S(t) > B\right) = \prod_{i=0,\dots,N} \mathbb{1}\left(S(t_i) > B\right),$$

the fundamental risk-neutral pricing equation for the European derivative with payoff $X_{DO}(T)$ in

(3.13) reads

$$\begin{aligned}
\pi^E_{\varphi,DO}(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(s)\mathrm{d}s}\varphi(S(T))\mathbb{1}\left(\min_{t\in\{t_0,\dots,t_N\}} S(t) > B\right)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(s)\mathrm{d}s}\varphi(S(T))\prod_{i=0,\dots,N}\mathbb{1}\left(S(t_i) > B\right)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{N-1}} r(s)\mathrm{d}s-\int_{t_{N-1}}^T r(s)\mathrm{d}s}\varphi(S(T))\mathbb{1}\left(S(T) > B\right)\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right)\right].
\end{aligned}$$

Exploiting the tower property of the conditional expectation and since both $\exp(-\int_0^{t_{N-1}} r(s)\mathrm{d}s)$ and $\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right)$ are $\mathcal{F}_{t_{N-1}}$-measurable, the previous expression becomes

$$\begin{aligned}
\pi^E_{\varphi,DO}(0) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{N-1}} r(s)\mathrm{d}s-\int_{t_{N-1}}^T r(s)\mathrm{d}s}\varphi(S(T))\mathbb{1}\left(S(T) > B\right)\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right)\middle|\mathcal{F}_{t_{N-1}}\right]\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{N-1}} r(s)\mathrm{d}s}\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_{N-1}}^T r(s)\mathrm{d}s}\varphi(S(T))\mathbb{1}\left(S(T) > B\right)\middle|\mathcal{F}_{t_{N-1}}\right]\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{N-1}} r(s)\mathrm{d}s}\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_{N-1}}^T r(s)\mathrm{d}s}v^E_{DO}(t_N)\middle|\mathcal{F}_{t_{N-1}}\right]\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{N-1}} r(s)\mathrm{d}s}\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right)v^E_{DO}(t_{N-1})\right].
\end{aligned}$$

Going backward, writing $e^{-\int_0^{t_{N-1}} r(s)\mathrm{d}s}$ as $e^{-\int_0^{t_{N-2}} r(s)\mathrm{d}s-\int_{t_{N-2}}^{t_{N-1}} r(s)\mathrm{d}s}$ and $\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right)$ as $\mathbb{1}\left(S(t_{N-1}) > B\right)\prod_{i=0,\dots,N-2}\mathbb{1}\left(S(t_i) > B\right)$ and exploiting again the tower property of the conditional expectation, $\pi^E_{\varphi}(0)$ becomes

$$\begin{aligned}
\pi^E_{\varphi,DO}(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{N-1}} r(s)\mathrm{d}s}\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right)v^E_{DO}(t_{N-1})\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{N-2}} r(s)\mathrm{d}s}\prod_{i=0,\dots,N-2}\mathbb{1}\left(S(t_i) > B\right)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_{N-2}}^{t_{N-1}} r(s)\mathrm{d}s}v^E_{DO}(t_{N-1})\mathbb{1}\left(S(t_{N-2}) > B\right)\middle|\mathcal{F}_{t_{N-2}}\right]\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_{N-2}} r(s)\mathrm{d}s}\prod_{i=0,\dots,N-2}\mathbb{1}\left(S(t_i) > B\right)v^E_{DO}(t_{N-2})\right].
\end{aligned}$$

The second-last step of this backward recursion will deliver

$$\pi^E_{\varphi,DO}(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_1} r(s)\mathrm{d}s}\prod_{i=0,\dots,1}\mathbb{1}\left(S(t_i)>B\right)v^E_{DO}(t_1)\right]$$
$$= v^E_{DO}(0)$$

that concludes the proof. ∎

Proposition 3.2. (Price of an American down-and-out option). The price at inception of the American down-and-out derivative with payoff $\varphi(S(t_i))$ is given by $v^A_{DO}(0)$ where

$$v^A_{DO}(t_N) = \varphi(S(T))\mathbb{1}\left(S(T)>B\right)$$

and

$$v^A_{DO}(t_i) = \max\left\{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_i}^{t_{i+1}} r(s)\mathrm{d}s}v^A_{DO}(t_{i+1})\middle|\left(S(t_i),r(t_i)\right)\right],\varphi(S(t_i))\right\}\mathbb{1}\left(S(t_i)>B\right)$$

for $i = N-1,\dots,0$.

Proof The fundamental risk-neutral pricing equation for the discretely monitored American option with payoff $\varphi(S(t_i))$ reads

$$\pi^A_{\varphi,DO}(0) = \max_{\tau\in\{t_0,\dots,t_N\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{\tau} r(s)\mathrm{d}s}X_{DO}(\tau)\right]$$
$$= \max_{\tau\in\{t_0,\dots,t_N\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{\tau} r(s)\mathrm{d}s}\varphi(S(\tau))\mathbb{1}\left(\min_{t\in\{t_0,\dots,\tau\}}S(t)>B\right)\right]$$

As explained in Section 21.4 of Björk (2009), $\pi^A_{\varphi,DO}(0)$ can be computed by the following backward recursion

$$V_{t_i} = \max\left\{X_{DO}(t_i),\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_i}^{t_{i+1}} r(s)\mathrm{d}s}V_{t_{i+1}}\middle|\mathcal{F}_{t_i}\right]\right\}\quad\text{for } i=0,\dots,N-1$$
$$V_{t_N} = X_{DO}(t_N) = \varphi(S(T))\mathbb{1}\left(\min_{t\in\{t_0,\dots,t_N\}}S(t)>B\right)\tag{3.A1}$$

as $\pi^A_{\varphi,DO}(0) = V_{t_0}$.

As shown below, it turns out that $V_{t_i} = v^A_{DO}(t_i)\prod_{j=0,\dots,i}\mathbb{1}\left(S(t_i)>B\right)$ for all $i=0,\dots,N$.

Hence, at t_0, $\pi^A_{\varphi,DO}(0) = V_{t_0} = v^A_{DO}(0)$ that concludes the proof.

I now show that $V_{t_i} = v^A_{DO}(t_i)\prod_{j=0,\dots,i}\mathbb{1}\left(S(t_i)>B\right)$ for all $i=0,\dots,N$. The first step of the backward recursion (3.A1) can be written as

$$V_{t_N} = \varphi(S(T))\mathbb{1}\left(S(T)>B\right)\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i)>B\right) = v^A_{DO}(t_N)\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i)>B\right)$$

since

$$\mathbb{1}\left(\min_{t\in\{t_0,\dots,t_N\}} S(t) > B\right) = \prod_{i=0,\dots,N} \mathbb{1}\left(S(t_i) > B\right).$$

The second backward step of the recursion reads

$$\begin{aligned} V_{t_{N-1}} &= \max\left\{X_{DO}(t_{N-1}), \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_{N-1}}^{t_N} r(s)\mathrm{d}s} V_{t_N}\middle|\mathcal{F}_{t_{N-1}}\right]\right\} \\ &= \max\left\{\varphi(S(t_{N-1}))\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right),\right. \\ &\qquad \left.\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_{N-1}}^{t_N} r(s)\mathrm{d}s} v_{DO}^A(t_N)\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right)\middle|\mathcal{F}_{t_{N-1}}\right]\right\}. \end{aligned}$$

Since $\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right)$ is $\mathcal{F}_{t_{N-1}}$-measurable, $V_{t_{N-1}}$ can be rewritten as

$$V_{t_{N-1}} = \max\left\{\varphi(S(t_{N-1})), \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_{N-1}}^{t_N} r(s)\mathrm{d}s} v_{DO}^A(t_N)\middle|\mathcal{F}_{t_{N-1}}\right]\right\}\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right).$$

Notice that, since $e^{-\int_{t_{N-1}}^{t_N} r(s)\mathrm{d}s}$ depends only on $r(t_{N-1})$ and $v_{DO}^A(t_N)$ only on $S(t_{N-1})$,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_{N-1}}^{t_N} r(s)\mathrm{d}s} v_{DO}^A(t_N)\middle|\mathcal{F}_{t_{N-1}}\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_{N-1}}^{t_N} r(s)\mathrm{d}s} v_{DO}^A(t_N)\middle|\left(S(t_{N-1}), r(t_{N-1})\right)\right]$$

that delivers

$$V_{t_{N-1}} = v_{DO}^A(t_{N-1})\prod_{i=0,\dots,N-1}\mathbb{1}\left(S(t_i) > B\right).$$

By induction,

$$V_{t_i} = v_{DO}^A(t_i)\prod_{j=0,\dots,i}\mathbb{1}\left(S(t_j) > B\right)$$

for all $i = 0, \dots, N$. ■

Proposition 3.3. (In-out parity). The price of the European knock-out option with payoff $\varphi(S(T))$ and of the European knock-in option with the same payoff satisfy

$$\pi_{\varphi,DO}^E(t) + \pi_{\varphi,DI}^E(t) = \pi_{\varphi}^E(t), \quad \forall t \in [0, T]$$

where $\pi_{\varphi}^E(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\mathrm{d}s}\varphi(S(T))\middle|\mathcal{F}_t\right]$ is the no-arbitrage price of the plain European derivative that pays $\varphi(S(T))$ at T.

Proof Since

$$1 = \left(\mathbb{1}\left(\min_{t\in\{t_0,\ldots,t_N\}} S(t) > B\right) + \mathbb{1}\left(\min_{t\in\{t_0,\ldots,t_N\}} S(t) < B\right)\right),$$

the linearity of the expectation delivers at any $t \in [0,T]$

$$\begin{aligned}
\pi^E_\varphi(t) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\mathrm{d}s}\varphi(S(T))\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\mathrm{d}s}\varphi(S(T))\left(\mathbb{1}\left(\min_{t\in\{t_0,\ldots,t_N\}} S(t) > B\right) + \mathbb{1}\left(\min_{t\in\{t_0,\ldots,t_N\}} S(t) < B\right)\right)\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\mathrm{d}s}\varphi(S(T))\mathbb{1}\left(\min_{t\in\{t_0,\ldots,t_N\}} S(t) > B\right)\middle|\mathcal{F}_t\right] + \\
&\qquad + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\mathrm{d}s}\varphi(S(T))\mathbb{1}\left(\min_{t\in\{t_0,\ldots,t_N\}} S(t) < B\right)\middle|\mathcal{F}_t\right] \\
&= \pi^E_{\varphi,DO}(t) + \pi^E_{\varphi,DI}(t).
\end{aligned}$$

■

Proposition 3.4. (Price of an American down-and-in option). The price at inception of the American down-and-in derivative with payoff $\varphi(S(t_i))$ is given by $v^A_{DI}(0)$ where

$$v^A_{DI}(t_N) = \varphi(S(T))\mathbb{1}\left(m(T) \le B\right)$$

and

$$v^A_{DI}(t_i) = \max\left\{\varphi(S(t_i))\mathbb{1}\left(m(t_i) \le B\right), \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_i}^{t_{i+1}} r(s)\mathrm{d}s} v^A_{DI}(t_{i+1})\middle|\left(S(t_i), r(t_i), m(t_i)\right)\right]\right\}$$

for $i = N-1, \ldots, 0$.

Proof As recalled in the proof of Proposition 3.2, the price of American down-and-in option $\pi^A_{\varphi,DI}(0)$, can be computed by the following backward recursion

$$\begin{aligned}
V_{t_i} &= \max\left\{X_{DI}(t_i), \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_i}^{t_{i+1}} r(s)\mathrm{d}s} V_{t_{i+1}}\middle|\mathcal{F}_{t_i}\right]\right\} \text{ for } i = 0,\ldots,N-1 \qquad (3.\text{A}2)\\
V_{t_N} &= X_{DI}(t_N) = \varphi(S(T))\mathbb{1}\left(m(T) \le B\right)
\end{aligned}$$

as $\pi^A_{\varphi,DI}(0) = V_{t_0}$.

The only difference between the backward recursions $\{v^A_{DI}(t_i)\}_{i=0,\ldots,N}$ and $\{V_{t_i}\}_{i=0,\ldots,N}$ is the conditioning of the expected value in the evaluation of the continuation value of the American option. In (3.A2), $\mathcal{F}_{t_i}$ is the standard sigma-algebra generated by $(S(t), r(t))$ and contains all the realizations of $(S(t_j), r(t_j))$ from $j = 0$ up to $j = i$. Consequently, the running minimum $m(t_i) = \min_{t \in \{t_0,\dots,t_i\}} S(t)$ is $\mathcal{F}_{t_i}$-measurable. As $m(t_{i+1}) = \min\{m(t_i), S(t_{i+1})\}$, the expected value in (3.A2) depends only on the current value of $(S(t_i), r(t_i), m(t_i))$ as its argument is now Markovian with respect to the two standard state variables and the running minimum. Therefore,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_i}^{t_{i+1}} r(s)\mathrm{d}s} V_{t_{i+1}} \middle| \mathcal{F}_{t_i} \right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_i}^{t_{i+1}} r(s)\mathrm{d}s} V_{t_{i+1}} \middle| (S(t_i), r(t_i), m(t_i)) \right]$$

that concludes the proof. ■

Proposition 3.5. (Convergence of the continuously monitored barrier options). Assume that $(S(t_i), r(t_i))_{i=0,\dots,N}$ is a discrete time stochastic process that converges in distribution to $(S(t), r(t))$ in (3.2) as $N \to +\infty$. Then the price of the down-and-out/down-and-in European/American barrier options discretely monitored at $\{S_i(t)\}_{i=0,\dots,N}$ evaluated by means of Propositions 3.1, 3.2, 3.3, 3.4 converges to the price of the related continuously monitored barrier options as $N \to +\infty$.

Proof The result is standard for European options. Indeed, if a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ converges in distribution to X so does its expected value, $\mathbb{E}[X_n] \overset{n\to+\infty}{\longrightarrow} \mathbb{E}[X]$.
On the contrary, Mulinacci and Pratelli (1998) prove the analogous result for American options showing that, if $\{X_n\}_{n\in\mathbb{N}}$ converges in distribution to X, then $\operatorname{ess\,sup} \mathbb{E}[X_n] \overset{n\to+\infty}{\longrightarrow} \operatorname{ess\,sup} \mathbb{E}[X]$. ■

3.B Details of the calibration

I calibrate my market model (3.1) under the risk-neutral measure $\mathbb{Q}$ to European market data as of November 2018 the 30$^{\text{th}}$.
The underlying equity price process I calibrate my model to is the Euro Stoxx 50. Introduced in February 1998 and designed by STOXX, an index provider owned by Deutsche Börse Group, the Euro Stoxx 50 collects fifty of the largest and most liquid European stocks. All the details about this index can be found on STOXX's official website at `https://www.stoxx.com`.
Prices and trading volumes of options on the Euro Stoxx 50 are provided by Eurex Exchange, the largest European futures and options market at `https://www.eurexchange.com`.
More specifically, I collect the weekly average of daily settlement prices of several European call

T	# opt.	# call	# put	# traded cnt.
1 m	28	14	14	304'868
3 m	45	17	28	21'697
6 m	29	15	14	18'525
1 y	27	10	17	2991

Table 3.B.1: Summary statistics of the options considered for the calibration exercise. T is the maturity of the option, "m", "y" stand for month and year respectively. # opt. (resp. # call, # put) are the number of options (resp. call, put) with maturity T and different strike price. Strike price increments are either 25 or 50 points large. # traded cnt. is the total number of options with maturity T traded on 30^{th}, 2018.

and put options on the index with moneyness (defined as $K/S(0)$) ranging from 0.8 to 1.2 and maturities equal to 1,3,6 months and 1 year. I neglect options with poor trading volume (less than 100 daily traded contracts) and this excludes most of the options with maturities longer than 1 year. The average number of contracts traded per option is roughly equal to 2000 per day. Summary statistics of the final sample of options exploited for the calibration are displayed in Table 3.B.1.

The interest rate process is calibrated to the AAA-rated zero coupon bonds of the Eurozone. The prices of the AAA-rated zero coupon bonds are collected and elaborated by the European Central Bank and are available for maturities equal to 3, 6, 9 months and 1,. . . , 30 years on its website at `https://www.ecb.europa.eu`. As the short-term interest rate has by definition an instantaneous holding period, the natural candidate for its approximation is the overnight rate. Nevertheless, this rate turns out to be quite volatile depending on the business day and poorly correlated to the even the short-term yields. Therefore, I proxy the short-term rate by the yield with shortest maturity, namely the three months one. As of November 2018 the 30^{th}, I have $r_0 = -0.6728\%$.

The parameters that have to be calibrated in (3.1) are $(\kappa, \theta, \sigma_r, \sigma_S, \rho) =: \Theta$. I set the continuous dividend yield equal to $q = 3.6\%$ as reported by Eurex Exchange as of the calibration date. First, I address the calibration of the baseline model.

Market prices of the zero coupon bonds $\{p^M(0,T)\}$, $T \in \{0.25, 0.5, 1, \ldots, 30\}$ are compared to the prices $\{p(0,T;\Theta)\}$ delivered by the model in (3.3). Market option prices[17] $\{\pi^M_{put}(t), \pi^M_{call}(t)\}$, $T \in \{1/12, 0.25, 0.5, 1\}$, $\{K/S_0 \in 0.8, \ldots, 1.2\}$ are compared to the prices $\{\pi^E_{put}(t;\Theta), \pi^E_{call}(t;\Theta)\}$

[17]For ease of reading, I drop the superscript ^{E}, characterizing the exercise style of the options and I leave only the superscript ^{M} that labels real market prices.

derived in (3.6) and (3.7).

To find the optimal vector of parameters Θ^* I solve

$$\Theta^* := \arg\min_{\Theta \in D} \left(\begin{bmatrix} \alpha\{p^M(0,T) - p(0,T;\Theta)\} \\ \beta\{\pi^M_{put}(t) - \pi^E_{put}(t;\Theta)\} \\ \beta\{\pi^M_{call}(t) - \pi^E_{call}(t;\Theta)\} \end{bmatrix}' \begin{bmatrix} \{p^M(0,T) - p(0,T;\Theta)\} \\ \{\pi^M_{put}(t) - \pi^E_{put}(t;\Theta)\} \\ \{\pi^M_{call}(t) - \pi^E_{call}(t;\Theta)\} \end{bmatrix} \right), \tag{3.B3}$$

where $\alpha := \#prices/\#bonds = 163/34$ and $\beta := \#prices/\#options = 163/129$ are two constant weights that compensate the different number of zero coupon bonds and options in the sample and D is a bounded domain where the optimal parameters are assumed to take values. More specifically, $D = (0,2) \times (0,0.1) \times (0,0.1) \times (0,0.2) \times (-1,1)$.

The minimization in (3.B3) is solved numerically exploiting 500 different initial points uniformly distributed in D. At the random initial points the objective function is equal to 12.3236 on average. The numerical solution to (3.B3) is

$$\Theta^* = \begin{bmatrix} \kappa^* \\ \theta^* \\ \sigma_r^* \\ \sigma_S^* \\ \rho^* \end{bmatrix} = \begin{bmatrix} 0.1199 \\ 0.0316 \\ 0.0154 \\ 0.1437 \\ 0.8542 \end{bmatrix}$$

that delivers a residual value of the objective function equal to 0.0028.

Now, the alternative models need to be calibrated as well. For each alternative model, the set of parameters that need to be calibrated, Θ, is modified accordingly. As an example, when calibrating the model with r assumed to be constant, the parameters κ, θ, σ_r and ρ do not belong to Θ, which boils down to $\Theta = (\sigma_S)$. When calibrating the CIR model, r is forced to be strictly positive. Despite the theoretical inapplicability of the CIR model in the present market conditions, it is left among the alternative models as it is extremely common among practitioners.

Table 3.B.2 collects the results of the calibration of the different models. We notice that if r is assumed to be constant, the volatility of the equity process is a little bit larger than in the baseline case. In other words, in these cases, σ_S accounts also for the possible variability of the neglected risk factor. When we assume that the two risk factors are uncorrelated, the calibration exercise delivers an extremely low level of mean-reversion speed associated to a remarkably large long-run mean. The volatility of the equity price process, on the contrary, is quite stable.

For sake of completeness, I also calibrate my market model (3.1) under the risk-neutral measure $\mathbb{Q}$ to US market data as of February 2019 the 21$^{\text{st}}$.

model	κ^*	θ^*	σ_r^*	σ_S^*	ρ^*
baseline	0.1199	0.0316	0.0154	0.1437	0.8542
$r(t) \equiv r_0$				0.1502	
$r(t) \equiv r^*$				0.1505	
$r(t) \equiv \theta^*$				0.1621	
$\rho = 0$	0.0230	0.1290	0.0130	0.1503	
$\rho = -\rho^*$	0.1334	0.0169	0.0081	0.1507	
CIR	0.0926	0.0381	0.0192	0.1473	0.7391

Table 3.B.2: Results of the calibration exercise for all the alternative models.

T	# opt.	# call	# put	# traded cnt.
1 m	45	25	30	57'146
3 m	42	23	29	13'277
6 m	22	9	13	12'745
1 y	21	8	1	6985

Table 3.B.3: Summary statistics of the options considered for the calibration exercise in the US market. T is the maturity of the option, "m", "y" stand for month and year respectively. # opt. (resp. # call, # put) are the number of options (resp. call, put) with maturity T and different strike price. Strike price increments are either 25 or 50 points large. # traded cnt. is the total number of options with maturity T traded on November 30$^{\text{th}}$, 2018.

The underlying equity price process I calibrate my model to is the S&P 500. Data on options on the index are taken from Option Metrics. The annualized average annual dividend yield is now equal to $q = 1.9\%$.
More specifically, I collect the weekly average of daily settlement prices of several European call and put options on the index with moneyness (defined as $K/S(0)$) ranging from 0.8 to 1.2 and maturities equal to 1,3,6 months and 1 year. I neglect options with poor trading volume (less than 100 daily traded contracts) and this excludes most of the options with maturities longer than 1 year. The average number of contracts traded per option is roughly equal to 1500 per day. Summary statistics of the final sample of options exploited for the calibration are displayed in Table 3.B.3.

The interest rate process is calibrated to the yield curve of Treasury Bills. The data on the US Government treasury yield curve are available at the US Department of the Treasury website[18]. As before, I proxy the short-term rate by the yield with shortest maturity, namely the three months one. As of February 2019 the 21st, I have $r_0 = 2.37\%$.
The numerical solution to (3.B3) for the US market is

$$\Theta^* = \begin{bmatrix} \kappa^* \\ \theta^* \\ \sigma_r^* \\ \sigma_S^* \\ \rho^* \end{bmatrix} = \begin{bmatrix} 0.0448 \\ 0.0565 \\ 0.0092 \\ 0.1708 \\ 0.7727 \end{bmatrix}.$$

We notice that, with respect to the optimal parameters for the European market, there are many differences. First of all, the speed of mean reversion κ is now small, only 4.48%. The long-term mean θ is two percentage points higher, which is coherent with the initial state of r as r_0 is 2 percentage points higher in the US market than in the European one. Along with a smaller mean-reversion, also the volatility of the interest rate σ_r is here smaller and equal to 0.92%. On the contrary, the volatility of the equity process and the correlation between r and S are comparable to the ones of the European market.
The results of the calibration of the other alternative US market are shown in Table 3.B.4. We notice that, as in the European case, the volatility of the equity price process is quite stable across all the alternative market models. As before, when we assume that the two risk factors are uncorrelated, we get a little smaller speed of mean reversion and a little higher mean reversion level.
Finally, we acknowledge that, dealing with the US market, we have no problem working with the CIR model as interest rates are non negative.

3.C Further details

3.C.1 The quadrinomial tree

The parameters of the quadrinomial tree introduced in (3.20) are

$$\begin{aligned} \Delta Y &= \sigma_S \sqrt{\Delta t} \\ \Delta r &= \sigma_r \sqrt{\Delta t} \end{aligned}$$

[18]The link is https://www.treasury.gov.

model	κ^*	θ^*	σ_r^*	σ_S^*	ρ^*
baseline	0.0448	0.0565	0.0092	0.1708	0.7727
$r(t) \equiv r_0$				0.1739	
$r(t) \equiv r^*$				0.1756	
$r(t) \equiv \theta^*$				0.1801	
$\rho = 0$	0.0343	0.0619	0.0078	0.1739	
$\rho = -\rho^*$	0.0371	0.0498	0.0087	0.1738	
CIR	0.1036	0.0582	0.0152	0.1721	0.7987

Table 3.B.4: Results of the calibration exercise for all the alternative models in the US market.

$$
\begin{aligned}
q_{uu} &= \frac{\mu_Y \mu_r \Delta t + \mu_Y \Delta r + \mu_r \Delta Y + (1+\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\
q_{ud} &= \frac{-\mu_Y \mu_r \Delta t + \mu_Y \Delta r - \mu_r \Delta Y + (1-\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\
q_{du} &= \frac{-\mu_Y \mu_r \Delta t - \mu_Y \Delta r + \mu_r \Delta Y + (1-\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S} \\
q_{dd} &= \frac{\mu_Y \mu_r \Delta t - \mu_Y \Delta r - \mu_r \Delta Y + (1+\rho)\sigma_r \sigma_S}{4\sigma_r \sigma_S}.
\end{aligned}
$$

where $\mu_Y := \left(r(t) - q - \frac{\sigma_S^2}{2}\right)$ and $\mu_r := \kappa(\theta - r(t))$. For the full derivation of the quadrinomial tree and for the complete discussion of the positivity of the four transition probabilities refer to Battauz and Rotondi (2019).

3.C.2 CEV models

Assume that the zero lower bound for the interest rate is present, namely assume $r(t) > 0$ for all $t \in [0, T]$. Allowing for local volatility for both the risky equity price and the interest rate the market model reads

$$
\begin{cases}
\mathrm{d}S(t) = S(t)(r(t) - q)\mathrm{d}t + \sigma_S S(t)^\alpha \mathrm{d}W_S^{\mathbb{Q}}(t) \\
\mathrm{d}r(t) = \kappa\left(\theta - r(t)\right)\mathrm{d}t + \sigma_r r(t)^\beta \mathrm{d}W_r^{\mathbb{Q}}(t)
\end{cases}
\tag{3.C4}
$$

with $\alpha, \beta > 0$. Notice that $\beta = 0.5$ leads to the celebrated CIR model for the short term interest rates proposed by Cox et al. (1885).

Following the derivativation of the quadrinomial tree in Section 2.2 of Battauz and Rotondi (2019), I first need to define two new processes $(Y(t), Z(t))$ with constant diffusion coefficients. Therefore, I set

$$
Y(t) := \frac{S(t)^{1-\alpha}}{1-\alpha},
$$

$$Z(t) = \frac{r(t)^{1-\beta}}{1-\beta}$$

that deliver also the inverse transformations

$$S(t) = \left[(1-\alpha)Y(t)\right]^{\frac{1}{1-\alpha}}$$

$$r(t) = \left[(1-\beta)Z(t)\right]^{\frac{1}{1-\beta}}.$$

Applying Itô's Lemma to both Y and Z leads to

$$\begin{cases} \mathrm{d}Y(t) &= \mu_Y \mathrm{d}t + \sigma_S \mathrm{d}W_S^{\mathbb{Q}}(t) \\ \mathrm{d}Z(t) &= \mu_Z \mathrm{d}t + \sigma_r \mathrm{d}W_r^{\mathbb{Q}}(t) \end{cases}$$

where

$$\mu_Y = S(t)^{1-\alpha}(r(t)-q) - \frac{\sigma_S}{2}\alpha S(t)^{\alpha-1},$$

$$\mu_Z = \kappa(\theta - r(t))r(t)^{-\beta} - \frac{\sigma_r^2}{2}\beta r(t)^{\beta-1}.$$

Therefore, I can exploit the transition probabilities and the increments recalled in Appendix 3.C.1 for the standard quadrinomial tree.

3.C.3 Additional plots

This Subsection contains additional plots for the two applications of Subsection 3.3.3.
Figure 3.C.1 shows the Greeks of the equity-linked note of the first example of Subsection 3.3.3.
Figure 3.C.2 shows Greeks of the American up-and-out call option of the second example of Subsection 3.3.3.

3.C.4 Additional numerical examples

This Section contains additional numerical examples within the same framework and with the same structure of the two main numerical examples of Subsection 3.3.2. All of the parameters of the two examples stay the same but the maturity of the options and the barrier level.
More precisely, Table 3.C.1 and Table 3.C.2 repeat the numerical exercise of Subsection 3.3.2. On average, we see that relative pricing errors are larger than the ones of Table 3.2 and Table 3.3 as the misspecification of the models worsens when the investment horizon moves forward.
The barrier events, if any, of the options priced in Table 3.C.3 and Table 3.C.4 (resp. Table 3.C.5 and Table 3.C.6) is set equal to $L = 60$ (resp. $L = 80$). The relative pricing errors are consistent with the benchmark numerical examples. The most relevant difference is that as the barrier is now higher (resp. lower) in contracts appreciate (resp. depreciate) as it is easier to cross the barrier

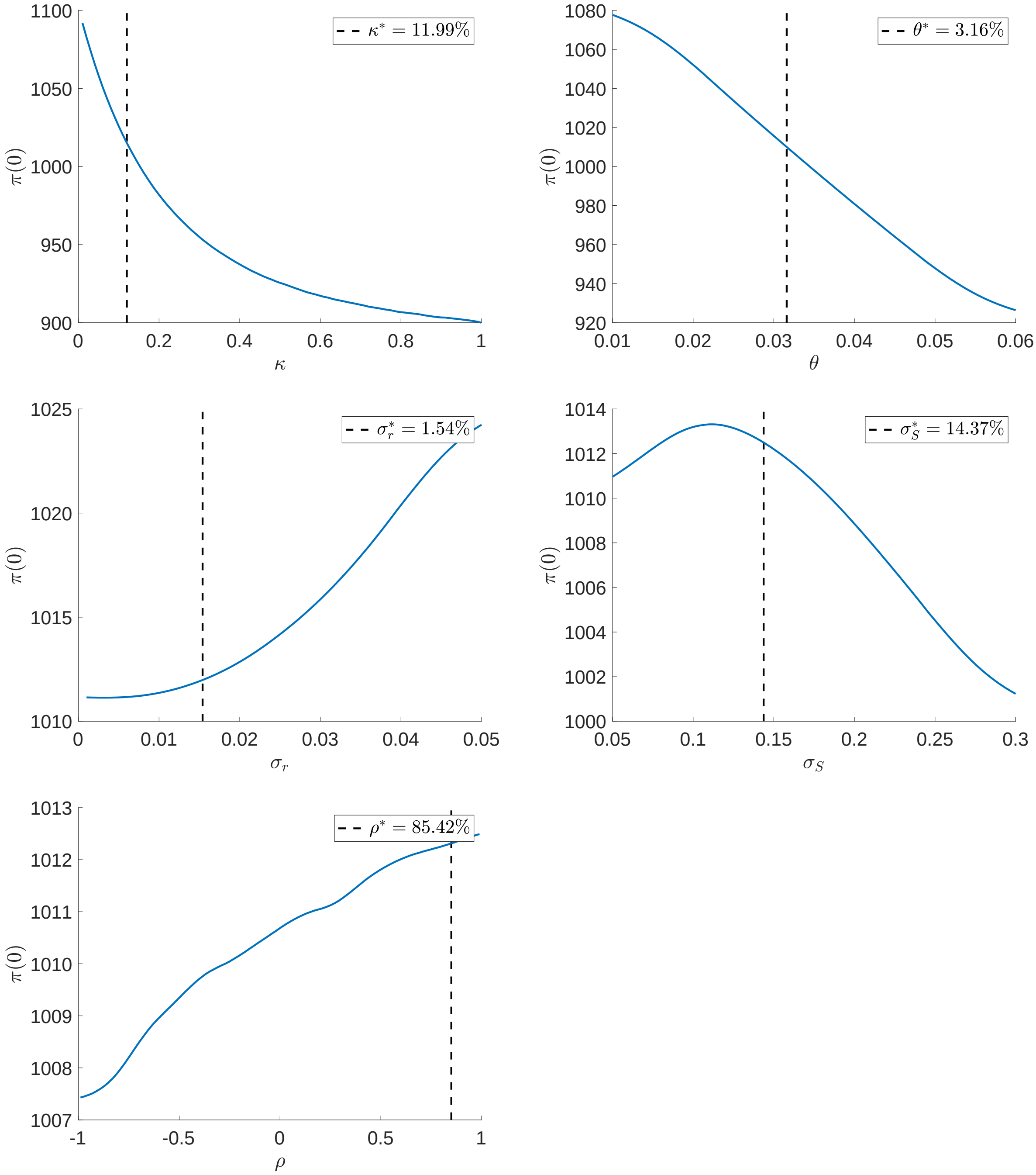

Figure 3.C.1: Sensitivities of the price at inception of the equity-linked note of the first example with respect to the five parameters of the market model.

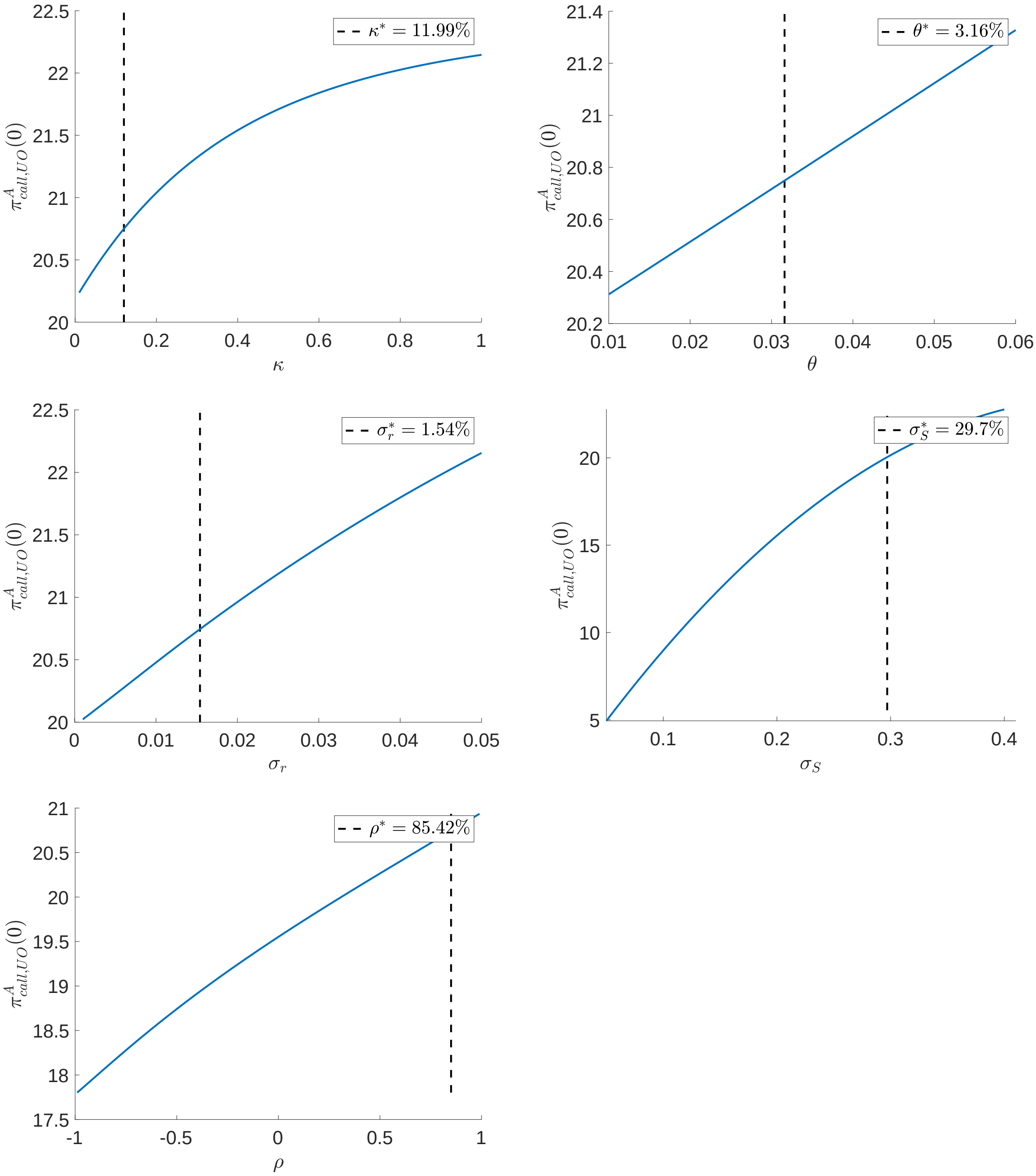

Figure 3.C.2: Sensitivities of the price at inception of the American up and out call option of the second example with respect to the five parameters of the market model.

whereas out contracts depreciate (resp. appreciate) as now the probability of being knocked out is larger (resp. smaller).

European Options

payoff	model	Vanilla price	Vanilla rel. error	Down-and-out price	Down-and-out rel. error	Down-and-in price	Down-and-in rel. error
Digital	baseline	0.9493		0.2715		0.6778	
	$r(t) \equiv r(0)$	1.0330	8.82 %	0.2892	6.52 %	0.7438	9.74 %
	$r(t) \equiv r^*$	1.0101	6.40 %	0.2929	7.88 %	0.7172	5.81 %
	$r(t) \equiv \theta^*$	0.8538	10.06 %	0.3147	15.91 %	0.5391	20.46 %
	$\rho = 0$	0.9493	0.00 %	0.2903	6.92 %	0.6590	2.77 %
	$\rho = -\rho^*$	0.9493	0.00 %	0.3135	15.47 %	0.6358	6.20 %
	CIR	0.9968	5.00 %	0.2931	7.96 %	0.7037	3.82 %
Put	baseline	27.4619		0.4495		27.0124	
	$r(t) \equiv r(0)$	29.4747	7.33 %	0.6161	37.06 %	28.8586	6.83 %
	$r(t) \equiv r^*$	27.9387	1.74 %	0.6049	34.57 %	27.3338	1.19 %
	$r(t) \equiv \theta^*$	18.3446	33.20 %	0.5083	13.08 %	17.8363	33.97 %
	$\rho = 0$	24.4079	11.12 %	0.5281	17.49 %	23.8798	11.60 %
	$\rho = -\rho^*$	20.8423	24.10 %	0.6416	42.74 %	20.2007	25.22 %
	CIR	27.1634	1.09 %	0.5925	31.81 %	26.5709	1.63 %
Call	baseline	27.6291		22.8470		4.7821	
	$r(t) \equiv r(0)$	21.2750	23.00 %	17.5728	23.08 %	3.7022	22.58 %
	$r(t) \equiv r^*$	22.0411	20.23 %	18.2544	20.10 %	3.7867	20.82 %
	$r(t) \equiv \theta^*$	28.0760	1.62 %	23.7215	3.83 %	4.3545	8.94 %
	$\rho = 0$	24.5746	11.06 %	20.1730	11.70 %	4.4016	7.96 %
	$\rho = -\rho^*$	21.0035	23.98 %	17.0224	25.49 %	3.9811	16.75 %
	CIR	22.5912	18.23 %	18.7298	18.02 %	3.8614	19.25 %

Table 3.C.1: European options prices. $S(0) = 100$, $q = 1\%$ and $T = 5$. The payoff at maturity of digital options is $\varphi(S(T)) = 1$; of (at the money) put options is $\varphi(S(T)) = (S(0) - S(T))^+$; of (at the money) call options is $\varphi(S(T)) = (S(T) - S(0))^+$. The baseline model is the Vasicek market model in (3.1) with the parameters specified in Table 3.1. The models $r(t) \equiv r(0)/r^* = -0.23\%/\theta^*$ are pure diffusive models (Black-Scholes ones) where the interest rate is constant and equal respectively to its initial value/ the yield on T-bond riskless bond/ the long-run mean. The models $\rho = 0/-\rho^*$ differ from the baseline one only with respect to the correlation which is assumed to be 0/ the opposite of the result from the calibration exercise. The CIR model is the one in (3.22) with $\gamma = 0.5$. The barrier is $L = 70$.

American Options

payoff	model	Vanilla			Down-and-out			Down-and-in			DFP
		price	rel. error	EEP	price	rel. error	EEP_{DO}	price	rel. error	EEP_{DI}	
Digital	baseline	1.0268		0.08	1.0247		0.75	0.6779		0.00	0.68
	$r(t) \equiv r(0)$	1.0330	0.60 %	0.00	1.0330	0.81 %	0.74	0.7439	9.74 %	0.00	0.74
	$r(t) \equiv r^*$	1.0101	1.63 %	0.00	1.0101	1.42 %	0.72	0.7172	5.80 %	0.00	0.72
	$r(t) \equiv \theta^*$	1.0000	2.61 %	0.15	1.0000	2.41 %	0.69	0.5392	20.46 %	0.00	0.54
	$\rho = 0$	1.0268	0.00 %	0.08	1.0217	0.29 %	0.73	0.6590	2.79 %	0.00	0.65
	$\rho = -\rho^*$	1.0268	0.00 %	0.08	1.0251	0.04 %	0.71	0.6359	6.20 %	0.00	0.63
	CIR	1.0000	2.61 %	0.00	1.0000	2.41 %	0.71	0.7037	3.81 %	0.00	0.70
Put	baseline	27.8110		0.35	18.5489		18.10	27.0124		0.00	17.75
	$r(t) \equiv r(0)$	29.4747	5.98 %	0.00	19.9257	7.42 %	19.31	28.8586	6.83 %	0.00	19.31
	$r(t) \equiv r^*$	27.9387	0.46 %	0.00	19.5808	5.56 %	18.98	27.3338	1.19 %	0.00	18.98
	$r(t) \equiv \theta^*$	20.1142	27.68 %	1.77	17.0076	8.31 %	16.50	17.8363	33.97 %	0.00	14.73
	$\rho = 0$	26.0891	6.19 %	1.68	19.0255	2.57 %	18.50	23.8798	11.60 %	0.00	16.82
	$\rho = -\rho^*$	24.4080	12.24 %	3.57	19.6001	5.67 %	18.96	20.2007	25.22 %	0.00	15.39
	CIR	27.1634	2.33 %	0.00	18.8956	1.87 %	18.30	26.5709	1.63 %	0.00	18.30
Call	baseline	27.6375		0.01	22.8550		0.01	4.7822		0.00	0.00
	$r(t) \equiv r(0)$	22.0788	20.11 %	0.80	18.2894	19.98 %	0.72	3.9622	17.15 %	0.26	0.17
	$r(t) \equiv r^*$	22.6938	17.89 %	0.65	18.8380	17.58 %	0.58	3.8766	18.94 %	0.09	0.02
	$r(t) \equiv \theta^*$	28.0979	1.67 %	0.02	23.7430	3.89 %	0.02	4.3645	8.73 %	0.01	0.01
	$\rho = 0$	25.0189	9.47 %	0.44	20.5696	10.00 %	0.40	4.6017	3.77 %	0.20	0.15
	$\rho = -\rho^*$	22.6444	18.07 %	1.64	18.4753	19.16 %	1.45	4.8711	1.86 %	0.89	0.70
	CIR	23.0585	16.57 %	0.47	21.0585	7.86 %	2.33	4.3728	8.56 %	0.51	2.37

Table 3.C.2: American options prices. $S(0) = 100$, $q = 1\%$ and $T = 5$. The immediate payoff of digital options is $\varphi(S(t)) = 1$; of (at the money) put options is $\varphi(S(t)) = (S(0) - S(t))^+$; of (at the money) call options is $\varphi(S(t)) = (S(t) - S(0))^+$. The baseline model is the Vasicek market model in (3.1) with the parameters specified in Table 3.1. The models $r(t) \equiv r(0)/r^* = -0.23\%/\theta^*$ are pure diffusive models (Black-Scholes ones) where the interest rate is constant and equal respectively to its initial value/ the yield on T-bond riskless bond/ the long-run mean. The models $\rho = 0/-\rho^*$ differ from the baseline one only with respect to the correlation which is assumed to be 0/ the opposite of the result from the calibration exercise. The CIR model is the one in (3.22) with $\gamma = 0.5$. The barrier is $L = 70$. The EEP is the Early Exercise Premium, namely the difference between the price of the American option and of its European counterpart. The deviation from parity, DFP, is defined in Equation (3.18).

payoff	model	European Options: Vanilla price	Vanilla rel. error	Down-and-out price	Down-and-out rel. error	Down-and-in price	Down-and-in rel. error
Digital	baseline	0.9950		0.6866		0.3084	
	$r(t) \equiv r(0)$	1.0131	1.82 %	0.7124	3.76 %	0.3007	2.50 %
	$r(t) \equiv r^*$	1.0121	1.72 %	0.7124	3.76 %	0.2997	2.82 %
	$r(t) \equiv \theta^*$	0.9388	5.65 %	0.7128	3.82 %	0.2260	26.72 %
	$\rho = 0$	0.9950	0.00 %	0.7097	3.36 %	0.2853	7.49 %
	$\rho = -\rho^*$	0.9950	0.00 %	0.7356	7.14 %	0.2594	15.89 %
	CIR	0.9997	0.47 %	0.7129	3.83 %	0.2868	7.00 %
Put	baseline	17.9580		5.0867		12.8713	
	$r(t) \equiv r(0)$	17.8640	0.52 %	5.6074	10.24 %	12.2566	4.78 %
	$r(t) \equiv r^*$	17.8021	0.87 %	5.5967	10.03 %	12.2054	5.17 %
	$r(t) \equiv \theta^*$	13.5771	24.40 %	4.7758	6.11 %	8.8013	31.62 %
	$\rho = 0$	16.8462	6.19 %	5.3543	5.26 %	11.4919	10.72 %
	$\rho = -\rho^*$	15.6479	12.86 %	5.6436	10.95 %	10.0043	22.27 %
	CIR	17.0602	5.00 %	5.4646	7.43 %	11.5956	9.91 %
Call	baseline	16.4712		16.3670		0.1042	
	$r(t) \equiv r(0)$	14.5723	11.53 %	14.4854	11.50 %	0.0869	16.60 %
	$r(t) \equiv r^*$	14.6117	11.29 %	14.5246	11.26 %	0.0871	16.41 %
	$r(t) \equiv \theta^*$	17.7203	7.58 %	17.6244	7.68 %	0.0959	7.97 %
	$\rho = 0$	15.3589	6.75 %	15.2648	6.73 %	0.0941	9.69 %
	$\rho = -\rho^*$	14.1599	14.03 %	14.0771	13.99 %	0.0828	20.54 %
	CIR	15.1125	8.25 %	15.0244	8.20 %	0.0881	15.45 %

Table 3.C.3: European options prices. $S(0) = 100$, $q = 1\%$ and $T = 5$. The payoff at maturity of digital options is $\varphi(S(T)) = 1$; of (at the money) put options is $\varphi(S(T)) = (S(0) - S(T))^+$; of (at the money) call options is $\varphi(S(T)) = (S(T) - S(0))^+$. The baseline model is the Vasicek market model in (3.1) with the parameters specified in Table 3.1. The models $r(t) \equiv r(0)/r^* = -0.23\%/\theta^*$ are pure diffusive models (Black-Scholes ones) where the interest rate is constant and equal respectively to its initial value/ the yield on T-bond riskless bond/ the long-run mean. The models $\rho = 0/-\rho^*$ differ from the baseline one only with respect to the correlation which is assumed to be 0/ the opposite of the result from the calibration exercise. The CIR model is the one in (3.22) with $\gamma = 0.5$. The barrier is $L = 60$.

American Options

payoff	model	Vanilla			Down-and-out			Down-and-in			
		price	rel. error	EEP	price	rel. error	EEP_{DO}	price	rel. error	EEP_{DI}	DFP
Digital	baseline	1.0138		0.02	1.0138		0.33	0.3984		0.09	0.40
	$r(t) \equiv r(0)$	1.0131	0.07 %	0.00	1.0131	0.07 %	0.30	0.3007	24.52 %	0.00	0.30
	$r(t) \equiv r^*$	1.0121	0.17 %	0.00	1.0121	0.17 %	0.30	0.2996	24.80 %	0.00	0.30
	$r(t) \equiv \theta^*$	1.0000	1.36 %	0.06	1.0000	1.36 %	0.29	0.2260	43.27 %	0.00	0.23
	$\rho = 0$	1.0138	0.00 %	0.02	1.0136	0.02 %	0.30	0.2854	28.36 %	0.00	0.29
	$\rho = -\rho^*$	1.0138	0.00 %	0.02	1.0138	0.00 %	0.28	0.2594	34.89 %	0.00	0.26
	CIR	1.0000	1.36 %	0.00	1.0000	1.36 %	0.29	0.2868	28.01 %	0.00	0.29
Put	baseline	17.9807		0.02	16.9953		11.91	12.9538		0.08	11.97
	$r(t) \equiv r(0)$	17.8640	0.65 %	0.00	17.4290	2.55 %	11.82	12.3468	4.69 %	0.09	11.91
	$r(t) \equiv r^*$	17.8021	0.99 %	0.00	17.3829	2.28 %	11.79	12.2955	5.08 %	0.09	11.88
	$r(t) \equiv \theta^*$	14.1292	21.42 %	0.55	14.1922	16.49 %	9.42	8.8880	31.39 %	0.09	8.95
	$\rho = 0$	17.1479	4.63 %	0.30	16.7921	1.20 %	11.44	11.5804	10.60 %	0.09	11.22
	$\rho = -\rho^*$	16.4252	8.65 %	0.78	16.4694	3.09 %	10.83	10.1002	22.03 %	0.10	10.14
	CIR	17.0602	5.12 %	0.00	17.1490	0.90 %	11.68	11.6845	9.80 %	0.09	11.77
Call	baseline	16.4753		0.00	16.4701		0.10	0.1082		0.00	0.10
	$r(t) \equiv r(0)$	14.8508	9.86 %	0.28	14.8476	9.85 %	0.36	0.0874	19.22 %	0.00	0.08
	$r(t) \equiv r^*$	14.8828	9.67 %	0.27	14.8800	9.65 %	0.36	0.0875	19.13 %	0.00	0.08
	$r(t) \equiv \theta^*$	17.7205	7.56 %	0.00	17.7245	7.62 %	0.10	0.0928	14.23 %	0.00	0.10
	$\rho = 0$	15.5759	5.46 %	0.22	15.5709	5.46 %	0.31	0.0943	12.85 %	0.00	0.09
	$\rho = -\rho^*$	14.8354	9.95 %	0.68	14.8303	9.96 %	0.75	0.0906	16.27 %	0.01	0.09
	CIR	15.2841	7.23 %	0.17	15.3643	6.71 %	0.34	0.0971	10.26 %	0.01	0.18

Table 3.C.4: American options prices. $S(0) = 100$, $q = 1\%$ and $T = 5$. The immediate payoff of digital options is $\varphi(S(t)) = 1$; of (at the money) put options is $\varphi(S(t)) = (S(0) - S(t))^+$; of (at the money) call options is $\varphi(S(t)) = (S(t) - S(0))^+$. The baseline model is the Vasicek market model in (3.1) with the parameters specified in Table 3.1. The models $r(t) \equiv r(0)/r^* = -0.23\%/\theta^*$ are pure diffusive models (Black-Scholes ones) where the interest rate is constant and equal respectively to its initial value/ the yield on T-bond riskless bond/ the long-run mean. The models $\rho = 0/-\rho^*$ differ from the baseline one only with respect to the correlation which is assumed to be 0/ the opposite of the result from the calibration exercise. The CIR model is the one in (3.22) with $\gamma = 0.5$. The barrier is $L = 60$. The EEP is the Early Exercise Premium, namely the difference between the price of the American option and of its European counterpart. The deviation from parity, DFP, is defined in Equation (3.18).

European Options

payoff	model	Vanilla price	Vanilla rel. error	Down-and-out price	Down-and-out rel. error	Down-and-in price	Down-and-in rel. error
Digital	baseline	0.9950		0.3264		0.6686	
	$r(t) \equiv r(0)$	1.0131	1.82 %	0.3293	0.89 %	0.6838	2.27 %
	$r(t) \equiv r^*$	1.0121	1.72 %	0.3297	1.01 %	0.6824	2.06 %
	$r(t) \equiv \theta^*$	0.9388	5.65 %	0.3591	10.02 %	0.5797	13.30 %
	$\rho = 0$	0.9950	0.00 %	0.3323	1.81 %	0.6627	0.88 %
	$\rho = -\rho^*$	0.9950	0.00 %	0.3387	3.77 %	0.6563	1.84 %
	CIR	0.9997	0.47 %	0.3344	2.45 %	0.6653	0.49 %
Put	baseline	17.9580		0.3361		17.6219	
	$r(t) \equiv r(0)$	17.8640	0.52 %	0.3827	13.86 %	17.4813	0.80 %
	$r(t) \equiv r^*$	17.8021	0.87 %	0.3823	13.75 %	17.4198	1.15 %
	$r(t) \equiv \theta^*$	13.5771	24.40 %	0.3471	3.27 %	13.2300	24.92 %
	$\rho = 0$	16.8462	6.19 %	0.3642	8.36 %	16.4820	6.47 %
	$\rho = -\rho^*$	15.6479	12.86 %	0.3983	18.51 %	15.2496	13.46 %
	CIR	17.0602	5.00 %	0.3764	11.99 %	16.6838	5.32 %
Call	baseline	16.4712		13.6285		2.8427	
	$r(t) \equiv r(0)$	14.5723	11.53 %	11.9884	12.03 %	2.5839	9.10 %
	$r(t) \equiv r^*$	14.6117	11.29 %	12.0231	11.78 %	2.5886	8.94 %
	$r(t) \equiv \theta^*$	17.7203	7.58 %	14.7922	8.54 %	2.9281	3.00 %
	$\rho = 0$	15.3589	6.75 %	12.6014	7.54 %	2.7575	3.00 %
	$\rho = -\rho^*$	14.1599	14.03 %	11.4971	15.64 %	2.6628	6.33 %
	CIR	15.1125	8.25 %	12.4618	8.56 %	2.6507	6.75 %

Table 3.C.5: European options prices. $S(0) = 100$, $q = 1\%$ and $T = 5$. The payoff at maturity of digital options is $\varphi(S(T)) = 1$; of (at the money) put options is $\varphi(S(T)) = (S(0) - S(T))^+$; of (at the money) call options is $\varphi(S(T)) = (S(T) - S(0))^+$. The baseline model is the Vasicek market model in (3.1) with the parameters specified in Table 3.1. The models $r(t) \equiv r(0)/r^* = -0.23\%/\theta^*$ are pure diffusive models (Black-Scholes ones) where the interest rate is constant and equal respectively to its initial value/ the yield on T-bond riskless bond/ the long-run mean. The models $\rho = 0/-\rho^*$ differ from the baseline one only with respect to the correlation which is assumed to be 0/ the opposite of the result from the calibration exercise. The CIR model is the one in (3.22) with $\gamma = 0.5$. The barrier is $L = 80$.

American Options

payoff	model	Vanilla			Down-and-out			Down-and-in			DFP
		price	rel. error	EEP	price	rel. error	EEP_{DO}	price	rel. error	EEP_{DI}	
Digital	baseline	1.0138		0.02	1.0137		0.69	0.6686		0.00	0.67
	$r(t) \equiv r(0)$	1.0131	0.07 %	0.00	1.0131	0.06 %	0.68	0.6838	2.27 %	0.00	0.68
	$r(t) \equiv r^*$	1.0121	0.17 %	0.00	1.0121	0.16 %	0.68	0.6823	2.05 %	0.00	0.68
	$r(t) \equiv \theta^*$	1.0000	1.36 %	0.06	1.0000	1.35 %	0.64	0.5797	13.30 %	0.00	0.58
	$\rho = 0$	1.0138	0.00 %	0.02	1.0124	0.13 %	0.68	0.6627	0.88 %	0.00	0.66
	$\rho = -\rho^*$	1.0138	0.00 %	0.02	1.0124	0.13 %	0.67	0.6563	1.84 %	0.00	0.65
	CIR	1.0000	1.36 %	0.00	1.0000	1.35 %	0.67	0.6653	0.49 %	0.00	0.67
Put	baseline	17.9807		0.02	17.9807		17.64	17.6220		0.00	17.62
	$r(t) \equiv r(0)$	17.8640	0.65 %	0.00	17.8640	0.65 %	17.48	17.4815	0.80 %	0.00	17.48
	$r(t) \equiv r^*$	17.8021	0.99 %	0.00	17.8021	0.99 %	17.42	17.4283	1.10 %	0.01	17.43
	$r(t) \equiv \theta^*$	14.1292	21.42 %	0.55	14.1292	21.42 %	13.78	13.2435	24.85 %	0.01	13.24
	$\rho = 0$	17.1479	4.63 %	0.30	17.1479	4.63 %	16.78	16.4821	6.47 %	0.00	16.48
	$\rho = -\rho^*$	16.4252	8.65 %	0.78	16.4252	8.65 %	16.03	15.2502	13.46 %	0.00	15.25
	CIR	17.0602	5.12 %	0.00	16.7843	6.65 %	16.41	16.6840	5.32 %	0.00	16.41
Call	baseline	16.4753		0.00	13.6321		0.00	2.8427		0.00	0.00
	$r(t) \equiv r(0)$	14.7508	10.47 %	0.18	12.2333	10.26 %	0.24	2.5839	9.10 %	0.00	0.07
	$r(t) \equiv r^*$	14.7828	10.27 %	0.17	12.2616	10.05 %	0.24	2.5885	8.94 %	0.00	0.07
	$r(t) \equiv \theta^*$	17.7205	7.56 %	0.00	14.7923	8.51 %	0.00	2.9282	3.01 %	0.00	0.00
	$\rho = 0$	15.4759	6.07 %	0.12	12.7928	6.16 %	0.19	2.7575	3.00 %	0.00	0.07
	$\rho = -\rho^*$	14.5354	11.77 %	0.38	11.9014	12.70 %	0.40	2.6628	6.33 %	0.00	0.03
	CIR	15.2841	7.23 %	0.17	13.2156	3.06 %	0.75	2.6508	6.75 %	0.00	0.58

Table 3.C.6: American options prices. $S(0) = 100$, $q = 1\%$ and $T = 5$. The immediate payoff of digital options is $\varphi(S(t)) = 1$; of (at the money) put options is $\varphi(S(t)) = (S(0) - S(t))^+$; of (at the money) call options is $\varphi(S(t)) = (S(t) - S(0))^+$. The baseline model is the Vasicek market model in (3.1) with the parameters specified in Table 3.1. The models $r(t) \equiv r(0)/r^* = -0.23\%/\theta^*$ are pure diffusive models (Black-Scholes ones) where the interest rate is constant and equal respectively to its initial value/ the yield on T-bond riskless bond/ the long-run mean. The models $\rho = 0/-\rho^*$ differ from the baseline one only with respect to the correlation which is assumed to be 0/ the opposite of the result from the calibration exercise. The CIR model is the one in (3.22) with $\gamma = 0.5$. The barrier is $L = 80$. The EEP is the Early Exercise Premium, namely the difference between the price of the American option and of its European counterpart. The deviation from parity, DFP, is defined in Equation (3.18).

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